Lecture C1. Discrete Time Markov Chain 1

Sim, Min Kyu, Ph.D., mksim@seoultech.ac.kr

SEOULTECH 서울과학기술대학교 데이터사이언스학과

I. Motivation

Motivation

- I drink a bottle of soda everyday. I drink either Coke or Pepsi everyday. When I choose what to drink for today, I only consider what I drank yesterday.

- Specifically,
  - Suppose I drank Coke yesterday, then the chance of drinking Coke again today is 0.7.
  - (What is the chance of drinking Pepsi today then?)
  - Suppose I drank Pepsi yesterday, then the chance of drinking Pepsi again today is 0.5.
  - (What is the chance of drinking Coke today then?)

Representation

- How would you describe this situation in diagram?

- How would you represent this situation to mathematical form?

Some intuitive approaches.

- Suppose I do this for an year. Which brand of soda I will drink more?

- If I drink Coke today, then what is the chance of drinking Pepsi two days later?

- If I drink Coke today, then what is the chance of drinking Pepsi three days later?

More questions that we may be interested in answering.

- Given I drink coke today, what is likely my consumption for upcoming 10 days? (Pepsi is \$1 and Coke is \$1.5)
- If I do this for 10 years (3650 days) from now, then how many days I will be drinking Pepsi?
- Suppose Pepsi is \$1 and Coke is \$1.5. How much on average I spend on soda in this month given today is the first day of the month and I drink Pepsi today?
- In answering above question, how much does what I drink today matter?
- Suppose there are a billion customers (who has same type of consuming pattern) like me in the world. You are working for Pepsi and like to boost Pepsi $\rightarrow$ Pepsi probability from 0.5 to 0.6 by marketing promotion. On average, how much additional revenue will be generated by this change for a month?

II. Definitions

- Stochastic process
  - Stochastic means *time* and *randomness* combined.
  - Stochastic process includes *multiple random variables indexed by time*.

- Discrete time stochastic process
  - *Discrete time* stochastic process includes multiple random variables indexed by *discrete time*.
  - For example,
    - $S_0, S_1, S_2, \ldots$, where each implies day-0, day-1, and day-2,$\cdots$
    - $S_t, S_{t+1}, S_{t+2}, \ldots$, where each implies year-t, year-$t+1$,$\cdots$
  - Formally, $\{S_t : t \leq 0, t \in \mathbb{Z}\}$

- Continuous time stochastic process
  - *Continuous time* stochastic process includes multiple random variables indexed by *continuous time*.
  - For example,
    - $\{S_t, t \in [0, \infty)\}$ where each implies daily or yearly evolution of certain quantity.
  - Formally, $\{S_t : t \in \mathbb{R}^+\}$

- State: value of $S_\cdot$.
  - It may be deterministic.
    - Ex) $S_t = c \Leftrightarrow$ I drink coke on day-t, or say, '*The state of S_t is c*'.
    - Ex) $S_1 = p \Leftrightarrow$ On day-1, I drink pepsi, or say, '*The state of S_1 is pepsi*'.
  - It may be random. (not deterministic)
    - Ex) $\mathbb{P}(S_2 = p) = 0.6 \Leftrightarrow$ The probability that I drink pepsi on day-2 is 0.6.
  - It may be random and often described as a distribution.
    - Ex) $(\mathbb{P}(S_3 = c), \mathbb{P}(S_3 = p)) = (0.3, 0.7) \Leftrightarrow$ The probability that I drink coke on day-3 is 0.3 and pepsi is 0.7.

- State space: a set of all possible states that $S_\cdot$ can take.
  - Ex) A set of all possible kind of sodas that I might drink, i.e. $S = (c, p)$.

Markov Property

- Intuitively,
  - The nearest future only depends on the present. Past does not matter.
  - S_{t+1} depends only on the state of S_t.
  - S_{t+1} is function of S_t and some randomness, i.e. $S_{t+1} = f(S_t, \textit{randomness})$.

- A bit rigorously,
  - The future only depends on the recent history that are known.
  - Future is independent of the past, give the present.

- Formally, Markov property holds if

$$\mathbb{P}(S_{t+1} = j | S_0 = i_0, S_1 = i_1, ..., S_t = i) = P(S_{t+1} = j | S_t = i)$$

- Transitions depend only on the nearest past.
- Transitions depend only on the recent history.

Discrete Time Markov Chain

Definition 1

Discrete Time Markov Chain (DTMC, hereafter) is a *a discrete time stochastic process with Markov Property*.

- To properly describe a DTMC, following components are essential:
  1. State space
  2. Transition probability matrix/diagram
  3. Initial distribution

- Transition probability matrix/diagram.
  - The probability that governs 'transition'.
  - $p_{ij} = P(S_{t+1} = j | S_t = i) = P(S_t = j | S_{t-1} = i) = P(S_1 = j | S_0 = i)$
  - The transition probability matrix $\mathbf{P}$ is a collection of p_{ij}, i.e. $\mathbf{P} = [p_{ij}]$.

