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Notes on Stochastic Finance

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Preface

These notes are an introduction to pricing and hedging in discrete and continuous time financial models, with an emphasis on the complementarity between analytical and probabilistic methods. The contents of the notes is mostly mathematical, and also aims at making the student aware of the power and limitations of mathematical models in finance, by taking into account their conditions of applicability. The text is targeted at the advanced undergraduate and graduate level in applied mathematics, financial engineering, and economics.

The descriptions of asset model, self-financing portfolio, arbitrage and market completeness are first given in a simple two time-step setting in Chapter 1. These notions are then extended to discrete time in Chapter 2. Here, the impossibility to know the future is formulated using the notion of adapted processes, which plays a central role in the construction of stochastic calculus.

In order to trade efficiently it would be useful to have a formula to estimate the "fair price" of a given asset, in order to determine whether it is currently undervalued or overvalued. Although such a formula is not available, we can instead derive formulas for the pricing of the option contracts that can act as insurance contracts to protect its holder against adverse changes of asset prices. Pricing and hedging of options in discrete time, including in the CRR model, are then considered in Chapter 3 with a description of the passage from discrete to continuous time.

A simplified presentation of Brownian motion, stochastic integrals and calculus, and the Itô formula, is given in Chapter 4. The Black-Scholes model is presented from the angle of PDEs methods in Chapter 5, with the derivation of the Black-Scholes formula by the heat kernel method. The martingale approach to pricing and hedging is presented in Chapter 6, including the treatment of some exotic options. An introduction to volatility estimation is given in Chapter 7.

Optimal stopping and exercise, with application to the pricing of American options, are considered in Chapter 8. The construction of forward measures

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by change of numéraire is given in Chapter 9 and is applied to the pricing of interest rate derivatives in Chapter 11, after an introduction to the modeling of forward rates in, based on material from [37]. The pricing of defaultable bonds is considered in Chapter 12.

Stochastic calculus with jumps is dealt with in Chapter 13 using compound Poisson processes which only have a finite number of jumps on any bounded interval. Those processes are used for option pricing in jump models in Chapter 14 and for hedging. Concerning hedging it is known that markets with jumps are generally incomplete, nevertheless risk minimizing strategies can still be computed. Chapter 15 contains an elementary introduction to finite difference methods for the resolution of PDEs and stochastic differential equations, dealing with the explicit and implicit finite difference schemes for the heat equations and the Black-Scholes PDE, as well as the Euler and Milshtein schemes for SDEs.

The text is completed with two appendices containing the probabilistic background required. and the solutions to some exercises. Chapters 8, 10 and 13 contain animated figures that can be viewed using Acrobat Reader on the complete pdf file.

The material in these notes has been used for teaching in the Masters of Science in Financial Engineering at City University of Hong Kong and at the Nanyang Technological University in Singapore. I thank Ju-Yi Yen (Vanderbilt University) for a detailed reading and for making several corrections to Chapters 1-7 and 15.

The cover graph represents the time evolution of the HSBC stock price from January to September 2009, plotted on the price surface of a European call option on that asset, expiring on October 05, 2009, cf. § 5.5.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Contents

Introduction 1

1 Assets, Portfolios and Arbitrage 9
1.1 Definitions and Formalism 9
1.2 Portfolio Allocation and Short-Selling 10
1.3 Arbitrage 11
1.4 Risk-Neutral Measures 13
1.5 Hedging of Contingent Claims 14
1.6 Market Completeness 16
1.7 Example 16
1.8 Exercises 22

2 Discrete-Time Model 25
2.1 Stochastic Processes 25
2.2 Portfolio Strategies 26
2.3 Arbitrage 29
2.4 Contingent Claims 29
2.5 Martingales and Conditional Expectation 31
2.6 Risk-Neutral Probability Measures 34
2.7 Market Completeness 35
2.8 The Cox-Ross-Rubinstein (CRR) Market Model 36
2.9 Exercises 38

3 Pricing and hedging in discrete time 39
3.1 Pricing of Contingent Claims 39
3.2 Hedging of Contingent Claims - Backward Induction 43
3.3 Pricing of Vanilla Options in the CRR Model 44
3.4 Hedging of Vanilla Options in the CRR model 47
3.5 Hedging of Exotic Options in the CRR Model 50
3.6 Convergence of the CRR Model 54
3.7 Exercises 58

4 Brownian Motion and Stochastic Calculus 61
4.1 Brownian Motion 61
4.2 Wiener Stochastic Integral 65
4.3 Itô Stochastic Integral 70
4.4 Deterministic Calculus 74
4.5 Stochastic Calculus 75
4.6 Geometric Brownian Motion 78
4.7 Stochastic Differential Equations 81
4.8 Exercises 82

5 The Black-Scholes PDE 87
5.1 Continuous-Time Market Model 87
5.2 Self-Financing Portfolio Strategies 87
5.3 Arbitrage and Risk-Neutral Measures 91
5.4 Completeness 92
5.5 The Black-Scholes PDE 93
5.6 The Heat Equation 102
5.7 Solution of the Black-Scholes PDE 104
5.8 Exercises 106

6 Martingale Approach to Pricing and Hedging 109
6.1 Martingale Property of the Itô Integral 109
6.2 Risk-neutral Measures 111
6.3 Pricing by the Martingale Method 115
6.4 Hedging Strategies 118
6.5 Pricing and Hedging of Some Exotic Options 123
6.6 Exercises 130

7 Estimation of Volatility 139
7.1 Historical Volatility 139
7.2 Implied Volatility 140
7.3 The Black-Scholes Formula vs Market Data 141
7.4 Local Volatility 146

8 American Options 149
8.1 Stopping Times and Martingales 149
8.2 Perpetual American Options 159
8.3 Finite Expiration American Options 170
8.4 Exercises 178

9 Change of Numéraire and Forward Measures 183
9.1 Notion of Numéraire 183
9.2 Change of Numéraire 185
9.3 Foreign Exchange 189
9.4 Pricing under Change of Numéraire 191
9.5 Pricing of Exchange Options 194

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

9.6 Self-Financing Hedging by Change of Numéraire 196
9.7 Exercises 198

10 Forward Rate Modeling 203
10.1 Short Term Models 203
10.2 Zero-Coupon Bonds 205
10.3 Forward Rates 209
10.4 The HJM Model 211
10.5 Forward Vasicek Rates 215
10.6 Modeling Issues 221
10.7 The LIBOR Model 229
10.8 Exercises 231

11 Pricing of Interest Rate Derivatives 237
11.1 Forward Measures and Tenor Structure 237
11.2 Caplets and Caps 240
11.3 Forward Swap Measures 242
11.4 Swaption Pricing on the LIBOR 244
11.5 Exercises 247

12 Credit Default 255
12.1 Exponential Distribution 255
12.2 Survival Probabilities 256
12.3 Stochastic Default 257
12.4 Defaultable Bonds 259
12.5 Credit Default Swaps 260
12.6 Exercises 262

13 Stochastic Calculus for Jump Processes 265
13.1 The Poisson Process 265
13.2 Compound Poisson Processes 270
13.3 Stochastic Integrals with Jumps 272
13.4 Itô Formula with Jumps 273
13.5 Stochastic Differential Equations with Jumps 275
13.6 Girsanov Theorem for Jump Processes 279
13.7 Exercises 283

14 Pricing and Hedging in Jump Models 285
14.1 Risk-Neutral Measures 285
14.2 Pricing in Jump Models 286
14.3 Black-Scholes PDE with Jumps 287
14.4 Exponential models 289
14.5 Self-Financing Hedging with Jumps 290
14.6 Exercises 293

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

15 Basic Numerical Methods 295
15.1 The heat equation 295
15.2 The Black-Scholes PDE 297
15.3 Euler discretization 301
15.4 Milshtein discretization 302

Appendix: Background on Probability Theory 305
16.1 Probability Spaces 305
16.2 Events 306
16.3 Probability Measures 308
16.4 Conditional Probabilities and Independence 309
16.5 Random Variables 310
16.6 Probability Distributions 312
16.7 Expectation of a Random Variable 317
16.8 Characteristic Functions and Laplace Transforms 321

Exercise Solutions 327

References 379

Index 383

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Introduction

Modern mathematical finance and quantitative analysis require a strong background in fields such as stochastic calculus, optimization, PDEs and numerical method, or infinite dimensional analysis. In addition, the emergence of new complex financial instruments on the markets makes it necessary to rely on increasingly sophisticated mathematics. Not all students in this type of course will work in quantitative financial analysis, nevertheless they may have to interact with quantitative analysts, and learning their language and the way they operate can be an advantage. In addition, although a lot of ready made financial calculators and software are available, it makes sense to be able oneself to design and implement such financial algorithms. This can be particularly useful under different types of conditions, including an eventual lack of trust in financial indicators, the possible unreliability of expert advice and buying/selling recommendations, or due to other factors such as market manipulation. To some extent we would like to have some form of control on the future behaviour of random (risky) assets, however, since knowledge of the future is not possible, the future prices of risky assets will be modelled by a random variable.

Historical Sketch

We now turn to a description of some of the steps, ideas and individuals that played a important role in the development of the field over the last century.

Robert Brown, botanist, 1827

Observed the movement of pollen particles.

Robert Brown, "A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the

pollen of plants; and on the general existence of active molecules in organic and inorganic bodies." Phil. Mag. 4, 161-173, 1828.

Philosophical Magazine, first published in 1798, is a journal that "publishes articles in the field of condensed matter describing original results, theories and concepts relating to the structure and properties of crystalline materials, ceramics, polymers, glasses, amorphous films, composites and soft matter."

Louis Bachelier, mathematician, PhD 1900

Used Brownian motion for the modelling of stock prices.

Louis Bachelier, "Théorie de la spéculation", Annales Scientifiques de l'Ecole Normale Supérieure 3 (17): 21-86, 1900.

Albert Einstein, physicist, Nobel Prize 1921

For investigations on the theory of Brownian movement: "in 1905 Einstein founded a kinetic theory to account for this movement".

Albert Einstein, Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen", Annalen der Physik 17 (1905) 223.

Norbert Wiener, mathematician, founder of cybernetics, 1923

Mathematical foundation of Brownian motion.

Constructed the Wiener space and Wiener measure on $\mathcal{C}_0([0,1])$ (the space of continuous functions from $[0,1]$ to $\mathbb{R}$ vanishing at 0).

Norbert Wiener, "Differential space", Journal of Mathematics and Physics of the Massachusetts Institute of Technology, 2, 131-174, 1923.

Kiyoshi Itô, mathematician, Gauss prize 2006

Constructed the Itô integral with respect to Brownian motion.
Itô, Kiyosi, Stochastic integral. Proc. Imp. Acad. Tokyo 20, (1944). 519-524.

Constructed the stochastic calculus with respect to Brownian motion.
Itô, Kiyoshi, "On stochastic differential equations", Mem. Amer. Math. Soc. 1951, (1951).

Renowned math wiz Itô, 93, dies.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

The Japan Times, Saturday, Nov. 15, 2008.

Kiyoshi Itô, an internationally renowned mathematician and professor emeritus at Kyoto University died Monday of respiratory failure at a Kyoto hospital, the university said Friday. He was 93.

Itô was once dubbed "the most famous Japanese in Wall Street" thanks to his contribution to the founding of financial derivatives theory.

He is known for his work on stochastic differential equations and the "Itô Formula", which laid the foundation for the Black-Scholes model, a key tool for financial engineering. His theory is also widely used in fields like physics and biology.

Paul Samuelson, economist, Nobel Prize 1970

Rediscovers Bachelier's ideas, proposes geometric Brownian motion as a model for stock prices (1965).

"In the early 1950s I was able to locate by chance this unknown [Bachelier's] book, rotting in the library of the University of Paris, and when I opened it up it was as if a whole new world was laid out before me."

Paul Samuelson, "Rational theory of warrant pricing", Industrial Management Review, p. 13-32, 1965.

Bachelier Finance Society: 1996.

World Bachelier Finance Congress holds every 2 years.

Robert Merton, Myron Scholes, economists, Nobel Prize 1997

"In collaboration with Fisher Black, developed a pioneering formula for the valuation of stock options ... paved the way for economic valuations in many areas ... generated new types of financial instruments and facilitated more efficient risk management in society."

Black, Fischer; Myron Scholes (1973). "The Pricing of Options and Corporate Liabilities". Journal of Political Economy 81 (3): 637-654.

Long Term Capital Management (LTCM):

- founded in 1994.

- with M. Scholes and R. Merton on its board of directors.

- yielded annualized returns of over 40% in its first years.

- in 1998, lost US$ 4.6 billion in less than four months.

- closed down in early 2000.

Oldrich Vasiček, economist, 1977

"An equilibrium characterisation of the term structure". Journal of Financial Economics 5: 177-188.

Interest rates behave differently from stock prices.

First model to capture the mean reversion property of interest rates, a property not possessed by geometric Brownian motion.

David Heath, Robert Jarrow, A. Morton (HJM), 1987

General framework to model the evolution of (forward) interest rates.

David Heath, Robert Jarrow, Andrew Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation", Econometrica, (January 1992), Vol. 60, No. 1, pp 77-105.

Alan Brace, Dariusz Gatarek, Marek Musiela (BGM), 1997

The BGM model is used for the pricing of interest rate derivatives such as caps and swaptions on the LIBOR market.

Alan Brace, Dariusz Gatarek, Marek Musiela, The Market Model of Interest Rate Dynamics. Mathematical Finance 7, page 127. Blackwell 1997.

Call and Put Options

We close this introduction with a description of vanilla call and put options.

As mentioned above, an important concern for the buyer of a stock at time t is whether its price S_T can fall down at some future date T. The buyer of the stock may seek protection from a market crash by purchasing a contract that allows him to sell his asset at time T at a guaranteed price K fixed at time t.

Fig. 0.1: A situation where holding put options might be useful.

This contract is called a put option with strike price K and exercise date T.

Definition. A (European) *put* option is a contract that gives its holder the right (but not the obligation) to *sell* a quantity of assets at a predefined price K called the strike and at a predefined date T called the maturity.

