# Lecture A6. Simulation 3

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# I. Recap on estimation of $\pi$

# Implementation - basic

- Following is the most basic example of Monte Carlo simulation.

```
set.seed(1234) # fix the random seed
MC_N <- 10^3
x <- runif(MC_N)*2-1 # runif() generates U(0,1)
y <- runif(MC_N)*2-1
t <- sqrt(x^2+y^2)
pi_hat <- 4*sum(t<=1)/MC_N
pi_hat
```

online

```
## [1] 3.188
```

- One caveat with this vectorized programming style is that we may not observe the intermediate results.
- This may become an issue when each MC repetition takes a long time. Often, observing intermediate result is beneficial.

# Implementation - "The first-timer would write"

- It is mentioned that the first timer would write as below.

```
set.seed(1234)
MC_N <- 10^6
count <- 0
for (MC_i in 1:MC_N) {
  x_i <- runif(1)*2-1
  y_i <- runif(1)*2-1
  t_i <- sqrt(x_i^2+y_i^2)
  if (t_i <= 1) count <- count + 1
}
pi_hat <- 4*count/MC_N
```

- This note will rewrite the above code in the way that the estimate for $\pi$ is iteratively updated as `MC_i` increases toward `MC_N`.

# II. Running estimate approach

# Development

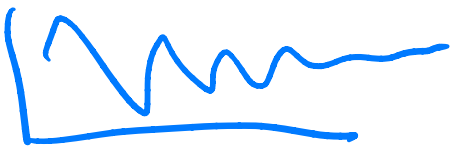
- Remind that from `A4, p8`,

$$\hat{\pi} = 4 \times \frac{\text{number of } \{t_i \leq 1\}}{N} = 4 \times \frac{\sum_{i=1}^{N} I_{\{t_i \leq 1\}}}{N}$$

- This is complete estimate after $N$ number of Monte Carlo repetitions are completed.
- Assume that we have currently completed $n (\leq N)$ number of Monte Carlo repetition, then using the completed $n$ samples, we can still generate an estimate as follows.

$$\hat{\pi}_n = \frac{\sum_{i=1}^{n} 4 \cdot I_{\{t_i \leq 1\}}}{n}$$

- Note that the notation $\hat{\pi}_n$ speaks itself as "estimate of $\pi$ using $n$ samples".
- In other words, the estimate of $\pi$ is updated as $\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \ldots, \hat{\pi}_n, \ldots, \hat{\pi}_N$.



- From the expression for $\hat{\pi}_n$, the numerator implies the running sum up to $n$ repetition, while the numerator counts the number of repetition so far.
- Let's notate the individual result of $i$-th experiment as $A_i$, and the running sum by $n$ repetition as $S_n$, as follows.

$$\hat{\pi}_n = \frac{\sum_{i=1}^{n} 4 \cdot I_{\{t_i \le 1\}}}{n} =: \frac{\sum_{i=1}^{n} A_i}{n} =: \frac{S_n}{n}$$

- Now, we can rewrite the above as a recursive form.

$$\begin{aligned}
\hat{\pi}_n &= \frac{A_1 + A_2 + \cdots + A_{n-1} + A_n}{n} = \frac{S_{n-1} + A_n}{n} \\
&= \frac{\frac{n-1}{n-1} \cdot S_{n-1} + A_n}{n} = \frac{(n-1)\hat{\pi}_{n-1} + A_n}{n} \\
&= \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n
\end{aligned}$$

$\hat{\pi}_{100} = \frac{99}{100}\hat{\pi}_{99} + \frac{1}{100}A_n$

- The last expression tells us that, the Monte Carlo updating occurs, from $\hat{\pi}_{n-1}$ to $\hat{\pi}_n$, in a way that the old estimate ($\hat{\pi}_{n-1}$) is updated with the feed of new information ($A_n$) as a weighted average of the two.

# Implementation

- Another possible benefit of this approach is that we can halt this experiment when $\hat{\pi}_{n-1}$ and $\hat{\pi}_n$ are close enough. (a.k.a. early stopping in deep learning domain)
- How would you impose the early stopping condition in this experiement?

$$\hat{\pi}_n = \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n$$

```
set.seed(1234)
beg_time <- Sys.time() # to time
old_est <- 0
n <- 1
MC_N <- 10^6
repeat{
  x_i <- runif(1)*2-1
  y_i <- runif(1)*2-1
  t_i <- sqrt(x_i^2+y_i^2)
  A_n <- 4*(t_i <= 1)
  new_est <- ((n-1)/n)*old_est + (1/n)*A_n
  if (n>MC_N) break
  n <- n+1
  old_est <- new_est
}
print(new_est)
```

```
## [1] 3.138345
```

