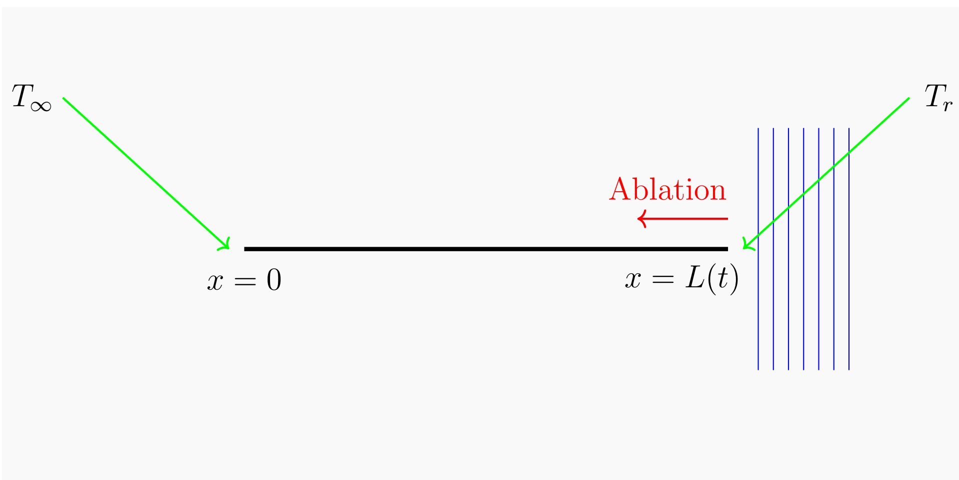
Hypersonic Aerothermodynamics: Radiation and Ablation on a 1D Leading Edge

Problem Statement: Suppose that you have an axisymmetric hypersonic rocket with a 1D-approximated cylindrical leading edge of length $L(t)$, radius r and temperature distribution $T(x,t)$.

The leading edge is subject to convection, radiation, and ablation. Assume a Sutton-Graves constant heat flux q_s' [W/m^2] at the front ($x = L(t)$). Ablation occurs at $x = L(t)$ when $T(L(t),t) \geq T_{abl}$, and $L(t)$ decreases over time. There are also Stefan-Boltzmann radiation effects at the stagnation point and leading edge sides. Assume the back is insulated. The freestream temperature is T_∞, and you must calculate the stagnation point temperature T_r using the Mach Number relation in your numerical solution.

1D Leading Edge



1 Constants

You will be provided with the following constants:

- r – leading edge radius, [m]
- ρ – material density, [kg/m^3]
- c_p – specific heat, [J/(kg·K)]
- k – thermal conductivity, [W/(m·K)]
- L_0 – initial leading edge length, [m]
- T_∞ – far-field gas temperature, [K]
- T_0 – initial temperature of the material, [K]
- M_∞ – freestream Mach number,
- γ – ratio of specific heats,
- ϵ – surface emissivity,

- σ – Stefan-Boltzmann constant, [W/(m^2·K^4)]
- T_{abl} – ablation onset temperature, [K]
- H – heat of ablation, [J/kg]
- q'_s – Sutton-Graves convective heat flux, [W/m^2]

2 Student Task

1. Derive all the mathematical equations and constraints using fundamental physics and hypersonic aerothermodynamics.

2. Use numerical methods to solve and model the temperature distribution $T(x,t)$ and leading edge length function $L(t)$.

3 Hints

- Consider the energy balance in the front.

- There is zero heat flux in the back ($x = 0$).

- Ablation will have a piecewise condition.

- You can express the stagnation temperature as T_r in your intermediate work. When you solve numerically though, you must express it in terms of the freestream temperature and Mach number, using the Mach Number Relation:

$$T_r = T_\infty \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)$$

- T_∞ and T_r are OF THE AIR, not of the 1-D leading edge.

- Keep everything in SI units.

- Do not try to model a 3D solution. You will waste a lot of time doing so. This problem should be doable on a laptop.

