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Pricing European average rate currency options

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This paper develops a simple methodology that yields closed-form analytical approximations for valuing European option claims involving the arithmetic average of future foreign exchange rates. The main advantage of the approach is that it avoids the need to adopt time-consuming numerical procedures. The precision of the resulting formula, and the distributional assumption that underlies it, is examined by way of Monte Carlo simulations. (JEL F31)

Average rate (or 'Asian') options (AROs) are path-dependent contingent claims which settle against the arithmetic average of prices calculated over a given time interval. This type of claim is being offered in a variety of markets and is receiving a significant amount of interest from banks and investors alike.

The rationale for the presence of such an instrument differs from market to market. Bergman (1985), in his introduction, favors such contracts as they represent an attractive specification for thinly traded asset markets where price manipulation on or near a maturity date is possible. In markets where prices are prone to periods of extreme volatility the averaging performs a smoothing operation. For example, options on oil are typically settled against an average of daily or monthly fixings from *Platts European Marketscan*. In the foreign exchange market, the ARO is offered over-the-counter to clients as a means to hedge a stream of foreign currency flows against adverse currency movements. Such options have become increasingly popular as (not surprisingly) they can be considerably cheaper than a European option of a similar maturity and often more relevant to a treasurer's needs. The alternative strategy of entering into a strip of individual options is unnecessarily costly if all that is required to be hedged is the resultant average of exchange rates.

Under standard assumptions, the valuation of such options presents certain difficulties. Apart from trivial circumstances, options involving the arithmetic average will not have closed-form solutions if the conventional assumption of a geometric diffusion is specified for the underlying price process. For unlike options

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on the geometric average, the density function for the arithmetic average is not log normal and has no explicit representation.

The few papers examining the valuation of this type of instrument fall into three main groups. Firstly, there are those that advocate a numerical approach: Kemna and Vorst (1990) propose a Monte Carlo methodology which employs the corresponding geometric option as a control variate; Carverhill and Clewlow (1990) adopt the Fast Fourier Transform to evaluate numerically the necessary convolutions of density functions. These methods can be accurate but time-consuming especially when the number of prices in the average is large. In addition, this shortcoming is compounded for traders examining the sensitivity of large portfolios of AROs to pertinent market parameters. A second approach is to modify the solution to the geometric average option: for example, Ruttiens (1990) and Vorst (1990). Employing the solution to the corresponding geometric average problem improves the speed of calculation but results in models which can systematically (and at times significantly) overprice put options and/or underprice call options, giving rise to misleading signals and inefficiencies in the dynamic hedging requirements. Furthermore, the methodology on which such solutions are based yields formulae which only satisfy the put-call parity condition for arithmetic options on the expiry date of the option. The final approach, and one which this paper follows, is to approximate the density function for the arithmetic average.

The simulations carried out in this paper demonstrate that for volatilities typically experienced in foreign exchange markets, the distribution of an arithmetic average is well-approximated by the log normal distribution when the underlying price process follows the conventional assumption of a geometric diffusion. As a result, the problem reduces to the less complicated task of determining the necessary parameters for the log normal density function. The reported simulations emphasize that in applying this approach to other underlying asset prices, users should be aware that the approximation is only valid for a limited range of volatilities and option maturities.

Turnbull and Wakeman (1991) also recognize the suitability of the log normal as a first-order approximation. These authors apply the Edgeworth series expansion (around the log normal) to the problem of valuing the ARO, and provide an algorithm to compute moments for the arithmetic average. However, Turnbull and Wakeman overlook the fact that when only the first two moments are taken into account in the approximation, the accuracy of the log normal assumption is acceptable making redundant the need to include additional terms in the expansion involving higher moments. This observation is important as closed-form expressions for the first two moments of the arithmetic average are relatively simple to derive.

This paper uses a straightforward approach (termed below as the 'Wilkinson approximation') to approximate the arithmetic density function. This can be interpreted as the first term in the Edgeworth expansion referred to above. Its main virtue, is that a closed-form analytical approximation for the valuation of AROs becomes possible which has the advantage of being, for typical ranges of volatility experienced, both accurate and easily implemented.[1] It is also shown how the approach can be extended to value other contingent claims involving the arithmetic average.

The paper is organized as follows. Section I outlines the option contract to be valued and the put-call parity condition for AROs. Section II introduces the 'Wilkinson approximation' and develops a formula for the valuation of the ARO. The performance of the approximation is examined in Section III via Monte Carlo simulations. Finally, the paper ends with a brief summary and some observations on possible modifications to the approach.

I. The average rate option

Let $S(t)$ be the spot price at time t, defined (as are all prices here) as units of domestic currency per unit of foreign currency. We suppose that the average is determined over the time interval $[t_0, t_N]$ and at points on this interval $t_i = t_0 + ih$ for $i = 0, 1, \ldots, N$ where $h = (t_N - t_0)/N$. The 'running average,' $A(t)$, is defined for a given point in time $t_0 \leqslant t \leqslant t_N$ by

$$A(t) = \frac{1}{m+1}\sum_{i=0}^{m} S(t_i)$$

for $0 \leqslant m \leqslant N$, and $A(t) = 0$ for $t < t_0$. Thus $A(t_N)$ represents the arithmetic average of $N + 1$ prices taken at equal intervals of time h between t_0 and t_N. Typically this time interval is specified to be a day, a week, or a month.[2]

The ARO is characterized by the payoff function at time t_N given by $\mathrm{Max}[A(t_N) - K, 0]$ for a call option, or $\mathrm{Max}[K - A(t_N), 0]$ for a put option. Here, K is the strike price of the option. It will be helpful later on to define the variable $M(t) = [A(t_N) - A(t)(m + 1)/(N + 1)]$ denoting the undetermined component of the final average.

The put-call parity condition is ignored (or at best implicit) in papers which examine the pricing of AROs, yet the condition is fundamental to the consistency of a pricing methodology. As for other European option claims, the condition is independent of volatility and should be embodied in all valuation models.

