Lecture A1. Math Review

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I. Differentiation and Integration

Differentiation

미분

Definition 1 (differentiation)

Differentiation is the action of computing a derivative. 찾기

Definition 2 (derivative)

The derivative of a function $y = f(x)$ of a variable x is a measure of the rate at which the value y of the function changes with respect to (wrt., hereafter) the change of the variable x. It is notated as $f'(x)$ and called derivative of f wrt. x.

순간변화율

Remark 1

If x and y are real numbers, and if the graph of f is plotted against x, the derivative is the slope of this graph at each point.

접선의 기울기.

Definition 3 (differentiable)

미분가능 하다

If $\lim_{h\to 0} \frac{f(x+h/2)-f(x-h/2)}{h}$ exists for a function f at x, we say the function f *is differentiable at* x. That is, $f'(x) = \lim_{h\to 0} \frac{f(x+h/2)-f(x-h/2)}{h}$. If f is differentiable for all x, then we say f *is differentiable (everywhere)*.

Remark 2

The followings are popular derivatives.

- $f(x) = x^p \Rightarrow f'(x) = px^{p-1}$ (polyomial)
- $f(x) = e^x \Rightarrow f'(x) = e^x$ (exponential) 그대로
- $f(x) = log(x) \Rightarrow f'(x) = 1/x$ (log function; not differentiable at $x = 0$)

Theorem 1

Differentiation is linear. That is, $h(x) = f(x) + g(x)$ *implies* $h'(x) = f'(x) + g'(x)$.

Theorem 2 (differentiation of product)

앞에꺼 뒤에꺼

If $h(x) = f(x)g(x)$, *then* $h'(x) = f'(x)g(x) + f(x)g'(x)$.

Exercise 1

Suppose $f(x) = xe^x$, *find* $f'(x)$.

$$f'(x) = e^x + xe^x = \underline{(1+x)e^x}$$

앞 뒤

Theorem 3 (differentiation of fraction)

분자 분모

If $h(x) = \frac{f(x)}{g(x)}$, *then* $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

Theorem 4 (composite function)

If $h(x) = f(g(x))$, *then* $h'(x) = f'(g(x)) \cdot g'(x)$.

Exercise 2

Suppose $f(x) = e^{2x}$, *find* $f'(x)$.

$$f'(x) = e^{2x} \cdot 2 = \underline{2e^{2x}},$$

Integration

적분

Definition 4 (integration)

Integration is the computation of an integral, which is a reverse operation of differentiation up to an additive constant.

Definition 5 (antiderivative)

Let's say a function f is a derivative of g, or $g'(x) = f(x)$, then we say g is an *antiderivative* of f, written as $g(x) = \int f(x)dx + C$, where C is a integration constant.

Remark 3

The followings are popular antiderivatives.

- For $p \neq 1$, $f(x) = x^p \Rightarrow \int f(x)dx = \frac{1}{p+1}x^{p+1} + C$ (polyomial)
- $f(x) = \frac{1}{x} \Rightarrow \int f(x)dx = log(x) + C$ (fraction)
- $f(x) = e^x \Rightarrow \int f(x)dx = e^x + C$ (exponential)
- $f(x) = \frac{g'(x)}{g(x)} \Rightarrow \int f(x)dx = log(g(x)) + C$ (See Theorem 4 above)

Exercise 3

Derive $\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g(x)'\ dx$. (Hint: Use Theorem 2 above.)

$h(x) = f(x)g(x) \rightarrow h'(x) = f'(x)g(x) + f(x)g'(x)$ } from theorem 1

$\int h'(x)\,dx = \int f'(x)g(x)\,dx + \int f(x)\cdot g'(x)\,dx$

$h(x) = f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)\cdot g'(x)\,dx$

$\Rightarrow \int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)\cdot g'(x)\,dx$

Exercise 4

Find $\int xe^x \; dx$*, and evaluate* $\int_0^1 xe^x \; dx$*. (Hint: Use Exercise 3 above.)*

$f'(x) = e^x \qquad g'(x) = 1$

$f(x) = e^x \longrightarrow g(x) = x$

$$\int xe^x \, dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

$$= e^x \cdot x - \int e^x \cdot 1 \, dx$$

$$= xe^x - e^x + C.$$

II. Numerical Methods

Differentiation

- Oftentimes, finding analytic derivative is hard, but finding numerical derivative is often possible.

