# Lecture A1. Math Review

Sim, Min Kyu, Ph.D., mksim@seoultech.ac.kr

SEOULTECH 서울과학기술대학교 데이터사이언스학과

# I. Differentiation and Integration

# Differentiation

### Definition 1 (differentiation)

Differentiation is the action of computing a derivative.

### Definition 2 (derivative)

The derivative of a function $y = f(x)$ of a variable $x$ is a measure of the rate at which the value $y$ of the function changes with respect to (wrt., hereafter) the change of the variable $x$. It is notated as $f'(x)$ and called derivative of $f$ wrt. $x$.

### Remark 1

If $x$ and $y$ are real numbers, and if the graph of $f$ is plotted against $x$, the derivative is the slope of this graph at each point.

## Definition 3 (differentiable)

If $\lim_{h\to 0} \frac{f(x+h/2)-f(x-h/2)}{h}$ exists for a function $f$ at $x$, we say the function $f$ *is differentiable at* $x$. That is, $f'(x) = \lim_{h\to 0} \frac{f(x+h/2)-f(x-h/2)}{h}$. If $f$ is differentiable for all $x$, then we say $f$ *is differentiable (everywhere)*.

## Remark 2

The followings are popular derivatives.

- $f(x) = x^p \Rightarrow f'(x) = px^{p-1}$ (polyomial)
- $f(x) = e^x \Rightarrow f'(x) = e^x$ (exponential)
- $f(x) = log(x) \Rightarrow f'(x) = 1/x$ (log function; not differentiable at $x = 0$)

## Theorem 1

*Differentiation is linear. That is,* $h(x) = f(x) + g(x)$ *implies* $h'(x) = f'(x) + g'(x)$.

### Theorem 2 (differentiation of product)

*If* $h(x) = f(x)g(x)$*, then* $h'(x) = f'(x)g(x) + f(x)g'(x)$.

### Exercise 1

*Suppose* $f(x) = xe^x$*, find* $f'(x)$.

$= e^x + x \times e^x$

### Theorem 3 (differentiation of fraction)

*If* $h(x) = \frac{f(x)}{g(x)}$*, then* $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

### Theorem 4 (composite function)

*If* $h(x) = f(g(x))$, *then* $h'(x) = f'(g(x)) \cdot g'(x)$.

### Exercise 2

*Suppose* $f(x) = e^{2x}$, *find* $f'(x)$.

$\rightarrow 2 \times e^{2x}$

# Integration

**Definition 4 (integration)**

Integration is the computation of an integral, which is a reverse operation of differentiation up to an additive constant.

**Definition 5 (antiderivative)**

Let's say a function $f$ is a derivative of $g$, or $g'(x) = f(x)$, then we say $g$ is an *antiderivative* of $f$, written as $g(x) = \int f(x)dx + C$, where $C$ is a integration constant.

## Remark 3

The followings are popular antiderivatives.

- For $p \neq 1$, $f(x) = x^p \Rightarrow \int f(x)dx = \frac{1}{p+1}x^{p+1} + C$ (polyomial)
- $f(x) = \frac{1}{x} \Rightarrow \int f(x)dx = log(x) + C$ (fraction)
- $f(x) = e^x \Rightarrow \int f(x)dx = e^x + C$ (exponential)
- $f(x) = \frac{g'(x)}{g(x)} \Rightarrow \int f(x)dx = log(g(x)) + C$ (See Theorem 4 above)

## Exercise 3

*Derive* $\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g(x)'\ dx$. *(Hint: Use Theorem 2 above.)*

$$\left(\int f'(x) \times g(x)\,dx\right)' = \left(f(x)g(x) - \int f(x) \times g(x)'\right)'$$

$$f'(x)g(x) = f'(x)g(x) + \cancel{g'(x)f(x)} - \cancel{f(x)g'(x)}$$

$$\underline{f'(x)g(x) = f'(x)g(x)}$$

## Exercise 4

*Find* $\int xe^x \ dx$, *and evaluate* $\int_0^1 xe^x \ dx$. *(Hint: Use Exercise 3 above.)*

$$\int xe^x = e^x(x-1) \Leftarrow (e^x - \int e^x)'$$

$$= xe^x - e^x$$

$$= [(x-1)e^x]_0^1$$

$$= -e^0 = -1 \times -1 = 1$$

# II. Numerical Methods

## Differentiation

- Oftentimes, finding analytic derivative is hard, but finding numerical derivative is often possible.

