- Pseudo-random Numbers (Chapter 11)
- What use are random sequences?
- What *are* "random sequences"?
- Pseudo-random sequences.
- How to get one.
- Relevant Java library classes and methods.
- Random permutations.

- Simulations
- Random algorithms
- Cryptography:
  - Choosing random keys
  - Generating streams of random bits (e.g., SSL xor's your data with a regeneratable, pseudo-random bit stream that only you and the recipient can generate).
- And, of course, games

occur with equal frequency"?

  - Like 1, 2, 3, 4, . . . ?
- Well then, how about: "an unpredictable sequence where all numbers occur with equal frequency?"
  - Like 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 3, 4, 4, 0, 1, 1, 1,. . . ?
- Besides, what is wrong with 0, 0, 0, 0, . . . anyway? Can't that occur by random selection?

is difficult for a computer (or human) to produce.

- For most purposes, need only a sequence that satisfies certain statistical properties, even if deterministic.
- Sometimes (e.g., cryptography) need sequence that is *hard* or *impractical* to predict.
- *Pseudo-random sequence:* deterministic sequence that passes some given set of statistical tests.
- For example, look at lengths of *runs:* increasing or decreasing contiguous subsequences.
- Unfortunately, statistical criteria to be used are quite involved. For details, see Knuth.

- Not as easy as you might think.
- Seemingly complex jumbling methods can give rise to bad sequences.
- *Linear congruential method* is a simple method used by Java:

$$X_0 = \textit{arbitrary seed}$$
$$X_i = (aX_{i-1} + c) \bmod m, \quad i > 0$$

- Usually, m is large power of 2.
- For best results, want $a \equiv 5 \bmod 8$, and a, c, m with no common factors.
- This gives generator with a *period of* m (length of sequence before repetition), and reasonable *potency* (measures certain dependencies among adjacent X_i.)
- Also want bits of a to "have no obvious pattern" and pass certain other tests (see Knuth).

cryptographically secure.

a, c, m even.

- Obvious patterns. E.g., just using lower 3 bits of X_i in Java's 48-bit generator, to get integers in range 0 to 7. By properties of modular arithmetic,

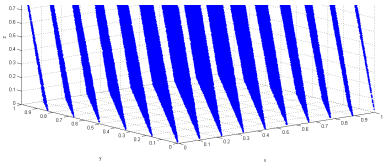
$$\begin{aligned} X_i \bmod 8 &= (25214903917 X_{i-1} + 11 \bmod 2^{48}) \bmod 8 \\ &= (5(X_{i-1} \bmod 8) + 3) \bmod 8 \end{aligned}$$

so we have a period of 8 on this generator; sequences like

$$0, 1, 3, 7, 1, 2, 7, 1, 4, \ldots$$

are impossible. This is why Java doesn't give you the raw 48 bits.

- The infamous IBM generator RANDU: $c = 0$, $a = 65539$, $m = 2^{31}$.
- When RANDU is used to make 3D points: $(X_i/S, X_{i+1}/S, X_{i+2}/S)$, where S scales to a unit cube, ...
- ... points will be arranged in parallel planes with voids between. So "random points" won't ever get near many points in the cube:



$$X_n = \begin{cases} \textit{arbitary value}, & n < 55 \\ (X_{n-24} + X_{n-55}) \bmod 2^e, & n \geq 55 \end{cases}$$

- Other choices than 24 and 55 possible.
- This one has period of $2^f(2^{55} - 1)$, for some $f < e$.
- Simple implementation with circular buffer:

```
    i = (i+1) % 55;
    X[i] += X[(i+31) % 55];  // Why +31 (55-24) instead
of -24?
    return X[i];   /* modulo 2^32 */
```

- where `X[0 .. 54]` is initialized to some "random" initial seed values.

- The simple form of linear congruential generators means that one can predict future values after seeing relatively few outputs.
- Not good if you want *unpredictable* output (think on-line games involving money or randomly generated keys for encrypting your web traffic.)
- A *cryptographic pseudo-random number generator (CPRNG)* has the properties that
  - Given k bits of a sequence, no polynomial-time algorithm can guess the next bit with better than 50% accuracy.
  - Given the current state of the generator, it is also infeasible to reconstruct the bits it generated in getting to that state.

- Start with a good *block cipher*—an encryption algorithm that encrypts blocks of N bits (not just one byte at a time as for Enigma). AES is an example.
- As a seed, provide a key, K, and an initialization value I.
- The j^{th} pseudo-random number is now $E(K, I+j)$, where $E(x, y)$ is the encryption of message y using key x.

above methods in range (e.g.) 0 to 2^{48}, how to get uniform random integers in range 0 to $n-1$?

