B1_Exercise

Son Min Sang

2021-01-11

Contents

Exercise

Exercise 1

Assume that D follows the following discrete distribution.

d	20	25	30	35
$P[D = d]$	0.1	0.2	0.4	0.3
$30 \wedge d$	20	25	30	30
$(30 - d)^+$	10	5	0	0
$24 \wedge d$	20	24	24	24
$(24 - d)^+$	4	0	0	0

Answer the followings.

- $E[30 \wedge D] = \sum_{d \in [20,25,30,35]} ((30 \wedge d) \cdot P(d)) = 20 \times 0.1 + 25 \times 0.2 + 30 \times 0.4 + 30 \times 0.3 = 28$
- $E[(30 - D)^+] = \sum_{d \in [20,25,30,35]} ((30 - d)^+ \cdot P(d)) = 10 \times 0.1 + 5 \times 0.2 + 0 \times 0.4 + 0 \times 0.3 = 2$
- $E[24 \wedge D] = \sum_{d \in [20,25,30,35]} ((24 \wedge d) \cdot P(d)) = 20 \times 0.1 + 24 \times 0.2 + 24 \times 0.4 + 24 \times 0.3 = 23.6$
- $E[(24 - D)^+] = \sum_{d \in [20,25,30,35]} ((24 - d)^+ \cdot P(d)) = 4 \times 0.1 + 0 \times 0.2 + 0 \times 0.4 + 0 \times 0.3 = 0.4$

Exercise 2

Find your brother's optimal stock level by the above Theorem 2. Then, find his expected profit using the Remark 1.

Theorem2

- If D is a continuous r.v, with cdf $F(\cdot)$, then find y s.t. $F(y) = \frac{c_u}{c_o+c_u}$.
- If D is a discrete r.v, with cdf $F(\cdot)$, then find smallest y such that $F(y) \geq \frac{c_u}{c_o+c_u}$.

Remark 1

- $E[Profit]$=E(Sale Rev.)+E(salvage Rev.)-E(material Cost)

Answer

1) Finding optimal stock level

C_o=(Material Cost - Salvage Price)=$(1-\frac{1}{2}) = \frac{1}{2}$
C_u=(Retail Price - Material Cost)=(2-1)=1

$F(y) \geq \frac{1}{1+1/2}$
$F(y) \geq \frac{2}{3}$

d	11	12	13	14	15
$P(D=d)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$F_D(11) = \frac{1}{5} < \frac{2}{3}$

$F_D(12) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} < \frac{2}{3}$

$F_D(13) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5} < \frac{2}{3}$

$F_D(14) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5} \geq \frac{2}{3}$

Thus, y^*=14

2) Finding expected profit

Sale Revenue=$2 \cdot (D \wedge Y)$

Salvage Revenue=$\frac{1}{2} \cdot (Y-D)^+$

Material Cost=$1 \cdot Y$

$E[Profit]$=E(Sale Rev.)+E(salvage Rev.)-E(material Cost)

$$=\sum_{d=11}^{15}(2 \cdot (d \wedge y) \cdot P(d)) + \sum_{d=11}^{15}(\frac{1}{2} \cdot (d-y)^+ \cdot P(d)) - \sum_{d=11}^{15}(y \cdot P(d))$$

Using $y^* = 14$,

$$
\begin{aligned}
E[Profit] &= \textstyle\sum_{D=11}^{15}(2 \cdot (D \wedge 14) \cdot P(D)) + \sum_{D=11}^{15}(\frac{1}{2} \cdot (D-14)^+ \cdot P(D)) - \sum_{D=11}^{15}(14 \cdot P(D)) \\
&= 2 \cdot (\textstyle\sum_{D=11}^{14}(D \cdot P(D)) + 14 \cdot P(15)) + \frac{1}{2} \cdot \sum_{D=11}^{14}((14-D) \cdot P(D)) - 14 \\
&= 2 \cdot (\tfrac{11+12+13+14}{5} + \tfrac{14}{5}) + \tfrac{1}{2} \cdot (\tfrac{3+2+1+0}{5}) - 14 \\
&= \tfrac{61}{5} \\
&= 12.2
\end{aligned}
$$

Thus, $E[Profit] = 12.2$

Exercise 3

Your brother is now selling milk. The customer demands follows $U(20, 40)$ gallons. Retail price is \$2 per gallon, material cost is \$1 per gallon, and salvage cost is \$ 0.5 per gallon. Find optimal stock level and expected profit

$D \sim U(20, 40)$

$$f(x) = \begin{cases} \frac{1}{20} & 20 \le x \le 40 \\ 0 & otherwise \end{cases}$$

$$F(x) = \begin{cases} 0 & 20 < x \\ \frac{x-20}{20} & 20 \le x \le 40 \\ 1 & x > 40 \end{cases}$$

Theorem2

- If D is a continuous r.v, with cdf $F(\cdot)$, then find y s.t. $F(y) = \frac{c_u}{c_o+c_u}$.
- If D is a discrete r.v, with cdf $F(\cdot)$, then find smallest y such that $F(y) \ge \frac{c_u}{c_o+c_u}$.

