Lecture A2. Probability Review

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I. Probability

Probability

Definition 1

Probability is a function from "Event" to real number between 0 and 1.

$$\mathbb{P} : S \to [0, 1],$$

where S is the set of all possible events.

Remark 1

- For any event E, $0 \le \mathbb{P}(E) \le 1$.
- For $E = \emptyset$, $\mathbb{P}(E) = 0$.
- $\mathbb{P}(S) = 1$, where S is whole space, or a space for all possible events.

Remark 2

- $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)$.
- If $E_1 \cap E_2 = \emptyset$, then $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$.

Conditional Probabilities 조건부확률

Definition 2

$\mathbb{P}(E|F)$, "probability of E given F", is the probability that event E occurs given that F has occurred. In a math notation,

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$$

- Ex) (from Ross) Suppose cards numbered one through ten are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten?

1. 10개 카드 중 1개뽑을 경우의 수 ⇒ $\frac{1}{10}$

2. 뽑은 카드 한 개 중 5 이상일 확률 ⇒ $\frac{6}{10}\left(\frac{3}{5}\right)$ $\left.\frac{\frac{1}{10}}{\frac{6}{10}}\right] = \frac{10}{60} = \frac{1}{6}$

3. 5 이상인 카드 중 10일 확률 ⇒ $\frac{1}{10}$

Independent Probability

Definition 3

If $\mathbb{P}(E) = \mathbb{P}(E|F)$, then we say the events E and F are independent events.

- That is, the occurence of E happening has nothing to do with the occurence of F.
- Just as Galilei once said *"And yet it rotates (even though people do not believe so)"*. People believing whether or not the earth rotates is independent event of earth's rotation.

Properties

1. $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$, if $E_1 \cap E_2 = \emptyset$.

2. $\mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c)$ $(\because (E \cap F) \cap (E \cap F^c) = \emptyset)$

3. If $F_i \cap F_j = \emptyset$ for all $i \neq j$ and $\mathbb{P}(F_1 \cup \cdots \cup F_n) = 1$ where $1 \leq i, j \leq n$, then $\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \cap F_i)$

Bayes' rule 베이즈

$F_1 \cap E, \ T_2 \cap E$

Theorem 1

상호   배반   사건

*Suppose $F_1, ..., F_n$ are **mutually exclusive** events (or, $F_i \cap F_j = \emptyset$, or, $\mathbb{P}(F_i \cap F_j) = 0$, for any $i \neq j$) and $\mathbb{P}(F_1 \cup \cdots \cup F_n) = 1$ (In other words, exactly one and only one events among $F_1, ..., F_n$ will occur.), then the following holds.*

$$
\begin{aligned}
\mathbb{P}(E) &= \mathbb{P}(E \cap F_1) + \cdots + \mathbb{P}(E \cap F_n) \\
&= \sum_{i=1}^{n} \mathbb{P}(E \cap F_i) \\
&= \sum_{i=1}^{n} \underbrace{\mathbb{P}(E|F_i)}_{\frac{\mathbb{P}(E \cap F_n)}{\mathbb{P}(F_n)}} \mathbb{P}(F_i)
\end{aligned}
$$

$E : 비가올 확률$

$F_1 : 온도$

$ex) F_1 : 0 \sim 10°C \quad 0.3$

$F_2 : 10 \sim 20°C \quad 0.5$

$0.3 \times 0.7 + 0.1 \times 0.3$

Exercise 1

Show that $\mathbb{P}(A|B \cap C)\mathbb{P}(B|C) = \mathbb{P}(A \cap B|C)$.

$$\frac{P(A \cap (B \cap C))}{P(B \cap C)} \times \frac{P(B \cap C)}{P(C)} = \frac{P(A \cap (B \cap C))}{P(C)} = \frac{P((A \cap B) \cap C)}{P(C)}$$

$$= P(A \cap B|C)$$

II. Random Variables

discrete vs continuous

Discrete r.v. 이산형

Definition 4

A random variable X is called a discrete random variable if (*complete sentence*)

- Example
  - flip a coin
  - throw a dice

Continuous r.v. 연속형

Definition 5

A random variable X is called a continuous random variable if (*complete sentence*)