In case the price S_T falls down below the level K, exercising the contract will give the holder of the option a gain equal to $K - S_T$ in comparison to those who did not subscribe the option and sell the asset at the market price S_T. In turn, the issuer of the option will register a loss also equal to $K - S_T$ (in the absence of transaction costs and other fees).

It S_T is above K then the holder will not exercise the option as he may choose to sell at the price S_T, in this case the profit derived from the option is 0.

In general, the payoff of a (so called European) put option will be of the form

$$(K - S_T)^+ = \begin{cases} K - S_T, & S_T \leq K, \\ 0, & S_T \geq K. \end{cases}$$

Two possible scenarios (S_T finishing above K or below K) are illustrated below.

Fig. 0.2: Sample price processes simulated according to a geometric Brownian motion.

On the other hand, if the trader aims at buying some stock or commodity, his interest will be in prices not going up and he might want to purchase a call option, which is a contract allowing him to buy the considered asset at time T at a price not higher than a level K fixed at time t.

Here, in the event that S_T goes above K, the buyer of the option will register a potential gain equal to $S_T - K$ in comparison to an agent who did not subscribe to the call option.

Definition. A (European) *call* option is a contract that gives its holder the right (but not the obligation) to *buy* a quantity of assets at a predefined price K called the strike and at a predefined date T called the maturity.

In general, a (European) call option is an option with payoff function

$$\phi(S_T) = (S_T - K)^+ = \begin{cases} S_T - K, \, S_T \geq K, \\ \\ 0, \qquad\quad S_T \leq K. \end{cases}$$

In order for an option contract to be fair, the buyer of the option should pay a fee (similar to an insurance fee) at the signature of the contract. The computation of this fee is an important issue, which is known as option *pricing*.

The second important issue is that of *hedging*, i.e. how to manage a given portfolio in such a way that it contains the required random payoff $(K - S_T)^+$ (for a put option) or $(S_T - K)^+$ (for a call option) at the maturity date T.

The next figure illustrates another type of situation where holding a call/put option might be useful.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Fig. 0.3: Stock price curve.

Chapter 1
Assets, Portfolios and Arbitrage

We consider a simplied financial model with only two time instants $t = 0$ and $t = 1$. In this simple setting we introduce the notions of portfolio, arbitrage, completeness, pricing and hedging using the notation of [21]. A binary asset price model is considered as an example.

1.1 Definitions and Formalism

We will use the following notation. An element $\bar{x}$ of $\mathbb{R}^{d+1}$ is a vector

$$\bar{x} = (x_0, x_1, \ldots, x_d)$$

made of $d + 1$ components.

We will often use the scalar product $\bar{x} \cdot \bar{y}$ of two vectors $\bar{x}$, $\bar{y} \in \mathbb{R}^{d+1}$, which is defined by

$$\bar{x} \cdot \bar{y} = x_0 y_0 + x_1 y_1 + \cdots x_d y_d.$$

The vector

$$\bar{\pi} = \left(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(d)}\right)$$

denotes the prices at time $t = 0$ of $d + 1$ assets numbered $0, 1, \ldots, d$.

The vector

$$\bar{S} = \left(S^{(0)}, S^{(1)}, \ldots, S^{(d)}\right)$$

denotes the (possibly random) values at time $t = 1$ of assets $0, 1, \ldots, d$.

We will assume that asset n° 0 is a riskless asset (of savings account type) that yields an interest rate $r > 0$. That means we have

$$S^{(0)} = (1+r)\pi^{(0)}.$$

1.2 Portfolio Allocation and Short-Selling

A *portfolio* based on the assets $0, 1, 2, \ldots, d$ is viewed as a vector $\bar{\xi} \in \mathbb{R}^{d+1}$ in which $\xi^{(i)}$ represents the number of units of asset n^o i owned by an investor, $i = 0, 1, \ldots, d$.

The *price* of such a portfolio is given by

$$\bar{\xi} \cdot \bar{\pi} = \sum_{i=0}^{d} \xi^{(i)} \pi^{(i)}$$

at time $t = 0$.

At time $t = 1$ the *value* of the portfolio is given by

$$\bar{\xi} \cdot \bar{S} = \sum_{i=0}^{d} \xi^{(i)} S^{(i)}.$$

If $\xi^{(0)} > 0$ the investor puts the amount $\xi^{(0)}\pi^{(0)} > 0$ on a savings account with interest rate r, while if $\xi^{(0)} < 0$ he borrows the amount $-\xi^{(0)}\pi^{(0)} > 0$ with the same interest rate.

If $\xi^{(i)} > 0$, $i = 1, \ldots, d$, the investor buys $\xi^{(i)} > 0$ units of the asset n^o i, while if $\xi^{(i)} < 0$ he borrows $-\xi^{(i)} > 0$ units of asset i and sells it to obtain the amount $-\xi^{(i)}\pi^{(i)} > 0$. In the latter case one says that the investor *short sells* $-\xi^{(i)} > 0$ units of the asset n^o i.

Traditionally, profits are made by first buying at a low price and then selling at a high price. Short-sellers apply the same rule but in the reverse time order: first sell high, and then buy low, by applying the following procedure.

1. Borrow the asset n^o i.

2. At time $t = 0$, sell the asset n^o i on the market at the price $\pi^{(i)}$.

3. Buy back the asset n^o i at time $t = 1$ at the price $S^{(i)}$, with hopefully $S^{(i)} < \pi^{(i)}$.

4. Return the asset to its owner, with possibly a (small) fee $p > 0$.

At the end of the operation the profit made on share n^o i is

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$$\pi^{(i)} - S^{(i)} - p > 0,$$

which is positive if $S^{(i)} < \pi^{(i)}$ and $p > 0$ is sufficiently small.

1.3 Arbitrage

As stated in the next definition, an arbitrage opportunity is the possibility to make a strictly positive amount of money starting from 0 or from a negative amount. In a sense, an arbitrage opportunity can be seen as a way to "beat" the market.

The short-selling procedure described in Section 1.2 represents one way to realize an arbitrage opportunity. One can proceed similarly by buying an asset.

1. Borrow the amount $-\xi^{(0)}\pi^{(0)} > 0$ on the riskless asset n° 0.

2. Use the amount $-\xi^{(0)}\pi^{(0)} > 0$ to buy the risky asset n^o i at time $t = 0$ and price $\pi^{(i)}$, for a quantity $\xi^{(i)} = -\xi^{(0)}\pi^{(0)}/\pi^{(i)}$, $i = 1, \ldots, d$.

3. At time $t = 1$, sell the risky asset n^o i at the price $S^{(i)}$, with hopefully $S^{(i)} > \pi^{(i)}$.

4. Refund the amount $-(1 + r)\xi^{(0)}\pi^{(0)} > 0$ with interest rate $r > 0$.

At the end of the operation the profit made is

$$S^{(i)} - (1 + r)\pi^{(i)} > 0,$$

which is positive if $S^{(i)} > \pi^{(i)}$ and r is sufficiently small.

Next we state a mathematical formulation of the concept of arbitrage.

Definition 1.1. A portfolio $\bar{\xi} \in \mathbb{R}^{d+1}$ constitutes an arbitrage opportunity if the three following conditions are satisfied:

i) $\bar{\xi} \cdot \bar{\pi} \leq 0$, [start from 0 or even with a debt]

ii) $\bar{\xi} \cdot \bar{S} \geq 0$, [finish with a non-negative amount]

iii) $\mathbb{P}(\bar{\xi} \cdot \bar{S} > 0) > 0$. [a profit is made with non-zero probability]

The are many real-life examples of situations where arbitrage does not occur, such as:

- assets with different returns (finance),

- servers with different speeds (queueing, networking, computing),

- highway lanes with different speeds (driving).

六合彩投注換算表
MARK SIX INVESTMENT TABLE

複式 Multiple		一胆拖 One Banker with		兩胆拖 Two Bankers with		三胆拖 Three Bankers with		四胆拖 Four Bankers with		五胆拖 Five Bankers with	
所選號碼總數 No. of Selections	HK$	配腳數目 No. of Legs	HK$	配腳數目 No. of Legs	HK$	配腳數目 No. of Legs	HK$	配腳數目 No. of Legs	HK$	配腳數目 No. of Legs	HK$
7	35	6	30	5	25	4	20	3	15	2	10
8	140	7	105	6	75	5	50	4	30	3	15
9	420	8	280	7	175	6	100	5	50	4	20
10	1,050	9	630	8	350	7	175	6	75	5	25
11	2,310	10	1,260	9	630	8	280	7	105	6	30
12	4,620	11	2,310	10	1,050	9	420	8	140	7	35
13	8,580	12	3,960	11	1,650	10	600	9	180	8	40
14	15,015	13	6,435	12	2,475	11	825	10	225	9	45
15	25,025	14	10,010	13	3,575	12	1,100	11	275	10	50
49	69,919,080	48	8,561,520	47	891,825	46	75,900	45	4,950	44	220

Fig. 1.1: Another example of absence of arbitrage.

In the table of Figure 1.1 the absence of arbitrage opportunities is materialized by the fact that the price of each combination is found to be proportional to its probability.

In the sequel we will work on the assumption that arbitrage does not occur and we will rely on this hypothesis for the pricing of financial instruments.

Let us give an example of pricing by A.O.A. (Absence of Arbitrage).

From March 24 to 31, 2009, HSBC issued rights to buy shares at the price of \$28. This right actually behaves like a call option since it gives the right but not the obligation to buy the stock at $K = \$28$. On March 24 the HSBC stock price finished at \$41.70.

The question is: how to value the price $\$R$ of the right to buy one share ? This question can be answered by looking for arbitrage opportunities. Indeed, there are two ways to buy the stock:

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1. buy on the market at \$41.70. Cost: \$41.70,
or:
2. buy the right at price $\$R$ and then the stock at price \$28. Total cost: \$R+\$28.

For an investor who owns no stock and no rights, in case $\$R + \$28 < \$41.70$, arbitrage would be possible by buying the right at a price $\$R$, then the stock at \$28, and finally selling the stock of the market at \$41.70. Profit would equal

$$\$41.70 - (\$R + \$28) > 0.$$

On the other hand, for an investor who owns the rights, in case $\$R + \$28 > \$41.70$, arbitrage would be possible by selling the right at price $\$R$, buying the stock on the market at \$41.70. At time $t = 1$ the stock could be sold at around \$28, and profit would equal

$$\$R + \$28 - \$41.70 > 0.$$

In the absence of arbitrage opportunities, this argument says that $\$R$ should satisfy

$$\$R + \$28 - \$41.70 = 0,$$

i.e. the *arbitrage* price of the right is

$$\$R = \$41.70 - \$28 = \$13.70. \tag{1.1}$$

Interestingly, the *market* price of the right was \$13.20 at the close of the session on March 24. The difference of \$0.50 can be explained by the presence of transaction costs, the time value of money, or other market factors. It may also represent a small arbitrage opportunity. Nevertheless, the absence of arbitrage (A.O.A.) argument (1.1) prices the right at \$13.70, which is quite close to its market value. Thus the A.O.A. hypothesis appears as an accurate tool for pricing.

1.4 Risk-Neutral Measures

In order to use absence of arbitrage in the general context of pricing financial derivatives we will need the notion of *risk-neutral measure*. The next definition says that under a risk-neutral (probability) measure, the risky assets n^o $1, \ldots, d$ have same *average* rate of return as the riskless asset n° 0.

Definition 1.2. A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if

$$\mathbb{E}^*[S^{(i)}] = (1 + r)\pi^{(i)}, \qquad i = 0, 1, \ldots, d. \tag{1.2}$$

Here, $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

In other words, $\mathbb{P}^*$ is called "risk neutral" because taking risks under $\mathbb{P}^*$ by buying a stock $S^{(i)}$ has a neutral effect: on average the expected yield of the risky asset equals the riskless rate obtained by investing on the savings account with interest rate r.

On the other hand, under a "risk premium" probability measure $\mathbb{P}^\#$, the expected return of the risky asset $S^{(i)}$ would be higher than r, i.e. we would have

$$\mathbb{E}^\#[S^{(i)}] > (1+r)\pi^{(i)}, \qquad i = 1, \ldots, d.$$

Note that in practice we will only consider risk-neutral measures $\mathbb{P}^*$ such that $d\mathbb{P}^*/d\mathbb{P}$ exists and is almost surely strictly positive, in other words $\mathbb{P}^*$ is said to be *equivalent* to $\mathbb{P}$.

The following result can be used to check for the existence of arbitrage opportunities, and is known as the first fundamental theorem of mathematical finance.

Theorem 1.1. A market is *without* arbitrage opportunity if and only if it admits at least one risk-neutral measure.

Proof. Relies on the theorem of separation of convex sets by hyperplanes, cf. Theorem 1.6 of [21]. □

1.5 Hedging of Contingent Claims

Definition 1.3. A contingent claim is any non-negative random variable $C \geq 0$.

In practice the random variable C represents the payoff of an (option) contract at time $t = 1$.

For example, in the case of a European call option with maturity $t = 1$ on the asset n^o i, the payoff C is given by

$$C = (S^{(i)} - K)^+$$

where K is called the *strike price*. The claim C is called "contingent" because its value may depend on various market conditions, such as $S^{(i)} > K$. A contingent claim is also called a "derivative" for the same reason.

Similarly, an European put option with maturity $t = 1$ on the asset n^o i is a contingent claim with payoff

$$C = (K - S^{(i)})^+.$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Definition 1.4. A contingent claim with payoff C is said to be attainable if there exists a portfolio strategy $\bar{\xi}$ such that

$$C = \bar{\xi} \cdot \bar{S}.$$

When a contingent claim C is attainable, a trader will be able to:

1. at time $t = 0$, make a portfolio allocation $\bar{\xi} = (\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)}) \in \mathbb{R}^{d+1}$,
2. invest the amount $\bar{\xi} \cdot \bar{\pi}$ in this portfolio at time $t = 0$,
3. at time $t = 1$, pay the claim C using the value $\bar{\xi} \cdot \bar{S}$ of the portfolio.

In order to attain the claim, an initial investment $\bar{\xi} \cdot \bar{\pi}$ is needed at time $t = 0$. This amount, to be paid by the buyer of the option, is called the *arbitrage price* of the contingent claim C, and denoted by

$$\pi(C) := \bar{\xi} \cdot \bar{\pi}. \tag{1.3}$$

Allocating a portfolio $\bar{\xi}$ such that

$$C = \bar{\xi} \cdot \bar{S} \tag{1.4}$$

is called *hedging*, or *replication*, of the contingent claim C.

