AN EXPLICIT DETERMINATION OF THE SPRINGER MORPHISM

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Abstract. Let G be a simply connected semisimple algebraic groups over $\mathbb{C}$ and let $\rho : G \to GL(V_\lambda)$ be an irreducible representation of G of highest weight λ. Suppose that ρ has finite kernel. Springer defined an adjoint-invariant regular map with Zariski dense image from the group to the Lie algebra, $\theta_\lambda : G \to \mathfrak{g}$, which depends on λ. This map, θ_λ, takes the maximal torus T of G to its Lie algebra $\mathfrak{t}$. Thus, for a given simple group G and an irreducible representation V_λ, one may write $\theta_\lambda(t) = \sum_{i=1}^{n} c_i(t)\check{\alpha}_i$, where we take the simple coroots $\{\check{\alpha}_i\}$ as a basis for $\mathfrak{t}$. We give a complete determination for these coefficients $c_i(t)$ for any simple group G as a sum over the weights of the torus action on V_λ.

1. Introduction

Let G be a connected reductive algebraic group over $\mathbb{C}$ with Borel subgroup B and maximal torus $T \subset B$ of rank n with character group $X^*(T)$. Let P be a standard parabolic subgroup with Levi subgroup L containing T. Let W (resp. W_L) be the Weyl group of G (resp. L). Let V_λ be an irreducible almost faithful representation of G with highest weight λ, i.e. λ is a dominant integral weight and the corresponding map $\rho_\lambda : G \to GL(V_\lambda)$ has finite kernel. Then, Springer defined an adjoint-invariant regular map with Zariski dense image from the group to its Lie algebra, $\theta_\lambda : G \to \mathfrak{g}$, which depends on λ (Sect. 2.1).

In recent work by Kumar [5], the Springer morphism is used in a crucial way to extend a classical result relating the polynomial representation ring of the general linear group GL_r and the singular cohomology ring $H^*(Gr(r,n))$ of the Grassmannian of r-dimensional complex linear subspaces of $\mathbb{C}^n$ to the Levi subgroups of any reductive group G and the

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cohomology of the corresponding flag varieties G/P. Computing $\theta_\lambda|_T$ is integral to this process. Importantly, θ_λ takes the maximal torus T to its Lie algebra $\mathfrak{t}$, thus inducing an injective $\mathbb{C}$-algebra homomorphism $(\theta_\lambda|_T)^* : \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[T]$ between the corresponding affine coordinate rings. Let L be the Levi subgroup of the parabolic P which contains the torus T. The Springer morphism is equivariant under the adjoint action and thus $(\theta_\lambda|_T)^*$ takes $\mathbb{C}[\mathfrak{t}]^{W_L}$ to $\mathbb{C}[T]^{W_L}$. One can then define the $\lambda -$ *polynomial subring* $Rep^{\mathbb{C}}_{\lambda-poly}(L)$ to be the image of $\mathbb{C}[\mathfrak{t}]^{W_L}$ under $(\theta_\lambda|_T)^*$ (as $Rep^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L}$). Here $Rep^{\mathbb{C}}(L)$ is the complex representation ring of L. This leads to a surjective $\mathbb{C}$-algebra homomorphism ξ^P_λ : $Rep^{\mathbb{C}}_{\lambda-poly}(L) \to H^*(G/P, \mathbb{C})$, as in [5]. The aim of this work is to compute $\theta_\lambda|_T$ in a uniform way for all simple algebraic groups G and any dominant integral weight λ.

As $\theta_\lambda|_T$ maps T into $\mathfrak{t}$, we have that for a given simple group G and an irreducible representation V_λ, one may write

$$\theta_\lambda(t) = \sum_{i=1}^{n} c_i(\lambda)\check{\alpha}_i,$$

where we take the simple coroots $\{\check{\alpha}_i\}$ as a basis for $\mathfrak{t}$. We give a complete determination for these coefficients $c_i(t)$ for any simple, simply-connected algebraic group G as a sum over the weights of the torus action on V_λ.

For a given representation V_λ, let Λ_λ be the set of weights appearing in the weight space decomposition of $V_\lambda = \bigoplus V_\lambda^\mu$, listed with multiplicity. Let $\omega_1, ..., \omega_n$ be the fundamental weights in $\mathfrak{t}^*$, and consider the weights $\mu \in \Lambda_\lambda$ written in the fundamental weight basis, i.e. $\mu = (\mu_1, ..., \mu_n) = \mu_1\omega_1 + ... + \mu_n\omega_n$. Let $e^\mu(t) \in X^*(T)$ be the corresponding character of T. Then we find (Sect. 3) that,

Theorem 1. *The coefficients $c_i(t)$ are determined by the following set of equations.*

$$\begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1 e^{\mu}(t) \\ \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_n e^{\mu}(t) \end{pmatrix} = S(G,\lambda) \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix},$$

where $S(G,\lambda) = \{ \sum\limits_{\mu\in\Lambda_\lambda} \mu_i\mu_j\}_{ij}$.

