Homework 3

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1

(a)

$$
\begin{array}{lllll}
\max & 10x_1 & +12x_2 & +20x_3 & \\
\text{s.t.} & 2x_1 & +3x_2 & +4x_3 & \leq 500 \\
 & 30x_1 & +20x_2 & +40x_3 & \leq 4800 \\
 & x_1 & & & \leq 100 \\
 & & x_2 & & \leq 80 \\
 & & & x_3 & \leq 50 \\
 & x_i \geq 0 & \forall i = 1,2,3 & &
\end{array}
$$

(b)

$$x^* = (48, 68, 50)$$

$$z^* = 2296$$

The bottle neck of this plan:

$$2x_1 + 3x_2 + 4x_3 = 500 \implies \text{No more materials.}$$

$$30x_1 + 20x_2 + 40x_3 = 4800 \implies \text{No more machine hours.}$$

$$x_3 = 50 \implies \text{product 3 has reached its daily demand.}$$

(c) dual LP:

$$\begin{array}{llllllr} \min & 500y_1 & +4800y_2 & +100y_3 & +80y_4 & +50y_5 & \\ \text{s.t.} & 2y_1 & +30y_2 & +y_3 & & & \geq 10 \\ & 3y_1 & +20y_2 & & +y_4 & & \geq 12 \\ & 4y_1 & +30y_2 & & & y_5 & \geq 20 \\ & y_i \geq 0 & \forall i = 1,2,3,4,5 & & & & \end{array}$$

Basis : $(x_1, x_2, x_3, s_3, s_4)$

$$c_B^T = \begin{bmatrix} 10 & 12 & 20 & 0 & 0 \end{bmatrix}$$

$$\text{A}_B = \begin{bmatrix} 2 & 3 & 4 & 0 & 0 \\ 30 & 20 & 40 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\bar{\text{y}}^T = c_B^T A_B^{-1} = \begin{bmatrix} 10 & 12 & 20 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{-2}{5} & \frac{3}{50} & 0 & 0 & \frac{-4}{5} \\ \frac{3}{5} & \frac{-1}{25} & 0 & 0 & \frac{-4}{5} \\ 0 & 0 & 0 & 0 & 1 \\ \frac{2}{5} & \frac{-3}{50} & 1 & 0 & \frac{4}{5} \\ \frac{-3}{5} & \frac{1}{25} & 0 & 1 & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{16}{5} & \frac{3}{25} & 0 & 1 & \frac{4}{5} \end{bmatrix}$$

$w = 1600 + 576 + 120 = 2296 = z^*$

Dual feasible:

$\frac{32}{5} + \frac{18}{5} + 0 = 10 \geq 10$

$\frac{48}{5} + \frac{12}{5} + 0 = 12 \geq 12$

$\frac{64}{5} + \frac{24}{5} + \frac{12}{5} = 20 \geq 20$

(d) The shadow price of constraint i is $(c_B^T A_B^{-1})$

constraint 1 : $\frac{16}{5}$

constraint 2 : $\frac{3}{25}$

constraint 3 : 0

constraint 4 : 0

constraint 5 : $\frac{12}{5}$

(e)

(1) materials : The shadow price is $\frac{16}{5}$, and the cost per unit is \$2.

$\frac{16}{5} - 2 \geq 0 \Longrightarrow$ Consider option 1.

(2) machine hours : The shadow price is $\frac{3}{25}$, and the cost per minute is $\$\frac{1}{6}$.

$\frac{3}{25} - \frac{1}{6} \leq 0 \Longrightarrow$ Don't consider option 2.

(3) Demands of product 1 : The shadow price is 0, and the cost per unit is \$5.

$0 - 5 \leq 0 \Longrightarrow$ Don't consider option 3.

(4) Demands of product 2 : The shadow price is 0, and the cost per unit is \$10.

$0 - 10 \leq 0 \Longrightarrow$ Don't consider option 4.

(5) Demands of product 3 : The shadow price is $\frac{12}{5}$, and the cost per unit is \$15.

$\frac{12}{5} - 15 \leq 0 \Longrightarrow$ Don't consider option 5.