Uniform distribution is continuous, so we shall use $F(x^*) = \frac{c_u}{c_o+c_u}$

c_o=(Material Cost - Salvage Price)=$(1 - \frac{1}{2}) = \frac{1}{2}$
c_u=(Retail Price - Material Cost)=(2-1)=1

$F(x^*) = \frac{1}{1+1/2} = \frac{2}{3}$

$\frac{x^*-20}{20} = \frac{2}{3},\ x^* = \frac{100}{3}$

$\mathbb{E}[Profit] = \mathbb{E}(SaleRev.) + \mathbb{E}(salvageRev.) - \mathbb{E}(materialCost)$

Sale Revenue=$2 \cdot (D \wedge \frac{100}{3})$

Salvage Revenue=$\frac{1}{2} \cdot (\frac{100}{3} - D)^+$

Material Cost=$1 \cdot \frac{100}{3}$

$\mathbb{E}[Profit] = \mathbb{E}[2 \cdot (D \wedge \frac{100}{3})] + \mathbb{E}[\frac{1}{2} \cdot (\frac{100}{3} - D)^+] - 1 \cdot \frac{100}{3}$

$= \int_{20}^{40} (2 \cdot (D \wedge \frac{100}{3}) \cdot \frac{1}{20})dD + \int_{20}^{40} (\frac{1}{2} \cdot (\frac{100}{3} - D)^+ \cdot \frac{1}{20})dD - \int_{20}^{40} (\frac{100}{30} \cdot \frac{1}{20})dD$

$= \frac{1}{10} \cdot (\int_{20}^{\frac{100}{30}} (D)dD + \int_{\frac{100}{3}}^{40} (\frac{100}{3})dD) + \frac{1}{40} \cdot (\int_{20}^{\frac{100}{3}} (\frac{100}{3} - D)dD + \int_{\frac{100}{3}}^{40} (0)dD) - \frac{100}{3}$

$= \frac{1}{10} \cdot ([\frac{1}{2}D^2]_{20}^{\frac{100}{3}} + \frac{100}{3}[D]_{\frac{100}{3}}^{40}) + \frac{1}{40} \cdot [\frac{100}{3}D - \frac{1}{2}D^2]_{20}^{\frac{100}{3}} - \frac{100}{3}$

$= \frac{80}{3}$

Exercise 4

Lemonade sells for \$18 per gallon but only costs \$3 per gallon to make. If we run out of lemonade, it will be impossible to get more. On the other hand, leftover lemonade has a value of \$1. Express the following quantity using sale as X and demand as D.

c_u=(Retail Price - Material Cost)=(18-3)=15

c_o=(Material Cost - Salvage Price)=(3-1)=2

c_v=Material Cost = 3

Expected economic cost $= 15 \times \mathbb{E}[(D - E)^+] + 2 \times \mathbb{E}[(X - D)^+]$

Expected profit $= 18 \times \mathbb{E}[min(X, D)] + 1 \times \mathbb{E}[(X - D)^+] - 3 \times \mathbb{E}[X]$

Exercise 5

Prove Theorem 1.

Theorem 1.

- Maximizing the expected profit is equivalent to minimizing the expected economic cost

Set

- Selling Price : p
- Buying Price : c
- Salvage Value : s
- Market Demand : D
- Order Quantity : X

Claim

$x^* = argmin_X((p-c)\cdot(D-X)^+ + (c-s)\cdot(X-D)^+)(\mathbf{LHS})$

$=argmax_X(pmin(D,X)+s(X-D)^+ - cX)(\mathbf{RHS})$

Suppose

$argmin_X((p-c)\cdot(D-X)^+ + (c-s)\cdot(X-D)^+)$

$=argmax_X(-(p-c)\cdot(D-X)^+ - (c-s)\cdot(X-D)^+)(\because argmin_X\ f(x) = argmax_X\ -f(x))$

(the goal is to work with above to match RHS up to constant)

