AVL Trees

Invented in 1962 **by**
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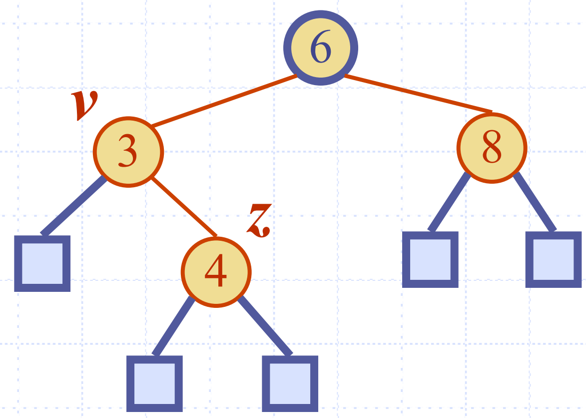
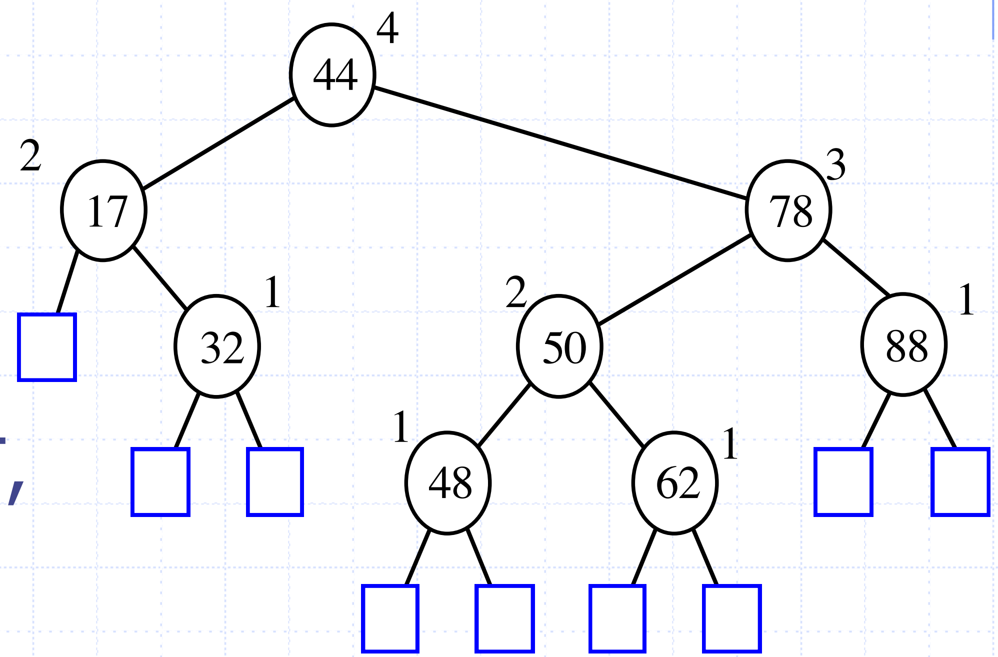
E. M. Landis



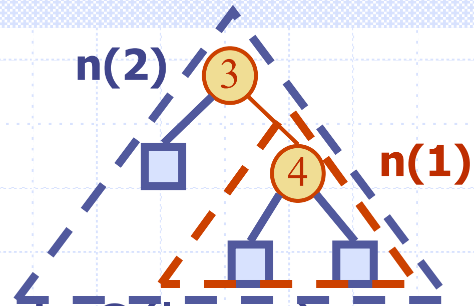
AVL Tree Definition

- AVL trees are balanced
- An AVL Tree is a binary search tree such that for every internal node v of T, the heights of the children of v can differ by at most 1



An example of an AVL tree where the heights are shown next to the nodes:

Height of an AVL Tree



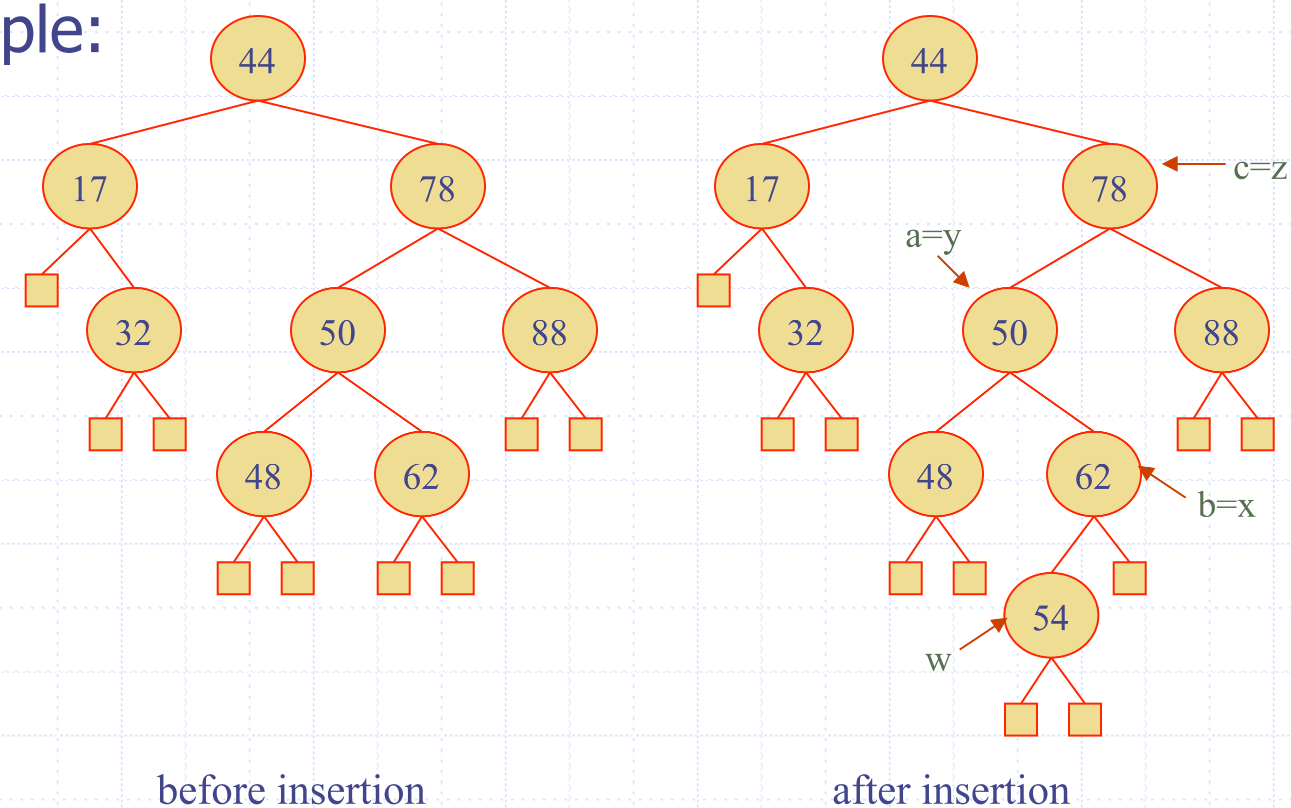
- Fact: The height of an AVL tree storing n keys is O(log n).
- Proof: Let us bound n(h): the minimum number of internal nodes of an AVL tree of height h.
- We easily see that n(1) = 1 and n(2) = 2
- For n > 2, an AVL tree of height h contains the root node, one AVL subtree of height n-1 and another of height n-2.
- That is, n(h) = 1 + n(h-1) + n(h-2)
- Knowing n(h-1) > n(h-2), we get n(h) > 2n(h-2). So

  n(h) > 2n(h-2), n(h) > 4n(h-4), n(h) > 8n(n-6), … (by induction), n(h) > 2^in(h-2i)

  base case, n(1) = 1; n(2) = 2; we take i = h/2-1 for h>=3
- Solving the base case we get: n(h) > 2 $^{h/2-1}$ (see book p. 482/3)
- Taking logarithms: h < 2 log n(h) + 2
- Thus the height of an AVL tree is O(log n)