Our main result (Sect. 4) determines that

Theorem 2. *The above matrix*

$$S(G,\lambda) := \{ \sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j\}_{ij} = (\frac{1}{2}\sum_{\mu\in\Lambda_\lambda} \mu_i^2) S \ ,$$

where S is a symmetrization of the Cartan matrix A for G, and μ_i is the coordinate of the fundamental weight corresponding to a long root (or any root in the simply-laced case).

In particular, for the simply-laced groups $S(G,\lambda) = (\frac{1}{2}\sum\limits_{\mu\in\Lambda_\lambda}\mu_1^2)A$. The determination of $S(G,\lambda)$ relies on the fact that Λ_λ is invariant under the action of the Weyl group W, and moreover that if $\sigma \in W$ then $dim(V_\mu) = dim(V_{\sigma.\mu})$.

2. Background

Let G be a simply-connected semi-simple algebraic group over $\mathbb{C}$ (though the constructions of this section are valid in the more general case of a connected reductive complex group). Denote its Lie algebra $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus\limits_{\alpha} \mathfrak{g}_\alpha$ of rank n, and fixed base of simple roots $\Delta = \{\alpha_j\}$. Take the set of simple co-roots $\check{\Delta} = \{\check{\alpha}_j\}$ as a basis for the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Then $\mathfrak{t}_{\mathbb{Z}} = \bigoplus\limits_{j=1}^{n} \mathbb{Z}\check{\alpha}_j$ is the co-root lattice. Further, the weight lattice is $\mathfrak{t}^*_{\mathbb{Z}} = \bigoplus\limits_{i=1}^{n} \mathbb{Z}\omega_i$, where $\omega_i \in \mathfrak{t}^*$ is the i^{th} fundamental weight of $\mathfrak{g}$ defined by $\omega_i(\check{\alpha}_j) = \delta_{ij}$. Then the maximal torus $T \subset G$

(with Lie algebra $\mathfrak{t}$) can be identified with $T = Hom_{\mathbb{Z}}(\mathfrak{t}^*_{\mathbb{Z}}, \mathbb{C}^*)$ as in [7]. Finally, let W be the Weyl group of G, generated by the simple reflections s_i. So for $\mu \in \mathfrak{t}^*$, $s_i(\mu) = \mu - \mu(\check{\alpha}_i)\alpha_i$.

Let V_λ be the irreducible representation of G with highest weight λ. Then V_λ has weight space decomposition

$$V_\lambda = \bigoplus V_\lambda^\mu$$

where $V_\lambda^\mu = \{v \in V_\lambda |\ t.v = ((\mu_1\omega_1 + ... + \mu_n\omega_n)(t))v\ \forall v \in V_\lambda\}$ is the weight space with weight $\mu = \mu_1\omega_1 + ... + \mu_n\omega_n$.

So for $t \in T$ and $v \in V_{\mu_1,\mu_2,...,\mu_n}$ we have that the action of t on v is given by

$$t.v = t(\mu_1, ..., \mu_n)v = e^\mu(t)v$$

where $(\mu_1, ...\mu_n) = \mu_1\omega_1 + ... + \mu_n\omega_n$. Additionally $\check{\alpha}_j \in \mathfrak{t}$ acts on v by

$$\check{\alpha}_j.v = (\mu_1\omega_1 + ... + \mu_n\omega_n)(\check{\alpha}_j)v = \mu_j v.$$

2.1. **Springer Morphism.** For a given almost faithful irreducible representation V_λ of G we define the Springer morphism as in [1]

$$\theta_\lambda : G \to \mathfrak{g}$$

given by

$$\begin{tikzcd} G \arrow[r] \arrow[drr, "\theta_\lambda"] & Aut(V(\lambda)) \subset End(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp \arrow[d, "\pi"] \\ & \mathfrak{g} \end{tikzcd}$$

where $\mathfrak{g}$ sits canonically inside $End(V_\lambda)$ via the derivative $d\rho_\lambda$, the orthogonal complement $\mathfrak{g}^\perp$ is taken via the adjoint invariant form $< A, B >= tr(AB)$ on $End(V_\lambda)$, and π is the projection onto the $\mathfrak{g}$ component. Note, that since $\pi \circ d\rho_\lambda$ is the the identity map, θ_λ is a local diffeomorphism at 1, and hence has Zariski dense image. Since the decomposition $End(V_\lambda) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ is G-stable, θ_λ is invariant under conjugation in G. Importantly, θ_λ restricts to $\theta_{\lambda|T} : T \mapsto \mathfrak{t}$