$= argmax_X(-(p-c)\cdot max(D-X,0) - c\cdot(X-D)^+ + s\cdot(X-D)^+)$

$= argmax_X((p-c)\cdot min(X-D,0) - c\cdot max(X-D,0)^+ s\cdot(X-D)^+)$

$(\because max(x,0) = -min(-x,0))$

$= argmax_X(pmin(X-D,0) - c\cdot[(X-D)+s(X-D)^+] + s\cdot(X-D)^+)$

$= argmax_X(pmin(X-D,0) - c\cdot(X-D) + s\cdot(X-D)^+)$

$(\because min(X,Y) + max(X,Y) = X+Y)$

$= argmax_X([pmin(X-D,0)+D-0] - c\cdot X + c\cdot D + s\cdot(X-D)^+)$

$= argmax_X(p\cdot[min(X-D,0)+D] + s\cdot(X-D)^+ - c\cdot X + c\cdot D)$

$= argmax_X(p\cdot min(X,D) + s\cdot(X-D)^+ - c\cdot X + -p\cdot D + c\cdot D)$

$= argmax_X(\underbrace{p\cdot min(X,D) + s\cdot(X-D)^+ - c\cdot X}_{(\mathbf{RHS})} - \underbrace{(p-c)\cdot D}_{\textbf{not affected by choice of X}})$

DaiPark Exercises

Exercise 1

Show that $(D \wedge Y) + (Y - D)^+ = Y$

i) $D > Y$

$(D \wedge Y) + (Y - D)^+ = Y + 0 = Y$

ii) $Y > D$

$(D \wedge Y) + (Y - D)^+ = D + (Y - D) = Y$

alternative

$$\begin{aligned} min(D, Y) + max(Y - D, 0) &= min(D - Y, 0) + Y + max(Y - D, 0) \\ &= D - Y + Y + Y - D \\ &= Y (\because min(x, y) + max(x, y) = x + y) \end{aligned}$$

Exercise 2

Let D **be a discrete a random variable with the following pmf.**

d	**5**	**6**	**7**	**8**	**9**
$P(D = d)$	0.1	0.3	0.4	0.1	0.1

Find

(a) $E[min(D, 7)]$

(b) $E[(7 - D)^+]$

where $x^+ = max(x, 0)$

(a) $E[min(D, 7)] = \sum_{d=5}^{7} \mathbb{P}(D = d) min(d, 7)$

$$= 0.1 \cdot min(5, 7) + 0.3 \cdot min(6, 7) \cdot +0.4 \cdot min(7, 7) + 0.1 \cdot min(8, 7) + 0.1 \cdot min(9, 7)$$

$$= 0.1 \cdot 5 + 0.3 \cdot 6 + 0.4 \cdot 7 + 0.1 \cdot 7 + 0.1 \cdot 7$$

$$= 0.5 + 1.8 + 2.8 + 0.7 + 0.7$$

$$= 6.5$$

(b) $E[(7-D)^+] = \sum_{d=5}^{7} \mathbb{P}(D=d) max(7-d,0)$

$= 0.1 \cdot max(7-5,0) + 0.3 \cdot max(7-6,0) \cdot +0.4 \cdot max(7-7,0) + 0.1 \cdot max(7-8,0) + 0.1 \cdot max(7-9,0)$

$= 0.1 \cdot 2 + 0.3 \cdot 1 + 0.4 \cdot 0 + 0.1 \cdot 0 + 0.1 \cdot 0$

$= 0.2 + 0.3$

$= 0.5$

Exercise 4

Let D be a continuous random variable and uniformly distributed between 5 and 10.

(a) $E[max(D,8)]$

(b) $E[(D-8)^-]$

where $x^- = min(x,0)$.

$D \sim U(5,10)$

$$f_d(x) = \begin{cases} \frac{1}{5} & 5 \leq x \leq 10 \\ 0 & otherwise \end{cases}$$

$$F_d(x) = \begin{cases} 0 & 5 < x \\ \frac{x-5}{5} & 5 \leq x \leq 10 \\ 1 & x > 10 \end{cases}$$

