A2_solution

Reinforcement Learning Study

2021-01-02

Contents

Exercise

Exercise 1

Show that $\mathbb{P}(A \mid B \cap C)\mathbb{P}(B \mid C) = \mathbb{P}(A \cap B \mid C)$

$$
\begin{aligned}
\mathbb{P}(A \mid B \cap C)\mathbb{P}(B \mid C) &= \tfrac{\mathbb{P}(A\cap(B\cap C))}{\mathbb{P}(B\cap C)} \times \tfrac{\mathbb{P}(B\cap C)}{\mathbb{P}(C)} \\
&= \tfrac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(C)} \\
&= \tfrac{\mathbb{P}((A\cap B)\cap C)}{\mathbb{P}(C)} \\
&= \mathbb{P}(A \cap B \mid C)
\end{aligned}
$$

$\therefore\ \mathbb{P}(A \mid B \cap C)\mathbb{P}(B \mid C) = \mathbb{P}(A \cap B \mid C)$

Exercise 2

X~U(10,20), then what is F(10)? and F(15)?

X~U(10,20)

$$f(x) = \begin{Bmatrix} \frac{1}{10} & (10 \leq x \leq 20) \\ 0 & otherwise \end{Bmatrix}$$

$$F(x) = \begin{Bmatrix} 0 & (10 \leq x) \\ \frac{x-10}{10} & (10 \leq x \leq 20) \\ 1 & (x > 20) \end{Bmatrix}$$

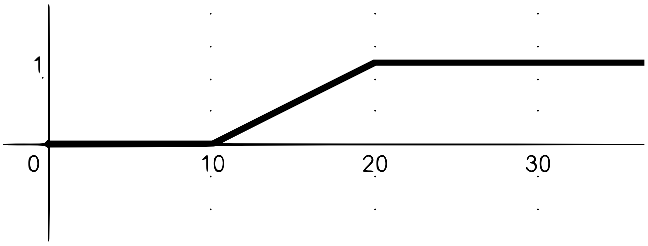

Figure 1: F(x) graph

$F(10) = 0$

$F(15) = \frac{15-10}{10}$
$= \frac{1}{2}$

$\therefore\ F(10) = 0,\ F(15) = \frac{1}{2}$

Exercise 3

Prove that pdf→cdf

$F(x) = \int_{-\infty}^{x} f(x)\ dx$

i) $if\ \ x < 0,\ \ then\ \ f(x) = 0,\ \ so\ \ F(x) = 0$

ii) $if\ \ x \geq 0,\ \ then\ \ F(x) = \int_{-\infty}^{x} f(x)\ dx$

$$
\begin{aligned}
&= \int_{-\infty}^{0} 0\ dx + \int_{0}^{x} \lambda e^{-\lambda x}\ dx \\
&= [-e^{\lambda x}]_{0}^{x} \\
&= -e^{-\lambda x} + 1
\end{aligned}
$$

$\therefore\ F(x) = 1 - e^{-\lambda x}$

Exercise 4

Show that EX=1/λ

$$
\begin{aligned}
\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)\ dx = \int_{-\infty}^{0} xf(x)\ dx + \int_{0}^{\infty} xf(x)\ dx \\
&= \int_0^\infty x\lambda e^{-\lambda x}\ dx \\
&= \int_0^\infty \lambda x e^{-\lambda x}\ dx \\
&= [xe^{-\lambda x}]_0^\infty - \int_0^\infty -e^{-\lambda x}\ dx \\
&= (0-0) + \int_0^\infty e^{-\lambda x}\ dx \\
&= [-\tfrac{e^{-\lambda x}}{\lambda}]_0^\infty \\
&= (0-(-\tfrac{1}{\lambda})) \\
&= \tfrac{1}{\lambda}
\end{aligned}
$$

$\therefore \mathbb{E}X = \frac{1}{\lambda}$

Exercise 5

Show that Var(X)=1/λ^2.(Hint: need to do EX^2 first)

