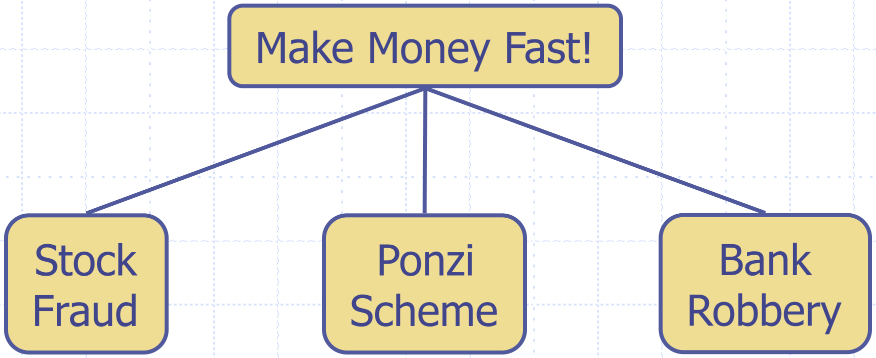
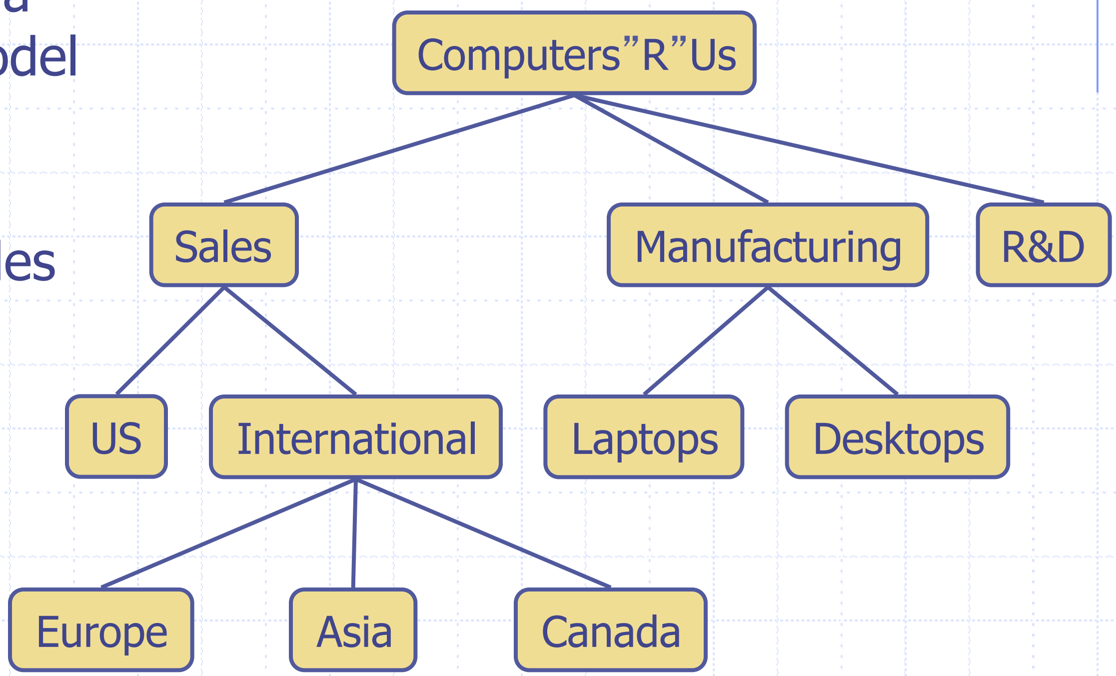
Trees



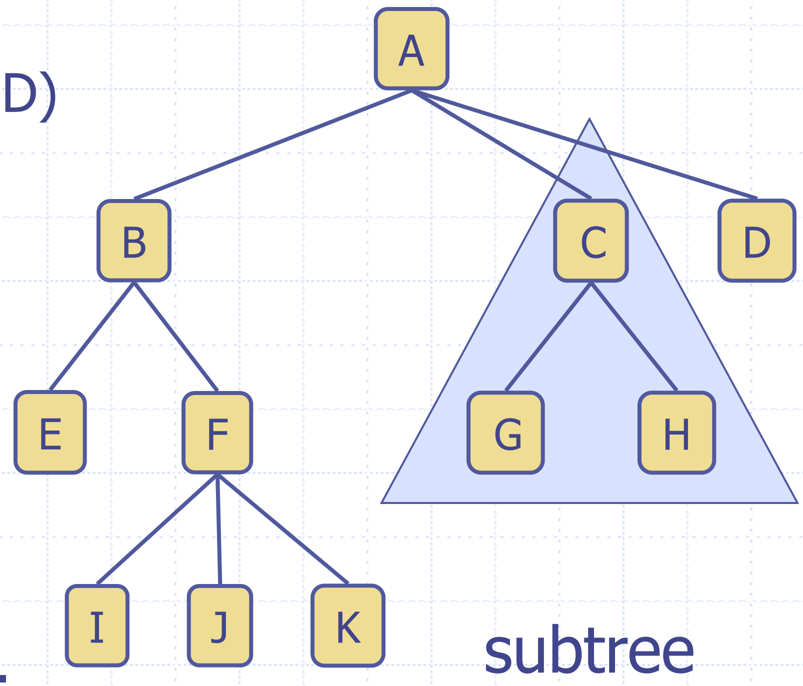
What is a Tree

- In computer science, a tree is an abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation
- Applications:
  - Organization charts
  - File systems
  - Programming environments



