DayPark - C3

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2021-01-13

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Exercise 5

Suppose each morning a factory posts the number of days worked in a row without any injuries. Assume that each day is injury free with probability 98/100. Furthermore, assume that whether tomorrow is injury free or not is independent of which of the preceding days were injury free. Let $X_0 = 0$ be the morning the factory first opened. Let X_n be the number posted on the morning after n full days of work.

(a) Is $\{X_n, n \geq 0\}$ a Markov chain? If so, give its state space, initial distribution, and transition matrix P. If not, show that it is not a Markov chain.

It is Markov chain. Because future state is only affected by present state.

- We can set the state as $S = \{\text{injury free, injury}\}$.
- Initial distribution : $a_0 = (1, 0)$
- and transition matrix P is

$$P = \begin{bmatrix} 0.98 & 0.02 \\ 0.02 & 0.98 \end{bmatrix}$$

(b) Is the Markov chain irreducible? Explain

It is irreducible. Because all the states communicate with each other.

(c) Is the Markov chain periodic or aperiodic? Explain and if it is periodic, also give the period.

It is aperiodic. Because it converges to

$$P^{\infty} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

(d) Find the stationary distribution.

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0.98 & 0.02 \\ 0.02 & 0.98 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

(1) $a + b = 1$

(2)

$$\begin{aligned} 0.98 \times a + 0.02 \times b &= a \\ 0.02 \times a + 0.98 \times b &= b \end{aligned}$$

$$\therefore a = \tfrac{1}{5}, b = \tfrac{1}{5}$$

(e) Is the Markov chain positive recurrent? If so, why? If not, why not?

Exercise 6.

Considering the following transition matrix

$$P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 & 0 \end{bmatrix}$$

(a) Is the Markov chain periodic? Give the period of each state.

$$P_1 = \begin{bmatrix} 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \end{bmatrix}$$

(b) Is $(\pi_1, \pi_2, \pi_3, \pi_4) = (33/96, 27/96, 15/96, 21/96)$ the stationary distribution of the Markov Chain?

1. $\pi_1 + \pi_2 + \pi_3 + \pi_4 = \frac{33+27+15+21}{96} = 1$
2.

$$(\pi_1, \pi_2, \pi_3, \pi_4) \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 & 0 \end{bmatrix} = (0.6\pi_2 + 0.8\pi_4, 0.5\pi_1 + 0.7\pi_3, 0.4\pi_2 + 0.2\pi_4, 0.5\pi_1 + 0.3\pi_3)$$

$$\begin{aligned} (0.6\pi_2 + 0.8\pi_4, 0.5\pi_1 + 0.7\pi_3, 0.4\pi_2 + 0.2\pi_4, 0.5\pi_1 + 0.3\pi_3) &= (\frac{33}{96}, \frac{27}{96}, \frac{15}{96}, \frac{21}{96}) \\ &= (\pi_1, \pi_2, \pi_3, \pi_4) \end{aligned}$$

$\therefore (\pi_1, \pi_2, \pi_3, \pi_4) \times P = (\pi_1, \pi_2, \pi_3, \pi_4)$

Therefore, $(\pi_1, \pi_2, \pi_3, \pi_4)$ is stationary distribution.

(c) Is $P_{11}^{100} = \pi_1$? $\quad P_{11}^{101} = \pi_1$? Give and expression for π_1 in terms of P_{11}^{100} and P_{11}^{101}.

$$P^{100} = \begin{pmatrix} 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \end{pmatrix}$$

$$P^{101} = \begin{pmatrix} 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \end{pmatrix}$$

