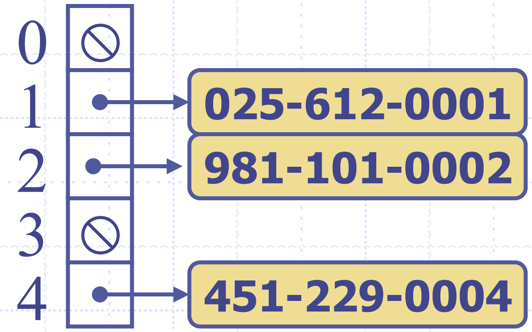
Hash Tables



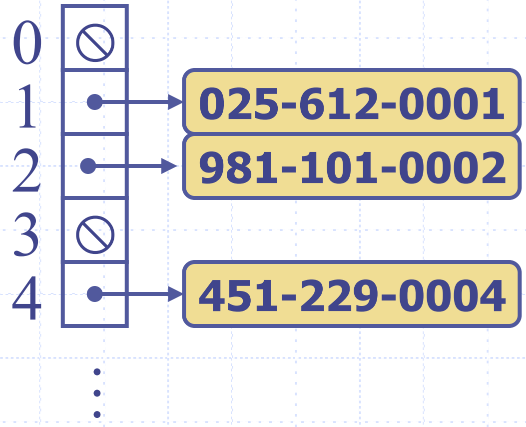
Recall the notion of a Map

- Intuitively, a map M supports the abstraction of using keys as indices with a syntax such as M[k].
- As a mental warm-up, consider a restricted setting in which a map with n items uses keys that are known to be integers in a range from 0 to $N - 1$, for some $N \geq n$.

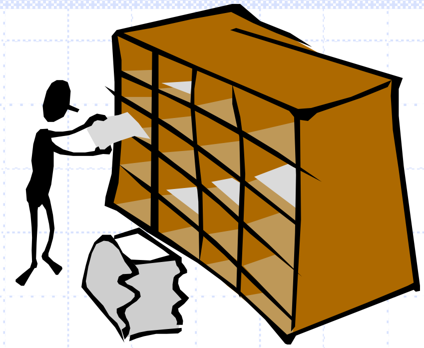
0	1	2	3	4	5	6	7	8	9	10
	D		Z			C	Q			

More General Kinds of Keys

- But what should we do if our keys are not integers in the range from 0 to N – 1?
  - Use a **hash function** to map general keys to corresponding indices in a table.
  - For instance, the last four digits of a Social Security number.



Hash Functions and Hash Tables



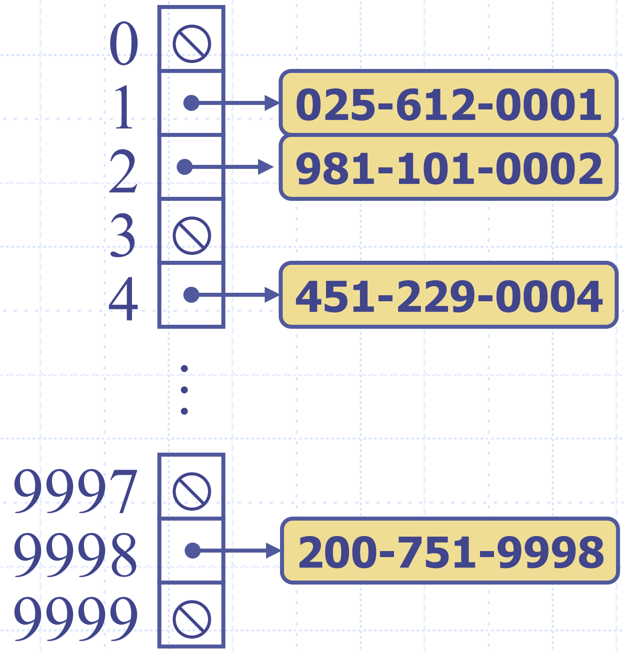
- A hash function h maps keys of a given type to integers in a fixed interval $[0, N-1]$
- Example:

  $$h(x) = x \bmod N$$

  is a hash function for integer keys
- The integer $h(x)$ is called the hash value of key x
- A hash table for a given key type consists of
  - Hash function h
  - Array (called table) of size N
- When implementing a map with a hash table, the goal is to store item (k, o) at index $i = h(k)$