- Initial distribution
  - The information of where the chain starts at time 0.
  - a_0 := distribution of S_0 in row vector.
  - Ex) $S_0 = c \Leftrightarrow \mathbb{P}(S_0 = c) = 1, \mathbb{P}(S_0 = p) = 0 \Leftrightarrow a_0 = (1\ 0)$
  - Ex) $\mathbb{P}(S_0 = c) = 0.6, \mathbb{P}(S_0 = p) = 0.4 \Leftrightarrow a_0 = (0.6\ 0.4)$

III. Exercises

Exercise 1

Let's revisit Coke & Pepsi DTMC. Describe the following.

1. *State Space*
2. *Transition Probability Matrix*
3. *Transition Diagram*
4. *Initial Distribution*

Remark 1

A few remarks on transition matrix:

1. The size of transition matrix is $|S| \times |S|$, where $|\cdot|$ implies the number of elements in a set.
2. Transition diagram and transition matrix carry exactly same information.
3. A legit transition matrix must have each row summing up to 1.

Exercise 2

Suppose $\mathbb{P}(S_0 = c) = 0.6$ *and* $\mathbb{P}(S_0 = p) = 0.4$*, then what is* $\mathbb{P}(S_1 = c) = ?$

Exercise 3

Suppose $\mathbb{P}(S_0 = c) = 0.6$ *and* $\mathbb{P}(S_0 = p) = 0.4$*, then what is* $\mathbb{P}(S_2 = c) = ?$

Transitions in DTMC

Exercise 4

Suppose $S_0 = p$, *then what is* $\mathbb{P}(S_2 = p) = ?$

IV. Simulating stochasic paths

Motivation

In this section, we will address the following two questions.

1. Given I drink coke today, what is likely my consumption for upcoming 10 days?
2. What is my expected spending for upcoming 10 days if I drink coke today? (Pepsi is \$1 and Coke is \$1.5)

DTMC Simulator

- For a transition between S_t and S_{t+1},
  - A *deterministic transition* can be formulated as
    $$S_{t+1} = f(S_t)$$
    for some function $f(\cdot)$.
  - A *stochastic transition* can be formulated as
    $$S_{t+1} = f(S_t, \text{randomness})$$
    for some function $f(\cdot)$.
- In this light, the soda DTMC's transition can be described as $S_{t+1} = f(S_t,\ u)$, where $u \sim U(0, 1)$.
- Specifically,
  - $f(S_t = c,\ u) = c$ if $u \leq 0.7$, and $= p$ otherwise.
  - $f(S_t = p,\ u) = c$ if $u \leq 0.5$, and $= p$ otherwise.

```r
soda_simul <- function(this_state) {
  u <- runif(1)
  if (this_state == "c") {
    if (u <= 0.7) {
      next_state <- "c"
    }
    else {
      next_state <- "p"
    }
  } else {
    if (u <= 0.5) {
      next_state <- "c"
    }
    else {
      next_state <- "p"
    }
  }
  return(next_state)
}
```

- Using the function `soda_simul()`, let's generate 5 possible paths for 10 days.

```r
library(stringr) # for str_sub() and str_count()
for (i in 1:5) {
  path <- "c" # coke today (day-0)
  for (n in 1:9) {
    this_state <- str_sub(path,-1,-1) # last elemen
    next_state <- soda_simul(this_state)
    path <- paste0(path, next_state)
  }
  print(path)
}
```

```
## [1] "cpcppppccc"
## [1] "cccccppcc"
## [1] "ccppppccpc"
## [1] "ccpccccpc"
## [1] "cccpppcccp"
```

- To address the second question regarding expected spending, we certainly need more than 5 paths.
- Let's do it with 10,000 Monte-Carlo simulation.
- We need cost evaluating function that calculates cost for each path.

```
cost_eval <- function(path) {
  cost_one_path <-
    str_count(path, pattern = "c")*1.5 +
    str_count(path, pattern = "p")*1
  return(cost_one_path)
}
```

```
MC_N <- 10000
spending_records <- rep(0, MC_N)
for (i in 1:MC_N) {
  path <- "c" # coke today (day-0)
  for (t in 1:9) {
    this_state <- str_sub(path, -1, -1)
    next_state <- soda_simul(this_state)
    path <- paste0(path, next_state)
  }
  spending_records[i] <- cost_eval(path)
}
```

```
mean(spending_records)
```

```
## [1] 13.3553
```

- The above simulation is characterized with
  - Each path has length of 10.
  - The 10,000 number of paths are generated for calculating expected cost.

- In the language of stochastic programming, we prefer to describe it as following
  - In this problem, *time horizon* is 10-days.
  - The MC simulation was conducted with 10,000 *episodes*.
  - Each *episode* is a full path simulation of *time horizon*.
  - In each *episode*, *a stochastic path* is generated and *total cost for a path* is evaluated.

```r
cat(str)
```

```
## If I only had an hour to chop down a tree, I would spend the first 45 minutes sharpening my axe. -
A. Lincoln
```