(a)

$$\begin{aligned}
E[max(D,8)] &= \int_{-\infty}^{\infty} max(x,8) f_D(x) dx \\
&= \int_{-\infty}^{0} max(x,8) f_D(x) dx + \int_{0}^{\infty} max(x,8) f_D(x) dx \\
&= 0 + \int_{0}^{\infty} max(x,8) f_D(x) dx \\
&= \int_{0}^{5} max(x,8) f_D(x) dx + \int_{5}^{10} max(x,8) f_D(x) dx + \int_{10}^{\infty} max(x,8) f_D(x) dx \\
&= 0 + \int_{5}^{10} max(x,8) \frac{1}{5} dx + 0 \\
&= \int_{5}^{8} max(x,8) \frac{1}{5} dx + \int_{8}^{10} max(x,8) \frac{1}{5} dx \\
&= \int_{5}^{8} 8 \frac{1}{5} dx + \int_{8}^{10} x \frac{1}{5} dx \\
&= \frac{8}{5} [x]_5^8 + \frac{1}{5} [\frac{1}{2} x^2]_8^{10} \\
&= \frac{24}{5} + \frac{18}{5} \\
&= \frac{42}{5}
\end{aligned}$$

(b)

$$
\begin{aligned}
E[(D-8)^-] &= \int_{-\infty}^{\infty} (x-8)^- f_D(x)dx \\
&= \int_{-\infty}^{0} (x-8)^- f_D(x)dx + \int_{0}^{\infty} (x-8)^- f_D(x)dx \\
&= 0 + \int_{0}^{\infty} (x-8)^- f_D(x)dx \\
&= \int_{0}^{5} (x-8)^- f_D(x)dx + \int_{5}^{10} (x-8)^- f_D(x)dx + \int_{10}^{\infty} (x-8)^- f_D(x)dx \\
&= 0 + \int_{5}^{10} (x-8)^- \frac{1}{5}dx + 0 \\
&= \int_{5}^{8} (x-8)^- \frac{1}{5}dx + \int_{8}^{10} (x-8)^- \frac{1}{5}dx \\
&= \int_{5}^{8} (x-8)\frac{1}{5}dx + 0 \\
&= \frac{1}{5}[\frac{1}{2}x^2 - 8x]_5^8 \\
&= -\frac{9}{2}
\end{aligned}
$$

Exercise 5

Let D be an exponential random variable with parameter 7. Find

(a) $E[max(D,3)]$

(b) $E[(D-4)^{-}]$

Note that pdf of an exponential random variable with parameter λ is

$$f_D(x) = \lambda e^{-\lambda x} for x \geq 0.$$

$D \sim exp(7)$

$$f_d(x) = \begin{cases} 7e^{-7x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$$F_d(x) = \begin{cases} 1 - e^{-7x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

(a)

$$\begin{aligned}
E[max(D,3)] &= \int_{-\infty}^{\infty} max(x,3) f_D(x)dx \\
&= \int_{-\infty}^{0} max(x,3) f_D(x)dx + \int_{0}^{\infty} max(x,3) f_D(x)dx \\
&= 0 + \int_{0}^{\infty} max(x,3) f_D(x)dx \\
&= \int_{0}^{3} max(x,3) f_D(x)dx + \int_{3}^{\infty} max(x,3) f_D(x)dx \\
&= \int_{0}^{3} 3 \cdot 7e^{-7x}dx + \int_{3}^{\infty} x \cdot 7e^{-7x}dx \\
&= 21 \int_{0}^{3} e^{-7x}dx + 7 \int_{3}^{\infty} x \cdot e^{-7x}dx \\
&= 21[-\frac{1}{7}e^{-7x}]_0^3 + 7([-\frac{-x}{7}e^{-7x}]_3^{\infty} - \int_{3}^{\infty} -\frac{1}{7}e^{-7x}dx) \\
&= [-3e^{-7x}]_0^3 + [-xe^{-7x}]_3^{\infty} + \int_{3}^{\infty} e^{-7x}dx \\
&= -3e^{-21} + 3 + 0 + 3e^{-21} - [\frac{1}{7}e^{-7x}]_3^{\infty} \\
&= 3 - (-\frac{1}{7}e^{-21}) \\
&= 3 + \frac{1}{7}e^{-21}
\end{aligned}$$

(b)

$$
\begin{aligned}
E[(D-4)^-] &= \int_{-\infty}^{\infty} (x-4)^- f_D(x)dx \\
&= \int_{-\infty}^{0} (x-4)^- f_D(x)dx + \int_{0}^{\infty} (x-4)^- f_D(x)dx \\
&= \int_{0}^{4} (x-4)^- f_D(x)dx + 0 \\
&= \int_{0}^{4} (x-4)^- 7 \cdot e^{-7x} dx \\
&= 7([-\frac{x-4}{7}e^{-7x}]_0^4 - \int_0^4 -\frac{1}{7}e^{-7x}dx) \\
&= [-(x-4)e^{-7x}]_0^4 + \int_0^4 e^{-7x}dx \\
&= -4 - \frac{1}{7}[e^{-7x}]_0^4 \\
&= -4 - \frac{1}{7}e^{-28} + \frac{1}{7} \\
&= -\frac{1}{7}e^{-28} - \frac{27}{7}
\end{aligned}
$$

Exercise 6

David buys fruits and vegetables wholesale and retails them at Davids Produce on La Vista Road. One of the more difficult decisions is the amount of bananas to buy. Let us make some simplifying assumptions, and assume that David purchases bananas once a week at 10 cents per pound and retails them at 30 cents per pound during the week. Bananas that are more than a week old are too ripe and are sold for 5 cents per pound.