- Example
  - weights and heights of a person
  - temperature of a room

pmf and pdf

확률 질량 함수
pmf(probability mass function) for a discrete r.v. X

Definition 6

A pmf $p(x)$ is a function that gives the **probability** that a discrete random variable is exactly equal to some value.

$$\mathbb{P}(X = x) = p(x)$$

- Suppose you throw a dice and X be a r.v. of the outcome. What is the pmf of X?
  1. *answer in a math form:* $P(x) = \frac{1}{6}$ $x = 1, 2, 3 \cdots, 6$
  2. *answer in a tabular form:*

개수	1	2	3	4	5	6
확률	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

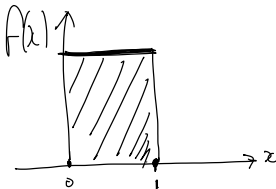**pdf(probability density function)** for a continuous r.v. X

Definition 7

A pdf $f(x)$ is a function that gives the **relative likelihood** for this continuous r.v. to take on a given value.

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$$

- Suppose a random variable takes a real values between 0 and 1 with equal likelihood, what is the pdf of X?
    1. *answer in a math form:* $P(0 \leq X \leq 1) = \int_0^1 f(x)dx$
    2. *draw a graph:*

Properties of pmf and pdf

- Mathematical property? *Something that always happen.*

Remark 3

The functions (pmf and pdf) are nonnegative everywhere. That is,

$$p(x) \geq 0 \text{ and } f(x) \geq 0$$

Remark 4

Its summation or integral over the entire area is equal to one. That is,

$$\sum p(x) = 1 \text{ and } \int f(x)dx = 1$$

- One can derive a cdf (cumulative distribution function) from a pmf or from a pdf.
  (See the upcoming definition of cdf.)

Expectation

$$x \mid 2 \ 3 \ 4 \ 5 \ 6$$
$$p(x) \ \frac{1}{6} \ \cdots \ \frac{1}{6}$$

이사값 $\quad Var(X) = E(X^2) - \left(E(X)\right)^2$

- For a discrete random variable X with pmf $p(x) \overset{6}{\underset{n=1}{\sum}} x_n p(x)$
  - $\mathbb{E}X = \sum xp(x)$
  - $\mathbb{E}[g(X)] = \sum g(x)p(x)$
  - Ex) $\mathbb{E}X^2 = \sum x^2 p(x)$
  - Ex) $\mathbb{E}(X^2 - 2X) = \sum$

- For a continuous$^{\text{연속적}}$ random variable X with pdf $f(x)$
  - $\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx$
  - $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$
  - Ex) $\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f(x)dx$

cdf 누적 분포

Definition 8

For a random variable X, the cdf(cumulative distribution function) $F(x)$ is a function of probability that the random variable X is found at a value less than or equal to x.

$$F(x) := \mathbb{P}(X \leq x) \quad \text{확률, 즉 도.}$$

- If discrete,

$$F(x) = \mathbb{P}(X \leq x) = \sum_{y=-\infty}^{x} \mathbb{P}(X = y) = \sum_{y=-\infty}^{x} p(y)$$

- If continuous,

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(y)dy$$

III. Uniform

Definition 9

연속 r.v

A continuous random variable X is said to follow uniform distribution with

균등 분포.

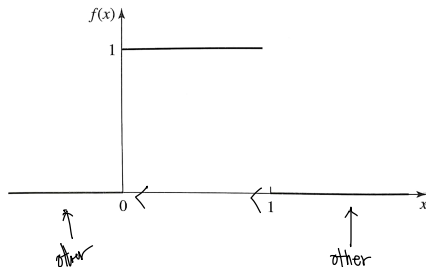parameter a and b, and write $X \sim U(a, b)$ if (*reads "X follows a uniform distribution with parameter a and b."*)

연속형 Pdf

$$\text{pdf } f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{cdf } F(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$
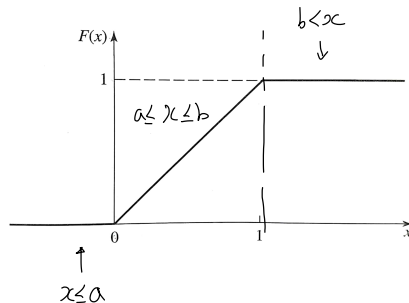
(The value $\frac{1}{b-a}$ is chosen as a constant so that it will make integrations over $-\infty$ to ∞ to be equal to 1.)