4 [SOLUTION] Solving the Physics

Side radiation losses – Stefan-Boltzmann Radiation Flux Equation

$$q''_{radiation} = \epsilon\sigma(T^4 - T_\infty^4) \quad [\text{W/m}^2] \tag{1}$$

Infinitesimal volume = $\pi r^2 \, dx$
Infinitesimal surface area = $2\pi r \, dx$

Multiply both sides by surface area: Total heat loss rate:

$$\dot{Q} = 2\pi r\epsilon\sigma(T^4 - T_\infty^4) \quad [\text{W}] \tag{2}$$

Divide both sides by volume: Heat loss rate per unit volume:

$$\frac{\dot{Q}}{\pi r^2 dx} = q''' = \frac{2}{r}\epsilon\sigma(T^4 - T_\infty^4) \quad [\text{W/m}^3] \tag{3}$$

To apply this, we need to add a forcing term to the Heat Equation. The equation for thermal diffusivity is $\alpha = \frac{k}{\rho c_p}$

Rearranged and Forced Heat Equation:

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} - \frac{2}{r} \epsilon \sigma (T^4 - T_\infty^4) \quad [\mathrm{W/m^3}] \tag{4}$$

Fourier's Law of Heat Conduction:

$$\vec{q''} = -k \nabla T \quad \Rightarrow \quad q''_{+x} = -k \frac{\partial T}{\partial x} \tag{5}$$

Left Boundary ($x = 0$) – Insulated (Zero Heat Flux):

$$k \frac{\partial T}{\partial x}\bigg|_{x=0} = 0 \tag{6}$$

Right Boundary ($x = L(t)$) – Energy Balance at Ablating Surface:

At the front surface, we apply conservation of energy. Heat fluxes entering the surface must equal heat fluxes leaving plus energy consumed by ablation:

$$\text{Convective Heat In} + \text{Radiation Heat In} = \text{Conduction Out} + \text{Ablation Energy Consumption} \tag{7}$$

Ablation Rate Equation:

$$-\frac{dL}{dt} = \frac{||q''_{abl}||}{\rho H} \tag{8}$$

where $||q''_{abl}||$ is the magnitude of heat flux consumed by ablation.

Complete Boundary Condition at $x = L(t)$:

$$\textbf{IF } T(L(t), t) \geq T_{abl}$$

$$q''_s + \epsilon \sigma \left(T_r^4 - [T(L(t), t)]^4 \right) = k \frac{\partial T}{\partial x}\bigg|_{x=L(t)} + \rho H \left(\frac{-dL}{dt} \right) \tag{9}$$

$$\textbf{ELSE}$$

$$q''_s + \epsilon \sigma \left(T_r^4 - [T(L(t), t)]^4 \right) = k \frac{\partial T}{\partial x}\bigg|_{x=L(t)} \quad \textbf{AND} \quad \frac{dL}{dt} = 0 \tag{10}$$

5 [SOLUTION] Solving the Math

Mathematical Problem Statement: Use numerical methods to find the temperature distribution $T(x, t)$ and leading edge length function $L(t)$.

Heat Equation:

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} - \frac{2}{r} \epsilon \sigma (T^4 - T_\infty^4), \quad 0 \leq x \leq L(t), \tag{11}$$

Initial Conditions:

$$T(x,0) = T_0, \quad L(0) = L_0. \tag{12}$$

Boundary Conditions:

$x = 0$:

$$k\frac{\partial T}{\partial x}\bigg|_{x=0} = 0 \tag{13}$$

$x = L(t)$:

$$\textbf{IF } T(L(t),t) \geq T_{abl}$$

$$q_s'' + \epsilon\sigma\left(T_r^4 - [T(L(t),t)]^4\right) = k\frac{\partial T}{\partial x}\bigg|_{x=L(t)} + \rho H\left(\frac{-dL}{dt}\right) \tag{14}$$

$$\textbf{ELSE}$$

$$q_s'' + \epsilon\sigma\left(T_r^4 - [T(L(t),t)]^4\right) = k\frac{\partial T}{\partial x}\bigg|_{x=L(t)} \quad \textbf{AND} \quad \frac{dL}{dt} = 0 \tag{15}$$

Coordinate Transformation: Transform the coordinates to normalize x with the current length $L(t)$.