As an example, in order to hedge oneself from the price rise of gasoline, one may buy oil related stocks. In this case a higher price of gasoline would result in a higher value of C, and should be correlated to an increase in the underlying stock value, so that the equality (1.4) would be maintained.

In case the value $\bar{\xi} \cdot \bar{S}$ exceeds the amount of the claim, i.e. if

$$\bar{\xi} \cdot \bar{S} \geq C,$$

we talk about *super-hedging*.

In this course we will focus on hedging (i.e. *replication* of the contingent claim C) and we will not consider super-hedging.

1.6 Market Completeness

Definition 1.5. A market model is said to be *complete* if every contingent claim C is attainable.

The next result is the second fundamental theorem of mathematical finance.

Theorem 1.2. A market model without arbitrage is complete if and only if it admits only one risk-neutral measure.

Proof. cf. Theorem 1.40 of [21]. □

Theorem 1.2 will give us a concrete way to check for market completeness by searching for a unique solution $\mathbb{P}^*$ to Equation (1.2).

1.7 Example

In this section we work out a simple example to check the use of Theorem 1.1 and Theorem 1.2.

We take $d = 1$, i.e. there is only a riskless asset n° 0 and a risky asset $S^{(1)}$. In addition we choose

$$\Omega = \{\omega^-, \omega^+\},$$

which is the simplest possible non-trivial choice of a probability space, made of only two possible outcomes.

In other words the behavior of the market is subject to only two possible outcomes. For example, the market expects an important binary decision of yes/no type, which can lead to two distinct scenarios called ω^- and ω^+.

In this context, the asset price $S^{(1)}$ is given by a random variable

$$S^{(1)} : \Omega \to \mathbb{R}$$

whose value depends whether the scenario ω^-, resp. ω^+, occurs.

Precisely, we set

$$S^{(1)}(\omega^-) = a, \qquad \text{and} \qquad S^{(1)}(\omega^+) = b,$$

i.e. the value of $S^{(1)}$ becomes equal a under the scenario ω^-, and equal to b under the scenario ω^+, where $0 < a \leq b$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

The first natural question we ask is:

- are there arbitrage opportunities in such a market ?

We will answer this question using Theorem 1.1, i.e. by searching for a risk-neutral measure $\mathbb{P}^*$. Recall that any measure $\mathbb{P}^*$ on $\Omega = \{\omega^-, \omega^+\}$ is given by the data of two numbers $\mathbb{P}^*(\{\omega^-\}) \in [0,1]$ and $\mathbb{P}^*(\{\omega^+\}) \in [0,1]$, such that

$$\mathbb{P}^*(\Omega) = \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^+\}) = 1, \tag{1.5}$$

and that according to Definition 1.2, a risk-neutral measure $\mathbb{P}^*$ should satisfy

$$\mathbb{E}^*[S^{(1)}] = (1+r)\pi^{(1)}. \tag{1.6}$$

Although we should solve this equation for $\mathbb{P}^*$, at this stage it is not clear how $\mathbb{P}^*$ appears in (1.6).

In order to make (1.6) more explicit we write

$$\mathbb{E}^*[S^{(1)}] = a\mathbb{P}^*(S^{(1)} = a) + b\mathbb{P}^*(S^{(1)} = b),$$

hence Condition (1.6) for the existence of a risk-neutral measure $\mathbb{P}^*$ reads

$$a\mathbb{P}^*(S^{(1)} = a) + b\mathbb{P}^*(S^{(1)} = b) = (1+r)\pi^{(1)}.$$

Using the Condition (1.5) we obtain the system of two equations

$$\begin{cases} a\mathbb{P}^*(\{\omega^-\}) + b\mathbb{P}^*(\{\omega^+\}) = (1+r)\pi^{(1)} \\ \\ \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^+\}) = 1, \end{cases} \tag{1.7}$$

with solution

$$\mathbb{P}^*(\{\omega^-\}) = \frac{b - (1+r)\pi^{(1)}}{b-a} \qquad \text{and} \qquad \mathbb{P}^*(\{\omega^+\}) = \frac{(1+r)\pi^{(1)} - a}{b-a}.$$

In order for a solution $\mathbb{P}^*$ to exist as a probability measure, the numbers $\mathbb{P}^*(\{\omega^-\})$ and $\mathbb{P}^*(\{\omega^+\})$ must be non-negative.

We deduce that if a, b and r satisfy the condition

$$a < (1+r)\pi^{(1)} < b, \tag{1.8}$$

then there exists a risk-neutral measure $\mathbb{P}^*$ which is unique, hence by Theorems 1.1 and 1.2 the market is without arbitrage and complete.

If $a = b = (1+r)\pi^{(1)}$ then (1.2) admits an infinity of solutions, hence the market is without arbitrage but it is not complete. More precisely, in this

case both the riskless and risky assets yield a deterministic interest rate r and the value of the portfolio becomes

$$\bar{\xi} \cdot \bar{S} = (1+r)\bar{\xi} \cdot \bar{\pi},$$

at time $t = 1$, hence the terminal value $\bar{\xi} \cdot \bar{S}$ is deterministic and this *single* value can not always coincide with a random contingent claim C that would be allowed to take *two* distinct values $C(\omega^-)$ and $C(\omega^+)$.

Note that if $a = (1+r)\pi^{(1)}$, resp. $b = (1+r)\pi^{(1)}$, then $\mathbb{P}^*(\{\omega^+\}) = 0$, resp. $\mathbb{P}^*(\{\omega^-\}) = 0$, and $\mathbb{P}^*$ is not equivalent to $\mathbb{P}$.

Under the conditions

$$a < b < (1+r)\pi^{(1)} \qquad \text{or} \qquad (1+r)\pi^{(1)} < a < b, \tag{1.9}$$

no risk neutral measure exists and as a consequence there are arbitrage opportunities in the market.

Let us give a financial interpretation of Conditions (1.9).

1. If $(1+r)\pi^{(1)} < a < b$, let $\xi^{(1)} = 1$ and choose $\xi^{(0)}$ such that $\xi^{(0)}\pi^{(0)} + \xi^{(1)}\pi^{(1)} = 0$, i.e.
$$\xi^{(0)} = -\xi^{(1)}\pi^{(1)}/\pi^{(0)} < 0.$$
This means that the investor borrows the amount $-\xi^{(0)}\pi^{(0)} > 0$ on the riskless asset and uses it to buy one unit $\xi^{(1)} = 1$ of the risky asset. At time $t = 1$ he sells the risky asset $S^{(1)}$ at a price at least equal to a and refunds the amount $-(1+r)\xi^{(0)}\pi^{(0)} > 0$ he borrowed, with interests. His profit is
$$\begin{aligned}\bar{\xi} \cdot \bar{S} &= (1+r)\xi^{(0)}\pi^{(0)} + \xi^{(1)}S^{(1)} \\ &\geq (1+r)\xi^{(0)}\pi^{(0)} + \xi^{(1)}a \\ &= -(1+r)\xi^{(1)}\pi^{(1)} + \xi^{(1)}a \\ &= \xi^{(1)}(-(1+r)\pi^{(1)} + a) \\ &> 0. \qquad \ddot{\smile}\end{aligned}$$

2. If $a < b < (1+r)\pi^{(1)}$, let $\xi^{(0)} > 0$ and choose $\xi^{(1)}$ such that $\xi^{(0)}\pi^{(0)} + \xi^{(1)}\pi^{(1)} = 0$, i.e.
$$\xi^{(1)} = -\xi^{(0)}\pi^{(0)}/\pi^{(1)} < 0.$$
This means that the investor borrows $-\xi^{(1)} > 0$ units of the risky asset, sells it for the amount $-\xi^{(1)}\pi^{(1)}$, and invests this money on the riskless account for the amount $\xi^{(0)}\pi^{(0)} > 0$. In this case one says that the investor *shortsells* the risky asset. At time $t = 1$ she obtains $(1+r)\xi^{(0)}\pi^{(0)} > 0$ from the riskless asset and she spends at most b to buy the risky asset and

return it to its original owner. Her profit is

$$\begin{aligned}\bar{\xi}\cdot\bar{S} &= (1+r)\xi^{(0)}\pi^{(0)} + \xi^{(1)}S^{(1)}\\ &\geq (1+r)\xi^{(0)}\pi^{(0)} + \xi^{(1)}b\\ &= -(1+r)\xi^{(1)}\pi^{(1)} + \xi^{(1)}b\\ &= \xi^{(1)}(-(1+r)\pi^{(1)} + b)\\ &> 0, \qquad \ddot{\smile}\end{aligned}$$

since $\xi^{(1)} < 0$. Note that here, $a \leq S^{(1)} \leq b$ became

$$\xi^{(1)}b \leq \xi^{(1)}S^{(1)} \leq \xi^{(1)}a$$

because $\xi^{(1)} < 0$.

Note that under Condition (1.8) there is absence of arbitrage and by Theorem 1.1, no portfolio strategy can yield $\bar{\xi}\cdot\bar{S} \geq 0$ and $\mathbb{P}(\bar{\xi}\cdot\bar{S} > 0) > 0$ starting from $\xi^{(0)}\pi^{(0)} + \xi^{(1)}\pi^{(1)} \leq 0$, although this is less simple to show directly.

Finally if $a = b \neq (1+r)\pi^{(1)}$ then (1.2) admits no solution as a probability measure $\mathbb{P}^*$ hence arbitrage opportunities exist and can be constructed by the same method as above.

The second natural question is:

- is the market complete, i.e. are all contingent claims attainable ?

In the sequel we work under the condition

$$a < (1+r)\pi^{(1)} < b,$$

under which Theorems 1.1 and 1.2 show that the market is without arbitrage and complete since the risk-neutral measure $\mathbb{P}^*$ exists and is unique.

Let us recover this fact by elementary calculations. For any contingent claim C we need to show that there exists a portfolio $\bar{\xi} = (\xi^{(0)}, \xi^{(1)})$ such that $C = \bar{\xi}\cdot\bar{S}$, i.e.

$$\begin{cases}\xi^{(0)}(1+r)\pi^{(0)} + \xi^{(1)}a = C(\omega^-)\\ \xi^{(0)}(1+r)\pi^{(0)} + \xi^{(1)}b = C(\omega^+).\end{cases} \tag{1.10}$$

These equations can be solved as

$$\xi^{(0)} = \frac{bC(\omega^-) - aC(\omega^+)}{\pi^{(0)}(1+r)(b-a)}, \qquad \xi^{(1)} = \frac{C(\omega^+) - C(\omega^-)}{b-a}, \tag{1.11}$$

and

$$\xi^{(0)}\pi^{(0)} = \frac{bC(\omega^-) - aC(\omega^+)}{(1+r)(b-a)}$$

represents the amount invested on the riskless asset. In this case we say that the portfolio $(\xi^{(0)}, \xi^{(1)})$ *hedges* the contingent claim C. In other words, any contingent claim C is attainable and the market is indeed complete.

Note that if $C(\omega^+) \geq C(\omega^-)$ then $\xi^{(1)} \geq 0$ and there is not short selling. This occurs in particular if C has the form $C = h(S^{(1)})$ with h a non-decreasing function, since

$$\begin{aligned}
\xi^{(1)} &= \frac{C(\omega^+) - C(\omega^-)}{b-a} \\
&= \frac{h(S^{(1)}(\omega^+)) - h(S^{(1)}(\omega^-))}{b-a} \\
&= \frac{h(b) - h(a)}{b-a} \\
&\geq 0,
\end{aligned}$$

thus there is no short-selling.

The *arbitrage price* $\pi(C)$ of the contingent claim C is defined in (1.3) as the initial value at $t = 0$ of the portfolio hedging C, i.e.

$$\pi(C) = \bar{\xi} \cdot \bar{\pi}, \tag{1.12}$$

where $(\xi^{(0)}, \xi^{(1)})$ are given by (1.11). Note that $\pi(C)$ can not be 0 since this would imply an arbitrage opportunity according to Definition 1.1.

The next proposition shows that the arbitrage price $\pi(C)$ of the claim can be computed as the expected value of its payoff C under the risk-neutral measure, after discounting at the rate $1 + r$ for the time value of money.

Proposition 1.1. *The arbitrage price $\pi(C) = \bar{\xi} \cdot \bar{\pi}$ of the contingent claim C satisfies*

$$\pi(C) = \frac{1}{1+r}\,\mathbb{E}^*[C]. \tag{1.13}$$

Proof. We have

$$\begin{aligned}
\pi(C) &= \bar{\xi} \cdot \bar{\pi} \\
&= \xi^{(0)}\pi^{(0)} + \xi^{(1)}\pi^{(1)} \\
&= \frac{bC(\omega^-) - aC(\omega^+)}{(1+r)(b-a)} + \pi^{(1)}\frac{C(\omega^+) - C(\omega^-)}{b-a} \\
&= \frac{1}{1+r}\left(C(\omega^-)\frac{b - \pi^{(1)}(1+r)}{b-a} + C(\omega^+)\frac{(1+r)\pi^{(1)} - a}{b-a}\right)
\end{aligned}$$

$$= \frac{1}{1+r}\Big(C(\omega^-)\mathbb{P}^*(S^{(1)}=a) + C(\omega^+)\mathbb{P}^*(S^{(1)}=b)\Big)$$
$$= \frac{1}{1+r}\,\mathbb{E}^*[C].$$

□

In the case of a European call with strike $K \in [a,b]$ we have $C = (S^{(1)} - K)^+$ and

$$\pi((S^{(1)} - K)^+) = \pi^{(1)}\frac{b-K}{b-a} - \frac{(b-K)a}{(1+r)(b-a)}.$$

Here, $(\pi^{(1)} - K)^+$ is called the *intrinsic value* at time 0 of the call option.

The simple setting described in this chapter raises several questions and remarks.

Remarks

1. The fact that $\pi(C)$ can be obtained by two different methods, i.e. an algebraic method via (1.11) and (1.12) and a probabilistic method from (1.13) is not a simple coincidence. It is actually a simple example of the deep connection that exists between probability and analysis.

