Notes on double pole 3pt integral

Consider length-3 subcycle integral,

$$Z_{\mathrm{DP3}} = \int dz_2 dz_3 \Omega(z_{12}, \beta, \tau)\Omega(z_{23}, \xi, \tau)\Omega(z_{31}, \gamma, \tau). \tag{1}$$

We use the Fay identity to reduce the length-3 subcycle as length-2 ones,

$$\Omega(z_{12}, \beta, \tau)\Omega(z_{23}, \xi, \tau)\Omega(z_{31}, \gamma, \tau) = \Big(\Omega(z_{12}, \beta+\gamma, \tau)\Omega(z_{32}, \gamma, \tau) - \Omega(z_{31}, \beta+\gamma, \tau)\Omega(z_{32}, -\beta, \tau)\Big)\Omega(z_{23}, \xi, \tau) \tag{2}$$

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where we have used $\Omega(-z, -\eta, \tau) = -\Omega(z, \eta, \tau)$ (Is this identity true?[1]). In the first term on the RHS, there is a length-2 subcycle $\Omega(z_{32}, \cdots)\Omega(z_{23}, \cdots)$ and a branch $\Omega(z_{12}, \cdots)$ planted on it through z_2. Inspired by the experience in dealing with tree-level analogs, we try to use the vanishing of the total derivative of z_3 to break the subcycle $\Omega(z_{32}, \cdots)\Omega(z_{23}, \cdots)$. Following the steps in the Oliver's draft on double pole 2pt integral, first we rewrite the subcycle via (2.21) in the Oliver's proceedings note

$$\Omega(z_{23}, \xi, \tau)\Omega(z_{32}, \gamma, \tau) = \partial_3\Omega(z_{23}, \xi-\gamma, \tau) + \Omega(z_{23}, \xi-\gamma, \tau)(\hat{g}^{(1)}(\gamma, \tau) - \hat{g}^{(1)}(\xi, \tau)). \tag{3}$$

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(It seems this equality holds locally no matter what the remaining is in the integrand, for example, $\Omega(z_{12}, \beta+\gamma, \tau)$ here. Is it true?) Then, eliminate $\partial_3\Omega(z_{23}, \cdots)$ via (2.12) in the Oliver's proceedings note,

$$\partial_3\Omega(z_{23}, \xi-\gamma, \tau) = \partial_\xi\Omega(z_{23}, \xi-\gamma, \tau) + \Big(\hat{g}^{(1)}(\xi-\gamma, \tau) - f^{(1)}(z_{23}, \tau)\Big)\Omega(z_{23}, \xi-\gamma, \tau), \tag{4}$$

with $f^{(1)}(z_{23}, \tau)$ to be further dealt with (Again, does this equality hold locally?). The above two equations have no analogs in the tree-level cases.

Now, we consider integration-by-parts to relate $\partial_3\Omega(z_{23}, \xi-\gamma, \tau)$ and $f^{(1)}(z_{23}, \tau)\Omega(z_{23}, \xi-\gamma, \tau)$,

$$\partial_3\Omega(z_{23}, \xi-\gamma, \tau)\Omega(z_{12}, \beta+\gamma, \tau)\mathrm{KN} \overset{\mathrm{IBP}}{\cong} \Big(s_{13}f^{(1)}(z_{13}, \tau) + s_{23}f^{(1)}(z_{23}, \tau)\Big)\Omega(z_{23}, \xi-\gamma, \tau)\Omega(z_{12}, \beta+\gamma, \tau)\mathrm{KN}\,, \tag{5}$$

where $\Omega(z_{12}, \beta+\gamma, \tau)$ doesn't contain z_3 and hence we only need to consider the derivative of KN. Different from the 2pt case, there are two terms appearing, with the first one $s_{13}f^{(1)}(z_{13}, \tau)$ very annoying.

