Moment-Matching Approximations for Asian Options

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On Moment-Matching Approximations for Asian Options

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Abstract

This study provides a generalized framework under which all types of Asian options can be priced, fixed and floating strike, forward-starting and in-progress. We not only extend the previous studies to our framework, but propose a new and theoretically supported closed-form approximation for the option prices. We utilize the moment-matching approach, providing a tractable, flexible, and efficient iterative method to calculate the moments. This study also suggests that the use of a Taylor expansion is unnecessary and exhibits the considerable improvement achieved by avoiding the truncation errors.

Key Words: Asian Option, Analytic Approximation, Floating strike, Moment Matching, Shifted Reciprocal Gamma.

JEL classification: G13

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1 Introduction

Asian options are common exotic derivatives in the commodity markets whose terminal payoff depends on some form of averaging of the underlying asset price over a part of or the whole of the option's life; they are also called *average options*. When the option payoff depends on the average of the underlying asset over a time interval, the option tends to be less expensive than its European counterpart. Furthermore, the averaging feature can lessen incentives for market manipulation, and the volatility of an average is lower than the volatility of the underlying asset, thus explaining their usage in risk management.

There has been discussion of Asian options in the literature for twenty years, and they and their average character are popular in practice. The Chicago Mercantile Exchange launched trading for three new cash-settled petroleum crack spread average price options contracts in July, 2009. These new average options are the gasoil-Brent crude oil crack spread options, the heating oil-crude oil crack spread options, and the RBOB-crude oil crack spread options. The London Metal Exchange (LME) also offers Traded Average Price Options based on the LME Monthly Average Settlement Price for several metals. The New York Mercantile Exchange and Intercontinental Exchange offers several average price options which are linked to energy products, e.g. Brent Average Price Options and WTI (West Texas Intermediate) Average Price Options. Other examples include commodity-linked bonds on average bond prices and Asian-style catastrophe (CAT) insurance options with payoffs depending on the accumulated catastrophic losses.[1]

Asian options come in numerous flavors. If the strike price is a fixed quantity, the option is referred to as a *fixed-strike* Asian option.[2] If instead the strike price is the asset price itself, the contract is called a *floating-strike* Asian option (see Exhibit 1).[3] If the starting

[1]See Bakshi and Madan (2002) and Chang et al. (2010).

[2]See also Turnbull and Wakeman (1991), Levy (1992), Rogers and Shi (1995), Milevsky and Posner (1998), Posner and Milevsky (1998), Zhang (2001), Thompson (2000), Ju (2002), and Nielsen and Sandmann (2003).

[3]See also Levy (1992), Rogers and Shi (1995), Bouaziz et al. (1994), Thompson (2000), Chung et al.

date of averaging is in the future, the option is *forward-starting*. If instead the averaging period has already begun, the option is *in-progress*. In particular, if the averaging period is exactly the time to maturity, the option is *plain-vanilla* (see Exhibit 2).

Exhibit 1
Payoff Structures of European and Asian Options

The symbol S_T denotes the price of underlying asset at maturity; K denotes a fixed strike price; A_T denotes some form of averaging of the underlying asset price.

Type	Call	Put
European Options	$(S_T - K)^+$	$(K - S_T)^+$
Fixed-Strike Asian Options	$(A_T - K)^+$	$(K - A_T)^+$
Floating-Strike Asian Options	$(S_T - A_T)^+$	$(A_T - S_T)^+$

Exhibit 2
Three Types of Averaging Period for Asian Options

The symbol t denotes the current time; t_0 denotes the starting date of averaging; t_1 denotes the ending date of averaging; and T denotes the option maturity.

Type	Definition
Plain-vanilla	$t = t_0 < t_1 = T$
Forward-starting	$t \leq t_0 < t_1 = T$
In-progress	$t_0 \leq t < t_1 = T$

Our first contribution is the integration of all the types of Asian options described above into a general framework. In view of the contributions of Henderson and Wojakowski (2002), Henderson et al. (2007) and others such as Eberlein et al. (2008), it is commonly thought that the floating strike case can always be reduced to the fixed strike case. However, Henderson and Wojakowski (2002) derive symmetries between floating-strike and fixed-strike Asian options only in the plain-vanilla case, and Henderson et al. (2007) mention that the symmetry

(2003), Tsao et al. (2003), Henderson et al. (2007), and Chang and Tsao (2011).

does not hold for the in-progress case.[4] This means that the floating strike in-progress case must be considered separately. In fact, what we do here is first derive a general formula (Theorem 2) for the floating-strike case which, using the symmetries just mentioned, also covers the forward-starting fixed strike case (Corollary 4). The in-progress fixed strike case must be considered separately as it cannot be reduced to the floating-strike case (Corollary 3).

The pricing of arithmetic average Asian options as considered here is analytically intractable because the distribution of the underlying variable is unknown. However it turns out that the moments of the variable can be easily calculated. Perhaps for this reason moment-matching has been commonly used to approximate the price of Asian options.[5] The basic idea here is to assume the underlying variable has a distribution depending on a certain number m of parameters and determine the parameters, hopefully by means of a closed form formula, so that the first m moments of the distribution coincide with those of the variable. The hope then is to also determine a closed form approximation for the option price. An advantage of this approach is that the price can be calculated practically instantaneously. The only issue is the accuracy.

The seminal work Levy (1992) proposes a two-parameter lognormal approximation. Later Bouaziz et al. (1994) proposed a two-parameter normal approximation, Posner and Milevsky (1998) propose four-parameter shifted-lognormal and arcsinh-normal approximations, and Chang and Tsao (2011) propose a three-parameter central chi-squared approximation.[6]

[4]Regarding the forward-starting case, Henderson et al. (2007) and Eberlein et al. (2008) show that a forward-starting floating-strike (resp. fixed-strike) Asian call option has the same price as a plain-vanilla fixed-strike (resp. floating-strike) Asian put option with interest rate and dividend yield reversing roles and a modified averaging period.

[5]Other methods in the literature include the Monte Carlo simulation method (e.g. Kemna and Vorst (1990)), tree and lattice techniques (e.g. Hull and White (1993)), Fourier and Laplace transform methods (e.g. Geman and Yor (1993)), and lower and upper bounds (e.g. Curran (1994), Rogers and Shi (1995), Thompson (2000), and Nielsen and Sandmann (2003)). However, our focus here is the moment-matching approach.

[6]Lam and Le-Ngoc (2007) and Nie and Chen (2007) argue that the log-shifted gamma and the Type-IV Pearson distributions approximate the sum of lognormal distributions well, but it does not seem possible to

Milevsky and Posner (1998) show that the probability density function of the infinite sum of correlated lognormal random variables is reciprocal gamma distributed and derive a two-parameter reciprocal gamma approximation. However, many studies give evidence that the accuracy of the reciprocal gamma approximation is worse than that of other *ad hoc* distributions. One would expect that matching more moments will lead to better accuracy. *Our second contribution is to use a three-parameter shifted reciprocal gamma approximation by adding a shift parameter.* We are able to find closed-form formulas for the parameters in terms of the moments and then use these to derive closed-form expressions for the option prices which yield, as expected, more precise approximations to the prices.[7] We also extend Levy (1992), Bouaziz et al. (1994), Posner and Milevsky (1998), and Chang and Tsao (2011), as mentioned above, to our generalized framework and price all flavors of Asian options.

Calculating the moments of the sum of lognormal random variables is the first step in applying the moment-matching approach. For fixed-strike Asian options, the moments are easy to obtain.[8] However, for floating-strike Asian options, the moments are not so straightforward to derive, especially the higher order moments. *Our third contribution is to give a simple recursive formula to calculate all the moments.* Comparing with those articles using three or four parameters, we note that Posner and Milevsky (1998) do not cover floating-strike Asian options and that Chang and Tsao (2011) need twenty-eight lemmas in order to obtain only the third moment.[9]

Our fourth contribution is to show that the use of Taylor expansions in a certain branch of the literature is unnecessary. Bouaziz et al. (1994) derive analytical expansions for the prices

derive closed-form formulas for the option prices using these distributions.

[7]Note that with the four-parameter arcsinh-normal and Type-IV Pearson distributions, and three-parameter log-shifted gamma distribution, one can not obtain closed-form solutions for the parameters in terms of the moments.

[8]See also Levy (1992). The moments of the underlying asset for fixed-strike Asian options can be found from the moment-generating function of the normal distribution. Fusai and Tagliani (2002) also provide many types of the moments to approximate fixed-strike Asian options.

[9]Let us remark here that Geman and Yor (1993) derive a formula for all the moments of the average.

of floating-strike Asian options using a first order Taylor approximation to the underlying variable. Their linearized model is simple but only accurate for low volatility, low interest rate, and short averaging period due to large truncation errors. To remedy this, Chung et al. (2003), Tsao et al. (2003) and Chang and Tsao (2011) add the second-order term in the Taylor expansion.[10] Their quadratic approximations reduce part of the truncation errors in Bouaziz et al. (1994); However, our recursive formula finds the exact moments, thus avoiding all the truncation errors arising from the Taylor approximation.

The remainder of this paper is organized as follows. In Section 2 we state the assumptions and set up our general pricing model for Asian options. In Section 3 we describe a recursive method for calculating the moments and find closed-form expressions for the option prices using the different approximations described above. Section 4 gives numerical comparisons of our method and demonstrates the substantial improvement over the truncated models. Section 5 concludes the paper. All proofs are in the Appendices.

2 The Model

Working in the standard Black-Scholes (1973) framework, we assume that trading takes place continuously, markets are perfectly competitive with no taxes and no transaction costs, there exists a risk-free asset paying a continuous flow at rate $r > 0$ per unit of time, and the underlying asset pays a known continuous dividend yield q.[11]

Under the risk-neutral measure $\mathbb{Q}$, the price of the underlying asset is generated by a geometric Brownian motion according to the stochastic differential equation:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t, \tag{1}$$

where $\sigma > 0$ denotes the constant volatility and W_t is a standard Wiener process under $\mathbb{Q}$.

[10] Unfortunately, evaluating the expectation of the quadratic approximation is as difficult as the original problem of approximating the sum of lognormal distributions. To handle this they assume the unknown distribution is normal or central chi-squared, and use moment-matching to determine the mean and variance.