$U(0,1)$

- pdf

$f(x)$

1

0

1

x

other

other

- cdf

$F(x)$

$b < x$

1

$a \leq x \leq b$

0

1

x

$x \leq a$

Exercise 2

$X \sim U(10, 20)$, *then what is* $F(10)$*? and* $F(15)$*?*

$$f(x) = \frac{1}{b-a} = \frac{1}{20-10} = \frac{1}{10}$$

$10 \leq x \leq 20$

$0 \quad$ otherwise

$F(10):$ $x \leq 10 \Rightarrow 0$

$10 \leq x \leq 20 \Rightarrow \frac{x-a}{b-a}$

$20 < x \Rightarrow 1$

$= \frac{10-10}{20-10}$

$= \frac{0}{10}$

$= 0$

$F(15):$ $x \leq 10 \Rightarrow 0$

$10 \leq x \leq 20 \Rightarrow \frac{x-a}{b-a}$

$20 < x \Rightarrow 1$

$= \frac{15-10}{20-10}$

$= \frac{5}{10}$

$= 0.5$

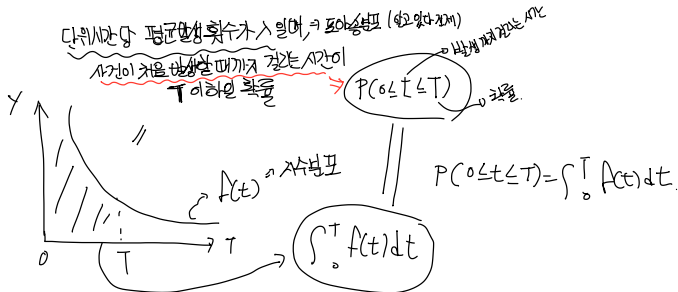# IV. Exponential

포아송분포

$$P(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

→ 단위시간동안
사건의평균발생횟수

↳
사건발생횟수

⇒ 길냥이를 마주칠때까지
걸리는 시간이 T이하일 확률

단위시간동안 평균발생 횟수가 λ 일때, ⇒ 포아송분포 (λ이 없다→개념)
사건이 처음 발생할때까지 걸리는 시간이
T 이하일 확률

→ 발생 전까지 걸리는 시간

$P(0 \le t \le T)$

↳ 확률.

Y

$f(t)$ ⇒ 지수분포

$P(0 \le t \le T)$

$P(0 \le t \le T) = \int_0^T f(t)\,dt.$

0   T   T

$\int_0^T f(t)\,dt$

Definition 10

아주 연속형 r.v

A nonnegative continuous random variable X is said to follow exponential 지수분포 distribution with parameter λ and write $X \sim exp(\lambda)$, if

ex) $X \sim exp(3)$
3분마다 한명씩.

$\longrightarrow$ λ시간안에 X가 생길 확률을
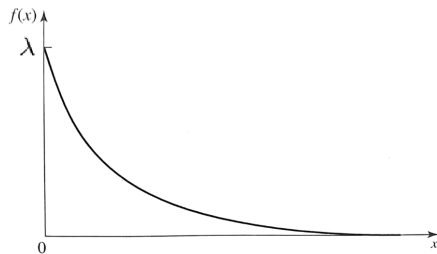
$$\text{pdf } f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$
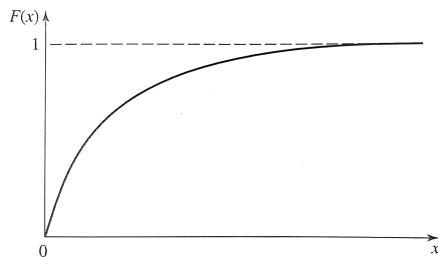
$$\text{cdf } F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- pdf

- cdf

- TFAE (The followings are equivalent - *If any one of them is true, the rest of them are all true. If any one of them is false, then the rest of them are all false.*)
  1. $X \sim exp(\lambda)$
  2. pdf $f(x) = \lambda e^{-\lambda x}$, if $x \geq 0$; $f(x) = 0$, otherwise
  3. cdf $F(x) = 1 - e^{-\lambda x}$, if $x \geq 0$; $F(x) = 0$, otherwise

Exercise 3

Prove that pdf $\rightarrow$ cdf

$$e^{2x} = 2e^{2x}$$

$$e^{x^2}$$

$$\int_{-\infty}^{x} f(x)\,dx = F(x)$$

$$\int_{0}^{x} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x} \right]_{0}^{x}$$

$$\text{if } x \geq 0$$

$$= -e^{-\lambda x} - (-1)$$

$$= 1 - e^{-\lambda x}.$$

More statistics (통계량) for r.v.