2

(a)

$$
\begin{array}{lccccl}
\max & 10x_1 & +12x_2 & +20x_3 & +30x_4 & \\
\text{s.t.} & 2x_1 & +3x_2 & +4x_3 & +5x_4 & \leq 500 \\
 & 30x_1 & +20x_2 & +40x_3 & +60x_4 & \leq 4800 \\
 & x_1 & & & & \leq 100 \\
 & & x_2 & & & \leq 80 \\
 & & & x_3 & & \leq 50 \\
 & x_i \geq 0 \quad \forall i = 1,2,3,4 & & & &
\end{array}
$$

Basis : $(x_1, x_2, x_3, s_3, s_4)$

$$
A_B^{-1}A_j = \begin{bmatrix} \frac{-2}{5} & \frac{3}{50} & 0 & 0 & \frac{-4}{5} \\ \frac{3}{5} & \frac{-1}{25} & 0 & 0 & \frac{-4}{5} \\ 0 & 0 & 0 & 0 & 1 \\ \frac{2}{5} & \frac{-3}{50} & 1 & 0 & \frac{4}{5} \\ \frac{-3}{5} & \frac{1}{25} & 0 & 1 & \frac{4}{5} \end{bmatrix} \times \begin{bmatrix} 5 \\ 60 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{3}{5} \\ 0 \\ \frac{-8}{5} \\ \frac{-3}{5} \end{bmatrix}
$$

$$c_B^T A_B^{-1} A_j - c_j = \begin{bmatrix} 10 & 12 & 20 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{8}{5} \\ \frac{3}{5} \\ 0 \\ \frac{-8}{5} \\ \frac{-3}{5} \end{bmatrix} - 30 = \frac{-34}{5}$$

Because $\frac{-34}{5} \leq 0$, the company should consider producing product 4.

$$\begin{array}{ccccccccc|c}
0 & 0 & 0 & 0 & 0 & \frac{-34}{5} & \frac{16}{5} & \frac{3}{25} & \frac{12}{5} & 2296 \\
\hline
1 & 0 & 0 & 0 & 0 & \boxed{\frac{8}{5}} & \frac{-2}{5} & \frac{3}{50} & \frac{-4}{5} & x_1 = 48 \\
0 & 1 & 0 & 0 & 0 & \frac{3}{5} & \frac{3}{5} & \frac{-1}{25} & \frac{-4}{5} & x_2 = 68 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 = 50 \\
0 & 0 & 0 & 1 & 0 & \frac{-8}{5} & \frac{2}{5} & \frac{-3}{50} & \frac{4}{5} & s_3 = 52 \\
0 & 0 & 0 & 0 & 1 & \frac{-3}{5} & \frac{-3}{5} & \frac{1}{25} & \frac{4}{5} & s_4 = 12 \\
\hline
\end{array}$$

x_4 enters, x_1 leaves:

New basis : $(x_4, x_2, x_3, s_3, s_5)$

$$x_B = A_B^{-1} b = \begin{bmatrix} 30 \\ 50 \\ 50 \\ 100 \\ 30 \end{bmatrix}$$

And the objective value of the new solution is 2500.

(b)

$$\begin{array}{lccccc}
\max & 10x_1 & +12x_2 & +20x_3 & & \\
\text{s.t.} & 2x_1 & +3x_2 & +4x_3 & \leq 500 & \\
 & 30x_1 & +20x_2 & +40x_3 & \leq 4800 & \\
 & x_1 & & & \leq 100 & \\
 & & x_2 & & \leq 80 & \\
 & & & x_3 & \leq 50 & \\
 & 90x_1 & +80x_2 & +200x_3 & \leq 12000 & x_i \geq 0 \quad \forall i = 1,2,3
\end{array}$$

Original plan : $x^* = (48, 68, 50)$

In the new constraint : $90 \times 48 + 80 \times 68 + 200 \times 50 = 19760 \geq 12000 \Longrightarrow$ the company should change the plan.

$B = (x_1, x_2, x_3, s_3, s_4, s_6)$

$N = (s_1, s_2, s_5)$

$$A_B = \begin{bmatrix} 2 & 3 & 4 & 0 & 0 & 0 \\ 30 & 20 & 40 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 90 & 80 & 200 & 0 & 0 & 1 \end{bmatrix} A_N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} A_B^{-1}A_N = \begin{bmatrix} \frac{-2}{5} & \frac{3}{50} & \frac{-4}{5} \\ \frac{3}{5} & \frac{-1}{25} & \frac{-4}{5} \\ 0 & 0 & 1 \\ \frac{2}{5} & \frac{-3}{50} & \frac{4}{5} \\ \frac{-3}{5} & \frac{1}{25} & \frac{4}{5} \\ -12 & \frac{-11}{5} & -64 \end{bmatrix} A_B^{-1}b = \begin{bmatrix} 48 \\ 68 \\ 50 \\ 52 \\ 12 \\ -7760 \end{bmatrix}$$

$$c_B^T A_B^{-1} A_N - c_N^T = \begin{bmatrix} \frac{16}{5} & \frac{3}{25} & \frac{12}{5} \end{bmatrix}$$
$$c_B^T A_B^{-1} b = 2296$$

Do the ratio test:

$\frac{-7760}{-64}$ has the smallest ratio, so s_5 enters and s_6 leaves.

