# Positional Games

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An extremely far-reaching generalisation of the simple game of Tic-Tac-Toe takes the following form. We have an arbitrary finite set as the board of the game, and some of its subsets are designated as winning sets. Two players alternately play on an unoccupied cell of the board, and the first player to complete a winning set is the winner — otherwise, the game ends in a draw. This is called the strong positional game (or strong game in short).

In the weak version, also called the Maker-Breaker version, the second player's aim is not to occupy a winning set but just to prevent the first player from doing so. For both versions, the interest is in the general results about the games. One may also focus on the more natural multidimensional generalisation of Tic-Tac-Toe: the Hales-Jewett game. Very roughly speaking, a fair amount is known for Maker-Breakers while almost nothing is known for strong games.

The aim of the course is to provide a brief introduction to this combinatorial discipline to those with interest and basic knowledge in combinatorics. We will be covering some very elegant results in the field, as well as many of its standing challenges and open problems. Among others, these include the Strategy Stealing argument, the Pairing Strategies, the Erdös-Selfridge Theorem, the Pairing Conjecture and the Neighbourhood Conjecture.

## Contents

# 1 Lecture 1: Introduction

We will set up the terminologies that will be used in future lectures.

### Definition 1.1 (Board & Winning Lines)

We define the **board** $X$ to be a finite set unless stated otherwise.
Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be the family of **winning lines** or **winning sets**.

This family can be represented as a hypergraph on $X$, where elements of $X$ are the vertices of the hypergraph while elements of $\mathcal{F}$ are the edges of the hypergraph.

Often, all $A \in \mathcal{F}$ have the same size $n$, i.e. $|A| = n$ for all $A \in \mathcal{F}$. In this case, $\mathcal{F}$ is a n-graph (a graph where all edges contain $n$ vertices).

For this course, we will cover two types of games:

### Definition 1.2 (Types of Games)

**Strong Games**
Two players takes turns to play until a player occupies all elements of some $A \in \mathcal{F}$. If $X$ is filled with no winning line occupied by either player, then the game is considered as a draw.
**Maker-breaker Games**
Two players take turn to play. Player 1 wins if they occupy all elements of some $A \in \mathcal{F}$ while player 2 wins if player 1 is unable to do so.

Note that by definition, maker-breaker games cannot end in a draw. We say that a game is a P1 win if player 1 has a winning strategy and likewise for player 2.

### Example 1.3 (Games)

1. Normal tic-tac-toe (insert drawing, Sri) This game is well-known to be a draw.
2. 3D tic-tac-toe
   - on a $3 \times 3 \times 3$ board (insert drawing, Sri)
     This game is known to be a P1 win.
   - on a $4 \times 4 \times 4$ board (insert drawing, Sri)
     This version is still known to be a P1 win but the explicit winning strategy is very complicated.

Let's take a look at the generalisation of Tic-tac-toe in higher dimensions, that is the Hales-Jewett game of the $[n]^d-$game.

### Definition 1.4 (Hales-Jewett / $[n]^d$−game)

Let the board $X = [n]^d = \{1, 2, \ldots, n\}^d = \{(a_1, a_2, \ldots, a_d) \mid a_1, a_2, \ldots, a_d \in [n]\}$.
The winning lines are the two type of lines defined as follows:

- A combinatorial line is a set of $n$ points of the form:

$$\left\{(x_1, x_2, \ldots, x_d) \;\middle|\; \begin{array}{l} x_i = x_j,\ \forall i,\ j \in I \\ x_i = a_i,\ \forall i \notin I \end{array}\right\}$$

  where $I \subseteq [d]$, $I \neq \phi$ and $a_i \in [n]$ for each $i \notin I$.

- A line is a set of the form:

$$\left\{(x_1, x_2, \ldots, x_d) \;\middle|\; \begin{array}{l} x_i = x_j,\ \forall i,\ j \in I \\ x_i = x_j,\ \forall i,\ j \in J \\ x_i = a_i,\ \forall i \notin I \cup J \end{array}\right\}$$

  where $I,\ J \subseteq [d]$, $I \cup J \neq \phi$, $I \cap J = \phi$ and $a_i \in [n]$ for each $i \notin I \cup J$.

Here, we refer $I$ as the active coordinates.

### Example 1.5 (Lines)