<A2 solution analyze>

Exercise 1~6,11,12번의 답안은 전 학생 동일하였습니다. (사칙연산 실수 제외)

일부 증명과정에서 디테일의 차이만 있을 뿐 모두 동일한 답안을 보였습니다.

그러나 8,9,10은 인원들 모두 상이한 답안을 제출하여 보고를 드립니다. (일부 사칙연산 과정에서 실수를 한 답안도 보여 이는 제외시켰습니다.)

#1번

Exercise 1

Show that $\mathbb{P}(A|B \cap C)\mathbb{P}(B|C) = \mathbb{P}(A \cap B|C)$.

$P(A|B\cap C) \cdot P(B|C)$

$= \dfrac{P(A\cap(B\cap C))}{\cancel{P(B\cap C)}} \cdot \dfrac{\cancel{P(B\cap C)}}{P(C)}$

$= \dfrac{P(A\cap B\cap C)}{P(C)}$

$= \dfrac{P((A\cap B)\cap C)}{P(C)}$

$= P(A\cap B|C)$

$\therefore P(A|B\cap C)\cdot P(B|C) = P(A\cap B|C)$

ok.

#2번

Exercise 2

$X \sim U(10, 20)$, then what is $F(10)$? and $F(15)$?

$X \sim U(10,20)$

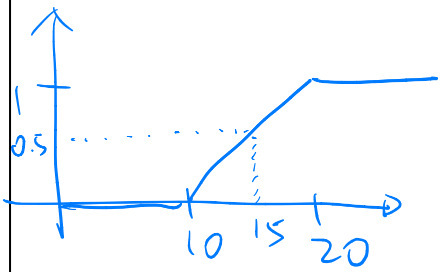
$$f(x) = \begin{cases} \frac{1}{10}, & \text{if } 10 \le x \le 20 \\ 0, & \text{otherwise} \end{cases}$$

$$F(10) = \frac{10-10}{10} = 0$$

$$F(x) = \begin{cases} 0, & \text{if } x < 10 \\ \frac{x-10}{10}, & \text{if } 10 \le x \le 20 \\ 1, & \text{if } x > 20 \end{cases}$$

$$F(15) = \frac{15-10}{10} = \frac{1}{2}$$

diagram 추가



#3번

Exercise 3

Prove that pdf $\rightarrow$ cdf

틀린풀이

$\underline{F(0) = 0}$

$\int f(x)\,dx = F(x) + C.$

$\int \lambda e^{-\lambda x}\,dx = -e^{-\lambda x} + C = F(x)$

$F(0) = 0 \Rightarrow \underbrace{-e^{\frac{-\lambda \times (0)}{0}}}_{-1} + C = 0 \quad \therefore \boxed{C = 1}$

$\therefore \underline{F(x) = 1 - e^{-\lambda x}} = \boxed{cdf}$

$\Uparrow$

$\therefore \underline{\text{pdf} \rightarrow \text{cdf}}$

$F(x) = \int_{-\infty}^{x} f(x)\,dx$

i) if $x < 0$, then $f(x) = 0$, so $F(x) = 0$

ii) if $x \ge 0$, then $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \lambda e^{-\lambda x}\,dx$

$= -e^{-\lambda x}\Big|_{0}^{x}$

$= -e^{-\lambda x} + 1$ ☑

$$EX = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cancel{f(x)}^{\,0}\,dx + \int_{0}^{\infty} x f(x)\,dx$$
$$= \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

#4번

Exercise 4

Show that $EX = 1/\lambda$

$$\lim_{x\to\infty} e^{-x} = 0$$

$$\cancel{\int_0^\infty x f(x)\,dx - E(X) = \int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{f(x)}\,dx}$$ 틀림

$$= \int_0^\infty \lambda \overset{u}{x} \overset{v'}{e^{-\lambda x}}\,dx \qquad \int uv' = uv - \int u'v$$

$$= \left[-\overset{u}{x}\overset{v}{e^{-\lambda x}}\right]_0^\infty - \int_0^\infty -e^{-\lambda x}\,dx$$

$$= (0 - 0) + \int_0^\infty e^{-\lambda x}\,dx$$

$$= \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = \left(0 - \left(-\frac{1}{\lambda}\right)\right) = \frac{1}{\lambda}$$

ok.