Note :

This construction was also studied by Kostant and Michor [4]. What we are calling the Springer morphism they referred to as a Generalized Cayley Map. As this work was inspired by Kumar's we adopt his nomenclature.

3. General Case

Let V_λ be a d dimensional almost faithful irreducible representation of G of highest weight λ. Let $\Lambda_\lambda = \{(\mu_1^i, ..., \mu_n^i)\}_{i=1}^d$ be an enumeration of the set of weights considered with their multiplicity that appear in the weight space decomposition of V_λ (so μ_j^i is the coordinate of the j^{th} fundamental weight for the i^{th} weight in the decomposition) Then we can take a basis of weight vectors $\{v_{\mu_1^i,...,\mu_n^i}\}_{i=1}^d$ on which the torus T and hence each simple co-root acts diagonally. Thus,

$$\rho_\lambda(t) = diag\{e^{\mu^1}(t), ..., e^{\mu^d}(t)\} \in Aut(V_\lambda)$$

and for a simple co-root $\check{\alpha}_j$ we have that

$$d\rho_\lambda(\check{\alpha}_j) = diag\{\mu_j^1, ..., \mu_j^d\} \in End(V_\lambda).$$

In order to compute the projection to $\mathfrak{g} \in End(V_\lambda)) \simeq \mathfrak{g} \oplus \mathfrak{g}^\perp$ we calculate $d\rho_\lambda(\mathfrak{g})^\perp \in End(V_\lambda)$ with respect to the symmetric bilinear form $tr(AB)$. Recall that $d\rho_\lambda$ is faithful so we identify $\mathfrak{g}$ with its image under $d\rho_\lambda$. Let $X = (x_{ij}) \in End(V_\lambda)$. Then for X to be contained in $d\rho_\lambda(\mathfrak{g})^\perp$ it follows that

$$tr(d\rho_\lambda(\check{\alpha}_j) \cdot X) = 0 \implies \sum_{i=1}^d \mu_j^i x_{ii} = 0$$

for all co-roots, $\check{\alpha}_j \in \mathfrak{t}$.

So $\sum_{\mu\in\Lambda_\lambda} \mu_1^i x_{ii} = \sum_{\mu\in\Lambda_\lambda} \mu_2^i x_{ii} = ... = \sum_{\mu\in\Lambda_\lambda} \mu_n^i x_{ii} = 0$. Now to project $\rho_\lambda(t)$ onto $d\rho_\lambda(\mathfrak{t})$ we write ρ_λ as a sum

$$\rho_\lambda(t) = \sum_{j=1}^{n} c_j(t) d\rho_\lambda(\check{\alpha}_j) + X(t).$$

where $c_j : T \mapsto \mathbb{C}$ is a function that depends on λ, and $X(t) \in d\rho_\lambda(\mathfrak{g})^\perp$. It follows then that

$$\theta_\lambda(t) = \sum c_j(t)\check{\alpha}_j$$

We aim to solve for the coefficients $c_j(t)$. Note that for the root space $\mathfrak{g}_\alpha$, we have that $\mathfrak{g}_\alpha . V_\mu \subset V_{\mu+\alpha}$. Thus, $d\rho_\lambda(e_\alpha)$ for $e_\alpha \in \mathfrak{g}_\alpha$ will only have off diagonal entries, and as such the condition $tr(d\rho_\lambda(e_\alpha) \cdot X) = 0$ will only add constraints to the off diagonal entries of $X \in d\rho_\lambda(\mathfrak{g})^\perp$. As the action of t and $\check{\alpha}_j$ are both diagonal, by comparing coordinates we have the following set of d equations

$$e^{\mu^1}(t) = c_1(t)\mu_1^1 + ... + c_n(t)\mu_n^1 + x_{11}(t)$$

$$e^{\mu^2}(t) = c_1(t)\mu_1^2 + ... + c_n(t)\mu_n^2 + x_{22}(t)$$

$$\vdots$$

$$e^{\mu^d}(t) = c_1(t)\mu_1^d + ... + c_n(t)\mu_n^d + x_{dd}(t).$$

This can be reduced to n equations by utilizing the fact that $\sum_{i=1}^{d} \mu_j^i x_{ii} = 0$, as follows. Multiply each equation above by μ_1^i and sum (then repeat with $\mu_2^i, ..., \mu_n^i$)