SSN Example

- We design a hash table for a map storing entries as (SSN, Name), where SSN (social security number) is a nine-digit positive integer
- Our hash table uses an array of size $N = 10{,}000$ and the hash function $h(x) = \text{last four digits of } x$



Hash Functions

- A hash function is usually specified as the composition of two functions:

  Hash code:

  h_1: keys → integers

  Compression function:

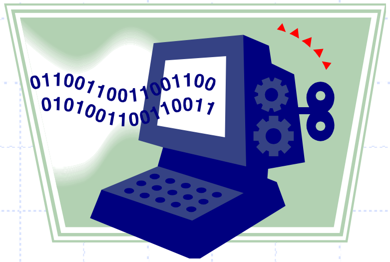
  h_2: integers → $[0, N-1]$

- The hash code is applied first, and the compression function is applied next on the result, i.e.,

  $$h(x) = h_2(h_1(x))$$

- The goal of the hash function is to "disperse" the keys in an apparently random way

Hash Codes



- Memory address:
  - We reinterpret the memory address of the key object as an integer Good in general, except for numeric and string keys
- Integer cast:
  - We reinterpret the bits of the key as an integer
  - Suitable for keys of length less than or equal to the number of bits of the integer
- Component sum:
  - We partition the bits of the key into components of fixed length (e.g., 16 or 32 bits) and we sum the components (ignoring overflows)
  - Suitable for numeric keys of fixed length greater than or equal to the number of bits of the integer type

Hash Codes (cont.)

- Polynomial accumulation:
  - We partition the bits of the key into a sequence of components of fixed length (e.g., 8, 16 or 32 bits)

    $$a_0\, a_1\, \ldots\, a_{n-1}$$

  - We evaluate the polynomial

    $$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots \\ \ldots + a_{n-1} z^{n-1}$$

    at a fixed value z, ignoring overflows
  - Especially suitable for strings (e.g., the choice $z = 33$ gives at most 6 collisions on a set of 50,000 English words)
- Polynomial $p(z)$ can be evaluated in $O(n)$ time using Horner's rule:
  - The following polynomials are successively computed, each from the previous one in $O(1)$ time

    $$p_0(z) = a_{n-1}$$

    $$p_i(z) = a_{n-i-1} + z p_{i-1}(z)$$
    $$(i = 1, 2, \ldots, n-1)$$
- We have $p(z) = p_{n-1}(z)$

Compression Functions

- Division:
  - $h_2(y) = y \bmod N$
  - The size N of the hash table is usually chosen to be a prime
  - The reason has to do with number theory and is beyond the scope of this course
- Multiply, Add and Divide (MAD):
  - $h_2(y) = (ay + b) \bmod N$
  - a and b are nonnegative integers such that
    $a \bmod N \neq 0$
  - Otherwise, every integer would map to the same value b

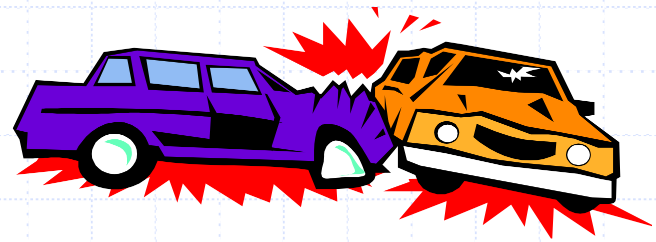
Abstract Hash Map Class

```
class HashMap( Map ):

    def __init__( self, cap = 11, p = 109345121 ):
        self._T = cap * [None]
        self._n = 0                              #number of entries in the map
        self._prime = p                          #prime for MAD compression
        self._scale = 1 + randrange( p - 1 ) #scale from 1 to p-1 for MAD
        self._shift = randrange( p )             #shift from 0 to p-1 for MAD

    def _hash_function(   self, k  ):
        return( hash( k )*self._scale+self._shift ) % self._prime % len( self._T )

    def __len__( self ):
        return self._n
```

