neering:

> "An engineer is someone who can do for a dime what any fool can do for a dollar."

- Cost can mean
  - *Operational cost* (for programs, time to run, space requirements).
  - *Development costs*: How much engineering time? When delivered?
  - *Maintenance costs*: Upgrades, bug fixes.
  - *Costs of failure*: How robust? How safe?
- Is this program *fast enough*? Depends on:
  - *For what purpose;*
  - *For what input data.*
- How much *space* (memory, disk space)?
  - Again depends on what input data.

so), and find and print the 20 most frequently used words, together with counts of how often they occur.

- Solution 1 (Knuth): Heavy-Duty data structures
  - Hash Trie implementation, randomized placement, pointers galore, several pages long.
- Solution 2 (Doug McIlroy): UNIX shell script:

```
tr -c -s '[:alpha:]' '[\n*]' < FILE | \
  sort | \
  uniq -c | \
  sort -n -r -k 1,1 | \
  sed 20q
```

- Which is better?
  - #1 is much faster,

- In very many cases, almost anything will do: Keep It Simple.

  - You can do this at home:

```
time java FindPrimes 1000
```

  - Advantages: easy to measure, meaning is obvious.
  - Appropriate where time is critical (real-time systems, e.g.).
  - Disadvantages: applies only to specific data set, compiler, machine, etc.
- *Dynamic statement counts* of # of times statements are executed:
  - Advantages: more general (not sensitive to speed of machine).
  - Disadvantages: doesn't tell you actual time, still applies only to specific data sets.
- *Symbolic execution times:*
  - That is, *formulas* for execution times as functions of input size.

approximate, may tell very little about actual time.

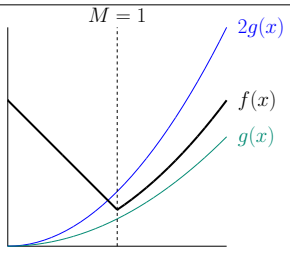
of the cost function.

- Since we are approximating anyway, pointless to be precise about certain things:
  - *Behavior on small inputs*:
    - Can always pre-calculate some results.
    - Times for small inputs not usually important.
    - Often more interested in *asymptotic behavior* as input size becomes very large.
  - *Constant factors* (as in "off by factor of 2"):
    - Just changing machines causes constant-factor change.
- How to abstract away from (i.e., ignore) these things?

tions that specify size, but rather *families of functions with similarly behaved magnitudes*.

- Then say something like "f is bounded by g if it is in g's family."
- For any function $g(x)$, the functions $2g(x)$, $0.5g(x)$, or for any $K > 0$, $K \cdot g(x)$, all have the same "shape". So put all of them into g's family.
- Any function $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some constant M) has g's shape "except for small values." So put all of these in g's family.
- For upper limits, throw in all functions whose absolute value is everywhere $\leq$ some member of g's family. Call this set $O(g)$ or $O(g(n))$.
- Or, for lower limits, throw in all functions

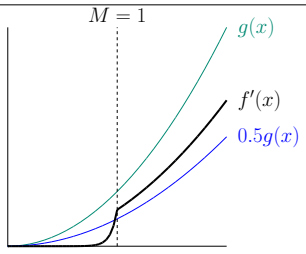
of functions *bracketed in magnitude by* two members of g's family.



- Here, $f(x) \leq 2g(x)$ as long as $x > 1$,
- So $f(x)$ is in g's "bounded-above family," written

$$f(x) \in O(g(x)),$$

- ...even though (in this case) $f(x) > g(x)$ everywhere.



- Here, $f'(x) \geq \frac{1}{2}g(x)$ as long as $x > 1$,
- So $f'(x)$ is in g's "bounded-below family," written

$$f'(x) \in \Omega(g(x)),$$

- ...even though $f(x) < g(x)$ everywhere.

$f(x) \in O(g(x))$ and $f'(x) \in \Omega(g(x))$,...

- ...but also $f(x) \in \Omega(g(x))$ and $f'(x) \in O(g(x))$.
- We can summarize this all by saying $f(x) \in \Theta(g(x))$ and $f'(x) \in \Theta(g(x))$.

- Technically, if I am going to talk about $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$ as sets of functions, I really should write, for example,

$$f \in O(g) \quad \text{instead of} \quad f(x) \in O(g(x))$$

- In effect, $f(x) \in O(g(x))$ is short for $\lambda\, x.\ f(x) \in O(\lambda\, x.\ g(x))$.
- The standard notation outside this course, in fact, is $f(x) = O(g(x))$, but personally, I think that's a serious abuse of notation.

etc., used generally to specify bounds on functions.

- For example,

$$\pi(N) = \Theta(\frac{N}{\ln N})$$

which I would prefer to write

$$\pi(N) \in \Theta(\frac{N}{\ln N})$$

(Here, $\pi(N)$ is the number of primes less than or equal to N.)

- Also, you'll see things like

$f(x) = x^3+x^2+O(x)$ (or $f(x) \in x^4+x^2+O(x)$),

meaning that $f(x) = x^3 + x^2 + g(x)$ where $g(x) \in O(x)$.

- For our purposes, the functions we will be bounding will be *cost functions:* functions

factors didn't matter at all, only the difference of $\Theta(N)$ vs. $\Theta(N^2)$.

- In reality they do matter, but at some point, constants always get swamped.

n	$16 \lg n$	$\sqrt{n}$	n	$n \lg n$	n^2	n^3	2^n
2	16	1.4	2	2	4	8	4
4	32	2	4	8	16	64	16
8	48	2.8	8	24	64	512	256
16	64	4	16	64	256	4,096	65,636
32	80	5.7	32	160	1024	32,768	4.2×10^9
64	96	8	64	384	4,096	262,144	1.8×10^{19}
128	112	11	128	896	16,384	2.1×10^9	3.4×10^{38}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1,024	160	32	1,024	10,240	1.0×10^6	1.1×10^9	1.8×10^{308}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{20}	320	1024	1.0×10^6	2.1×10^7	1.1×10^{12}	1.2×10^{18}	$6.7 \times 10^{315,652}$

time?