Definition 6

For a function f and small constant h,

- $f'(x) \approx \frac{f(x+h)-f(x)}{h}$ (*forward difference formula*)
- $f'(x) \approx \frac{f(x)-f(x-h)}{h}$ (*backward difference formula*)
- $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ (*centered difference formula*)

Solving an equation

- For the rest of this section, we consider a nonlinear and differentiable (thus, continuous) function $f : \mathbb{R} \to \mathbb{R}$, we aim to find a point $x^* \in \mathbb{R}$ such that $f(x^*) = 0$. We call such x^* as a *solution* or a *root*.

기울기 를 0 으로 만드는 x 값 → 꼭짓점

(극값 → 극대 (극소))
위로볼록 아래로볼록

Bisection Method

이분법

- The *bisection* method aims to find a very short interval $[a, b]$ in which f changes a sign.
- Why? Changing a sign from a to b means the function crosses the $\{y = 0\}$-axis, (a.k.a. x-axis), at least once. It means x^* such that $f(x^*) = 0$ is in this interval. Since $[a, b]$ is a very short interval, We may simply say $x^* = \frac{a+b}{2}$.

Definition 7 (sign function)

$sgn(\cdot)$ is called a *sign function* that returns 1 if the input is positive, -1 if negative, and 0 if zero.

Bisection algorithm

- Let tol be the maximum allowable length of the *short interval* and an initial interval $[a, b]$ be such that $sgn(f(a)) \neq sgn(f(b))$.
- The *bisection algorithm* is the following.

$f(a) \leftrightarrow f(b)$

while $((b-a) > tol)$ do
    $m = \frac{a+b}{2}$
    if $sgn(f(a)) = sgn(f(m))$ then
        $a = m$
    else
        $b = m$
    end
end

1/2 (half)

- At each *iteration*, the interval length is halved. As soon as the interval length becomes smaller than tol, then the algorithm stops.

Newton Method

- The bisection technique makes no used of the function values other than their signs, resulting in slow but sure convergence.
- More rapid convergence can be achieved by using the function values to obtain a more accurate approximation to the solution *at each iteration*.
- Newton method is a method that use both the function value and derivative value.

- Newton method approximates the function f near x_k by the tangent line at $f(x_k)$.

x_0 = initial guess
for k=0,1,2,...
$\quad x_{k+1} = x_k - f(x_k)/f'(x_k)$
$\quad$ break if $|x_{k+1} - x_k| < tol$
end

$x_0 = \sim$
$k=0 \quad x_1 = x_0 - f(x_0)/f'(x_0)$
$k=1 \quad x_2 = x_1 - f(x_1)/f'(x_1)$
$\vdots$

x^* such that $f(x^*) = 0$

- Root-finding numerical methods such as bisection method and newton method has a few common properties.
  1. It is characterized as a *iterative process* (such as $x_0 \to x_1 \to x_2 \to \cdots$).
  2. In each *iteration*, the current candidate *gets closer* to the true value.
  3. It converges. That is, it is theoretically reach the *exact value* up to tolerance.

     smaller interval

- Many iterative numerical methods share the properties above.
- The famous back propagation in deep neural network is also motivated by Newton method.
- Major algorithms for dynamic programming are called *policy iteration* and *value iteration* that also share the properties above.

  DP

III. Matrix Algebra

Matrix multiplication

Exercise 5

Solve the followings.

$(1,2) \times (2,2) = (1,2)$

$$(.6 \quad .4)\begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = (0.42 + 0.2 \quad 0.18 + 0.2)$$

$$= \underline{(0.62 \quad 0.38)}$$

Exercise 6

What is P^2?

$$P = \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix}$$

$$P^2 = P \cdot P = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.49 + 0.15 & 0.21 + 0.15 \\ 0.35 + 0.25 & 0.15 + 0.25 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 0.64 & 0.36 \\ 0.6 & 0.4 \end{bmatrix}}$$

Solution to system of linear equations

Exercise 7

Solve the followings.

$$
\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}
$$

$$
\pi_1 + \pi_2 = 1
$$

$$
[\pi_1 \quad \pi_2] \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} = [\overset{①}{0.7\pi_1 + 0.5\pi_2} \quad \overset{②}{0.3\pi_1 + 0.5\pi_2}] = [\pi_1 \quad \pi_2]
$$