(a) **Suppose the demand for the good bananas follows the same distribution as D given in Problem 2. What is the expected profit of David in a week if he buys 7 pounds of banana?**

d	5	6	7	8	9
$P(D = d)$	0.1	0.3	0.4	0.1	0.1

E[Sale Revenue]=$30 \cdot (D \wedge 7)$

Salvage Revenue=$5 \cdot (7 - D)^+$

Material Cost=$10 \cdot 7$

$E[Profit] = E(SaleRev.) + E(salvageRev.) - E(materialCost)$

$$= \sum_{D=5}^{9}(30 \cdot (D \wedge 7) \cdot P(D)) + \sum_{D=5}^{9}(5 \cdot (7 - D)^+ \cdot P(D)) - 10 \cdot 7$$

$$= 30 \cdot (5 * 0.1 + 6 * 0.3 + 7 * 0.4 + 7 * 0.1 + 7 * 0.1)$$

$$+5 \cdot (2 * 0.1 + 1 * 0.3 + 0 * 0.4 + 0 * 0.1 + 0 * 0.1) - 70$$

$$= 127.5$$

(b) **Now assume that the demand for the good bananas is uniformly distributed between 5 and 10 like in Problem 4. What is the expected profit of David in a week if he buys 7 pounds of banana?**

$D \sim U(5, 10)$

$$f_D(x) = \begin{cases} \frac{1}{5} & 5 \le x \le 10 \\ 0 & otherwise \end{cases}$$

$E[Profit] = E(SaleRev.) + E(salvageRev.) - E(materialCost)$

$$= \int_5^{10} 30 \cdot (D \wedge 7) \cdot P(D)dD + \int_5^{10} 5 \cdot (7 - D)^+ \cdot P(D)dD - 10 \cdot 7$$

$$= \int_5^{7} 30 \cdot D \cdot \tfrac{1}{5}dD + \int_7^{10} 30 \cdot 7 \cdot \tfrac{1}{5}dD + \int_5^{7} 5 \cdot (7 - D) \cdot \tfrac{1}{5}dD + \int_7^{10} 5 \cdot 0 \cdot \tfrac{1}{5}dD - 10 \cdot 7$$

$$= \int_5^{7} 6DdD + \int_7^{10} 42dD + \int_5^{7} 7dD - \int_5^{7} DdD - 70$$

$$= \int_5^{7} 5DdD + [42D]_7^{10} + [7D]_5^{7} - 70$$

$$= 6.25 \cdot [\tfrac{1}{2}D^2]_5^{7} + 105$$

$$= 130$$

(c) Find the expected profit if David's demand for the good bananas follows an exponential distribution with mean 7 and if he buys 7 pounds of banana.

$D \sim exp(\frac{1}{7})$

$$f_D(x) = \begin{cases} \frac{1}{7}e^{-\frac{1}{7}x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$$F_D(x) = \begin{cases} 1 - e^{-\frac{1}{7}x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$$E[Profit] = 30 \cdot \int_0^\infty (D \wedge 7) \cdot f(D)dD + 5 \cdot \int_0^\infty (7 - D)^+ \cdot f(D)dD - 10 \cdot 7$$

$$= 30 \cdot \int_0^7 D \cdot f(D)dD + 30 \cdot \int_7^\infty 7 \cdot f(D)dD + 5 \cdot \int_0^7 (7 - D) \cdot f(D)dD + 5 \cdot \int_7^\infty 0 \cdot f(D)dD - 10 \cdot 7$$

$$= 25 \cdot \int_0^7 D \cdot f(D)dD + 30 \cdot \int_7^\infty 7 \cdot f(D)dD + 35 \cdot \int_0^7 f(D)dD - 70$$

$$= 25 \cdot ([-De^{-\frac{1}{7}D}]_0^7 - [7e^{-\frac{1}{7}D}]_0^7) + 30 \cdot 7 \cdot [-e^{-\frac{1}{7}D}]_7^\infty + 35[-e^{-\frac{1}{7}D}]_0^7 - 70$$