#5번

Exercise 5

Show that $Var(X) = 1/\lambda^2$. *(Hint: need to do* EX^2 *first)*

$$Var(X) = E(X^2) - (E(X))^2 \Rightarrow \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\int_0^\infty x^2 f(x)\,dx = \int_0^\infty \lambda x^2 e^{-\lambda x}\,dx = E(X^2)$$

$$= \left[-x^2 e^{-\lambda x}\right]_0^\infty - \int_0^\infty -2x e^{-\lambda x}\,dx$$

$$= (0 - 0) + \int_0^\infty 2x e^{-\lambda x}\,dx$$

$$= 2\int_0^\infty x e^{-\lambda x}\,dx = 2\left\{\left[\frac{x e^{-\lambda x}}{\lambda}\right]_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{\lambda}\,dx\right\}$$

$$= 2\left\{(0 - 0) - \int_0^\infty \frac{e^{-\lambda x}}{\lambda}\,dx\right\}$$

$$= 2\left\{-\left[\frac{e^{-\lambda x}}{\lambda^2}\right]_0^\infty\right\} = 2 \times \left(0 - \left(-\frac{1}{\lambda^2}\right)\right) = 2 \times \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$$

4번과 마찬가지.
$EX^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx$ 로부터 시작.

#6번

Exercise 6

Prove the previous theorem.

Claim, $P(X>s+t \mid X>t) = P(X>s)$

$P(X>s+t \cap X>t) = P(X>s+t)$

$$P(X>s+t \mid X>t) = \frac{P(X>s+t)}{P(X>t)}$$

$$P(X>t) = \int_t^\infty f(x)\,dx = \left[1 - e^{-1/\lambda x}\right]_t^\infty = 1 - (1 - e^{-t/\lambda}) = e^{-t/\lambda}$$

$$P(X>s+t) = e^{-(s+t)/\lambda}$$

$$\therefore \frac{e^{-(s+t)/\lambda}}{e^{-t/\lambda}} = e^{-s/\lambda}$$

$P(X>s) = e^{-s/\lambda}$

$\therefore P(X>s+t \mid X>t) = P(X>s)$

Ans) 좌변에서 시작해서 우변에서 끝나야 가장 깔끔한 증명

$$\mathbb{P}(X>s+t \mid X>t) = \frac{\mathbb{P}(X>s+t,\, X>t)}{\mathbb{P}(X>t)} = \frac{\mathbb{P}(X>s+t)}{\mathbb{P}(X>t)}$$

$$= \frac{1 - \mathbb{P}(X \le s+t)}{1 - \mathbb{P}(X \le t)} = \frac{1 - F(s+t)}{1 - F(t)}$$

$$= \frac{1-(1-e^{-\lambda(s+t)})}{1-(1-e^{-\lambda t})} = e^{-\lambda s}$$

$$= 1-(1-e^{-\lambda s}) = 1-F(s)$$

$$= 1-\mathbb{P}(X \le s)$$

$$= \mathbb{P}(X > s)$$

#7번

Exercise 7

For $X \sim poi(\lambda)$, prove that $\mathbb{E}X = \lambda$.

- cf) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- pf) We have $\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$, and

$$\mathbb{E}X = \sum_{x=-\infty}^{\infty} xp(x) \text{ (this is common for all discrete r.v.)}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} \cdot x$$

$$= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} \qquad \boxed{\text{Let, } k = x-1} \quad \left\{ \begin{array}{l} x: 1 \to \infty \\ k: 0 \to \infty \end{array} \right\}$$

$$= \lambda \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} = \lambda e^{-\lambda} \times e^{\lambda} = \lambda$$

good

#8번

(아래와 동일한 방법으로 계산하여 동일한 답안이 나온 인원은 3명이었습니다. 나머지 인원은 문제에 교수님이 알려주신 방법과는 다른 방법으로 계산하거나 사칙연산에서 실수가 있었던 인원으로 제외시켰습니다.)

Exercise 8

For $X \sim U(20, 40)$, evaluate $\mathbb{E}[X \wedge 25]$ and $\mathbb{E}[(25 - X)^{+}]$.

$$f(x) = \begin{cases} \frac{1}{20} & (20 \le x \le 40) \\ 0 & \text{otherwise.} \end{cases}$$

① $E[X \wedge 25] = \int_{-\infty}^{\infty} (x \wedge 25)\frac{1}{20}\,dx = \int_{20}^{25} \frac{1}{20}x\,dx + \int_{25}^{40} 25 \times \frac{1}{20}\,dx.$

② $E[(25-X)^{+}] = \int_{-\infty}^{\infty} (25-x)^{+}\frac{1}{20}\,dx = \int_{20}^{25} (25-x)\frac{1}{20}\,dx + \int_{25}^{40} 0 \times \frac{1}{20}\,dx.$