[1] What really puzzles me is $\partial_{z_2}\Omega(z_{23}, \xi, \tau)$. According to differential link rules, it seems we have $\partial_{z_2}\Omega(z_{23}, \xi, \tau) = -\partial_{z_3}\Omega(z_{23}, \xi, \tau)$. However, we can write $\Omega(z_{23}, \xi, \tau)$ as $-\Omega(z_{32}, -\xi, \tau)$ at first, then $\partial_{z_2}\Big(-\Omega(z_{32}, -\xi, \tau)\Big) = -\partial_{z_2}\Omega(z_{32}, -\xi, \tau) = \partial_{z_3}\Omega(z_{32}, -\xi, \tau)$, which leads to $\partial_{z_2}\Omega(z_{23}, \xi, \tau) = \partial_{z_3}\Omega(z_{32}, -\xi, \tau)$. Sorry for my shortness of related background knowledge.

According to the above two simultaneous equations, we can get

$$\begin{aligned} \partial_3\Omega(z_{23},\xi-\gamma,\tau) &\cong \Big[\frac{s_{23}}{1+s_{23}}\Big(\partial_\xi+\hat{g}^{(1)}(\xi-\gamma,\tau)\Big)+\frac{s_{13}}{1+s_{23}}f^{(1)}(z_{13},\tau)\Big]\Omega(z_{23},\xi-\gamma,\tau)\,, \\ s_{23}f^{(1)}(z_{23},\tau)\Omega(z_{23},\xi-\gamma,\tau) &\cong \Big[\frac{1}{1+s_{23}}\Big(\partial_\xi+\hat{g}^{(1)}(\xi-\gamma,\tau)\Big)-\frac{s_{13}}{1+s_{23}}f^{(1)}(z_{13},\tau)\Big]\Omega(z_{23},\xi-\gamma,\tau)\,. \end{aligned} \tag{6}$$

With this solution, we can only write the first term on the RHS of (2) as

$$\begin{aligned} &\Omega(z_{12},\beta+\gamma,\tau)\Omega(z_{32},\gamma,\tau)\Omega(z_{23},\xi,\tau) \\ \overset{\text{IBP}}{\cong}\ &\Big[\frac{s_{23}}{1+s_{23}}\Big(\partial_\xi+\hat{g}^{(1)}(\xi-\gamma,\tau)\Big)+\frac{s_{13}}{1+s_{23}}f^{(1)}(z_{13},\tau)+\Big(\hat{g}^{(1)}(\gamma,\tau)-\hat{g}^{(1)}(\xi,\tau)\Big)\Big] \\ &\times\Omega(z_{23},\xi-\gamma,\tau)\Omega(z_{12},\beta+\gamma,\tau)\,, \end{aligned} \tag{7}$$

with a new length-3 subcycle $f^{(1)}(z_{13},\tau)\Omega(z_{12},\beta+\gamma,\tau)\Omega(z_{23},\xi-\gamma,\tau)$ to be further dealt with.

Similarly, for the second term on the RHS of (2), we have

$$\begin{aligned} &-\Omega(z_{31},\beta+\gamma,\tau)\Omega(z_{23},\xi,\tau)\Omega(z_{32},-\beta,\tau) \\ \overset{\text{IBP}}{\cong}\ &\Big[\frac{s_{23}}{1+s_{23}}\Big(\partial_\xi+\hat{g}^{(1)}(-\xi-\beta,\tau)\Big)+\frac{s_{12}}{1+s_{23}}f^{(1)}(z_{12},\tau)+(\hat{g}^{(1)}(-\beta,\tau)-\hat{g}^{(1)}(\xi,\tau))\Big] \\ &\times\Omega(z_{31},\beta+\gamma,\tau)\Omega(z_{23},\xi+\beta,\tau)\,, \end{aligned} \tag{8}$$

with $f^{(1)}(z_{12},\tau)\Omega(z_{31},\beta+\gamma,\tau)\Omega(z_{23},\xi+\beta,\tau)$ to be further dealt with.

I have tried many methods to eliminate the new kind of length-3 subcycles $f^{(1)}(\cdots)\Omega(\cdots)\Omega(\cdots)$ including transforming them as the form $\Omega(z_{12},\cdots)\Omega(z_{23},\cdots)\Omega(z_{31},\cdots)$ but all failed. New insights are needed.

