Lecture B2_solution

Reinforcement Lerining Study

2021-01-04

차 례

Exercise 1

Assume that D follows the following discrete distribution.

d	20	25	30	35
$P[D=d]$	0.1	0.2	0.4	0.3
$30 \wedge d$	20	25	30	30
$(30-d)^+$	10	5	0	0
$24 \wedge d$	20	24	24	24
$(24-d)^+$	4	0	0	0

Answer the followings.

Solution:

- $\mathbb{E}[30 \wedge D] = \sum((30 \wedge d) \cdot P(d))\ 0.1(20) + 0.2(25) + 0.4(30) + 0.3(30) = 26$
- $\mathbb{E}[(30-D)^+] = \sum((30-d)^+ \cdot P(d))\ 0.1(10) + 0.2(5) + 0.4(0) + 0.3(0) = 2$
- $\mathbb{E}[24 \wedge D] = \sum((24 \wedge d) \cdot P(d))\ 0.1(20) + 0.2(24) + 0.4(24) + 0.3(24) = 23.6$
- $\mathbb{E}[(24-D)^+] = \sum((24-d)^+ \cdot P(d))\ 0.1(4) + 0.2(0) + 0.4(0) + 0.3(0) = 0.4$

Exercise 2

Find your brother's optimal stock level by the above Theorem 2. Then find his expected profit using the Remark 1

solution for optimal stock:

Demand	11	12	13	14	15
$P[D = d]$	0.2	0.2	0.2	0.2	0.2
$P[D \leq d]$	0.2	0.4	0.6	0.8	1.0

Theorem2

- If D is a continuous r.v, with cdf $F(\cdot)$, then find y s.t. $F(y) = \frac{c_u}{c_o + c_u}$.
- If D is a discrete r.v, with cdf $F(\cdot)$, then find smallest y such that $F(y) \geq \frac{c_u}{c_o + c_u}$.

Remark 1

- $\mathbb{E}(proit) = \mathbb{E}(sales.rev) + \mathbb{E}(slavage.rev) - \mathbb{E}(material)$

Solution

1) Finding optimal stock level

C_o=(Material Cost - Salvage Price)=$(1 - \frac{1}{2}) = \frac{1}{2}$
C_u=(Retail Price - Material Cost)=(2-1)=1

$F(y) \geq \frac{1}{1+1/2}$
$F(y) \geq \frac{2}{3}$

d	11	12	13	14	15
$P(D = d)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$F_D(11) = \frac{1}{5} < \frac{2}{3}$

$F_D(12) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} < \frac{2}{3}$

$F_D(13) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5} < \frac{2}{3}$

$F_D(14) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5} \geq \frac{2}{3}$

Thus, y^*=14

2) Finding expected profit

Sale Revenue=$2 \cdot (D \wedge Y)$

Salvage Revenue=$\frac{1}{2} \cdot (Y - D)^+$

Material Cost=$1 \cdot Y$

$\mathbb{E}(proit) = \mathbb{E}(sales.rev) + \mathbb{E}(slavage.rev) - \mathbb{E}(material)$

=$\sum_{d=11}^{15}(2 \cdot (d \wedge y) \cdot P(d)) + \sum_{d=11}^{15}(\frac{1}{2} \cdot (d - y)^+ \cdot P(d)) - \sum_{d=11}^{15}(y \cdot P(d))$

Using, $y^* = 14$

$\mathbb{E}[Profit] = \sum_{D=11}^{15}(2 \cdot (D \wedge 14) \cdot P(d)) + \sum_{D=11}^{15}(\frac{1}{2} \cdot (D - 14)^+ \cdot P(d)) - \sum_{D=11}^{15}(14 \cdot P(d))$

=$2 \cdot (\sum_{D=11}^{14}(D \cdot P(d)) + 14 \cdot P(15)) + \frac{1}{2} \cdot \sum_{D=11}^{14}((14 - D) \cdot P(d)) - 14$