Define r_d and r_f as, respectively, the domestic and foreign (continuously compounding) interest rates and assume them constant. We also assume frictionless markets.[3] Consider a position of being long one call option and short one put option each with strike K. This is equivalent to a position which pays

$$\frac{1}{N+1}\sum_{i=0}^{N} S(t_i) - K$$

units of domestic currency at time t_N. Suppose during the averaging period at some time $t = t_m + \xi h$, $0 \leqslant m \leqslant N$, we wish to value this position. At this time the uncertainty in the position is due only to the unknown spot prices $S(t_i)$, $i = m + 1, \ldots, N$ which can be hedged in the forward market.[4] Arrange at time t to sell forward $\frac{1}{N+1} e^{-r_d(t_N - t_i)}$ units of foreign currency for delivery at t_i for $i = m + 1, \ldots, N$. Note that the value of such a hedge is zero at t. Subsequently, at each t_i, these forward positions are closed at the prevailing spot rates $S(t_i)$ for net value

$$\frac{1}{N+1} e^{-r_d(t_N - t_i)}[F(t, t_i) - S(t_i)],$$

where, $F(t, t_i)$ denotes the arbitrage-free forward price for foreign exchange at

$t_m + \xi h$ for delivery at t_i given by

$$F(t, t_i) = S(t_m)e^{(r_d - r_f)(t_i - t_m - \xi h)}.$$

These flows may be placed on deposit (or borrowed) for $(t_N - t_i)$ realizing a total value of

$$\frac{1}{N+1}\sum_{i=m+1}^{N}[F(t, t_i) - S(t_i)].$$

Finally, setting off this amount against the maturity value of the call and put position, we have a known domestic currency amount which can be discounted at the risk-free rate to give

$$\begin{aligned} V(t) &= e^{-r_d(t_N - t)}\left[\frac{1}{N+1}\sum_{i=0}^{m} S(t_i) + \frac{S(t)}{N+1}\sum_{i=m+1}^{N} e^{(r_d - r_f)(t_i - t)} - K\right] \\ &= e^{-r_d\tau}\left\{\frac{m+1}{N+1}A(t) + \frac{S(t)e^{g(1-\xi)h}}{N+1}\frac{[1 - e^{g(N-m)h}]}{1 - e^{gh}} - K\right\}, \end{aligned}$$

where, $g = (r_d - r_f)$ and $\tau = (t_N - t)$ and where we have made use of $(t_i - t) = (i - m - \xi)h$. It follows that if $C(t)$ and $P(t)$ denote the value of, respectively, the call and put options at t, then to avoid the possibility of arbitrage we require

$$\langle 1a \rangle \qquad C(t) - P(t) = V(t).$$

A similar argument in the case of $t < t_0$ results in

$$\langle 1b \rangle \qquad C(t) - P(t) = e^{-r_d\tau}\left\{\frac{S(t)}{N+1}\frac{[e^{g(t_0 - t)} - e^{g[t_0 - t + h(N+1)]}]}{1 - e^{gh}} - K\right\}.$$

Thus if an ARO call option is valued, we may use expressions $\langle 1 \rangle$ to value the corresponding put option.

II. A pricing formula

In the remainder of this paper, we will assume that the spot price process is the familiar geometric diffusion

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dz,$$

where dz is a Wiener process, and μ and σ are constant. Applying the risk-neutrality transformation of Cox and Ross (1976), we may characterize the value of the ARO as:

$$\langle 2 \rangle \qquad C[S(t), A(t), t] = e^{-r_d\tau}\mathbf{E}^*\,\mathrm{Max}[A(t_N) - K, 0],$$

where $\mathbf{E}^*$ is the expectation operator conditioned on $[A(t), S(t)]$ at time t under the risk-adjusted density function so that $S(t)$ is now described by the diffusion

$$\langle 3 \rangle \qquad dS(t) = gS(t)\,dt + \sigma S(t)\,dz.$$

For $\eta > t$, $\langle 3 \rangle$ implies that $\ln S(\eta)$ is normally distributed with mean $\ln S(t) + (g - \frac{1}{2}\sigma^2)(\eta - t)$ and standard deviation $\sigma\sqrt{(\eta - t)}$.

For values of $K \neq 0$, the evaluation of $\langle 2 \rangle$ is not straightforward as it requires knowledge of the conditional distribution of $M(t) = [A(t_N) - A(t)(m+1)/(N+1)]$. Specifically, $M(t)$ is a sum of correlated log normal random variables. Although expressions for the moment generating function, mean, and variance for the sum of two log normal variates exist,[5] closed-form expressions for the density function of sums of log normals are not possible. As a result, unlike conventional European options which do possess closed-form solutions, solving $\langle 2 \rangle$ will require numerical procedures.[6]

There is however a large body of evidence suggesting that the distribution of such sums is well-approximated by another log normal distribution.[7] Let us, for the moment, take $\ln M(t)$ as normally distributed with unknown mean $\alpha(t)$ and variance $v(t)$.[2] Then a simple approach to determining these parameters is to make use of the moment generating function for $X(t) = \ln M(t)$, $\Psi_x(k)$, given by

$$\langle 4 \rangle \qquad \Psi_x(k) = \mathbf{E}^*[M(t)^k] = e^{k\alpha(t) + 1/2k^2 v(t)^2}.$$

Thus we see that the first two moments of $M(t)$ are jointly complete and sufficient statistics for $\alpha(t)$ and $v(t)$. Regarding $\langle 4 \rangle$ for $k = 1$ and $k = 2$ as simultaneous equations in the unknown $\alpha(t)$ and $v(t)$ yields the following expressions:

$$\langle 5a \rangle \qquad \alpha(t) = 2 \ln \mathbf{E}^*[M(t)] - \tfrac{1}{2} \ln \mathbf{E}^*[M(t)^2],$$

$$\langle 5b \rangle \qquad v(t) = \sqrt{\ln \mathbf{E}^*[M(t)^2] - 2 \ln \mathbf{E}^*[M(t)]}.$$

Such a procedure has previously been employed in the field of communication engineering and is sometimes referred to as the 'Wilkinson approximation.'[8]