   In a continuous time setting, (1.11) will be replaced with a *partial differential equation* (PDE) and (1.13) will be computed via the *Monte Carlo* method. In practice, the quantitative analysis departments of major financial institutions are split into the *PDE team* and the *Monte Carlo* team, often trying to determine the same option prices by two different methods.

2. For example, what if we have three possible scenarios, i.e. $\Omega = \{\omega^-, \omega^o, \omega^+\}$ and the random asset $S^{(1)}$ is allowed to take more than two values, e.g. $S^{(1)} \in \{a,b,c\}$ according to each scenario ? In this case the system (1.7) would be rewritten as

$$\begin{cases} a\mathbb{P}^*(\{\omega^-\}) + b\mathbb{P}^*(\{\omega^o\}) + c\mathbb{P}^*(\{\omega^+\}) = (1+r)\pi^{(1)} \\ \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^o\}) + \mathbb{P}^*(\{\omega^+\}) = 1, \end{cases}$$

   and this system of two equations for three unknowns does not have a unique solution, hence the market can be without arbitrage but it can not be complete. Completeness can be reached by adding a second risky asset, i.e. taking $d = 2$, in which case we will get three equations and three unknowns. More generally, when Ω has $n \geq 2$ elements, completeness of the market can be reached provided we consider d risky assets with

$d+1 \geq n$. This is related to the Meta-theorem 8.3.1 of [4] in which the number d of traded underlying risky assets is linked to the number of random sources through arbitrage and completeness.

1.8 Exercises

Exercise 1.1. Consider a financial model with two instants $t = 0$ and $t = 1$ and two assets:

- a riskless asset π with price π_0 at time $t = 0$ and value $\pi_1 = \pi_0(1+r)$ at time $t = 1$,
- a risky asset S with price S_0 at time $t = 0$ and random value S_1 at time $t = 1$.

We assume that S_1 can take only the values $S_0(1+a)$ and $S_0(1+b)$, where $-1 < a < r < b$.

The *return* of the risky asset is defined as

$$R = \frac{S_1 - S_0}{S_0}.$$

1. What are the possible values of R ?
2. Show that under the probability measure $\mathbb{P}^*$ defined by

$$\mathbb{P}^*(R = a) = \frac{b-r}{b-a}, \qquad \mathbb{P}^*(R = b) = \frac{r-a}{b-a},$$

   the expected return $\mathbb{E}^*[R]$ of S is equal to the return r of the riskless asset.
3. Does there exist arbitrage opportunities in this model ? Explain why.
4. Is this market model complete ? Explain why.
5. Consider a contingent claim with payoff C given by

$$C = \begin{cases} \alpha & \text{if } R = a, \\ \beta & \text{if } R = b. \end{cases}$$

   Show that the portfolio (η, ξ) defined[1] by

$$\eta = \frac{\alpha(1+b) - \beta(1+a)}{\pi_0(1+r)(b-a)} \quad \text{and} \quad \xi = \frac{\beta - \alpha}{S_0(b-a)},$$

[1] Here, η is the number of units invested in asset π and ξ is the number of units invested in asset S.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

hedges the contingent claim C, i.e. show that at time $t = 1$ we have

$$\eta\pi_1 + \xi S_1 = C.$$

Hint: distinguish two cases $R = a$ and $R = b$.

6. Compute the arbitrage price $\pi(C)$ of the contingent claim C using η, π_0, ξ, and S_0.
7. Compute $\mathbb{E}^*[C]$ in terms of a, b, r, α, β.
8. Show that the arbitrage price $\pi(C)$ of the contingent claim C satisfies

$$\pi(C) = \frac{1}{1+r}\,\mathbb{E}^*[C]. \tag{1.14}$$

9. What is the interpretation of Relation (1.14) above ?
10. Let C denote the payoff at time $t = 1$ of a put option with strike $K = \$11$ on the risky asset. Give the expression of C as a function of S^1 and K.
11. Compute a portfolio (ξ^0, ξ^1) hedging the contingent claim C.
12. Compute the arbitrage price $\pi(C)$ of the claim C.

Chapter 2
Discrete-Time Model

An important limitation of the two time step model considered in the preceding chapter is that it does not allow for trading until the end of the time period is reached. In order to be able to re-allocate the portfolio over time, we consider a discrete-time financial model with $N+1$ time instants $t = 0, 1, \ldots, N$. The practical importance of this model lies in its direct computer implementability.

2.1 Stochastic Processes

A *stochastic process* is a family $(X_t)_{t \in T}$ of random variables $X_t : \Omega \to \mathbb{R}$ indexed by a set T.

Two-instant model: $T = \{0, 1\}$.

Discrete-time model: $T = \{0, 1, 2, \ldots, N\}$ (finite horizon).

Discrete-time model: $T = \mathbb{N}$ (infinite horizon).

Continuous-time model: $T = \mathbb{R}_+$.

Real life examples of stochastic processes include:

- the time evolution of a risky asset - in this case X_t represents the price of the asset at time $t \in T$.

- the time evolution of a physical parameter - for example, X_t represents a temperature observed at time $t \in T$.

In this chapter we will focus on the finite horizon discrete-time model $T = \{0, 1, 2, \ldots, N\}$.

The vector
$$\bar{\pi} = (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(d)})$$
denotes the prices at time $t = 0$ of $d + 1$ assets numbered $0, 1, \ldots, d$.

The *random* vector
$$\bar{S}_t = (S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)})$$
on Ω denotes the values at time t of assets $0, 1, \ldots, d$, and forms a stochastic process $(\bar{S}_t)_{t=0,1,\ldots,N}$.

Here we still assume that asset 0 is a riskless asset (of savings account type) yielding an interest rate r, i.e. we have
$$S_t^{(0)} = (1 + r)^t \pi^{(0)}, \qquad t = 0, 1, \ldots, N.$$

2.2 Portfolio Strategies

A portfolio strategy is a stochastic process $(\bar{\xi}_t)_{t=1,\ldots,N} \subset \mathbb{R}^{d+1}$ such that $\xi_t^{(i)}$ denotes the number of units (quantity) of asset i held in the portfolio over the period $[t - 1, t]$, $t = 1, \ldots, N$.

Note that the portfolio allocation
$$\xi_t = (\xi_t^{(0)}, \xi_t^{(1)}, \ldots, \xi_t^{(d)})$$
remains constant over the period $[t - 1, t]$ while the stock price changes from S_{t-1} to S_t over this period.

In other terms,
$$\xi_t^{(i)} S_{t-1}^{(i)}$$
represents the amount invested in asset i at the beginning of $[t - 1, t]$, and
$$\xi_t^{(i)} S_t^{(i)}$$
represents the value of this investment at the end of $[t - 1, t]$.

The value of the porfolio at the beginning of the time period $[t - 1, t]$ is

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\bar{\xi}_t \cdot \bar{S}_{t-1} = \sum_{i=0}^{d} \xi_t^{(i)} S_{t-1}^{(i)},$$

when the market "opens", and becomes

$$\bar{\xi}_t \cdot \bar{S}_t = \sum_{i=0}^{d} \xi_t^{(i)} S_t^{(i)} \tag{2.1}$$

at the end of $[t-1, t]$, i.e. when the market "closes".

At the beginning of the next trading period $[t, t+1]$ the value of the portfolio becomes

$$\bar{\xi}_{t+1} \cdot \bar{S}_t = \sum_{i=0}^{d} \xi_{t+1}^{(i)} S_t^{(i)}. \tag{2.2}$$

Note that the stock price $\bar{S}_t$ is assumed to remain constant "overnight", i.e. from the end of $[t-1, t]$ to the beginning of $[t, t+1]$.

Obviously the question arises whether (2.1) should be identical to (2.2). In the sequel we will need the following consistency hypothesis on the portfolio strategy $\bar{\xi}_t$.

Definition 2.1. We say that the portfolio strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$ is *self-financing*, i.e.

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_t, \qquad t = 1, \ldots, N-1. \tag{2.3}$$

The meaning of the self-financing condition is simply that one can not take any money in or out of the portfolio during the transition "overnight" period at time t. In other words, at the beginning of the new trading period $[t, t+1]$ one should re-invest the totality of the portfolio value obtained at the end of period $[t-1, t]$. The next figure is an illustration of the self-financing condition.

Fig. 2.1: Illustration of the self-financing condition (2.3).

Note that the portfolio re-allocation happens "overnight" while the portfolio global value remains the same due to the self-financing condition. The portfolio allocation ξ_t remains the same throughout the day, however the portfolio value changes from morning to evening due to a change in the stock price. Also, $\bar{\xi}_0$ does not exist and is not needed in this framework.

Of course the chosen unit of time may not be the day, and it can be replaced by weeks, hours, minutes, seconds, etc.

We will denote by

$$V_t := \bar{\xi}_t \cdot \bar{S}_t$$

the value of the portfolio at time $t = 1, \ldots, N$, with

$$V_0 = \bar{\xi}_1 \cdot \bar{S}_0.$$

Let

$$\bar{X}_t = (X_t^{(0)}, X_t^{(1)}, \ldots, X_t^{(d)})$$

denote vector of discounted asset prices defined as:

$$X_t^{(i)} = \frac{1}{(1+r)^t} S_t^{(i)}, \qquad i = 0, 1, \ldots, d, \quad t = 0, 1, \ldots, N,$$

or

$$\bar{X}_t := \frac{1}{(1+r)^t} \bar{S}_t, \qquad t = 0, 1, \ldots, N.$$

The *discounted* value at time 0 of the portfolio is defined by

$$\tilde{V}_t = \frac{1}{(1+r)^t} V_t, \qquad t = 0, 1, \ldots, N.$$

We have

$$\begin{aligned}
\tilde{V}_t &= \frac{1}{(1+r)^t} \bar{\xi}_t \cdot \bar{S}_t \\
&= \frac{1}{(1+r)^t} \sum_{i=0}^{d} \xi_t^{(i)} S_t^{(i)} \\
&= \sum_{i=0}^{d} \xi_t^{(i)} X_t^{(i)} \\
&= \bar{\xi}_t \cdot \bar{X}_t, \qquad t = 1, \ldots, N,
\end{aligned}$$

and

$$\tilde{V}_0 = \bar{\xi}_1 \cdot X_0 = \bar{\xi}_1 \cdot S_0.$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

The effect of discounting from time t to time 0 is to divide prices by $(1+r)^t$, making all prices comparable at time 0.

2.3 Arbitrage

In discrete time, the definition of arbitrage follows the lines of its analog in the two-step model.

Definition 2.2. A portfolio strategy $(\bar{\xi}_t)_{t=1,\dots,N}$ constitutes an arbitrage opportunity if all *three* following conditions are satisfied:

(i) $V_0 \leq 0$, [start from 0 or even with a debt]

(ii) $V_N \geq 0$, [finish with a non-negative amount]

(iii) $\mathbb{P}(V_N > 0) > 0$. [a profit is made with non-zero proability]

2.4 Contingent Claims

A contingent claim is defined as an option contract that yields a non-negative random payoff C at time $t = N$. For example, in the case of a European call the payoff C is given by $X = (S_N - K)^+$ where K is called the strike price.

In a discrete-time setting we are able to consider path-dependent options in addition to European type options. One can distinguish between vanilla options such as simple European contracts and exotic or path-dependent options such as Asian, barrier, or lookback options.

The list provided below is actually very restricted and there exists many more option types, with new ones appearing constantly on the markets.

European options

The payoff of a European call on the underlying asset no i with maturity N and strike K is

$$C = (S_N^{(i)} - K)^+.$$

The payoff of a European put on the underlying asset no i with exercise date N and strike K is

$$C = (K - S_N^{(i)})^+.$$

Let us mention also the existence of binary, or digital options, also called cash-or-nothing options, whose payoffs are

$$C = \mathbf{1}_{[K,\infty)}(S_N^{(i)}) = \begin{cases} \$1 \text{ if } S_N^{(i)} \geq K, \\ \\ 0 \quad \text{if } S_N^{(i)} < K, \end{cases}$$

for call options, and

$$C = \mathbf{1}_{(-\infty,K]}(S_N^{(i)}) = \begin{cases} \$1 \text{ if } S_N^{(i)} \leq K, \\ \\ 0 \quad \text{if } S_N^{(i)} > K, \end{cases}$$

for put options.

Asian options

The payoff of an Asian call option (also called average value option) on the underlying asset no i with exercise date N and strike K is

$$C = \left(\frac{1}{N+1} \sum_{t=0}^{N} S_t^{(i)} - K \right)^+ .$$

The payoff of an Asian put option on the underlying asset no i with exercise date N and strike K is

$$C = \left(K - \frac{1}{N+1} \sum_{t=0}^{N} S_t^{(i)} \right)^+ .$$

Barrier options

The payoff of a down-an-out barrier call option on the underlying asset no i with exercise date N, strike K and barrier B is

$$C = \left(S_N^{(i)} - K \right)^+ \mathbf{1}_{\left\{ \min\limits_{t=0,1,\ldots,N} S_t^{(i)} > B \right\}} = \begin{cases} S_N^{(i)} - K \text{ if } \min\limits_{t=0,1,\ldots,N} S_t^{(i)} > B, \\ \\ 0 \qquad\qquad \text{if } \min\limits_{t=0,1,\ldots,N} S_t^{(i)} \leq B. \end{cases}$$

This option is also called a Callable Bull Contract with no residual value, in which B denotes the call price $B \geq K$. It is also called a turbo warrant with

no rebate.

The payoff of an up-and-out barrier put option on the underlying asset no i with exercise date N, strike K and barrier B is

$$C = \left(K - S_N^{(i)}\right)^+ \mathbf{1}_{\left\{\max\limits_{t=0,1,\ldots,N} S_t^{(i)} < B\right\}} = \begin{cases} K - S_N^{(i)} & \text{if } \max\limits_{t=0,1,\ldots,N} S_t^{(i)} < B, \\ 0 & \text{if } \max\limits_{t=0,1,\ldots,N} S_t^{(i)} \geq B. \end{cases}$$

This option is also called a Callable Bear Contract with no residual value, in which the call price B usually satisfies $B \leq K$. See [19], [53] for recent results on the pricing of CBBCs, or turbo warrants.

