Probability

Definition 1

Probability is a function from "Event" to real number between 0 and 1.

$$\mathbb{P} : S \to [0, 1],$$

where S is the set of all possible events.

Remark 1

- For any event E, $0 \leq \mathbb{P}(E) \leq 1$.
- For $E = \emptyset$, $\mathbb{P}(E) = 0$.
- $\mathbb{P}(S) = 1$, where S is whole space, or a space for all possible events.

Remark 2

- $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)$.
- If $E_1 \cap E_2 = \emptyset$, then $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$.

Conditional Probabilities

Definition 2

$\mathbb{P}(E|F)$, "probability of E given F", is the probability that event E occurs given that F has occurred. In a math notation,

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$$

- Ex) (from Ross) Suppose cards numbered one through ten are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten?

Independent Probability

Definition 3

If $\mathbb{P}(E) = \mathbb{P}(E|F)$, then we say the events E and F are independent events.

- That is, the occurence of E happening has nothing to do with the occurence of F.
- Just as Galilei once said *"And yet it rotates (even though people do not believe so)"*. People believing whether or not the earth rotates is independent event of earth's rotation.

Properties

- $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$, if $E_1 \cap E_2 = \emptyset$.
- $\mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c)$ $(\because (E \cap F) \cap (E \cap F^c) = \emptyset)$
- If $F_i \cap F_j = \emptyset$ for all $i \neq j$ and $\mathbb{P}(F_1 \cup \cdots \cup F_n) = 1$ where $1 \leq i, j \leq n$, then $\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \cap F_i)$

Bayes' rule

Theorem 1

Suppose $F_1, \ldots, F_n$ are ***mutually exclusive*** *events (or, $F_i \cap F_j = \emptyset$, or, $\mathbb{P}(F_i \cap F_j) = 0$, for any $i \neq j$) and $\mathbb{P}(F_1 \cup \cdots \cup F_n) = 1$ (In other words, exactly one and only one events among $F_1, \ldots, F_n$ will occur.), then the following holds.*

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(E \cap F_1) + \cdots + \mathbb{P}(E \cap F_n) \\ &= \sum_{i=1}^{n} \mathbb{P}(E \cap F_i) \\ &= \sum_{i=1}^{n} \mathbb{P}(E|F_i)\mathbb{P}(F_i)\end{aligned}$$

Exercise 1

Show that $\mathbb{P}(A|B \cap C)\mathbb{P}(B|C) = \mathbb{P}(A \cap B|C)$.

$$P(A|B\cap C)\cdot P(B|C) = \frac{P(A\cap B\cap C)}{P(B\cap C)} \cdot \frac{P(B\cap C)}{P(C)} = \frac{P(A\cap B\cap C)}{P(C)} = P(A\cap B|C)$$

II. Random Variables

discrete vs continuous

Discrete r.v.

Definition 4

A random variable X is called a discrete random variable if (*complete sentence*)

- Example
  - flip a coin
  - throw a dice

Continuous r.v.

Definition 5

A random variable X is called a continuous random variable if (*complete sentence*)

- Example
  - weights and heights of a person
  - temperature of a room

pmf and pdf

pmf(probability mass function) for a discrete r.v. X

Definition 6

A pmf $p(x)$ is a function that gives the **probability** that a discrete random variable is exactly equal to some value.

$$\mathbb{P}(X = x) = p(x)$$

- Suppose you throw a dice and X be a r.v. of the outcome. What is the pmf of X?
  1. *answer in a math form:*
  2. *answer in a tabular form:*

pdf(probability density function) for a continuous r.v. X

Definition 7

A pdf $f(x)$ is a function that gives the **relative likelihood** for this continuous r.v. to take on a given value.

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$$

- Suppose a random variable takes a real values between 0 and 1 with equal likelihood, what is the pdf of X?
  1. *answer in a math form:*
  2. *draw a graph:*

Properties of pmf and pdf

* pmf = 이산확률변수 X의 분포함수, 함수값이 확률
pdf = 연속확률변수 X의 분포함수, 함수넓이가 확률

- Mathematical property? *Something that always happen.*

Remark 3

The functions (pmf and pdf) are nonnegative everywhere. That is,

$$p(x) \geq 0 \text{ and } f(x) \geq 0$$

Remark 4

Its summation or integral over the entire area is equal to one. That is,

$$\sum p(x) = 1 \text{ and } \int f(x)dx = 1$$

- One can derive a cdf (cumulative distribution function) from a pmf or from a pdf. (See the upcoming definition of cdf.)