$$\sum_{i=1}^{d} \mu_1^i e^{(\mu_1^i,...,\mu_n^i)}(t) = \sum_{i=1}^{d} (\mu_1^i)^2 c_1(t) + \sum_{i=1}^{d} \mu_1^i \mu_2^i c_2(t) + ... + \sum_{i=1}^{d} \mu_1^i \mu_n^i c_n(t)$$

$$\vdots$$

$$\sum_{i=1}^{d} \mu_n^i e^{(\mu_1^i,...,\mu_n^i)} = \sum_{i=1}^{d} \mu_1^i \mu_n^i c_1(t) + \sum_{i=1}^{d} \mu_2^i \mu_n^i c_2(t) + ... + \sum_{i=1}^{d} (\mu_n^i)^2 c_n(t)$$

More concisely this can be written as

$$\begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1 e^{\mu}(t) \\ \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_n e^{\mu}(t) \end{pmatrix} = S(G,\lambda) \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix}$$

where

$$S(G,\lambda) := \begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1^2 & \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_2 & \cdots & \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_n \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_2 & \sum\limits_{\mu\in\Lambda_\lambda} \mu_2^2 & \cdots & \sum\limits_{\mu\in\Lambda_\lambda} \mu_2\mu_n \\ \vdots & \ddots & & \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_n & \cdots & \sum\limits_{\mu\in\Lambda_\lambda} \mu_{n-1}\mu_n & \sum\limits_{\mu\in\Lambda_\lambda} \mu_n^2 \end{pmatrix}$$

In the next section we will show that $S(G,\lambda)$ is a multiple of a symmetrization of the Cartan matrix for G, and is thus invertible. So, we have that

$$\begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix} = S^{-1}(G,\lambda) \begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1 e^{\mu}(t) \\ \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_n e^{\mu}(t) \end{pmatrix}$$

We calculate the matrix $S(G,\lambda)$ for the classical and exceptional simple algebraic groups. In the following sections, we continue the notation

$$\Lambda_\lambda = \{(\mu_1, ...\mu_n)|\ \mu_1\omega_1 + ... + \mu_n\omega_n \ is\ a\ weight\ of V_\lambda\}$$

counted with multiplicity.

4. Main Result

Our main result will be calculating the matrix $S(G,\lambda)$ as defined in section 3, for the simple algebraic groups. We use the convention that the Cartan matrix associated to the root system of $\mathfrak{g}$ is $A = (A_{ij})$, where $A_{ij} = \alpha_i(\check{\alpha}_j)$. Then A is a change-of-basis matrix for $\mathfrak{t}^*$ between the fundamental weights and the simple roots. Furthermore, A satisfies the following properties

- For diagonal entries $A_{ii} = 2$

- For non-diagonal entries $A_{ij} \leq 0$
- $A_{ij} = 0$ iff $A_{ji} = 0$
- A can be written as DS, where D is a diagonal matrix, and S is a symmetric matrix.

Let D be the diagonal matrix defined by $D_{ij} = \frac{\delta_{ij}}{2}(\alpha_i, \alpha_j)$, where if we realize the root system R associated to $\mathfrak{g}$ as a set of vectors in a Euclidean space E, then $(\cdot, \cdot)$ is the standard inner product. In this framework we can write $A_{ij} = \alpha_i(\check{\alpha}_j) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ Then, writing $A = DS$, we find that the matrix S has coordinate entries given by

$$S_{ij} = \frac{4(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$$

and is clearly symmetric.

$(\cdot, \cdot)$ is an invariant bilinear form on $\mathfrak{t}^*$, normalized so that so that $(\alpha_i, \alpha_i) = 2$ where α_i is the highest root. Note that under this formulation, if G is of simply-laced type then D is the identity matrix and S is the Cartan matrix. We find that in general for a given simple group G that $S(G, \lambda)$ is a multiple of S. Before stating our result precisely we fix the following notation. If α_j is any long simple root (for the simply laced case α_j can be any simple root), consider the corresponding fundamental weight ω_j. Let $x_j(\lambda) := \sum\limits_{\mu \in \Lambda_\lambda} \mu_j^2$, where μ_j is the j^{th} coordinate of the weight $\mu \in \Lambda_\lambda$ in the fundamental weight basis.