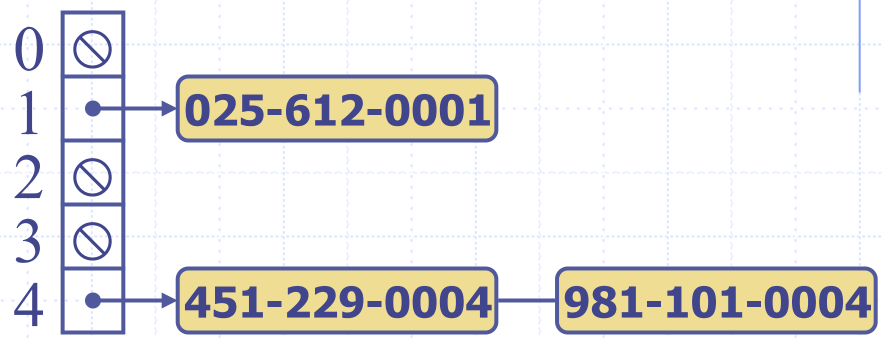
HashMap Class

```
def __len__( self ):
    return self._n

def __getitem__( self, k ):
    j = self._hash_function( k )
    return self._bucket_getitem( j,   k )

def __setitem__( self, k, v   ):
    j = self._hash_function( k )
    self._bucket_setitem( j, k, v )
    if self._n > len( self._T ) // 2:
        self._resize( 2 * len( self._T ) - 1 )

def __delitem__( self, k ):
    j = self._hash_function( k )
    self._bucket_delitem( j, k )
    self._n  -= 1

def _resize( self, c ):
    old = list( self.items() )
    self._T  = c * [None]
    self._n  = 0
    for (k,v) in old:
        self[k] = v
```

Collision Handling



- Collisions occur when different elements are mapped to the same cell



- Separate Chaining: let each cell in the table point to a linked list of entries that map there
- Separate chaining is simple, but requires additional memory outside the table

Map with Separate Chaining

Delegate operations to a list-based map at each cell:

Algorithm get(k):
return A[h(k)].get(k)

Algorithm put(k,v):
t = A[h(k)].put(k,v)
if t = **null then** {k is a new key}
    n = n + 1
return t

Algorithm remove(k):
t = A[h(k)].remove(k)
if t ≠ **null then** {k was found}
    n = n - 1
return t

ChainHashMap

```python
class ChainHashMap( HashMap ):

    def _bucket_getitem( self, j, k ):
        bucket = self._T[j]
        if bucket is None:
            return False
        return bucket[k]

    def _bucket_setitem( self, j, k, v ):
        if self._T[j] is None:
            self._T[j] = UnsortedListMap()
        oldsize = len( self._T[j] )
        self._T[j][k] = v
        if len( self._T[j] ) > oldsize:
            self._n += 1

    def _bucket_delitem( self, j, k ):
        bucket = self._T[j]
        if bucket is None:
            return False
        del bucket[k]

    def __iter__( self ):
        for bucket in self._T:
            if bucket is not None:
                for key in bucket:
                    yield key
```

Linear Probing

- Open addressing: the colliding item is placed in a different cell of the table
- Linear probing: handles collisions by placing the colliding item in the next (circularly) available table cell
- Each table cell inspected is referred to as a "probe"
- Colliding items lump together, causing future collisions to cause a longer sequence of probes

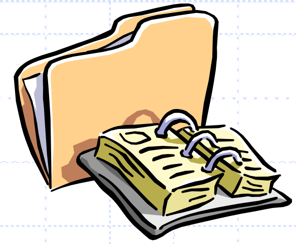
- Example:
  - $h(x) = x \bmod 13$
  - Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

0	1	2	3	4	5	6	7	8	9	10	11	12

$\Downarrow$

		41			18	44	59	32	22	31	73	
0	1	2	3	4	5	6	7	8	9	10	11	12

Search with Linear Probing



- Consider a hash table A that uses linear probing
- get(k)
  - We start at cell $h(k)$
  - We probe consecutive locations until one of the following occurs
    - An item with key k is found, or
    - An empty cell is found, or
    - N cells have been unsuccessfully probed

Algorithm *get*(k)
  $i = h(k)$
  $p = 0$
  repeat
    $c = A[i]$
    if $c = \varnothing$
      return *null*
    else if $c.getKey() = k$
      return $c.getValue()$
    else
      $i = (i + 1) \bmod N$
      $p = p + 1$
  until $p = N$
  return *null*

Updates with Linear Probing

- To handle insertions and deletions, we introduce a special object, called ***AVAILABLE***, which replaces deleted elements
- remove(k)
  - We search for an entry with key k
  - If such an entry (k, o) is found, we replace it with the special item ***AVAILABLE*** and we return element o
  - Else, we return ***null***
- put(k, o)
  - We throw an exception if the table is full
  - We start at cell $h(k)$
  - We probe consecutive cells until one of the following occurs
    - A cell i is found that is either empty or stores ***AVAILABLE***, or
    - N cells have been unsuccessfully probed
  - We store (k, o) in cell i