Accepting that $M(t)$ is distributed as log normal with mean $\alpha(t)$ and variance $v(t)^2$ given by $\langle 5 \rangle$, we can immediately evaluate $\langle 2 \rangle$ and express the call option as:

$$\langle 6 \rangle \qquad C[S(t), A(t), t] = e^{-r_d \tau}\{\mathbf{E}^*[M(t)]N(d_1) - [K - A(t)(m+1)/(N+1)]N(d_2)\},$$

where

$$d_1 = \frac{\tfrac{1}{2} \ln \mathbf{E}^*[M(t)^2] - \ln[K - A(t)(m+1)/(N+1)]}{v(t)},$$

$$d_2 = d_1 - v(t),$$

$v(t)$ is given by $\langle 5b \rangle$, and $N(.)$ is the cumulative normal distribution function. Expression $\langle 6 \rangle$ is the solution to a European option paying $\mathrm{Max}[M(t) - K^*, 0]$ at time t_N where $K^* = [K - A(t)(m+1)/(N+1)]$ and $\ln M(t)$ is distributed as $N[\alpha(t), v(t)]$. Thus the strike price of our ARO is reduced by the known component of $A(t_N)$.[9] Closed-form expressions for $\mathbf{E}^*[M(t)]$ and $\mathbf{E}^*[M(t)^2]$ are derived in the Appendix.

The value for the corresponding put option, $P[S(t), A(t), t]$, can be found by either using the above expression for the call option and put-call parity (expression $\langle 1 \rangle$) or by evaluating $e^{-r_d \tau}\mathbf{E}^* \mathrm{Max}[K^* - M(t), 0]$ under the assumption that $M(t)$ is distributed as log normal with parameters $\alpha(t)$ and $v(t)$ given by $\langle 5 \rangle$. Both approaches yield:[10]

$$P[S(t), A(t), t] = e^{-r_d \tau}\{\mathbf{E}^*[M(t)][N(d_1) - 1] - [K - A(t)(m+1)/(N+1)][N(d_2) - 1]\}.$$

A simple extension of the ideas here is the valuation of the 'floating ARO' or 'average strike option.' This is an option whose payoff on maturity is $\mathrm{Max}[A(t_N) - S(t_N), 0]$. That is, the strike price of the ARO is the spot rate on maturity [or alternatively, the strike price of a conventional European option is $A(t_N)$]. To provide a reliable approximation to the valuation of such an option, we assume $\mathbf{X}(t) = [\ln A(t_N), \ln S(t_N)]$ is approximately bivariate normal and hence the option valuation problem can be solved in the manner of Margrabe (1978).

To determine the necessary parameters to value this option, extend the Wilkinson approximation to vector processes. Under the bivariate lognormal distribution assumption, we use of the moment generating function for $\mathbf{X}(t)$, $\Psi_x(k_1, k_2)$, to produce sets of simultaneous equations. The moment generating function for $\mathbf{X}(t)$ is:

$$\Psi_x(k_1, k_2) = \mathbf{E}^*[A(t_N)^{k_1} S(t_N)^{k_2}] = e^{\alpha_1 k_1 + \alpha_2 k_2 + 1/2(v_1^2 k_1^2 + 2\rho v_1 v_2 k_1 k_2 + v_2^2 k_2^2)}.$$

Substituting $(k_1, k_2) = (0, 1)$ and $(0, 2)$ yields $\alpha_2 = \ln S(t) + (g - \frac{1}{2}\sigma^2)\tau$ and $v_2 = \sigma\sqrt{\tau}$ as parameters for $\ln S(t_N)$. For $(k_1, k_2) = (1, 0)$ and $(2, 0)$ the mean, α_1, and standard deviation, v_1, for $\ln A(t_N)$ are determined as:

$$\alpha_1 = 2 \ln \mathbf{E}^*[A(t_N)] - \tfrac{1}{2} \ln \mathbf{E}^*[A(t_N)^2],$$

$$v_1 = \sqrt{\ln \mathbf{E}^*[A(t_N)^2] - 2 \ln \mathbf{E}^*[A(t_N)]}.$$

Finally, the covariance term, $\rho v_1 v_2$, is determined as

$$\rho v_1 v_2 = \ln \mathbf{E}^*[A(t_N) S(t_N)] - (\alpha_1 + \alpha_2) - \tfrac{1}{2}(v_1^2 + v_2^2),$$

having substituted $(k_1, k_2) = (1, 1)$. The expression for $\mathbf{E}^*[A(t_N)S(t_N)]$ is derived in the Appendix. Those for $\mathbf{E}^*[A(t_N)]$ and $\mathbf{E}^*[A(t_N)^2]$ may be readily found using $A(t_N) = A(t) + M(t)$.

The expressions for $\alpha(t)$ and $v(t)$ in $\langle 5 \rangle$ represent reliable estimates for the mean and standard deviation of $\ln M(t)$ to the extent that $M(t)$ is approximately log normally distributed. Some idea as to the seriousness of the error in the approximation, and hence the reliability of these estimates, can be gauged by comparing the Wilkinson approximation with outcomes generated using Monte Carlo methods.

III. Empirical results

In this section we investigate the approximation error in the Wilkinson procedure and evaluate the adequacy of the ARO pricing equation (expression $\langle 6 \rangle$) using Monte Carlo methods.

The empirical moments for $\ln M(t)$ are compared with those implied by the Wilkinson approximation. If $X(t) = \ln M(t)$ has mean $\alpha(t)$ and variance $v(t)^2$ then, under our assumption, the kth moment of $X(t)$ is given by

$$\mathbf{E}^*[X(t)^k] = \left[\frac{\partial^k \Psi_x(\phi)}{\partial \phi^k}\right]_{\phi = 0},$$

where $\Psi_x(\phi) = \exp(\phi\alpha(t) + \frac{1}{2}\phi^2 v(t)^2)$ is again the moment generating function for $X(t)$. The approximation for $\alpha(t)$ and $v(t)$ can be examined by substituting the discrete-form expressions for the first two moments of $M(t)$ given in the Appendix.

