Machine Learning Homework 3

Random Data Generator

a. Univariate Gaussian data generator

- Input
  - Expectation value or mean: m
  - Variance: s
- Output
  - a data point from $N(m, s)$
- HINT
  - [Generating values from normal distribution]
  - You have to handcraft your generator based on one of the approaches given in the hyperlink
  - You can use uniform distribution function (e.g., NumPy)

b. Polynomial basis linear model data generator

- $y = W^T\phi(x) + e$
  - W is a $n \times 1$ vector
  - $e \sim N(0, a)$
- Input
  - n (basis number), a, w
  - e.g., $n = 2 \rightarrow y = w_0x^0 + w_1x^1$
- Output
  - a point (x, y)
- Internal constraint
  - $-1.0 < x < 1.0$
  - x is uniformly distributed

Sequential Estimator

- Sequential estimate the mean and variance
  - Data is given from the univariate Gaussian data generator (1.a)
- Input
  - m, s as in (1.a)
- Function
  - Call (1.a) to get a new data point from $N(m, s)$
  - Use sequential estimation to find the current estimates to m and s
  - Repeat steps above until the estimates converge
- Output

    - Print the new data point and the current estimates of m and s in each iteration
- Notes
    - You should derive the recursive function of mean and variance based on the sequential estimation
    - Hint: [Online algorithm]
- Sample input & output **(for reference only)**

```
Data point source function: N(3.0, 5.0)

Add data point: 1.220492527761238
Mean = 1.220492527761238    Variance = 0.0
Add data point: 3.6967805272943366
Mean = 2.458636527527787    Variance = 1.53300056415791
Add data point: 2.7258100985704146
Mean = 2.5476943845419964   Variance = 1.0378629798971994
Add data point: 2.2138523069477527
Mean = 2.4642338651434352   Variance = 0.7992942098177336
Add data point: 2.2113035958584453
Mean = 2.4136478112864372   Variance = 0.6496711632334788
Add data point: 0.05399706095719692
Mean = 2.020372686231564    Variance = 1.3147192559625305
Add data point: 4.3538771826058
Mean = 2.3537304714278835   Variance = 1.7936666971024264

...

Add data point: 4.233592159021013
Mean = 2.961576104513964    Variance = 5.045715437349161
Add data point: 3.529990930040463
Mean = 2.961883688294010    Variance = 5.043159812425648
Add data point: 1.125210345431449
Mean = 2.960890354955524    Variance = 5.042255747918937
```

Mathematical Derivation

Prove Gamma-Poisson conjugation

- Show that the Gamma distribution acts as a conjugate prior to the Poisson likelihood, including deriving the posterior distribution
- Notes
    - During the demo, you will be required to explain the entire mathematical proof
    - Upload the handwritten file to e3 (PDF or any image format)

Posterior mean and variance with Gaussian prior

- Derive the posterior mean and variance for a prior given by $w \sim N\left(\mu_0, \Lambda_0^{-1}\right)$
- Notes
    - During the demo, you will be required to explain the entire mathematical proof
    - Upload the handwritten file to e3 (PDF or any image format)