Proposition 4.1. *Let G be a simple algebraic group. Let $S(G, \lambda)$ be defined as in section 3. Set $x_j(\lambda) := \sum\limits_{\mu \in \Lambda_\lambda} \mu_j^2$ for a long root α_j. Let S be a symmetrization of the Cartan matrix as above. Then $S(G, \lambda)$ is a multiple of S. More precisely,*

$$S(G, \lambda) = \frac{1}{2} x_j(\lambda) \cdot S$$

and this is independent of the choice of long root α_j.

Proof. The proof will rely on the fact that the set of weights Λ_λ of V_λ is invariant under the action of the Weyl group on $\mathfrak{t}^*$, i.e. for $w \in W$, $w.\Lambda_\lambda = \Lambda_\lambda$. The following Lemma is true for all simple groups. The following two lemmas are sufficient to prove the simply-laced case but also hold for the non-simply laced cases.

Lemma 4.2. *For a given simple group G, if the Cartan matrix entry $A_{ij} = 0$, i.e the nodes representing the simple roots α_i and α_j are not connected on the associated Dynkin diagram, then*

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = 0,$$

where $\mu = (\mu_1, ..., \mu_n)$.

Proof. Consider the simple reflection s_i acting on a weight $\mu = (\mu_1, ...\mu_n) \in \Lambda_\lambda$. Then

$$s_i(\mu) = (\mu_1, ...\mu_n) - ((\mu_1, ...\mu_n)(\check{\alpha}_i))(\alpha_i)$$

where $(\mu_1, ...\mu_n)(\check{\alpha}_i) = (\mu_1\omega_1 + ...\mu_n\omega_n)(\check{\alpha}_i) = \mu_i$. Using the Cartan matrix to write the simple roots α_i in the fundamental weight basis gives $\alpha_i = (A_{i,1}, ..., A_{i,n})$. Then the above reflection yields

$$s_i(\mu) = (\mu_1, ...\mu_n) - \mu_i(A_{i,1}, ..., A_{i,n}) = (\mu_1 - \mu_i A_{i1}, ..., \mu_n - \mu_i A_{in})$$

Now note that $A_{ii} = 2$ and $A_{ij} = 0$. So the i^{th} coordinate of $s_i(\mu)$ is $[s_i(\mu)]_i = \mu_i - \mu_i A_{ii} = -\mu_i$ and the j^{th} coordinate of $s_i(\mu)$ is $[s_i(\mu)]_j = \mu_j - \mu_i A_{ij} = \mu_j$. Thus we find that

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = \sum_{s_i(\mu) \in \Lambda_\lambda} \mu_i \mu_j = \sum_{\mu \in \Lambda_\lambda} [s_i(\mu)]_i \cdot [s_i(\mu)]_j = \sum_{\mu \in \Lambda_\lambda} -\mu_i \mu_j,$$

by invariance of Λ_λ under s_i. Thus, the result follows. □

Lemma 4.3. *If simple roots α_i and α_j of G are connected via the Dynkin diagram and have the same length then*

$$\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = \sum_{\mu \in \Lambda_\lambda} \mu_j^2.$$

Furthermore,

$$\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = -\frac{1}{2}\sum_{\mu\in\Lambda_\lambda} \mu_i^2$$

Proof. We have that $A_{ij} = A_{ji} = -1$. Then as above with $\mu = (\mu_1, ...\mu_n) \in \Lambda_\lambda$, we have that $s_i(\mu) = (\mu_1 - \mu_i A_{i1}, ..., \mu_n - \mu_i A_{in})$. Now consider

$$s_j s_i(\mu) = ((\mu_1 - \mu_i A_{i1}) - (\mu_j - \mu_i A_{ij})A_{j1}, ..., (\mu_n - \mu_i A_{in}) - (\mu_j - \mu_i A_{ij})A_{jn})$$

Thus, $[s_j s_i(\mu)]_i = (\mu_i - \mu_i A_{ii}) - (\mu_j - \mu_i A_{ij})A_{ji} = -\mu_i - (\mu_j + \mu_i)(-1) = \mu_j$. Thus,

$$\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_i = \sum_{\mu\in\Lambda_\lambda} [s_j s_i(\mu)]_i \cdot [s_j s_i(\mu)]_i = \sum_{\mu\in\Lambda_\lambda} \mu_j\mu_j$$