Insertion

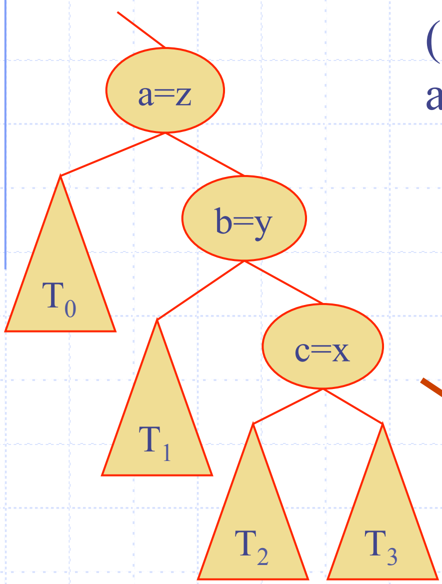
- Insertion is as in a binary search tree
- Always done by expanding an external node.
- Example:

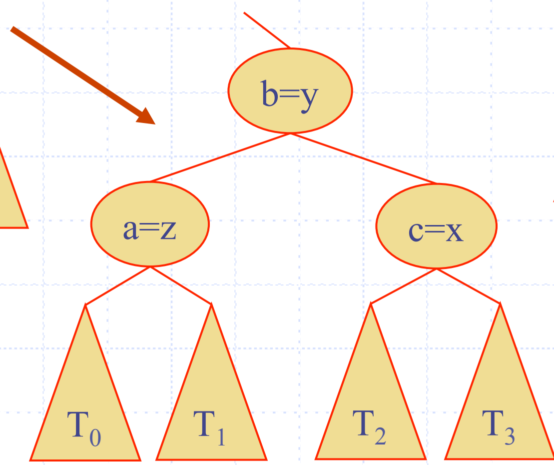


Trinode Restructuring

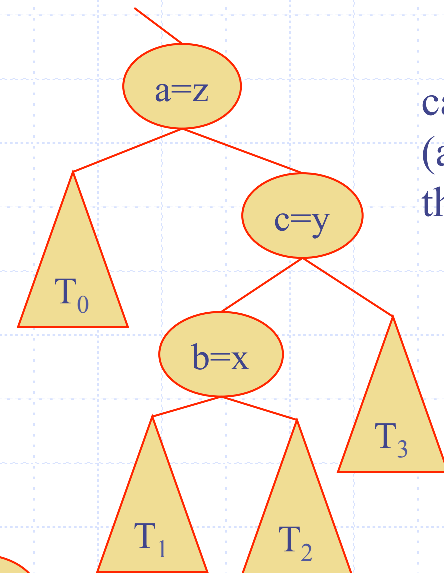
- let (a,b,c) be an inorder listing of x, y, z
- perform the rotations needed to make b the topmost node of the three

(other two cases are symmetrical)