[11] In this study, we assume $r \neq q$ for convenience.

The intertemporal uncertainty is captured by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where Ω consists of all possible states of the world which could exist at the maturity of the option T, $\mathcal{F}$ is a $\sigma-$ field of subsets of Ω, $\mathbb{F}$ is a filtration and $\mathbb{P}$ a probability measure. The filtration $\mathbb{F}$ is given by $\mathbb{F} = \{\mathcal{F}_t | t \in [0, T]\}$ which is right-continuous and such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. For an initial condition $S_{t_0} > 0$, Ito's lemma implies that equation (1) has the solution:

$$S_t = S_{t_0} \exp\left\{(r - q - \sigma^2/2)(t - t_0) + \sigma(W_t - W_{t_0})\right\}. \tag{2}$$

An Asian option contract is to be priced at time t and expires at maturity $T > t$ and its payoff depends on the quantity $A(t_0, T)$ where $\{A(t_0, t_1)\}_{t_1 > t_0}$ is the arithmetic average:

$$A(t_0, t_1) \equiv \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} S_u du \ \times I_{\{t_1 > t_0\}}. \tag{3}$$

where t_0 denotes the starting date of averaging, t_1 denotes the ending date of averaging, and $I_{\{\cdot\}}$ denotes the indicator function.

Let $D \equiv T - t_0$ denote the duration of the averaging. If $t_0 = t$, then the option is called a *plain-vanilla Asian option*; if $t_0 > t$, it is called *forward-starting* and if $t_0 < t$, it is called *in-progress* as the averaging has already begun. In this section, we derive general formulas for all cases.

Definition 1. *Define*

$$\begin{aligned}
AC_t^{fx}(K; r, q; t_0, t_1) &= e^{-r(T-t)}\mathbb{E}\Big[\Big(A(t_0, t_1) - K\Big)^+ \Big| \mathcal{F}_t\Big], \\
AP_t^{fx}(K; r, q; t_0, t_1) &= e^{-r(T-t)}\mathbb{E}\Big[\Big(K - A(t_0, t_1)\Big)^+ \Big| \mathcal{F}_t\Big], \\
AC_t^{fl}(\beta; r, q; t_0, t_1) &= e^{-r(T-t)}\mathbb{E}\Big[\Big(\beta S_T - A(t_0, t_1)\Big)^+ \Big| \mathcal{F}_t\Big], \\
AP_t^{fl}(\beta; r, q; t_0, t_1) &= e^{-r(T-t)}\mathbb{E}\Big[\Big(A(t_0, t_1) - \beta S_T\Big)^+ \Big| \mathcal{F}_t\Big].
\end{aligned}$$

where $\mathbb{E}[\cdot]$ denotes the risk-neutral expectation under the martingale measure $\mathbb{Q}$, K denotes the strike price for fixed-strike cases, $\beta > 0$ denotes a multiplier for floating-strike cases, and $t_0 < t_1 \leq T$.

Remark 1: Note that the payoffs above are called Asian options only if $\beta = 1$ and $t_1 = T$ and the formulas give the time t prices of the Asian call and put with fixed strike K and Asian call and put with floating strike respectively. Nevertheless, the cases $\beta \neq 1$ and $t_1 < T$ are needed below in Corollary 4.

Remark 2: Without loss of generality, this study only deals with Asian call options since the corresponding put option prices can be derived from Asian put-call parity:[12]

$$\begin{aligned} & AP_t^{fx}(K; r, q; t_0, t_1) + S_t e^{-r(T-t)} \left[\frac{e^{(r-q)(t_1-t)} - e^{(r-q)(t_*-t)}}{(r-q)(t_1-t_0)} + \frac{t_* - t_0}{t_1 - t_0} \frac{A(t_0, t_*)}{S_t} \right] \\ = \; & AC_t^{fx}(K; r, q; t_0, t_1) + K e^{-r(T-t)}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} & AP_t^{fl}(\beta; r, q; t_0, t_1) + \beta S_t e^{-q(T-t)} \\ = \; & AC_t^{fl}(\beta; r, q; t_0, t_1) + S_t e^{-r(T-t)} \left[\frac{e^{(r-q)(t_1-t)} - e^{(r-q)(t_*-t)}}{(r-q)(t_1-t_0)} + \frac{t_* - t_0}{t_1 - t_0} \frac{A(t_0, t_*)}{S_t} \right], \end{aligned} \tag{5}$$

where $t_* \equiv \max\{t, t_0\}$.[13]

First we show that the prices of floating-strike options can be written in terms of the expectation of a certain variable. Corollary 3 shows there is a similar formula for the prices of in-progress fixed-strike Asian call options. Corollary 4 shows that the prices of forward-starting fixed-strike options follow from the floating-strike case with a different averaging period.

Theorem 2. *The price of a floating-strike Asian call option (both forward-starting and in-progress) at time t is $AC_t^{fl}(1; r, q; t_0, T)$, which is a special case of:*

$$AC_t^{fl}(\beta; r, q; t_0, t_1) = S_t e^{-r(T-t)+(r-q)(t_*-t)} \mathbb{E}\left[\left(\xi_t(t_0, t_1; \beta) - K_t(t_0, t_1) \right)^+ \Big| \mathcal{F}_t \right], \tag{6}$$

[12] Bouaziz et al. (1994) and Tsao et al. (2003) derive Asian put-call parity for the floating-strike case, and we can easily extend it to the fixed-strike case.

[13] If $r = q$, (4) and (5) can be easily rewritten as their limits as $(r - q)$ tends to zero.

where

$$\xi_t(t_0, t_1; \beta) \equiv \beta \frac{S_T}{S_{t_*}} - \frac{t_1 - t_*}{t_1 - t_0} \frac{A(t_*, t_1)}{S_{t_*}}, \tag{7}$$

$$K_t(t_0, t_1) \equiv \frac{t_* - t_0}{t_1 - t_0} \frac{A(t_0, t_*)}{S_t}. \tag{8}$$

Proof: See Appendix A.

Corollary 3. *The price of an in-progress fixed-strike Asian call option at time t is:*

$$AC_t^{fx}(K; r, q; t_0, T) = S_t e^{-r(T-t)} \mathbb{E}\Big[\big(-\xi_t(t_0, T; 0) - \bar{K}_t\big)^+ \Big| \mathcal{F}_t\Big], \tag{9}$$

where $\bar{K}_t \equiv K/S_t - K_t(t_0, T)$ and $t > t_0$.

Proof: See Appendix A.

Corollary 4. *The price of a forward-starting fixed-strike Asian call option at time t is:*

$$\begin{aligned} AC_t^{fx}(K; r, q; t_0, T) &= AC_t^{fl}(K/S_t; q, r; t, t + T - t_0) \\ &\quad - K e^{-r(T-t)} + S_t e^{-q(T-t)} \left[\frac{1 - e^{-(r-q)(T-t_0)}}{(r-q)(T-t_0)}\right]. \end{aligned} \tag{10}$$

Proof: See Appendix A.[14]

Unfortunately the distribution of the variable $\xi_t(t_0; t_1; \beta)$ is unknown. In the next section we assume $\xi_t(t_0, t_1; \beta)$ has a certain distribution and determine the parameters in the distribution by matching moments.

3 Analytic Approximations

In this section our chief contributions are to (i) show that the moments of the underlying variable (7) can be calculated by a recursive formula, (ii) demonstrate that the Taylor series approximation used for floating strike options is unnecessary, (iii) observe that there are only three independent parameters in the four-parameter shifted lognormal distribution used by

[14] If $r = q$, (10) can be easily rewritten as its limit as $(r - q)$ tends to zero.

Posner and Milevsky (1998), and (iv) approximate Asian option prices by applying the moment matching approach using a three-parameter shifted reciprocal gamma approximation, obtained by adding a shift parameter to the reciprocal gamma distribution. This distribution has not been employed before in this context. Note we can restrict consideration to floating-strike options and in-progress fixed-strike options, their prices being given by the formulas (6) and (9) involving the variable $\xi_t(t_0, t_1; \beta)$, as the price of the forward-starting fixed-strike options (10) can be derived from (6).

The idea is to approximate the unknown distribution of $\xi_t(t_0, t_1; \beta)$ by one of the distributions mentioned above and determine the parameters in the distribution by matching moments. In Section 3.1 we show that the moments of $\xi_t(t_0, t_1; \beta)$ can be calculated by a recursive formula. Next we choose the distribution, derive formulas for the parameters in terms of the moments and then the approximation formula for the option price. *Note in all cases we find closed-form formulas for both the parameters and the option prices.* In Section 3.2, we employ the two parameter normal approximation. In Sections 3.3, 3.4 and 3.5, we employ the three-parameter distributions - the shifted gamma (for comparison with the results of Chang and Tsao (2011)), the shifted lognormal (essentially equivalent to the four-parameter shifted lognormal distribution in Posner and Milevsky (1998)), and the shifted reciprocal gamma.

3.1 The Moments

For fixed-strike Asian options the moments of $\xi_t(t_0, T; 0)$ can be easily obtained from the moment-generating function of the normal distribution. However, for floating-strike Asian options, it is not so straightforward to derive the moments of the variable $\xi_t(t_0, T; 1)$. In Proposition 5 we give a simple recursive method to calculate all the moments of $\xi_t(t_0, t_1; \beta)$.