The second part of the lemma follows from the fact that $[s_i(\mu)]_j = \mu_j - \mu_i A_{ij}$ with $A_{ij} = -1$. It follows that

$$\sum_{\mu\in\Lambda_\lambda} \mu_j^2 = \sum_{\mu\in\Lambda_\lambda} [s_i(\mu)]_j^2 = \sum_{\mu\in\Lambda_\lambda} (\mu_j + \mu_i)^2$$

Thus, $\sum\limits_{\mu\in\Lambda_\lambda} \mu_i\mu_i = -2\sum\limits_{\mu\in\Lambda_\lambda} \mu_i\mu_j$ ∎

With the above results we see that for groups of simply-laced type that

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \frac{2}{\sum\limits_{\mu\in\Lambda_\lambda} \mu_j^2} A^{-1} \begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1 e^{\mu}(t) \\ \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_n e^{\mu}(t) \end{pmatrix}$$

The inverses of the Cartan matrices for the simply laced root systems are in the Appendix.

4.1. **Non-simply laced groups.** Recall that the root systems of simple groups of type B_n, C_n, G_2, F_4 contain long and short simple roots. Our convention will be the same as in [2]. That is, for B_n that $\alpha_1, ..., \alpha_{n-1}$ are the long roots and α_n is short, for C_n that $\alpha_1, ...\alpha_{n-1}$

are short and α_n is long, for G_2 that α_1 is short and α_2 is long, and for F_4 that the first and second are long and that the third and fourth are short.

4.1.1. *G of type B,C or F.*

Proposition 4.1.1. *Let G be a rank n simple group of types B_n, C_n, or F_4. For any long root α_j, set $x_j(\lambda) = \sum_{\mu\in\Lambda_\lambda} \mu_j^2$. If α_i is a short root, then $\sum_{\mu\in\Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$. If either or both of α_i and α_j are short, then $\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = -x_j(\lambda)$*

Proof. Note that if α_i and α_j are both long roots, connected via the Dynkin diagram, then $A_{ij} = A_{ji} = -1$ So Lemma 4.3 shows that

$$\sum_{\mu\in\Lambda_\lambda} \mu_i^2 = \sum_{\mu\in\Lambda_\lambda} \mu_j^2,$$

and that $\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = -\frac{1}{2}\sum_{\mu\in\Lambda_\lambda} \mu_i^2$. The same is true for the short roots as $A_{ij} = A_{ji} = -1$ for connected short roots. So we need to show that if α_i and α_j are short and long roots respectively and connected via the Dynkin diagram, then $\sum_{\mu\in\Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$, and that $\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = -x_j(\lambda)$. To show this we first note that $A_{ij} = -1$ and $A_{ji} = -2$ and then compare $[s_i(\mu)]_i, [s_j(\mu)]_j, [s_j(\mu)]_i$ and $[s_i(\mu)]_j$. Note that $[s_i(\mu)]_i = -\mu_i$ and $s_j(\mu_j) = -\mu_j$ as before. Also, $[s_i(\mu)]_j = \mu_j - \mu_i A_{i,j} = \mu_j + \mu_i$ and $[s_j(\mu)]_i = \mu_i - \mu_j A_{ji} = \mu_i + 2\mu_j$. Thus, we have that

$$\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = \sum_{\mu\in\Lambda_\lambda} [s_j(\mu)]_i \cdot [s_j(\mu)]_j = \sum_{\mu\in\Lambda_\lambda} (\mu_i + 2\mu_j)(-\mu_j) = \sum_{\mu\in\Lambda_\lambda} -\mu_i\mu_j - 2\mu_j^2$$

Thus $\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = -\sum_{\mu\in\Lambda_\lambda} \mu_j^2 = -x_j(\lambda)$. Applying, s_i to μ gives

$$\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = \sum_{\mu\in\Lambda_\lambda} [s_i(\mu)]_i \cdot [s_i(\mu)]_j = \sum_{\mu\in\Lambda_\lambda} -\mu_i\mu_j - \mu_i^2$$

Thus, $\sum\limits_{\mu\in\Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$ ∎

So it follows that with $x_j(\lambda) = \sum\limits_{\mu \in \Lambda_\lambda} \mu_j^2$, where α_j is a long root, then

$$S(B_n, \lambda) = \frac{x_j(\lambda)}{2} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & & & \\ & & & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -2 & 4 \end{pmatrix}, S(C_n, \lambda) = \frac{x_j(\lambda)}{2} \begin{pmatrix} 4 & -2 & & & & \\ -2 & 4 & -2 & & & \\ & -2 & \ddots & & & \\ & & & 4 & -2 & \\ & & & -2 & 4 & -2 \\ & & & & -2 & 2 \end{pmatrix}$$

$$S(F_4, \lambda) = \frac{x_j(\lambda)}{2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

We give inverses of these matrices in the appendix.

