B1 - Exercise

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Contents

Exercise 1

Assume that D follows the following discrete distribution.

d	20	25	30	35
$P[D=d]$	0.1	0.2	0.4	0.3
$30 \wedge d$	20	25	30	30
$(30-d)^+$	10	5	0	0
$24 \wedge d$	20	24	24	24
$(24-d)^+$	4	0	0	0

Answer the followings.

- $E[30 \wedge D] = \sum((30 \wedge D) \cdot P(D)) = 20 \times 0.1 + 25 \times 0.2 + 30 \times 0.4 + 30 \times 0.3 = 28$
- $E[(30-D)^+] = \sum((30-D)^+ \cdot P(D)) = 10 \times 0.1 + 5 \times 0.2 + 0 \times 0.4 + 0 \times 0.3 = 2$
- $E[24 \wedge D] = \sum((24 \wedge D) \cdot P(D)) = 20 \times 0.1 + 24 \times 0.2 + 24 \times 0.4 + 24 \times 0.3 = 23.6$
- $E[(24-D)^+] = \sum((24-D)^+ \cdot P(D)) = 4 \times 0.1 + 0 \times 0.2 + 0 \times 0.4 + 0 \times 0.3 = 0.4$

Exercise 2

Find your brother's optimal stock level by the above Theorem 2. Then, find his expected profit using the Remark 1.

Theorem2

- If D is a continuous r.v, with cdf $F(\cdot)$, then find y s.t. $F(y) = \frac{c_u}{c_o+c_u}$.
- If D is a discrete r.v, with cdf $F(\cdot)$, then find smallest y such that $F(y) \geq \frac{c_u}{c_o+c_u}$.

Remark 1

- $E[Profit]$=E(Sale Rev.)+E(salvage Rev.)-E(material Cost)

Answer

1) Finding optimal stock level

C_o=(Material Cost - Salvage Price)=$(1-\frac{1}{2}) = \frac{1}{2}$
C_u=(Retail Price - Material Cost)=(2-1)=1

$F(y) \geq \frac{1}{1+1/2}$
$F(y) \geq \frac{2}{3}$

d	11	12	13	14	15
$P(D=d)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$F_D(11) = \frac{1}{5} < \frac{2}{3}$

$F_D(12) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} < \frac{2}{3}$

$F_D(13) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5} < \frac{2}{3}$

$F_D(14) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5} \geq \frac{2}{3}$

Thus, y^*=14

2) Finding expected profit

Sale Revenue=$2 \cdot (D \wedge Y)$

Salvage Revenue=$\frac{1}{2} \cdot (Y-D)^+$

Material Cost=Y

$E[Profit]$=E(Sale Rev.)+E(salvage Rev.)-E(material Cost)

$$=\sum_{D=11}^{15}(2 \cdot (D \wedge Y) \cdot P(D)) + \sum_{D=11}^{15}(\frac{1}{2} \cdot (D-Y)^+ \cdot P(D)) - \sum_{D=11}^{15}(Y \cdot P(D))$$

Then, $y^* = 14$

$$E[Profit] = \sum_{D=11}^{15}(2 \cdot (D \wedge 14) \cdot P(D)) + \sum_{D=11}^{15}(\tfrac{1}{2} \cdot (D-14)^{+} \cdot P(D)) - \sum_{D=11}^{15}(14 \cdot P(D))$$
$$= 2 \cdot (\sum_{D=11}^{14}(D \cdot P(D)) + 14 \cdot P(15)) + \tfrac{1}{2} \cdot \sum_{D=11}^{14}((14-D) \cdot P(D)) - 14$$
$$= 2 \cdot (\tfrac{11+12+13+14}{5} + \tfrac{14}{5}) + \tfrac{1}{2} \cdot (\tfrac{3+2+1+0}{5}) - 14$$
$$= \tfrac{61}{5}$$
$$= 12.2$$

Thus, $E[Profit] = 12.2$

Exercise 3

Your brother is now selling milk. The customer demands follow $U(20, 40)$ gallons. Retail price is \$2 per gallon, materia cost is \$1 per gallon, and salvage cost is \$0.5 per gallon. Find optimal stock level and expected profit.