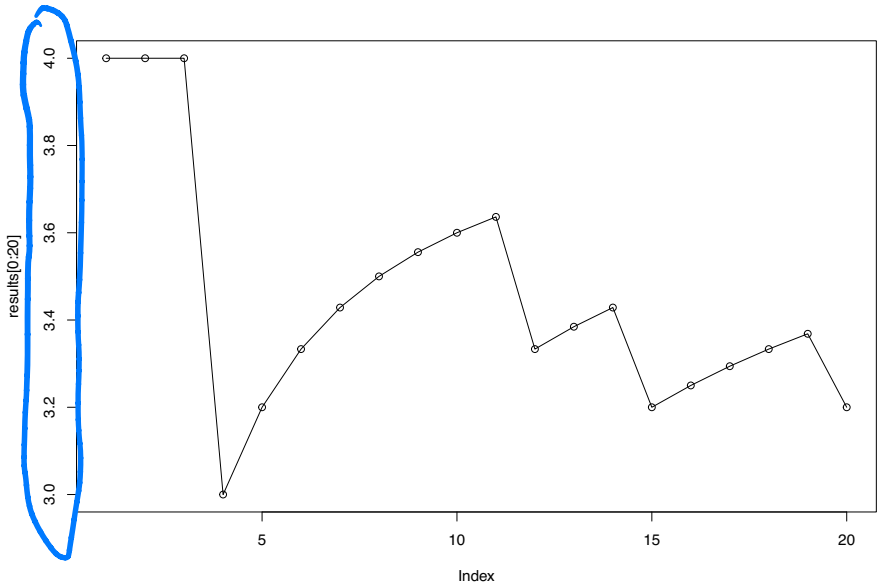
```
end_time <- Sys.time() # to time
print(end_time-beg_time) # to time
```

```
## Time difference of 3.049618 secs
```

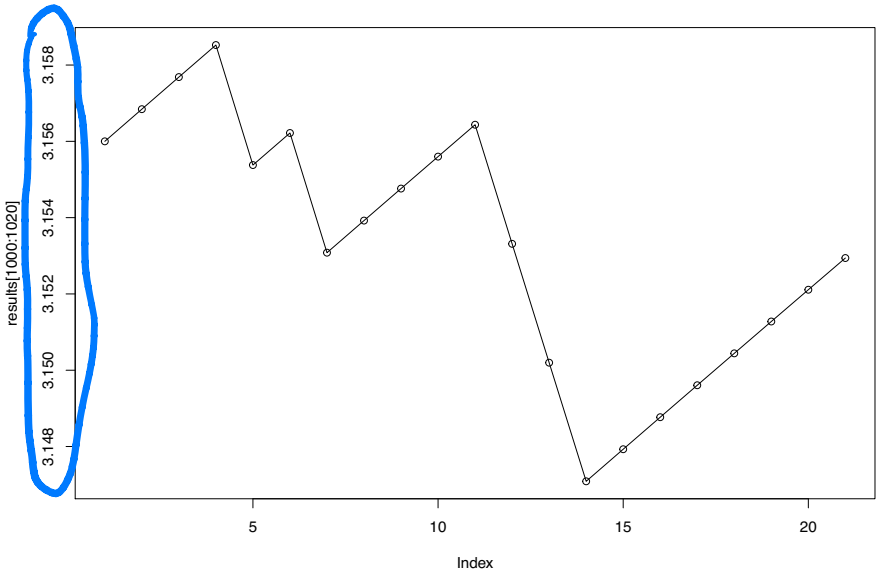
# Convergence trajectory

```r
set.seed(1234)
beg_time <- Sys.time() # to time
old_est <- 0
n <- 1
MC_N <- 10^6
results <- rep(0, MC_N) # to save
repeat{
  x_i <- runif(1)*2-1
  y_i <- runif(1)*2-1
  t_i <- sqrt(x_i^2+y_i^2)
  A_n <- 4*(t_i <= 1)
  new_est <- ((n-1)/n)*old_est + (1/n)*A_n
  results[n] <- new_est # to save
  if (n>MC_N) break
  n <- n+1
  old_est <- new_est
}
```