- Variance: $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$
- Standard Deviation: $sd(X) = \sqrt{Var(X)}$
- Coefficient of Variation (cv): $cv(X) = \frac{sd(X)}{\mathbb{E}X}$
- (*A cv measures 'relative' variation and often more meaningful.*)
- Examples
  1. cv of $N(10, \sqrt{10}^2)$ vs $N(10000, 10^2)$. $\frac{\sqrt{10}}{10}$ $\gg$ $\frac{1}{1000}$
  2. For an exponential r.v., $X \sim exp(\lambda)$, then $cv(X) =$ (*next slide*)
  3. For a normal r.v. $X \sim N(\mu, \sigma^2)$, $cv(X) = \frac{\sqrt{\sigma^2}}{\mu} = \frac{\sigma}{\mu}$
  4. For deterministic variable X, $c_x = 0$

  $Cv(X)$

Properties of exponential distribution

- Statistics for a r.v. $X \sim exp(\lambda)$,
  1. $EX = 1/\lambda$
  2. $Var(X) = 1/\lambda^2$ $\sqrt{Var(X)} = Sd(X) = \frac{1}{\lambda}$
  3. $cv(X) = 1$ *(Exponential r.v. has c.v. equal to 1, always.)*

- Theorems
  1. Memoryless property
  2. Suppose $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$, and independent, then
     $\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
  3. If $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$, and independent, then
     $min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$

Exercise 4

Show that $EX = 1/\lambda$

$$E(x) = \int_0^\infty x \cdot f(x)\, dx = \int_0^\infty x\, \lambda e^{-\lambda t}\, dt \quad \text{(부분적분)}$$

$$\text{if, } x \geq 0 \qquad = \left[-x e^{-\lambda x} \right]_0^\infty - \int_0^\infty -e^{-\lambda t}\, dt$$

$$= \left[-x \cdot e^{-\lambda x} \right]_0^\infty + \left[-\tfrac{1}{\lambda} e^{-\lambda x} \right]_0^\infty$$

$$= \lim_{x \to \infty} \left(-x\, e^{-\lambda x} \right) - 0 + \lim_{x \to \infty} \left(-\tfrac{1}{\lambda} e^{-\lambda x} \right) - \left(-\tfrac{1}{\lambda} e^{0} \right)$$

$$\qquad\qquad 0 \qquad\qquad = +\tfrac{1}{\lambda}$$

$$= \lim_{x \to \infty} \left(-x \cdot e^{-\lambda x} \right) + \tfrac{1}{\lambda}$$

$$= -\lim_{t \to \infty} \left(\frac{x}{e^{\lambda x}} \right) + \tfrac{1}{\lambda}$$

$$\text{로피탈의 정리} \quad = -\tfrac{1}{\lambda} \left(\lim \frac{\lambda x}{e^{\lambda x}} \right) + \tfrac{1}{\lambda} \longrightarrow 0 \qquad = \left(\tfrac{1}{\lambda} \right)$$

Exercise 5

Show that $Var(X) = 1/\lambda^2$. (Hint: need to do EX^2 first)

$$Var(x) = E(x^2) - \{E(x)\}^2$$

$$Var(x) = E(x^2) - \frac{1}{\lambda^2}$$

$$E(x)^2 = \int_0^\infty x^2 f(x) dx$$

$$E(x^2) = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} dx = \left[x^2(-e^{-\lambda x}) \right]_0^\infty - \int_0^\infty 2x \cdot (-e^{-\lambda x}) dx \quad \text{(부분적분)}$$

$$= \left[x^2 \cdot (-e^{\lambda x}) \right]_0^\infty + \frac{2}{\lambda} \cdot \overline{\int_0^\infty 2x \underline{\lambda} e^{-\lambda x} dx}$$