From a practical viewpoint, only discrete averages are relevant. However, as a simplification to ease computations, and to provide upper bounds (for $t \leqslant t_0$) to the approximate values, we might substitute continuous-form expressions $\lim \mathbf{E}^*[M(t)]$ and $\lim \mathbf{E}^*[M(t)^2]$ as $N \to \infty$:

$$\langle 7a \rangle \qquad \lim \mathbf{E}^*[M(t)] = \frac{S(t)}{(t_N - t_0)g}(e^{g\tau} - 1) \quad \text{for } t > t_0,$$

and

$$\langle 7b \rangle \qquad \lim \mathbf{E}^*[M(t)] = \frac{S(t)e^{g(t_0 - t)}}{(t_N - t_0)g}(e^{g(t_N - t_0)} - 1) \quad \text{for } t \leqslant t_0,$$

$$\langle 8a \rangle \qquad \lim \mathbf{E}^*[M(t)^2] = \frac{2S(t)^2}{(t_N - t_0)^2(g + \sigma^2)}\left[\frac{e^{(2g+\sigma^2)\tau} - 1}{(2g + \sigma^2)} - \frac{e^{g\tau} - 1}{g}\right] \quad \text{for } t > t_0,$$

and

$$\langle 8b \rangle \qquad \lim \mathbf{E}^*[M(t)^2] = \frac{2S(t)^2 e^{(2g+\sigma^2)(t_0 - t)}}{(t_N - t_0)^2(g + \sigma^2)}\left[\frac{e^{(2g+\sigma^2)(t_N - t_0)} - 1}{(2g + \sigma^2)} - \frac{e^{g(t_N - t_0)} - 1}{g}\right] \quad \text{for } t \leqslant t_0.$$

This will also allow us to consider in these experiments the rate of convergence to the continuous values as $N \to \infty$ and thus the importance of allowing for discretization when valuing average options. The tables below refer to moments for $\ln M(t)$ implied by the discrete and continuous approximations as *WD* and *WC*, respectively, with the empirical moments from the Monte Carlo outcomes denoted by *MC*.

The distribution for $\ln M(t)$ is examined for values of $N = 4$, 32, and 256, and a range of values for σ between 0.10 and 1.00. Simulations of $M(t)$ were achieved by generating a sequence of spot rates $S(t_i)$ for $i = 1, 2, \ldots, \mathrm{N}$ using the following expression:

$$S(t_i) = S(t_{i-1})e^{(r_d - r_f - 1/2\sigma^2)h + \omega_i \sigma \sqrt{h}},$$

where ω_i is distributed as normal with zero mean and unit variance.

Monte Carlo estimates for $\alpha(t)$ and $v(t)$ were calculated by averaging 10 000 replications of $\ln M(t)$. In each replication, $r_d = 0.08$, $r_f = 0.16$, $S(t) = 1.5$, $t = t_0 - 0.001$ (thus the current timepoint is a fraction before the first fixing), and $\tau = 1$. To reduce the standard error of the estimates further, moments of the logarithm of the corresponding geometric average were employed as control variates for the moments of $\ln M(t)$.

Tables 1, 2, and 3 present a sample of outcomes of these experiments. The tables display the first four moments of $\ln M(t)$ together with their standard errors (in parentheses). In addition, statistics γ_1 and γ_2 are presented which test for, respectively, the null hypotheses of zero skewness and zero kurtosis (see Madansky, 1988, p. 37). Under the null, both statistics are distributed as normal

TABLE 1. Analysis of moments for $X(t) = \ln M_N(t)$, $N = 4$.

	$\sigma = 0.20$			$\sigma = 0.50$			$\sigma = 0.80$		
Moment	MC	WD	WC	MC	WD	WC	MC	WD	WC
$E[X(t)]$	0.3601	0.3600	0.3592	0.3307	0.3283	0.3240	0.2772	0.2642	0.2554
	(4×10^{-5})			(3×10^{-4})			(7×10^{-4})		
$E[X(t)^2]$	0.1412	0.1413	0.1421	0.1794	0.1829	0.1884	0.2429	0.2732	0.2858
	(3×10^{-5})			(3×10^{-4})			(0.0012)		
$E[X(t)^3]$	0.0593	0.0593	0.0604	0.1114	0.1094	0.1151	0.1937	0.1796	0.1857
	(2×10^{-5})			(4×10^{-4})			(0.0020)		
$E[X(t)^4]$	0.0264	0.0263	0.0273	0.0812	0.0772	0.0844	0.2129	0.2141	0.2366
	(2×10^{-5})			(6×10^{-4})			(0.0044)		
γ_1	0.1222			0.3072			0.5075		
γ_2	0.0297			0.1998			0.3504		

Notes: MC = Monte Carlo estimate (standard error in parentheses).
WD = Wilkinson discrete approximation.
WC = Wilkinson continuous approximation.

TABLE 2. Analysis of moments for $X(t) = \ln M_N(t)$, $N = 32$.

	$\sigma = 0.20$			$\sigma = 0.50$			$\sigma = 0.80$		
Moment	*MC*	*WD*	*WC*	*MC*	*WD*	*WC*	*MC*	*WD*	*WC*
$\mathbf{E}[X(t)]$	0.3593	0.3593	0.3592	0.3262	0.3247	0.3240	0.2665	0.2568	0.2554
	(3×10^{-5})			(2×10^{-4})			(5×10^{-4})		
$\mathbf{E}[X(t)^2]$	0.1419	0.1420	0.1421	0.1839	0.1876	0.1884	0.2618	0.2838	0.2858
	(2×10^{-5})			(3×10^{-4})			(0.0010)		
$\mathbf{E}[X(t)^3]$	0.0603	0.0603	0.0604	0.1147	0.1142	0.1151	0.1963	0.1848	0.1857
	(2×10^{-5})			(3×10^{-4})			(0.0016)		
$\mathbf{E}[X(t)^4]$	0.0272	0.0272	0.0273	0.0845	0.0833	0.0844	0.2285	0.2329	0.2366
	(1×10^{-5})			(5×10^{-4})			(0.0032)		
γ_1	0.0786			0.1991			0.2977		
γ_2	−0.0051			0.0231			0.1792		

Notes: See Table 1.

Table 3. Analysis of moments for $X(t) = \ln M_N(t)$, $N = 256$.