Lookback options

The payoff of a floating strike lookback call option on the underlying asset no i with exercise date N is

$$C = S_N^{(i)} - \min_{t=0,1,\ldots,N} S_t^{(i)}.$$

The payoff of a floating strike lookback put option on the underlying asset no i with exercise date N is

$$C = \left(\max_{t=0,1,\ldots,N} S_t^{(i)}\right) - S_N^{(i)}.$$

2.5 Martingales and Conditional Expectation

Before proceeding to the definition of risk-neutral probability measures in discrete time we need to introduce more mathematical tools.

Conditional expectations

Clearly the expected value of any risky asset or random variable is dependent on the amount of available information.

For example, the expected return of a real estate investment typically depends on the location of this investment.

In the probabilistic framework the available information is formalized as a collection $\mathcal{G}$ of events, which may be smaller than the collection $\mathcal{F}$ of all avail-

able events, i.e. $\mathcal{G} \subset \mathcal{F}$.

The notation $\mathbb{E}[F|\mathcal{G}]$ represents the expected value of a random variable F given (or conditionally to) the information contained in $\mathcal{G}$, and it is read "the conditional expectation of F given $\mathcal{G}$". In a certain sense, $\mathbb{E}[F|\mathcal{G}]$ represents the best possible estimate of F given the information contained in $\mathcal{G}$.

The conditional expectation satisfies the following five properties:

(i) $\mathbb{E}[F|\mathcal{G}] = F$ when F depends only on the information contained in $\mathcal{G}$.

(ii) $\mathbb{E}[FG|\mathcal{G}] = F\,\mathbb{E}[G|\mathcal{G}]$ if F depends only on the information contained in $\mathcal{G}$.

(iii) $\mathbb{E}[\mathbb{E}[F|\mathcal{H}] \mid \mathcal{G}] = \mathbb{E}[F|\mathcal{G}]$ if $\mathcal{G} \subset \mathcal{H}$, called the *tower property.*

(iv) $\mathbb{E}[F|\mathcal{G}] = \mathbb{E}[F]$ when F does not depend on the information contained in $\mathcal{G}$.

(v) If G depends only on $\mathcal{G}$ and F is independent of $\mathcal{G}$, then

$$\mathbb{E}[h(F,G)|\mathcal{G}] = \mathbb{E}[h(x,F)]_{x=G}.$$

Filtrations

The total amount of "information" present in the market at time $t = 0, 1, \ldots, N$ is denoted by $\mathcal{F}_t$. We assume that

$$\mathcal{F}_t \subset \mathcal{F}_{t+1}, \qquad t = 0, 1, \ldots, N-1,$$

which means that the amount of information available on the market increases over time.

In practice, $\mathcal{F}_t$ corresponds to the knowledge of the values $S_0^{(i)}, \ldots, S_t^{(i)}$, $i = 1, \ldots, d$, of the risky assets up to time t. In mathematical notation we say that $\mathcal{F}_t$ is generated by $S_0^{(i)}, \ldots, S_t^{(i)}$, and we usually write

$$\mathcal{F}_t = \sigma\left(S_0^{(i)}, \ldots, S_t^{(i)}\right), \qquad t = 0, 1, \ldots, N.$$

The notation $\mathcal{F}_t$ is useful to represent a quantity of information available at time t. Note that different agents or traders may work with distinct filtration. For example, an insider will have access to a filtration $(\mathcal{G}_t)_{t=0,1,\ldots,N}$ larger than the filtration $(\mathcal{F}_t)_{t=0,1,\ldots,N}$ available to an ordinary agent, in the sense that

$$\mathcal{F}_t \subset \mathcal{G}_t, \qquad t = 0, 1, \ldots, N.$$

This version: March 8, 2011
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The notation $\mathbb{E}[F|\mathcal{F}_t]$ represents the expected value of a random variable F given (or conditionally to) the information contained in $\mathcal{F}_t$. Again, $\mathbb{E}[F|\mathcal{F}_t]$ denotes the best possible estimate of F given the information known up to time t.

We will assume that no information is available at time $t = 0$, hence

$$\mathbb{E}[F|\mathcal{F}_0] = \mathbb{E}[F]$$

for any integrable random variable F.

As above, the conditional expectation with respect to $\mathcal{F}_t$ satisfies the following five properties:

(i) $\mathbb{E}[F|\mathcal{F}_t] = F$ when F depends only on the information known at time t and contained in $\mathcal{F}_t$.

(ii) $\mathbb{E}[FG|\mathcal{F}_t] = F\,\mathbb{E}[G|\mathcal{F}_t]$ if F depends only on the information contained in $\mathcal{F}_t$.

(iii) $\mathbb{E}[\mathbb{E}[F|\mathcal{F}_{t+1}] \mid \mathcal{F}_t] = \mathbb{E}[F|\mathcal{F}_t]$ if $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ (also called the *tower property*).

(iv) $\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F]$ when F does not depend on the information contained in $\mathcal{F}_t$.

(v) If F depends only on $\mathcal{F}_t$ and G is independent of $\mathcal{F}_t$, then

$$\mathbb{E}[h(F,G)|\mathcal{F}_t] = \mathbb{E}[h(x,G)]_{x=F}.$$

Note that by the tower property *(iii)* the process $t \mapsto \mathbb{E}[F \mid \mathcal{F}_t]$ is a martingale.

Martingales

A martingale is a stochastic process whose value at time $t+1$ can be estimated using conditional expectation given its value at time t. Recall that a process $(M_t)_{t=0,1,\ldots,N}$ is said to be $\mathcal{F}_t$-adapted if the value of M_t depends only on the information available at time t in $\mathcal{F}_t$, $t = 0, 1, \ldots, N$.

Definition 2.3. A stochastic process $(M_t)_{t=0,1,\ldots,N}$ is called a discrete time *martingale* with respect to the filtration $(\mathcal{F}_t)_{t=0,1,\ldots,N}$ if $(M_t)_{t=0,1,\ldots,N}$ is $\mathcal{F}_t$-adapted and satisfies the property

$$\mathbb{E}[M_{t+1}|\mathcal{F}_t] = M_t, \qquad t = 0, 1, \ldots, N-1.$$

Note that the above definition implies that $M_t \in \mathcal{F}_t$, $t = 0, 1, \ldots, N$. In other words, a random process $(M_t)_{t=0,1,\ldots,N}$ is a martingale if the best possible prediction of M_{t+1} given $\mathcal{F}_t$ is simply M_t.

As an example of the use of martingales we can mention weather forecasting. If M_t denotes the random temperature observed at time t, this process is a martingale when the best possible forecast of tomorrow's temperature M_{t+1} given information known up to time t is just today's temperature M_t, $t = 0, 1, \ldots, N-1$.

2.6 Risk-Neutral Probability Measures

As in the two time step model, the concept of risk neutral measures will be used to price financial claims under the absence of arbitrage hypothesis.

Definition 2.4. A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if under $\mathbb{P}^*$, the expected return of each risky asset equals the return r of the riskless asset, that is

$$\mathbb{E}^*[S_{t+1}^{(i)}|\mathcal{F}_t] = (1+r)S_t^{(i)}, \qquad t = 0, 1, \ldots, N-1, \tag{2.4}$$

$i = 0, 1, \ldots, d$. Here, $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

Since $S_t^{(i)} \in \mathcal{F}_t$, Relation (2.4) can be rewritten in terms of asset returns as

$$\mathbb{E}^*\left[\frac{S_{t+1}^{(i)} - S_t^{(i)}}{S_t^{(i)}}\Big|\mathcal{F}_t\right] = r, \qquad t = 0, 1, \ldots, N-1.$$

In other words, taking risks under $\mathbb{P}^*$ by buying the risky asset n^o i has a neutral effect, as the expected return is that of the riskless asset. The measure $\mathbb{P}^*$ would be represent a risk premium if we had

$$\mathbb{E}^*[S_{t+1}^{(i)}|\mathcal{F}_t] = (1+\tilde{r})S_t^{(i)}, \qquad t = 0, 1, \ldots, N-1,$$

with $\tilde{r} > r$.

The definition of risk-neutral probability measure can be reformulated using the notion of martingale.

Proposition 2.1. *A probability measure $\mathbb{P}^*$ on Ω is a risk-neutral measure if and only if the discounted price process $X_t^{(i)}$ is a martingale* under $\mathbb{P}^*$*, i.e.*

$$\mathbb{E}^*[X_{t+1}^{(i)}|\mathcal{F}_t] = X_t^{(i)}, \qquad t = 0, 1, \ldots, N-1, \tag{2.5}$$

$i = 0, 1, \ldots, d$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Proof. It suffices to check that Conditions (2.4) and (2.5) are equivalent since

$$\mathbb{E}^*[S_{t+1}^{(i)}|\mathcal{F}_t] = (1+r)^{t+1}\,\mathbb{E}^*[X_{t+1}^{(i)}|\mathcal{F}_t] \qquad \text{and} \qquad S_t^{(i)} = (1+r)^t X_t^{(i)},$$

$t = 0, 1, \ldots, N-1$, $i = 1, \ldots, d$. □

Next we restate the first fundamental theorem of mathematical finance in discrete time, which can be used to check for the existence of arbitrage opportunities.

Theorem 2.1. A market is *without* arbitrage opportunity if and only if it admits at least one risk-neutral measure.

Proof. cf. Theorem 5.17 of [21]. □

2.7 Market Completeness

Definition 2.5. A contingent claim with payoff C is said to be attainable (at time N) if there exists a portfolio strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$ such that

$$C = \bar{\xi}_N \cdot \bar{S}_N. \tag{2.6}$$

In case $(\bar{\xi}_t)_{t=1,\ldots,N}$ is a portfolio that attains the claim C at time N, i.e. if (2.6) is satisfied, we also say that $(\bar{\xi}_t)_{t=1,\ldots,N}$ *hedges* the claim C. In case (2.6) is replaced by the condition

$$\bar{\xi}_N \cdot \bar{S}_N \geq C,$$

we talk of super-hedging. When $(\bar{\xi}_t)_{t=1,\ldots,N}$ hedges C, the arbitrage price $\pi_t(C)$ of the claim at time t will be given by the value

$$\pi_t(C) = \bar{\xi}_t \cdot \bar{S}_t$$

of the portfolio at time $t = 0, 1, \ldots, N$. Note that at time $t = N$ we have

$$\pi_N(C) = \bar{\xi}_N \cdot \bar{S}_N = C,$$

i.e. since exercise of the claim occurs at time N, the price $\pi_N(C)$ of the claim equals the value C of the payoff.

Definition 2.6. A market model is said to be *complete* if every contingent claim is attainable.

The next result can be viewed as the second fundamental theorem of mathematical finance.

Theorem 2.2. A market model without arbitrage is complete if and only if it admits only one risk-neutral measure.

Proof. cf. Theorem 5.38 of [21]. □

2.8 The Cox-Ross-Rubinstein (CRR) Market Model

We consider the discrete time Cox-Ross-Rubinstein model [12] with $N+1$ time instants $t = 0, 1, \ldots, N$ and $d = 1$ risky asset, also called the *binomial model*. The price $S_t^{(0)}$ of the riskless asset evolves as

$$S_t^{(0)} = \pi^{(0)}(1+r)^t, \qquad t = 0, 1, \ldots, N.$$

Let the *return* of the risky asset $S = S^{(1)}$ be defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, \ldots, N.$$

In the CRR model the return R_t is random and allowed to take only two values a and b at each time step, i.e.

$$R_t \in \{a, b\}, \qquad t = 1, \ldots, N.$$

That means, the evolution of S_{t-1} to S_t is random and given by

$$S_t = \left\{ \begin{array}{ll} (1+b)S_{t-1} & \text{if } R_t = b \\ \\ (1+a)S_{t-1} & \text{if } R_t = a \end{array} \right\} = (1+R_t)S_{t-1}, \qquad t = 1, \ldots, N,$$

and

$$S_t = S_0 \prod_{j=1}^{t} (1+R_j), \qquad t = 0, 1, \ldots, N.$$

Note that the price process $(S_t)_{t=0,1,\ldots,N}$ evolves on a binary recombining (or binomial) tree. The discounted asset price is

$$X_t = \frac{S_t}{(1+r)^t}, \qquad t = 0, 1, \ldots, N,$$

with

$$X_t = \left\{ \begin{array}{ll} \frac{1+b}{1+r} X_{t-1} & \text{if } R_t = b \\ \\ \frac{1+a}{1+r} X_{t-1} & \text{if } R_t = a \end{array} \right\} = \frac{1+R_t}{1+r} X_{t-1}, \qquad t = 1, \ldots, N,$$

and

$$X_t = \frac{S_0}{(1+r)^t} \prod_{j=1}^{t} (1+R_j) = X_0 \prod_{j=1}^{t} \frac{1+R_j}{1+r}.$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

In this model the discounted value at time t of the portfolio is given by

$$\bar{\xi}_t \cdot \bar{X}_t = \xi_t^{(0)}\pi_0 + \xi_t^{(1)} X_t, \qquad t = 1, \ldots, N.$$

The information $\mathcal{F}_t$ known in the market up to time t is given by the knowledge of $S_1, \ldots, S_t$, which is equivalent to the knowledge of $X_1, \ldots, X_t$ or $R_1, \ldots, R_t$, i.e. we write

$$\mathcal{F}_t = \sigma(S_1, \ldots, S_t) = \sigma(X_1, \ldots, X_t) = \sigma(R_1, \ldots, R_t), \qquad t = 0, 1, \ldots, N,$$

where as a convention $\mathcal{F}_0 = \{\emptyset, \Omega\}$ contains no information.

Theorem 2.3. The CRR model is without arbitrage if and only if $a < r < b$. In this case the market is complete.

