Dynamics of Self-Driving Cars

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Consider a car of length L moving at speed V in the +y direction from the origin of the xy-plane. Suppose the steering wheel is turned an angle ϕ CW from the neutral position. The path of the car lies on a circle with turn radius ρ. The car lies on the sector of a circle, which for small turn angles has arc length $\approx L$ and sector angle ϕ. Using the relationship $s = r\theta$ for circle arc length, we obtain the car turn radius as $\rho = L/\phi$. The car therefore lies on the path $r = \frac{L}{\phi} < 1 - cos\omega t, sin\omega t >$, which represents a circle through the origin. The angular frequency on the circle is $\omega = 2\pi/T$, with period $T = C/V$ where C is the circumference of the circle $C = 2\pi\rho = 2\pi L/\phi$. Therefore, we have $\omega = V\phi/L$ and the dynamics of the car become the following.

$$r(t) = \frac{L}{\phi} < 1 - \cos\left(\frac{V\phi t}{L}\right), \sin\left(\frac{V\phi t}{L}\right) >$$

However, cars often drive straight with $\phi = 0$ which produces a singularity. However, as $\phi \to 0$, we perform limits of the functions $\frac{\sin(x)}{x} = sinc(x)$ and $\frac{1-\cos(x)}{x} = \sin\left(\frac{x}{2}\right) sinc\left(\frac{x}{2}\right) \stackrel{\text{def}}{=} \text{cosc(x)}$, which are known power series. Rewriting the original trajectory in this form gives us:

$$x(t) = Vt \operatorname{cosc}\left(\frac{V\phi t}{L}\right) + x_c$$

$$y(t) = Vt \operatorname{sinc}\left(\frac{V\phi t}{L}\right) + y_c$$

A plot of these trajectories for different turn angles are shown below, including a perfect straight-line path! The remaining dynamics of the system involve forward acceleration and turning:

$$\begin{Bmatrix} \dot{V}(t) = u_1 \\ \dot{\phi}(t) = u_2 \end{Bmatrix}$$