	$\sigma = 0.20$			$\sigma = 0.50$			$\sigma = 0.80$		
Moment	MC	WD	WC	MC	WD	WC	MC	WD	WC
$\mathbf{E}[X(t)]$	0.3593	0.3592	0.3592	0.3254	0.3241	0.3240	0.2641	0.2556	0.2554
	(4×10^{-5})			(2×10^{-4})			(5×10^{-4})		
$\mathbf{E}[X(t)^2]$	0.1420	0.1421	0.1421	0.1851	0.1883	0.1884	0.2648	0.2856	0.2858
	(2×10^{-5})			(3×10^{-4})			(0.0010)		
$\mathbf{E}[X(t)^3]$	0.0604	0.0604	0.0604	0.1160	0.1150	0.1151	0.1975	0.1856	0.1857
	(2×10^{-5})			(3×10^{-4})			(0.0017)		
$\mathbf{E}[X(t)^4]$	0.0274	0.0273	0.0273	0.0861	0.0843	0.0844	0.2319	0.2361	0.2366
	(2×10^{-5})			(5×10^{-4})			(0.0035)		
γ_1	0.0720			0.1896			0.2840		
γ_2	0.0127			0.0299			0.1415		

Notes: See Table 1.

with zero mean. For γ_1 the asymptotic standard error under the null hypothesis with 10 000 replications is 0.0245, whilst that for γ_2 is 0.0490.

The results indicate that, for values of σ less than 0.20, the discrete Wilkinson approximation is very accurate. Beyond this value for σ the approximation deteriorates across all values of N. Secondly, the continuous Wilkinson approximation (WC) is found to be acceptable for values of $N > 30$. This implies that the value of an ARO with weekly fixings is likely to have a similar value to one with daily fixings.[11] Finally, the tests for lack of skewness are comprehensively rejected. Values for γ_1 indicate a general tendency for the right hand tail of the distribution to be heavier than the left hand tail. The tests for kurtosis, however, are significant only for $\sigma > 0.40$ for $N = 4$ and $\sigma > 0.60$ for $N = 32$ and $N = 256$.

Apart from the evidence for the presence of skewness, these results are very encouraging. In the foreign exchange market, options are rarely quoted with an implied annualized volatility (σ) greater than 20 per cent. However, the experiments clearly demonstrate the unreliability of the procedure for average options on an underlying asset whose volatility is in excess of 30 per cent. To assess the significance of the evidence for skewness on valuation, we now examine the performance of the pricing formula itself.

To determine values for the call option, experiments (each of 10 000 replications) of $e^{-r_d\tau}\,\mathrm{Max}[\mathrm{A}(\mathrm{t}_N) - \mathrm{K}, 0]$ were generated with $S(t) = 1.5$, $r_d = 0.15$, $r_f = 0.10$, and $(t_N - t_0) = 1$. The results are presented in Tables 4, 5, and 6 for; $t = t_0 - 0.5$, t_0, and $t_0 + 0.5$; $\sigma = 0.10$, 0.20, and 0.30; and $N = 4$, 12, and 256. In each experiment, the replications were averaged and an estimated standard error calculated (the maximum standard error corresponding to σ and N is reported in the tables). Following Kemna and Vorst (1990), the corresponding geometric average rate option was used as a control variate to improve the accuracy of each estimated option value. In order to assess the behaviour of the Wilkinson approximation in the region of the tails of the distribution for $M(t)$, we examined AROs over a range of values for K.

The standard errors indicate that the Monte Carlo estimates of the ARO values are accurate to within 0.0001 with 95 per cent confidence. As expected from the above analysis of the distribution for $M(t)$, the Wilkinson approximation (WD) is generally good across all values for N and K for $\sigma \leqslant 0.20$. There is a slight tendency for the approximation to undervalue out-of-the-money calls and overvalue in-the-money calls. For the case of $\sigma = 0.10$, the accuracy is to within 0.007 per cent of the underlying spot price. For $\sigma = 0.20$ the greatest error is 0.02 per cent of the underlying spot price and for $\sigma = 0.30$ it is 0.06 per cent.

The precision of this formula improves as the ARO contract tends in nature to a conventional European option. This occurs (more quickly for small values of N) as the ARO reaches maturity, or for AROs where the averaging period $(t_N - t_0)$ is much smaller than the option horizon $(t_N - t)$. Imprecision primarily occurs for deep in-the-money options and large values of σ. The simulations indicate, however, that fine accuracy is generally secured for $\sigma\sqrt{\tau} \leqslant 0.20$. Further simulations (not reported) indicate that the interest differential, g, has little or no bearing on the success of the Wilkinson approximation. Finally, the tables demonstrate the success of the continuous time limit expressions $\langle 7 \rangle$ and $\langle 8 \rangle$ (denoted in the tables as WC) for evaluating AROs with daily fixings.

A curious feature displayed by these tables is the comparative cheapness of AROs with different values of N. We know that conventional European options

Table 4. Average rate call option values, 10 000 replications.

$t = t_0 - 0.5$		$N = 4$		$N = 12$		$N = 256$		$N \to \infty$
σ	$K/S(t)$	MC	WD	MC	WD	MC	WD	WC
	1.2	0.0039	0.0038	0.0040	0.0040	0.0041	0.0041	0.0041
	1.1	0.0228	0.0228	0.0232	0.0232	0.0235	0.0235	0.0235
0.10	1.0	0.0814	0.0815	0.0818	0.0819	0.0821	0.0822	0.0822
	0.9	0.1831	0.1832	0.1832	0.1832	0.1832	0.1833	0.1833
	0.8	0.3012	0.3012	0.3012	0.3012	0.3011	0.3011	0.3011
Std error		2.0×10^{-5}		1.7×10^{-5}		1.6×10^{-5}		
	1.2	0.0326	0.0325	0.0334	0.0333	0.0340	0.0338	0.0338
	1.1	0.0662	0.0661	0.0671	0.0671	0.0677	0.0677	0.0677
0.20	1.0	0.1220	0.1222	0.1231	0.1231	0.1236	0.1237	0.1237
	0.9	0.2036	0.2038	0.2042	0.2044	0.2046	0.2048	0.2048
	0.8	0.3065	0.3067	0.3067	0.3069	0.3069	0.3071	0.3071
Std error		7.6×10^{-5}		6.4×10^{-5}		5.3×10^{-5}		
	1.2	0.0729	0.0727	0.0742	0.0741	0.0750	0.0750	0.0750
	1.1	0.1110	0.1112	0.1126	0.1127	0.1134	0.1136	0.1137
0.30	1.0	0.1645	0.1650	0.1661	0.1664	0.1669	0.1673	0.1673
	0.9	0.2342	0.2363	0.2367	0.2374	0.2376	0.2381	0.2382
	0.8	0.3243	0.3252	0.3254	0.3259	0.3260	0.3264	0.3264
Std error		1.7×10^{-4}		1.5×10^{-4}		1.3×10^{-4}		