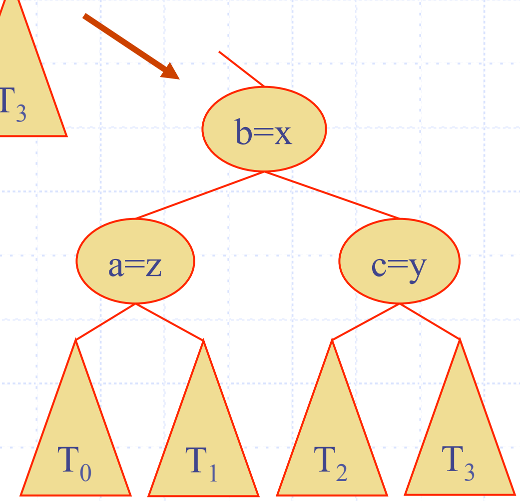




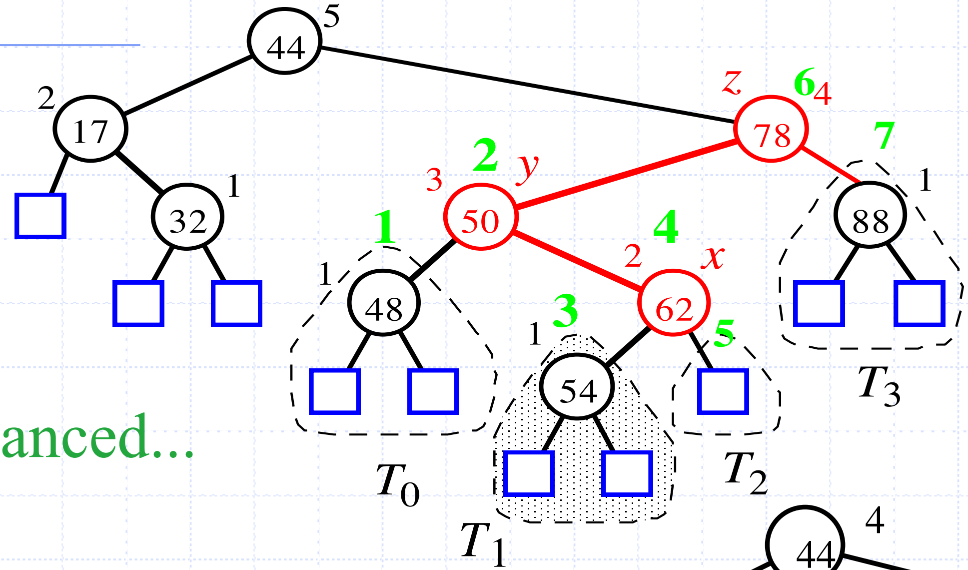
case 1: single rotation
(a left rotation about a)



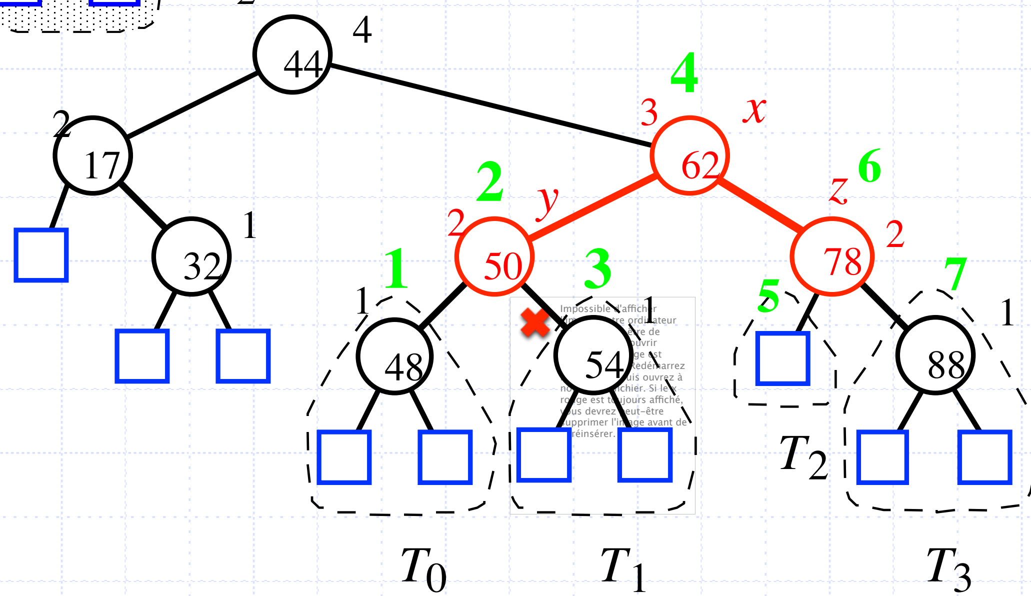
case 2: double rotation
(a right rotation about c,
then a left rotation about a)