The new basis : $x_B = (x_1, x_2, x_3, s_3, s_4, s_5)$

$$A_B^{-1}b = \begin{bmatrix} 145 \\ 165 \\ \frac{-285}{4} \\ -45 \\ -85 \\ \frac{485}{4} \end{bmatrix}$$

However, the basis is not feasible after one iteration, which means we need more iterations to get a feasible solution.

3

(a)

$$f(x) = x_1^2 + 3x_2^2 + x_1x_2 - x_1 - 2x_2$$

$$\nabla f(x) = \begin{bmatrix} 2x_1 + x_2 - 1 \\ 6x_2 + x_1 - 2 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

$$2 \geq 0, \quad 6 \geq 0, \quad 2 \times 6 - 1 = 11 \geq 0 \Longrightarrow \text{This is a convex program.}$$

(b)

$$f(x) = -x_3^2 + x_1x_3 + x_2x_3$$

$$\nabla f(x) = \begin{bmatrix} x_3 \\ x_3 \\ x_1 + x_2 - 2x_3 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

-2 $\leq 0 \Longrightarrow$ This is not a convex program.

(c)

$$f(x) = x_1^2 + 2^{x_2}$$

$$\nabla f(x) = \begin{bmatrix} 2x_1 \\ \ln 2 \times 2^{x_2} \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & (\ln 2)^2 x^{x_2} \end{bmatrix}$$

$2 \geq 0, \quad (\ln 2)^2 x^{x_2} \geq 0, \quad 2(\ln 2)^2 x^{x_2} \geq 0 \Longrightarrow$ f(x) is convex.

But the program is maximizing f, so the program is unbounded.

(d)

$$f(x) = x_1^2 + 3x_2^2 + x_1x_2 - x_1 - 2x_2 \quad \text{over} \quad x_1 \geq 0 \quad x_2 \geq 0 \quad x_1x_2 \leq 1$$

$$\nabla f(x) = \begin{bmatrix} 2x_1 + x_2 - 1 \\ 6x_2 + x_1 - 2 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

$\nabla^2 f(x)$ is semi-definite over

$$\{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, x_1x_2 \leq 1\}$$

$\Longrightarrow$ It's a convex program.

4

$$\begin{aligned} \min_{x\in R} \quad & x_1^2 + 3x_2^2 + 2x_1x_2 - 4x_2 \\ \text{s.t.} \quad & x_1 \geq 2 \end{aligned}$$

(a)

$$f(x) = x_1^2 + 3x_2 + 2x_1x_2 - 4x_2$$

$$\boldsymbol{\nabla} f(x) = \begin{bmatrix} 2x_1 + 2x_2 \\ 2x_1 + 6x_2 - 4 \end{bmatrix}$$

$$\boldsymbol{\nabla}^2 f(x) = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$$

$$2 \geq 0, \quad 6 \geq 0, \quad 2 \times 6 - 2 \times 2 = 8 \geq 0 \Longrightarrow f \text{ is convex.}$$

(b)

Let $\boldsymbol{\nabla} f(x) = 0$:

$$\boldsymbol{\nabla} f(x) = 0 \Longrightarrow \begin{cases} 2x_1 + 2x_2 = 0 \\ 2x_1 + 6x_2 = 4 \end{cases}$$

$x^* = (-1, 1)$

$-1 \leq 2 \Longrightarrow$ Doesn't satisfy $x_1 \geq 2$

(c)

$z^L = \min_{x\in R^2} x_1^2 + 3x_2 + 2x_1x_2 - 4x_2 - \lambda(2 - x_1) \quad$ where $\lambda \leq 0$

(d)

KKT condition :

Primal feasibility : $x_1 \geq 2$

Dual feasibility : $\lambda \leq 0$ and $\boldsymbol{\nabla} L(x_1, x_2|\lambda) = 0$