Expectation

- For a discrete random variable X with pmf $p(x)$
  - $\mathbb{E}X = \sum xp(x)$
  - $\mathbb{E}[g(X)] = \sum g(x)p(x)$
  - Ex) $\mathbb{E}X^2 = \sum x^2 p(x)$
  - Ex) $\mathbb{E}(X^2 - 2X) = \sum$
- For a continuous random variable X with pdf $f(x)$
  - $\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx$
  - $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$
  - Ex) $\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f(x)dx$

cdf

Definition 8

For a random variable X, the cdf(cumulative distribution function) $F(x)$ is a function of probability that the random variable X is found at a value less than or equal to x.

$$F(x) := \mathbb{P}(X \le x)$$

- If discrete,

$$F(x) = \mathbb{P}(X \le x) = \sum_{y=-\infty}^{x} \mathbb{P}(X = y) = \sum_{y=-\infty}^{x} p(y)$$

- If continuous,

$$F(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f(y)dy$$

III. Uniform

Definition 9

A continuous random variable X is said to follow uniform distribution with parameter a and b, and write $X \sim U(a,b)$ if (*reads "X follows a uniform distribution with parameter a and b."*)

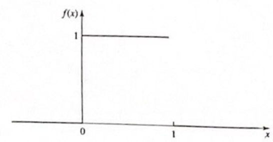
$$\text{pdf } f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{cdf } F(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

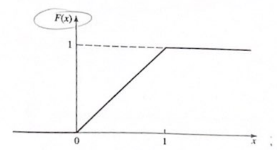
- *(The value $\frac{1}{b-a}$ is chosen as a constant so that it will make integrations over $-\infty$ to ∞ to be equal to 1.)*

$U(0,1)$

- pdf

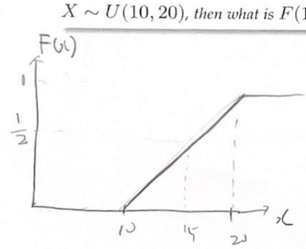


- cdf



Exercise 2

$X \sim U(10,20)$, *then what is* $F(10)$? *and* $F(15)$?



$F(10) = 0$

$F(15) = \frac{15-10}{20-10} = \frac{1}{2}$

IV. Exponential

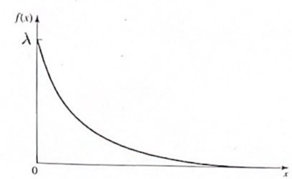
Definition 10

A nonnegative continuous random variable X is said to follow exponential distribution with parameter λ and write $X \sim exp(\lambda)$, if

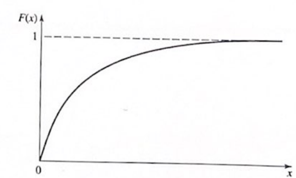
$$\text{pdf } f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{cdf } F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- pdf



- cdf



- TFAE (The followings are equivalent - *If any one of them is true, the rest of them are all true. If any one of them is false, then the rest of them are all false.*)
  - $X \sim exp(\lambda)$
  - pdf $f(x) = \lambda e^{-\lambda x}$, if $x \geq 0$; $f(x) = 0$, otherwise
  - cdf $F(x) = 1 - e^{-\lambda x}$, if $x \geq 0$; $F(x) = 0$, otherwise

Exercise 3

Prove that pdf → cdf

$\int f(x) = F(x)$ & $F'(x) = f(x)$

$\text{pdf} \underset{\text{integral calculus}}{\overset{\text{differential}}{\rightleftarrows}} \text{cdf}$

let $\text{pdf} = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$\text{cdf} = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

& $\text{pdf} \xrightarrow[X]{} \text{cdf}$

$$\int_0^\infty \lambda e^{-\lambda x} = [-e^{-\lambda x}]_0^\infty = -e^{-\infty} + e^0 = 1$$

$\Rightarrow$ The sum of the width of pdf is 1 (Contradiction)

$\therefore$ pdf $\to$ cdf

More statistics (통계량) for r.v.

