Differentiation

Definition 1 (differentiation)

Differentiation is the action of computing a derivative.

Definition 2 (derivative)

The derivative of a function $y = f(x)$ of a variable x is a measure of the rate at which the value y of the function changes with respect to (wrt., hereafter) the change of the variable x. It is notated as $f'(x)$ and called derivative of f wrt. x.

Remark 1

If x and y are real numbers, and if the graph of f is plotted against x, the derivative is the slope of this graph at each point.

Definition 3 (differentiable)

If $\lim_{h\to 0} \frac{f(x+h/2)-f(x-h/2)}{h}$ exists for a function f at x, we say the function f *is differentiable at* x. That is, $f'(x) = \lim_{h\to 0} \frac{f(x+h/2)-f(x-h/2)}{h}$. If f is differentiable for all x, then we say f *is differentiable (everywhere).*

Remark 2

The followings are popular derivatives.

- $f(x) = x^p \Rightarrow f'(x) = px^{p-1}$ (polyomial)
- $f(x) = e^x \Rightarrow f'(x) = e^x$ (exponential)
- $f(x) = log(x) \Rightarrow f'(x) = 1/x$ (log function; not differentiable at $x = 0$)

Theorem 1

Differentiation is linear. That is, $h(x) = f(x) + g(x)$ *implies* $h'(x) = f'(x) + g'(x)$.

Theorem 2 (differentiation of product)

If $h(x) = f(x)g(x)$, *then* $h'(x) = f'(x)g(x) + f(x)g'(x)$.

Exercise 1

Suppose $f(x) = xe^x$, *find* $f'(x)$.

$f(x) = xe^x \quad f'(x) = (x)' \cdot e^x + x(e^x)'$

$\therefore f'(x) = (x+1)e^x$

Theorem 3 (differentiation of fraction)

If $h(x) = \frac{f(x)}{g(x)}$, *then* $h'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.

Theorem 4 (composite function)

If $h(x) = f(g(x))$, *then* $h'(x) = f'(g(x)) \cdot g'(x)$.

Exercise 2

Suppose $f(x) = e^{2x}$, *find* $f'(x)$.

$f(x) = e^{2x} \quad$ let $h(x) = e^x, \; g(x) = 2x$

$f'(x) = h'(g(x)) \cdot g'(x) = e^{2x} \cdot (2x)' = 2 \cdot e^{2x}$

$\therefore f'(x) = 2 \cdot e^{2x}$

Integration

Definition 4 (integration)

Integration is the computation of an integral, which is a reverse operation of differentiation up to an additive constant.

Definition 5 (antiderivative)

Let's say a function f is a derivative of g, or $g'(x) = f(x)$, then we say g is an *antiderivative* of f, written as $g(x) = \int f(x)dx + C$, where C is a integration constant.

Remark 3

The followings are popular antiderivatives.

- For $p \neq 1$, $f(x) = x^p \Rightarrow \int f(x)dx = \frac{1}{p+1}x^{p+1} + C$ (polyomial)
- $f(x) = \frac{1}{x} \Rightarrow \int f(x)dx = log(x) + C$ (fraction)
- $f(x) = e^x \Rightarrow \int f(x)dx = e^x + C$ (exponential)
- $f(x) = \frac{g'(x)}{g(x)} \Rightarrow \int f(x)dx = log(g(x)) + C$ (See Theorem 4 above)

Exercise 3

Derive $\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g(x)'\ dx$. (Hint: Use Theorem 2 above.)

let $\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = h(x)$

$h'(x) = f'(x)g(x) + f(x)g'(x) = (f(x)g(x))'$

$\therefore f(x) \cdot g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$

Exercise 4

Find $\int xe^x\ dx$, and evaluate $\int_0^1 xe^x\ dx$. (Hint: Use Exercise 3 above.)

$f(x)g(x) = \int f'(x)g(x) + \int f(x)g'(x)$

$\Rightarrow x \cdot e^x = \int (x)'e^x dx + \int x(e^x)'dx$

$\therefore \int xe^x dx = (x-1)e^x + C$

$\int_0^1 xe^x dx = 0 \cdot e^1 + C - (-1 \cdot e^0 + C) = 1$

II. Numerical Methods

Differentiation

- Oftentimes, finding analytic derivative is hard, but finding numerical derivative is often possible.