$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

$$
\begin{aligned}
\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\ dx &= \int_{-\infty}^{0} x^2 f(x)\ dx + \int_{0}^{\infty} x^2 f(x)\ dx \\
&= \int_{0}^{\infty} x^2 \lambda e^{-\lambda x}\ dx \\
&= \int_{0}^{\infty} \lambda x^2 e^{-\lambda x}\ dx \\
&= [-x^2 e^{-\lambda x}]_0^{\infty} - \int_{0}^{\infty} -2x e^{-\lambda x}\ dx \\
&= (0-0) + \int_{0}^{\infty} 2x e^{-\lambda x}\ dx \\
&= 2\int_{0}^{\infty} x e^{-\lambda x}\ dx \\
&= 2\{[-\tfrac{x e^{-\lambda x}}{\lambda}]_0^{\infty} - \int_{0}^{\infty} \tfrac{e^{-\lambda x}}{\lambda}\ dx\} \\
&= 2\{(0-0) - \int_{0}^{\infty} \tfrac{e^{-\lambda x}}{\lambda}\ dx\} \\
&= 2(-[\tfrac{e^{-\lambda x}}{\lambda^2}]_0^{\infty}) \\
&= 2 \times (0 - (-\tfrac{1}{\lambda^2})) \\
&= \tfrac{2}{\lambda^2}
\end{aligned}
$$

$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \frac{1}{\lambda^2}$

$\therefore Var(X) = \frac{1}{\lambda^2}$

Exercise 6

Prove the previous theorem

Claim, $\mathbb{P}(X > s + t \mid X > t) = \mathbb{P}(X > s)$

$$\begin{aligned}
\mathbb{P}(X > s + t \mid X > t) &= \tfrac{\mathbb{P}(X>s+t, X>t)}{\mathbb{P}(X>t)} \\
&= \tfrac{\mathbb{P}(X>s+t)}{\mathbb{P}(X>t)} \\
&= \tfrac{1-\mathbb{P}(X\leq s+t)}{1-\mathbb{P}(X\leq t)} \\
&= \tfrac{1-F(s+t)}{1-F(t)} \\
&= \tfrac{1-(1-e^{-\lambda(s+t)})}{1-(1-e^{-\lambda(t)})} \\
&= e^{-\lambda s} \\
&= 1 - (1 - e^{-\lambda s}) = 1 - F(s) \\
&= 1 - \mathbb{P}(X \leq s) \\
&= \mathbb{P}(X > s)
\end{aligned}$$

$\therefore \mathbb{P}(X > s + t \mid X > t) = \mathbb{P}(X > s)$

Exercise 7

For $X \sim poi(\lambda)$, prove that $\mathbb{E}X = \lambda$.

$\mathbb{E}X = \sum_{x=-\infty}^{\infty} xp(x)$ *(this is common for all discrete r.v.)*

$= \sum_{x=-\infty}^{\infty} \frac{x \times \lambda^x e^{-\lambda}}{x!}$

$= \lambda \sum_{x=-\infty}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!}$ $\quad$ *Let,* $k = x - 1 \quad (x : 1 -> \infty, \quad k : 0 -> \infty)$

$= \lambda \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$

$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$ $\quad$ $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$

$= \lambda e^{-\lambda} \times e^{\lambda}$

$= \lambda$

$\therefore \mathbb{E}X = \lambda$

Exercise 8

For X~U(20,40), evaluate$\mathbb{E}[X \wedge 25]$*and*$\mathbb{E}[(25-X)^{+}]$.

$$f(x) = \begin{Bmatrix} \frac{1}{20} \ (20 \leq x \leq 40) \\ \\ 0 \qquad otherwise \end{Bmatrix}$$

1. $\mathbb{E}[X \wedge 25] = \int_{-\infty}^{\infty} (X \wedge 25) \frac{1}{20} \ dx$

$$= \int_{20}^{25} \frac{1}{20} x \ dx + \int_{25}^{40} 25 \times \frac{1}{20} \ dx$$
$$= [\tfrac{1}{40} x^2]_{20}^{25} + [\tfrac{5}{4} x]_{25}^{40}$$
$$= \tfrac{225}{40} + \tfrac{75}{4}$$
$$= \tfrac{195}{8}$$

2. $\mathbb{E}[(25-X)^{+}] = \int_{-\infty}^{\infty} (25-X)^{+} \frac{1}{20} \ dx$

$$= \int_{20}^{25} (25-X) \frac{1}{20} \ dx + \int_{25}^{40} 0 \times \frac{1}{20} \ dx$$
$$= [\tfrac{5}{4} x - \tfrac{1}{40} x^2]_{20}^{25} + 0$$
$$= \tfrac{25}{4} - \tfrac{225}{40}$$
$$= \tfrac{5}{8}$$