4.1.2. *G of type G_2.* Let α_1 be the short root, and α_2 the long root of G_2.

Proposition 4.1.2. $\sum\limits_{\mu \in \Lambda_\lambda} \mu_1^2 = -2 \sum\limits_{\mu \in \Lambda_\lambda} \mu_1 \mu_2 = 3 \sum\limits_{\mu \in \Lambda_\lambda} \mu_2^2$

Proof. Let $\mu = (\mu_1, \mu_2) \in \Lambda_\lambda$. Then since $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, we find that $s_1(\mu) = (-\mu_1, \mu_1 + \mu_2)$ and that $s_2(\mu) = (\mu_1 + 3\mu_2, -\mu_2)$. So,

$$\sum_{\mu \in \Lambda_\lambda} \mu_1^2 = \sum_{\mu \in \Lambda_\lambda} (\mu_1 + 3\mu_2)^2$$

from which it follows that $\sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_2 = -\frac{3}{2} \sum_{\mu \in \Lambda_\lambda} \mu_2^2$. Additionally, we have that

$$\sum_{\mu \in \Lambda_\lambda} \mu_2^2 = \sum_{\mu \in \Lambda_\lambda} (\mu_1 + \mu_2)^2$$

from which we can see that $\sum_{\mu \in \Lambda_\lambda} \mu_1^2 = -2 \sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_2 = 3 \sum_{\mu \in \Lambda_\lambda} \mu_2^2$. Thus,

$$S(G_2, \lambda) = \frac{1}{2} \sum_{\mu \in \Lambda_\lambda} \mu_2^2 \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$$

□

In particular, we can solve for $c_1(t)$ and $c_2(t)$ as

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = S(G_2,\lambda)^{-1} \begin{pmatrix} \sum\limits_{\Lambda_\lambda} \mu_1 e^\mu(t) \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_2 e^\mu(t) \end{pmatrix}$$

then, letting $x = \sum\limits_{\mu\in\Lambda_\lambda} \mu_2^2$ we have that $S^{-1}(G,\lambda) = \frac{2}{3x}\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$. Thus,

$$c_1(t,\lambda) = \frac{2}{3x} \sum_{\mu\in\Lambda_\lambda} (2\mu_1 + 3\mu_2) e^\mu(t)$$

$$c_2(t,\lambda) = \frac{2}{3x} \sum_{\mu\in\Lambda_\lambda} (3\mu_1 + 6\mu_2) e^\mu(t)$$

. □

5. Example($G = C_n$, Defining Representation)

Consider $G = Sp(2n,\mathbb{C})$=$\{A \in GL(2n) | M = A^t M A\}$ where $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ where I_n is the $n\times n$ identity matrix, and $\mathfrak{sp}(2n,\mathbb{C})$=$\{X \in \mathfrak{gl}(2n) | X^t M + MX = 0\}$.

Let $\lambda = \omega_1$,the defining representation. Then we have that Λ_λ=$\{\pm\omega_1$ and $\pm(\omega_i - \omega_{i+1})$ for $1 \le i \le n-1\}$. So, $x = \sum\limits_{\Lambda_\lambda} \mu_n^2 = 2$. Let $T = diag\{t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1}\}$. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = 2\epsilon_n$. The simple coroots in $\mathfrak{t}$ are then $\check{\alpha}_i = E_i - E_{i+1} - E_{n+i} + E_{n+i+1}$ for $1 \le 1 \le n-1$ and $\check{\alpha}_n$=$E_n - E_{2n}$ where E_i is the diagonal matrix with a 1 in the i^{th} slot and 0's elsewhere [3]. In the orthogonal basis for $\mathfrak{t}$, $\omega_i = \epsilon_1 + ... + \epsilon_i$. Thus, the character $e^\mu(t)$ is given by $e^\mu(t) = t_1^{\mu_1+...\mu_n} \cdot t_2^{\mu_2+...+\mu_n} \cdot ... \cdot t_n^{\mu_n}$.