=$2 \cdot (\frac{11+12+13+14}{5} + \frac{14}{5}) + \frac{1}{2} \cdot (\frac{3+2+1+0}{5}) - 14$

=$\frac{61}{5}$

=12.2

Thus, $\mathbb{E}[Profit] = 12.2$

Excercise 3

Your brother is now selling milk. The customer demands follows $U(20,40)$ gallons. Retail price is \$2 per gallon, material cost is \$1 per gallon, and salvage cost is \$ 0.5 per gallon. Find optimal stock level and expected profit

D~$U(20,40)$

$$f(x)=\begin{cases}\frac{1}{20} & 20\le x\le 40\\ 0 & otherwise\end{cases}$$

$$F(x)=\begin{cases}0 & 20<x\\ \frac{x-20}{20} & 20\le x\le 40\\ 1 & x>40\end{cases}$$

Theorem2

- If D is a continuous r.v, with cdf $F(\cdot)$, then find y s.t. $F(y)=\frac{c_u}{c_o+c_u}$.
- If D is a discrete r.v, with cdf $F(\cdot)$, then find smallest y such that $F(y)\ge\frac{c_u}{c_o+c_u}$.

Uniform distribution is continuous,so we sahll use $F(x^*)=\frac{c_u}{c_o+c_u}$

c_o=(Material Cost - Salvage Price)=$(1-\frac{1}{2})=\frac{1}{2}$
c_u=(Retail Price - Material Cost)=(2-1)=1

$F(x^*)=\frac{1}{1+1/2}=\frac{2}{3}$
$\frac{x^*-20}{20}=\frac{2}{3},\ x^*=\frac{100}{3}$

$\mathbb{E}[Profit]=\mathbb{E}(SaleRev.)+\mathbb{E}(salvageRev.)-\mathbb{E}(materialCost)$

Sale Revenue=$2\cdot(D\wedge\frac{100}{3})$

Salvage Revenue=$\frac{1}{2}\cdot(\frac{100}{3}-D)^+$

Material Cost=$1\cdot\frac{100}{3}$

$\mathbb{E}[Profit]=\mathbb{E}[2\cdot(D\wedge\frac{100}{3})]+\mathbb{E}[\frac{1}{2}\cdot(\frac{100}{3}-D)^+]-1\cdot\frac{100}{3}$
$=\int_{20}^{40}(2\cdot(X\wedge\frac{100}{3})\cdot\frac{1}{20})dx+\int_{20}^{40}(\frac{1}{2}\cdot(\frac{100}{3}-X)^+\cdot\frac{1}{20})dx-\int_{20}^{40}(\frac{100}{30}\cdot\frac{1}{20})dx$
$=\frac{1}{10}\cdot(\int_{20}^{\frac{100}{30}}(X)dx+\int_{\frac{100}{3}}^{40}(\frac{100}{3})dx)+\frac{1}{40}\cdot(\int_{20}^{\frac{100}{3}}(\frac{100}{3}-X)dx+\int_{\frac{100}{3}}^{40}(0)dx)-\frac{100}{3}$
$=\frac{1}{10}\cdot([\frac{1}{2}X^2]_{20}^{\frac{100}{3}}+\frac{100}{3}[X]_{\frac{100}{3}}^{40})+\frac{1}{40}\cdot[\frac{100}{3}X-\frac{1}{2}X^2]_{20}^{\frac{100}{3}}-\frac{100}{3}$
$=\frac{80}{3}$

Excercise 4

Lemonade sells for $18 per gallon but only costs $3 per gallon to make. if we run out of lemonade. it will be impossible to get more. On the other hand, leftover lemonade has a value of $1. Express The following quantity using sale as X and demand as D

Solution :

$c_u =$ lemonade($\$18$) - material cost($\$3$) = 15

$c_o =$ material cost($\$3$) - salvage cost($\$1$) = 2

$\mathbb{E}$(economic cost) = 15 $\times \mathbb{E}[(D-E)^+] + 2 \times \mathbb{E}[(X-D)^+]$