- Variance: $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$
- Standard Deviation: $sd(X) = \sqrt{Var(X)}$
- Coefficient of Variation (cv): $cv(X) = \frac{sd(X)}{\mathbb{E}X}$
- *(A cv measures 'relative' variation and often more meaningful.)*
- Examples
  - cv of $N(10, \sqrt{10}^2)$ vs $N(10000, 10^2)$.
  - For an exponential r.v., $X \sim exp(\lambda)$, then $cv(X) =$ *(next slide)*
  - For a normal r.v. $X \sim N(\mu, \sigma^2)$, $cv(X) =$
  - For deterministic variable X, $c_x =$

Properties of exponential distribution

- Statistics for a r.v. $X \sim exp(\lambda)$,
  1. $EX = 1/\lambda$
  2. $Var(X) = 1/\lambda^2$
  3. $cv(X) = 1$ *(Exponential r.v. has c.v. equal to 1, always.)*
- Theorems
  1. Memoryless property
  2. Suppose $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$, and independent, then $\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$
  3. If $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$, and independent, then $min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$

Exercise 4

Show that $EX = 1/\lambda$

$f(x)=\lambda e^{-\lambda x}$

$f(x)\cdot x = E(x) = \int_0^\infty \lambda x e^{-\lambda x} dx = -\int 1\cdot e^{-\lambda x}\,dx + x e^{-\lambda x} = \left[\left(x-\frac{1}{\lambda}\right) e^{\lambda x}\right]_0^\infty$

$= \frac{1}{\lambda}$

Exercise 5

Show that $Var(X) = 1/\lambda^2$. *(Hint: need to do* EX^2 *first)*

$E[X^2] = \int_0^\infty \lambda x^2 e^{-\lambda x}\,dx = \left[-x^2 e^{-\lambda x}\right]_0^\infty + \int_0^\infty 2x e^{-\lambda x}\,dx$

$= 0 + \frac{2}{\lambda}E(x) = \frac{2}{\lambda^2}$

$\therefore Var(x) = E(x^2) - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Common usage of exponential distribution

- Since exponential r.v. is continuous and nonnegative, it is frequently used to describe time.
- In such cases, the parameter λ can be understood as "rate" as an inverse value of the time.
- For example, $X \sim exp(\lambda)$ and $\mathbb{E}X = 1/\lambda = 5$ yrs imply $\lambda = 1/5$ per year.
- For example, consider the series of events that occur in every time interval that follows exponential distribution with mean 3 minutes. (In post office, you expect one customer arrival in 3 minutes on average and the time follows exponential distribution). We say that it follows exponential arrival times with rate 1/3 per minute.

Memoryless property

Definition 11

A r.v. X is *memoryless*, if $\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s)$, for $s, t \geq 0$.

Theorem 2

Exponential random variable is memoryless.

- For example, we shall assume that bus arrival time follows $exp(1/5)$. In other words, it follows an exponential distribution and its expected arrival time is 5 minutes. You are waiting for bus, and have been waiting for 3 minutes, what is the probability that bus will not come in 5 minutes from now.

Exercise 6

Prove the previous theorem.

$$\Rightarrow P(X > x+a | X > a) = P(X > x)$$

$$P(X > x+a | X > a) = \frac{P(X > x+a) \cap P(X > a)}{P(X > a)} = \frac{P(X > x+a)}{P(X > a)}$$

$$= \frac{1 - F_X(x+a)}{1 - F_X(a)} = \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} = e^{-\lambda x} = P(X > x)$$

$\ast$ $F(x) = $ cdf

Theorem 3

Suppose $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$ and they are independent, then $\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

- Smith and Jones came to post office together and they are served by two clerks, A and B, respectively. Server A has service time following $exp(1/3)$ and server B has service time following $exp(1/5)$. What is the chance that Smith will be done first?
- Suppose that Smith came to post office earlier than Jones by 2 minutes, but Smith was still being served at the moment that Jones started to being served. Would this assumption change your previous answer? Why or why not?

V. Poisson

포아송 모수: 단위시간 또는 단위공간에서 평균발생횟수 (λ)

포아송분포: 단위시간, 단위공간에 어떤 사건이 몇번 발생할 것인가를 표현하는 이산확률분포

* 조건: 독립성, 일정성, 비집락성 (매우 짧은 시간에 동시에 일어날 확률 0)

→ 시간에 따라 확률도 바뀐다

Definition 12

A discrete random variable X is said to follow Poisson distribution with parameter λ, and write $X \sim poi(\lambda)$, if its pmf is

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k = 0, 1, 2, \ldots$$

- What is cdf of $poi(\lambda)$?