Tree Terminology

- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- External node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, grand-grandparent, etc.
- Depth of a node: number of ancestors
- Height of a tree: maximum depth of any node (3)
- Descendant of a node: child, grandchild, grand-grandchild, etc.
- Subtree: tree consisting of a node and its descendants



Tree ADT

- We use positions to abstract nodes
- Generic methods:
  - Integer len()
  - Boolean is_empty()
  - Iterator positions()
  - Iterator iter()
- Accessor methods:
  - position root()
  - position parent(p)
  - Iterator children(p)
  - Integer num_children(p)

- Query methods:
  - Boolean is_leaf(p)
  - Boolean is_root(p)
- Update method:
  - element replace (p, o)
- Additional update methods may be defined by data structures implementing the Tree ADT

Abstract Nested Position Class in Python

```
#ADT Tree "interface"
class Tree:

    class Position:

        def element( self ):
            pass

        def __eq__( self, other ):
            pass

        def __ne__( self, other):
            return not( self == other )
```

Abstract Tree Class in Python…

```python
#ADT Tree "interface"
class Tree:

    def root( self ):
        pass

    def parent( self, p ):
        pass

    def num_children( self, p ):
         pass

    def children( self, p ):
         pass

    def __len__( self ):
         pass

    def is_root( self, p ):
         return self.root() == p

    def is_leaf( self, p ):
         return self.num_children() == 0

    def is_empty( self ):
         return len( self ) == 0
```

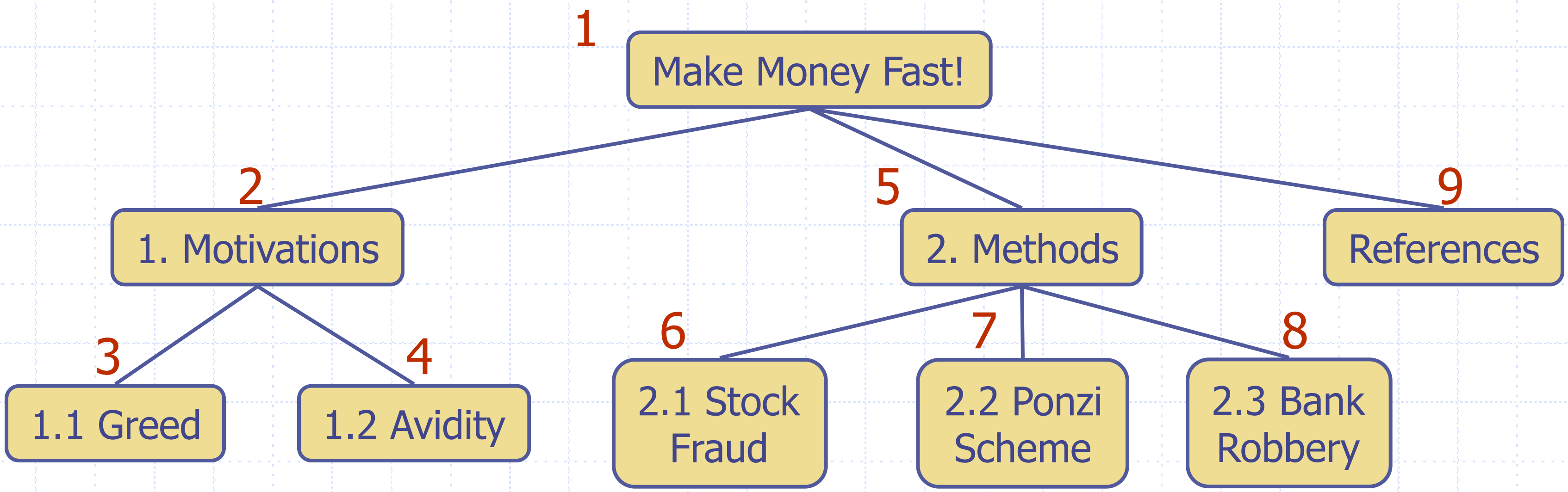
Abstract Tree Class in Python

```
def depth( self, p ):
      #returns the number of ancestors of p
      if self.is_root( p ):
        return 0
      else:
        return 1 + self.depth( self.parent() )

def height1( self, p ):
      #returns the maximum depth of the leaf positions
      return max( self.depth( p ) for p in self.positions() if self.is_leaf( p ))

def height2( self, p ):
      #returns the height of the subtree at Position p
      if self.is_leaf( p ):
        return 0
      else:
        return 1 + max( self.height2( c ) for c in self.children( p ) )

def height( self, p = None ):
      #returns the height of the subtree rooted at Position p
      #if p is None, then the height of the entire tree
      if p is None:
        p = self.root()
      return self.height2( p )
```

Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

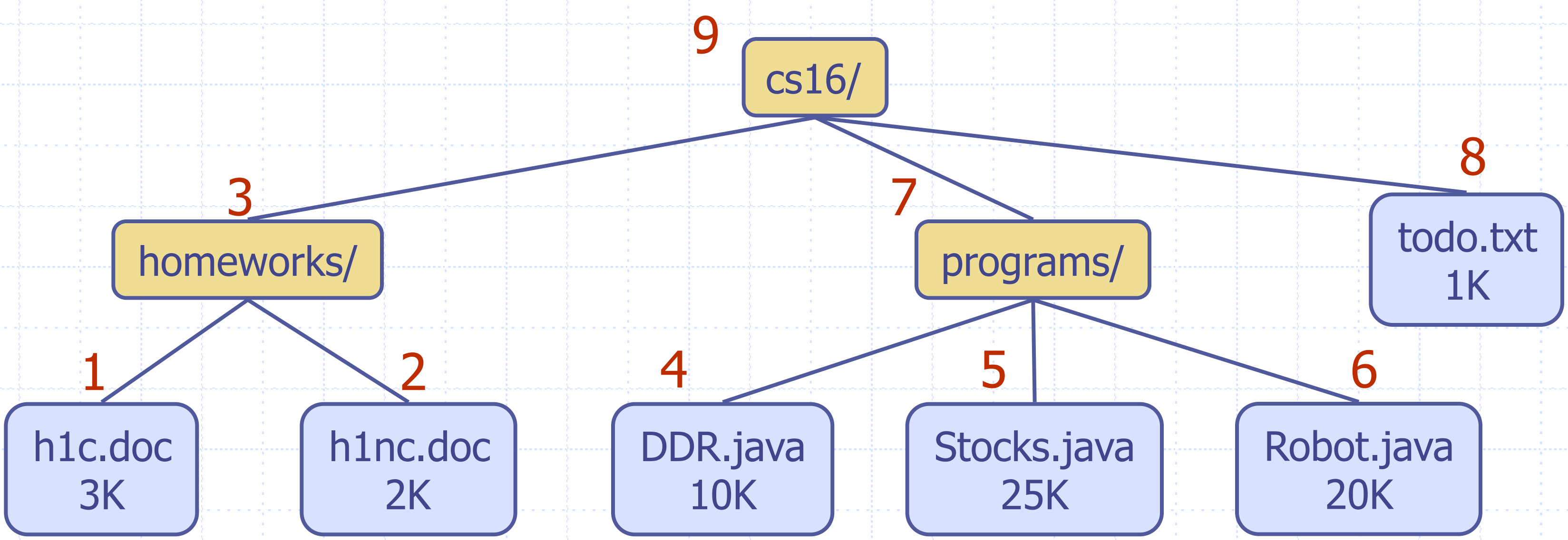
Algorithm *preOrder*(v)
  visit(v)
  for each child w of v
    preorder (w)



Postorder Traversal

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

Algorithm *postOrder*(v)
    for each child w of v
        postOrder (w)
    visit(v)

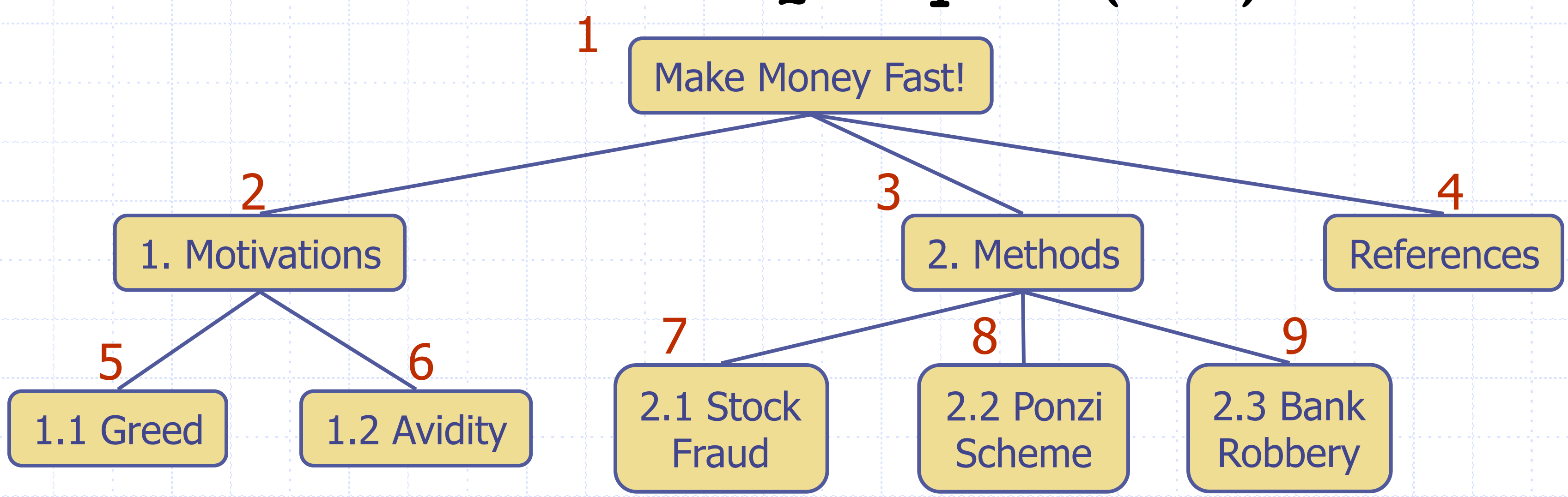


pre- and postorder traversal in Python

```
def preorder_print( self, p ):
    print( p )
 for c in self.children( p ):
        preorder_print( c )

def postorder_print( self, p ):
 for c in self.children( p ):
        preorder_print( c )
 print( p )
```