$\therefore \ \mathbb{E}[x \wedge 25] = \frac{195}{8}, \ \ \mathbb{E}[(25-X)^{+}] = \frac{5}{8}$

Exercise 9

For X~Poi(8)

1. $\mathbb{P}(X = 0)$
2. $\mathbb{P}(2 \leq X \leq 4)$
3. $\mathbb{P}(X > 2)$

$* \; \mathbb{P}(X) = \frac{8^x e^{-8}}{x!} \; *$

1. $\mathbb{P}(X = 0) = \frac{8^0 e^{-8}}{0!}$

   $= e^{-8}$

2. $\mathbb{P}(2 \leq X \leq 4) = \mathbb{P}(X \leq 4) - \mathbb{P}(X < 2)$

   $= \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4)$
   $= \frac{8^2 e^{-8}}{2!} + \frac{8^3 e^{-8}}{3!} + \frac{8^4 e^{-8}}{4!}$
   $= 288e^{-8}$

3. $\mathbb{P}(X > 2) = 1 - (\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2))$

   $= 1 - (\frac{8^0 e^{-8}}{0!} + \frac{8^1 e^{-8}}{1!} + \frac{8^2 e^{-8}}{2!})$
   $= 1 - 41e^{-8}$

$$\therefore Answer : \left\{ \begin{array}{c} \mathbb{P}(X = 0) = e^{-8} \\ \\ \mathbb{P}(2 \leq X \leq 4) = 288e^{-8} \\ \\ \mathbb{P}(X > 2) = 1 - 41e^{-8} \end{array} \right\}$$

Exercise 10

For X~exp(7), evaluate$\mathbb{E}[max(X,7)]$.

$* \ f(x) = 7e^{-7x}$

$$
\begin{aligned}
\mathbb{E}[max(X,7)] &= \int_{-\infty}^{\infty} xf(x)\ dx \\
&= \int_{-\infty}^{7} 7 \times 7e^{-7x}\ dx + \int_{7}^{\infty} x \times 7e^{-7x}\ dx \\
&= \int_{0}^{7} 49e^{-7x}\ dx + \int_{7}^{\infty} 7xe^{-7x}\ dx \\
&= [-7e^{-7x}]_0^7 + [-xe^{-7x}]_7^{\infty} - \int_{7}^{\infty} -e^{-7x}\ dx \\
&= (-7e^{-49} - (-7)) + (0 - (-7e^{-49})) + \int_{7}^{\infty} e^{-7x}\ dx \\
&= 7 + [-\tfrac{e^{-7x}}{7}]_7^{\infty} \\
&= 7 + (0 - (-\tfrac{e^{-49}}{7})) \\
&= 7 + \tfrac{e^{-49}}{7}
\end{aligned}
$$

$\therefore \ \mathbb{E}[max(X,7)] = 7 + \frac{e^{-49}}{7}$

Exercise 11

For X~exp(8),findx^* *such that* $F(x^*) = 0.6$.

$$f(x) = \begin{cases} 8e^{-8x} & (x \leq 0) \\ 0 & otherwise \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-8x} & (x \leq 0) \\ 0 & otherwise \end{cases}$$

$F(x^*) = 1 - e^{-8x} = 0.6$

$e^{-8x^*} = 0.4$

$\frac{1}{e^{8x^*}} = \frac{2}{5}$

$e^{8x^*} = \frac{5}{2}$

Taking the ln(x) on both side

$8x^* = ln(\frac{5}{2}) = ln5 - ln2$

$\therefore x^* = \frac{ln5 - ln2}{8}$

Exercise 12

For X~U(10,20),find x^* *such that* $F(x^*) = 0.7$.

$$F(x) = \begin{Bmatrix} 0 & (10 \leq x) \\ \frac{x-10}{10} & (10 \leq x \leq 20) \\ 1 & (x > 20) \end{Bmatrix}$$

$*F(x^*) = 0.7$

For getting the value 0.7,x^**should be between 10 and 20, Because the other ranges have value only 0 or 1*

$F(x^*) = \frac{x^*-10}{10} = 0.7$

$x^* - 10 = 7$

$\therefore x^* = 17$