$\mathbb{E}$(Expected profit) = 18 $\times \mathbb{E}[min(X, D)] + 1 \times \mathbb{E}[(X-D)^+] - 3 \times \mathbb{E}[X]$

Excercise 5

Prove Theorem 1. (Hint: you may use formulation from Exercise 4)

Solutuon :

clam that, $x^* = argmin_X(p-c)(D-x)^+ + (c-s)(X-D)^+$ (LHS)

= $argmax_X\ pmin(D,X) + s(X-D)^+ - cX$ (RHS)

suppose,x^*=$argmin_X\ (p-c)(D-x)^+ + (c-s)(X-D)^+$

= $argmax_X\ -(p-c)(D-x)^+ - (c-s)(X-D)^+$

($\because argmin_X\ f(x)$ = $argmax_X\ -f(x)$)

the goal is to wrok with above to match RHS up to constant

= $argmax_X\ -(p-c)max(D-X,0) - c(X-D)^+ + s(X-D)^+$

= $argmax_X\ (p-c)min(X-D,0) - cmax(X-D,0)^+ s(X-D)^+$

($\because max(x,0) = -min(-x,0)$)

= $argmax_X\ pmin(X-D,0) - c[(X-D) + s(X-D)^+] + s(X-D)^+$

= $argmax_X\ pmin(X-D,0) - c(X-D) + s(X-D)^+$

($\because min(X,Y) + max(X,Y) = X+Y$)

= $argmax_X\ [pmin(X-D,0) + D - 0] - cX + cD + s(X-D)^+$

= $argmax_X\ p[min(X-D,0) + D] + s(X-D)^+ - cX + cD$

= $argmax_X\ pmin(X,D) + s(X-D)^+ - cX + -pD + cD$

= $argmax_X\ pmin(X,D) + s(X-D)^+$ (RHS)

$-cX - (p-c)D$, not affected by choice of x

DaiPark Excercise 4

Let D be a continuous random variable and uniformly distributed between 5 and 10. Find

(a) $\mathbb{E}[max(D, 8)]$

(b) $\mathbb{E}[(D-8)^{-}]$

where $x^{-1} = min(x, 0)$

Solution:

$D \sim U(5, 10)$

$$f_d(x) = \begin{cases} \frac{1}{5} & 5 \leq x \leq 10 \\ 0 & otherwise \end{cases}$$

$$F_d(x) = \begin{cases} 0 & 5 < x \\ \frac{x-5}{5} & 5 \leq x \leq 10 \\ 1 & x > 10 \end{cases}$$

(a)

$$\begin{aligned}
\mathbb{E}[max(D, 8)] &= \int_{-\infty}^{\infty} max(x, 8) f_D(x) dx \\
&= \int_{-\infty}^{0} max(x, 8) f_D(x) dx + \int_{0}^{\infty} max(x, 8) f_D(x) dx \\
&= 0 + \int_{0}^{\infty} max(x, 8) f_D(x) dx \\
&= \int_{0}^{5} max(x, 8) f_D(x) dx + \int_{5}^{10} max(x, 8) f_D(x) dx + \int_{10}^{\infty} max(x, 8) f_D(x) dx \\
&= 0 + \int_{5}^{10} max(x, 8) \frac{1}{5} dx + 0 \\
&= \int_{5}^{8} max(x, 8) \frac{1}{5} dx + \int_{8}^{10} max(x, 8) \frac{1}{5} dx \\
&= \int_{5}^{8} 8 \frac{1}{5} dx + \int_{8}^{10} x \frac{1}{5} dx \\
&= \frac{8}{5} [x]_5^8 + \frac{1}{5} [\frac{1}{2} x^2]_8^{10} \\
&= \frac{24}{5} + \frac{18}{5} \\
&= \frac{42}{5}
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathbb{E}[(D-8)^-] &= \int_{-\infty}^{\infty}(x-8)^- f_D(x)dx \\
&= \int_{-\infty}^{0}(x-8)^- f_D(x)dx + \int_{0}^{\infty}(x-8)^- f_D(x)dx \\
&= 0 + \int_{0}^{\infty}(x-8)^- f_D(x)dx \\
&= \int_{0}^{5}(x-8)^- f_D(x)dx + \int_{5}^{10}(x-8)^- f_D(x)dx + \int_{10}^{\infty}(x-8)^- f_D(x)dx \\
&= 0 + \int_{5}^{10}(x-8)^- \frac{1}{5}dx + 0 \\
&= \int_{5}^{8}(x-8)^- \frac{1}{5}dx + \int_{8}^{10}(x-8)^- \frac{1}{5}dx \\
&= \int_{5}^{8}(x-8)\frac{1}{5}dx + 0 \\
&= \frac{1}{5}[\frac{1}{2}x^2 - 8x]_5^8 \\
&= -\frac{9}{2}
\end{aligned}
$$