Proposition 5. *If j is a positive integer, the j-th moment of $\xi_t(t_0, t_1; \beta)$ in (7) is given by:*

$$\mathbb{E}\Big[\Big(\xi_t(t_0,t_1;\beta)\Big)^j\Big] = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \frac{i!\beta^{j-i}}{(t_1-t_0)^i} e^{(j-i)\psi(i-1)(T-t_*)} G_i(\alpha_1, ..., \alpha_i), \tag{11}$$

where $\psi(k) \equiv r - q + \frac{k}{2}\sigma^2$, $\alpha_i \equiv \psi\big(2(j-i)\big)$ for $i = 1, ..., j$, and the functions G_i can be found recursively according to

$$G_i(x_1, x_2, ..., x_i) = \frac{1}{x_1}\Big[G_{i-1}(x_1 + x_2, x_3, ..., x_i) - G_{i-1}(x_2, x_3, ..., x_i)\Big], \tag{12}$$

starting with

$$G_1(x_1) = \frac{\exp\big(x_1(t_1 - t_*)\big) - 1}{x_1}. \tag{13}$$

In particular, the mean, variance, skewness, and kurtosis of $\xi_t(t_0, t_1; \beta)$ are:

$$\begin{aligned}
m_* &= \mathbb{E}\Big[\big(\xi_t(t_0,t_1;\beta)\big)\Big], \\
v_* &= \mathbb{E}\Big[\big(\xi_t(t_0,t_1;\beta)\big)^2\Big] - m_*^2, \\
s_* &= \Big[\mathbb{E}\Big[\big(\xi_t(t_0,t_1;\beta)\big)^3\Big] - 3m_*v_* - m_*^3\Big]\Big/ v_*^{3/2}, \\
k_* &= \Big[\mathbb{E}\Big[\big(\xi_t(t_0,t_1;\beta)\big)^4\Big] - 4m_*v_*^{3/2}s_* - 6m_*^2v_* - m_*^4\Big]\Big/ v_*^2.
\end{aligned}$$

Proof: See Appendix B, where we also give the explicit formulas for the first three moments.

Remark 3: For in-progress fixed-strike Asian options (Corollary 3), (11) reduces to

$$\mathbb{E}\big[\big(-\xi_t(t_0,t_1;0)\big)^j\big] = \frac{j!}{(t_1-t_0)^j} G_j(\alpha_1, ..., \alpha_j).$$

3.2 Normal Approximation

Bouaziz et al. (1994), Chung et al. (2003) and Tsao et al. (2003) derive analytical approximations via Taylor expansion of the underlying variable, with different truncation errors. In the case of Bouaziz et al. (1994), their linear approximation is normally distributed. Chung et al. (2003) and Tsao et al. (2003) use a quadratic approximation to reduce the truncation errors in Bouaziz et al. (1994) but, unfortunately, their quadratic approximation is not normally distributed. For convenience they still assume the quadratic approximation is normally distributed, resulting in additional distribution errors.

In this subsection, we do not approximate the variable but assume that it is normally distributed. The two parameters, the risk-neutral mean and variance of the variable $\xi_t(t_0, t_1; \beta)$, are calculated exactly using Proposition 5. Using Theorem 2 and Corollary 3, we price the Asian option as follows.

Proposition 6. *The approximate price of the Asian call option given by the normal distribution is:*

$$AC_t^N = S_t e^{-r(T-t)+(r-q)(t_*-t)} \left[(m_* - K_*) N\left(\frac{m_* - K_*}{\sqrt{v_*}} \right) + \sqrt{\frac{v_*}{2\pi}} \exp\left\{ -\frac{(m_* - K_*)^2}{2v_*} \right\} \right], \tag{14}$$

where $N(\cdot)$ denotes the standard normal cumulative distribution function, and

$$K_* = \begin{cases} K_t(t_0, T) & \text{for floating-strike Asian options,} \\ \bar{K}_t & \text{for in-progress fixed-strike Asian options.} \end{cases} \tag{15}$$

Proof: See Appendix C.

3.3 Shifted Gamma Approximation

Chang and Tsao (2011) propose a three-parameter central chi-squared distribution to approximate the sum of lognormal distributions.[15] In fact, an exponential distribution, a central chi-squared distribution, and an Erlang distribution are all special cases of the gamma distribution. In this subsection we reformulate the central chi-squared approximation of Chang and Tsao (2011) as an equivalent shifted gamma approximation to our generalized framework, and we apply moment-matching to the untruncated variable, thus avoiding their truncation errors.

[15] Chang et al. (2011) extend Chang and Tsao (2011) using the same distribution and truncation to approximate the prices of forward-starting floating-strike quanto Asian options.

A shifted gamma distributed random variable Z has the probability density function

$$f(z;\alpha,\beta,\eta) = \frac{(z-\eta)^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} \exp\Big\{-\frac{z-\eta}{\beta}\Big\}, \qquad z>\eta,\ \beta>0, \tag{16}$$

with the parameters α, β, and η, where $\Gamma(\cdot)$ denotes the Gamma function

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt. \tag{17}$$

The mean, variance, and skewness are

$$m_* = \eta + \alpha\beta, \qquad v_* = \alpha\beta^2, \qquad s_* = \frac{2}{\sqrt{\alpha}}. \tag{18}$$

First we solve for the parameters in terms of the moments. The solution of the non-linear system (18) for the parameters α, β, and η in terms of m_*, v_*, and s_* is:

$$\alpha = \frac{4}{s_*^2}, \qquad \beta = \sqrt{\frac{v_*}{\alpha}}, \qquad \eta = m_* - \alpha\beta. \tag{19}$$

The risk-neutral mean, variance, and skewness of the function $\xi_t(t_0,t_1;\beta)$ are calculated using Proposition 5. Using Theorem 2 and Corollary 3, we can price the Asian option as follows.

Proposition 7. *The approximate price of the Asian call option given by the shifted gamma distribution is:*

$$AC_t^{SG} = S_t e^{-r(T-t)+(r-q)(t_*-t)}\left[(m_*-\eta)\left(1-\frac{\Gamma_{\omega_1}(\alpha+1)}{\Gamma(\alpha+1)}\right) - (K_*-\eta)\left(1-\frac{\Gamma_{\omega_1}(\alpha)}{\Gamma(\alpha)}\right)\right] \tag{20}$$

where K_* as in (15), $\omega_1 = (K_*-\eta)/\beta$, and $\Gamma_x(\cdot)$ is the incomplete gamma function

$$\Gamma_x(\alpha) = \int_x^{\infty} t^{\alpha-1}e^{-t}dt. \tag{21}$$

Proof: See Appendix C.[16]

[16]Note that taking $f = 2\alpha$, $k_1 = \eta$ and $k_2 = \beta/2$ reduces (20) to formulas (16) and (18) of Chang and Tsao (2011) but without their truncated moments.

3.4 Shifted Lognormal Approximation

Posner and Milevsky (1998) use a four-parameter shifted lognormal distribution to fit the sum of lognormal distributions, but we observe here that there are only three independent parameters, as Johnson (1949) had already pointed out. In this subsection we use a three-parameter shifted lognormal distribution, essentially equivalent to Posner and Milevsky's, and derive the closed-form expression for the relation between the moments and the parameters.[17]

A shifted lognormally distributed random variable Z has the probability density function

$$f(z;\mu,\sigma,\eta) = \frac{1}{\sigma(z-\eta)\sqrt{2\pi}} \exp\left\{-\frac{\big(\ln(z-\eta)-\mu\big)^2}{2\sigma^2}\right\}, \qquad z > \eta, \tag{22}$$

with parameters μ, σ, and η. The mean, variance, and skewness are

$$m_* = \eta + e^{\mu+\frac{1}{2}\sigma^2}, \qquad v_* = e^{2\mu+\sigma^2}\big(e^{\sigma^2}-1\big), \qquad s_* = \big(e^{\sigma^2}+2\big)\sqrt{e^{\sigma^2}-1}. \tag{23}$$

To price the options, we follow the same procedure as in Section 3.3.

Lemma 8. *The solution of the non-linear system (23) for the parameters μ, σ^2, and η in terms of m_*, v_*, and s_* is:*

$$\mu = \ln(m_*-\eta) - \frac{\sigma^2}{2}, \quad \sigma^2 = \ln\left|1+\frac{v_*}{(m_*-\eta)^2}\right|, \quad \eta = m_* - \frac{\sqrt{v_*}}{s_*}\left[1+(B)^{\frac{1}{3}}+(B)^{-\frac{1}{3}}\right], \tag{24}$$

where $B \equiv \frac{1}{2}\Big(s_*^2+2-\sqrt{s_*^4+4s_*^2}\Big) \in (0,1]$.

Proof: See Appendix C.

Proposition 9. *The approximate price of the Asian call option given by the shifted lognormal distribution is:*

$$AC_t^{SL} = S_t e^{-r(T-t)+(r-q)(t_*-t)}\Big[(m_*-\eta)N(d_+) - (K_*-\eta)N(d_-)\Big], \tag{25}$$

where $N(\cdot)$ and K_* are defined as in (15), and $d_\pm = \left[\ln\left|\frac{m_*-\eta}{K_*-\eta}\right| \pm \frac{\sigma^2}{2}\right]\Big/\sigma$.

[17] Borovkova, Permana, and Weide (2007) also propose the shifted lognormal approximation for basket and spread options, but they did not derive the closed-form relationship between the moments and the parameters.

Proof: See Appendix C.

Remark 4: Setting $\eta = 0$, (25) reduces to the lognormal approximation of Levy (1992).

3.5 Shifted Reciprocal Gamma Approximation

This subsection extends the work of Milevsky and Posner (1998) by adding a shift parameter and uses the resultant shifted reciprocal gamma distribution to be the new approximation for the sum of lognormals. To the best of our knowledge, this is the first study that uses the shifted reciprocal gamma distribution as an approximation in this way. The additional shift parameter captures the skewness and improves the accuracy.

A shifted reciprocal gamma distributed random variable Z has the probability density function

$$f(z;\alpha,\beta,\eta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{z-\eta}\right)^{\alpha-1}\exp\left\{-\frac{\beta}{z-\eta}\right\}, \qquad z > \eta,\ \beta > 0, \tag{26}$$

with the parameters α, β, and η. The mean, variance, and skewness of the shifted reciprocal gamma distribution are

$$m_* = \eta + \frac{\beta}{\alpha-1}, \qquad \alpha > 1; \tag{27}$$

$$v_* = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}, \qquad \alpha > 2; \tag{28}$$

$$s_* = \frac{4\sqrt{\alpha-2}}{\alpha-3}, \qquad \alpha > 3. \tag{29}$$

To price the options, we follow the same procedure as in Section 3.3.