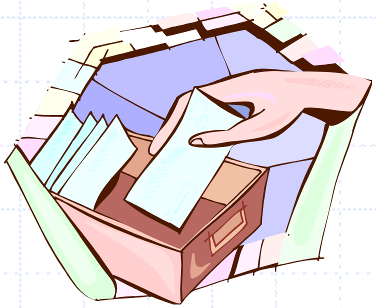
Hash Map with Linear Probing

```
class ProbeHashMap( HashMap ):

    _AVAIL = object()

    def _is_available( self, j ):
        return self._T[j] is None or self._T[j] is ProbeHashMap._AVAIL

    def _find_slot( self, j, k ):
        firstAvail = None
        while True:
            if self._is_available( j ):
                if firstAvail is None:
                    firstAvail = j
                if self._T[j] is None:
                    return (False,firstAvail)
            elif k == self._T[j]._key:
                return (True,j)
            j = (j + 1) % len( self._T )
```

ProbeHashMap

```python
def _bucket_getitem( self, j, k ):
    found, s = self._find_slot( j, k )
    if not found:
        return False
    return self._T[s]._value

def _bucket_setitem( self, j, k, v ):
    found, s = self._find_slot( j, k )
    if not found:
        self._T[s] = self._Item( k, v )
        self._n += 1
    else:
        self._T[s]._value = v

def _bucket_delitem( self, j, k ):
    found, s = self._find_slot( j, k )
    if not found:
        return False
    self._T[s] = ProbeHashMap._AVAIL

def __iter__( self ):
    for j in range( len( self._T ) ):
        if not self._is_available( j ):
            yield self._T[j]._key
```

Double Hashing



- Double hashing uses a secondary hash function $d(k)$ and handles collisions by placing an item in the first available cell of the series
  $$(i + jd(k)) \bmod N$$
  for $j = 0, 1, \ldots, N-1$
- The secondary hash function $d(k)$ cannot have zero values
- The table size N must be a prime to allow probing of all the cells
- Common choice of compression function for the secondary hash function:
  $$d_2(k) = q - k \bmod q$$
  where
  - $q < N$
  - q is a prime
- The possible values for $d_2(k)$ are
  $$1, 2, \ldots, q$$

Example of Double Hashing

- Consider a hash table storing integer keys that handles collision with double hashing
  - $N = 13$
  - $h(k) = k \bmod 13$
  - $d(k) = 7 - k \bmod 7$
- Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

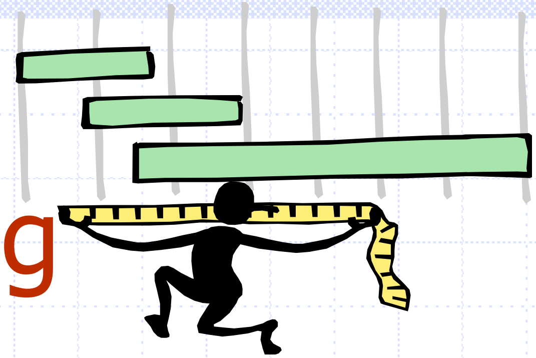
k	$h(k)$	$d(k)$	Probes		
18	5	3	5		
41	2	1	2		
22	9	6	9		
44	5	5	5	10	
59	7	4	7		
32	6	3	6		
31	5	4	5	9	0
73	8	4	8		

0	1	2	3	4	5	6	7	8	9	10	11	12

⇓

0	1	2	3	4	5	6	7	8	9	10	11	12
31		41			18	32	59	73	22	44		

Performance of Hashing



- In the worst case, searches, insertions and removals on a hash table take $O(n)$ time
- The worst case occurs when all the keys inserted into the map collide
- The load factor $a = n/N$ affects the performance of a hash table
- Assuming that the hash values are like random numbers, it can be shown that the expected number of probes for an insertion with open addressing is
  $$1 / (1 - a)$$
- The expected running time of all the dictionary ADT operations in a hash table is $O(1)$
- In practice, hashing is very fast provided the load factor is not close to 100%
- Applications of hash tables:
  - small databases
  - compilers
  - browser caches