Theorem

- If D is a continuous r.v, with cdf $F(\cdot)$, then find y s.t. $F(y) = \frac{c_u}{c_o+c_u}$.
- If D is a discrete r.v, with cdf $F(\cdot)$, then find smallest y such that $F(y) \geq \frac{c_u}{c_o+c_u}$.

Answer

1) Finding optimal stock level

$D \sim U(20, 40)$

$$f_d(x) = \begin{cases} \frac{1}{20} & 20 \leq x \leq 40 \\ 0 & otherwise \end{cases}$$

$$F_d(x) = \begin{cases} 0 & 20 < x \\ \frac{x-20}{20} & 20 \leq x \leq 40 \\ 1 & x > 40 \end{cases}$$

$F(Y) = \frac{C_u}{C_o+C_u}$

C_o=(Material Cost - Salvage Price)=$(1 - \frac{1}{2}) = \frac{1}{2}$
C_u=(Retail Price - Material Cost)=(2-1)=1

$F(Y) = \frac{1}{1+1/2}$

$F(Y) = \frac{2}{3}$

$\frac{Y-20}{20} = \frac{2}{3}$

Thus, $Y^* = \frac{100}{3}$

2) Finding expected profit

Sale Revenue=$2 \cdot (D \wedge Y)$

Salvage Revenue=$\frac{1}{2} \cdot (Y-D)^+$

Material Cost=Y

$E[Profit]$=E(Sale Rev.)+E(salvage Rev.)-E(material Cost)

$=E[2 \cdot (D \wedge Y)] + E[\frac{1}{2} \cdot (Y-D)^+] - E[Y]$

Then, $y^* = \frac{100}{3}$

$$E[Profit] = \int_{20}^{40}(2 \cdot (D \wedge \tfrac{100}{3}) \cdot \tfrac{1}{20})d_D + \int_{20}^{40}(\tfrac{1}{2} \cdot (\tfrac{100}{3} - D)^+ \cdot \tfrac{1}{20})d_D - \int_{20}^{40}(\tfrac{100}{30} \cdot \tfrac{1}{20})d_D$$

$$=\tfrac{1}{10} \cdot \int_{20}^{40}(D \wedge \tfrac{100}{3})d_D + \tfrac{1}{40} \cdot \int_{20}^{40}(\tfrac{100}{3} - D)^+ d_D - \tfrac{3}{5} \cdot [D]_{20}^{40}$$

$$=\tfrac{1}{10} \cdot (\int_{20}^{\frac{100}{30}}(D)d_D + \int_{\frac{100}{3}}^{40}(\tfrac{100}{3})d_D) + \tfrac{1}{40} \cdot \int_{20}^{\frac{100}{3}}(\tfrac{100}{3} - D)d_D - \tfrac{100}{3}$$

$$=\tfrac{1}{10} \cdot ([\tfrac{1}{2}D^2]_{20}^{\frac{100}{3}} + \tfrac{100}{3}[D]_{\frac{100}{3}}^{40}) + \tfrac{1}{40} \cdot [\tfrac{100}{3}D - \tfrac{1}{2}D^2]_{20}^{\frac{100}{3}} - \tfrac{100}{3}$$

$$=\tfrac{80}{3}$$

Thus, $E[Profit] = \frac{80}{3}$

Exercise 4

Lemonade sells for \$18 per gallon but only costs \$3 per gallon to mak. If we run out of lemonade, it will be impossible to get more. On the other hand, leftover lemonade has a value of \$1. Express the following quantity using sale as X and demand as D.