Notes: MC = Monte Carlo estimate; Std error = maximum standard error for MC corresponding to σ and N. WD = Wilkinson discrete approximation; WC = Wilkinson continuous approximation; $S(t) = 1.5$; $A(t) = 0$; $r_d = 0.15$; $r_f = 0.10$; $t_N - t_0 = 1$.

TABLE 5. Average rate call option values. 10 000 replications.

$t = t_0$		$N = 4$		$N = 12$		$N = 256$		$N \to \infty$
σ	$K/S(t)$	*MC*	*WD*	*MC*	*WD*	*MC*	*WD*	*WC*
	1.2	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
	1.1	0.0038	0.0038	0.0042	0.0042	0.0045	0.0044	0.0044
0.10	1.0	0.0481	0.0482	0.0489	0.0490	0.0494	0.0495	0.0495
	0.9	0.1622	0.1622	0.1622	0.1622	0.1622	0.1622	0.1622
	0.8	0.2911	0.2911	0.2911	0.2911	0.2910	0.2910	0.2910
Std error		2.0×10^{-5}		1.7×10^{-5}		1.6×10^{-5}		
	1.2	0.0062	0.0060	0.0067	0.0065	0.0071	0.0068	0.0068
	1.1	0.0248	0.0247	0.0261	0.0259	0.0270	0.0267	0.0268
0.20	1.0	0.0753	0.0755	0.0772	0.0774	0.0781	0.0784	0.0785
	0.9	0.1694	0.1696	0.1703	0.1707	0.1710	0.1714	0.1714
	0.8	0.2914	0.2915	0.2916	0.2917	0.2917	0.2918	0.2918
Std error		7.1×10^{-5}		6.2×10^{-5}		5.6×10^{-5}		
	1.2	0.0235	0.0231	0.0249	0.0243	0.0259	0.0251	0.0252
	1.1	0.0517	0.0515	0.0540	0.0537	0.0551	0.0550	0.0551
0.30	1.0	0.1034	0.1038	0.1061	0.1067	0.1077	0.1084	0.1085
	0.9	0.1858	0.1864	0.1882	0.1890	0.1893	0.1905	0.1905
	0.8	0.2958	0.2961	0.2966	0.2974	0.2974	0.2982	0.2982
Std error		1.7×10^{-4}		1.4×10^{-4}		1.3×10^{-4}		

Notes: See Table 4. $A(t)$.

Table 6. Average rate call option values, 10 000 replications.

$t = t_0 + 0.5$		$N = 4$		$N = 12$		$N = 256$		$N \to \infty$
σ	$K/S(t)$	MC	WD	MC	WD	MC	WD	WC
	1.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	1.0	0.0185	0.0185	0.0172	0.0172	0.0164	0.0164	0.0164
	0.9	0.1497	0.1497	0.1486	0.1486	0.1480	0.1480	0.1479
	0.8	0.2889	0.2889	0.2878	0.2878	0.2871	0.2871	0.2871
Std error		1.0×10^{-5}		8.0×10^{-6}		8.0×10^{-6}		
	1.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1.1	0.0010	0.0009	0.0006	0.0006	0.0005	0.0004	0.0004
0.20	1.0	0.0307	0.0307	0.0288	0.0288	0.0275	0.0276	0.0275
	0.9	0.1498	0.1498	0.1486	0.1487	0.1480	0.1480	0.1480
	0.8	0.2889	0.2889	0.2878	0.2878	0.2871	0.2871	0.2871
Std error		3.9×10^{-5}		3.1×10^{-5}		2.8×10^{-5}		
	1.2	0.0005	0.0004	0.0003	0.0002	0.0002	0.0001	0.0001
	1.1	0.0057	0.0057	0.0045	0.0044	0.0039	0.0037	0.0036
0.30	1.0	0.0430	0.0431	0.0406	0.0406	0.0388	0.0390	0.0389
	0.9	0.1508	0.1508	0.1495	0.1496	0.1486	0.1488	0.1488
	0.8	0.2889	0.2889	0.2878	0.2878	0.2871	0.2871	0.2871
Std error		8.5×10^{-5}		6.8×10^{-5}		6.1×10^{-5}		

Notes: See Table 4. $A(t)$.

are relatively more expensive than otherwise identical AROs. This is largely explained by the fact that the variance of an average is smaller than the variance of the underlying price process. However, compare two AROs, daily and quarterly fixings (say), both with one year maturity and identical strike prices. In general, we should expect an ARO to be cheaper than an otherwise identical ARO with larger N because a relatively higher proportion of the exposure is taken out by the first fixing. Thus the quarterly ARO will initially be less expensive than the daily ARO. However, with one quarter remaining and assuming $A(t)$ is identical for both AROs, the quarterly ARO is a European option on $S(t_N)/5$ with strike K^*. The payout on the daily ARO, however, is still determined by the difference between the average of prices over this remaining period and K^* and thus will be less expensive. In general, as the maturity date approaches, *ceteris paribus*, a given ARO will always subsequently become more expensive than one with larger N.

In sum, under typical market conditions, the Wilkinson approximation provides a quick and accurate method for valuing AROs in foreign exchange and has the further benefit of ensuring that put-call parity is maintained. The approach is simple to implement and circumvents the need for time-consuming numerical procedures.