$$= 25 \cdot (-14e^{-1} + 7) + 210 \cdot e^{-1} + 35 \cdot (-e^{-1} + 1) - 70$$

$$= 140 - 175e^{-1}$$

Exercise 7

Suppose we are selling lemonade during a football game. The lemonade sells for $18 per gallon but only costs $3 per gallon to make. If we run out of lemonade during the game, it will be impossible to get more. On the other hand, leftover lemonade has a value of $1. Assume that we believe the fans would buy 10 gallons with probability 0.1, 11 gallons with probability 0.2, 12 gallons with probability 0.4, 13 gallons with probability 0.2, and 14 gallons with probability 0.1.

(a) What is the mean demand?

(b) If 11 gallons are prepared, what is the expected profit?

(c) What is the best amount of lemonade to order before the game?

(d) Instead, suppose that the demand was normally distributed with mean 1000 gallons and variance 200 gallons2 How much lemonades hould be ordered?

Solution:

Demand	10	11	12	13	14
P[D=d]	0.1	0.2	0.4	0.2	0.1

(a)

mean demand $= \sum xp(x) = 10(0.1) + 11(0.2) + 12(0.4) + 13(0.2) + 14(0.1)$ **=12**

(b)

Demand	**10**	**12**	**13**	**14**	**Expected Profit**	
14	$10(18) + 1(1) - 11(3) = 148$	$11(18) + 0(0) - 11(3) = 147$	$11(18) + 0(0) - 11(3) = 147$	$11(18) + 0(0) - 11(3) = 147$	$11(18) + 0(0) - 11(3) = 147$	$0.1(148) + 0.2(147) + 0.4(147) + 0.2(147) + 0.1(147) = 147.1$

Thus, Expected profit is $ 147.1 If 11 gallons are prepared

(c)

$c_o = 3 - 1 = 2$

$c_u = 18 - 3 = 15$

If D is a discrete r.v with cdf F(x), then find smallest y s.t $F(y) \geq \frac{c_u}{c_u+c_o} = \frac{15}{15+2} = \frac{15}{17} = 0.88235$

Demand	**10**	**11**	**12**	**13**	**14**
$P[D = d]$	**0.1**	**0.2**	**0.4**	**0.2**	**0.1**
$P[D \leq d]$	**0.1**	**0.3**	**0.7**	**0.9**	**1.0**

Thus, 13 is the best amount of lemonade before the game to order

(d)

Demand~$N(1000, 200^2)$

If D is a continuous r.v with cdf F(x), then find y s.t $F(y) = \frac{c_u}{c_u+c_o} = \frac{15}{15+2} = \frac{15}{17} = 0.88235$

then in standard Normal Distribution $P(Z \leq 1.175) = 0.88$

thus, $\frac{X-100}{200} = 1.175$

$X = 1.175(200) + 1000 = 1235$

we should prepare 1235 gallons.

Exercise 8

Suppose that a bakery specializes in chocolate cakes. Assume the cakes retail at \$20 per cake, but it takes \$10 to prepare each cake. Cakes cannot be sold after one week, and they have a negligible salvage value. It is estimated that the weekly demand for cakes is: 15 cakes in 5% of the weeks, 16 cakes in 20% of the weeks, 17 cakes in 30% of the weeks, 18 cakes in 25% of the weeks, 19 cakes in 10% of the weeks, and 20 cakes in 10% of the weeks. How many cakes should the bakery prepare each week? What is the bakery's expected optimal weekly profit?

optimal weekly profit $= 20 \cdot (\sum_{D=15}^{20} (D \wedge X) \cdot P(D)) - 10 \cdot X$

$= 20 \cdot ((15 \wedge X) \cdot \frac{5}{100} + (16 \wedge X) \cdot \frac{20}{100} + (17 \wedge X) \cdot \frac{30}{100} + (18 \wedge X) \cdot \frac{25}{100} + (19 \wedge X) \cdot \frac{10}{100} + (20 \wedge X) \cdot \frac{10}{100}) - 10 \cdot X$

$= (15 \wedge X) + 4(16 \wedge X) + 6(17 \wedge X) + 5(18 \wedge X) + 2(19 \wedge X) + 2(20 \wedge X) - 10 \cdot X$

$X = 15,\ 15 + 4 \cdot 15 + 6 \cdot 15 + 5 \cdot 15 + 2 \cdot 15 + 2 \cdot 15 - 10 \cdot 15 = 150$

