Lecture C4. Discrete Time Markov Chain 4

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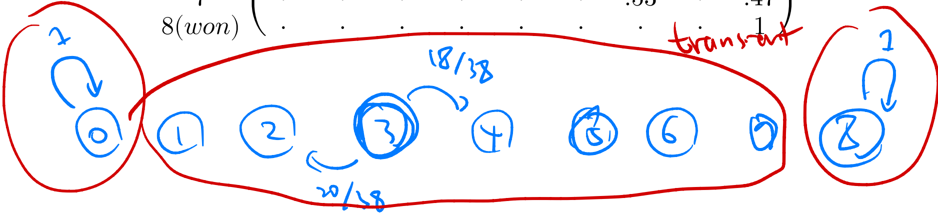
I. Gambler's ruin probability

Gambler's ruin

- Suppose you have 3\$(=x), and bet 1\$ with winning probability $p = 18/38$ until your wealth becomes 0\$(=a) or your wealth becomes 8\$(=b). What is chance of you will leave Casino with 8\$?

$$
\mathbf{P} = \begin{array}{r} 0(lose) \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8(won) \end{array}
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
.53 & \cdot & .47 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & .53 & \cdot & .47 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & .53 & \cdot & .47 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & .53 & \cdot & .47 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & .53 & \cdot & .47 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & .53 & \cdot & .47 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & .53 & \cdot & .47 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

absorbing



- Result of $a = 0, b = 8, p = 18/38$

$$
\begin{array}{c}
\\
\\
\checkmark \quad \mathbf{P}^{\infty} =
\end{array}
\begin{array}{c}
\\
0(lose) \\ 1 \\ 2 \\ \underline{3} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8(won)
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 0 & & & & & & & & 8 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .08 \\
.82 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .18 \\
\underline{.72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{.28} \\
.60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .40 \\
.48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .52 \\
.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .67 \\
.18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .82 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

$\checkmark$ 0 100 1000

- Result of $a = 0, b = 1000, p = 18/38, x = 100.$
- What is the quantity for $\mathbf{P}^{\infty}_{100\$\rightarrow win}$?

6.5×10^{-42}

보사-투자

- Result of $a = 0, b = 1000, p = 19/38, x = 100.$
- What is the quantity for $\mathbf{P}^{\infty}_{100\$\rightarrow win}$?

10%

0%

1% 1% 1.5%

$\boxed{EX_T = 100 = (1-p) \times 0 + p \times 1000}$

- Result of $a = 0, b = 10 \times 100\$, p = 18/38, x = 1 \times 100\,\$$ (bet 100$ for each)
- What is the quantity for $\mathbf{P}^{\infty}_{1\times 100\$\rightarrow win}$?

6%

5% 10% 1.5%

1%

II. Squash

Squash

- Racket sports (court number 5 in CRC)
- Rules
  - Two players, three or five games.
  - Only the server scores points.
  - The server, on winning a rally, scores a point
  - The receiver, on winning a rally, becomes the server.
  - The player who scores nine points wins the game

- Rules (cont'd)
  - Suppose A and B are playing for the first set and $8 : \overline{7}$ now.
    (A's score is 8, B's score is 7, and B is serving)
  - Suppose B wins this play so that it becomes $8 : \overline{8}$.
  - Because A got to 8 first, A can decide either
    - i) This set ends at 9
    - ii) This set ends at 10
- Questions
  - Suppose the chance of A winning a play is 0.6, then should A choose i) or ii)?

- Suppose A decides "i) This set ends at 9".
- DTMC
  - Transition diagram and matrix
  - Classification of states
  - What is the chance of A winning this game?

- Suppose A decides "ii) This set ends at 10".
- DTMC

$$
\mathbf{P} = \begin{array}{c} lose \\ 8:\overline{8} \\ \overline{8}:8 \\ 8:\overline{9} \\ \overline{8}:9 \\ 9:\overline{8} \\ \overline{9}:8 \\ 9:\overline{9} \\ \overline{9}:9 \\ win \end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

- What is the chance of A winning this game?

- What if the chance of A winning a rally is not 0.6, but for general p?