CamScanner로 스캔하기

① $\left[\frac{1}{40}x^2\right]_{20}^{25} + \left[\frac{5}{4}x\right]_{25}^{40} = \frac{625-400}{40} + \frac{5}{4}(40-25)$

$= \frac{225}{40} + \frac{75}{4}$

$= \frac{975}{40} = \boxed{\frac{195}{8}}$

② $\left[\frac{5}{4}x - \frac{1}{40}x^2\right]_{20}^{25} + 0$

$\left(\frac{125}{4} - \frac{625}{40}\right) - \left(\frac{100}{4} - \frac{400}{40}\right)$

$= \frac{25}{4} - \frac{225}{40} = \frac{250-225}{40} = \frac{25}{40} = \boxed{\frac{5}{8}}$

CamScanner로 스캔하기

#9번

(9번은 2번3번 문제를 해결할 때 2가지 방법의 답안이 나왔습니다)

correct 1. 2번문제 P(2<=x<=4) = p(x=2)+p(x=3)+p(x=4)로 계산하고, 3번 문제 또한 (1-P(x=0+P(x=1)+P(x=2))로 계산하는 방법

2. 2번문제 P(2<=x<=4) = P(x=4)-P(x=2)로 계산하며, 3번 문제 또한 (1-P(x<=2))로 계산하는 방법

주어진것이 cdf이지 pdf가 아니므로 사용 x

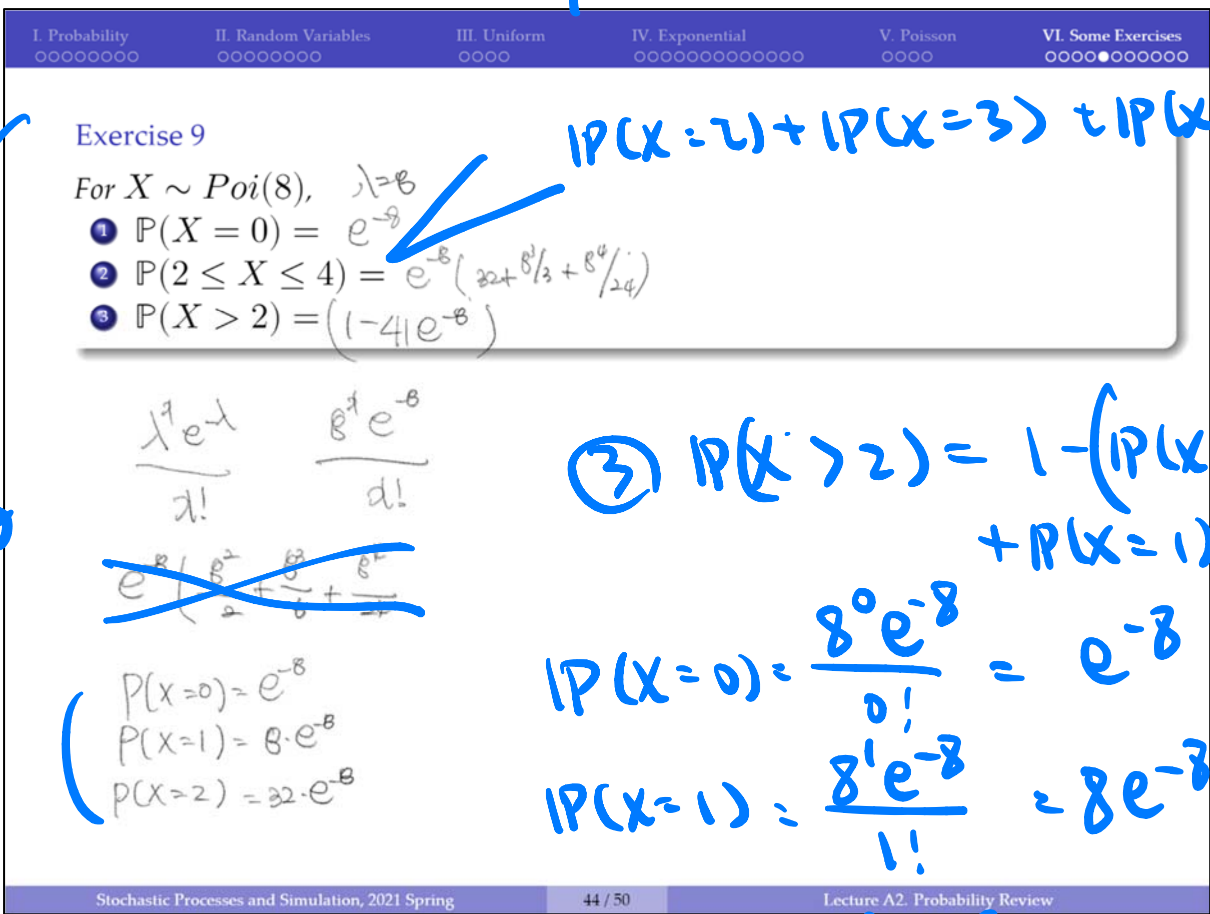
3. 2번문제 P(~~2~~<=x<=4) = F(4)-F(2)=f(3)+f(4) 로 계산하고, 3번은 1번방법과 동일하게 계산하는 방법