Lemma 10. *The solution of the non-linear system (27), (28), and (29) for the parameters α, β, and η in terms of the m_*, v_*, and s_* is given by:*

$$\eta = m_* - \frac{\sqrt{v_*}}{s_*}\left[2 + \sqrt{4 + s_*^2}\right], \tag{30}$$

$$\alpha = 2 + \frac{(m_* - \eta)^2}{v_*}, \tag{31}$$

$$\beta = (m_* - \eta)(\alpha - 1). \tag{32}$$

Proof: See Appendix C.

Proposition 11. *The approximate price of the Asian call option given by the shifted reciprocal gamma distribution is:*

$$AC_t^{SRG} = S_t e^{-r(T-t)+(r-q)(t_*-t)} \left[(m_* - \eta) \frac{\Gamma_{\omega_2}(\alpha - 1)}{\Gamma(\alpha - 1)} - (K_* - \eta) \frac{\Gamma_{\omega_2}(\alpha)}{\Gamma(\alpha)} \right] \tag{33}$$

where K_* as in (15) and $\omega_2 = \beta/(K_* - \eta)$.

Proof: See Appendix C.

Remark 5: Setting $\eta = 0$ and $K_* = \bar{K}_t$, (33) reduces to the reciprocal gamma approximation of Milevsky and Posner (1998).

4 Numerical Comparisons

This section examines the performance of the approximations described in Section 3. The approximate distributions used have the property that the first two or three moments coincide with those of the unknown distribution. So first we examine how well the approximate distributions fit the unknown distribution of $\xi_t(t_0, t_1; \beta)$ by comparing the third and the fourth moments. Then we investigate the pricing errors in the Asian option prices calculated under the different distributions. In order to compare with previous studies, we only consider the plain-vanilla fixed-strike and in-progress floating-strike Asian options as no results are available for other flavors. The computation is almost instantaneous as see from the average computing time obtained using a Matlab package in Microsoft Windows XP, running on a machine equipped with Intel Pentium Dual CPU T2390 with 1.86GHz CPU and 3.00GB RAM. Notice that the computing time may differ with different hardware and software environments.

4.1 Higher Order Moments

Exhibit 3 shows the skewness and kurtosis under six distributions, the normal (N) as in Section 3.2,[18] shifted gamma (SG) as in Section 3.3,[19] lognormal (LN) as in Levy (1992),[20] shifted lognormal (SL) as in Section 3.4,[21] reciprocal gamma (RG) as in Milevsky and Posner (1998),[22] and shifted reciprocal gamma (SRG) as in Section 3.5, respectively.

Panel A displays the fixed-strike case and Panel B displays the floating-strike case.

First we consider the two-parameter approximations N, LN and RG. Consistent with Bouaziz et al. (1994), the normal distribution performs worse in the case of longer maturity and higher volatility since it cannot capture the skewness and kurtosis. The lognormal distribution fits well in the fixed-strike case, but is the worst of all the distributions in the floating-strike case. This is because the random variable $\xi_t(t_0, T; 1)$ in the floating-strike case may have negative values but the lognormal assumption forces $\xi_t(t_0, T; 1)$ to be positive. The reciprocal gamma fits well in the fixed-strike case but gives negative skewness and very small kurtosis in the floating-strike case. These results also indicate that with two parameters it is more difficult to approximate the unknown distribution in the floating-strike case than in the fixed-strike case.

Next we turn to the three-parameter approximations. Exhibit 3 shows, as expected, that SL, SG and SRG all fit the skewness exactly in both the fixed-strike and floating-strike cases. In regard to the kurtosis, SL underestimates the true value, the underestimation being worse

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[18]Bouaziz et al. (1994) derive an analytic approximation for floating-strike Asian option prices under the normal distribution; the difference is that the moments here do not contain the truncation errors.

[19]Chang and Tsao (2011) propose an analytic approximation for floating-strike Asian option prices under the shifted central chi-squared distribution; the difference is that the moments here do not contain the truncation errors.

[20]Levy(1992) propose an analytic approximation for fixed-strike Asian option prices under the lognormal distribution; the difference is that he considers the discrete average.

[21]Posner and Milevsky (1998) propose an analytic approximation for forward-starting fixed-strike Asian option prices under the shifted lognormal distribution; the difference is that one of their parameters is redundant.

[22]Milevsky and Posner (1998) propose an analytic approximation for forward-starting fixed-strike Asian option prices under the reciprocal gamma distribution.

than with LN in the fixed-strike case. However, both SG and SRG capture the kurtosis well and SRG performs best across maturities and volatilities among all distributions in Exhibit 3. In particular, we observe that SL does much better than LN as the shift parameter enables us to allow negative values and thus reduce the errors.

<< Insert Exhibit 3 >>

4.2 Fixed-Strike Asian Options

In regard to the accuracy of the Asian option prices produced by the different approximating distributions, Exhibit 4 and Exhibit 5 show the prices of plain-vanilla fixed-strike Asian options for 0.08 and 1 year, respectively. The parameters and benchmark values are from Zhang (2001). We observe by comparing the root mean squared errors (RMSE) and the maximum absolute errors (MAE) that SL and SRG approximate the benchmark values significantly better than others. For example, for $T = 0.08$, the RMSEs of SL and SRG are only 0.0002, and MAEs of them are 0.0005 and 0.0006, respectively. On the other hand, the RMSE and MAE of RG are 0.0041 and 0.0114. It follows that adding the shifted parameter can reduces the pricing errors.

<< Insert Exhibit 4 >>

<< Insert Exhibit 5 >>

4.3 Floating-Strike Asian Options

Exhibit 6 and Exhibit 7 show the prices of in-progress floating-strike Asian options. The parameters and benchmark values (MC) are obtained from Henderson et al. (2007) using the Monte Carlo control variate method with 100,000 simulated paths and 3,000 sampling points. We also examine whether the approximate prices stay within the upper bounds derived by Henderson et al. (2007).

As in the fixed-strike case, SL and SRG give more accurate prices of floating-strike Asian options than the other four. For example, for $\sigma = 30\%$, the RMSEs of SL and SRG are only 0.0058 and 0.0057, and MAEs of them are 0.0155 and 0.0117, respectively. In particular, SL gives accurate prices for small time to maturity (e.g. $t = 0.9$). It is worth emphasizing that SL in Posner and Milevsky (1998) is only used to price fixed-strike Asian options, but this study also uses SL to price floating-strike Asian options. On the other hand, the approximate option prices found with LN and RG are far away from the benchmark values. It was already clear from Exhibit 3 that LN and RG are not suitable for floating-strike cases. We also observe that most prices calculated using N exceed the upper bound.

<< Insert Exhibit 6 >>

<< Insert Exhibit 7 >>

4.4 Truncation Errors

To price floating-strike Asian options, some studies use Taylor expansion to approximate the underlying variable. We have suggested above that the use of a Taylor expansion is unnecessary. In Exhibit 8 we compare our non-truncated approximations to the Taylor approximations and demonstrate significant improvements. The parameters and benchmark values (MC) are as in Exhibit 6 and Exhibit 7 obtained from Henderson et al. (2007). Panel A displays the case $\sigma = 30\%$ and Panel B displays the case $\sigma = 50\%$.

<< Insert Exhibit 8 >>

First, we compare the non-truncated normal approximation N with the linear approximation of Bouaziz et al. (1994). Note the linear approximation to the underlying asset is exactly normally distributed under their model. So their pricing error is due to truncation error only but, on the other hand, our error is due only to distribution error. Exhibit 8 shows that the truncation errors are more crucial than the distribution errors. For example, the RMSEs of N and Bouaziz et al. (1994) are 0.3283 and 1.1793 for $\sigma = 30\%$, respectively.

Next, we compare the non-truncated shifted gamma approximation SG with the quadratic approximation of Chang and Tsao (2011) who use the shifted central chi-squared distribution as a proxy. Their quadratic approximation reduces the truncation errors but introduces distribution error. Exhibit 8 demonstrates that the RMSEs and MAEs of SG are significantly smaller than those using the method of Chang and Tsao (2011).

In short, our non-truncated approximations are more accurate. However, even though N and SG do better than the truncations, we see from the previous Exhibits that they are less accurate for pricing Asian options than some of the other methods.

5 Conclusion

Up to the present time most of the literature on Asian options deals with fixed-strike Asian options. In this paper we give a generalized framework under which all types (fixed- and floating-strike, forward-starting and in-progress) of Asian options can be priced. This generalized framework also overcomes the difficulty in the in-progress case that the fixed-floating symmetry does not hold.

As is well-known, pricing arithmetic average Asian options is analytically intractable under the Black-Scholes economy. In this paper we use moment-matching to derive several closed-form approximations. We not only extend the previous studies to our generalized framework, but propose a new and theoretically supported shifted reciprocal gamma approximation. The numerical comparisons show that the shifted lognormal and the shifted reciprocal gamma approximations are more accurate than previous approximations used in the literature. In calculating the moments, we suggest that the use of a Taylor expansion, as employed by some authors, is unnecessary and provide a simple recursive method to calculates moments of all orders. The numerical results also indicate that the improvement obtained by avoiding truncation errors is considerable. Moreover, our recursive formula is potentially useful and easily extendable if higher moments are required.

Our approximations are simple, accurate, and easy to implement. The moment-matching approach could be applied to other financial derivatives involving unknown distributions. For example, our approximations could be employed to price basket options and quanto Asian options. Our closed-form approximations should also be useful for pricing embedded Asian-style options such as CAT-linked securities and guaranteed minimum withdrawal annuities, etc..