Then, we have that

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & 2 & 2 & ... & 2 \\ 1 & 2 & 3 & ... & 3 \\ ... & ... & ... & ... & ... \\ 1 & 2 & 3 & ... & n \end{pmatrix} \begin{pmatrix} t_1 - t_1^{-1} - t_2 + t_2^{-1} \\ t_2 - t_2^{-1} - t_3 + t_3^{-1} \\ \vdots \\ t_{n-1} - t_{n-1}^{-1} - t_n + t_n^{-1} \\ t_n - t_n^{-1} \end{pmatrix}$$

which gives

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} t_1 - t_1^{-1} \\ \vdots \\ t_{n-1} - t_{n-1}^{-1} \\ t_1 - t_1^{-1} + ... + t_n - t_n^{-}1 \end{pmatrix}$$

Thus,

$$\theta_\lambda(t) = c_1(t)\check{\alpha}_1 + ... + c_n(t)\check{\alpha}_n = diag(\frac{t_1 - t_1^{-1}}{2}, ..., \frac{t_n - t_n^{-1}}{2}, -\frac{t_1 - t_1^{-1}}{2}, ..., -\frac{t_n - t_n^{-1}}{2}).$$

Note that this is equivalent to the Cayley transform as in §6 of [5]. Similar results hold for $\theta_{\omega_1}(t)$ for the standard maximal tori of $SO(2n+1, \mathbb{C})$ and $SO(2n, \mathbb{C})$.

Appendix A. inverse of the cartan matrices and their symmetrizations S

The inverses of the Cartan matrices for A_n, D_n, E_6, E_7, E_8 respectively have the form (as in [6])

$$\frac{1}{n+1} \begin{pmatrix} n & n-1 & n-2 & ... & 3 & 2 & 1 \\ n-1 & 2(n-1) & 2(n-3) & ... & 6 & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & ... & 9 & 6 & 3 \\ ... & ... & ... & ... & ... & ... & ... \\ 2 & 4 & 6 & ... & (2n-2) & 2(n-1) & n-1 \\ 1 & 2 & 3 & ... & n-2 & n-1 & n \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & ... & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & ... & 2 & 1 & 1 \\ 1 & 2 & 3 & ... & 3 & \frac{3}{2} & \frac{3}{2} \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & 2 & 3 & ... & n-2 & \frac{n-2}{2} & \frac{n-2}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & ... & \frac{n-2}{2} & \frac{n}{4} & \frac{n-2}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & ... & \frac{n-2}{2} & \frac{n-2}{4} & \frac{n}{4} \end{pmatrix}$$

$$
\begin{pmatrix}
\frac{4}{3} & 1 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\
1 & 2 & 2 & 3 & 2 & 1 \\
\frac{5}{3} & 2 & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} \\
2 & 3 & 4 & 6 & 4 & 2 \\
\frac{4}{3} & 2 & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} \\
\frac{2}{3} & 1 & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3}
\end{pmatrix},
\begin{pmatrix}
2 & 2 & 3 & 4 & 3 & 2 & 1 \\
2 & \frac{2}{2} & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\
3 & 4 & 6 & 8 & 6 & 4 & 2 \\
4 & 6 & 8 & 12 & 9 & 6 & 3 \\
3 & \frac{9}{2} & 6 & 9 & \frac{15}{2} & 5 & \frac{5}{2} \\
2 & 3 & 4 & 6 & 5 & 4 & 2 \\
1 & \frac{3}{2} & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2}
\end{pmatrix},
\begin{pmatrix}
4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\
5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\
7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\
10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\
8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\
6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\
4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\
2 & 3 & 4 & 6 & 5 & 4 & 3 & 2
\end{pmatrix}
$$

The inverse of the matrix S for types C_n, B_n, G_2, F_4 have the form

$$
\frac{1}{2}\begin{pmatrix}
1 & 1 & 1 & ... & 1 \\
1 & 2 & 2 & ... & 2 \\
1 & 2 & 3 & ... & 3 \\
... & ... & ... & ... & ... \\
1 & 2 & 3 & ... & n
\end{pmatrix},
\frac{1}{2}\begin{pmatrix}
2 & 2 & 2 & ... & 2 & 1 \\
2 & 4 & 4 & ... & 4 & 2 \\
2 & 4 & 6 & ... & 6 & 3 \\
... & ... & ... & ... & ... & ... \\
2 & 4 & 6 & ... & 2(n-1) & n-1 \\
1 & 2 & 3 & ... & n-1 & 2
\end{pmatrix},
\begin{pmatrix}
\frac{2}{3} & 1 \\
1 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 3 & 2 & 1 \\
3 & 6 & 4 & 2 \\
2 & 4 & 3 & \frac{3}{2} \\
1 & 2 & \frac{3}{2} & 1
\end{pmatrix}
$$

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