Proof. In order to check for arbitrage opportunities we may use Theorem 2.1 and look for a risk-neutral measure $\mathbb{P}^*$. According to the definition of a risk-neutral measure this probability $\mathbb{P}^*$ should satisfy Condition (2.4), i.e.

$$\mathbb{E}^*[S_{t+1}^{(i)} | \mathcal{F}_t] = (1+r) S_t^{(i)}, \qquad t = 0, 1, \ldots, N-1.$$

Rewriting $\mathbb{E}^*[S_{t+1}|\mathcal{F}_t]$ as

$$\mathbb{E}^*[S_{t+1}|\mathcal{F}_t] = (1+a)S_t\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + (1+b)S_t\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t),$$

it follows that any risk-neutral measure $\mathbb{P}^*$ should satisfy the equations

$$\begin{cases} (1+b)S_t\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + (1+a)S_t\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = (1+r)S_t \\ \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = 1, \end{cases}$$

i.e.

$$\begin{cases} b\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + a\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = r \\ \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = 1, \end{cases}$$

with solution

$$\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a} \qquad \text{and} \qquad \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a}. \tag{2.7}$$

Clearly, $\mathbb{P}^*$ can be a non singular probability measure only if $r - a > 0$ and $b - r > 0$. In this case the solution $\mathbb{P}^*$ of the problem is unique hence the market is complete by Theorem 2.2. □

Note that the values of $\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t)$ and $\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t)$ computed in (2.7) are non random, hence they are independent of the information contained in $\mathcal{F}_t$. As a consequence, under $\mathbb{P}^*$, the random variable R_{t+1} is independent of the information $\mathcal{F}_t$ up to time t, which is generated by $R_1, \ldots, R_t$.

In other words, R_{t+1} is independent of $R_1, \ldots, R_t$ for all $t = 1, \ldots, N-1$, the random variables $R_1, \ldots, R_N$ are *independent* under $\mathbb{P}^*$, and by (2.7) we have

$$\mathbb{P}^*(R_{t+1} = b) = \frac{r-a}{b-a} \qquad \text{and} \qquad \mathbb{P}^*(R_{t+1} = a) = \frac{b-r}{b-a}.$$

As a consequence, letting $p^* := (r-a)/(b-a)$, when $(k_1, \ldots, k_n) \in \{a, b\}^{N+1}$ we have

$$\mathbb{P}^*(R_1 = k_1, \ldots, R_N = k_n) = (p^*)^l (1-p^*)^{N-l},$$

where l, resp. $N-l$, denotes the number of times the term "a", resp. "b", appears in the sequence $\{k_1, \ldots, k_N\}$.

2.9 Exercises

Exercise 2.1. We consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$, and the price π_t of the riskless asset evolves as $\pi_t = \pi_0(1+r)^t$, $t = 0, 1, \ldots, N$. The evolution of S_{t-1} to S_t is given by

$$S_t = \begin{cases} (1+b)S_{t-1} \\ \\ (1+a)S_{t-1} \end{cases}$$

with $-1 < a < r < b$. The *return* of the risky asset S is defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, \ldots, N,$$

and $\mathcal{F}_t$ is generated by $R_1, \ldots, R_t$, $t = 1, \ldots, N$.

1. What are the possible values of R_t ?
2. Show that under the probability measure P^* defined by

$$P^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a}, \qquad P^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a},$$

   $t = 0, 1, \ldots, N-1$, the expected return $\mathbb{E}^*[R_{t+1}|\mathcal{F}_t]$ of S is equal to the return r of the riskless asset.
3. Show that under P^* the process $(S_t)_{t=0,\ldots,N}$ satisfies

$$\mathbb{E}^*[S_{t+k}|\mathcal{F}_t] = (1+r)^k S_t, \qquad t = 0, \ldots, N-k, \quad k = 0, \ldots, N.$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Chapter 3
Pricing and hedging in discrete time

We consider the pricing and hedging of options in a discrete time financial model with $N+1$ time instants $t = 0, 1, \ldots, N$. Vanilla options are treated using backward induction, and exotic options with arbitrary payoff functions are considered using the Clark-Ocone formula in discrete time.

3.1 Pricing of Contingent Claims

Let us consider an attainable contingent claim with payoff $C \geq 0$ and maturity N. Recall that by the definition of attainability there exists a hedging portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ such that

$$\bar{\xi}_N \cdot \bar{S}_N = C \tag{3.1}$$

at time N.

Clearly, if (3.1) holds, then investing the amount

$$V_t = \bar{\xi}_t \cdot \bar{S}_t$$

into a self-financing hedging portfolio at any time $t = 0, 1, \ldots, N$ will allow one to hedge the option and obtain the perfect replication (3.1) at time N.

The value at time t of such a self-financing portfolio is called the arbitrage price of the claim C at time t, and denoted by $\pi_t(C)$, with

$$\pi_t(C) = V_t = \bar{\xi}_t \cdot \bar{S}_t, \qquad t = 0, 1, \ldots, N.$$

Next we develop the other approach to the pricing of contingent claims, based on conditional expectations. We will need the following lemma.

Lemma 3.1. The following statements are equivalent:

(i) The portfolio strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$ is self-financing.

(ii) $\bar{\xi}_t \cdot \bar{X}_t = \bar{\xi}_{t+1} \cdot \bar{X}_t$, $t = 1, \ldots, N-1$.

(iii) We have

$$\tilde{V}_t = \tilde{V}_0 + \sum_{j=1}^{t} \bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}), \qquad t = 0, 1, \ldots, N. \tag{3.2}$$

Proof. First, the self-financing condition (i)

$$\bar{\xi}_{t-1} \cdot \bar{S}_{t-1} = \bar{\xi}_t \cdot \bar{S}_{t-1}, \qquad t = 1, \ldots, N,$$

is clearly equivalent to (ii) by division of both sides by $(1+r)^{t-1}$.

Next, assuming that (ii) holds we have

$$\begin{aligned}
\tilde{V}_t &= \tilde{V}_0 + \sum_{j=1}^{t} (\tilde{V}_j - \tilde{V}_{j-1}) \\
&= \tilde{V}_0 + \sum_{j=1}^{t} \bar{\xi}_j \cdot \bar{X}_j - \bar{\xi}_{j-1} \cdot \bar{X}_{j-1} \\
&= \tilde{V}_0 + \sum_{j=1}^{t} \bar{\xi}_j \cdot \bar{X}_j - \bar{\xi}_j \cdot \bar{X}_{j-1} \\
&= \tilde{V}_0 + \sum_{j=1}^{t} \bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}), \qquad t = 1, \ldots, N.
\end{aligned}$$

Finally, assuming that (iii) holds we get

$$\tilde{V}_t - \tilde{V}_{t-1} = \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}),$$

hence

$$\bar{\xi}_t \cdot \bar{X}_t - \bar{\xi}_{t-1} \cdot \bar{X}_{t-1} = \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}),$$

and

$$\bar{\xi}_{t-1} \cdot \bar{X}_{t-1} = \bar{\xi}_t \cdot \bar{X}_{t-1}, \qquad t = 1, \ldots, N.$$

□

In Relation (3.2), the term $\bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1})$ represents the profit and loss

$$\tilde{V}_t - \tilde{V}_{t-1} = \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}),$$

of the self-financing portfolio strategy $(\bar{\xi}_j)_{j=1,\ldots,N}$ over the time period $[t-1, t]$, computed by multiplication of the portfolio allocation $\bar{\xi}_t$ with the

change of price $\bar{X}_t - \bar{X}_{t-1}$, $t = 1, \ldots, N$.

Relation (3.2) admits an obvious interpretation by saying that when a portfolio is self-financing the value $\tilde{V}_t$ of the (discounted) portfolio at time t is given by adding the (discounted) profits and losses registered over all time periods from time 0 to time t.

The sum (3.2) is also referred to as a *stochastic integral* in discrete time of the portfolio process $(\bar{\xi}_t)_{t=1,\ldots,N}$ with respect to the random process $(\bar{X}_t)_{t=0,1,\ldots,N}$. In particular, this expression shows that $(\tilde{V}_t)_{t=0,1,\ldots,N}$ is a martingale under $\mathbb{P}^*$.

As a consequence of Lemma 3.1, if a contingent claim C with discounted payoff

$$\tilde{C} := \frac{C}{(1+r)^N}$$

is attainable by a self-financing portfolio strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$ then we have

$$\tilde{C} = \bar{\xi}_N \cdot \bar{X}_N = \tilde{V}_N = \tilde{V}_0 + \sum_{t=1}^{N} \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}).$$

Note that in the above formula, discounting at time 0 also allows one to make the sum of profits and losses $\bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1})$ which become comparable on each time period, since \$1 at time $t = 0$ and \$1 at time $t = 1$ can not be added without discounting.

Theorem 3.1. The arbitrage price $\pi_t(C)$ of a contingent claim C is given by

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0, 1, \ldots, N, \tag{3.3}$$

where $\mathbb{P}^*$ denotes any risk-neutral probability measure.

Proof. Let $\tilde{C} = C/(1+r)^N$ denote the discounted payoff of the claim C. We will show that for any risk-neutral measure $\mathbb{P}^*$ the discounted value of a self-financing portfolio hedging C is given by

$$\tilde{V}_t = \mathbb{E}^*[\tilde{C} \mid \mathcal{F}_t], \qquad t = 0, 1, \ldots, N, \tag{3.4}$$

which shows that

$$V_t = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t]$$

after multiplication of both sides by $(1+r)^t$. Then we note that at any time t the price $\pi_t(C)$ of the claim coincides with the value V_t of the corresponding self-financing portfolio.

It now remains to prove (3.4). Since the portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ is self-financing, from Lemma 3.1 we have

$$\begin{aligned}
&\mathbb{E}^*[\tilde{C} \mid \mathcal{F}_t] = \mathbb{E}^*[\tilde{V}_N \mid \mathcal{F}_t] \\
&= \mathbb{E}^*\left[\tilde{V}_0 + \sum_{j=1}^N \bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\tilde{V}_0 \mid \mathcal{F}_t\right] + \sum_{j=1}^N \mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t\right] \\
&= \tilde{V}_0 + \sum_{j=1}^t \mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t\right] + \sum_{j=t+1}^N \mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t\right] \\
&= \tilde{V}_0 + \sum_{j=1}^t \bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) + \sum_{j=t+1}^N \mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t\right] \\
&= \tilde{V}_t + \sum_{j=t+1}^N \mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t\right],
\end{aligned}$$

where we used Relation 3.2 of Lemma 3.1. To conclude the proof we need to show that

$$\sum_{j=t+1}^N \mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t\right] = 0.$$

Let us show that

$$\mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t\right] = 0,$$

for all $j = t+1, \ldots, N$. We have $0 \leq t \leq j-1$ hence $\mathcal{F}_t \subset \mathcal{F}_{j-1}$, and by the tower property of conditional expectations we get

$$\mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t\right] = \mathbb{E}^*\left[\mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_{j-1}\right] \mid \mathcal{F}_t\right],$$

therefore it suffices to show that

$$\mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_{j-1}\right] = 0.$$

We note that the porfolio allocation $\bar{\xi}_j$ over the time period $[j-1, j]$ is decided at time $j-1$, thus it depends[1] only on the information $\mathcal{F}_{j-1}$ known up to time $j-1$, and using Property (ii) of conditional expectations we may pull out $\bar{\xi}_j$ out of the expectation since it behaves as a constant parameter given $\mathcal{F}_{j-1}$, $j = 1, \ldots, N$. We then obtain

$$\mathbb{E}^*\left[\bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_{j-1}\right] = \bar{\xi}_j \cdot \mathbb{E}^*\left[\bar{X}_j - \bar{X}_{j-1} \mid \mathcal{F}_{j-1}\right].$$

[1] One says that the process $(\bar{\xi}_j)_{j=1,\ldots,N}$ is *predictable*.

Finally we note that

$$\begin{aligned}\mathbb{E}^*\left[\bar{X}_j - \bar{X}_{j-1} \mid \mathcal{F}_{j-1}\right] &= \mathbb{E}^*\left[\bar{X}_j \mid \mathcal{F}_{j-1}\right] - \mathbb{E}^*\left[\bar{X}_{j-1} \mid \mathcal{F}_{j-1}\right] \\ &= \mathbb{E}^*\left[\bar{X}_j \mid \mathcal{F}_{j-1}\right] - \bar{X}_{j-1} \\ &= 0, \qquad j = 1, \ldots, N,\end{aligned}$$

because $(\bar{X}_t)_{t=0,1,\ldots,N}$ is a martingale under the risk-neutral measure $\mathbb{P}^*$, and this concludes the proof. □

Note that (3.3) admits an interpretation in the framework of insurance, in which $\pi_t(C)$ represents an insurance premium and C represents the random value of an insurance claim made by a subscriber. It this context, the insurance contract premium is written as the average value of the random claims after time discounting. In addition, the discounted price process $((1+r)^{-t}\pi_t(C))_{t=0,1,\ldots,N}$ is a martingale under $\mathbb{P}^*$.

As a consequence of Theorem 3.1, the discounted portfolio process $(\tilde{V}_t)_{t=0,1,\ldots,N}$ is a martingale under $\mathbb{P}^*$, since

$$\begin{aligned}\mathbb{E}^*\left[\tilde{V}_{t+1} \mid \mathcal{F}_t\right] &= \mathbb{E}^*\left[\mathbb{E}^*\left[\tilde{C} \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[\tilde{C} \mid \mathcal{F}_t\right] \\ &= \tilde{V}_t, \qquad t = 0, \ldots, N-1,\end{aligned}$$

from the "tower property" of conditional expectations.

In particular for $t = 0$ we obtain the price of the contingent claim C at time 0:

$$\pi_0(C) = \mathbb{E}^*[\tilde{C} \mid \mathcal{F}_0] = \mathbb{E}^*[\tilde{C}] = \frac{1}{(1+r)^N}\,\mathbb{E}^*[C].$$

3.2 Hedging of Contingent Claims - Backward Induction

The basic idea of hedging is to allocate assets in a portfolio in order to protect oneself from a given risk. For example, a risk of increasing oil prices can be hedged by buying oil-related stocks, whose value should be positively correlated with the oil price. In this way, a loss connected to increasing oil prices could be compensated by an increase in the value of the corresponding portfolio.