① $0.7\pi_1 + 0.5\pi_2 = \pi_1$

$3\pi_1 = 5\pi_2 \qquad \pi_2 = \frac{3}{5}\pi_1$

② $0.3\pi_1 + 0.5\pi_2 = \pi_2$

$3\pi_1 = 5\pi_2 \qquad \pi_2 = \frac{3}{5}\pi_1$

$\pi_1 + \frac{3}{5}\pi_1 = \frac{8}{5}\pi_1 = 1$

$\therefore \pi_1 = \frac{5}{8}, \quad \pi_2 = \frac{3}{5} \times \frac{5}{8} = \frac{3}{8}$ //

Exercise 8

Solve the following system of equations.

$$x = y$$
$$y = 0.5z$$
$$z = 0.6 + 0.4x$$
$$x + y + z = 1$$

$$x - y = 0$$
$$y - 0.5z = 0$$
$$0.4x - z = -0.6$$

$$y = x \qquad z = 0.4x + 0.6$$

$$x + y + z = x + x + 0.4x + 0.6$$
$$= 2.4x + 0.6 = 1$$
$$2.4x = 0.4 \qquad x = \frac{0.4}{2.4} = \frac{1}{6}$$

$$\therefore \begin{cases} x = \frac{1}{6} \\ y = \frac{1}{6} \\ z = \frac{0.4}{6} + 0.6 = \frac{0.4+3.6}{6} = \frac{4}{6} = \frac{2}{3} \end{cases}$$

Exercise 9

Solve the following system of equations.

$$
\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix}
\begin{pmatrix} -2 & 2 & \\ 3 & -5 & 2 \\ & 3 & -3 \end{pmatrix}
= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}
$$

$$
\pi_0 + \pi_1 + \pi_2 = 1
$$

$-2\pi_0 + 3\pi_1 = 0 \rightarrow \pi_0 = \frac{3}{2}\pi_1$

$2\pi_0 - 5\pi_1 + 3\pi_2 = 0 \longrightarrow$

$2\pi_1 - 3\pi_2 = 0 \rightarrow \pi_2 = \frac{2}{3}\pi_1$

$\pi_0 + \pi_1 + \pi_2 = \frac{3}{2}\pi_1 + \pi_1 + \frac{2}{3}\pi_1$

$= \frac{9+6+4}{6}\pi_1 = \frac{19}{6}\pi_1 = 1$

$\therefore \begin{cases} \underline{\pi_1 = \frac{6}{19}}, \quad \pi_2 = \frac{2}{3} \times \frac{6}{19} = \underline{\frac{4}{19}}. \\ \pi_0 = \frac{3}{2} \times \frac{6}{19} = \underline{\frac{9}{19}} \end{cases}$

∘ verification (검산)

$2 \cdot \frac{9}{19} - 5 \cdot \frac{6}{19} + 3 \cdot \frac{4}{19}$

$= \frac{18 - 30 + 12}{19} = 0.$ ok..

Exercise 10

Solve the following system of equations.

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}\begin{pmatrix} .7 & .3 & & \\ .5 & .5 & & \\ & & .6 & .4 \\ & & .3 & .7 \end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$$\left.\begin{aligned} 0.7\pi_1 + 0.5\pi_2 &= \pi_1 \to 5\pi_2 = 3\pi_1 \\ 0.3\pi_1 + 0.5\pi_2 &= \pi_2 \to 3\pi_1 = 5\pi_2 \end{aligned}\right\} \pi_2 = \frac{3}{5}\pi_1$$

$$\left.\begin{aligned} 0.6\pi_3 + 0.3\pi_4 &= \pi_3 \to 3\pi_4 = 4\pi_3 \\ 0.4\pi_3 + 0.7\pi_4 &= \pi_4 \to 4\pi_3 = 3\pi_4 \end{aligned}\right\} \pi_4 = \frac{4}{3}\pi_3$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = \pi_1 + \frac{3}{5}\pi_1 + \pi_3 + \frac{4}{3}\pi_3$$

$$= \frac{8}{5}\pi_1 + \frac{7}{3}\pi_3 = 1$$

$$\to \frac{7}{3}\pi_3 = 1 - \frac{8}{5}\pi_1$$

$$\pi_3 = \frac{3}{7} - \frac{24}{35}\pi_1 \to \pi_4 = \frac{4}{3}\left(\frac{3}{7} - \frac{24}{35}\pi_1\right) = \frac{4}{7} - \frac{32}{35}\pi_1$$

$$\therefore \begin{cases} \pi_1 = \text{free} \\ \pi_2 = \frac{3}{5}\pi_1 \\ \pi_3 = \frac{3}{7} - \frac{24}{35}\pi_1 \\ \pi_4 = \frac{4}{7} - \frac{32}{35}\pi_1 \end{cases}$$