- C_u
- C_o
- Expected economic cost
- Expected Profit

Answer

C_u=(Retail Price - Material Cost)=(18-3)=15

C_o=(Material Cost - Salvage Price)=(3-1)=2

$$\begin{aligned}\text{Expected Economic Cost} &= \text{E[Manufacturing Cost]} + \text{E[Cost associated with understock Risk]} \\ &\quad + \text{E[Cost associated with overstock Risk]} \\ &= C_v \cdot E[X] + C_u \cdot E[(D-X)^+] + C_o \cdot E[(X-D)^+] \\ &= 3E[X] + 15E[(D-X)^+] + 2E[(X-D)^+]\end{aligned}$$

$$\begin{aligned}\text{Expected Profit} &= \text{E[Revenue] - E[Cost]} \\ &= 18E[(X \wedge D)] - (3E[X] + 15E[(D-X)^+] + 2E[(X-D)^+])\end{aligned}$$

Thus,

- $C_u = 15$
- $C_o = 2$
- $E[cost] = 3X + 15E[(D-X)^+] + 2E[(X-D)^+]$
- $E[profit] = 18E[(X \wedge D)] - (3E[X] + 15E[(D-X)^+] + 2E[(X-D)^+])$

Exercise 5

Prove Theorem 1.

Theorem 1.

- Maximizing the expected profit is equivalent to minimizing the expected economic cost

Answer

Set

- Selling Price : p
- Buying Price : c
- Salvage Value : s
- Holding Cost : h
- Market Demand : D
- Order Quantity : X

$$
\begin{aligned}
\text{Expected Economic Cost} &= \text{E[Manufacturing Cost]} + \text{E[Cost associated with understock Risk]} \\
&\quad + \text{E[Cost associated with overstock Risk]} \\
&= c \cdot E[X] + (p - c) \cdot E[(D - X)^+] + (c - s) \cdot E[(X - D)^+]
\end{aligned}
$$

$$
\begin{aligned}
\text{Expected Profit} &= \text{E[Revenue] - E[Cost]} \\
&= \text{E[Revenue] - E[Ordering Cost] - E[Holding Cost] - E[Backorder Cost]} \\
&= p \cdot E[(X \wedge D)] - c \cdot E[X] - h \cdot E[(X - D)^+] - (p - c) \cdot E[(D - X)^+] \\
&= p \cdot E[(X \wedge D)] - (c \cdot E[X] + h \cdot E[(X - D)^+] + (p - c) \cdot E[(D - X)^+]) \\
&= p \cdot E[(X \wedge D)] - (c \cdot E[X] + (c - s) \cdot E[(X - D)^+] + (p - c) \cdot E[(D - X)^+]) \\
&= p \cdot E[(X \wedge D)]\text{-E[Cost]}
\end{aligned}
$$

Thus,

$E[profit] = \alpha - E[cost]$, $(\alpha \geq 0)$.

DaiPark Exercises

1. Show that $(D \wedge Y) + (Y - D)^+ = Y$

i) $D > Y$

$$(D \wedge Y) + (Y - D)^+ = Y + 0 = Y$$

ii) $Y > D$

$$(D \wedge Y) + (Y - D)^+ = D + (Y - D) = Y$$

2. Let D be a discrete a random variable with the following pmf.

d	5	6	7	8	9
$P(D = d)$	0.1	0.3	0.4	0.1	0.1

(a)

$$\begin{aligned}
E[min(D,7)] &= E[min(5,7)] + E[min(6,7)] + E[min(7,7)] + E[min(8,7)] + E[min(9,7)] \\
&= E[5] + E[6] + E[7] + E[7] + E[7] \\
&= 5 \cdot \frac{1}{10} + 6 \cdot \frac{3}{10} + 7 \cdot \frac{4}{10} + 7 \cdot \frac{1}{10} + 7 \cdot \frac{1}{10} \\
&= \frac{5 + 18 + 28 + 7 + 7}{10} \\
&= \frac{65}{10} \\
&= 6.5
\end{aligned}$$