*대다수의 학생이 Exercise 9의 1번 문제는 맞추었고, 2번문제는 상이했으며, 3번문제는 몇몇의 학생이 1번 3번과 방법과 같은 답안이 도출되었습니다.

note 애초에 $\mathbb{P}(2 \le X \le 4)$ 이므로

$$= \mathbb{P}(X \le 4) - \mathbb{P}(X < 2)$$
$$\ne \mathbb{P}(X \le 4) - \mathbb{P}(X \le 2)$$

<1번 방법>



이게 정답

$\mathbb{P}(X=2) + \mathbb{P}(X=3) + \mathbb{P}(X=4)$

③ $\mathbb{P}(X > 2) = 1 - (\mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2))$

$$\mathbb{P}(X=0) = \frac{8^0 e^{-8}}{0!} = e^{-8}$$
$$\mathbb{P}(X=1) = \frac{8^1 e^{-8}}{1!} = 8e^{-8}$$
$$\mathbb{P}(X=2) = \frac{8^2 e^{-8}}{2!} = 32e^{-8}$$

<2번 방법>

Exercise 9

For $X \sim Poi(8)$,

- $\mathbb{P}(X = 0) =$
- $\mathbb{P}(2 \leq X \leq 4) =$
- $\mathbb{P}(X > 2) =$

$$P(X=0) = \frac{e^{-8} \cdot 8^0}{0!} = \frac{1}{e^8}$$

$$P(2 \leq X \leq 4) = \frac{e^{-8} \cdot 8^4}{4!} - \frac{e^{-8} \cdot 8^2}{2!} = \frac{2^5 \times 29}{3} \times e^{-8}$$

$$P(X > 2) = 1 - \frac{e^{-8} \times 64}{2!} = 1 - \frac{32}{e^8}$$

틀림

<3번 방법>

Exercise 9

$$P(X=x) = \frac{8^k e^{-8}}{k!} \quad (k = 1, 2, 3 ..)$$

For $X \sim Poi(8)$,

1. $\mathbb{P}(X = 0) =$ $\cancel{\frac{8^0 \cdot e^{-8}}{0!} = 1}$
2. $\mathbb{P}(2 \leq X \leq 4) =$ $\longrightarrow F(4) - F(2) = P(3) + P(4) = \cancel{\frac{8^3 e^{-8}}{3!} + \frac{8^4 e^{-8}}{4!}}$
3. $\mathbb{P}(X > 2) =$

$$1 - P(X \leq 2)$$
$$= 1 - F(2) = 1 - (f(0) + f(1) + f(2))$$
$$= 1 - \left(\frac{8^0 e^{-8}}{0!} + \frac{8^1 e^{-8}}{1!} + \frac{8^2 e^{-8}}{2!}\right)$$

이거 맞음

#10번

(10번문제는 특정구간의 적분과정에서 상이한 점을 발견했고, 다른 답안이 나왔습니다.)

<case 1>

$$E[\max(X,7)] = \int_{-\infty}^{\infty} \max(X,7)\cdot f(x)\,dx$$

$$= \int_{-\infty}^{0} \max(x,7) f(x)\,dx + \int_{0}^{7} \max(x,7) f(x)\,dx + \int_{7}^{\infty} \max(x,7) f(x)\,dx$$

$$= 0 + \int_0^7 7 f(x)\,dx + \int_7^\infty x\cdot f(x)\,dx$$

$$= 7\int_0^7 7\cdot e^{-7x}\,dx + \int_7^\infty x\cdot 7e^{-7x}\,dx$$

$$= 49\int_0^7 e^{-7x}\,dx + 7\int_7^\infty x\cdot e^{-7x}\,dx$$

$$= 49\left[-\tfrac{1}{7}e^{-7x}\right]_0^7 + 7\left(\left[-\tfrac{1}{7}xe^{-7x}\right]_7^\infty - \int_7^\infty -\tfrac{1}{7}e^{-7x}\,dx\right)$$

