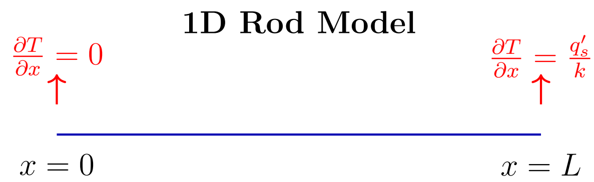
1D Neumann BVP for Heat Equation

Problem Statement: Solve for the temperature distribution $T(x,t), \quad 0 \le x \le L, \quad t \ge 0$ given the Neumann boundary conditions $\frac{\partial T}{\partial x}|_{x=0} = 0$ and $\frac{\partial T}{\partial x}|_{x=L} = \frac{q_s'}{k}$, and the initial condition $T(x,0) = T_0$.

- $T(x,t)$ – temperature in the rod, [K]
- T_0 – initial temperature [K]
- ρ – material density, [kg/m^3]
- c_p – specific heat, [J/(kg·K)]
- k – thermal conductivity, [W/(m·K)]
- α – thermal diffusivity, [m^2/s]
    - $\alpha = \frac{k}{\rho c_p}$
- q_s' – Sutton-Graves convective heat flux, [W/m^2]



$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \quad \longrightarrow \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

1 Principle of Superposition + Separation of Variables

$$T(x,t) = T_P(x) + u(x,t) \tag{2}$$

Homogeneous $u(x,t)$	Particular $T_P(x,t)$
Satisfies Heat Equation	Satisfies Heat Equaion
Satisfies Zero-Flux BC	Satisfies Problem BC
Satisfies IC	Doesn't satisfy IC

1.1 Particular Solution $T_P(x,t)$

Unlike most heat-equation BVPs, this one does not have a steady state distribution. There is a constant net heat flux entering the rod.

Let $\langle T \rangle = \frac{1}{L}\int_0^L T dx$ be the average temperature in the whole rod.

Using the specific heat formula, $Q = mc\Delta T \quad$ [J], we can take the derivative to obtain $\frac{dQ}{dt} = \underbrace{\rho \overbrace{AL}^{\text{Volume}}}_{\text{Mass}} c_p \frac{d\langle T \rangle}{dt} \quad$ [W]

Substituting and rearranging, we get $q_s' = \rho L c_p \frac{d\langle T \rangle}{dt} \quad [\mathrm{W/m^2}] \quad \longrightarrow \quad \frac{d\langle T \rangle}{dt} = \frac{q_s'}{\rho L c_p} \quad$ [K/s].
Since $\frac{d\langle T \rangle}{dt} =$ Constant, $\langle T \rangle$ is positive-linear with respect to t.

The homogeneous solution $u(x,t)$ has zero-flux boundary conditions (heat cannot enter or exit). It cannot model the temperature increase. For large values of t, the homogeneous solution will approach a steady state.

Thus, the particular solution will have a linear in t to model the temperature increase.

$$T_P = f(x) + g(x)\cdot t \tag{3}$$

Applying the Heat Equation:

$$\frac{\partial T_P}{\partial t} = \alpha \frac{\partial^2 T_P}{\partial x^2} \tag{4}$$

$$g(x) = \alpha[f''(x) + g''(x)t] \tag{5}$$

Equating t-polynomial coefficients of T_P:

$$g(x) = \alpha f''(x) \tag{6}$$

$$g''(x) = 0 \tag{7}$$

Set up the undetermined coefficients:

$$g'(x) = C_1 \tag{8}$$

$$g(x) = C_1 x + C_2 \tag{9}$$

$$f''(x) = \frac{C_1 x + C_2}{\alpha} \tag{10}$$

$$f'(x) = \frac{C_1 x^2}{2\alpha} + \frac{C_2 x}{\alpha} + C_3 \tag{11}$$

$$f(x) = \frac{C_1 x^3}{6\alpha} + \frac{C_2 x^2}{2\alpha} + C_3 x + C_4 \tag{12}$$

