1 Lecture 1: Introduction

We will set up the terminologies that will be used in future lectures.

Definition 1.1 (Board & Winning Lines)

We define the **board** X to be a finite set unless stated otherwise.
Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be the family of **winning lines** or **winning sets**.

This family can be represented as a hypergraph on X, where elements of X are the vertices of the hypergraph while elements of $\mathcal{F}$ are the edges of the hypergraph.

Often, all $A \in \mathcal{F}$ have the same size n, i.e. $|A| = n$ for all $A \in \mathcal{F}$. In this case, $\mathcal{F}$ is a n-graph (a graph where all edges contain n vertices).

For this course, we will cover two types of games:

Definition 1.2 (Types of Games)

Strong Games
Two players takes turns to play until a player occupies all elements of some $A \in \mathcal{F}$. If X is filled with no winning line occupied by either player, then the game is considered as a draw.
Maker-breaker Games
Two players take turn to play. Player 1 wins if they occupy all elements of some $A \in \mathcal{F}$ while player 2 wins if player 1 is unable to do so.

Note that by definition, maker-breaker games cannot end in a draw. We say that a game is a P1 win if player 1 has a winning strategy and likewise for player 2.

Example 1.3 (Games)

1. Normal tic-tac-toe (insert drawing, Sri) This game is well-known to be a draw.
2. 3D tic-tac-toe
   - on a $3 \times 3 \times 3$ board (insert drawing, Sri)
     This game is known to be a P1 win.
   - on a $4 \times 4 \times 4$ board (insert drawing, Sri)
     This version is still known to be a P1 win but the explicit winning strategy is very complicated.

Let's take a look at the generalisation of Tic-tac-toe in higher dimensions, that is the Hales-Jewett game of the $[n]^d-$game.

Definition 1.4 (Hales-Jewett / $[n]^d$−game)

Let the board $X = [n]^d = \{1, 2, \dots, n\}^d = \{(a_1, a_2, \dots, a_d) \mid a_1, a_2, \dots, a_d \in [n]\}$.
The winning lines are the two type of lines defined as follows:

- A combinatorial line is a set of n points of the form:

$$\left\{(x_1, x_2, \dots, x_d) \;\middle|\; \begin{array}{l} x_i = x_j,\ \forall i,\ j \in I \\ x_i = a_i,\ \forall i \notin I \end{array}\right\}$$

  where $I \subseteq [d]$, $I \neq \phi$ and $a_i \in [n]$ for each $i \notin I$.

- A line is a set of the form:

$$\left\{(x_1, x_2, \dots, x_d) \;\middle|\; \begin{array}{l} x_i = x_j,\ \forall i,\ j \in I \\ x_i = x_j,\ \forall i,\ j \in J \\ x_i = a_i,\ \forall i \notin I \cup J \end{array}\right\}$$

  where $I,\ J \subseteq [d]$, $I \cup J \neq \phi$, $I \cap J = \phi$ and $a_i \in [n]$ for each $i \notin I \cup J$.

Here, we refer I as the active coordinates.

Example 1.5 (Lines)