Transform	Inverse
$\xi = \frac{x}{L(t)}$	$x = \xi L(\tau)$
$\tau = t$	$t = \tau$

Chain rules:

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \tau}\frac{\partial \tau}{\partial x} = \frac{\partial T}{\partial \xi}\frac{1}{L(\tau)} \tag{16}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial \xi}\frac{1}{L(\tau)}\right) = \frac{1}{L(\tau)}\frac{\partial^2 T}{\partial \xi^2}\frac{\partial \xi}{\partial x} = \frac{1}{L(\tau)^2}\frac{\partial^2 T}{\partial \xi^2} \tag{17}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial T}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial T}{\partial \xi}\frac{-x}{L(\tau)^2}\frac{dL}{d\tau} + \frac{\partial T}{\partial \tau} = \frac{\partial T}{\partial \tau} - \frac{\xi}{L(\tau)}\frac{\partial T}{\partial \xi}\frac{dL}{d\tau} \tag{18}$$

5.1 Transformed Equations:

New Heat Equation:

$$\rho c_p\left(\frac{\partial T}{\partial \tau} - \frac{\xi}{L(\tau)}\frac{\partial T}{\partial \xi}\frac{dL}{d\tau}\right) = \frac{k}{L(\tau)^2}\frac{\partial^2 T}{\partial \xi^2} - \frac{2}{r}\epsilon\sigma(T^4 - T_\infty^4), \quad 0 \leq \xi \leq 1, \tag{19}$$

Initial Conditions:

$$T(\xi,0) = T_0, \quad L(0) = L_0 \tag{20}$$

Boundary Conditions:
$\xi = 0$:

$$\left.\frac{\partial T}{\partial \xi}\right|_{\xi=0} = 0 \tag{21}$$

$\xi = 1$:

$$\textbf{IF } T(1,\tau) \geq T_{abl}$$

$$q_s' + \epsilon\sigma\left(T_r^4 - [T(1,\tau)]^4\right) = \frac{k}{L(\tau)}\left.\frac{\partial T}{\partial \xi}\right|_{\xi=1} + \rho H\left(\frac{-dL}{d\tau}\right) \tag{22}$$

$$\textbf{ELSE}$$

$$q_s' + \epsilon\sigma\left(T_r^4 - [T(1,\tau)]^4\right) = \frac{k}{L(\tau)}\left.\frac{\partial T}{\partial \xi}\right|_{\xi=1} \quad \textbf{AND} \quad \frac{dL}{d\tau} = 0 \tag{23}$$

5.2 [SOLUTION] Numerical Implementation

Pick one:

- Method of Lines - Finite Differences
- Method of Lines - Spectral
- Discontinuous Galerkin

Scope:
Honors Project will include an analytical solution (See the 1D Neumann BVP File) and FD MOL.
AIAA conference project will compare all 3 numerical methods.

5.2.1 Method of Lines - Finite Differences

Discretize the ξ-axis into points $\xi_0, \xi_1, \xi_2, \cdots, \xi_N$

Nomenclature (using a general function $f(x)$):

- h is the separation between x-axis points.
- Let $x_{n+m} = x_n + mh$
- Let $f_n = f(x_n)$

By Taylor Series:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots \tag{24}$$

Applying to f_n:

$$f_{n+1} = f_n + hf_n' + \frac{h^2}{2}f_n'' + \frac{h^3}{6}f_n''' + \frac{h^4}{24}f_n^{(4)} + \cdots \tag{25}$$

$$f_{n-1} = f_n - hf_n' + \frac{h^2}{2}f_n'' - \frac{h^3}{6}f_n''' + \frac{h^4}{24}f_n^{(4)} - \cdots \tag{26}$$

1st Partial Centered Difference:

$$f_{n+1} - f_{n-1} = 2hf'_n + \frac{h^3}{3}f'''_n + \frac{h^5}{60}f_n^{(5)} + \cdots \tag{27}$$

$$f'_n = \frac{f_{n+1} - f_{n-1}}{2h} + O(h^2) \tag{28}$$

2nd Partial Centered Difference:

$$f_{n+1} + f_{n-1} = 2f_n + h^2 f''_n + \frac{h^4}{12}f_n^{(4)} + \cdots \tag{29}$$

$$f''_n = \frac{f_{n-1} - 2f_n + f_{n+1}}{h^2} + O(h^2) \tag{30}$$

Discretizing the PDE:

Handling the Boundaries:
CHOOSE:
A) Apply lateral radiation to all points (faithful to the continuous PDE)

- Faithful to Continuous PDE
- Argues that each discrete point represents a finite volume
- Discretization Philosophy: Discrete points inherit all properties of the continuous domain.

B) Apply it only to interior points (geometrically consistent with discrete boundaries)

- Avoids potential double counting of radiation (T_{inf} and T_r) at $x = L$
- Argues that each discrete point represents a single point in space
- Discretization Philosophy: Discrete points should represent specific geometric entities.