Insertion Example, continued



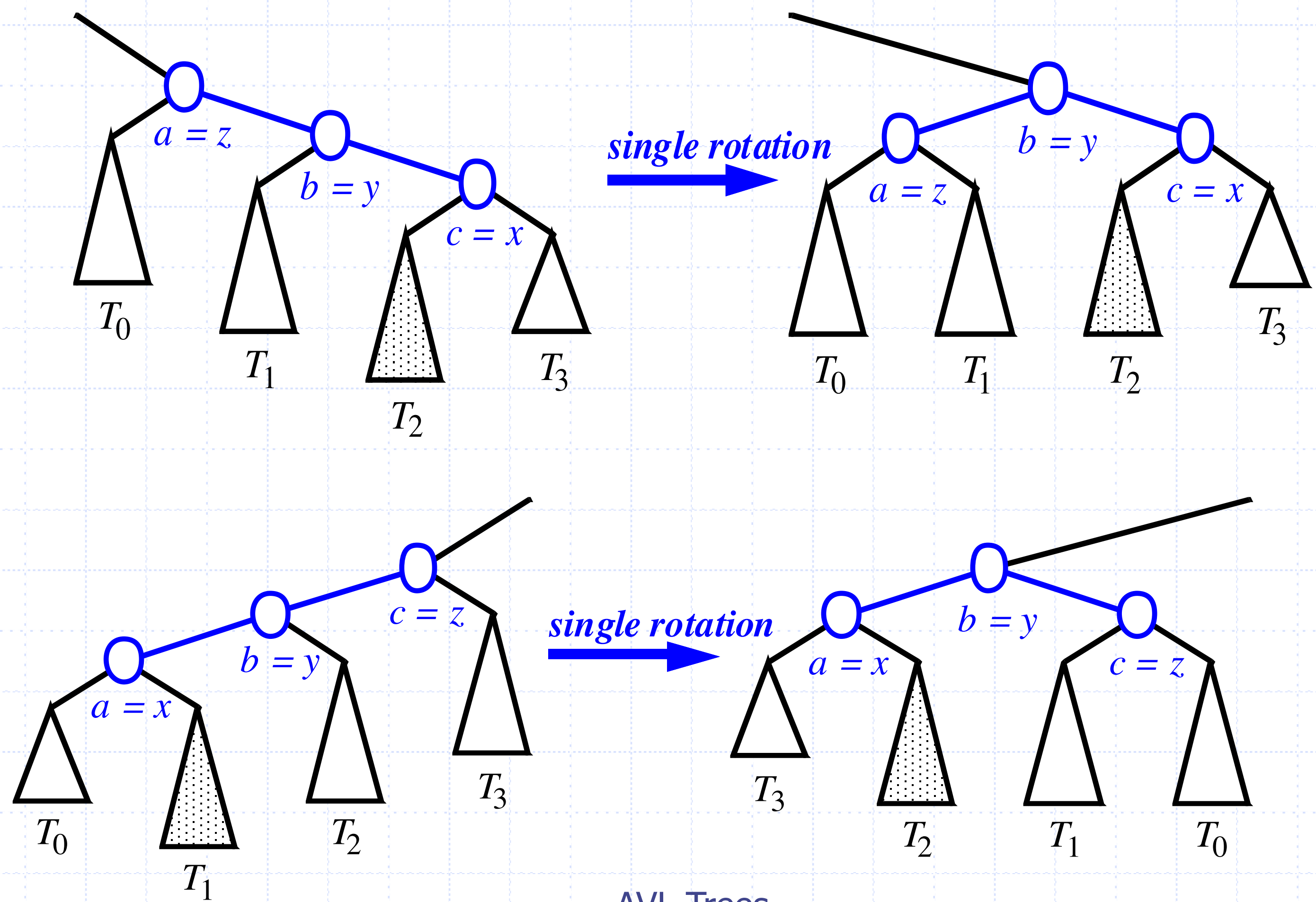
unbalanced...