$$= \left[x^2 \cdot (-e^{\lambda x}) \right]_0^\infty + \frac{2}{\lambda^2}$$

$$= \lim_{x \to \infty} x^2 \cdot (-e^{-\lambda x}) + \frac{2}{\lambda^2}$$

$$= \lim_{x \to \infty} \frac{x^2}{e^{\lambda x}} + \frac{2}{\lambda^2}$$

$$E(x^2) = \frac{2}{\lambda^2}, \quad E(x) = \frac{1}{\lambda}$$

$$Var(x) = E(x^2) - \{E(x)\}^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

$$\hookrightarrow E(x) = \frac{1}{\lambda}$$

Common usage of exponential distribution

- Since exponential r.v. is continuous and nonnegative, it is frequently used to describe time.
- In such cases, the parameter λ can be understood as "rate" as an inverse value of the time.
- For example, $X \sim exp(\lambda)$ and $\mathbb{E}X = 1/\lambda = 5$ yrs imply $\lambda = 1/5$ per year.
- For example, consider the series of events that occur in every time interval that follows exponential distribution with mean 3 minutes. (In post office, you expect one customer arrival in 3 minutes on average and the time follows exponential distribution). We say that it follows exponential arrival times with rate 1/3 per minute.

    $2 \sim 5$ 시간 $2 \sim 5$ 년

    $0 \sim 3$ 시간 $0 \sim 3$ 월

Memoryless property 무기억성

$$0 \sim 2$$
$$2 \sim 5 = 0 \sim 3$$

Definition 11

A r.v. X is *memoryless*, if $\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s)$, for $s, t \geq 0$.

Theorem 2

Exponential random variable is memoryless.

- For example, we shall assume that bus arrival time follows $exp(1/5)$. In other words, it follows an exponential distribution and its expected arrival time is 5 minutes. You are waiting for bus, and have been waiting for 3 minutes, what is the probability that bus will not come in 5 minutes from now.

$$0 \sim 3 \text{ 안 옴} , \quad 0 \sim 5 \text{ 을 확률 같음}$$
$$3 \sim 8$$

Exercise 6

Prove the previous theorem.

$$P(X > s+t \mid X > t) = \frac{P(x > s+t \text{ and } x > s)}{P(x > t)} = \frac{P(x > s+t)}{P(x > t)}$$

① $P(x > s+t) = 1 - P(x < s+t)$

$= 1 - F(s+t)$

$= 1 - \left(1 - e^{-\lambda(s+t)}\right)$

$= e^{-\lambda(s+t)}$

② $P(x > t) = 1 - P(x < t)$

$= 1 - F(t)$

$= 1 - \left(1 - e^{-\lambda t}\right)$

$= e^{-\lambda t}$

$$\frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(x > s)$$

한편일이나는데 걸리는 시간

Theorem 3

Suppose $X_1 \sim exp(\lambda_1), X_2 \sim exp(\lambda_2)$ *and* ~~they are independent~~, *then*
$\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

- Smith and Jones came to post office together and they are served by two clerks, A and B, respectively. Server A has service time following $exp(1/3)$ and server B has service time following $exp(1/5)$. What is the chance that Smith will be done first?

$$\frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{5}} = \frac{\frac{1}{3}}{\frac{8}{15}} = \frac{15}{24}$$

- Suppose that Smith came to post office earlier than Jones by 2 minutes, but Smith was still being served at the moment that Jones started to being served. Would this assumption change your previous answer? Why or why not?

Memoryless

변하지 않습니다. Smith는 사이에 발생하지 않기 때문에
재분포의 비자생으로 영향을 주지 않습니다.

V. Poisson

Definition 12

시간안에 일어나는 횟수.

A discrete random variable X is said to follow Poisson distribution with parameter λ, and write $X \sim poi(\lambda)$, if its pmf is

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k = 0, 1, 2, ...$$

- What is cdf of $poi(\lambda)$? $\sum\limits_{k=0}^{x} \dfrac{\lambda^k e^{-\lambda}}{k!}$

Remark 5

$$\mathbb{E}X = Var(X) = \lambda$$

기댓값 = 분산 $= \lambda$

Exercise 7

For $X \sim poi(\lambda)$, prove that $\mathbb{E}X = \lambda$.