Appendix A

Proof of Theorem 2 : For the forward-starting case, $t < t_0$, $t_* = t_0$, and $K_t(t_0, t_1) = 0$, so that

$$\begin{aligned} AC_t^{fl}(\beta; r, q; t_0, t_1) &= e^{-r(T-t)}\mathbb{E}\Big[\Big(\beta S_T - A(t_0, t_1)\Big)^+\Big|\mathcal{F}_t\Big] && \text{(A.1)} \\ &= e^{-r(T-t)}\mathbb{E}\Big[S_{t_0}\mathbb{E}\Big[\big(\xi_t(t_0, t_1; \beta)\big)^+\big|\mathcal{F}_{t_0}\Big]\Big|\mathcal{F}_t\Big] && \text{(A.2)} \end{aligned}$$

by the tower property of conditional expectation. Furthermore, $\xi_t(t_0, t_1; \beta)$ is independent of $\mathcal{F}_{t_0}$ since $W_u - W_{t_0}$ is independent of $\mathcal{F}_{t_0}$ for all $u \geq t_0$. It follows that:

$$\begin{aligned} AC_t^{fl}(\beta; r, q; t_0, t_1) &= e^{-r(T-t)}\mathbb{E}\big[S_{t_0}\big|\mathcal{F}_t\big]\mathbb{E}\Big[\big(\xi_t(t_0, t_1; \beta)\big)^+\Big] && \text{(A.3)} \\ &= S_t e^{-r(T-t)}e^{(r-q)(t_0-t)}\mathbb{E}\Big[\big(\xi_t(t_0, t_1; \beta)\big)^+\Big]. && \text{(A.4)} \end{aligned}$$

For the in-progress case, $t_0 < t$ and $t_* = t$, the average can be separated into two parts: the past information $A(t_0, t)$ and the random term $A(t, t_1)$. Then

$$\begin{aligned} AC_t^{fl}(\beta; r, q; t_0, t_1) &= e^{-r(T-t)}\mathbb{E}\Big[\Big(\beta S_T - \frac{t_1 - t}{t_1 - t_0}A(t, t_1) - \frac{t - t_0}{t_1 - t_0}A(t_0, t)\Big)^+\Big|\mathcal{F}_t\Big] && \text{(A.5)} \\ &= S_t e^{-r(T-t)}\mathbb{E}\Big[\Big(\xi_t(t_0, t_1; \beta) - K(t_0, t_1)\Big)^+\Big|\mathcal{F}_t\Big]. && \text{(A.6)} \end{aligned}$$

From (A.4) and (A.6), we conclude that:

$$AC_t^{fl}(\beta; r, q; t_0, t_1) = S_t e^{-r(T-t)+(r-q)(t_*-t)} \mathbb{E}\Big[\big(\xi_t(t_0, t_1; \beta) - K(t_0, t_1)\big)^+ \Big| \mathcal{F}_t\Big]. \tag{A.7}$$

Proof of Corollary 3 : For the in-progress cases, $t_0 < t$, the average can be separated into two parts: the past information $A(t_0, t)$ and the random term $A(t, T)$. It follows that:

$$\begin{aligned}
AC_t^{fx}(K; r, q; t_0, t_1) &\equiv e^{-r(T-t)} \mathbb{E}\Big[\Big(A(t_0, T) - K\Big)^+ \Big| \mathcal{F}_t\Big] && \text{(A.8)}\\
&= e^{-r(T-t)} \mathbb{E}\Bigg[\Bigg(\frac{T-t}{T-t_0} A(t, T) + \frac{t-t_0}{T-t_0} A(t_0, t) - K\Bigg)^+ \Bigg| \mathcal{F}_t\Bigg] && \text{(A.9)}\\
&= e^{-r(T-t)} \mathbb{E}\Bigg[\Bigg(\frac{T-t}{T-t_0}\ A(t, T) - \Big[K - \frac{t-t_0}{T-t_0} A(t_0, t)\Big]\Bigg)^+ \Bigg| \mathcal{F}_t\Bigg] && \text{(A.10)}\\
&= S_t e^{-r(T-t)} \mathbb{E}\Big[\big(- \xi_t(t_0, T; 0) - \bar{K}_t\big)^+ \Big| \mathcal{F}_t\Big], && \text{(A.11)}
\end{aligned}$$

where $\bar{K}_t = K/S_t - K_t(t_0, T)$.

Proof of Corollary 4 : This follows from fixed-floating symmetry:[23]

$$AC_t^{fx}(K; r, q; t_0, T) \;=\; AP_t^{fl}(K/S_t; q, r; t, t + T - t_0), \tag{A.12}$$

and Asian put-call parity (5).

Appendix B

Proof of Proposition 5 : Define

$$F(i, k) \equiv \mathbb{E}\Bigg[\bigg(\frac{S_T}{S_{t_*}}\bigg)^k \bigg(\frac{A(t_*, t_1)}{S_{t_*}}\bigg)^i\ \Bigg], \tag{B.1}$$

and $\rho \equiv \frac{t_1 - t_*}{t_1 - t_0}$. Since $\xi_t(t_0, t_1; \beta) = \beta \frac{S_T}{S_{t_*}} - \rho \frac{A(t_*, t_1)}{S_{t_*}}$, it follows that the j-th moment of $\xi_t(t_0, t_1; \beta)$ is given by:

$$\mathbb{E}\Big[\Big(\xi_t(t_0, t_1; \beta)\Big)^j\Big] = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \beta^{j-i} \rho^i F(i, j-i). \tag{B.2}$$

In the following lemma we derive a formula for $F(i, k)$ and then, (B.2) can be rewritten as (11).

[23] See Henderson et al. (2007) for details.

Lemma 12. *We define*

$$G_i(x_1, x_2, ..., x_i) \equiv \int_0^{t_1-t_*} \int_0^{u_i} \int_0^{u_{i-1}} \cdots \int_0^{u_2} \exp\left(\sum_{n=1}^{i} x_n u_n\right) du_1 \cdots du_i \tag{B.3}$$

and $\alpha_{in}^k \equiv \psi\big(2(k+i-n)\big)$.[24] *Then* $F(i,k)$ *can be written as*

$$F(i,k) = \frac{i!}{(t_1-t_*)^i} \exp\Big(k\psi(k-1)(T-t_*)\Big) G_i\big(\alpha_{i1}^k, ..., \alpha_{ii}^k\big). \tag{B.4}$$

Proof: For $i = 0$,

$$\begin{aligned} F(0,k) &\equiv \mathbb{E}\left[\left(\frac{S_T}{S_{t_*}}\right)^k\right] = \exp\left\{k\psi(-1)(T-t_*) + \frac{1}{2}k^2\sigma^2(T-t_*)\right\} \\ &= \exp\Big\{k\psi(k-1)(T-t_*)\Big\}. \end{aligned} \tag{B.5}$$

For $i \geq 1$,

$$\begin{aligned} & F(i,k) \\ \equiv\ & \mathbb{E}\left[\exp\left\{k\Big(\psi(-1)(T-t_*) + \sigma(W_T - W_{t_*})\Big)\right\}\left(\frac{1}{t_1-t_*}\int_0^{t_1-t_*}\exp\Big(\psi(-1)u + \sigma W_u\Big)du\right)^i\right] \\ =\ & \frac{e^{k\psi(-1)(T-t_*)}}{(t_1-t_*)^i}\mathbb{E}\left[e^{k\sigma(W_T-W_{t_*})}\int_0^{t_1-t_*}\cdots\int_0^{t_1-t_*}\exp\left(\psi(-1)\sum_{n=1}^{i}u_n + \sigma\sum_{n=1}^{i}W_{u_n}\right)du_1\cdots du_i\right] \\ =\ & \frac{e^{k\psi(-1)(T-t_*)}}{(t_1-t_*)^i}\int_0^{t_1-t_*}\cdots\int_0^{t_1-t_*} e^{\psi(-1)\sum_{n=1}^{i}u_n}\ \mathbb{E}\left[\exp\left(k\sigma W_{T-t_*} + \sigma\sum_{n=1}^{i}W_{u_n}\right)\right]du_1\cdots du_i \\ =\ & \frac{i!e^{k\psi(-1)(T-t_*)}}{(t_1-t_*)^i}\int_0^{t_1-t_*}\int_0^{u_i}\cdots\int_0^{u_2} e^{\psi(-1)\sum_{n=1}^{i}u_n}\ \mathbb{E}\left[\exp\left(k\sigma W_{T-t_*} + \sigma\sum_{n=1}^{i}W_{u_n}\right)\right]du_1\cdots du_i. \end{aligned} \tag{B.6}$$

If $0 \equiv u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_i \leq u_{i+1} \equiv T - t_*$, then

$$k\sigma W_{T-t_*} + \sigma\sum_{n=1}^{i}W_{u_n} = \sigma\left[\sum_{n=1}^{i+1}(k+i-n+1)\Big(W_{u_n} - W_{u_{n-1}}\Big)\right]. \tag{B.7}$$

Hence, the random variable $k\sigma W_{T-t_*} + \sigma\sum_{n=1}^{i}W_{u_n}$ is normally distributed with mean 0 and variance

$$\sigma^2\left[k^2(T-t_*) + \sum_{n=1}^{i}\Big(1 + 2(k+i-n)\Big)u_n\right] \tag{B.8}$$

[24]Note that when $k = j - i$, we write $\alpha_n \equiv \alpha_{in}^{j-i} = \psi\big(2(j-n)\big)$ as in (11).

since $W_{u_n} - W_{u_{n-1}}$, $n = 1, ..., i+1$, are independent. Therefore,

$$\begin{aligned} F(i,k) &= \frac{i!}{(t_1 - t_*)^i} \times \exp\Big(k\psi(-1)(T - t_*) + \frac{\sigma^2}{2}k^2(T - t_*)\Big) \\ &\quad \times \int_0^{t_1 - t_*}\int_0^{u_i} \cdots \int_0^{u_2} \exp\left(\sum_{n=1}^{i}\Big[\psi(-1) + \frac{\sigma^2}{2}\Big(1 + 2(k + i - n)\Big)\Big]u_n\right)du_1 \cdots du_i \\ &\equiv \frac{i!}{(t_1 - t_*)^i} \times \exp\Big(k\psi(k-1)(T - t_*)\Big) \times G_i\big(\alpha_{i1}^k, ..., \alpha_{ii}^k\big), \end{aligned} \tag{B.9}$$

where

$$\alpha_{in}^k \equiv \psi(-1) + \frac{\sigma^2}{2}\Big(1 + 2(k + i - n)\Big) \;=\; \psi\big(2(k + i - n)\big). \tag{B.10}$$

Finally, observing that

$$G_i(x_1, x_2, ..., x_i) = \int_0^{t_1 - t_*}\int_0^{u_i} \cdots \int_0^{u_3} \exp\left(\sum_{n=2}^{i} x_n u_n\right)\frac{\exp(x_1 u_2) - 1}{x_1}\; du_2 \cdots du_i, \tag{B.11}$$

we see that the function $G_i(x_1, x_2, ..., x_i)$ can be calculated recursively by (12) and (13). This can be done with programs such as Maple or Maxima.