**Definition 6**

For a function $f$ and small constant $h$,

- $f'(x) \approx \frac{f(x+h)-f(x)}{h}$ *(forward difference formula)*
- $f'(x) \approx \frac{f(x)-f(x-h)}{h}$ *(backward difference formula)*
- $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ *(centered difference formula)*

## Solving an equation

- For the rest of this section, we consider a nonlinear and differentiable (thus, continuous) function $f : \mathbb{R} \to \mathbb{R}$, we aim to find a point $x^* \in \mathbb{R}$ such that $f(x^*) = 0$. We call such $x^*$ as a *solution* or a *root*.

# Bisection Method

- The *bisection* method aims to find a very short interval $[a, b]$ in which $f$ changes a sign.
- Why? Changing a sign from $a$ to $b$ means the function crosses the $\{y = 0\}$-axis, (a.k.a. $x$-axis), at least once. It means $x^*$ such that $f(x^*) = 0$ is in this interval. Since $[a, b]$ is a very short interval, We may simply say $x^* = \frac{a+b}{2}$.

**Definition 7 (sign function)**

$sgn(\cdot)$ is called a *sign function* that returns 1 if the input is positive, -1 if negative, and 0 if zero.

# Bisection algorithm

- Let $tol$ be the maximum allowable length of the *short interval* and an initial interval $[a, b]$ be such that $sgn(f(a)) \neq sgn(f(b))$.
- The *bisection algorithm* is the following.

```
while ((b − a) > tol) do
    m = (a+b)/2
    if sgn(f(a)) = sgn(f(m)) then
        a = m
    else
        b = m
    end
end
```

- At each *iteration*, the interval length is halved. As soon as the interval length becomes smaller than $tol$, then the algorithm stops.

## Newton Method

- The bisection technique makes no used of the function values other than their signs, resulting in slow but sure convergence.
- More rapid convergence can be achieved by using the function values to obtain a more accurate approximation to the solution *at each iteration*.
- Newton method is a method that use both the function value and derivative value.

- Newton method approximates the function $f$ near $x_k$ by the tangent line at $f(x_k)$.

$x_0 =$ initial guess
for $k$=0,1,2,...
    $x_{k+1} = x_k - f(x_k)/f'(x_k)$
    break if $|x_{k+1} - x_k| < tol$
end

- Root-finding numerical methods such as bisection method and newton method has a few common properties.
  1. It is characterized as a *iterative process* (such as $x_0 \to x_1 \to x_2 \to \cdots$).
  2. In each *iteration*, the current candidate *gets closer* to the true value.
  3. It converges. That is, it is theoretically reach the *exact value* up to tolerance.

- Many iterative numerical methods share the properties above.
- The famous back propagation in deep neural network is also motivated by Newton method.
- Major algorithms for dynamic programming are called *policy iteration* and *value iteration* that also share the properties above.

# III. Matrix Algebra

# Matrix multiplication

### Exercise 5

*Solve the followings.*

$$(.6 \quad .4)\begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = (0.622, -0.38)$$

## Exercise 6

*What is* $P^2$*?*

$$P = \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix}$$

$$\begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.64 & -0.36 \\ 0.6 & 0.4 \end{pmatrix}$$

# Solution to system of linear equations

## Exercise 7

*Solve the followings.*

$$\begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}.7 & .3 \\ .5 & .5\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}$$

$$\pi_1 + \pi_2 = 1$$

$\pi_2 = 1 - \pi_1$

② $-0.3\pi_1 + 0.3\pi_2 = 0$

$0.5\pi_1 - 0.5\pi_2 = 0$

$\Rightarrow 0.5\pi_1 - 0.5(1 - \pi_1) = 0$

$$\boxed{\begin{array}{l}\pi_1 = 0.5 \\ \pi_2 = 0.5\end{array}}$$

## Exercise 8

*Solve the following system of equations.*

$$
\begin{aligned}
x &= y \\
y &= 0.5z \\
z &= 0.6 + 0.4x \\
x + y + z &= 1
\end{aligned}
$$

- No · Answer

There are n+1 equation but there are just n knowledge!

## Exercise 9

*Solve the following system of equations.*

$$\begin{pmatrix}\pi_0 & \pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}-2 & 2 & \\ 3 & -5 & 2 \\ & 3 & -3\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\end{pmatrix}$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$-2\pi_0 + 2\pi_1 = 0$