DaiPark Excercise 7

Suppose we are selling lemonade during a football game. The lemonade sells for $18 per gallon but only costs $3 per gallon to make. If we run out of lemonade during the game, it will be impossible to get more. On the other hand, leftover lemonade has a value of $1. Assume that we believe the fans would buy 10 gallons with probability 0.1, 11 gallons with probability 0.2, 12 gallons with probability 0.4, 13 gallons with probability 0.2, and 14 gallons with probability 0.1.

(a) What is the mean demand?

(b) If 11 gallons are prepared, what is the expected profit?

(c) What is the best amount of lemonade to order before the game?

(d) Instead, suppose that the demand was normally distributed with mean 1000 gallons and variance 200 gallons2 How much lemonades hould be ordered?

Solution:

Demand	10	11	12	13	14
P[D=d]	0.1	0.2	0.4	0.2	0.1

(a)

mean demand = $\sum xp(x) = 10(0.1) + 11(0.2) + 12(0.4) + 13(0.2) + 14(0.1)$ =12

(b)

Demand	10	12	13	14	Expected Profit	
14	$10(18) + 1(1) - 11(3) = 148$	$11(18) + 0(0) - 11(3) = 147$	$11(18) + 0(0) - 11(3) = 147$	$11(18) + 0(0) - 11(3) = 147$	$11(18) + 0(0) - 11(3) = 147$	$0.1(148) + 0.2(147) + 0.4(147) + 0.2(147) + 0.1(147) = 147.1$

Thus, Expected profit is $ 147.1 If 11 gallons are prepared

(c)

$c_o = 3 - 1 = 2$ $c_u = 18 - 3 = 15$

If D is a discrete r.v with cdf F(x), then find smallest y s.t $F(y) \geq \frac{c_u}{c_u+c_o} = \frac{15}{15+2} = \frac{15}{17} = 0.88235$

Demand	10	11	12	13	14
$P[D = d]$	0.1	0.2	0.4	0.2	0.1
$P[D \leq d]$	0.1	0.3	0.7	0.9	1.0

Thus, 13 is the best amount of lemonade before the game to order

(d)

Demand~$N(1000, 200^2)$

If D is a continuous r.v with cdf F(x), then find y s.t $F(y) = \frac{c_u}{c_u + c_o} = \frac{15}{15+2} = \frac{15}{17} = 0.88235$

then in standard Normal Distribution $P(Z \leq 1.175) = 0.88$

thus, $\frac{X-100}{200=1.175}$ $X = 1.175(200) + 1000 = 1235$

we should prepare 1235 gallons.

DaiPark Excercise 14

A store sells a particular brand of fresh juice. By the end of the day, any unsold juice is sold at a discounted price of $2 per gallon. The store gets the juice daily from a local producer at the cost of $5 per gallon, and it sells the juice at $10 per gallon. Assume that the daily demand for the juice is uniformly distributed between 50 gallons to 150 gallons.