In the setting of this chapter, hedging an attainable contingent claim C means computing a self-financing portfolio strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$ such that $\bar{\xi}_N \cdot \bar{S}_N = C$, i.e.

$$\bar{\xi}_N \cdot \bar{X}_N = \tilde{C}. \tag{3.5}$$

Equation (3.5) can be solved for $\bar{\xi}_N$. The idea is then to work by *backward* induction and to compute successively $\bar{\xi}_{N-1}$, $\bar{\xi}_{N-2}$, ... , $\bar{\xi}_3$, down to $\bar{\xi}_3$, $\bar{\xi}_2$ and finally $\bar{\xi}_1$.

In order to implement this algorithm we may use the self-financing condition which yields $N-1$ equations

$$\bar{\xi}_t \cdot \bar{X}_t = \bar{\xi}_{t+1} \cdot \bar{X}_t, \qquad t = 1, \ldots, N-1, \tag{3.6}$$

and allows us in principle to compute the portfolio strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$.

After solving (3.5) for $\bar{\xi}_N$, we then use $\bar{\xi}_N$ to solve the self-financing condition

$$\bar{\xi}_{N-1} \cdot \bar{S}_{N-1} = \bar{\xi}_N \cdot \bar{S}_{N-1}$$

for $\bar{\xi}_{N-1}$, then

$$\bar{\xi}_{N-2} \cdot \bar{S}_{N-2} = \bar{\xi}_{N-1} \cdot \bar{S}_{N-2}$$

for $\bar{\xi}_{N-2}$, and successively $\bar{\xi}_2$ down to $\bar{\xi}_1$.

Then the discounted value $\tilde{V}_t$ at time t of the portfolio claim can be obtained from

$$\tilde{V}_0 = \bar{\xi}_1 \cdot \bar{X}_0 \qquad \text{and} \qquad \tilde{V}_t = \bar{\xi}_t \cdot \bar{X}_t, \qquad t = 1, \ldots, N. \tag{3.7}$$

In the proof of Theorem 3.1 we actually showed that the price $\pi_t(C)$ of the claim at time t coincides with the value V_t of any self-financing portfolio hedging the claim C, i.e.

$$\pi_t(C) = V_t, \qquad t = 0, 1, \ldots, N.$$

Therefore we have

$$\bar{\xi}_t \cdot \bar{S}_t = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0, 1, \ldots, N. \tag{3.8}$$

and the price of the claim can be computed either by algebra via (3.6) and (3.7), or by a probabilistic method by (3.8).

3.3 Pricing of Vanilla Options in the CRR Model

In this section we consider the hedging of contingent claims in the discrete time Cox-Ross-Rubinstein model, with $d = 1$. Recall that the portfolio value process $(V_t)_{t=0,1,\ldots N}$ and the discounted portfolio value process are respectively defined as

$$V_t = \bar{\xi}_t \cdot \bar{S}_t \qquad \text{and} \qquad \tilde{V}_t = \frac{1}{(1+r)^t} V_t = \bar{\xi}_t \cdot \bar{X}_t, \qquad t = 0, 1, \ldots, N.$$

Here we will be concerned with the pricing of vanilla options with payoffs of the form

$$C = f(S_N),$$

e.g. $f(x) = (x-K)^+$ in the case of a European call. Equivalently, the discounted claim

$$\tilde{C} = \frac{C}{(1+r)^N}$$

satisfies $\tilde{C} = \tilde{f}(S_N)$ with $\tilde{f}(x) = f(x)/(1+r)^N$, i.e. $\tilde{f}(x) = \dfrac{1}{(1+r)^N}(x-K)^+$ in the case of a European call with strike K.

From Theorem 3.1, the discounted value of a portfolio hedging the attainable (discounted) claim $\tilde{C}$ is given by

$$\tilde{V}_t = \mathbb{E}^*[\tilde{f}(S_N) \mid \mathcal{F}_t], \qquad t = 0, 1, \ldots, N,$$

under the risk-neutral measure $\mathbb{P}^*$. Equivalently, the arbitrage price $\pi_t(C)$ of the contingent claim $C = f(S_N)$ is given by

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[f(S_N) \mid \mathcal{F}_t], \qquad t = 0, 1, \ldots, N. \tag{3.9}$$

Proposition 3.1. *The price $\pi_t(C)$ of the contingent claim $C = f(S_N)$ is given by*

$$\pi_t(C) = v(t, S_t),$$

where the function $v(t,x)$ is given by

$$v(t,x) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^* \left[f\left(x \prod_{j=t+1}^{N} (1+R_j) \right) \right]$$
$$= \frac{1}{(1+r)^{N-t}} \sum_{j=0}^{N-t} \frac{(N-t)!}{j!(N-t-j)!} (p^*)^j (1-p^*)^{N-t-j} f\left(x\,(1+b)^j\,(1+a)^{N-t-j} \right).$$

Proof. From the relations

$$S_N = S_t \prod_{j=t+1}^{N} (1+R_j),$$

and (3.9) we have, using Property (v) of the conditional expectation and the independence of $\{R_1, \ldots, R_t\}$ with $\{R_{t+1}, \ldots, R_N\}$,

$$
\begin{aligned}
\pi_t(C) &= \frac{1}{(1+r)^{N-t}}\,\mathbb{E}^*[f(S_N)\mid \mathcal{F}_t] \\
&= \frac{1}{(1+r)^{N-t}}\,\mathbb{E}^*\left[f\left(S_t\prod_{j=t+1}^{N}(1+R_j)\right)\Big|\mathcal{F}_t\right] \\
&= \frac{1}{(1+r)^{N-t}}\,\mathbb{E}^*\left[f\left(x\prod_{j=t+1}^{N}(1+R_j)\right)\right]_{x=S_t} \\
&= \frac{1}{(1+r)^{N-t}}\sum_{j=0}^{N-t}\frac{(N-t)!}{j!(N-t-j)!}(p^*)^j(1-p^*)^{N-t-j}f\left(S_t\,(1+b)^j\,(1+a)^{N-t-j}\right),
\end{aligned}
$$

with

$$
p^* = \frac{r-a}{b-a} \qquad \text{and} \qquad 1-p^* = \frac{b-r}{b-a},
$$

where we used the fact that the set of paths from time $t+1$ to time N containing j times "$(1+b)$" has cardinal $\binom{N-t}{j}$, each such path having the probability $(p^*)^j(1-p^*)^{N-t-j}$, $j=0,\ldots,N-t$. □

Note that the price $\pi_t(C)=V_t$ at time t of the claim C is written as the average the values of the contingent claim over all possible paths starting from S_t.

The discounted price $\tilde{V}_t$ of the portfolio can also be computed via a *backward induction* procedure. Namely we have

$$
\begin{aligned}
\tilde{V}_t &= \tilde{v}(t,S_t) \\
&= \mathbb{E}^*[\tilde{f}(S_N)\mid\mathcal{F}_t] \\
&= \mathbb{E}^*[\mathbb{E}^*[\tilde{f}(S_N)\mid\mathcal{F}_{t+1}]\mid\mathcal{F}_t] \\
&= \mathbb{E}^*[\tilde{V}_{t+1}\mid\mathcal{F}_t] \\
&= \mathbb{E}^*[\tilde{v}(t+1,S_{t+1})\mid\mathcal{F}_t] \\
&= \tilde{v}\left(t+1,(1+a)S_t\right)\mathbb{P}^*(R_{t+1}=a)+\tilde{v}\left(t+1,(1+b)S_t\right)\mathbb{P}^*(R_{t+1}=b) \\
&= (1-p^*)\tilde{v}\left(t+1,(1+a)S_t\right)+p^*\tilde{v}\left(t+1,(1+b)S_t\right),
\end{aligned}
$$

hence

$$
\tilde{v}(t,x) = (1-p^*)\tilde{v}\left(t+1,x(1+a)\right)+p^*\tilde{v}\left(t+1,x(1+b)\right),
$$

under the terminal condition $\tilde{V}_N=\tilde{f}(S_N)$, i.e.

$$
\tilde{v}(N,x)=\tilde{f}(x).
$$

Note also that from Theorem 3.1, the discounted portfolio process $(\tilde{V}_t)_{t=0,1,\ldots,N}$ is a martingale under $\mathbb{P}^*$. Indeed, from the tower property of conditional expectations we have

$$\begin{aligned}\mathbb{E}^*[\tilde{V}_{t+1} \mid \mathcal{F}_t] &= \mathbb{E}^*[\mathbb{E}^*[\tilde{C} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \\ &= \mathbb{E}^*[\tilde{C} \mid \mathcal{F}_t] \\ &= \tilde{V}_t, \qquad t = 0, 1, \ldots, N-1.\end{aligned}$$

3.4 Hedging of Vanilla Options in the CRR model

In this section we consider the hedging of contingent claims in the discrete time Cox-Ross-Rubinstein model. Our aim is to compute a self-financing portfolio strategy hedging a vanilla option with payoff of the form

$$C = f(S_N).$$

Proposition 3.2. *The replicating portfolio* $(\xi_t^{(0)}, \xi_t^{(1)})_{t=1,\ldots,N}$ *hedging the contingent claim* $C = f(S_N)$ *is given by*

$$\xi_t^{(1)} = \frac{\tilde{v}\,(t, (1+b)S_{t-1}) - \tilde{v}\,(t, (1+a)S_{t-1})}{X_{t-1}(b-a)/(1+r)}, \qquad t = 1, \ldots, N,$$

and

$$\xi_t^{(0)} = \frac{\tilde{v}(t-1, S_{t-1}) - \xi_t^{(1)} X_{t-1}}{\pi^{(0)}}, \qquad t = 1, \ldots, N.$$

Proof. Recall that by Lemma 3.1 the following statements are equivalent:

(i) The portfolio strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$ is self-financing.

(ii) We have

$$\tilde{V}_t = \tilde{V}_0 + \sum_{j=1}^{t} \bar{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}), \qquad t = 1, \ldots, N.$$

As a consequence, any self-financing hedging strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$ should satisfy

$$\tilde{v}(t, S_t) - \tilde{v}(t-1, S_{t-1}) = \tilde{V}_t - \tilde{V}_{t-1} = \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}).$$

Note that since the discounted price $X_t^{(0)}$ of the riskless asset satisfies

$$X_t^{(0)} = (1+r)^{-t} S_t^{(0)} = \pi^{(0)},$$

we have

$$\begin{aligned}\bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}) &= \xi_t^{(0)}(X_t^{(0)} - X_{t-1}^{(0)}) + \xi_t^{(1)}(X_t^{(1)} - X_{t-1}^{(1)}) \\ &= \xi_t^{(0)}(\pi^{(0)} - \pi^{(0)}) + \xi_t^{(1)}(X_t^{(1)} - X_{t-1}^{(1)})\end{aligned}$$

$$
\begin{aligned}
&= \xi_t^{(1)}(X_t^{(1)} - X_{t-1}^{(1)}) \\
&= \xi_t^{(1)}(X_t - X_{t-1}), \qquad t = 1, \ldots, N.
\end{aligned}
$$

Hence we have

$$
\tilde{v}(t, S_t) - \tilde{v}(t-1, S_{t-1}) = \xi_t^{(1)}(X_t - X_{t-1}), \qquad t = 1, \ldots, N,
$$

and from this we deduce two equations

$$
\begin{cases}
\tilde{v}\left(t, (1+a)S_{t-1}\right) - \tilde{v}(t-1, S_{t-1}) = \xi_t^{(1)}\left(\dfrac{1+a}{1+r}X_{t-1} - X_{t-1}\right), \\[2ex]
\tilde{v}\left(t, (1+b)S_{t-1}\right) - \tilde{v}(t-1, S_{t-1}) = \xi_t^{(1)}\left(\dfrac{1+b}{1+r}X_{t-1} - X_{t-1}\right), \qquad t = 1, \ldots, N,
\end{cases}
$$

i.e.

$$
\begin{cases}
\tilde{v}\left(t, (1+a)S_{t-1}\right) - \tilde{v}(t-1, S_{t-1}) = \xi_t^{(1)}\dfrac{a-r}{1+r}X_{t-1}, \\[2ex]
\tilde{v}\left(t, (1+b)S_{t-1}\right) - \tilde{v}(t-1, S_{t-1}) = \xi_t^{(1)}\dfrac{b-r}{1+r}X_{t-1}, \qquad t = 1, \ldots, N,
\end{cases}
$$

hence

$$
\xi_t^{(1)} = \frac{\tilde{v}\left(t, (1+a)S_{t-1}\right) - \tilde{v}\left(t-1, S_{t-1}\right)}{X_{t-1}(a-r)/(1+r)}, \qquad t = 1, \ldots, N,
$$

and

$$
\xi_t^{(1)} = \frac{\tilde{v}\left(t, (1+b)S_{t-1}\right) - \tilde{v}\left(t-1, S_{t-1}\right)}{X_{t-1}(b-r)/(1+r)}, \qquad t = 1, \ldots, N.
$$

From the obvious relation

$$
\xi_t^{(1)} = \frac{b-r}{b-a}\xi_t^{(1)} - \frac{a-r}{b-a}\xi_t^{(1)},
$$

we get

$$
\xi_t^{(1)} = \frac{\tilde{v}\left(t, (1+b)S_{t-1}\right) - \tilde{v}\left(t, (1+a)S_{t-1}\right)}{X_{t-1}(b-a)/(1+r)}, \qquad t = 1, \ldots, N,
$$

which only depends on S_{t-1} as expected. This is consistent with the fact that $\xi_t^{(1)}$ represents the number of units of the risky asset to be present in the portfolio over the time period $[t-1, t]$ in order to hedge the claim C at time N, and is decided at time $t-1$.