Explicit Formulas for the first three Moments

In this subsection we illustrate the use of Proposition 5 by deriving the first three moments.

The first moment of $\xi_t(t_0, t_1; \beta)$ is

$$\mathbb{E}\Big[\xi_t(t_0, t_1; \beta)\Big] = \beta F(0,1) - \rho F(1,0), \tag{B.12}$$

where $F(0,1) = e^{\psi(0)(T - t_*)}$ and $F(1,0) = G_1\big(\alpha_{11}^0\big)/(t_1 - t_*)$.

The second moment of $\xi_t(t_0, t_1; \beta)$ is

$$\mathbb{E}\Big[\big(\xi_t(t_0, t_1; \beta)\big)^2\Big] = \beta^2 F(0,2) - 2\beta\rho F(1,1) + \rho^2 F(2,0), \tag{B.13}$$

where $F(0,2) = e^{2\psi(1)(T - t_*)}$, $F(1,1) = e^{\psi(0)(T - t_*)}G_1\big(\alpha_{11}^1\big)/(t_1 - t_*)$, and

$$F(2,0) = 2\frac{G_1\big(\alpha_{21}^0 + \alpha_{22}^0\big) - G_1\big(\alpha_{22}^0\big)}{\alpha_{21}^0 (t_1 - t_*)^2}.$$

The third moment of $\xi_t(t_0, t_1; \beta)$ is

$$\mathbb{E}\Big[\big(\xi_t(t_0, t_1; \beta)\big)^3\Big] = \beta^3 F(0,3) - 3\beta^2 \rho F(1,2) + 3\beta\rho^2 F(2,1) - \rho^3 F(3,0), \tag{B.14}$$

where $F(0,3) = e^{3\psi(2)(T-t_*)}$, $F(1,2) = e^{2\psi(1)(T-t_*)} G_1\big(\alpha_{11}^2\big)/(t_1 - t_*)$,

$$F(2,1) = 2e^{\psi(0)(T-t_*)} \frac{G_1\big(\alpha_{21}^1 + \alpha_{22}^1\big) - G_1\big(\alpha_{22}^1\big)}{\alpha_{21}^1 (t_1 - t_*)^2},$$

and

$$F(3,0) = \frac{6}{\alpha_{31}^0} \left[\frac{G_1\big(\alpha_{31}^0 + \alpha_{32}^0 + \alpha_{33}^0\big) - G_1\big(\alpha_{33}^0\big)}{(\alpha_{31}^0 + \alpha_{32}^0)(t_1 - t_*)^3} - \frac{G_1\big(\alpha_{32}^0 + \alpha_{33}^0\big) - G_1\big(\alpha_{33}^0\big)}{\alpha_{32}^0 (t_1 - t_*)^3} \right].$$

Appendix C

Proof of Proposition 6 : Suppose $\xi_t(t_0, t_1; \beta) \sim N(m, v)$, then

$$\begin{aligned}
& \mathbb{E}\Big[\big(\xi_t(t_0, t_1; \beta) - K_*\big)^+ \Big| \mathcal{F}_t\Big] && \text{(C.1)} \\
= & \int_{K_*}^{\infty} (z - K_*) \frac{1}{\sqrt{2\pi v}} e^{-\frac{(z-m)^2}{2v}} dz && \\
= & \int_{K_*}^{\infty} (z - m) \frac{1}{\sqrt{2\pi v}} e^{-\frac{(z-m)^2}{2v}} dz + \int_{K_*}^{\infty} (m - K_*) \frac{1}{\sqrt{2\pi v}} e^{-\frac{(z-m)^2}{2v}} dz && \text{(C.2)} \\
= & \sqrt{\frac{v}{2\pi}} \exp\left\{ -\frac{(K_* - m)^2}{2v} \right\} + (m - K_*) N\left(\frac{m - K_*}{\sqrt{v}} \right) && \text{(C.3)}
\end{aligned}$$

since K_* is $\mathcal{F}_t$-measurable.

Proof of Proposition 7 : Suppose $\xi_t(t_0, t_1; \beta) \equiv \eta + Z$, $Z \sim Gamma(\alpha, \beta)$, and set $C \equiv \big(K_t(t_0, T) - \eta\big)$. Then

$$\begin{aligned}
\mathbb{E}\Big[\big(Z - C\big)^+ \Big| \mathcal{F}_t\Big] &= \int_C^{\infty} (z - C) \frac{z^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-z/\beta} dz && \text{(C.4)} \\
&= \frac{1}{\Gamma(\alpha)} \int_C^{\infty} \left(\frac{z}{\beta}\right)^\alpha e^{-z/\beta} dz - C\Big(1 - F_\Gamma(C; \alpha, \beta)\Big), && \text{(C.5)}
\end{aligned}$$

where $F_\Gamma(C;\alpha,\beta) = \Gamma_{C/\beta}(\alpha)/\Gamma(\alpha)$ is the cumulative gamma distribution function and

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_C^\infty \left(\frac{z}{\beta}\right)^\alpha e^{-z/\beta}dz &= \frac{\beta}{\Gamma(\alpha)}\int_{C/\beta}^\infty z^\alpha e^{-z}dz &\text{(C.6)}\\ &= \frac{\beta}{\Gamma(\alpha)}\Big(\Gamma(\alpha+1)-\Gamma_{C/\beta}(\alpha+1)\Big) &\text{(C.7)}\\ &= \alpha\beta\left(1-\frac{\Gamma_{C/\beta}(\alpha+1)}{\Gamma(\alpha+1)}\right). &\text{(C.8)}\end{aligned}$$

Applying (18) to replace $\alpha\beta$ by $m_*-\eta$, we obtain (20).

Proof of Lemma 8 : Defining $x = e^\mu > 0$ and $y = e^{\sigma^2} > 1$, (23) can be re-expressed as

$$m_* - \eta = x\sqrt{y}, \tag{C.9}$$

$$v_* = x^2y(y-1) = (m_*-\eta)^2(y-1), \tag{C.10}$$

$$s_* = (y+2)\sqrt{y-1}. \tag{C.11}$$

Then

$$\begin{aligned}s_*^2(m_*-\eta)^6 &= \Big[(m_*-\eta)^2(y-1+3)\Big]^2(m_*-\eta)^2(y-1) = \Big[v_*+3(m_*-\eta)^2\Big]^2 v_*\\ &= 9v_*(m_*-\eta)^4+6v_*^2(m_*-\eta)^2+v_*^3. &\text{(C.12)}\end{aligned}$$

Note that the cubic equation (C.12) has only one real root which is positive when we regard $(m_*-\eta)^2$ as an unknown with coefficients dependening on v_* and s_*. It follows that we can solve η from (C.12), and then (C.10) and (C.9) can be solved for y and x obtaining (24).

Proof of Proposition 9: We assume that $\xi_t(t_0,t_1;\beta)$ has a shifted lognormal distribution. Then $\xi_t(t_0,t_1;\beta) \equiv \eta + Z$, where η is a constant and Z is lognormally distributed with mean μ and variance σ^2. Then it can be easily derived that

$$\begin{aligned}\mathbb{E}\Big[\big(\xi_t(t_0,t_1;\beta)-K_*\big)^+\Big|\mathcal{F}_t\Big] &= \mathbb{E}\Big[\Big(Z-\big(K_*-\eta\big)\Big)^+\Big|\mathcal{F}_t\Big]\\ &= \mathbb{E}[Z]N(d_+)-\big(K_*-\eta\big)N(d_-) &\text{(C.13)}\end{aligned}$$

since K_* is $\mathcal{F}_t$-measurable, and

$$\mathbb{E}[Z] = \exp(\mu+\sigma^2/2) = m_*-\eta, \tag{C.14}$$

using (23). Thus we obtain (25).

Proof of Lemma 10: We can re-express (27), (28), and (29) as

$$
\begin{aligned}
m_* - \eta &= \frac{\beta}{\alpha - 1}, && \text{(C.15)}\\
v_* &= \left[\frac{\beta}{\alpha-1}\right]^2 \frac{1}{\alpha - 2} = (m_* - \eta)^2 \frac{1}{\alpha - 2}, && \text{(C.16)}\\
s_* &= \frac{4\sqrt{\alpha - 2}}{(\alpha - 2) - 1} = \frac{4}{\sqrt{v_*}}(m_* - \eta) \Big/ \left(\frac{(m_* - \eta)^2}{v_*} - 1\right). && \text{(C.17)}
\end{aligned}
$$

Next, (C.17) can be rewritten as

$$
(m_* - \eta)^2 - \frac{4\sqrt{v_*}}{s_*}(m_* - \eta) - v_* = 0. \tag{C.18}
$$

It follows that

$$
m_* - \eta = \frac{1}{2}\left(\frac{4\sqrt{v_*}}{s_*} \pm \sqrt{\frac{16 v_*}{s_*^2} + 4 v_*}\right) = \frac{\sqrt{v_*}}{s_*}\left(2 \pm \sqrt{4 + s_*^2}\right). \tag{C.19}
$$

Since $\alpha > 1$ in (27) and $\beta > 0$, $m_* - \eta$ must be positive so that

$$
m_* - \eta = \frac{\sqrt{v_*}}{s_*}\left(2 + \sqrt{4 + s_*^2}\right). \tag{C.20}
$$

Hence, we obtain (30). Then (31) and (32) can be easily derived from (C.16) and (C.15).

Proof of Lemma 11: Similar to Proposition 7.

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Exhibit 3
The Skewness and Kurtosis under Alternative Distributions

The symbol ξ is the random variable with unknown distribution related to the average, and the parameters are chosen as $S_t = 100$, $r = 10\%$, $q = 0$, and $t_0 = t = 0$. N, SG, LN, SL, RG, and SRG exhibit the values of the skewness and kurtosis using normal, shifted gamma, lognormal, shifted lognormal, reciprocal gamma, and shifted reciprocal gamma approximations, respectively. The symbol $\star$ indicates that this parameter is matched.