(a) What is the optimal number of gallons that the store should order from the distribution each day in order to maximize the expected profit each day?

(b) If 100 gallons are ordered, what is the expected profit per day?

Solution:

information

- Salvage Value= s= $2 per gallon
- Wholesale price= c =$5 per gallon
- retail price = p = $10 per gallon
- Demand distribution = U(50,150)

$c_o = c - s = 5 - 2 = 3$

$c_u = p - c = 10 - 5 = 5$

$$f(y) = \begin{cases} \frac{1}{100} & 50 \leq x \leq 150 \\ 0 & otherwise \end{cases}$$

$$F(y) = \begin{cases} 0 & y \leq 50 \\ \frac{y-50}{150-50} & 50 \leq y \leq 150 \\ 1 & y > 150 \end{cases}$$

(a) If D is a continous r.v with cdf F(x), then find y s.t $F(y) = \frac{c_u}{c_u+c_o} = \frac{5}{5+3} = \frac{5}{8}$

$F(\frac{5}{8})^{-1} = (150 - 50)\frac{5}{8} + 50 = 112.5$

Thus optimal number of gallon is 112.5 gallon to maximize the expected profit each day.

(b)

$\mathbb{E}(proit) = \mathbb{E}(sales.rev) + \mathbb{E}(slavage.rev) - \mathbb{E}(material)$

when x=100,

$\mathbb{E}(profit)$ = $10\mathbb{E}[(D \wedge 100)] + 2\mathbb{E}[(100 - D)^+] - (10)100$

(1)

$$
\begin{aligned}
\mathbb{E}[(D \wedge 100)]] &= \int_{-\infty}^{\infty} \mathbb{E}[(D \wedge 100)] f_D(x) dx \\
&= \int_{-\infty}^{50} \mathbb{E}[(D \wedge 100)] f_D(x) dx + \int_{50}^{150} \mathbb{E}[(D \wedge 100)] f_D(x) dx + \int_{150}^{\infty} \mathbb{E}[(D \wedge 100)] f_D \\
&= 0 + \int_{50}^{150} \mathbb{E}[(D \wedge 100)] f_D(x) dx + 0 \\
&= \int_{50}^{150} \mathbb{E}[(D \wedge 100)] f_D(x) dx \\
&= \int_{50}^{100} \mathbb{E}[(D \wedge 100)] f_D(x) dx + \int_{100}^{150} \mathbb{E}[(D \wedge 100)] f_D(x) dx \\
&= \int_{50}^{100} (X) \frac{1}{100} dx + \int_{100}^{150} (100) \frac{1}{100} dx = 87.5
\end{aligned}
$$

(2)

$$
\begin{aligned}
\mathbb{E}[(100 - X)^{+}] &= \int_{-\infty}^{\infty} \mathbb{E}[(100 - X)^{+}] f_D(x) dx \\
&= \int_{-\infty}^{50} \mathbb{E}[(100 - X)^{+}] f_D(x) dx + \int_{50}^{150} \mathbb{E}[(100 - X)^{+}] f_D(x) dx + \int_{150}^{\infty} \mathbb{E}[(100 - X \\
&= 0 + \int_{50}^{150} \mathbb{E}[(100 - X)^{+}] f_D(x) dx + 0 \\
&= \int_{50}^{150} \mathbb{E}[(100 - X)^{+}] f_D(x) dx \\
&= \int_{50}^{100} \mathbb{E}[(100 - X)^{+}] f_D(x) dx + \int_{100}^{150} \mathbb{E}[(100 - X)^{+}] f_D(x) dx \\
&= \int_{50}^{100} (100 - X)(\frac{1}{100}) dx + \int_{100}^{150} (0) \frac{1}{100} = 12.5
\end{aligned}
$$

(3)

$\mathbb{E}[100] = 100$

Thus, Expected Profit is $10(87.5) + 2(12.5) - 5(100) = 400$

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"Newsvendor"
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## [1] "Newsvendor"
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