Exercise 11

Solve following and express π_i *for* $i = 0, 1, 2, \ldots$

$0.02(\underbrace{\pi_0+\pi_1+\cdots}_{1}) = \pi_0 \leftarrow$

$\pi_0 = 0.02$

$$
\begin{aligned}
\pi_0 + \pi_1 + \pi_2 + \ldots &= 1 \\
0.02\pi_0 + 0.02\pi_1 + 0.02\pi_2 + \ldots &= \pi_0 \\
0.98\pi_0 &= \pi_1 \\
0.98\pi_1 &= \pi_2 \\
0.98\pi_2 &= \pi_3 \\
\ldots &= \ldots
\end{aligned}
$$

$\pi_i = (0.98)^i \pi_0$

$\pi_0 = 0.02$

$\pi_i = (0.98)^i \pi_0 \quad (i \geq 1)$

$= (0.98)^i \times (0.02)$

$\therefore \underline{\pi_i = (0.98)^i \times (0.02)}$

IV. Series and Others

수렴 등비급수

Exercise 12 (Infinite geometric series)

Simplify the following. When $|r| < 1$, $S = a + ar + ar^2 + ar^3 + \ldots$

$|r|<1$, $S = a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} a \cdot r^{n-1} = \underline{\frac{a}{1-r}}$

등비수열의 합

Exercise 13 (Finite geometric series)

Simplify the following. When $r \neq 1$, $S = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$

$S = a + ar + \cdots + ar^{n-1}$

$r \cdot S = ar + ar^2 + ar^3 + \cdots + ar^n$

$S - r \cdot S = a - ar^n$

$S = \dfrac{a(1-r^n)}{(1-r)}$

멱급수

Exercise 14 (Power series)

Simplify the following. When $|r| < 1$, $S = r + 2r^2 + 3r^3 + 4r^4 + \ldots$

$$S = r + 2r^2 + 3r^3 + 4r^4 + \cdots$$

$$rS = r^2 + 2r^3 + 3r^4 + \cdots$$

$$S - rS = r + r^2 + r^3 + r^4 + \cdots$$

$$(1-r)S = (1 + r + r^2 + r^3 + \cdots) - 1$$

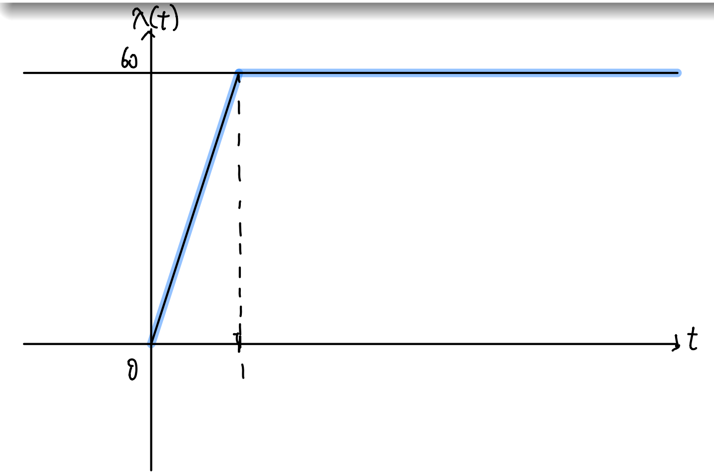
$$(1-r)S = \frac{1}{1-r} - 1 = \frac{r}{1-r}$$

$$S = \frac{r}{(1-r)^2}$$

Formulation of time varying function

Exercise 15

During the first hour $(0 \leq t \leq 1)$, $\lambda(t)$ *increases linearly from* 0 *to* 60. *After the first hour,* $\lambda(t)$ *is constant at* 60. *Draw plot for* $\lambda(t)$ *and express the function in math form.*



$$\lambda(t) = \begin{cases} 60t & \text{if } t \leq 60 \\ 60 & \text{if } t > 60 \end{cases}$$

$(t \geq 0)$

time always positive

(hour)

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"Man can learn nothing unless he proceeds from the known to the unknown. - Claude Bernard"
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