Definition 6

For a function f and small constant h,

- $f'(x) \approx \frac{f(x+h)-f(x)}{h}$ *(forward difference formula)*
- $f'(x) \approx \frac{f(x)-f(x-h)}{h}$ *(backward difference formula)*
- $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ *(centered difference formula)*

Solving an equation

- For the rest of this section, we consider a nonlinear and differentiable (thus, continuous) function $f : \mathbb{R} \to \mathbb{R}$, we aim to find a point $x^* \in \mathbb{R}$ such that $f(x^*) = 0$. We call such x^* as a *solution* or a *root*.

Bisection Method

- The *bisection* method aims to find a very short interval $[a, b]$ in which f changes a sign.
- Why? Changing a sign from a to b means the function crosses the $\{y = 0\}$-axis, (a.k.a. x-axis), at least once. It means x^* such that $f(x^*) = 0$ is in this interval. Since $[a, b]$ is a very short interval, We may simply say $x^* = \frac{a+b}{2}$.

Definition 7 (sign function)

$sgn(\cdot)$ is called a *sign function* that returns 1 if the input is positive, -1 if negative, and 0 if zero.

Bisection algorithm

- Let tol be the maximum allowable length of the *short interval* and an initial interval $[a, b]$ be such that $sgn(f(a)) \neq sgn(f(b))$.
- The *bisection algorithm* is the following.

```
while ((b − a) > tol) do
    m = (a+b)/2
    if sgn(f(a)) = sgn(f(m)) then
        a = m
    else
        b = m
    end
end
```

- At each *iteration*, the interval length is halved. As soon as the interval length becomes smaller than tol, then the algorithm stops.

- The bisection technique makes no used of the function values other than their signs, resulting in slow but sure convergence.
- More rapid convergence can be achieved by using the function values to obtain a more accurate approximation to the solution *at each iteration*.
- Newton method is a method that use both the function value and derivative value.

- Newton method approximates the function f near x_k by the tangent line at $f(x_k)$.

1: $x_0 =$ initial guess
2: for k=0,1,2,...
3: $\quad x_{k+1} = x_k - f(x_k)/f'(x_k)$
4: $\quad$ break if $|x_{k+1} - x_k| < tol$
5: end

- Root-finding numerical methods such as bisection method and newton method has a few common properties.
  1. It is characterized as a *iterative process* (such as $x_0 \to x_1 \to x_2 \to \cdots$).
  2. In each *iteration*, the current candidate *gets closer* to the true value.
  3. It converges. That is, it is theoretically reach the *exact value* up to tolerance.
- Many iterative numerical methods share the properties above.
- The famous back propagation in deep neural network is also motivated by Newton method.
- Major algorithms for dynamic programming are called *policy iteration* and *value iteration* that also share the properties above.

*BIsecton= 해가 반드시 존재하는 폐구간 에서
중점을 구한뒤 그 중점의 함수값이 0이 되는거 계속해서 찾기
※ 0이 아닐경우 해가 반드시 존재하도록 [음,양], [양,음] 이렇게 다시 세워서 루프

*Newton= 해당 함수값의 접선 구하여 그 선의 x절편을 찾는다
만약 그 x절편의 함수값이 0이 될때까지 계속해서 찾기
※ 0이 아닐경우 ~~x~~ x절편의 함수값을 찾아 다시 접선구하면서 루프

III. Matrix Algebra

Matrix multiplication

Exercise 5

Solve the followings.

$$(.6 \quad .4)\begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} =$$

$(.6\times.7+.4\times.5, .6\times.3+.4\times.5) = (.62, .38)$

Exercise 6

What is P^2?

$$P = \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix}$$

$\begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix}\begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = \begin{pmatrix} .7\times.7+.3\times.5 & .7\times.3+.3\times.5 \\ .7\times.5+.5\times.5 & .5\times.3+.5\times.5 \end{pmatrix}$

$= \begin{pmatrix} .64 & .36 \\ .60 & .40 \end{pmatrix}$

Solution to system of linear equations

Exercise 7

Solve the followings.

$$(\pi_1 \quad \pi_2)\begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = (\pi_1 \quad \pi_2)$$

$$\pi_1 + \pi_2 = 1$$

$\begin{cases} .7\pi_1 + .5\pi_2 = \pi_1 \\ .3\pi_1 + .5\pi_2 = \pi_2 \end{cases}$

$\Rightarrow \begin{matrix} -.3\pi_1 + .5\pi_2 = 0 \\ .3\pi_1 - .5\pi_2 = 0 \end{matrix}$

$\Rightarrow \pi_1 : \pi_2 = 5 : 3$

$\therefore \pi_1 + \pi_2 = 5K + 3K = 1$

$K = \frac{1}{8}$

$\therefore \pi_1 = \frac{5}{8}, \quad \pi_2 = \frac{3}{8}$

Exercise 8

Solve the following system of equations.