Remark 5

$\mathbb{E}X = Var(X) = \lambda$

Exercise 7

For $X \sim poi(\lambda)$, prove that $\mathbb{E}X = \lambda$.

- cf) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- pf) We have $\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$, and

$$\mathbb{E}X = \sum_{x=-\infty}^{\infty} xp(x) \text{ (this is common for all discrete r.v.)}$$
$$=$$

$$E(x) = \sum_{x=0}^{\infty} x \cdot f(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} = \sum_{x=0}^{\infty} \lambda \cdot \frac{e^{-\lambda} \cdot \lambda^{x-1}}{(x-1)!}$$

$$= \lambda \cdot e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$ by tailor series $\sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{\lambda}$

$$\therefore E(x) = \lambda$$

VI. Some Exercises

Maximum and minimum

Definition 13

$$x \wedge y = min(x, y) = \begin{cases} x & \text{, if } x \le y \\ y & \text{, otherwise} \end{cases}$$

$$x \vee y = max(x, y) = \begin{cases} x & \text{, if } x \ge y \\ y & \text{, otherwise} \end{cases}$$

$$x^+ = max(x, 0) \text{ (positive part)}$$

- Ex) $x \wedge 25 = min(x, 25)$
- Ex) $(25 - x)^+ = max(25 - x, 0)$

Exercise 8

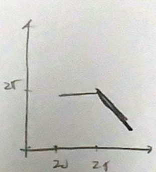
For $X \sim U(20, 40)$, evaluate $\mathbb{E}[X \wedge 25]$ and $\mathbb{E}[(25 - X)^+]$.

$x \le 25$ $\quad X + max(0, 25-x)$

$\Rightarrow X + 25 - x = 25$

$x \ge 25$ $\quad 25 + max(25-x, 0)$

$25 + 0 \quad = 25$



$\boxed{25}$

$$E[X \wedge 25] = \int_{-\infty}^{\infty} (x \wedge 25) f(x)\, dx \qquad (\because U(20, 40))$$

$$= \int_{-\infty}^{20} \underbrace{\cdot}_{0}\, dx + \int_{20}^{40} \underbrace{\cdot}_{\frac{1}{20}}\, dx + \int_{40}^{\infty} \underbrace{\cdots}_{0}\, dx$$

$$= \int_{20}^{40} (x \wedge 25) \frac{1}{20} dx \Rightarrow \int_{20}^{25} \underbrace{(x \wedge 25)}_{x} \frac{1}{20} dx + \int_{25}^{40} \underbrace{(x \wedge 25)}_{25} \frac{1}{20} dx$$

$$= \int_{20}^{25} x \cdot \frac{1}{20} dx + \int_{25}^{40} 25 \cdot \frac{1}{20} dx = \frac{195}{8}$$

Exercise 9

For $X \sim Poi(8)$,

- $\mathbb{P}(X = 0) =$
- $\mathbb{P}(2 \le X \le 4) =$
- $\mathbb{P}(X > 2) =$

$$P(X=0) = \frac{e^{-8} \cdot 8^0}{0!} = \frac{1}{e^8}$$

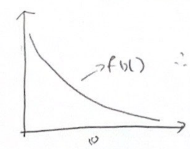
$$P(2 \le X \le 4) = \frac{e^{-8} \cdot 8^4}{4!} - \frac{e^{-8} \cdot 8^2}{2!} = \frac{2^5 \times 29}{3} \times e^{-8}$$

$$P(X > 2) = 1 - \frac{e^{-8} \times 64}{2!} = 1 - \frac{32}{e^8}$$

Exercise 10

For $X \sim exp(7)$, *evaluate* $\mathbb{E}[max(X, 7)]$.

$$f(x) = 7 \cdot e^{-7x}$$



$$\therefore \mathbb{E}[max(X,7)] = E[9,7]$$

$$\therefore \mathbb{E}[max(X,7)] = \frac{7}{e^{49}}$$

Exercise 11

For $X \sim exp(8)$, *find* x^* *such that* $F(x^*) = 0.6$.

$$1 - e^{-8x} = 0.6$$

$$e^{-8x} = 0.4$$

$$-8x = \ln 0.4$$

$$\therefore x = \frac{\ln 5 - \ln 2}{8}$$

Exercise 12

For $X \sim U(10, 20)$, *find* x^* *such that* $F(x^*) = 0.7$.

$$F(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x \ge b \end{cases}$$

$$\therefore \quad \frac{x-10}{20-10} = 0.7$$

$$x = 17$$