1 Log-normal approximation for Asian option in the BS model

A very popular approach for pricing Asian options is the log-normal approximation, or the Levy approximation. This assumes that the arithmetic time-average of the price (either continuous-time or discrete-time averaging)

$$A_T = \frac{1}{T}\int_0^T dt S_t \tag{1}$$

is log-normally distributed. Under this assumption, the time-average has the form

$$A_T = \mathbb{E}[A_T]\exp(\Sigma_{\rm LN}\sqrt{T}Z - \frac{1}{2}\Sigma_{\rm LN}^2 T) \tag{2}$$

with $Z \sim N(0,1)$ and volatility

$$\Sigma_{\rm LN}^2 T = \log\left(\frac{\mathbb{E}[A_T^2]}{\mathbb{E}[A_T]^2}\right). \tag{3}$$

2 Explicit results in the BS model

In the BS model, the asset price follows a geometric Brownian motion

$$S_t = S_0 e^{\sigma W_t + (r-\frac{1}{2}\sigma^2)t}. \tag{4}$$

Strictly speaking one should replace $r \to r-q$, but for simplicity I took $q=0$.

The first two moments of the arithmetic time-average can be obtained in closed form

$$\mathbb{E}[A_T] = S_0\frac{1}{rT}(e^{rT}-1) \tag{5}$$

$$\mathbb{E}[A_T^2] = S_0^2\frac{2}{(\sigma^2+r)T}\left\{\frac{1}{(\sigma^2+2r)T}\left(e^{(\sigma^2+2r)T}-1\right) - \frac{1}{rT}\left(e^{rT}-1\right)\right\}. \tag{6}$$

Remark. Denoting $\rho = rT$, we observe that the expectation of the time-average is just A_∞ in the notations of our paper

$$\mathbb{E}[A_T] = S_0\frac{1}{\rho}(e^\rho - 1) = A_\infty. \tag{7}$$

We will show that in the limit $\sigma^2 T \to 0$ and keeping ρ constant, the log-normal volatility Σ_{LN} defined by (3) approaches the asymptotic ATM implied volatility of the Asian option given in (109) of the paper

$$\lim_{n\to\infty}\frac{\Sigma_{\rm BS}(A_\infty, n)}{\sigma} = \frac{S_0}{A_\infty}\sqrt{v(\rho)} \tag{8}$$

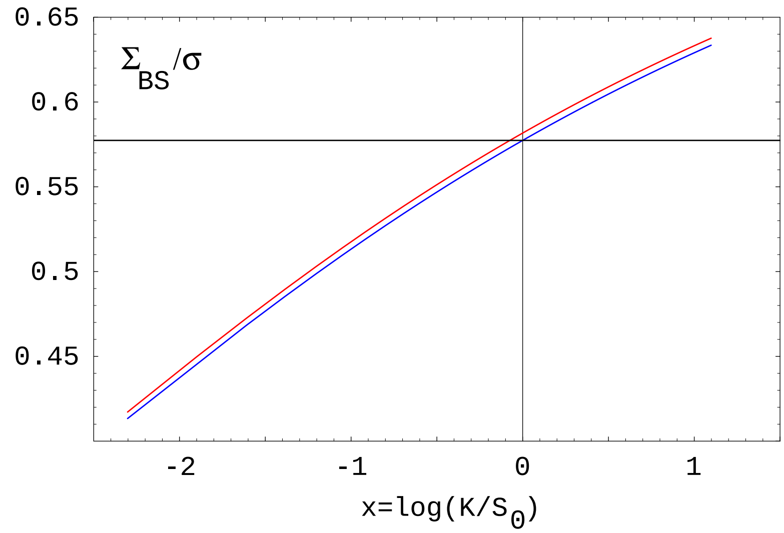

Figure 1: Plot of the asymptotic log-normal volatility of an Asian option $\Sigma_{\rm BS}(K/S_0,\rho)/\sigma$ vs $x=\log(K/S_0)$. The blue curve corresponds to $\rho=0$ and the red curve to $\rho=0.1$. The horizontal line shows the $T\to 0$ limit of the log-normal approximation $1/\sqrt{3}$.

with

$$v(\rho)=\frac{1}{\rho^3}\left\{\left(\rho-\frac{3}{2}\right)e^{2\rho}+2e^{\rho}-\frac{1}{2}\right\}\,. \tag{9}$$

Proposition. *Denote $x=\sigma^2 T$ and $\rho=rT$. Then we have the following limiting result*

$$\lim_{x\to 0}\Sigma_{\rm LN}|_{\rho\ \rm fixed}=\sigma\frac{\sqrt{v(\rho)}}{(e^{\rho}-1)/\rho}\,. \tag{10}$$

This means that the log-normal approximation becomes exact in the $\sigma^2 T\to 0$ limit (small maturity or small volatility) for the ATM strike $K=A_\infty$.

For $\rho=0$ (zero interest rate) this reduces to

$$\lim_{x\to 0}\Sigma_{\rm LN}|_{\rho=0}=\sigma\frac{1}{\sqrt{3}}\,. \tag{11}$$

A plot of the function multiplying σ in (10) vs ρ is shown below.

Proof. Follows from the limiting result

$$\lim_{\sigma^2T\to 0}\frac{1}{x}\log\frac{\mathbb{E}[A_T^2]}{\mathbb{E}[A_T]^2}=\frac{(2\rho-3)e^{2\rho}+4e^{\rho}-1}{2\rho(e^{\rho}-1)^2}=\frac{v(\rho)}{(A_\infty/S_0)^2}\,. \tag{12}$$

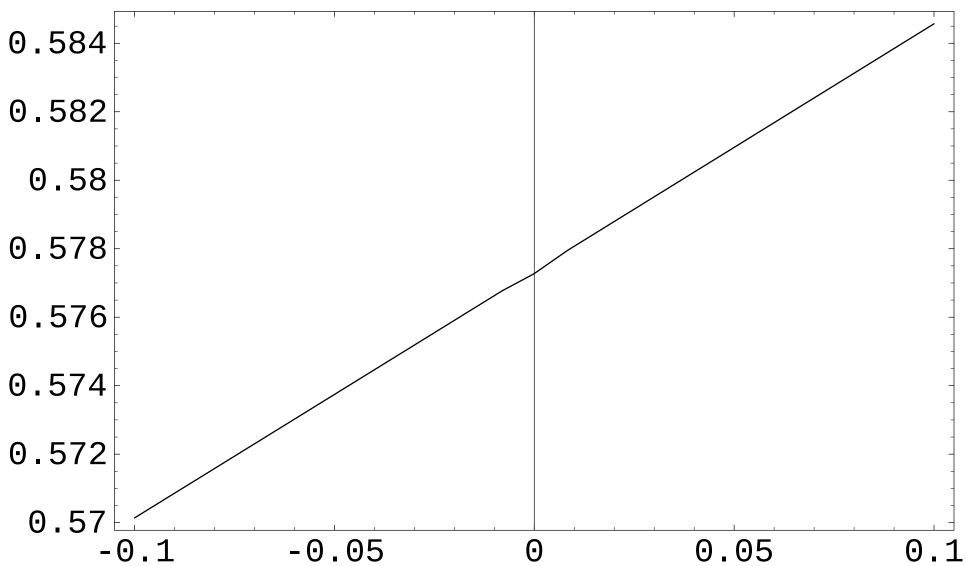

Figure 2: Plot of the log-normal volatility of an Asian option in the log-normal approximation in the $\sigma^2 T \to 0$ limit, as function of $\rho = rT$. At $\rho = 0$ this is $1/\sqrt{3}$.