1)

$$\begin{aligned}
(\pi_1, \pi_2, \pi_3, \pi_4)P^{100} &= (\pi_1, \pi_2, \pi_3, \pi_4) \begin{pmatrix} 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \end{pmatrix} \\
&= (0.6875\pi_2 + 0.6875\pi_4, 0.5625\pi_1 + 0.5625\pi_3, \\
&\quad 0.3125\pi_2 + 0.3125\pi_4, 0.4375\pi_1 + 0.4375\pi_3) \\
&= (\frac{33}{96}, \frac{27}{96}, \frac{15}{96}, \frac{21}{96})
\end{aligned}$$

$$P_{11}^{100} = \tfrac{33}{96}$$

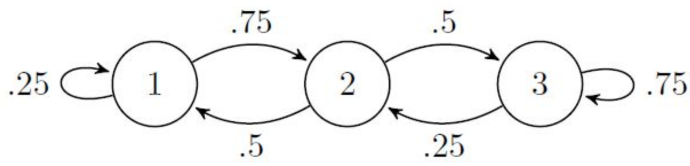
2)

$$\begin{aligned}
(\pi_1, \pi_2, \pi_3, \pi_4)P^{101} &= (\pi_1, \pi_2, \pi_3, \pi_4) \begin{pmatrix} 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \end{pmatrix} \\
&= (0.6875\pi_1 + 0.6875\pi_3, 0.5625\pi_2 + 0.5625\pi_4, \\
&\quad 0.3125\pi_1 + 0.3125\pi_3, 0.4375\pi_2 + 0.4375\pi_4) \\
&= (\frac{33}{96}, \frac{27}{96}, \frac{15}{96}, \frac{21}{96})
\end{aligned}$$

$$P_{11}^{101} = \tfrac{33}{96}$$

Exercise 14

Suppose you are working as an independent consultant for a company. Your performance is evaluated at the end of each month and the evaluation falls into one of three categories: 1, 2, and 3. Your evaluation can be modeled as a discrete-time Markov chain and its transition diagram is as follows.



Denote your evaluation at the end of nth month by X_n and assume that $X_0 = 2$.

(a) What are state space, transition probability matrix and initial distribution of X_n?

- State space : $S = \{1, 2, 3\}$
- Initial distribution of X_n : $a_0 = (0, 1, 0)$
- Transition probability matrix

$$P = \begin{pmatrix} .25 & .75 & 0 \\ .5 & 0 & .5 \\ 0 & .25 & .75 \end{pmatrix}$$

(b) What is the stationary distribution?

1) $V = (v_1, v_2, v_3)$, $v_1 + v_2 + v_3 = 1$
2) $VP = V$

$$(v_1, v_2, v_3) \begin{pmatrix} .25 & .75 & 0 \\ .5 & 0 & .5 \\ 0 & .25 & .75 \end{pmatrix} = (v_1, v_2, v_3)$$

$$\begin{aligned} 0.25v_1 + 0.5v_2 &= v_1 \\ 0.5v_1 + 0.5v_3 &= v_2 \\ 0.25v_2 + 0.75v_3 &= v_3 \end{aligned}$$

$$\therefore v_1 = \tfrac{2}{11}\,, v_2 = \tfrac{3}{11}\,, v_3 = \tfrac{6}{11}$$

$$v_1 + v_2 + v_3 = \frac{2+3+6}{11} = 1$$

Thus, stationary distribution is $V = (\frac{2}{11}, \frac{3}{11}, \frac{6}{11})$.

(c) What is the long-run fraction of time when your evaluation is either 2 or 3?

$$P^{\infty} = \begin{pmatrix} 0.1818 & 0.2727 & 0.5454 \\ 0.1818 & 0.2727 & 0.5454 \\ 0.1818 & 0.2727 & 0.5454 \end{pmatrix}$$

Thus, When the evaluation is 2, the long-run fraction of time is about 0.2727.

And when the evaluation is 3, the long-run fraction of time is about 0.5454.

(d) What is the long-run average monthly salary?

Your monthly salary is determined by the evaluation of each month in the following way.

Salary when your evaluation is $n = \$5000 + n^2 \times \$5000$, $n = 1, 2, 3$.

$$\begin{aligned} Salary &= (5000 + 1^2 \times 5000, 5000 + 2^2 \times 5000, 5000 + 3^2 \times 5000) \\ &= (10000, 25000, 50000) \end{aligned}$$

$$Salary \times longRunProb = \begin{pmatrix} 10000 & 25000 & 50000 \end{pmatrix} \begin{pmatrix} 0.1818 \\ 0.2727 \\ 0.5454 \end{pmatrix} = 35905.5$$

Thus, 'Long-run average monthly salary' is about $\$35905.5$.