- If $n = 2^k$, is easy: use top k bits of next X_i (bottom k bits not as "random")
- For other n, be careful of slight biases at the ends. For example, if we compute $X_i/(2^{48}/n)$ using all integer division, and if $(2^{48}/n)$ gets rounded down, then you can get n as a result (which you don't want).
- If you try to fix that by computing $(2^{48}/(n-1))$ instead, the probability of getting $n-1$ will be wrong.

evenly divide 2^{48}, Java throws out values after the largest multiple of n that is less than 2^{48}:

```
/** Random integer in the range 0 .. n-1, n>0.
*/
int nextInt(int n) {
  long X = next random long (0 ≤ X < 2^48);
  if (n is 2^k for some k)
      return top k bits of X;

  int MAX = largest multiple of n that is < 2^48;
  while (X_i >= MAX)
       X = next random long (0 ≤ X < 2^48);
  return X_i / (MAX/n);
}
```


to U)?

- To get random `float`, x in range $0 \le x < d$, compute

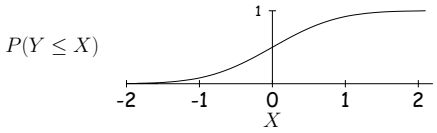
```
return d*nextInt(1<<24) / (1<<24);
```

- Random `double` a bit more complicated: need two integers to get enough bits.

```
long bigRand = ((long) nextInt(1<<26) << 27)
+ (long) nextInt(1<<27);
return d * bigRand / (1L << 53);
```

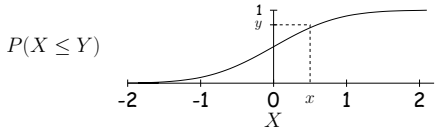

distribution function, and want to get random numbers that are distributed according to that distribution. How can we do this?

- Example: the normal distribution:



- Curve is the desired probability distribution. $P(Y \le X)$ is the probability that random variable Y is $\le X$.

and the corresponding x will be distributed according to P.



- Class `java.util.Random`: a random number generator with constructors:

  Random() generator with "random" seed (based on time).

  Random(seed) generator with given starting value (reproducible).

- Methods

  next(k) k-bit random integer

  nextInt(n) `int` in range $[0..n)$.

  nextLong() random 64-bit integer.

  nextBoolean(), nextFloat(), nextDouble() Next random values of other primitive types.

  nextGaussian() normal distribution with mean 0 and standard deviation 1 ("bell curve").

- `Collections.shuffle(`L, R`)` for list R and `Random` R permutes L randomly (using R).

sequence.

- Obvious dumb technique for sorting N-element list:
  - Generate N random numbers
  - Attach each to one of the list elements
  - Sort the list using random numbers as keys.
- Can do quite a bit better:

```
  void shuffle(List L, Random R) {
     for (int i = L.size(); i > 0; i -= 1)
         swap element i-1 of L with element R.nextInt(i)
of L;
  }
```

- Example:

$4 \Longleftrightarrow 2$ | A♣ | 3♡ | 2♡ | A♡ | 3♣ | 2♣ | $1 \Longleftrightarrow 0$ | 3♡ | 2♡ | A♣ | A♡ | 3♣ |

items from list:

```
  /** Permute L and return sublist of K>=0 randomly
   *  chosen elements of L, using R as random source.
*/
  List select(List L, int k, Random R) {
    for (int i = L.size(); i+k > L.size(); i -= 1)
      swap element i-1 of L with element
        R.nextInt(i) of L;
    return L.sublist(L.size()-k, L.size());
  }
```

- Not terribly efficient for selecting random sequence of K distinct integers from $[0..N)$, with $K \ll N$.


```
/** Random sequence of K
distinct integers
 *  from 0..N-1, 0<=K<=N. */
IntList selectInts(int N, int
K, Random R)
{
  IntList S = new IntList();

  for (int i = N-K; i < N; i +=
1) {
    // All values in S are < i
    int s = R.randInt(i+1); //
0 <= s <= i < N
    if (s == S.get(j) for some
j)
      // Insert value i (which
can't be there
      // yet) after the s
(i.e., at a random
      // place other than the
front)
      S.add(j+1, i);
    else
      // Insert random value s
at front
      S.add(0, s);
  }
```

Example

i	s	S
5	4	[4]
6	2	[2, 4]
7	5	[5, 2, 4]
8	5	[5, 8, 2, 4]
9	4	[5, 8, 2, 4, 9]

```
selectRandomIntegers(10, 5, R)
```