- cf) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- pf) We have $\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, ...$, and

$$
\begin{aligned}
\mathbb{E}X &= \sum_{x=-\infty}^{\infty} x p(x) \text{ (this is common for all discrete r.v.)} \\
&= \sum_{k=0}^{\infty} x \frac{\lambda^k e^{-\lambda}}{k!} \text{ (x이 0을 넣으면 지워짐에 0)} \\
&= \sum_{k=1}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&\quad \text{k-1을 n으로 치환} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda e^{-\lambda} \cdot e^{\lambda} \\
&= \lambda
\end{aligned}
$$

VI. Some Exercises

Maximum and minimum

Definition 13

$$x \wedge y = min(x, y) = \begin{cases} x & \text{, if } x \leq y \\ y & \text{, otherwise} \end{cases}$$

$$x \vee y = max(x, y) = \begin{cases} x & \text{, if } x \geq y \\ y & \text{, otherwise} \end{cases}$$

$$x^+ = max(x, 0) \text{ (positive part)}$$

- Ex) $x \wedge 25 = min(x, 25)$
- Ex) $(25 - x)^+ = \underset{\sim}{max(25 - x, 0)}$

$$\mathbb{E}[U] = \frac{1}{b-a}$$

Exercise 8

$$x \wedge y = min(x, y) = \begin{cases} x & \text{, if } x \leq y \\ y & \text{, otherwise} \end{cases}$$

For $X \sim U(20, 40)$, evaluate $\mathbb{E}[X \wedge 25]$ and $\mathbb{E}[(25 - X)^+]$.

$x^+ = max(x, 0)$ *(positive part)*

$X \wedge 25 \begin{bmatrix} X & (X \leq 25) \\ 25 & (X > 25) \end{bmatrix}$

$\mathbb{E}[X \wedge 25] = \int_{-\infty}^{\infty} (X \wedge 25) \underline{f_X(x)} dx$

$= \int_{20}^{25} (X \wedge 25) \frac{1}{20} dx + \int_{25}^{40} (X \wedge 25) \frac{1}{20} dx$

$= \int_{20}^{25} X(\frac{1}{20}) dx + \int_{25}^{40} 25 \cdot \frac{1}{20} dx$

$= \left[\frac{1}{40} x^2 \right]_{20}^{25} + \left[\frac{5}{4} X \right]_{25}^{40}$

$= \frac{1}{40} (25^2 - 20^2) + \frac{5}{4} (40 - 25)$

$= \frac{225}{40} + \frac{75}{4}$

$\approx \frac{300}{40}$

$= \frac{15}{2}$

$(25 - X)^+ = max((25 - X), 0)$

$max((25 - x), 0) \begin{bmatrix} 25 - x & (25 \geq x) \\ 0 & (25 < x) \end{bmatrix}$

$\mathbb{E}[(25 - X)^+] = \int_{-\infty}^{\infty} (25 - x)^+ f_X(x) dx$

$= \int_{20}^{40} (25 - x)^+ \frac{1}{20} dx$

$= \int_{20}^{25} (25 - x)^+ \frac{1}{20} dx + \int_{25}^{40} (25 - x)^+ \frac{1}{20} dx$

$= \int_{20}^{25} 0 \cdot \frac{1}{20} dx + \int_{25}^{40} (25 - x) \cdot \frac{1}{20} dx$

$= 0 + \frac{1}{20} \left[25x - \frac{1}{2} x^2 \right]_{25}^{40}$

$= \frac{1}{20} \left[\left(\underbrace{25 \cdot 40}_{1000} - \underbrace{\frac{1}{2} 40^2}_{800} \right) - \left(25 \cdot 25 - \frac{1}{2} 25^2 \right) \right]$

$\phantom{= \frac{1}{20}} \underbrace{}_{625} - \underbrace{\frac{625}{2}}$

$= \frac{1}{20} (200 - \frac{625}{2}) = -\frac{45}{8}$

$\frac{1}{20} = \frac{225}{2} \cdot \frac{45}{}$

Exercise 9

$\lambda = 8$

For $X \sim Poi(8)$,

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k = 0, 1, 2, ...$$

1. $\mathbb{P}(X = 0) =$
2. $\mathbb{P}(2 \leq X \leq 4) =$
3. $\mathbb{P}(X > 2) =$