$X = 16,\ 15 + 4 \cdot 16 + 6 \cdot 16 + 5 \cdot 16 + 2 \cdot 16 + 2 \cdot 16 - 10 \cdot 16 = 159$

$X = 17,\ 15 + 4 \cdot 16 + 6 \cdot 17 + 5 \cdot 17 + 2 \cdot 17 + 2 \cdot 17 - 10 \cdot 17 = 164$

$X = 18,\ 15 + 4 \cdot 16 + 6 \cdot 17 + 5 \cdot 18 + 2 \cdot 18 + 2 \cdot 18 - 10 \cdot 18 = 163$

$X = 19,\ 15 + 4 \cdot 16 + 6 \cdot 17 + 5 \cdot 18 + 2 \cdot 19 + 2 \cdot 19 - 10 \cdot 19 = 157$

$X = 20,\ 15 + 4 \cdot 16 + 6 \cdot 17 + 5 \cdot 18 + 2 \cdot 19 + 2 \cdot 20 - 10 \cdot 20 = 149$

therefore, the bakery should prepare 17 cakes each week. and the bakery's expected optimal weekly profit is 164\$

Exercise 14

A store sells a particular brand of fresh juice. By the end of the day, any unsold juice is sold at a discounted price of \$2 per gallon. The store gets the juice daily from a local producer at the cost of \$5 per gallon, and it sells the juice at \$10 per gallon. Assume that the daily demand for the juice is uniformly distributed between 50 gallons to 150 gallons.

(a) What is the optimal number of gallons that the store should order from the distribution each day in order to maximize the expected profit each day?

(b) If 100 gallons are ordered, what is the expected profit per day?

Solution:

information

- **Salvage Value= s= \$2 per gallon**
- **Wholesale price= c =\$5 per gallon**
- **retail price = p = \$10 per gallon**
- **Demand distribution = U(50,150)**

$c_o = c - s = 5 - 2 = 3$

$c_u = p - c = 10 - 5 = 5$

$$f(y) = \begin{cases} \frac{1}{100} & 50 \leq x \leq 150 \\ 0 & otherwise \end{cases}$$

$$F(y) = \begin{cases} 0 & y \leq 50 \\ \frac{y-50}{150-50} & 50 \leq y \leq 150 \\ 1 & y > 150 \end{cases}$$

(a) If D is a continous r.v with cdf F(x), then find y s.t $F(y) = \frac{c_u}{c_u+c_o} = \frac{5}{5+3} = \frac{5}{8}$

$F(\frac{5}{8})^{-1} = (150 - 50)\frac{5}{8} + 50 = 112.5$

Thus optimal number of gallon is 112.5 gallon to maximize the expected profit each day.

(b)

$\mathbb{E}(proit) = \mathbb{E}(sales.rev) + \mathbb{E}(slavage.rev) - \mathbb{E}(material)$ **when x=100,**

$\mathbb{E}(profit) = 10\mathbb{E}[(D \wedge 100)] + 2\mathbb{E}[(100 - D)^+] - (10)100$

(1)

$$\begin{aligned}
\mathbb{E}[(D \wedge 100)]] &= \int_{-\infty}^{\infty} \mathbb{E}[(D \wedge 100)] f_D(x)dx \\
&= \int_{-\infty}^{50} \mathbb{E}[(D \wedge 100)] f_D(x)dx + \int_{50}^{150} \mathbb{E}[(D \wedge 100)] f_D(x)dx + \int_{150}^{\infty} \mathbb{E}[(D \wedge 100)] f_D(x)dx \\
&= 0 + \int_{50}^{150} \mathbb{E}[(D \wedge 100)] f_D(x)dx + 0 \\
&= \int_{50}^{150} \mathbb{E}[(D \wedge 100)] f_D(x)dx \\
&= \int_{50}^{100} \mathbb{E}[(D \wedge 100)] f_D(x)dx + \int_{100}^{150} \mathbb{E}[(D \wedge 100)] f_D(x)dx \\
&= \int_{50}^{100} (X)\frac{1}{100}dx + \int_{100}^{150} (100)\frac{1}{100}dx = 87.5
\end{aligned}$$