- In the following table, left column shows time in microseconds to solve a given problem as a function of problem size N.
- Entries show the *size of problem* that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.
- N = problem size.

Time (μsec) for problem size N	Max N Possible in 1 second	1 hour	1 month	1 century
$\lg N$	10^{300000}	$10^{1000000000}$	$10^{8 \cdot 10^{11}}$	$10^{10^{14}}$
N	10^6	$3.6 \cdot 10^9$	$2.7 \cdot 10^{12}$	$3.2 \cdot 10^{15}$
$N \lg N$	63000	$1.3 \cdot 10^8$	$7.4 \cdot 10^{10}$	$6.9 \cdot 10^{13}$
N^2	1000	60000	$1.6 \cdot 10^6$	$5.6 \cdot 10^7$
N^3	100	1500	14000	150000
2^N	20	32	41	51

real-valued function.

- We will use them to describe cost functions. Example:

```
  /** Find position of X in list L, or
-1 if not found. */
  int find(List L, Object X) {
     int c;
     for (c = 0; L != null; L = L.next,
c += 1)
        if (X.equals(L.head)) return c;
     return -1;
  }
```

- Choose representative operation: number of `.equals` tests.
- If N is length of L, then loop does *at most* N tests: *worst-case time* is N tests.
- In fact, total # of instructions executed is

- Use $N > M$ provision (in defn. of $O(\cdot)$) to ignore empty list.

$O(N^2)$, since $N \in O(N^2)$ also: Big-Oh bounds are loose.

- The worst-case time is $\Omega(N)$, since $N \in \Omega(N)$, but that does *not* mean that the loop *always* takes time N, or even $K \cdot N$ for some K.
- Instead, we are just saying something about the *function* that maps N into the *largest possible* time required to process any array of length N.
- To say as much as possible about our worst-case time, we should try to give a Θ bound: in this case, we can: $\Theta(N)$.
- But again, that still tells us nothing about *best-case* time, which happens when we find `X` at the beginning of the loop. Best-case time is $\Theta(1)$.


```
  for (int i = 0; i < A.length; i += 1)
     for (int j = 0; j < A.length; j +=
1)
        if (i != j && A[i] == A[j])
           return true;
  return false;
```

- Clearly, time is $O(N^2)$, where $N =$ `A.length`. *Worst-case time* is $\Theta(N^2)$.
- Loop is inefficient though:

```
  for (int i = 0; i < A.length; i += 1)
     for (int j = i+1; j < A.length; j
+= 1)
        if (A[i] == A[j]) return true;
  return false;
```

- Now worst-case time is proportional to

$$N-1+N-2+\ldots+1 = N(N-1)/2 \in \Theta(N^2)$$

- Silly example of recursion. In the worst case, both recursive calls happen:

```
/** True iff X is a substring of S */
boolean occurs(String S, String X) {
  if (S.equals(X)) return true;
  if (S.length() <= X.length()) return false;
  return
    occurs(S.substring(1), X) ||
    occurs(S.substring(0, S.length()-1), X);
}
```

- Define $C(N)$ to be the worst-case cost of `occurs(S,X)` for `S` of length N, `X` of fixed size N_0, measured in # of calls to `occurs`. Then

$$C(N) = \begin{cases} 1, & \text{if } N \le N_0, \\ 2C(N-1)+1 & \text{if } N > N_0 \end{cases}$$

- So $C(N)$ grows exponentially:

$$C(N) = 2C(N-1)+1 = 2(2C(N-2)+1)+1 = \ldots = \underbrace{2(\cdots 2}_{N-N_0}\cdot 1+1)+\ldots+$$

```
   * S in ascending order, 0 <= L <= U+1 <= S.length.
*/
  boolean isIn(String X, String[] S, int L, int U)
{
    if (L > U) return false;
    int M = (L+U)/2;
    int direct = X.compareTo(S[M]);
    if (direct < 0) return isIn(X, S, L, M-1);
    else if (direct > 0) return isIn(X, S, M+1, U);
    else return true;
  }
```

- Here, worst-case time, $C(D)$, (as measured by # of calls to `.compareTo`), depends on size $D = U - L + 1$.
- We eliminate `S[M]` from consideration each time and look at half the rest. Assume $D = 2^k - 1$ for simplicity, so:

$$C(D) = \begin{cases} 0, & \text{if } D \le 0, \\ 1 + C((D-1)/2), & \text{if } D > 0. \end{cases}$$


```
  List sort(List L) {
    if (L.length() < 2)
return L;
    Split L into L0 and L1 of
about equal size;
    L0 = sort(L0);  L1 =
sort(L1);
    return Merge of L0 and
L1
  }
```

Merge ("combine into a single ordered list") takes time proportional to size of its result.

- Assuming that size of `L` is $N = 2^k$, worst-case cost function, $C(N)$, counting just merge time (which is proportional to # items merged):

$$\begin{aligned} C(N) &= \begin{cases} 0, & \text{if } N < 2; \\ 2C(N/2) + N, & \text{if } N \ge 2. \end{cases} \\ &= 2(2C(N/4) + N/2) + N \\ &= 4C(N/4) + N + N \\ &= 8C(N/8) + N + N + N \end{aligned}$$

- In general, can say it's $\Theta(N \lg N)$ for arbitrary N (not just 2^k).