Breadth-first traversal in Python

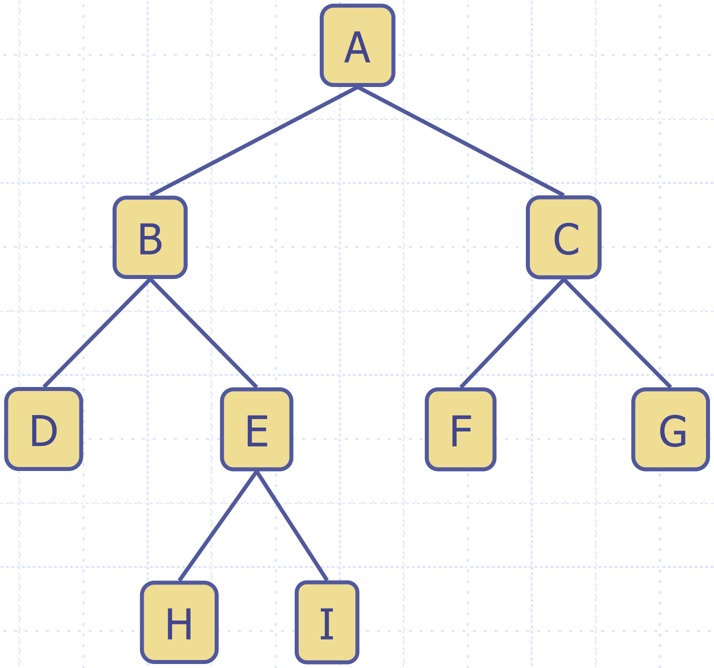
```
def breadth_first_print( self ):
    Q = ArrayQueue()
    Q.enqueue( self.root() )
        while not Q.is_empty():
            p = Q.dequeue()
            print( p )
            for c in self.children( p ):
                Q.enqueue( c )
```



Binary Trees

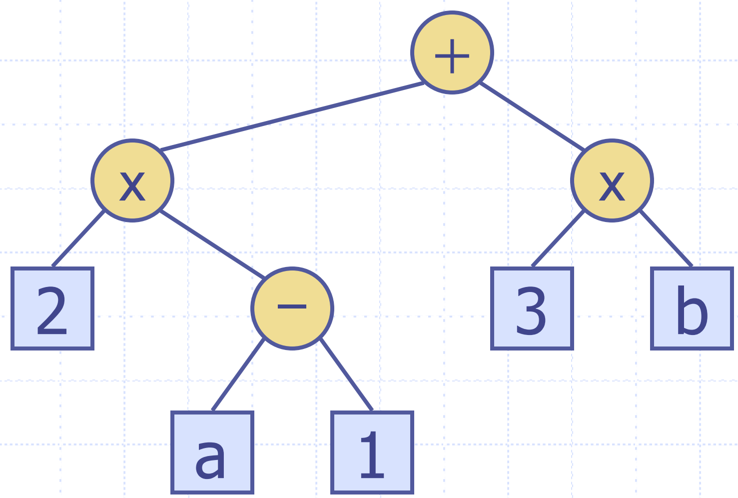
- A binary tree is a tree with the following properties:
  - Each internal node has at most two children (exactly two for proper binary trees)
  - The children of a node are an ordered pair
- We call the children of an internal node left child and right child
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree

- Applications:
  - arithmetic expressions
  - decision processes
  - searching



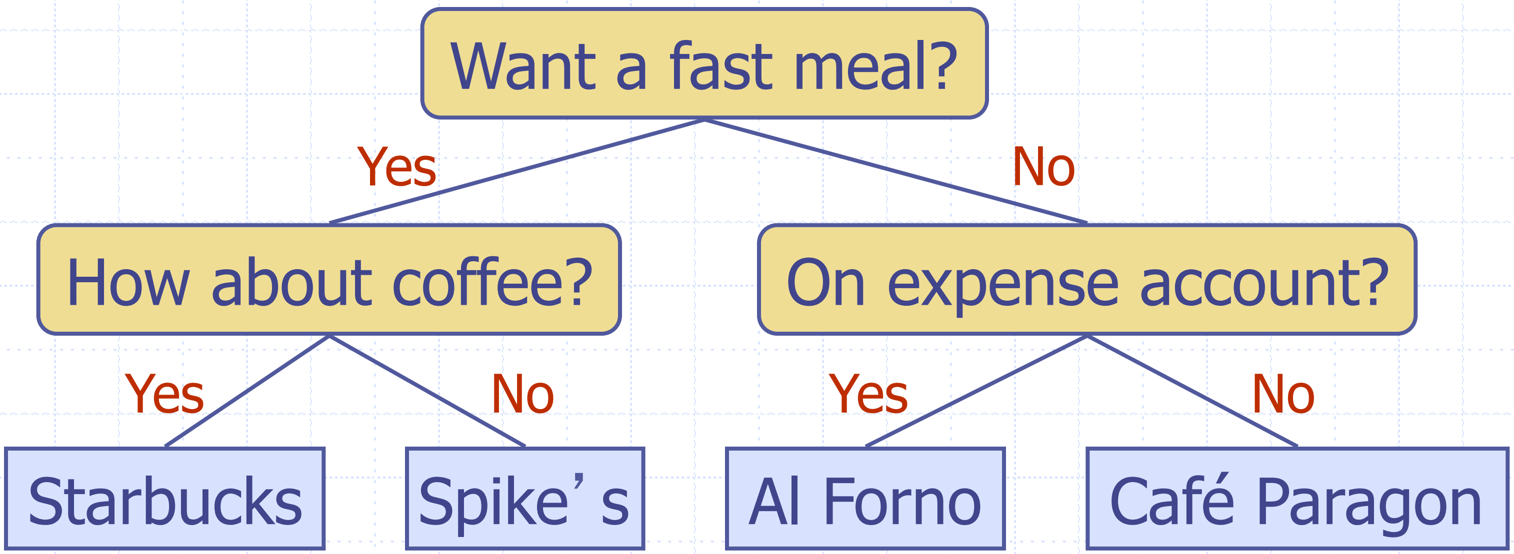
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