(b)

$$\begin{aligned}
E[(7 - D)^+] &= E[(7-5)^+] + E[(7-6)^+] + E[(7-7)^+] + E[(7-8)^+] + E[(7-9)^+] \\
&= E[2] + E[1] + E[0] + E[0] + E[0] \\
&= 2 \cdot \frac{1}{10} + 1 \cdot \frac{3}{10} + 0 \cdot \frac{4}{10} + 0 \cdot \frac{1}{10} + 0 \cdot \frac{1}{10} \\
&= \frac{2 + 3}{10} \\
&= \frac{5}{10} \\
&= 0.5
\end{aligned}$$

3. Let D be a Poisson random variable with parameter 3. $\setminus D \sim Poi(3)$

$P(D = k) = \frac{3^k e^{-3}}{k!}$ for $k = 0, 1, 2, ...$
$E(X) = Var(X) = 3$

(a)

$$\begin{aligned}
E[min(D,2)] &= \sum_{x=-\infty}^{\infty} min(D,2)p(x) \\
&= \sum_{x=-\infty}^{0} min(D,2)p(x) + \sum_{x=0}^{\infty} min(D,2)p(x) \\
&= 0 + \sum_{x=0}^{\infty} min(D,2)p(x) \\
&= 0 + 3 \\
&= 3
\end{aligned}$$

(b)

$$\begin{aligned}
E[(3-D)^+] &= \sum_{D=-\infty}^{\infty} (3-D)^+ p(D) \\
&= \sum_{D=-\infty}^{0} (3-D)^+(D) + \sum_{D=0}^{\infty} (3-D)^+ p(D) \\
&= 0 + \sum_{D=0}^{\infty} (3-D)^+ p(D) \\
&= \sum_{D=0}^{3} (3-D)^+ p(D) + \sum_{D=4}^{\infty} (3-D)^+ p(D) \\
&= \sum_{D=0}^{3} (3-D) p(D) + 0 \\
&= 3
\end{aligned}$$

4. Let D be a continuous random variable and uniformly distributed between 5 and 10. \
$D \sim U(5, 10)$

$$f_d(x) = \begin{cases} \frac{1}{5} & 5 \leq x \leq 10 \\ 0 & otherwise \end{cases}$$

$$F_d(x) = \begin{cases} 0 & 5 < x \\ \frac{x-5}{5} & 5 \leq x \leq 10 \\ 1 & x > 10 \end{cases}$$

(a)

$$\begin{aligned}
E[max(D, 8)] &= \int_{-\infty}^{\infty} max(D, 8) f_D X d_x \\
&= \int_{-\infty}^{0} max(D, 8) f_D X d_x + \int_{0}^{\infty} max(D, 8) f_D X d_x \\
&= 0 + \int_{0}^{\infty} max(D, 8) f_D X d_x \\
&= \int_{0}^{5} max(D, 8) f_D X d_x + \int_{5}^{10} max(D, 8) f_D X d_x + \int_{10}^{\infty} max(D, 8) f_D X d_x \\
&= 0 + \int_{5}^{10} max(D, 8) \frac{1}{5} d_x + 0 \\
&= \int_{5}^{8} max(D, 8) \frac{1}{5} d_x + \int_{8}^{10} max(D, 8) \frac{1}{5} d_x \\
&= \int_{5}^{8} 8 \frac{1}{5} d_x + \int_{8}^{10} D \frac{1}{5} d_x \\
&= \frac{8}{5} [x]_5^8 + \frac{1}{5} [\frac{1}{2} D^2]_8^{10} \\
&= \frac{24}{5} + \frac{18}{5} \\
&= \frac{42}{5}
\end{aligned}$$