(2)

$$\begin{aligned}
\mathbb{E}[(100 - X)^+] &= \int_{-\infty}^{\infty} \mathbb{E}[(100 - X)^+] f_D(x)dx \\
&= \int_{-\infty}^{50} \mathbb{E}[(100 - X)^+] f_D(x)dx + \int_{50}^{150} \mathbb{E}[(100 - X)^+] f_D(x)dx + \int_{150}^{\infty} \mathbb{E}[(100 - X)^+] f_D(x)dx \\
&= 0 + \int_{50}^{150} \mathbb{E}[(100 - X)^+] f_D(x)dx + 0 \\
&= \int_{50}^{150} \mathbb{E}[(100 - X)^+] f_D(x)dx \\
&= \int_{50}^{100} \mathbb{E}[(100 - X)^+] f_D(x)dx + \int_{100}^{150} \mathbb{E}[(100 - X)^+] f_D(x)dx \\
&= \int_{50}^{100} (100 - X)(\frac{1}{100})dx + \int_{100}^{150} (0)\frac{1}{100} = 12.5
\end{aligned}$$

(3)

$\mathbb{E}[100] = 100$ **Thus, Expected Profit is** $10(87.5) + 2(12.5) - 5(100) = 400$

Exercise 16

A company is obligated to provide warranty service for Product A to its customers next year. The warranty demand for the product follows the following distribution.

d	100	200	300	400
$Pr(D = d)$	.2	.4	.3	.1

The company needs to make one production run to satisfy the warranty demand for entire next year. Each unit costs \$100 to produce; the penalty cost of a unit is \$500. By the end of the year, the salvage value of each unit is \$50.

(a) **Suppose that the company has currently 0 units. What is the optimal quantity to produce in order to minimize the expected total cost? Find the optimal expected total cost.**

(b) **Suppose that the company has currently 100 units at no cost and there is \$20000 fixed cost to start the production run. What is the optimal quantity to produce in order to minimize the expected total cost? Find the optimal expected total cost.**

(a)

- C_o**=(Material Cost + Salvage Cost, overstock Cost)=(100+50)=150**
- C_u**=(penalty Cost + Material Cost, understock Cost)=(500+100)=600**

Unit have discrete properties that can count. $\therefore F(y) \geq \frac{c_u}{c_o+c_u}$

$F(y) \geq \frac{600}{150+600} = 0.8$

From problem's table,

$F_D(100) = 0.2 < 0.8$

$F_D(200) = 0.2 + 0.4 < 0.8$

$F_D(300) = 0.2 + 0.4 + 0.3 \geq 0.8$

Thus, x^***=300**

Expected Economic Cost = E**[Cost associated with understock Risk] +** E**[Cost associated with overstock Risk]**

$= E[cost] = (600 \times E[(D - X)^+] + 150 \times E[(X - D)^+])$

$= (\sum_{D=100}^{400}(600 \times (D \wedge 300) \times P(D)) + \sum_{D=100}^{400}(150 \times (D - 300)^+ \times P(D)))$

$= 600 \times (0.2 \times 100 + 0.4 \times 200 + 0.3 \times 300 + 0.1 \times 300) + 150 \times (0.2 \times 200 + 0.4 \times 100 + 0.3 \times 0)$

$= 126,000$

$\therefore$ **Expected Economic Cost = 126,000\$**

(b)

- C_o**=(Material Cost + Salvage Cost, Overstock Cost)=(100+50)=150**
- C_u**=(Penalty Cost + Material Cost, Understock Cost)=(500+100)=600**

Unit have discrete properties that can count. $\therefore F(y) \geq \frac{c_u}{c_o+c_u}$

$F(y) \geq \frac{600}{150+600} = 0.8$

From problem's table,

$F_D(100) = 0.2 < 0.8$

$F_D(200) = 0.2 + 0.4 < 0.8$

$F_D(300) = 0.2 + 0.4 + 0.3 \geq 0.8$

Thus, x^***=300**

Expected Economic Cost = E**[Cost associated with understock Risk] +** E**[Cost associated with overstock Risk] +** E**[Fixed cost]**

$= E[cost] = (600 \times E[(D-X)^+] + 150 \times E[(X-D)^+] + 20000 \times (D-100) \vee 1)$

$= \sum_{D=100}^{400}(600 \times (D \wedge 300) \times P(D)) + \sum_{D=100}^{400}(150 \times (D-300)^+ \times P(D)) + \sum_{D=100}^{400}(20000 \times (D-100) \vee 1)$

$= 600 * (0 \cdot P(100) + \sum_{D=200}^{300}(D \cdot P(D)) + 300 \cdot P(400)) + 150 * \sum_{D=200}^{300}(14-D) \cdot P(D) - 150$

$+20000 * \sum_{D=200}^{400} P(D)$

$= 600 \times (0.4 \times 200 + 0.3 \times 300 + 0.1 \times 300) + 150 \times (0.2 \times 200 + 0.4 \times 100 + 0.3 \times 0) + 20000 * 0.9$

$= 150000$