$$= -7\left[e^{-7x}\right]_0^7 + 7\left(e^{-49} + \tfrac{1}{7}\int_7^\infty e^{-7x}\,dx\right)$$

$$= -7(e^{-49}-1) + 7e^{-49} + \int_7^\infty e^{-7x}\,dx$$

$$= 7 + \int_7^\infty e^{-7x}\,dx$$

$$= 7 + \left[-\tfrac{1}{7}e^{-7x}\right]_7^\infty$$

$$= 7 + (0 + e^{-49})$$

여기 틀림.

$$= 7 + e^{-49}$$

<case 2>

Exercise 10

For $X \sim exp(7)$, evaluate $\mathbb{E}[max(X,7)]$.

이게 정답같음.

$$f(x) = 7e^{-7x}$$

$$\int_{-\infty}^{7} 7\times 7e^{-7x}\,dx + \int_7^\infty x\times 7e^{-7x}\,dx$$

$$= \int_0^7 49e^{-7x}\,dx + \int_7^\infty 7xe^{-7x}\,dx.$$

$$\left[-7e^{-7x}\right]_0^7 + \left[-xe^{-7x}\right]_7^\infty - \int_7^\infty -e^{-7x}\,dx$$

$$(-7e^{-49} - (-7)) + (0 - (-7e^{-49})) + \int_7^{\infty} e^{-7x} dx$$

$$= \cancel{-7e^{-49}} + 7 + \cancel{7e^{-49}} + \left[-\frac{e^{-7x}}{7}\right]_7^{\infty}$$

$$= 7 + \left(0 - \left(-\frac{e^{-49}}{7}\right)\right)$$

$$= \boxed{7 + \frac{e^{-49}}{7}}$$

<case 3>

Exercise 10

For $X \sim exp(7)$, *evaluate* $\mathbb{E}[max(X, 7)]$.

$f(x) = 7e^{-7x} (x \geq 0)$

0 (otherwise)

$$E[\max(X, 7)] = \int_{-\infty}^{\infty} \max(X, 7) f(x) dx$$

$$= \int_{-\infty}^{0} max(X, 7) \cdot 0dx + \int_0^7 7 \cdot 7e^{-7x} dx + \int_7^{\infty} X \cdot 7e^{-7x} dx$$

$$= 49\left[-\frac{1}{7}e^{-7x}\right]_0^7 + 7\left(\left[-\frac{1}{7}xe^{-7x}\right]_7^{\infty} - \int_7^{\infty} e^{-7x} dx\right)$$

$$= -7e^{-49} + 7 + 7e^{-49} - 7\left[-\frac{1}{7}e^{-7x}\right]_7^{\infty}$$

$$= 7 - e^{-49}$$

틀렸음.

$\mathbb{E}[max(X, 7)] \geq \mathbb{E}[7]$

$\because max(X, 7) \geq 7$ always

#11번

Exercise 11

For $X \sim exp(8)$*, find* x^* *such that* $F(x^*) = 0.6$.

pdf $f(x) = \begin{cases} 8e^{-8x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

cdf $F(x) = \begin{cases} 1 - e^{-8x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$F(x^*) = \underline{1 - e^{-8x^*} = 0.6}$ equation

$e^{-8x^*} = 0.4$

$\frac{1}{e^{8x^*}} = \frac{2}{5}$

$e^{8x^*} = \frac{5}{2}$

ln x

$8x^* = \ln\left(\frac{5}{2}\right) = \ln 5 - \ln 2$

$\therefore \underline{x^* = \frac{\ln 5 - \ln 2}{8}}$

#12번

Exercise 12

For $X \sim U(10, 20)$*, find* x^* *such that* $F(x^*) = 0.7$.

$F(x) = \begin{cases} 0 & \text{, if } x < 10 \\ \frac{x-10}{10} & \text{, if } 10 \leq x \leq 20 \\ 1 & \text{, if } x > 20 \end{cases}$

$F(x^*) = 0.7$

$F(\max(x, 0)) = 0.7$

$F(0) = 0 \neq 0.7$

$\therefore F(x^*) = F(x) = 0.7$

For getting the value 0.7, x should be between 10 and 20

$\therefore F(x) = \frac{x-10}{10} = 0.7$

$\frac{x-10}{10} = \frac{7}{10}$

$\therefore x = 17$

$x^* = \max(0, 17) = 17$