(b)

$$
\begin{aligned}
E[(D-8)^-] &= \int_{-\infty}^{\infty} (D-8)^- f_D(x) d_x \\
&= \int_{-\infty}^{0} (D-8)^- f_D(x) d_x + \int_{0}^{\infty} (D-8)^- f_D(x) d_x \\
&= 0 + \int_{0}^{\infty} (D-8)^- f_D(x) d_x \\
&= \int_{0}^{5} (D-8)^- f_D(x) d_x + \int_{5}^{10} (D-8)^- f_D(x) d_(x) + \int_{10}^{\infty} (D-8)^- f_D(x) d_x \\
&= 0 + \int_{5}^{10} (D-8)^- \frac{1}{5} d_x + 0 \\
&= \int_{5}^{8} (D-8)^- \frac{1}{5} d_x + \int_{8}^{10} (D-8)^- \frac{1}{5} d_x \\
&= \int_{5}^{8} (D-8) \frac{1}{5} d_x + 0 \\
&= \frac{1}{5}[\frac{1}{2}D^2 - 8D]_5^8 \\
&= -\frac{9}{2}
\end{aligned}
$$

5. Let D be an exponential random variable with parameter 7. $\setminus D \sim exp(7)$

$$f_d(x) = \begin{cases} 7e^{-7x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$$F_d(x) = \begin{cases} 1 - e^{-7x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

(a)

$$\begin{aligned}
E[max(D,3)] &= \int_{-\infty}^{\infty} max(D,3) f_D(x) d_x \\
&= \int_{-\infty}^{0} max(D,3) f_D(x) d_x + \int_{0}^{\infty} max(D,3) f_D(x) d_x \\
&= 0 + \int_{0}^{\infty} max(D,3) f_D(x) d_x \\
&= \int_{0}^{3} max(D,3) f_D(x) d_x + \int_{3}^{\infty} max(D,3) f_D(x) d_x \\
&= \int_{0}^{3} 3 \cdot 7e^{-7x} d_x + \int_{3}^{\infty} x \cdot 7e^{-7x} d_x \\
&= 21 \int_{0}^{3} e^{-7x} d_x + 7 \int_{3}^{\infty} x \cdot e^{-7x} d_x \\
&= 3 - 3e^{-21} - 3e^{-21} + \frac{1}{7} e^{-21} \\
&= 3 - \frac{41}{7} e^{-21}
\end{aligned}$$

(b)

$$\begin{aligned}
E[(D-4)^-] &= \int_{-\infty}^{\infty} (D-4)^- f_D(x) d_x \\
&= \int_{-\infty}^{0} (D-4)^- f_D(x) d_x + \int_{0}^{\infty} (D-4)^- f_D(x) d_x \\
&= 0 + \int_{0}^{\infty} (D-4)^- f_D(x) d_x \\
&= \int_{0}^{4} (D-4)^- f_D(x) d_x + \int_{4}^{\infty} (D-4)^- f_D(x) d_x \\
&= 0 + \int_{4}^{\infty} (D-4) f_D(x) d_x \\
&= \int_{4}^{\infty} (x-4) 7 \cdot e^{-7x} d_x \\
&= \frac{1}{7} e^{-28}
\end{aligned}$$

6. David buys fruits and vegetables wholesale and retails them at Davids Produce on La vista Road. One of the more difficult decisions is the amount of bananas to by. Let us make some simplifying assumptions, and assume that David purchases bananas once a week at 10 cents per pounds and retails them at 30 cents per pound during the week. Bananas that are more than a week old are too ripe and are sold for 5 cents per pound. \

(a) Suppose the demand for the good bananas follows the same distribution as D given in Problem 2. What is the expected profit of David in a week if he buys 7 pounds of banana?