Panel A: Fixed-strike Asian Options

	$-\xi_t(t,T;0)$	N	SG	LN	SL	RG	SRG
σ=30%, T=30/365							
Skewness	0.1793	0.0000	0.1793$\star$	0.1494	0.1793$\star$	0.1995	0.1793$\star$
Kurtosis	3.0571	3.0000	3.0482	3.0397	2.1326	3.0748	3.0604
σ=30%, T=1							
Skewness	0.6484	0.0000	0.6484$\star$	0.5378	0.6484$\star$	0.7326	0.6484$\star$
Kurtosis	3.7788	3.0000	3.6306	3.5186	1.3168	4.0337	3.8050
σ=50%, T=30/365							
Skewness	0.3002	0.0000	0.3002$\star$	0.2498	0.3002$\star$	0.3345	0.3002$\star$
Kurtosis	3.1650	3.0000	3.1352	3.1111	1.7501	3.2110	3.1697
σ=50%, T=1							
Skewness	1.1445	0.0000	1.1445$\star$	0.9332	1.1445$\star$	1.3286	1.1445$\star$
Kurtosis	5.5034	3.0000	4.9649	4.5874	1.9016	6.6117	5.6208

Panel B: Floating-strike Asian Options

	$\xi_t(t,T;1)$	N	SG	LN	SL	RG	SRG
σ=30%, T=30/365							
Skewness	0.2703	0.0000	0.2703$\star$	1.84×10^3	0.2703$\star$	-0.3311	0.2703$\star$
Kurtosis	3.1433	3.0000	3.1096	4.99×10^8	1.8316	0.0718	3.1375
σ=30%, T=1							
Skewness	1.0263	0.0000	1.0263$\star$	6.40×10	1.0263$\star$	-1.1484	1.0263$\star$
Kurtosis	5.0939	3.0000	4.5800	5.90×10^4	1.6578	0.8420	5.0810
σ=50%, T=30/365							
Skewness	0.4554	0.0000	0.4554$\star$	8.51×10^3	0.4554$\star$	-0.1968	0.4554$\star$
Kurtosis	3.4127	3.0000	3.3110	3.01×10^{10}	1.4492	0.0254	3.3928
σ=50%, T=1							
Skewness	1.9539	0.0000	1.9539$\star$	3.12×10^2	1.9539$\star$	-0.6169	1.9539$\star$
Kurtosis	11.0352	3.0000	8.7263	4.30×10^6	5.1681	0.2477	11.6395

Exhibit 4
Option Prices and Pricing Errors
Plain-vanilla Fixed-strike Asian Options (T=0.08)

The parameters and benchmark values are from Zhang (2001). $S_t = 100$, $r = 9\%$, $q = 0$, $t_0 = t = 0$, and $T = 0.08$. N, SG, LN, SL, RG, and SRG exhibit the option values using normal, shifted gamma, lognormal, shifted lognormal, reciprocal gamma, and shifted reciprocal gamma approximations, respectively. RMSE and MAE are the root of mean squared errors and the maximum absolute error. Time exhibits the average computing time obtained by the Matlab package on Microsoft Windows XP and based on Intel Pentium Dual CPU T2390 with 1.86GHz CPU and 3.00GB RAM

σ	K	Zhang	N	SG	LN	SL	RG	SRG
0.05	95	5.3224	5.3224	5.3224	5.3224	5.3224	5.3224	5.3224
	100	0.5343	0.5349	0.5343	0.5344	0.5343	0.5342	0.5342
	105	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	95	5.3226	5.3226	5.3226	5.3226	5.3226	5.3225	5.3226
	100	0.8431	0.8446	0.8432	0.8434	0.8431	0.8430	0.8431
	105	0.0015	0.0011	0.0015	0.0014	0.0015	0.0015	0.0015
0.2	95	5.3816	5.3924	5.3815	5.3834	5.3816	5.3805	5.3817
	100	1.4835	1.4869	1.4838	1.4842	1.4836	1.4830	1.4835
	105	0.1293	0.1160	0.1292	0.1270	0.1293	0.1208	0.1293
0.3	95	5.6305	5.6657	5.6303	5.6364	5.6305	5.6265	5.6305
	100	2.1290	2.1353	2.1298	2.1304	2.1291	2.1279	2.1289
	105	0.4880	0.4546	0.4882	0.4827	0.4881	0.4916	0.4880
0.4	95	6.0222	6.0852	6.0223	6.0332	6.0223	6.0148	6.0222
	100	2.7755	2.7861	2.7774	2.7782	2.7758	2.7734	2.7753
	105	0.9752	0.9230	0.9762	0.9772	0.9754	0.9803	0.9750
0.5	95	6.4946	6.5866	6.4956	6.5111	6.4948	6.4832	6.4945
	100	3.4223	3.4390	3.4260	3.4268	3.4229	3.4186	3.4218
	105	1.5271	1.4588	1.5298	1.5172	1.5275	1.5331	1.5266
RMSE			0.0357	0.0012	0.0060	0.0002	0.0041	0.0002
MAE			0.0920	0.0037	0.0165	0.0005	0.0114	0.0006
Time (second)			0.0018	0.0034	0.0014	0.0010	0.0019	0.0020

Exhibit 5
Option Prices and Pricing Errors
Plain-vanilla Fixed-strike Asian Options (T=1)

The parameters and benchmark values are from Zhang (2001). $S_t = 100$, $r = 9\%$, $q = 0$, $t_0 = t = 0$, and $T = 1$. N, SG, LN, SL, RG, and SRG exhibit the option values using normal, shifted gamma, lognormal, shifted lognormal, reciprocal gamma, and shifted reciprocal gamma approximations, respectively. RMSE and MAE are the root of mean squared errors and the maximum absolute error. Time exhibits the average computing time obtained by the Matlab package on Microsoft Windows XP and based on Intel Pentium Dual CPU T2390 with 1.86GHz CPU and 3.00GB RAM.

σ	K	Zhang	N	SG	LN	SL	RG	SRG
0.05	95	8.8088	8.8092	8.8088	8.8089	8.8088	8.8088	8.8088
	100	4.3082	4.3175	4.3082	4.3097	4.3082	4.3072	4.3082
	105	0.9584	0.9565	0.9585	0.9582	0.9584	0.9585	0.9584
0.1	95	8.9119	8.9461	8.9111	8.9172	8.9118	8.9082	8.9120
	100	4.9151	4.9618	4.9153	4.9231	4.9151	4.9094	4.9151
	105	2.0701	2.0688	2.0714	2.0705	2.0703	2.0695	2.0699
0.2	95	9.9957	10.1942	9.9949	10.0304	9.9955	9.9705	9.9959
	100	6.7773	6.9127	6.7836	6.8035	6.7782	6.7572	6.7765
	105	4.2965	4.3142	4.3068	4.3041	4.2979	4.2889	4.2949
0.3	95	11.6559	12.0573	11.6628	11.7329	11.6566	11.5973	11.6555
	100	8.8288	9.0895	8.8519	8.8858	8.8318	8.7822	8.8257
	105	6.5178	6.5955	6.5509	6.5463	6.5224	6.4903	6.5129
0.4	95	13.5107	14.1694	13.5327	13.6479	13.5131	13.4017	13.5095
	100	10.9238	11.3708	10.9758	11.0311	10.9304	10.8322	10.9176
	105	8.7299	8.9301	8.8025	8.7996	8.7397	8.6630	8.7198
0.5	95	15.4427	16.4396	15.4798	15.6649	15.4462	15.2598	15.4426
	100	13.0282	13.7462	13.1159	13.2120	13.0388	12.8669	13.0195
	105	10.9296	11.3380	11.0551	10.0675	10.9458	11.7973	10.9139
RMSE			0.3834	0.0442	0.0911	0.0056	0.0779	0.0053
MAE			0.9969	0.1255	0.2221	0.0162	0.1829	0.0158
Time (second)			0.0019	0.0027	0.0017	0.0014	0.0015	0.0015

Exhibit 6
Option Prices and Pricing Errors
In-progress Floating-strike Asian Options ($\sigma = 30\%$)

The parameters and benchmark values (MC) are obtained from Henderson et al. (2007). $S_t = 100$, $r = 10\%$, $q = 0$, $\sigma = 30\%$ $t_0 = 0$, and $T = 1$. $A(0,t)$, the average realized up to time t, is assumed to be 90, 100 and 110. N, SG, LN, SL, RG, and SRG exhibit the option values using normal, shifted gamma, lognormal, shifted lognormal, reciprocal gamma, and shifted reciprocal gamma approximations, respectively. UB exhibits the upper bounds of Henderson et al. (2007). The symbol $\star$ indicates that those option prices exceed UB. RMSE and MAE are the root of mean squared errors and the maximum absolute error. Time exhibits the average computing time obtained by the Matlab package on Microsoft Windows XP and based on Intel Pentium Dual CPU T2390 with 1.86GHz CPU and 3.00GB RAM.