IV. Summary and observations

This paper has introduced a convenient and accurate method for valuing average rate currency options. The approach relies on the assumption that the distribution of sums of log normal variates is itself well approximated at least to a first order by the log normal. Our simulations have shown that this assumption is acceptable particularly for values of $\sigma\sqrt{\tau}$ less than 0.20. For larger values the accuracy of the pricing formula deteriorates and we might consider modifying the approach to estimating the necessary parameters by correcting the Wilkinson estimates.

Augmenting the pricing formula along the lines of Turnbull and Wakeman (1991) is one way to correct for the effects of higher volatility. Some useful results that should help in formulating correction factors are given in Fenton (1960) and Janos (1970). Fenton proposes that, where interest is in the tails of the distribution one should pay more attention to higher moments. Thus a better approximation for higher volatility conditions could be achieved by equating higher moments in equation $\langle 4 \rangle$, say $k = 3$ and $k = 4$. Janos shows that the behavior of the logarithm of the tail of the distribution of sums of log normals is dominated by that term with largest logarithmic variance. His results therefore suggest that we should concentrate attention to the distribution of $S(t_N)$ when correcting the tails. Lastly, further improvements could be made by adopting the more precise procedures in Schwartz and Yeh (1982). These ideas no doubt will lead to more satisfactory approximations to the valuation of AROs but their further consideration is beyond the scope of this paper. It is hoped that the results here provide some direction and encouragement for future research.

Appendix

In this appendix we provide expressions for $\mathbf{E}^*[M(t)]$, $\mathbf{E}^*[M(t)^2]$, and $\mathbf{E}^*[A(t_N)S(t_N)]$ conditional on time t when $S(t)$ satisfies the diffusion $\langle 3 \rangle$.

Let ξh be the time interval from the last spot rate fixing at t_m for some m, $0 \leqslant m \leqslant N$. Thus ξ is defined as $\xi = (t - t_m)/h$. Then from the definition for $M(t)$, we have

$$\mathbf{E}^*[M(t)] = \frac{1}{N+1} \sum_{i=m+1}^{N} \mathbf{E}^*[S(t_i)]$$
$$= \frac{1}{N+1} \sum_{i=m+1}^{N} S(t) e^{g(i-m-\xi)h}$$
$$= \frac{S(t)}{N+1} e^{g(1-\xi)h} \left[\frac{1 - e^{g(N-m)h}}{1 - e^{gh}} \right].$$

For $t < t_0$, $\mathbf{E}^*[M(t)]$ is given as:

$$\mathbf{E}^*[M(t)] = \frac{S(t)}{N+1} e^{g(t_0 - t)} \left[\frac{1 - e^{g(N+1)h}}{1 - e^{gh}} \right].$$

We can express $\mathbf{E}^*[M(t)^2]$ for $t \geqslant t_0$ as

$$\mathbf{E}^*[M(t)^2] = \frac{1}{(N+1)^2} \sum_{i=m+1}^{N} \sum_{j=m+1}^{N} \mathbf{E}^*[S(t_i)S(t_j)].$$

The expectation term in this expression is given by

$$\mathbf{E}^*[S(t_i)S(t_j)] = e^{g(i+j-2\xi)h + \sigma^2[\mathrm{Min}(i,j) - \xi]h} S(t)^2,$$

and hence (after some tedious algebra) we can evaluate $\mathbf{E}^*[M(t)^2]$ as

$$\mathbf{E}^*[M(t)^2] = \frac{S(t)^2}{(N+1)^2} e^{-2\xi(g + 1/2\sigma^2)h} (A_1 - A_2 + A_3 - A_4),$$

where

$$A_1 = \frac{e^{(2g+\sigma^2)h} - e^{(2g+\sigma^2)(N-m+1)h}}{(1 - e^{gh})(1 - e^{(2g+\sigma^2)h})},$$

$$A_2 = \frac{e^{[g(N-m+2)+\sigma^2]h} - e^{(2g+\sigma^2)(N-m+1)h}}{(1 - e^{gh})(1 - e^{(g+\sigma^2)h})},$$

$$A_3 = \frac{e^{(3g+\sigma^2)h} - e^{[g(N-m+2)+\sigma^2]h}}{(1 - e^{gh})(1 - e^{(g+\sigma^2)h})},$$

$$A_4 = \frac{e^{2(2g+\sigma^2)h} - e^{(2g+\sigma^2)(N-m+1)h}}{(1 - e^{(g+\sigma^2)h})(1 - e^{(2g+\sigma^2)h})}.$$

For $t < t_0$

$$\mathbf{E}^*[M(t)^2] = \frac{S(t)^2}{(N+1)^2} e^{(2g+\sigma^2)(t_0 - t)} (B_1 - B_2 + B_3 - B_4),$$

where

$$B_1 = \frac{1 - e^{(2g+\sigma^2)(N+1)h}}{(1 - e^{gh})(1 - e^{(2g+\sigma^2)h})},$$

$$B_2 = \frac{e^{g(N+1)h} - e^{(2g+\sigma^2)(N+1)h}}{(1 - e^{gh})(1 - e^{(g+\sigma^2)h})},$$

$$B_3 = \frac{e^{gh} - e^{g(N+1)h}}{(1 - e^{gh})(1 - e^{(g+\sigma^2)h})},$$

$$B_4 = \frac{e^{(2g+\sigma^2)h} - e^{(2g+\sigma^2)(N+1)h}}{(1 - e^{(g+\sigma^2)h})(1 - e^{(2g+\sigma^2)h})}.$$

Finally, for $t \geqslant t_0$, $\mathbf{E}^*[A(t_N)S(t_N)]$ can be found as follows

$$\begin{aligned}
\mathbf{E}^*[A(t_N)S(t_N)] &= \frac{m+1}{N+1} A(t)\mathbf{E}^*[S(t_N)] + \frac{1}{N+1} \sum_{i=m+1}^{N} \mathbf{E}^*[S(t_i)S(t_N)] \\
&= \frac{m+1}{N+1} A(t)S(t)e^{g(N-m-\xi)h} + \frac{S(t)^2}{N+1} \sum_{i=1}^{N-m} e^{g(i+N-m-2\xi)h+\sigma^2(i-\xi)h} \\
&= \frac{m+1}{N+1} A(t)S(t)e^{g(N-m-\xi)h} \\
&\quad + \frac{S(t)^2}{N+1} e^{g(N-m+1-2\xi)h+\sigma^2(1-\xi)h} \left[\frac{1-e^{(g+\sigma^2)(N-m)h}}{1-e^{(g+\sigma^2)h}} \right].
\end{aligned}$$