...balanced

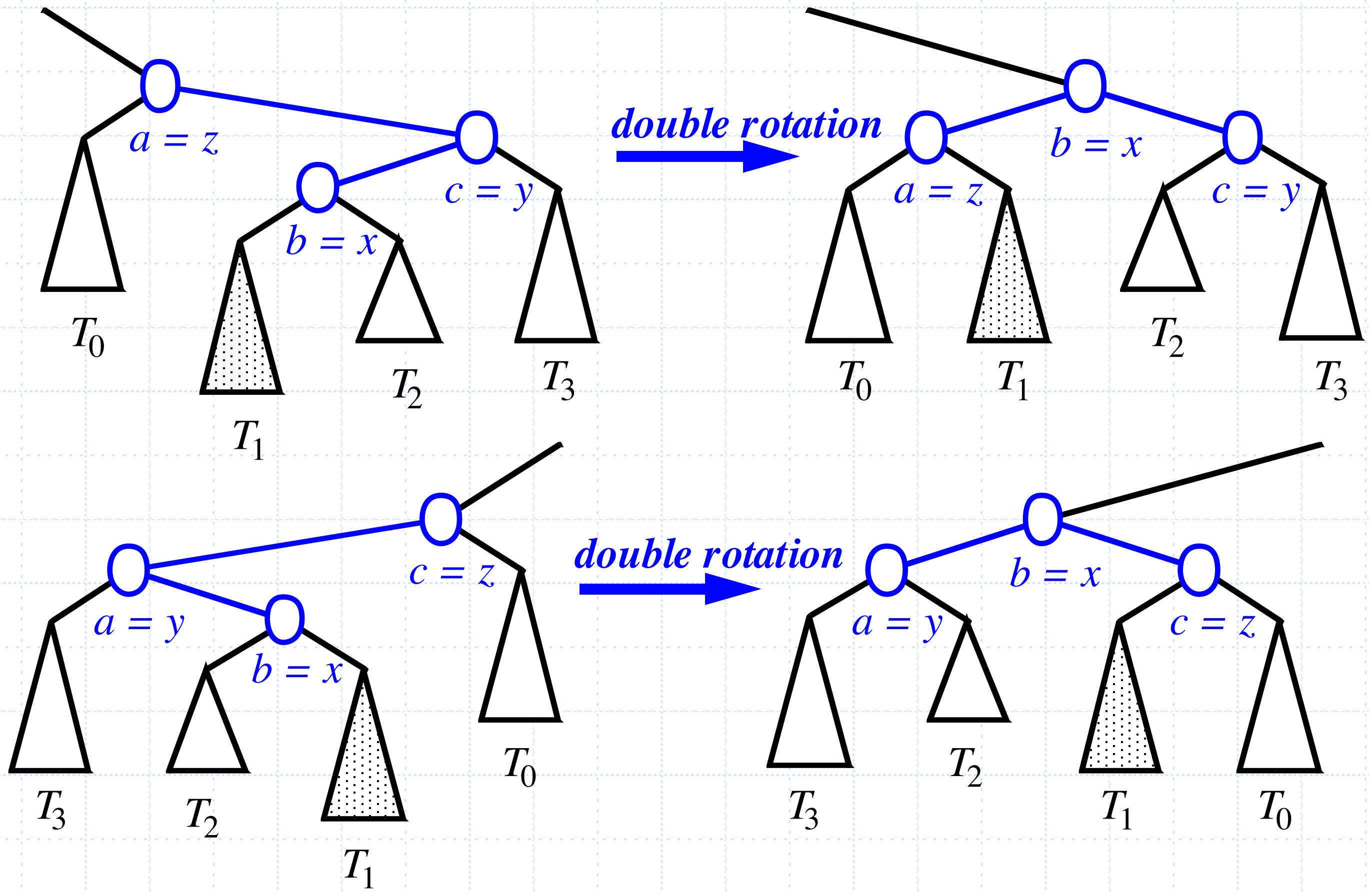
Restructuring (as Single Rotations)

- Single Rotations:



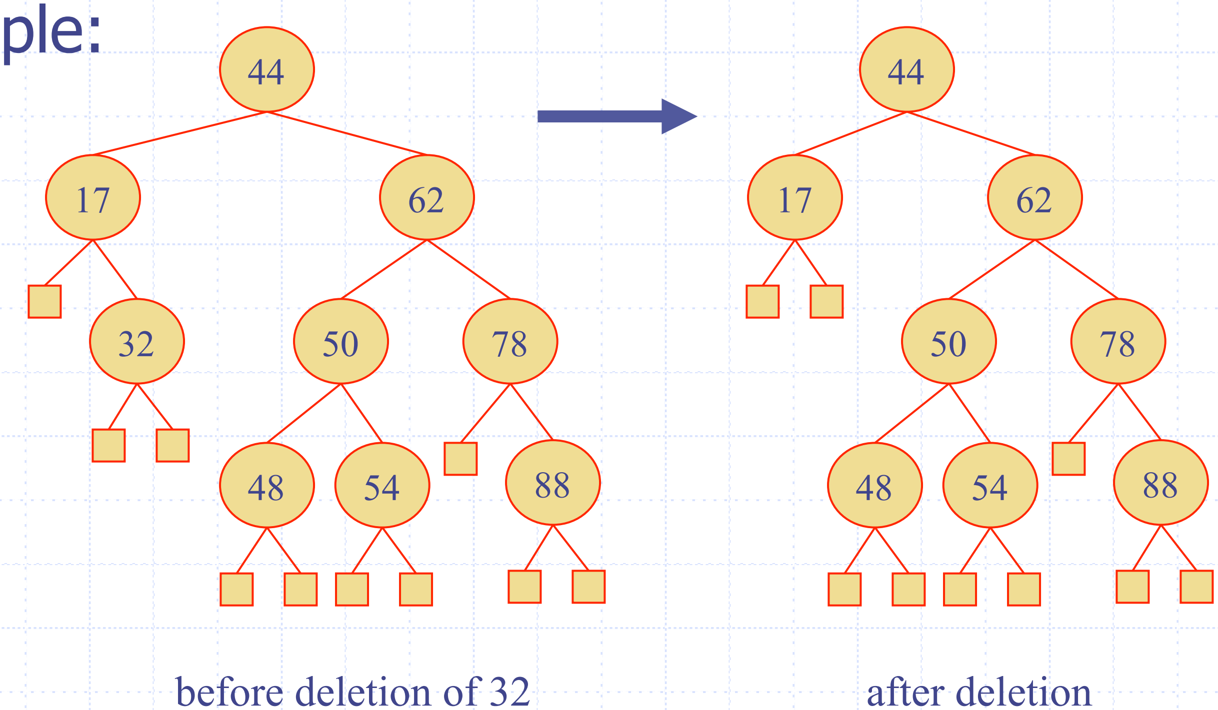
Restructuring (as Double Rotations)

- double rotations:



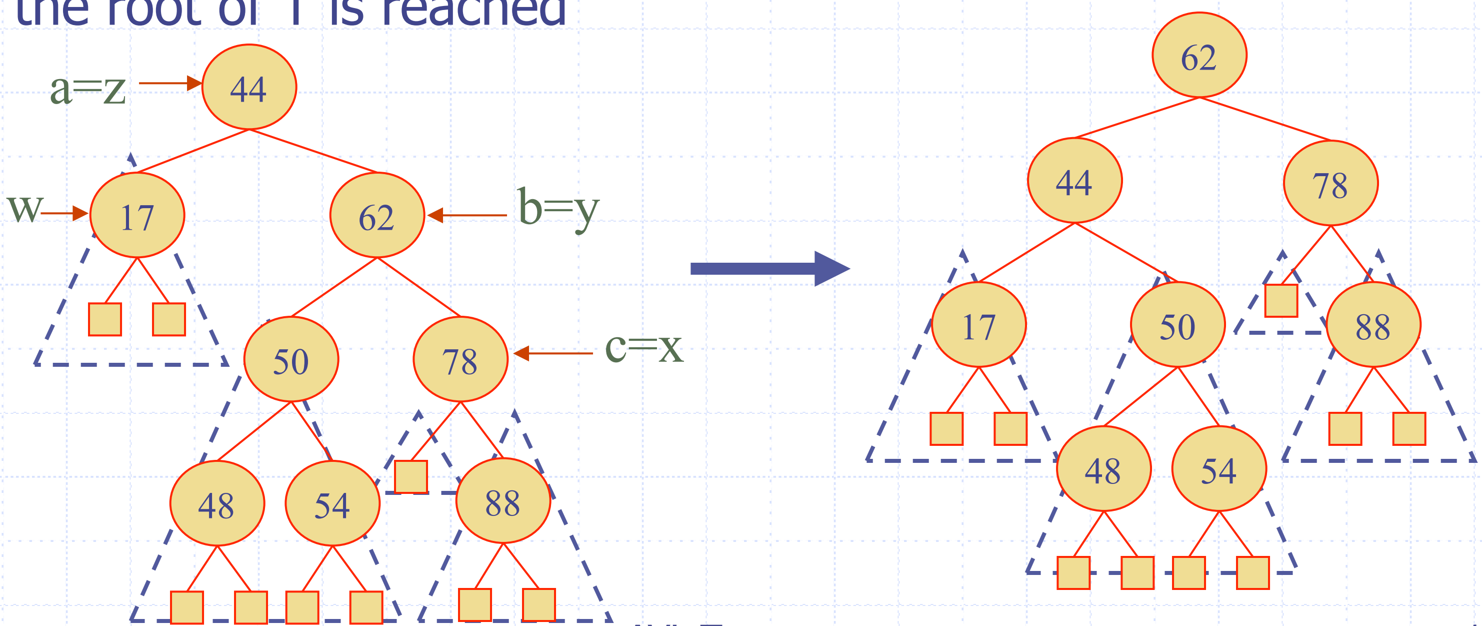
Removal

- Removal begins as in a binary search tree, which means the node removed will become an empty external node. Its parent, w, may cause an imbalance.
- Example:

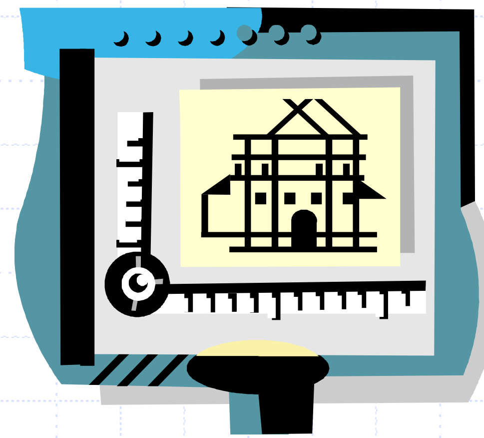


Rebalancing after a Removal

- Let z be the first unbalanced node encountered while travelling up the tree from w. Also, let y be the child of z with the larger height, and let x be the child of y with the larger height
- We perform restructure(x) to restore balance at z
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached



AVL Tree Performance



- a single restructure takes O(1) time
  - using a linked-structure binary tree
- Searching takes O(log n) time
  - height of tree is O(log n), no restructures needed
- Insertion takes O(log n) time
  - initial find is O(log n)
  - Restructuring up the tree, maintaining heights is O(log n)
- Removal takes O(log n) time
  - initial find is O(log n)
  - Restructuring up the tree, maintaining heights is O(log n)

Python Implementation

```
class AVLTreeMap(TreeMap):
  """Sorted map implementation using an AVL tree."""

  #------------------------- nested _Node class -------------------------
  class _Node(TreeMap._Node):
    """Node class for AVL maintains height value for balancing."""
    __slots__ = '_height'         # additional data member to store height

    def __init__(self, element, parent=None, left=None, right=None):
      super().__init__(element, parent, left, right)
      self._height = 0            # will be recomputed during balancing

    def left_height(self):
      return self._left._height if self._left is not None else 0

    def right_height(self):
      return self._right._height if self._right is not None else 0
```

Python Implementation, Part 2

```
#------------------------ positional-based utility methods ------------------------
def _recompute_height(self, p):
  p._node._height = 1 + max(p._node.left_height(), p._node.right_height())

def _isbalanced(self, p):
  return abs(p._node.left_height() - p._node.right_height()) <= 1

def _tall_child(self, p, favorleft=False): # parameter controls tiebreaker
  if p._node.left_height() + (1 if favorleft else 0) > p._node.right_height():
    return self.left(p)
  else:
    return self.right(p)

def _tall_grandchild(self, p):
  child = self._tall_child(p)
  # if child is on left, favor left grandchild; else favor right grandchild
  alignment = (child == self.left(p))
  return self._tall_child(child, alignment)
```

Python Implementation, end

```
def _rebalance(self, p):
    while p is not None:
        old_height = p._node._height          # trivially 0 if new node
        if not self._isbalanced(p):           # imbalance detected!
            # perform trinode restructuring, setting p to resulting root,
            # and recompute new local heights after the restructuring
            p = self._restructure(self._tall_grandchild(p))
            self._recompute_height(self.left(p))
            self._recompute_height(self.right(p))
        self._recompute_height(p)             # adjust for recent changes
        if p._node._height == old_height:     # has height changed?
            p = None                          # no further changes needed
        else:
            p = self.parent(p)                # repeat with parent

#------------------------- override balancing hooks -------------------------
def _rebalance_insert(self, p):
    self._rebalance(p)

def _rebalance_delete(self, p):
    self._rebalance(p)
```