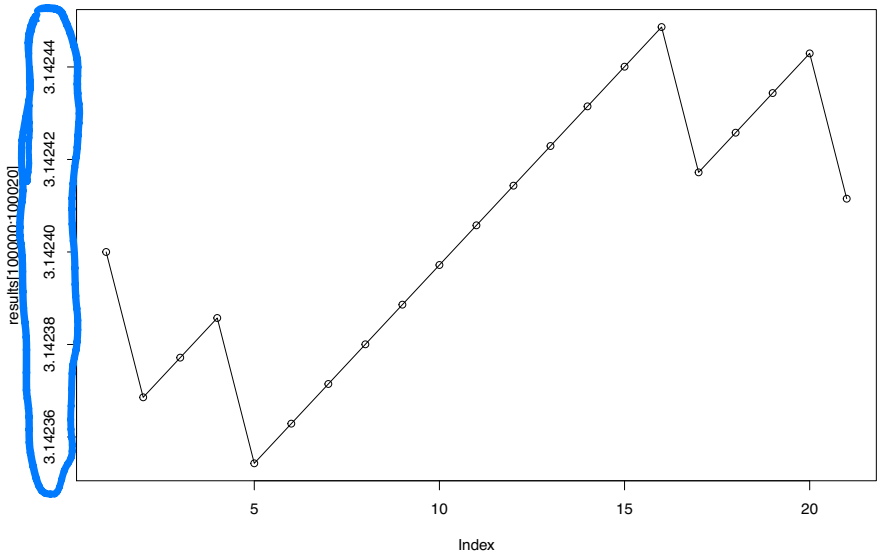
```r
plot(results[0:20], type='o')
```



```
plot(results[1000:1020], type='o')
```




```
plot(results[100000:100020], type='o')
```



# Discussion

$$\hat{\pi}_n = \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n \tag{1}$$

- The above discussion can be generalized into:

$$\hat{\pi}_n = (1-\alpha)\hat{\pi}_{n-1} + \alpha A_n \tag{2}$$

weight (importance) of new info

- The previous implementation had $\alpha = \frac{1}{n}$. The quantity $\alpha \in (0,1)$ implies the importance of the last observation.
- Oftentimes, setting $\alpha > \frac{1}{n}$ makes sense. When?

$$\hat{\pi}_n = \frac{A_1 + A_2 + \cdots + A_{n-1}}{n} + \frac{A_n}{n}$$

$$= (1-\alpha)\hat{\pi}_{n-1} + \alpha A_n$$

$\alpha = \frac{1}{n}$

# Discussion

$$\begin{aligned}
\hat{\pi}_n &= \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n \\
&= \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n - \left(\frac{1}{n}\right)\hat{\pi}_{n-1} \\
&= \hat{\pi}_{n-1} + (1/n)(A_n - \pi_{n-1})
\end{aligned}$$

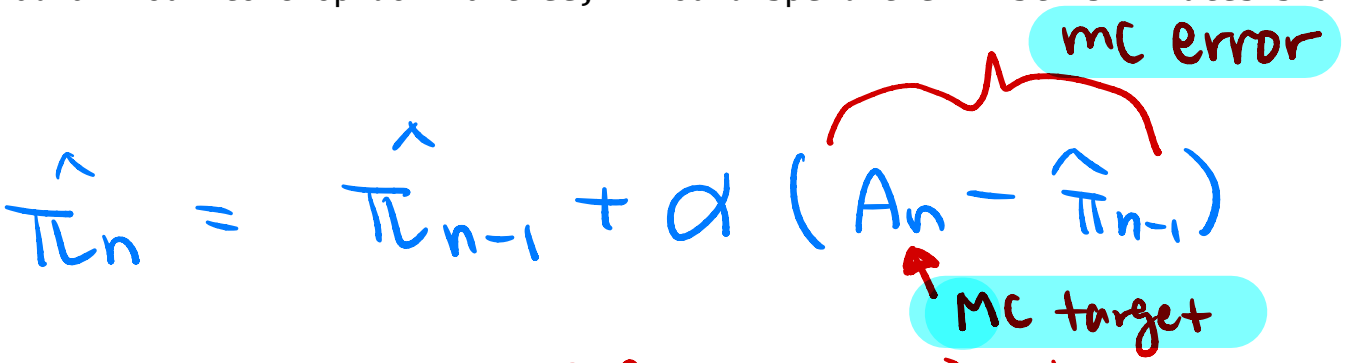
new est, old est., importance of new info, MC error

- Likewise,

$$\begin{aligned}
\hat{\pi}_n &= (1-\alpha)\hat{\pi}_{n-1} + \alpha A_n \\
&= \hat{\pi}_{n-1} + \alpha(A_n - \hat{\pi}_{n-1})
\end{aligned}$$

- What is your intepretation on the quantity, $(A_n - \hat{\pi}_{n-1})$? MC error

## If I only had an hour to chop down a tree, I would spend the first 45 minutes sharpening my axe. - A. Lincoln




⊛ mc error를 이용해서 est를 보정한다.

⊛ old est를 $A_n$의 $\alpha$ 만큼의 비율로 가깝게 한다.