Notes

1. Ritchken *et al.* (1990) have recently independently advanced a similar approach to value the ARO. Their paper does not allow for an underlying asset which pays a dividend (the foreign rate of interest for currency AROs).
2. Ritchken *et al.* (1990) and Carverhill and Clewlow (1990) define the average as determined in the interval $[t_0, t_N]$ but at points on this interval $t_i = t_0 + ih$ for $i = 1, \ldots, N$. Thus the first fixing is at time t_1 with $N - 1$ remaining.
3. That is, no transaction costs, unrestricted borrowing or lending at a given interest rate, and continuous trading.
4. For our purposes, it is reasonable to assume that markets are continuously arbitraged. Thus synthetic forward prices can always be constructed by appropriate positions in the foreign exchange and money markets (see, for example, Walmsley, 1983).
5. See for example, Naus (1969), Hamdan (1971), and Schwartz and Yeh (1982).
6. Garman and Kohlhagen (1983) and Grabbe (1983) examine the valuation of conventional European currency options.
7. For example, Janos (1970), Barakat (1976), and Schwartz and Yeh (1982).
8. Marlow (1967) and Nasell (1967) refer to an unpublished work in 1934 by R.I. Wilkinson at Bell Telephone Laboratories. As an alternative to $\langle 5 \rangle$, we could have adopted the procedure in Schwartz and Yeh (1982), who provide exact expressions for the mean and variance of the logarithm of the sum of two log normal variates. Their approach can be applied sequentially to determine more precise estimates for $\alpha(t)$ and $v(t)$ if needed.
9. Levy (1990) approximates $A(t_N)$ by the log normal using continuous time expressions.
10. The term $e^{-r_d\tau}\{\mathbf{E}^*[M(t)] + A(t)(m+1)/(N+1)\}$ is the value at time t of the 'averaging claim' considered by Bergman (1985). Note however that, given the above argument determining the put-call parity condition, we do not require $S(t)$ to be described by an Itô process to value this claim, only that the spot rates be defined at the points where the averaging rates are collected.
11. This finding is also noted in Turnbull and Wakeman (1991).

References

Barakat, Richard, 'Sums of Independent Lognormally Distributed Random Variables,' *Journal of the Optical Society of America*, March 1976, **66:** 211–216.

Bergman, Yaacov Z., 'Pricing Path Contingent Claims,' in Haim Levy, ed., *Research in Finance*, Vol. 5, JAI Press Inc., 1985, 229–241.

Carverhill, Andrew P., and Les J. Clewlow, 'Flexible Convolution,' *Risk*, April 1990, **3:** 25–29.

Cox, John C., and Stephen A. Ross, 'The Valuation of Options for Alternative Stochastic Processes,' *Journal of Financial Economics*, September 1976, **3:** 145–166.

Fenton, Lawrence F., 'The Sum of Log-Normal Probability Distributions in Scatter Transmission Systems,' *IRE Transactions on Communications Systems*, March 1960, **CS-8:** 57–67.

GARMAN, MARK B., AND STEVEN W. KOHLHAGEN, 'Foreign Currency Option Values,' *Journal of International Money and Finance*, December 1983, **2:** 231–237.

GRABBE, J. ORLIN, 'The Pricing of Call and Put Options on Foreign Exchange,' *Journal of International Money and Finance*, December 1983, **2:** 239–253.

HAMDAN, M.A., 'The Logarithm of the Sum of Two Correlated Log-normal Variates,' *Journal of the American Statistical Association*, March 1971, **66:** 105–106.

JANOS, WILLIAM A., 'Tail of the Distribution of Sums of Log-Normal Variates,' *IEEE Transactions on Information Theory*, May 1970, **IT-16:** 299–302.

KEMNA, A.G.Z., AND A.C.F. VORST, 'A Pricing Method for Options Based on Average Asset Values,' *Journal of Banking and Finance*, March 1990, **14:** 113–129.

LEVY, EDMOND, 'Asian Arithmetic,' *Risk*, May 1990, **3:** 7–8.

MADANSKY, ALBERT, *Prescriptions for Working Statisticians*, Springer-Verlag New York Inc., 1988.

MARGRABE, WILLIAM, 'The Value of an Option to Exchange One Asset for Another,' *Journal of Finance*, March 1978, **33:** 177–186.

MARLOW, N.A., 'A Normal Limit Theorem for Power Sums of Independent Random Variables,' *Bell System Technical Journal*, November 1967, **46:** 2081–2089.

NASELL, I.E.O., 'Some Properties of Power Sums of Truncated Normal Random Variables,' *Bell System Technical Journal*, November 1967, **46:** 2091–2110.

NAUS, J.I., 'The Distribution of the Logarithm of the Sum of Two Log-Normal Variates,' *Journal of the American Statistical Association*, June 1969, **64:** 655–659.

RITCHKEN, PETER, L. SANKARASUBRAMANIAN, AND ANAND M. VIJH, 'The Valuation of Path Dependent Contracts on the Average,' unpublished manuscript, School of Business Administration, University of Southern California, September 1990.

RUTTIENS, ALAIN, 'Classical Replica,' *Risk*, February 1990, **3:** 33–36.

SCHWARTZ, STUART C., AND Y.S. YEH, 'On the Distribution Function and Moments of Power Sums with Log-Normal Components,' *Bell System Technical Journal*, September 1982, **61:** 1441–1462.

TURNBULL, STUART M., AND LEE MACDONALD WAKEMAN, 'A Quick Algorithm for Pricing European Average Options,' *Journal of Financial and Quantitative Analysis*, September 1991, **26:** 377–389.

VORST, TON, 'Analytic Boundaries and Approximations of the Prices and Hedge Ratios of Average Exchange Rate Options,' unpublished manuscript, Econometric Institute, Erasmus University Rotterdam, February 1990.

WALMSLEY, JULIAN K., *The Foreign Exchange Handbook*, New York: John Wiley, 1983.