Complementary slackness : $\lambda(2 - x_1) = 0$

- $\lambda \leq 0 \Longrightarrow x_1 = 2$

  $\nabla L = 0 \Longrightarrow 2x_1 + 2x_2 = -\lambda \quad \text{and} \quad 2x_1 + 6x_2 = 4$

  $(x_1, x_2) = (2, 0) \quad \text{and} \quad \lambda = -4$

- $\lambda = 0 \Longrightarrow 2x_1 + 2x_2 = 0 \quad \text{and} \quad 2x_1 + 6x_2 = 4$

  $(x_1, x_2) = (-1, 1)$ which contradicts $x_1 \geq 2$

The optimal solution is $\bar{x} = (2, 0)$ with $z^* = 4$

5

Ridge regression :

First consider $i = 1, j = 1$ and collect terms with x_{11} or y_1

$f_1(\beta_0, \beta_1) = (y_1 - (\beta_0 + \beta_1 x_{11}))^2 + \alpha\beta_1^2 = y_1^2 - 2y_1\beta_0 - 2x_{11}y_1\beta_1 + \beta_0^2 + 2x_{11}\beta_0\beta_1 + x_{11}^2\beta_1^2 + \alpha\beta_1^2$

$\nabla f_1(\beta_0, \beta_1) = \begin{bmatrix} -2y_1 + 2\beta_0 + 2x_{11}\beta_1 \\ -2x_{11}y_1 + 2x_{11}\beta_0 + 2x_{11}^2 + 2\alpha\beta_1 \end{bmatrix}$

$\nabla^2 f_1(\beta_0, \beta_1) = \begin{bmatrix} 2 & 2x_{11} \\ 2x_{11} & 2x_{11}^2 + 2\alpha \end{bmatrix}$

$2 \geq 0, \quad 2x_{11}^2 + 2\alpha \geq 0, \quad 2(2x_{11}^2 + 2\alpha) - (2x_{11})^2 = 4\alpha \geq 0$

It means that f_1 is a convex program. As the summation of convex functions is also a convex function, the ridge regression is a convex program.

Lasso regression :

First consider $i = 1, j = 1$ and collect terms with x_{11} or y_1

If $\beta_1 \geq 0$

$f_1(\beta_0, \beta_1) = (y_1 - (\beta_0 + \beta_1 x_{11}))^2 + \alpha\beta_1 = y_1^2 - 2y_1\beta_0 - 2x_{11}y_1\beta_1 + \beta_0^2 + 2x_{11}\beta_0\beta_1 + x_{11}^2\beta_1^2 + \alpha\beta_1$

$\nabla f_1(\beta_0, \beta_1) = \begin{bmatrix} -2y_1 + 2\beta_0 + 2x_{11}\beta_1 \\ -2x_{11}y_1 + 2x_{11}\beta_0 + 2x_{11}^2 + \alpha \end{bmatrix}$

$\nabla^2 f_1(\beta_0, \beta_1) = \begin{bmatrix} 2 & 2x_{11} \\ 2x_{11} & 2x_{11}^2 \end{bmatrix}$

If $\beta_1 < 0$

$f_1(\beta_0, \beta_1) = (y_1 - (\beta_0 + \beta_1 x_{11}))^2 + \alpha\beta_1 = y_1^2 - 2y_1\beta_0 - 2x_{11}y_1\beta_1 + \beta_0^2 + 2x_{11}\beta_0\beta_1 + x_{11}^2\beta_1^2 - \alpha\beta_1$

$\nabla f_1(\beta_0, \beta_1) = \begin{bmatrix} -2y_1 + 2\beta_0 + 2x_{11}\beta_1 \\ -2x_{11}y_1 + 2x_{11}\beta_0 + 2x_{11}^2 - \alpha \end{bmatrix}$

$\nabla^2 f_1(\beta_0, \beta_1) = \begin{bmatrix} 2 & 2x_{11} \\ 2x_{11} & 2x_{11}^2 \end{bmatrix}$

No matter what the sigh of β_1 is , the second order derivative would be the same.

$2 \geq 0, \quad 2x_{11}^2 \geq 0, \quad 2(2x_{11}^2) - (2x_{11})^2 = 0 \geq 0$

Thus, f_1 is a convex program, which means the Lasso regression is a convex function because the summation of convex functions is also a convex funcion.