Applying the Boundary Conditions for T_P:

$$\left.\frac{\partial T_P}{\partial x}\right|_{x=0} = 0 \quad \rightarrow \quad f'(0) + g'(0)\cdot t = 0 \quad \rightarrow \quad C_3 + C_1 t = 0 \tag{13}$$

$$\left.\frac{\partial T_P}{\partial x}\right|_{x=L} = \frac{q_s'}{k} \quad \rightarrow \quad f'(L) + g'(L)\cdot t = \frac{q_s'}{k} \quad \rightarrow \quad \frac{C_1 L^2}{2\alpha} + \frac{C_2 L}{\alpha} + C_3 + C_1 t = \frac{q_s'}{k} \tag{14}$$

Since the BC must hold for all t, $C_1 = 0$. Then, $C_3 = 0$, and $C_2 = \frac{\alpha q_s'}{kL}$

The constant C_4 represents the initial spatial distribution of T_P. Since T_P does not need to satisfy an initial condition (which will be handled by the homogeneous solution $u(x,t)$), we can set $C_4 = 0$ without loss of generality. The initial condition for $u(x,t)$ will then be $u(x,0) = T_0 - T_P(x,0)$.

$$T_P(x,t) = \frac{q_s'}{2kL}x^2 + \frac{\alpha q_s'}{kL}t \tag{15}$$

1.2 Homogenous Solution $u(x,t)$

Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{16}$$

with boundary conditions: $\frac{\partial u}{\partial x}|_{x=0} = \frac{\partial u}{\partial x}|_{x=L} = 0$ and initial condition: $u(x,0) = T_0 - \frac{q_s'}{2kL}x^2$

2 Separation of variables:

$$u(x,t) = X(x)\Theta(t) \tag{17}$$

$$X(x)\Theta'(t) = \alpha X''(x)\Theta(t) \longrightarrow \frac{\Theta'}{\alpha\Theta} = \frac{X''}{X} = -\lambda \tag{18}$$

$$\Theta' = -\alpha\lambda\Theta \longrightarrow \Theta = e^{-\alpha\lambda t} \tag{19}$$

$$X'' + \lambda X = 0 \longrightarrow X = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x) \tag{20}$$

3 Applying Boundary Conditions:

$$X'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x) \tag{21}$$

$$X'(0) = C_2\sqrt{\lambda} = 0 \Rightarrow C_2 = 0 \tag{22}$$

$$X'(L) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0 \tag{23}$$

This gives us: $\sin(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda}L = n\pi$ for $n = 0, 1, 2, \ldots$
Note that we don't want $C_1 = 0$ because that would be a trivial solution.
The eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 0, 1, 2 \ldots$
The eigenfunctions are $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

The homogeneous solution is:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \tag{24}$$

4 Applying Initial Conditions:

Since T_P doesn't depend on t and $T(x,0) = T_0$, $u(x,0) = T_0 - T_P = T_0 - \frac{q_s'}{2kL}x^2$

As $e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} = 1$ for $t = 0$,

$$T_0 - \frac{q_s'}{2kL}x^2 = \sum_{n=\mathbf{0}}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \tag{25}$$

4.1 Orthogonality of Cosines

The cosine functions $\cos\left(\frac{n\pi x}{L}\right)$ form an orthogonal set on $[0, L]$ with the following properties:

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} L & \text{if } m = n = 0 \\ \frac{L}{2} & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n \end{cases} \tag{26}$$

Proof: Use the cosine product-to-sum identity:

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx = \int_0^L \frac{1}{2}\cos\left(\frac{\pi x}{L}(m-n)\right) + \frac{1}{2}\cos\left(\frac{\pi x}{L}(m+n)\right)dx \tag{27}$$

$$\begin{cases} = \int_0^L 1 dx = L & \text{if } m = n = 0 \\ = \int_0^L \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi n x}{L}\right) = [\frac{1}{2}x - \frac{L}{4\pi n}\sin\left(\frac{2\pi n x}{L}\right)]|_0^L = \left(\frac{1}{2}L - 0\right) - (0 - 0) = \frac{L}{2} & \text{if } m = n \neq 0 \\ = \left[\frac{L}{2\pi(m-n)}\sin\left(\frac{\pi x}{L}(m-n)\right) + \frac{L}{2\pi(m+n)}\sin\left(\frac{\pi x}{L}(m+n)\right)\right]\Bigg|_0^L = (0+0) - (0+0) = 0 & \text{if } m \neq n \end{cases}$$

4.2 Applying Orthogonality

$$\left(T_0 - \frac{q_s'}{2kL}x^2\right)\cos\left(\frac{m\pi x}{L}\right) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) \tag{28}$$

Integrating both sides:

$$\int_0^L \left(T_0 - \frac{q_s'}{2kL}x^2\right)\cos\left(\frac{0\pi x}{L}\right)dx = A_0 \cdot (L + 0 + 0 + \cdots) \quad m = 0 \tag{29}$$

$$\int_0^L \left(T_0 - \frac{q_s'}{2kL}x^2\right)\cos\left(\frac{m\pi x}{L}\right)dx = A_m \cdot \left(0 + 0 + \cdots + \frac{L}{2} + 0 + 0 + \cdots\right) \quad m \geq 1 \tag{30}$$

Rearranging and swapping the subscript m for n, we get:

$$A_0 = \frac{1}{L}\int_0^L \left(T_0 - \frac{q_s'}{2kL}x^2\right)dx = T_0 - \frac{q_s'L}{6k} \tag{31}$$

$$A_n = \frac{2}{L}\int_0^L \left(T_0 - \frac{q_s'}{2kL}x^2\right)\cos\left(\frac{n\pi x}{L}\right)dx \tag{32}$$

Taking the integral $\int_0^L \left(T_0 - \frac{q_s'}{2kL}x^2\right)\cos\left(\frac{n\pi x}{L}\right)dx$:

D		I		$+/-$
$T_0 - \frac{q_s'}{2kL}x^2$	$\searrow$	$\cos\left(\frac{n\pi x}{L}\right)$		$+$
$-\frac{q_s'}{kL}x$	$\searrow$	$\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$	$\rightarrow$	-
$-\frac{q_s'}{kL}$	$\searrow$	$-\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$	$\rightarrow$	$+$
0		$-\frac{L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)$		

$$\int_0^L \left(T_0 - \frac{q_s'}{2kL}x^2\right)\cos\left(\frac{n\pi x}{L}\right)dx \tag{33}$$

$$= \left[\left(T_0 - \frac{q_s'}{2kL}x^2\right)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{q_s'}{kL}x\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right) + \frac{q_s'}{kL}\frac{L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L \tag{34}$$

$$= \left[0 - \frac{q_s'}{kL}L\frac{L^2}{n^2\pi^2}\cos(n\pi) + 0\right] - \left[0 - 0 + 0\right] \tag{35}$$

$$= -\frac{q_s'L^2}{kn^2\pi^2}(-1)^n \tag{36}$$

$$A_n = -\frac{2q_s'L}{kn^2\pi^2}(-1)^n \tag{37}$$

5 Putting it all together:

$$u(x,t) = T_0 - \frac{q_s'L}{6k} - \sum_{n=1}^{\infty}\frac{2q_s'L}{kn^2\pi^2}(-1)^n\cos\left(\frac{n\pi x}{L}\right)e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \tag{38}$$

$$T(x,t) = T_P + u \tag{39}$$

$$T(x,t) = \frac{q_s'}{2kL}x^2 + \frac{\alpha q_s'}{kL}t + T_0 - \frac{q_s'L}{6k} - \sum_{n=1}^{\infty}\frac{2q_s'L}{kn^2\pi^2}(-1)^n\cos\left(\frac{n\pi x}{L}\right)e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \tag{40}$$