- Selling Price : 30
- Buying Price : 10
- Salvage Value : 5
- C_u=Selling Price - Buying Price = 30 - 10 = 20
- C_o=Buying Price - Salvage Value = 10 - 5 = 5

Thus,

$$
\begin{aligned}
F(y) &>= \frac{C_u}{C_o + C_u} \\
&>= \frac{20}{20+5} \\
&>= \frac{4}{5}
\end{aligned}
$$

$F_D(5) = \frac{1}{10} < \frac{1}{10}$
$F_D(6) = \frac{1}{10} + \frac{3}{10} = \frac{2}{5} < \frac{2}{5}$
$F_D(7) = \frac{1}{10} + \frac{3}{10} + \frac{4}{10} = \frac{4}{5} \geq \frac{4}{5}$

Thus, $Y^* = 7$

$$
\begin{aligned}
E[profit] &= E(salesrev.) + E(salvagerev.) - E(materialcost) \\
&= E[30(Y \wedge D)] + E[5(Y-D)^+] - E[10Y] \\
&= 30\sum_{D=5}^{9}(7 \wedge D)p(D) + 5\sum_{D=5}^{9}(7-D)^+p(D) - 10\sum_{D=5}^{9}7p(D) \\
&= 30\sum_{D=5}^{9}(7 \wedge 7)p(D) + 5\sum_{D=5}^{9}(7-7)^+p(D) - 10\sum_{D=5}^{9}7p(D) \\
&= 30\sum D = 5^9 7 \cdot p(D) - 10\sum D = 5^9 7 \cdot p(D) \\
&= 30 \times 7\frac{1+3+4+1+1}{10} - 10 \times 7\frac{1+3+4+1+1}{10} \\
&= 210 - 70 \\
&= 140
\end{aligned}
$$

(b) Now assume that the demand for the good bananas is uniformly distributed between 5 and 10 like in Problem 4. What is the expected profit of David in a week if he buys 7 pounds of banana?

$F_D(x) = \begin{cases} 0 & 5 < x \\ \frac{x-5}{5} & 5 \leq x \leq 10 \\ 1 & x > 10 \end{cases}$

$F_D(Y) = \frac{Y-5}{5} = \frac{4}{5}$

Thus, $Y^* = 9$

$$\begin{aligned} E[profit] &= E(salesrev.) + E(salvagerev.) - E(materialcost) \\ &= E[30(Y \wedge D)] + E[5(Y - D)^+] - E[10Y] \\ &= E[30(7 \wedge 9)] + E[5(9 - 7)^+] - E[10(9)] \\ &= 30E[7] + 5E[2] - 10E[9] \\ &= 30 \int_5 107 \cdot \frac{1}{5} d_x + 5 \int_5 102 \cdot \frac{1}{5} d_x - 10 \int_5 109 \cdot \frac{1}{5} d_x \\ &= 210 + 10 - 90 \\ &= 130 \end{aligned}$$

(c) Find the expected profit if David's demand for the good bananas follows an exponential distribution with mean 7 and if the buys 7 pounds of banana.

$D \sim exp(\frac{1}{7})$

$f_D(x) = \begin{cases} \frac{1}{7} e^{-\frac{1}{7}x} & x \geq 0 \\ 0 & otherwise \end{cases}$

$F_D(x) = \begin{cases} 1 - e^{-\frac{1}{7}x} & x \geq 0 \\ 0 & otherwise \end{cases}$

$F_D(Y) = 1 - e^{-\frac{1}{7}Y} = \frac{4}{5}$

Thus, $Y^+ = 7ln5$

$$
\begin{aligned}
E[profit] &= E(salesrev.) + E(salvagerev.) - E(materialcost) \\
&= E[30(Y \wedge D)] + E[5(Y-D)^+] - E[10Y] \\
&= E[30(7ln5 \wedge 7)] + E[5(7ln5-7)^+] - E[10(7ln5)] \\
&= 30E[7] + 5E[7ln5-7] - 10E[7ln5] \\
&= 30\int_0^\infty 7 \cdot \frac{1}{7}e^{-\frac{1}{7}x}d_x + 5\int_0^\infty (7ln5-7)\cdot\frac{1}{7}e^{-\frac{1}{7}x}d_x - 10\int_0^\infty 7ln5\cdot\frac{1}{7}e^{-\frac{1}{7}x}d_x \\
&= 210 + 35ln5 - 35 - 70ln5 \\
&= 175 - 35ln5
\end{aligned}
$$