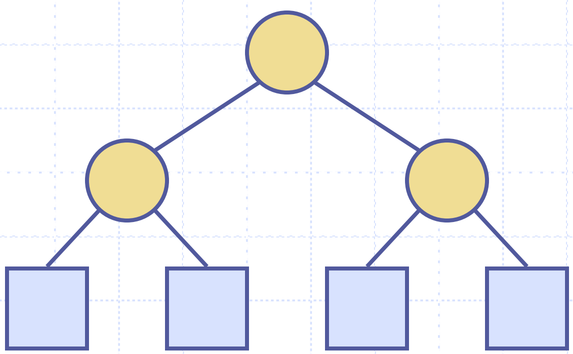
Decision Tree

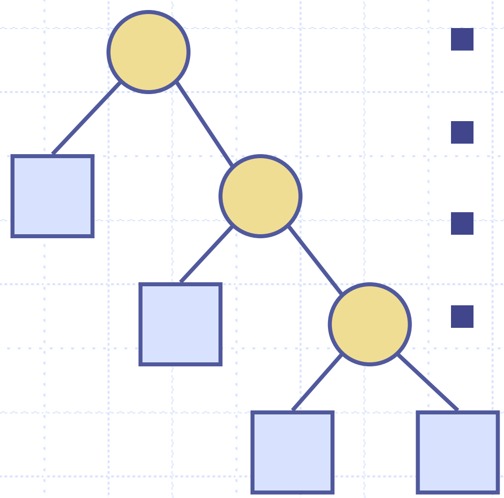
- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision



Properties of Proper Binary Trees

- Notation
  - n number of nodes
  - e number of external nodes
  - i number of internal nodes
  - h height





- Properties:
  - $e = i + 1$
  - $n = 2e - 1$
  - $h \leq i$
  - $h \leq (n - 1)/2$
  - $e \leq 2^h$
  - $h \leq \log_2 e$
  - $h \geq \log_2 (n + 1) - 1$

BinaryTree ADT

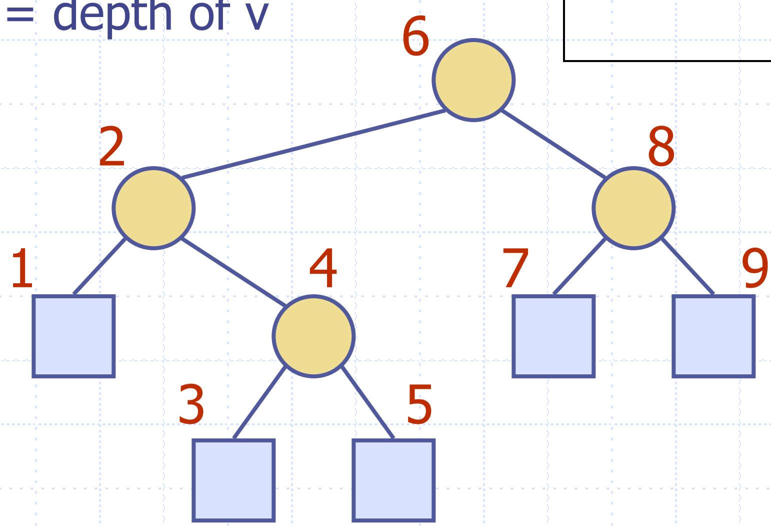
- The BinaryTree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional methods:
  - position left(p)
  - position right(p)
  - position sibling(p)
- Update methods may be defined by data structures implementing the BinaryTree ADT

Abstract BinaryTree Class in Python

```
from Tree import Tree

class BinaryTree( Tree ):

    def left( self, p ):
        pass

    def right( self, p ):
        pass

    def sibling( self, p ):
        #return the sibling Position
        parent = self.parent()
         if parent is None:
            return None
        else:
            if p == self.left( parent ):
                return self.right( parent )
            else:
                return self.left( parent )

    def children( self, p ):
         if self.left( p ) is not None:
            yield self.left( p )
        if self.right( p ) is not None:
            yield self.right( p )
```

Inorder Traversal

- In an inorder traversal a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - x(v) = inorder rank of v
  - y(v) = depth of v



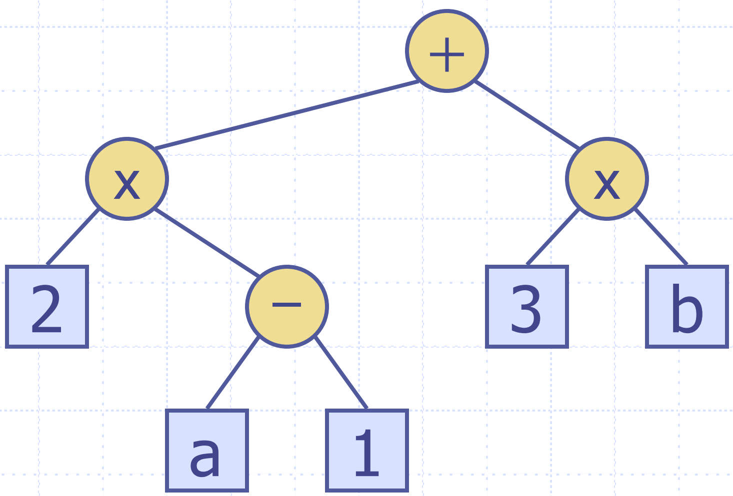
Algorithm *inOrder*(*v*)
    if *v* **has a left child**
        inOrder (*left* (*v*))
    visit(*v*)
    if *v* **has a right child**
        inOrder (*right* (*v*))

inorder_print in Python

```
def inorder_print( self, p ):
    if self.left( p ) is not None:
        self.inorder_print( self.left( p ) )
    print( p )
    if self.right( p ) is not None:
        self.inorder_print( self.right( p ) )
```

Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree



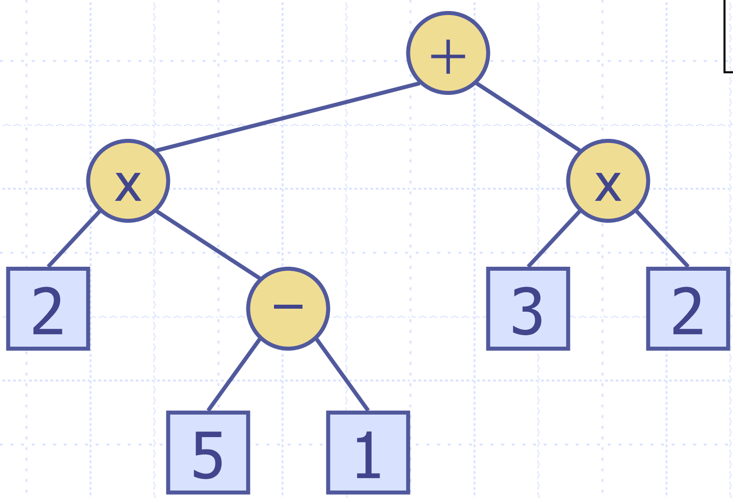
Algorithm *printExpression*(*v*)
  if *v* **has a left child**
    print("(")
    inOrder (*left*(*v*))
  print(*v.element* ())
  if *v* **has a right child**
    inOrder (*right*(*v*))
    print (")")

$((2 \times (a - 1)) + (3 \times b))$

Evaluate Arithmetic Expressions

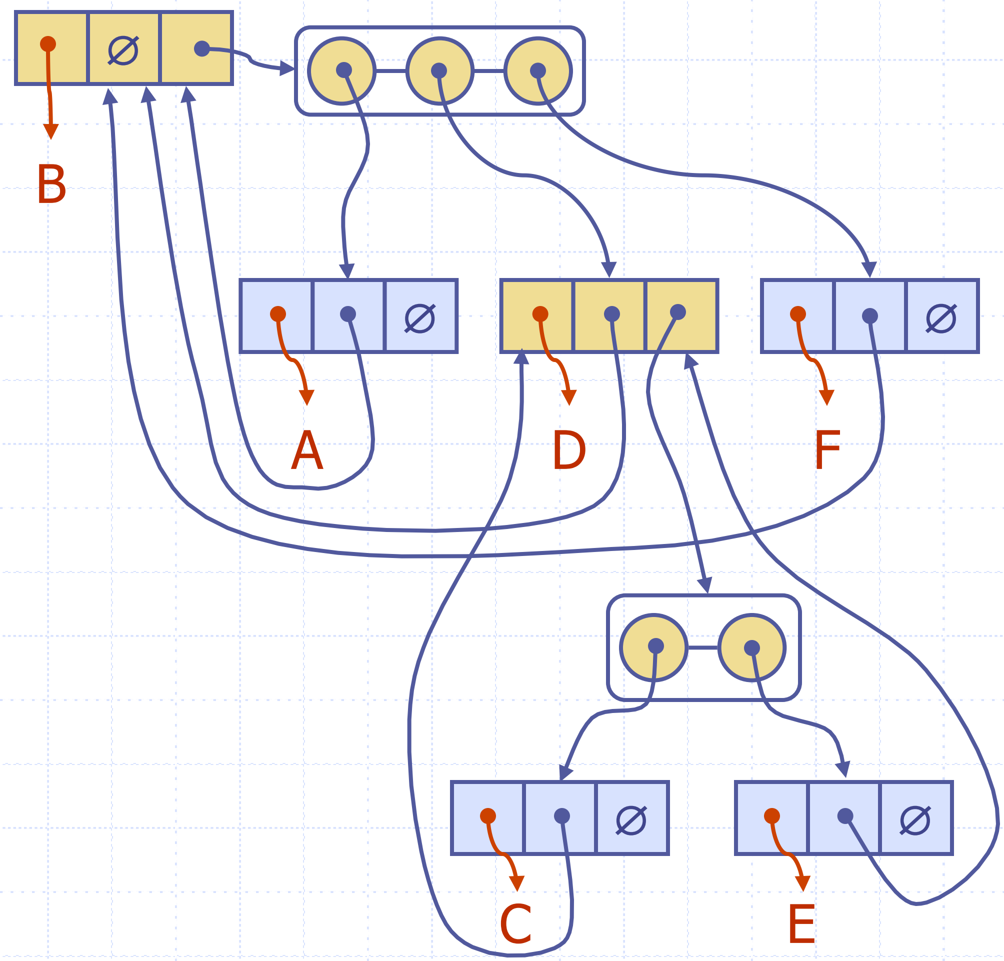
- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees

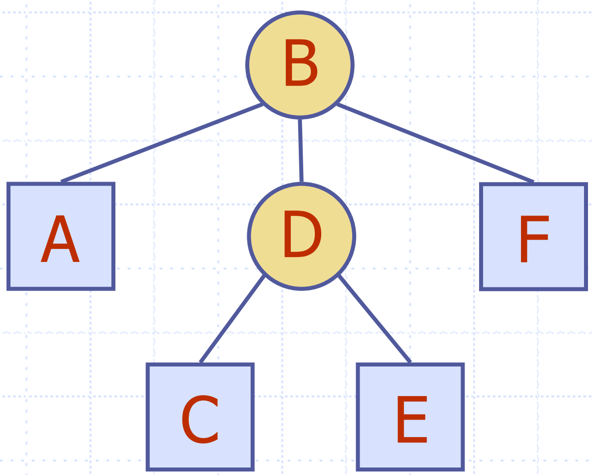
Algorithm *evalExpr*(v)
  if *is_leaf* (v)
    return *v.element* ()
  else
    x = *evalExpr*(*left* (v))
    y = *evalExpr*(*right* (v))
    op = operator stored at v
    return x **op** y



Linked Structure for Trees

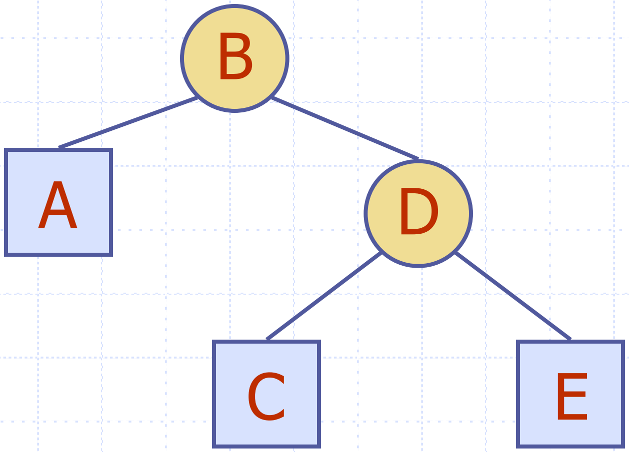
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT

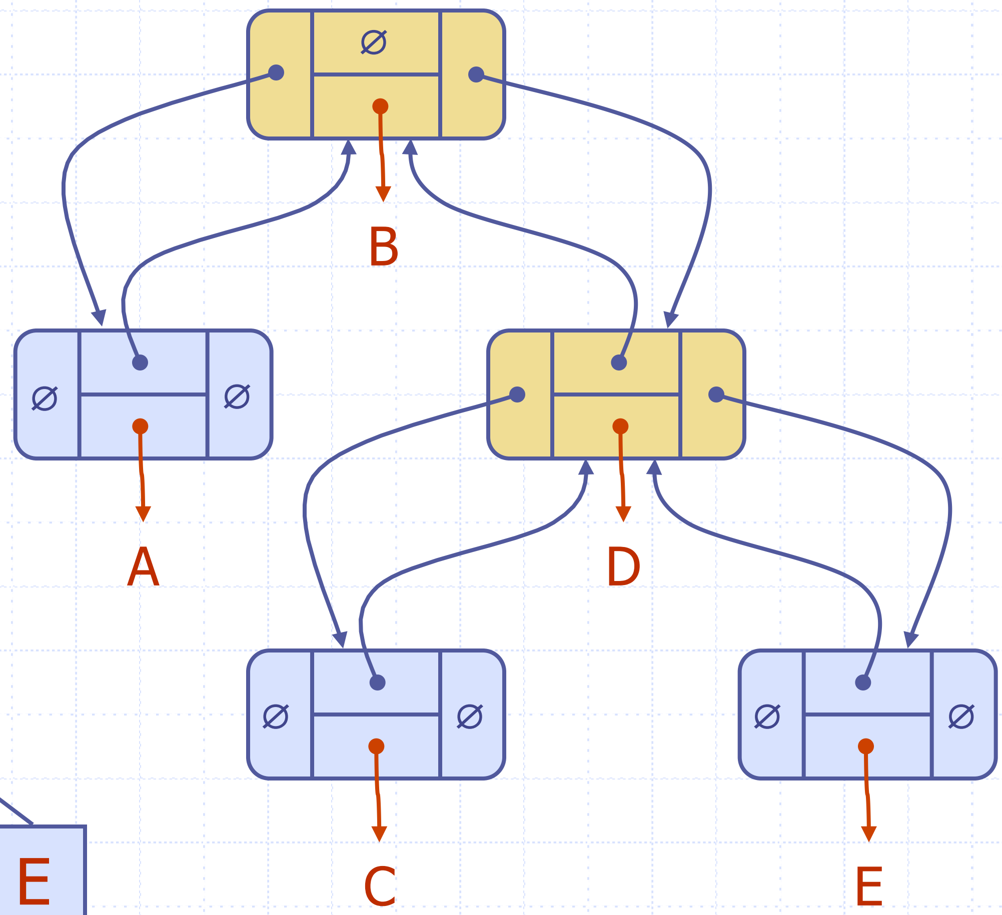




Linked Structure for Binary Trees

- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node
- Node objects implement the Position ADT





LinkedBinaryTree Class in Python

```
from BinaryTree import BinaryTree

class LinkedBinaryTree( BinaryTree ):

    class _Node:
        def __init__( self, element,
                      parent = None,
                      left = None,
                      right = None ):
            self._element = element
            self._parent = parent
            self._left = left
            self._right = right
```