$$x = y$$
$$y = 0.5z$$
$$z = 0.6 + 0.4x$$
$$x + y + z = 1$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -0.5 \\ -0.4 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.6 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -0.5 & 0 \\ -0.4 & 0 & 1 & 0.6 \\ 1 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & -0.4 & 1 & 0.6 \\ 0 & 2 & 1 & 1 \end{array}\right]$$

Solution doesn't exist

Exercise 9

Solve the following system of equations.

$$(\pi_0 \;\; \pi_1 \;\; \pi_2) \begin{pmatrix} -2 & 2 & \\ 3 & -5 & 2 \\ & 3 & -3 \end{pmatrix} = (0 \;\; 0 \;\; 0)$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$3.8\pi_2 = 0.8 \quad \therefore \pi_2 = \frac{4}{19}$$

$$\pi_1 = \frac{6}{19}, \quad \pi_0 = \frac{9}{19}$$

$$\left(\frac{9}{19}, \frac{6}{19}, \frac{4}{19}\right)$$

Exercise 10

Solve the following system of equations.

$$(\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4) \begin{pmatrix} .7 & .3 & & \\ .5 & .5 & & \\ & & .6 & .4 \\ & & .3 & .7 \end{pmatrix} = (\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4)$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$$X = \begin{pmatrix} \frac{1}{2} - \pi_4 \\ \frac{1}{2} - \pi_4 \\ \pi_4 \\ \pi_4 \end{pmatrix}$$

$$\pi_1 = \tfrac{1}{2} - \pi_4, \quad \pi_2 = \tfrac{1}{2} - \pi_4, \quad \pi_3 = \pi_4, \quad \pi_4 = \pi_4$$

Exercise 11

Solve following and express π_i for $i = 0, 1, 2, \ldots$

$$\begin{aligned} \pi_0 + \pi_1 + \pi_2 + \ldots &= 1 \\ \cancel{0.02\pi_0 + 0.02\pi_1 + 0.02\pi_2 + \ldots} &= \pi_0 \\ 0.98\pi_0 &= \pi_1 \\ 0.98\pi_1 &= \pi_2 \\ 0.98\pi_2 &= \pi_3 \\ \ldots &= \ldots \end{aligned}$$

$$\boxed{\pi_i = 0.02 \times (0.98)^{i-1}}$$

$$\frac{a}{1-0.98} = 1, \quad a = 0.02$$

$$\frac{\pi_0}{1-0.98} = 1 \quad \therefore \pi_0 = 0.02$$

IV. Series and Others

Exercise 12 (Infinite geometric series)

Simplify the following. When $|r| < 1$, $S = a + ar + ar^2 + ar^3 + \ldots$

$|r|<1 \Rightarrow \lim_{n\to\infty} r^n = 0$, $S = \dfrac{a(1-r^n)}{1-r}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{a(1-r^n)}{1-r} = \dfrac{a}{1-r}$ $(\because \lim r^n = 0)$

Exercise 13 (Finite geometric series)

Simplify the following. When $r \neq 1$, $S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

$S = a + ar + ar^2 + \cdots + ar^{n-1}$

$rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

$\Rightarrow (1-r)S = a - ar^n$

$\therefore S = \dfrac{a(1-r^n)}{1-r}$

Exercise 14 (Power series)

Simplify the following. When $|r| < 1$, $S = r + 2r^2 + 3r^3 + 4r^4 + \ldots$

$$S = (r + r^2 + r^3 + r^4 + \cdots) + (r^2 + r^3 + r^4 + \cdots) + (r^3 + r^4 + \cdots) + (r^4 + \cdots)$$

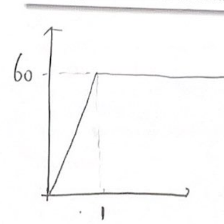
$$= \frac{r}{1-r} + \frac{r^2}{1-r} + \frac{r^3}{1-r} + \frac{r^4}{1-r} + \cdots$$

$$= \frac{r + r^2 + r^3 + \cdots}{1-r} = \frac{\frac{r}{1-r}}{1-r} = \frac{r}{(1-r)^2}$$

Formulation of time varying function

Exercise 15

During the first hour ($0 \le t \le 1$), $\lambda(t)$ increases linearly from 0 to 60. After the first hour, $\lambda(t)$ is constant at 60. Draw plot for $\lambda(t)$ and express the function in math form.



$\lambda(t) = 60t \quad (0 \le t \le 1)$

$\lambda(t) = 60 \quad (t \ge 1)$