$\pi_0 = \pi_1$

$3\pi_0 + (-5)\pi_0 + 2\pi_2 = 0$

$\Rightarrow \pi_0 = \pi_2$

$\pi_0 + \pi_1 + \pi_2 = 1$

$\therefore \pi_0 = \pi_1 = \pi_2 = \frac{1}{3}$

## Exercise 10

*Solve the following system of equations.*

$$
\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}
\begin{pmatrix}
.7 & .3 & & \\
.5 & .5 & & \\
 & & .6 & .4 \\
 & & .3 & .7
\end{pmatrix}
= \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}
$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$$
(\pi_1\ \pi_2\ \pi_3\ \pi_4)\left(\begin{pmatrix}0.7 & 0.3 & & \\ 0.5 & 0.5 & & \\ & & 0.6 & 0.4 \\ & & 0.3 & 0.7\end{pmatrix} - \begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}\right) = (\pi_1\ \pi_2\ \pi_3\ \pi_4)\begin{pmatrix}-0.3 & 0.3 & & \\ 0.5 & -0.5 & & \\ & & -0.4 & 0.4 \\ & & 0.3 & -0.3\end{pmatrix} = (0)
$$

combine with second equation

$$
(\pi_1\ \pi_2\ \pi_3\ \pi_4)\begin{pmatrix}-0.3 & 0.3 & & & 1 \\ 0.5 & -0.5 & & & 1 \\ & & -0.4 & 0.4 & 1 \\ & & 0.3 & -0.3 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 1\end{pmatrix}
$$

① $\pi_1 = \pi_2$

② $\pi_3 = \pi_4$

$2\pi_1 + 2\pi_3 = 1$

$\pi_1 + \pi_3 = \frac{1}{2}$

$\therefore \boxed{\pi_1 = \frac{1}{2} - \pi_3} \to \boxed{\pi_3 = \frac{1}{2} - \pi_1}$

$\therefore$
$$\boxed{\begin{aligned}\pi_1 &= \pi_1 \\ \pi_2 &= \pi_1 \\ \pi_3 &= \tfrac{1}{2} - \pi_1 \\ \pi_4 &= \tfrac{1}{2} - \pi_1\end{aligned}}$$

## Exercise 11

*Solve following and express* $\pi_i$ *for* $i = 0, 1, 2, \ldots$

$$
\begin{aligned}
\pi_0 + \pi_1 + \pi_2 + \ldots &= 1 \\
0.02\pi_0 + 0.02\pi_1 + 0.02\pi_2 + \ldots &= \pi_0 \\
0.98\pi_0 &= \pi_1 \\
0.98\pi_1 &= \pi_2 \\
0.98\pi_2 &= \pi_3 \\
\ldots &= \ldots
\end{aligned}
$$

$$\underbrace{0.02(\pi_0 + \cdots + \pi_n)}_{= \pi_0} + 0.98\pi_0 + (0.98) \times \pi_1 + 0.98\pi_2 \cdots$$

$$= \pi_0 + 0.98\big(\underbrace{0.02(\pi_0 + \cdots + \pi_n)}_{\pi_0} + 0.98 \times (\pi_0 + \cdots + \pi_n)\big)$$

$\vdots$ → it can be repeated

$$\pi_0 + 0.98\pi_0 + (0.98)^2\pi_0 + \cdots$$

$$\frac{\pi_0}{1-0.98} = 50\pi_0$$

# IV. Series and Others

## Exercise 12 (Infinite geometric series)

*Simplify the following. When* $|r| < 1$, $S = a + ar + ar^2 + ar^3 + \dots$

$$\frac{a}{1-r}$$

## Exercise 13 (Finite geometric series)

*Simplify the following. When* $r \neq 1$, $S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

$$\frac{a(1-r^n)}{1-r}$$

## Exercise 14 (Power series)

*Simplify the following. When* $|r| < 1$, $S = r + 2r^2 + 3r^3 + 4r^4 + \ldots$

$$r + r^2 + \cdots + r^n = \frac{r(1-r^n)}{1-r}$$

Differential

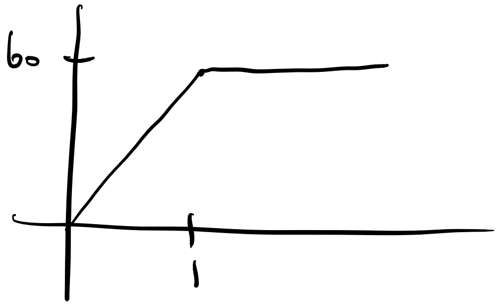
$$r \times (1 + 2r + \cdots + n \times r^{n-1}) = \left(\frac{n \times r^{n+2} - (n+1)r^n + 1}{(1-r)^2}\right) \times r$$

$$r + 2r^2 + 3r^3 \cdots + n \times r^n = \frac{n \times r^{n+3} - (n+1)r^{n+1} + r}{(1-r)^2}$$

# Formulation of time varying function

## Exercise 15

*During the first hour* $(0 \leq t \leq 1)$, $\lambda(t)$ *increases linearly from* $0$ *to* $60$. *After the first hour,* $\lambda(t)$ *is constant at* $60$. *Draw plot for* $\lambda(t)$ *and express the function in math form.*




$60t$ if $t \leq 1$

$60$ if $t > 1$

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"Man can learn nothing unless he proceeds from the known to the unknown. - Claude Bernard"
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