- This part may help you solve the next question

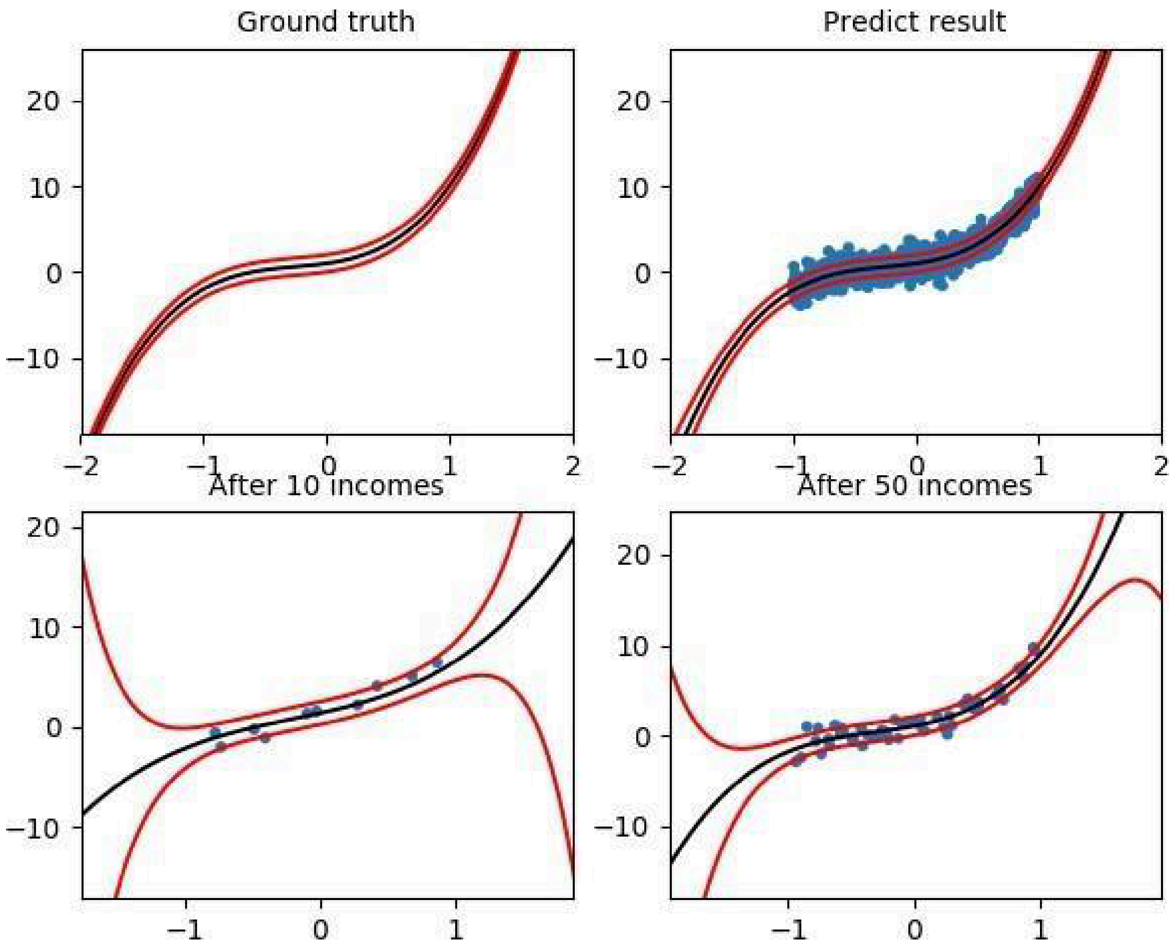
Bayesian Linear Regression

- Input
  - The precision (i.e., b) for initial prior $w \sim N\left(0, b^{-1}I\right)$
  - All other required inputs for the polynomial basis linear model generator (1.b)
- Function
  - Call (1.b) to generate one data point
  - Update the prior, and calculate the parameters of predictive distribution
  - Repeat steps above until the posterior probability converges
- Output
  - Print the new data point and the current parameters for posterior and predictive distribution
  - After probability converged, do the visualization
    - Ground truth function (from linear model generator)
    - Final predict result
    - At the time that have seen 10 data points
    - At the time that have seen 50 data points
  - Notes
    - Except ground truth, you have to draw those data points which you have seen before
    - Draw a black line to represent the mean of function at each point
    - Draw two red lines to represent the variance of function at each point
      - In other words, distance between red line and mean is **ONE variance**
    - Hint: Online learning
- Sample input & output **(for reference only)**
  - Case 1: $b = 1$, $n = 4$, $a = 1$, $w = [1, 2, 3, 4]$

```
Add data point (-0.64152, 0.19039):

Posterior mean:
     0.0718294547
    -0.0460797888
     0.0295609502
    -0.0189638408

Posterior variance:
     0.6227289276,   0.2420256620,  -0.1552634839,   0.0996041049
     0.2420256620,   0.8447365161,   0.0996041049,  -0.0638976884
    -0.1552634839,   0.0996041049,   0.9361023116,   0.0409914289
     0.0996041049,  -0.0638976884,   0.0409914289,   0.9737033172

Predictive distribution ~ N(0.00000, 2.65061)

Add data point (0.07122, 1.63175):
```