Nested Abstract Position

```
class Position( BinaryTree.Position ):

  def __init__( self, container, node ):
    self._container = container
    self._node = node

  def __str__( self ):
    return str( self._node._element )

  def element( self ):
    return self._node._element

  def __eq__( self, other ):
    return type( other ) is type( self ) and other._node is self._node
```

Validate and MakePosition

```
def _validate( self, p ):
  #return associated node if position is valid
  if not isinstance( p, self.Position ):
    raise TypeError( 'p must be proper Position type' )
  if p._container is not self:
    raise ValueError( 'p does not belong to this container' )
  if p._node._parent is p._node:
    raise ValueError( 'p is no longer valid' )
  return p._node

def _make_position( self, node ):
  #return Position instance for given node (None if no node)
  return self.Position( self, node ) if node is not None else None
```

LinkedBinaryTree…

```python
def __init__( self ):
    #create an initially empty binary tree
    self._root = None
    self._size = 0

def __len__( self ):
    return self._size

def root( self ):
    return self._make_position( self._root )

def parent( self, p ):
    node = self._validate( p )
    return self._make_position( node._parent )

def left( self, p ):
    node = self._validate( p )
    return self._make_position( node._left )

def right( self, p ):
    node = self._validate( p )
    return self._make_position( node._right )
```

LinkedBinaryTree…

```
def num_children( self, p ):
    node = self._validate( p )
    count = 0
    if node._left is not None:
        count += 1
    if node._right is not None:
        count += 1
    return count

def _add_root( self, e ):
    if self._root is not None:
      raise ValueError( 'Root exists' )
    self._size = 1
    self._root = self._Node( e )
    return self._make_position( self._root )
```

LinkedBinaryTree…

```
def _add_left( self, p, e ):
    node = self._validate( p )
    if node._left is not None:
      raise ValueError( 'Left child exists' )
    self._size += 1
    node._left = self._Node( e, node )
    return self._make_position( node._left )

def _add_right( self, p, e ):
    node = self._validate( p )
    if node._right is not None:
      raise ValueError( 'Right child exists' )
    self._size += 1
    node._right = self._Node( e, node )
    return self._make_position( node._right )
```

LinkedBinaryTree...

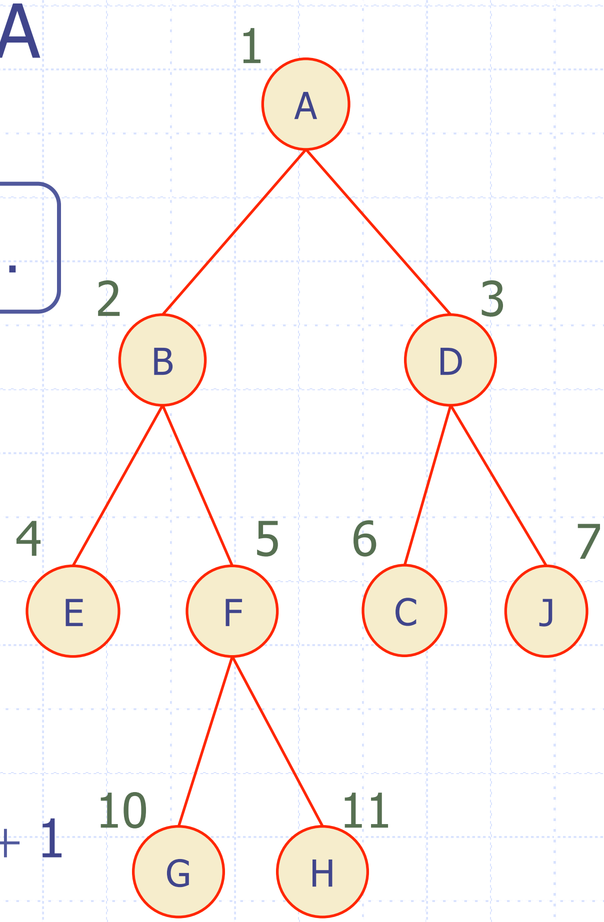
```
def _replace( self, p, e ):
    node = self._validate( p )
    old = node._element
    node._element = e
    return old

def _delete( self, p ):
    #remove the external node p
    node = self._validate( p )
    if self._num_children( p ) == 2:
        raise ValueError( 'p has two children' )
    child = node._left if node._left else node._right
    if child is not None:
        child._parent = node._parent
    if node is self._root:
        self._root = child
    else:
        parent = node._parent
        if node is parent._left:
            parent._left = child
        else:
            parent._right = child
    self._size -= 1
    node._parent = node
    return node._element
```

LinkedBinaryTree...

```
def _attach( self, p, t1, t2 ):
    node = self._validate( p )
    if not self.is_left( p ):
        raise ValueError( 'position must be leaf' )
    if not type( self ) is type( t1 ) is type( t2 ):
        raise TypeError( 'Tree types must match' )
    self._size += len( t1 ) + len( t2 )
    if not t1.is_empty():
        t1._root._parent = node
        node._left = t1._root
        t1._root = None
        t1._size = 0
    if not t2.is_empty():
        t2._root._parent = node
        node._left = t2._root
        t2._root = None
        t2._size = 0
```

Array-Based Representation of Binary Trees

- Nodes are stored in an array A



- Node v is stored at A[rank(v)]
  - rank(root) = 1
  - if node is the left child of parent(node), rank(node) = 2 x rank(parent(node))
  - if node is the right child of parent(node), rank(node) = 2 x rank(parent(node)) + 1