$A(0,t)$	t	MC	N	SG	LN	SL	RG	SRG	UB
90	0.1	9.8614	10.4049$\star$	9.8971	7.5836	9.8741	6.6687	9.8695	9.9666
	0.2	10.3054	10.8443$\star$	10.3316	9.1475	10.3143	8.4059	10.3112	10.4776
	0.3	10.6948	11.2095$\star$	10.7112	10.0612	10.6985	9.5292	10.6964	10.8925
	0.4	11.0028	11.4798$\star$	11.0124	10.6443	11.0034	10.2620	11.0020	11.1978
	0.5	11.2041	11.6333$\star$	11.2106	11.0034	11.2046	10.7261	11.2039	11.3775
	0.6	11.2749	11.6448$\star$	11.2789	11.1678	11.2756	10.9680	11.2753	11.4128
	0.7	11.1851	11.4835$\star$	11.1862	11.1330	11.1853	10.9945	11.1854	11.2793
	0.8	10.8962	11.1101$\star$	10.8955	10.8758	10.8962	10.7894	10.8967	10.9466
	0.9	10.3797	10.4880$\star$	10.3787	10.3757	10.3799	10.3378	10.3804	10.3915
100	0.1	9.3457	9.8433$\star$	9.3897	7.1335	9.3599	6.1437	9.3530	9.4627
	0.2	9.2441	9.6951$\star$	9.2835	8.1788	9.2539	7.4113	9.2465	9.4348
	0.3	9.0576	9.4487$\star$	9.0894	8.5112	9.0607	7.9872	9.0529	9.2778
	0.4	8.7564	9.0840$\star$	8.7833	8.4704	8.7566	8.1166	8.7490	8.9778
	0.5	8.3147	8.5773$\star$	8.3380	8.1686	8.3145	7.9314	8.3075	8.5161
	0.6	7.7003	7.8975$\star$	7.7193	7.6306	7.7002	7.4762	7.6942	7.8659
	0.7	6.8647	6.9981$\star$	6.8782	6.8353	6.8643	6.7417	6.8598	6.9845
	0.8	5.7215	5.7972$\star$	5.7298	5.7122	5.7214	5.6637	5.7186	5.7930
	0.9	4.0707	4.0985	4.0739	4.0692	4.0707	4.0526	4.0695	4.1165
110	0.1	8.8508	9.2995$\star$	8.9024	6.7202	8.8663	5.6753	8.8570	8.9790
	0.2	8.2670	8.6193$\star$	8.3173	7.3178	8.2772	6.5525	8.2658	8.4735
	0.3	7.6121	7.8594$\star$	7.6566	7.1801	7.6155	6.6976	7.6030	7.8495
	0.4	6.8623	7.0077	6.9017	6.6748	6.8630	6.3867	6.8506	7.0992
	0.5	6.0006	6.0516	6.0340	5.9315	6.0006	5.7740	5.9895	6.2128
	0.6	5.0082	4.9769	5.0335	4.9931	5.0077	4.9243	4.9988	5.1782
	0.7	3.8619	3.7686	3.8781	3.8669	3.8612	3.8572	3.8553	3.9802
	0.8	2.5370	2.4141	2.5449	2.5451	2.5368	2.5693	2.5339	2.6021
	0.9	1.0369	0.9396	1.0381	1.0403	1.0369	1.0696	1.0364	1.0557
RMSE			0.3240	0.0269	0.8437	0.0058	1.2997	0.0057	
MAE			0.5435	0.0516	2.2778	0.0155	3.2020	0.0117	
Time (second)			0.0016	0.0024	0.0015	0.0009	0.0012	0.0013	

Exhibit 7
Option Prices and Pricing Errors
In-progress Floating-strike Asian Options ($\sigma = 50\%$)

The parameters and benchmark values (MC) are obtained from Henderson et al. (2007). $S_t = 100$, $r = 10\%$, $q = 0$, $\sigma = 50\%$ $t_0 = 0$, and $T = 1$. $A(0,t)$, the average realized up to time t, is assumed to be 90, 100 and 110. N, SG, LN, SL, RG, and SRG exhibit the option values using normal, shifted gamma, lognormal, shifted lognormal, reciprocal gamma, and shifted reciprocal gamma approximations, respectively. UB exhibits the upper bounds of Henderson et al. (2007). The symbol $\star$ indicates that those option prices exceed UB. RMSE and MAE are the root of mean squared errors and the maximum absolute error. Time exhibits the average computing time obtained by the Matlab package on Microsoft Windows XP and based on Intel Pentium Dual CPU T2390 with 1.86GHz CPU and 3.00GB RAM.

$A(0,t)$	t	MC	N	SG	LN	SL	RG	SRG	UB
90	0.1	14.0854	15.6883$\star$	14.1416	8.7387	14.1462	6.9938	14.1729	14.2607
	0.2	14.4306	15.9385$\star$	14.4754	11.3422	14.4714	9.5152	14.4914	14.7292
	0.3	14.7114	16.0887$\star$	14.7443	12.9249	14.7299	11.4063	14.7424	15.0582
	0.4	14.8765	16.1056$\star$	14.9048	13.8516	14.8820	12.6845	14.8876	15.2240
	0.5	14.8806	15.9509$\star$	14.9102	14.3114	14.8834	13.4493	14.8838	15.1970
	0.6	14.6794	15.5763$\star$	14.7066	14.3808	14.6809	13.7663	14.6784	14.9375
	0.7	14.2050	14.9148$\star$	14.2242	14.0624	14.2044	13.6470	14.2012	14.3875
	0.8	13.3462	13.8590$\star$	13.3567	13.2913	13.3461	13.0374	13.3440	13.4512
	0.9	11.9074	12.2005$\star$	11.9093	11.8953	11.9077	11.7758	11.9075	11.9435
100	0.1	13.6521	15.1712$\star$	13.7354	8.3976	13.7161	6.5246	13.7347	13.8452
	0.2	13.5400	14.8842$\star$	13.6278	10.5917	13.5820	8.6848	13.5878	13.8625
	0.3	13.3326	14.4780$\star$	13.4180	11.6908	13.3509	10.1485	13.3451	13.7055
	0.4	12.9760	13.9196$\star$	13.0606	12.0794	12.9811	10.9350	12.9667	13.3502
	0.5	12.4235	13.1692$\star$	12.5064	11.9558	12.4247	11.1497	12.4056	12.7643
	0.6	11.6225	12.1745$\star$	11.6956	11.3964	11.6218	10.8592	11.6018	11.9019
	0.7	10.4886	10.8579$\star$	10.5446	10.3929	10.4869	10.0643	10.4697	10.6900
	0.8	8.8749	9.0821$\star$	8.9108	8.8445	8.8741	8.6740	8.8625	8.9949
	0.9	6.4442	6.5197$\star$	6.4588	6.4395	6.4441	6.3814	6.4391	6.4900
110	0.1	13.2319	14.6645$\star$	13.3404	8.0819	13.2982	6.1057	13.3089	13.4418
	0.2	12.6993	13.8730$\star$	12.8269	9.9107	12.7429	7.9591	12.7350	13.0442
	0.3	12.0728	12.9688$\star$	12.2006	10.5940	12.0880	9.0682	12.0655	12.4643
	0.4	11.2887	11.9236$\star$	11.4187	10.5408	11.2936	9.4621	11.2620	11.6814
	0.5	10.3158	10.7044$\star$	10.4367	9.9665	10.3153	9.2610	10.2804	10.6685
	0.6	9.1014	9.2695	9.2034	8.9622	9.0995	8.5465	9.0671	9.3872
	0.7	7.5770	7.5608	7.6506	7.5376	7.5744	7.3380	7.5491	7.7793
	0.8	5.6295	5.4858	5.6722	5.6291	5.6285	5.5799	5.6132	5.7464
	0.9	3.0510	2.8704	3.0642	3.0559	3.0508	3.0884	3.0459	3.0916
RMSE			0.9286	0.0742	2.1088	0.0262	3.1722	0.0358	
MAE			1.6029	0.1300	5.3467	0.0663	7.1275	0.0875	
Time (second)			0.0016	0.0020	0.0012	0.0009	0.0011	0.0012	

Exhibit 8
Option Prices and Truncation Errors
In-progress Floating-Strike Asian Options

The parameters and benchmark values (MC) are obtained from Henderson et al. (2007). $S_t = 100$, $r = 10\%$, $q = 0$, $\sigma = 30\%$ and 50%, $t_0 = 0$, and $T = 1$. $A(0,t)$, the average realized up to time t, is taken to be 90, 100 and 110. BBC represents the normal approximation in the linear model of Bouaziz et al. (1994). CT represents the shifted central chi-squared approximation in the quadratic model of Chang and Tsao (2011). N and SG represent the normal and shifted gamma approximations without truncation errors. RMSE and MAE are the root of mean squared errors and the maximum absolute error. Time exhibits the average computing time obtained by the Matlab package on Microsoft Windows XP and based on Intel Pentium Dual CPU T2390 with 1.86GHz CPU and 3.00GB RAM.

		$\sigma = 30\%$					$\sigma = 50\%$				
$A(0,t)$	t	MC	BBC	N	CT	SG	MC	BBC	N	CT	SG
90	0.1	9.8614	8.1584	10.4049	9.5482	9.8971	14.0854	10.4118	15.6883	13.0999	14.1416
	0.3	10.6948	9.2472	11.2095	10.4626	10.7112	14.7114	11.4829	16.0887	13.9128	14.7443
	0.5	11.2041	10.0825	11.6333	11.0568	11.2106	14.8806	12.2952	15.9509	14.3334	14.9102
	0.7	11.1851	10.4578	11.4835	11.1165	11.1862	14.2050	12.4789	14.9148	13.9260	14.2242
	0.9	10.3797	10.0954	10.4880	10.3689	10.3787	11.9074	11.2546	12.2005	11.8535	11.9093
100	0.1	9.3457	7.6340	9.8433	9.0312	9.3897	13.6521	9.9642	15.1712	12.6605	13.7354
	0.3	9.0576	7.5899	9.4487	8.8191	9.0894	13.3326	10.0701	14.4780	12.5177	13.4180
	0.5	8.3147	7.1791	8.5773	8.1600	8.3380	12.4235	9.8076	13.1692	11.8584	12.5064
	0.7	6.8647	6.1429	6.9981	6.7894	6.8782	10.4886	8.7529	10.8579	10.1951	10.5446
	0.9	4.0707	3.8260	4.0985	4.0561	4.0739	6.4442	5.8187	6.5197	6.3828	6.4588
110	0.1	8.8508	7.1303	9.2995	8.5350	8.9024	13.2319	9.5292	14.6645	12.2338	13.3404
	0.3	7.6121	6.1259	7.8594	7.3701	7.6566	12.0728	8.7747	12.9688	11.2426	12.2006
	0.5	6.0006	4.8520	6.0516	5.8437	6.0340	10.3158	7.6700	10.7044	9.7408	10.4367
	0.7	3.8619	3.1457	3.7686	3.7866	3.8781	7.5570	5.8313	7.5608	7.2805	7.6506
	0.9	1.0369	0.8324	0.9396	1.0246	1.0381	3.0510	2.4501	2.8704	2.9924	3.0642
RMSE			1.1793	0.3283	0.1920	0.0274		2.6267	0.9507	0.6405	0.0722
MAE			1.7205	0.5435	0.3158	0.0516		3.7027	1.6029	0.9981	0.1278
Time (second)			0.0018	0.0002	0.0026	0.0012		0.0018	0.0002	0.0025	0.0011