As a desperate try, I made use of the vanishing of the total derivative of other punctures, for example, z_2 for the first term on the RHS of (2). I rewrote $\Omega(z_{23},\xi,\tau)\Omega(z_{32},\gamma,\tau)$ as $\Omega(z_{32},-\xi,\tau)\Omega(z_{23},-\gamma,\tau)$ and similar to (3) we had

$$\Omega(z_{32},-\xi,\tau)\Omega(z_{23},-\gamma,\tau)=\partial_2\Omega(z_{32},-\xi+\gamma,\tau)+\Omega(z_{32},-\xi+\gamma,\tau)(\hat{g}^{(1)}(-\gamma,\tau)-\hat{g}^{(1)}(-\xi,\tau)), \tag{9}$$

with

$$\partial_2\Omega(z_{32},-\xi+\gamma,\tau)=\partial_\beta\Omega(z_{32},-\xi+\gamma,\tau)+\Big(\hat{g}^{(1)}(-\xi+\gamma,\tau)-f^{(1)}(z_{32},\tau)\Big)\Omega(z_{32},-\xi+\gamma,\tau). \tag{10}$$

The IBP calculation became tougher but I finished it by brute-force,

$$\begin{aligned} \partial_2\Omega(z_{32},-\xi-\gamma,\tau)\Omega(z_{12},\beta+\gamma,\tau)\text{KN} \overset{\text{IBP}}{\cong}\ &\Big(s_{12}f^{(1)}(z_{12},\tau)+s_{23}f^{(1)}(z_{32},\tau)\Big)\Omega(z_{32},-\xi-\gamma,\tau)\Omega(z_{12},\beta+\gamma,\tau)\text{KN} \\ &-\Omega(z_{32},-\xi-\gamma,\tau)\partial_2\Omega(z_{12},\beta+\gamma,\tau)\text{KN}\,, \end{aligned} \tag{11}$$

with

$$\partial_2\Omega(z_{12},\beta+\gamma,\tau)=\partial_\beta\Omega(z_{12},\beta+\gamma,\tau)+\Big(\hat{g}^{(1)}(\beta+\gamma,\tau)-f^{(1)}(z_{12},\tau)\Big)\Omega(z_{12},\beta+\gamma,\tau). \tag{12}$$

According to the above three simultaneous equations, I solved $\partial_2\Omega(z_{32},-\xi-\gamma,\tau), \partial_2\Omega(z_{12},\beta+\gamma,\tau)$ and $f^{(1)}(z_{32},\tau)$ at the price of introducing $f^{(1)}(z_{12},\tau)$ and finally expressed the first term on the RHS of (2) as

$$\begin{aligned}
&\Omega(z_{12},\beta+\gamma,\tau)\Omega(z_{32},\gamma,\tau)\Omega(z_{23},\xi,\tau) \\
\overset{\text{IBP}}{\cong}\quad &\frac{s_{23}}{1+s_{23}}\Big[\Big(\partial_\gamma+\hat{g}^{(1)}(-\xi-\gamma,\tau)-\hat{g}^{(1)}(-\xi,\tau)\Big)\Omega(z_{23},\xi+\gamma,\tau)\Big]\Omega(z_{12},\beta+\gamma,\tau) \\
&-\frac{1}{1+s_{23}}\Big[\Big(\partial_\gamma+\hat{g}^{(1)}(\beta+\gamma,\tau)+\hat{g}^{(1)}(-\xi,\tau)\Big)\Omega(z_{12},\beta+\gamma,\tau)\Big]\Omega(z_{23},\xi+\gamma,\tau) \\
&+\Big[\frac{1+s_{12}}{1+s_{23}}f^{(1)}(z_{12},\tau)+\hat{g}^{(1)}(\gamma,\tau)\Big]\Omega(z_{12},\beta+\gamma,\tau)\Omega(z_{23},\xi+\gamma,\tau)\,.
\end{aligned} \tag{13}$$

We can see we have introduced another equality (13) in addition to (7) but another new variable came out as well. There are $f^{(1)}(z_{12},\tau)$ factor in both (8) and (13), but many things are left to be done to eliminate this factor because the two equalities are not really identities but only hold on the support of IBP. I will read Oliver's proceedings note more carefully, which mainly focuses on dealing with the subcycles of $f^{(k)}$ factors and check if I can learn some skills to handle the present problems.