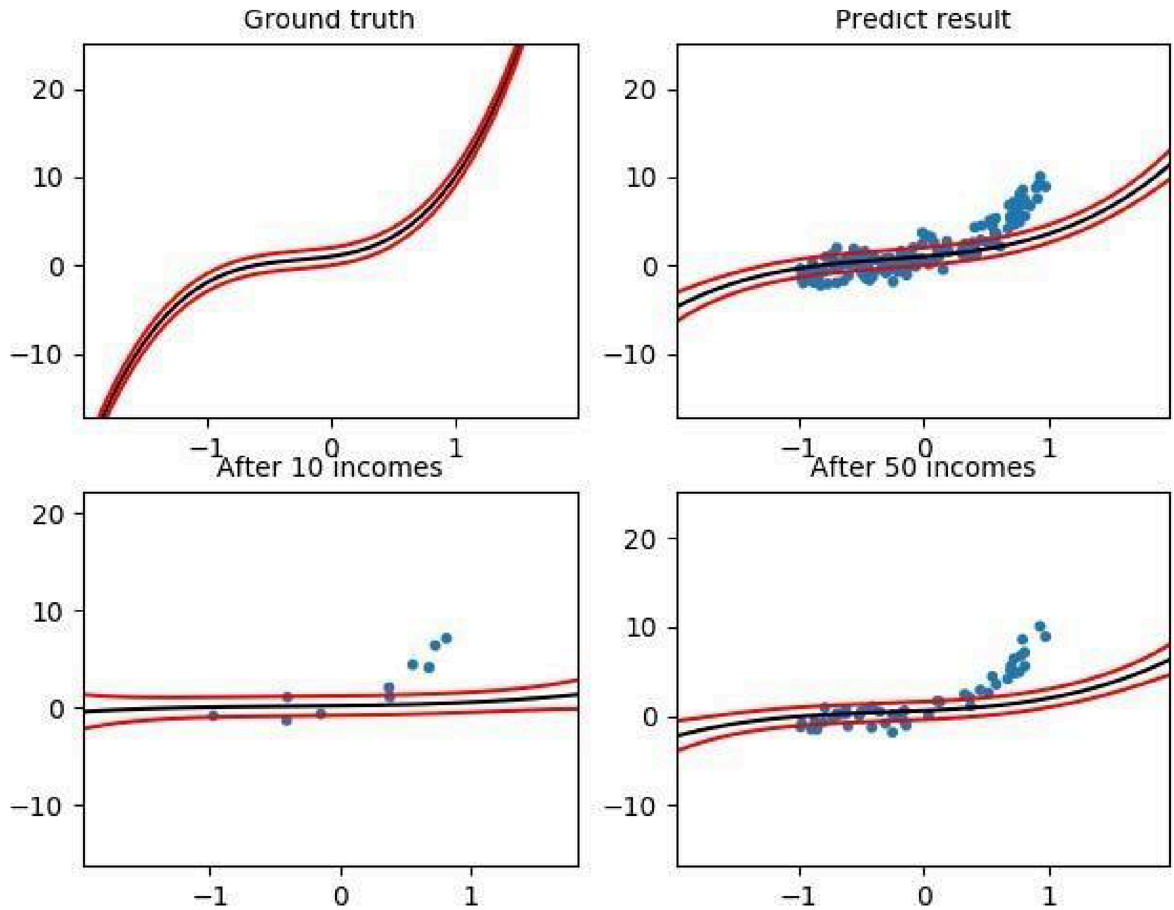
```
Posterior mean:
     0.6736864869
     0.2388980107
    -0.1054659080
     0.0710615952

Posterior variance:
     0.3765992302,   0.1254838660,  -0.1000441911,   0.0627881634
     0.1254838660,   0.7895542671,   0.1257503020,  -0.0813299447
    -0.1000441911,   0.1257503020,   0.9237138418,   0.0492510997
     0.0627881634,  -0.0813299447,   0.0492510997,   0.9681964094

Predictive distribution ~ N(0.06869, 1.66008)

Add data point (-0.19330, 0.24507):

Posterior mean:
     0.5760972313
     0.2450231522
    -0.0801842453
     0.0504992402

Posterior variance:
     0.2867129751,   0.1311255325,  -0.0767580827,   0.0438488542
     0.1311255325,   0.7892001707,   0.1242887609,  -0.0801412282
    -0.0767580827,   0.1242887609,   0.9176812972,   0.0541575540
     0.0438488542,  -0.0801412282,   0.0541575540,   0.9642058389

Predictive distribution ~ N(0.62305, 1.34848)

......

Add data point (-0.76990, -0.34768):

Posterior mean:
     0.9107496675
     1.9265499885
     3.1119297129
     4.1312375189

Posterior variance:
     0.0051883836,  -0.0004416700,  -0.0086000319,   0.0008247001
    -0.0004416700,   0.0401966605,   0.0012708906,  -0.0554822477
    -0.0086000319,   0.0012708906,   0.0265353911,  -0.0031205875
     0.0008247001,  -0.0554822477,  -0.0031205875,   0.0937197255

Predictive distribution ~ N(-0.61566, 1.00921)

Add data point (0.36500, 2.22705):

Posterior mean:
     0.9107404583
     1.9265225090
     3.1119408740
     4.1312734131
```

```
Posterior variance:
     0.0051731092,  -0.0004872471,  -0.0085815201,   0.0008842340
    -0.0004872471,   0.0400606628,   0.0013261280,  -0.0553046044
    -0.0085815201,   0.0013261280,   0.0265129556,  -0.0031927398
     0.0008842340,  -0.0553046044,  -0.0031927398,   0.0934876838

Predictive distribution ~ N(2.22942, 1.00682)
```



- Case 2: $b = 100$, $n = 4$, $a = 1$, $w = [1, 2, 3, 4]$

```
(Console output omitted)
```



- Case 3: $b = 1$, $n = 3$, $a = 3$, $w = [1, 2, 3]$

```
(Console output omitted)
```