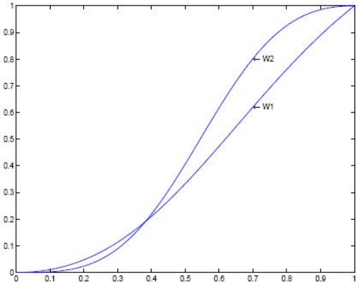


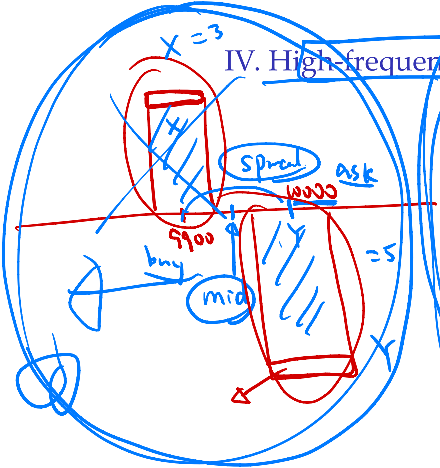
Figure 1: Probability of winning

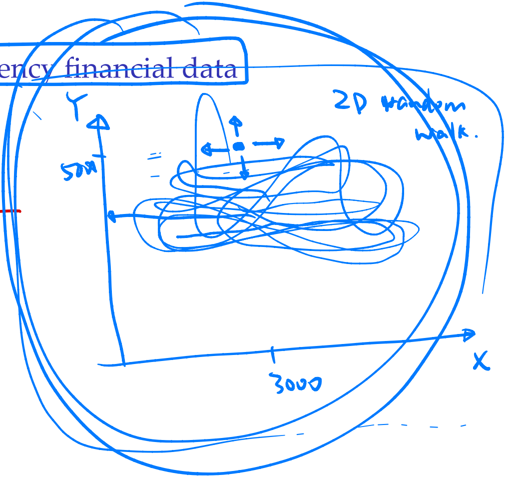
- optimal decision
  - If $p \le$, then choose i) ends at 9
  - Otherwise, choose ii) ends at 10
- Upon your decision, you are choosing one DTMC among the two different DTMC.

- Reference
  - Optimal Decision for the Squash Player
  - Jan Vecer, Columbia University, Department of Statistics
  - Journal of Chinese Statistical Association, 2004.
  - www.stat.columbia.edu/∼vecer/squash.ps

III. Tennis

IV. High-frequency financial data

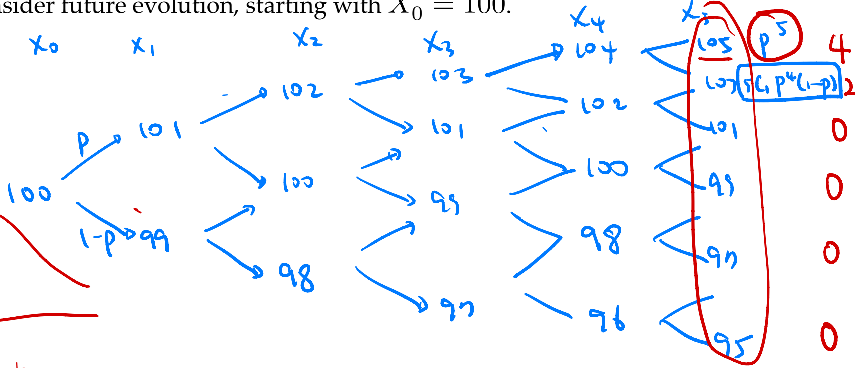


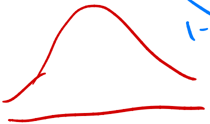


V. Stock price - binomial tree

Stock price - binomial tree

- Let X_n be the closing price of the stock at n-th day.
- Let $p = \mathbb{P}(X_{n+1} = x + 1 | X_n = x)$, and $1 - p = \mathbb{P}(X_{n+1} = x - 1 | X_n = x)$
- Consider future evolution, starting with $X_0 = 100$.





Option의 행사가격

Black Scholes

$4 \times p^5 + 2 \times 5C_1 \times p^4(1-p)$

- Consider an European call option which matures at day 5 with exercise price 101.
- (If you possess one unit of the call option, then at the day 5, you have a right to buy the stock at 101 dollars.)
- If $X_5 = 103$, then you can buy the stock at 101 and sell at 103. In this case, you earn 2 dollar.
- If $X_5 = 99$, then you still can buy the stock at 101. But you would not do it because you can buy a stock at 99 dollars. (Possessing call option is the "right" not the "obligation")
- i.e., the payoff of a call option is $\boxed{(X_5 - 101)^+}$

- What is the expected payoff for the option, when $p = 0.6$?

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cat(str)
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## If I only had an hour to chop down a tree, I would spend the first 45 minutes sharpening my axe. -
A. Lincoln
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