The Fundamental Question:

- When should physical reality override mathematical formalism?
- How do you handle boundary effects in discretization?
- What's the "correct" interpretation of discrete grid points?
- When reducing dimensionality, what should be preserved?

I choose to apply option B. According to the Finite Difference theory, each (ξ_i, T_i) is treated as a specific point in space rather than a finite volume. Therefore, including lateral radiation at the boundary points would introduce a double-counting problem. The boundary $\xi = 0$ should be adiabatic while $\xi = 1$ already has its stagnation radiation term T_r. These boundary nodes therefore represent flat end surfaces, with their heat transfer fully accounted for by the prescribed boundary conditions, while lateral radiation is applied only to interior points representing cylindrical segments. Ultimately, it comes down to the philosophical problem of applied vs. theoretical mathematics. Given the goal of ultimately creating a hypersonic simulation software with applied PDEs, this choice reflects the applied mathematics principle that discretization schemes should preserve physical validity while maintaining mathematical rigor.

5.2.2 Picard Iteration Loop

At the moving boundary ($\xi = 1$), the coupled system requires simultaneous evaluation of the temperature gradient $\frac{\partial T}{\partial \xi}\big|_{\xi=1}$ and the boundary recession rate $\frac{dL}{dt}$. These two unknowns appear in each other's boundary conditions, which prevents direct evaluation. To handle this coupling, a fixed-point (Picard) iteration with relaxation is employed.

Initial guesses:

$$\left(\frac{\partial T}{\partial \xi}\bigg|_{\xi=1}\right)^{(0)} = \frac{T_N - T_{N-1}}{h}, \quad \left(\frac{dL}{dt}\right)^{(0)} = 0, \tag{31}$$

Simplifying Variable:

$$q_{\text{in}} = q_s + \epsilon\sigma\left(T_r^4 - T_N^4\right) \tag{32}$$

Smooth Activation Function for ablation boundary:

$$f(T) = \begin{cases} 0, & T < T_{\text{abl}} - \frac{\Delta T}{2} \\ \frac{1}{2}\left[1 + \sin\left(\frac{\pi}{\Delta T}\left(T - T_{\text{abl}}\right)\right)\right], & |T - T_{\text{abl}}| \le \frac{\Delta T}{2} \\ 1, & T > T_{\text{abl}} + \frac{\Delta T}{2} \end{cases} \tag{33}$$

At iteration m:

$$\left(\frac{dL}{dt}\right)^{(m+1)} = \frac{\alpha}{-\rho H}\left[q_{\text{in}} - \frac{k}{L}\left(\frac{\partial T}{\partial \xi}\bigg|_{\xi=1}\right)^{(m)}\right] \tag{34}$$

$$\left(\frac{\partial T}{\partial \xi}\bigg|_{\xi=1}\right)^{(m+1)} = \frac{L}{k}\left[q_{\text{in}} + \rho H\left(\frac{dL}{dt}\right)^{(m+1)}\right] \tag{35}$$

To promote convergence, relaxation is applied:

$$\left(\frac{dL}{dt}\right)^{(m+1)} \leftarrow \omega\left(\frac{dL}{dt}\right)^{(m+1)} + (1-\omega)\left(\frac{dL}{dt}\right)^{(m)} \tag{36}$$

$$\left(\frac{\partial T}{\partial \xi}\bigg|_{\xi=1}\right)^{(m+1)} \leftarrow \omega\left(\frac{\partial T}{\partial \xi}\bigg|_{\xi=1}\right)^{(m+1)} + (1-\omega)\left(\frac{\partial T}{\partial \xi}\bigg|_{\xi=1}\right)^{(m)} \tag{37}$$

The iteration continues until both variables satisfy the convergence criterion

$$\left|\left(\frac{\partial T}{\partial \xi}\bigg|_{\xi=1}\right)^{(m+1)} - \left(\frac{\partial T}{\partial \xi}\bigg|_{\xi=1}\right)^{(m)}\right| < \text{tolerance}, \quad \left|\left(\frac{dL}{dt}\right)^{(m+1)} - \left(\frac{dL}{dt}\right)^{(m)}\right| < \text{tolerance}. \tag{38}$$

This iterative procedure ensures that the nonlinear boundary conditions are satisfied in a numerically stable manner, while the relaxation factor $\omega \in (0, 1)$ damps oscillations and improves robustness.