1. $P(X=0) = \dfrac{8^0 e^{-8}}{0!} = \dfrac{1 \times e^{-8}}{1} = e^{-8}$

2. $P(2 \leq X \leq 4) = P(X=2) + P(X=3) + P(X=4)$

$P(X=2) = \dfrac{8^2 e^{-8}}{2!} = \dfrac{\cancel{64}^{32} \times e^{-8}}{\cancel{2}} = 32e^{-8}$

$P(X=3) = \dfrac{8^3 e^{-8}}{3!} = \dfrac{8^3 \times e^{-8}}{6} = \dfrac{256}{3}e^{-8}$

$P(X=4) = \dfrac{8^4 e^{-8}}{4!} = \dfrac{8^4 \times e^{-8}}{24} = \dfrac{512}{3}e^{-8}$

3. $P(X > 2) = 1 - P(X=0) - P(X=1) - P(X=2)$

$\underset{\underset{e^{-8}}{\Downarrow}}{} \quad \underset{\underset{8e^{-8}}{\Downarrow}}{} \quad \underset{\underset{32e^{-8}}{\Downarrow}}{}$

$= 1 - 25e^{-8}$

$\therefore \dfrac{96}{3}e^{-8} + \dfrac{256}{3}e^{-8} + \dfrac{512}{3}e^{-8}$

$= \dfrac{864}{3}e^{-8}$

$= 288e^{-8}$

Exercise 10 $\text{nonnegative random variable}$

$$\text{기댓값의 기댓값} = \frac{1}{\lambda}$$

For $X \sim exp(7)$, *evaluate* $\mathbb{E}[max(X, 7)]$.

$$max(X, 7) = \begin{bmatrix} X & (X \geq 7) \\ 7 & (X < 7) \end{bmatrix}$$

$$\mathbb{E}[max(X,7)] = \int_{-\infty}^{\infty} (X \vee 7) f_X(x) \, dx$$

$$= \int_0^7 (x \vee 7) \cdot 7 e^{-7x} \, dx + \int_7^{\infty} (x \vee 7) \cdot 7 e^{-7x} \, dx$$

$$= \int_0^7 \left(7 \cdot 7 e^{-7x}\right) dx + \int_7^{\infty} \left(x \cdot 7 e^{-7x}\right) dx$$

$$= 7 \int_0^7 e^{-7x} \, dx + 7 \int_7^{\infty} x e^{-7x} \, dx$$

$$= 7 \times -\frac{1}{7} \left[e^{-7x} \right]_0^7 + 7 \times -\frac{1}{7} \cdot \left[x e^{-7x} \right]_7^{\infty}$$

$$= -e^{-49} - 0 + -7 e^{-49}$$

$$= -8 e^{-49}$$

pdf $f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

cdf $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$x \wedge y = min(x, y) = \begin{cases} x & \text{, if } x \leq y \\ y & \text{, otherwise} \end{cases}$

$x \vee y = max(x, y) = \begin{cases} x & \text{, if } x \geq y \\ y & \text{, otherwise} \end{cases}$

$x^+ = max(x, 0)$ *(positive part)*

Exercise 11

$\ln = \log_e e.$

For $X \sim exp(8)$, find x^* such that $F(x^*) = 0.6$.

제약 $\lambda = 8$

$F(x) = 1 - e^{-\lambda x}$

cdf $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$F(x^*) = 1 - e^{-8x^*} = 0.6$

$= e^{-8x^*} = 0.4$

$= \log_e e^{-8x^*} = \log_e 0.4$

$-8x^* = \log_e 0.4$

$x^* = -\frac{1}{8} \log_e 0.4$

Exercise 12

For $X \sim U(10, 20)$, find x^* such that $F(x^*) = 0.7$.

$$F(x) = \frac{x - a}{b - a}$$

$$F(x^*) = \frac{x^* - a}{b - a}$$

$$= \frac{x^* - 10}{20 - 10} = 0.7$$

$$= x^* - 10 = 7$$

$$= x^* = 17$$

"Man can learn nothing unless he proceeds from the known to the unknown. - Claude Bernard"