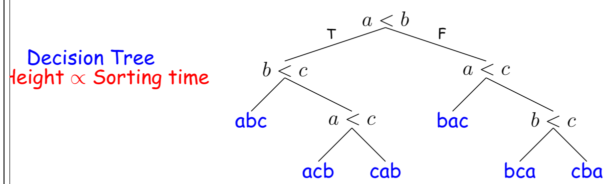
- Lower bounds on sorting by comparison
- Distribution counting, radix sorts

Readings: Today: *DS(IJ)*, Chapter 8; Next topic: Chapter 9.

compare them, then sorting must take $\Omega(N \lg N)$.

- Basic idea: there are $N!$ possible ways the input data could be scrambled.
- Therefore, your program must be prepared to do $N!$ different combinations of data-moving operations.
- Therefore, there must be $N!$ possible combinations of outcomes of all the **if**-tests in your program, since those determine what move gets moved where (we're assuming that comparisons are 2-way).



possible different outcomes for k **if**-tests is 2^k.

- Thus, need enough tests so that $2^k \geq N!$, which means $k \in \Omega(\lg N!)$.
- Using Stirling's approximation,

$$N! \in \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \left(1 + \Theta\left(\frac{1}{N}\right)\right),$$

$$\lg(N!) \in 1/2(\lg 2\pi + \lg N) + N \lg N - N \lg e + \lg\left(1 + \Theta\left(\frac{1}{N}\right)\right)$$

$$= \Theta(N \lg N)$$

- This tells us that k, the worst-case number of tests needed to sort N items by comparison sorting, is in $\Omega(N \lg N)$: there must be cases where we need (some multiple of) $N \lg N$ comparisons to sort N things.

- For example, how can we sort a set of N integer keys whose values range from 0 to kN, for some small constant k?
- One technique: put the integers into N buckets, with an integer p going to bucket $\lfloor p/k \rfloor$.
- At most k keys per bucket, so catenate and use insertion sort, which will now be fast.
- E.g., $k = 2, N = 10$:

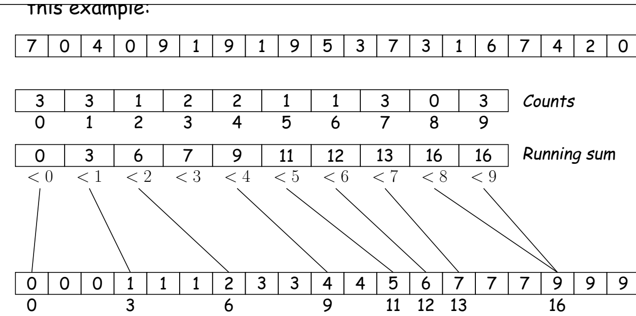
```
Start:
  14  3   10  13  4   2   19  17  0   9
In buckets:
  | 0 | 3  2 |  4 |    |  9  | 10 | 13 | 14 |  17 |  19  |
```

- Now insertion sort is fast. Putting in buckets takes time $\Theta(N)$, and insertion sort takes $\Theta(kN)$. When k is fixed (constant), we have sorting in time $\Theta(N)$.

< 1, < 2, etc.

- If M_p = #items with value $< p$, then in sorted order, the j^{th} item with value p must be item $\#M_p + j$.
- Gives another *linear-time* algorithm.

This example:



- "Counts" line gives # occurrences of each key.
- "Running sum" gives cumulative count of keys $<$ each value...
- ...which tells us where to put each key:

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Next positions*
0	1	2	3	4	5	6	7	8	9	

																			Output
0			3			6			9			12			15			18	

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
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1	3	6	7	9	11	12	14	16	16	*Next positions*
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0	0		1	1					4				7			9	9	9	Output
0			3			6			9			12			15			18	

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0	0		1	1					4		5		7			9	9	9	Output
0			3			6			9			12			15			18	

Counts	3	3	1	2	2	1	1	3	0	3
	0	1	2	3	4	5	6	7	8	9
Running sum of Counts	0	3	6	7	9	11	12	13	16	16
	0	1	2	3	4	5	6	7	8	9
Next positions	2	5	6	8	10	12	12	14	16	19
	0	1	2	3	4	5	6	7	8	9

Output	0	0		1	1			3		4		5		7			9	9	9
	0			3			6			9			12			15			18

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	0	1	2	3	4	5	6	7	8	9
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	0	1	2	3	4	5	6	7	8	9
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	0	1	2	3	4	5	6	7	8	9

Output	0	0		1	1			3		4		5		7	7		9	9	9
	0			3			6			9			12			15			18

Counts	3	3	1	2	2	1	1	3	0	3
	0	1	2	3	4	5	6	7	8	9
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	0			3			6			9			12			15			18

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	0	1	2	3	4	5	6	7	8	9
Running sum of Counts	0	3	6	7	9	11	12	13	16	16
	0	1	2	3	4	5	6	7	8	9
Next positions	2	6	6	9	10	12	13	16	16	19
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Output	0	0		1	1	1		3	3	4		5	6	7	7	7	9	9	9
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0	0	0	1	1	1	2	3	3	4	4	5	6	7	7	7	9	9	9	Output
0			3			6			9			12			15			18	

- Can use distribution counting for each digit.
- Can work either right to left (LSD radix sort) or left to right (MSD radix sort)
- LSD radix sort is venerable: used for punched cards.

Initial: set, cat, cad, con, bat, can, be, let, bet

Pass 1 (by char #2)

'␣'	'd'	'n'	't'
			bet
			let
			bat
		can	cat
be	cad	con	set

be, cad, con, can, set, cat, bat, let, bet

Pass 2 (by char #1)

'a'	'e'	'o'
bat	bet	
cat	let	
can	set	
cad	be	con

cad, can, cat, bat, be, set, let, bet, c

Pass 3 (by char #0)

'b'	'c'	'l'	's'
	con		
bet	cat		
be	can		
bat	cad	let	set

bat, be, bet, cad, can, cat, con, let, set

each step separate

- But, can stop processing 1-element lists

A	posn
⋆ set, cat, cad, con, bat, can, be, let, bet	0
⋆ bat, be, bet / cat, cad, con, can / let / set	1
bat / ⋆ be, bet / cat, cad, con, can / let / set	2
bat / be / bet / ⋆ cat, cad, con, can / let / set	1
bat / be / bet / ⋆ cat, cad, can / con / let / set	2
bat / be / bet / cad / can / cat / con / let / set	

size of the key data.

- Have measured other sorts as function of #records.
- How to compare?
- To have N different records, must have keys at least $\Theta(\lg N)$ long [why?]
- Furthermore, comparison actually takes time $\Theta(K)$ where K is size of key in worst case [why?]
- So $N \lg N$ comparisons really means $N(\lg N)^2$ operations.
- While radix sort would take $B = N \lg N$ time with minimal-length keys.
- On the other hand, must work to get good constant factors with radix sort.

read in inorder.

- Need *balance* to really use for sorting [next topic].
- Given balance, same performance as heapsort: N insertions in time $\lg N$ each, plus $\Theta(N)$ to traverse, gives

$$\Theta(N + N \lg N) = \Theta(N \lg N)$$

where k is maximum amount data is displaced from final position.

  - Good for small datasets or almost ordered data sets.
- Quicksort: $\Theta(N \lg N)$ with good constant factor if data is not pathological. Worst case $O(N^2)$.
- Merge sort: $\Theta(N \lg N)$ guaranteed. Good for external sorting.
- Heapsort, treesort with guaranteed balance: $\Theta(N \lg N)$ guaranteed.
- Radix sort, distribution sort: $\Theta(B)$ (number of bytes). Also good for external sorting.