Concerning the quantity $\xi_t^{(0)}$ of the riskless asset in the portfolio at time t, recall that we have

$$\tilde{V}_t = \bar{\xi}_t \cdot \bar{X}_t = \xi_t^{(0)} X_t^{(0)} + \xi_t^{(1)} X_t^{(1)}, \qquad t = 1, \ldots, N,$$

hence

$$\begin{aligned}
\xi_t^{(0)} &= \frac{\tilde{V}_t - \xi_t^{(1)} X_t^{(1)}}{X_t^{(0)}} \\
&= \frac{\tilde{V}_t - \xi_t^{(1)} X_t^{(1)}}{\pi^{(0)}} \\
&= \frac{\tilde{v}(t, S_t) - \xi_t^{(1)} X_t^{(1)}}{\pi^{(0)}}, \qquad t = 1, \ldots, N.
\end{aligned}$$

Note that we have

$$\begin{aligned}
\xi_t^{(0)} &= \frac{\tilde{v}(t-1, S_{t-1}) + (\tilde{v}(t, S_t) - \tilde{v}(t-1, S_{t-1})) - \xi_t^{(1)} X_t^{(1)}}{\pi^{(0)}} \\
&= \frac{\tilde{v}(t-1, S_{t-1}) + \xi_t^{(1)} (X_t - X_{t-1}) - \xi_t^{(1)} X_t^{(1)}}{\pi^{(0)}} \\
&= \frac{\tilde{v}(t-1, S_{t-1}) - \xi_t^{(1)} X_{t-1}}{\pi^{(0)}}, \qquad t = 1, \ldots, N.
\end{aligned}$$

□

Hence the discounted amount $\xi_t^{(0)} \pi^{(0)}$ invested on the riskless asset is

$$\xi_t^{(0)} \pi^{(0)} = \tilde{v}(t-1, S_{t-1}) - \frac{\tilde{v}\left(t, (1+b) S_{t-1}\right) - \tilde{v}\left(t, (1+a) S_{t-1}\right)}{(b-a)/(1+r)}, \qquad t = 1, \ldots, N,$$

and we recover the fact that $\xi_t^{(0)}$ depends only on S_{t-1} and not on S_t.

Without discount this amount reads

$$\begin{aligned}
\xi_t^{(0)} S_t^{(0)} &= (1+r) v(t-1, S_{t-1}) - (1+r) \frac{v\left(t, (1+b) S_{t-1}\right) - v\left(t, (1+a) S_{t-1}\right)}{b-a} \\
&= \frac{1+r}{b-a} \left((b-a) v(t-1, S_{t-1}) - v\left(t, (1+b) S_{t-1}\right) + v\left(t, (1+a) S_{t-1}\right) \right),
\end{aligned}$$

$t = 1, \ldots, N$. Recall that this portfolio strategy $(\xi_t^{(0)}, \xi_t^{(1)})_{t=1,\ldots,N}$ hedges the claim $C = f(S_N)$, i.e. at time N we have

$$V_N = f(S_N),$$

and it is self-financing by Lemma 3.1.

3.5 Hedging of Exotic Options in the CRR Model

In this section we consider the hedging of path dependent options using the finite difference gradient and the discrete Clark formula of stochastic analysis, see also [21], [30], [38], Chapter 1 of [39], [46], or §15-1 of [52]. Given

$$\omega = (\omega_1, \ldots, \omega_N) \in \Omega = \{-1, 1\}^N,$$

and $k \in \{1, 2, \ldots, N\}$, let

$$\omega_+^k = (\omega_1, \ldots, \omega_{k-1}, +1, \omega_{k+1}, \ldots, \omega_N)$$

and

$$\omega_-^k = (\omega_1, \ldots, \omega_{k-1}, -1, \omega_{k+1}, \ldots, \omega_N).$$

We also assume that the return $R_t(\omega)$ is constructed as

$$R_t(\omega_+^t) = b \qquad \text{and} \qquad R_t(\omega_-^t) = a, \qquad t = 1, \ldots, N, \quad \omega \in \Omega.$$

The operator D_t, $t \geq 1$, is defined on any random variable F by

$$D_t F(\omega) = F(\omega_+^t) - F(\omega_-^t), \qquad t \geq 1. \tag{3.10}$$

Recall the following predictable representation formula for the functionals of the binomial process.

Definition 3.1. Let the centered and normalized return Y_t be defined by

$$Y_t = \frac{R_t - r}{b - a} = \begin{cases} \dfrac{a - r}{b - a}, \\[2ex] \dfrac{b - r}{b - a}, \end{cases} \qquad t = 1, \ldots, N.$$

Note that we have $\mathbb{E}[Y_t] = 0$ and

$$\operatorname{var}[Y_t] = pq, \qquad t = 1, \ldots, N,$$

and that the discounted asset price increment reads

$$\begin{aligned} X_t - X_{t-1} &= \frac{1}{1+r} X_{t-1}(R_t - r) \\ &= \frac{b-a}{1+r} Y_t X_{t-1}, \qquad t = 1, \ldots, N. \end{aligned}$$

The next proposition is the Clark predictable representation formula, cf. e.g. [39], Proposition 1.7.1.

Proposition 3.3. *For all square-integrable random variables F on Ω we have*

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http://www.ntu.edu.sg/home/nprivault/indext.html

$$F = \mathbb{E}[F] + \sum_{t=1}^{\infty} \mathbb{E}[D_t F | \mathcal{F}_{t-1}] Y_t. \tag{3.11}$$

In addition to the Clark formula we also state a discrete-time analog of Itô's change of variable formula, which will be useful for option hedging, cf. Proposition 1.13.1 of [39].

Proposition 3.4. *Let $(M_n)_{n\in\mathbb{N}}$ be a martingale and $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$. We have*

$$\begin{aligned} f(M_t, t) = f(M_0, 0) &+ \sum_{k=1}^{t} D_k f(M_k, k) Y_k \\ &+ \sum_{k=1}^{t} \left(\mathbb{E}[f(M_k, k) | \mathcal{F}_{k-1}] - f(M_{k-1}, k-1) \right). \end{aligned} \tag{3.12}$$

Note that if $(M_t)_{t\in\mathbb{N}}$ is a martingale in $L^2(\Omega)$ with respect to $(\mathcal{F}_t)_{t\in\mathbb{N}}$ and written as

$$M_t = M_0 + \sum_{k=1}^{t} u_k Y_k, \qquad t \in \mathbb{N},$$

where $(u_t)_{t\in\mathbb{N}}$ is a predictable process locally in $L^2(\Omega\times\mathbb{N})$, (i.e. $u(\cdot)\mathbf{1}_{[0,N]}(\cdot) \in L^2(\Omega \times \mathbb{N})$ for all $N > 0$), then we have

$$D_t f(M_t, t) = f\left(M_{t-1} + u_t \sqrt{\frac{q}{p}}, t\right) - f\left(M_{t-1} - u_t \sqrt{\frac{p}{q}}, t\right), \tag{3.13}$$

$t = 1, \ldots, N$. On the other hand, the term

$$\mathbb{E}[f(M_t, t) - f(M_{t-1}, t-1) | \mathcal{F}_{t-1}]$$

is analog to the finite variation part in the continuous time Itô formula, and can be written as

$$pf\left(M_{t-1} + u_t \sqrt{\frac{q}{p}}, t\right) + qf\left(M_{t-1} - u_t \sqrt{\frac{p}{q}}, t\right) - f\left(M_{t-1}, t-1\right).$$

Naturally, if $(f(M_t, t))_{t\in\mathbb{N}}$ is a martingale we have

$$\begin{aligned} f(M_t, t) &= f(M_0, 0) \\ &\quad + \sum_{k=1}^{t} \left(f\left(M_{k-1} + u_k \sqrt{\frac{q}{p}}, k\right) - f\left(M_{k-1} - u_k \sqrt{\frac{p}{q}}, k\right) \right) Y_k \\ &= f(M_0, 0) + \sum_{k=1}^{t} Y_k D_k f(M_k, k). \end{aligned}$$

In this case the Clark formula and the change of variable formula all coincide and we have in particular

$$\begin{aligned} D_k f(M_k, k) &= \mathbb{E}[D_k f(M_t, t)|\mathcal{F}_{k-1}] \\ &= \mathbb{E}[D_k f(M_k, k)|\mathcal{F}_{k-1}], \qquad k = 1, \ldots, N. \end{aligned}$$

For example we recover the martingale representation

$$\begin{aligned} X_t &= S_0 + \sum_{k=1}^{t} Y_k D_k X_k \\ &= S_0 + \frac{b-a}{1+r} \sum_{k=1}^{t} X_{k-1} Y_k \\ &= S_0 + \sum_{k=1}^{t} X_{k-1} \frac{R_k - r}{1+r} \\ &= S_0 + \sum_{k=1}^{t} (X_k - X_{k-1}), \end{aligned}$$

of the discounted asset price.

Our goal is to hedge an arbitrary claim C on Ω, i.e. given an $\mathcal{F}_N$-measurable random variable C we search for a portfolio $(\xi_t, \eta_t)_{t=1,\ldots,N}$ such that the equality

$$C = V_N = \eta_N A_N + \xi_N S_N \tag{3.14}$$

holds at time $N \in \mathbb{N}$.

The next proposition provides a solution to the hedging problem under the constraint (3.14).

Proposition 3.5. *Given C a contingent claim, let*

$$\xi_t = (1+r)^{-(N-t)} \frac{1}{S_{t-1}(b-a)} \mathbb{E}^*[D_t C|\mathcal{F}_{t-1}], \tag{3.15}$$

$t = 1 \ldots, N$, and

$$\eta_t = \frac{1}{A_t} \left((1+r)^{-(N-t)} \mathbb{E}^*[C|\mathcal{F}_t] - \xi_t S_t \right), \tag{3.16}$$

$t = 1 \ldots, N$. Then the portfolio $(\xi_t, \eta_t)_{t=1\ldots,N}$ is self financing and satisfies

$$V_t = \eta_t A_t + \xi_t S_t = (1+r)^{-(N-t)} \mathbb{E}^*[C|\mathcal{F}_t], \qquad t = 1 \ldots, N,$$

in particular we have $V_N = C$, hence $(\xi_t, \eta_t)_{t=1\ldots,N}$ is a hedging strategy leading to C.

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Proof. Let $(\xi_t)_{t=1\ldots,N}$ be defined by (3.15), and consider the process $(\eta_t)_{t=0,1\ldots,N}$ defined by

$$\eta_0 = (1+r)^{-N}\frac{\mathbb{E}^*[C]}{S_0} \quad \text{and} \quad \eta_{t+1} = \eta_t - \frac{(\xi_{t+1}-\xi_t)S_t}{A_t},$$

$t = 0, \ldots, N-1$. Then $(\xi_t, \eta_t)_{t=1,\ldots,N}$ satisfies the self-financing condition

$$A_t(\eta_{t+1}-\eta_t) + S_t(\xi_{t+1}-\xi_t) = 0, \qquad t = 1, \ldots, N-1.$$

Let now

$$V_0 = \mathbb{E}^*[C](1+r)^{-N}, \quad \text{and} \quad V_t = \eta_t A_t + \xi_t S_t, \qquad t = 1, \ldots, N,$$

and

$$\tilde{V}_t = V_t(1+r)^{-t} \qquad t = 0, \ldots, N.$$

Since $(\xi_t, \eta_t)_{t=1,\ldots,N}$ is self-financing, by Lemma 3.1 we have

$$\tilde{V}_t = \tilde{V}_0 + (b-a)\sum_{k=1}^{t} Y_k \xi_k S_{k-1}(1+r)^{-k}, \tag{3.17}$$

$t = 1, \ldots, N$. On the other hand, from the Clark formula (3.11) and the definition of $(\xi_t)_{t=1,\ldots,N}$ we have

$$\begin{aligned}
&(1+r)^{-N}\,\mathbb{E}^*[C|\mathcal{F}_t] \\
&\quad = \mathbb{E}^*\left[\mathbb{E}^*[C](1+r)^{-N} + \sum_{i=0}^{N} Y_i\,\mathbb{E}^*[D_iC|\mathcal{F}_{i-1}](1+r)^{-N}\Big|\mathcal{F}_t\right] \\
&\quad = \mathbb{E}^*[C](1+r)^{-N} + \sum_{i=0}^{t} Y_i\,\mathbb{E}^*[D_iC|\mathcal{F}_{i-1}](1+r)^{-N} \\
&\quad = \mathbb{E}^*[C](1+r)^{-N} + (b-a)\sum_{i=0}^{t} Y_i\xi_i S_{i-1}(1+r)^{-i} \\
&\quad = \tilde{V}_t
\end{aligned}$$

from (3.17). Hence

$$\tilde{V}_t = \mathbb{E}^*[C|\mathcal{F}_t](1+r)^{-N}, \qquad t = 0, 1, \ldots, N,$$

and

$$V_t = \mathbb{E}^*[C|\mathcal{F}_t](1+r)^{-(N-t)}, \qquad t = 0, 1, \ldots, N. \tag{3.18}$$

In particular, (3.18) shows that we have $V_N = C$. To conclude the proof we note that from the relation $V_t = \eta_t A_t + \xi_t S_t$, $t = 1, \ldots, N$, the process $(\eta_t)_{t=1,\ldots,N}$ coincides with $(\eta_t)_{t=1,\ldots,N}$ defined by (3.16). □

From Proposition 3.1, when $C = f(S_N)$, the price $\pi_t(C)$ of the contingent claim $C = f(S_N)$ is given by

$$\pi_t(C) = v(t, S_t),$$

where the function $v(t, x)$ is given by

$$v(t, S_t) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C|\mathcal{F}_t] = \frac{1}{(1+r)^{N-t}} \mathbb{E}^* \left[f\left(x \prod_{j=t+1}^{N} (1 + R_j) \right) \right]_{x=S_t}.$$

Note that in this case we have $C = v(N, S_N)$, $\mathbb{E}[C] = v(0, M_0, 0)$, and

$$\begin{aligned} C &= \mathbb{E}[C] + \sum_{t=1}^{n} \mathbb{E}[D_t v(N, S_N)|\mathcal{F}_{t-1}]Y_t \\ &= \mathbb{E}[C] + \sum_{t=1}^{n} D_t v(t, S_t) Y_t \\ &= \mathbb{E}[C] + \sum_{t=1}^{n} D_t \, \mathbb{E}[v(N, S_N)|\mathcal{F}_t] Y_t. \end{aligned}$$

Hence the hedging strategy for $C = f(S_N)$ is given by

$$\begin{aligned} \xi_t &= \frac{(1+r)^{-(N-t)}}{S_{t-1}\sqrt{pq}(b-a)} D_t v(t, S_t) \\ &= \frac{(1+r)^{-(N-t)}}{S_{t-1}(b-a)} \left(v\left(t, S_{t-1}(1+b)\right) - v\left(t, S_{t-1}(1+a)\right) \right), \end{aligned}$$

$t = 1, \ldots, N$, which recovers Proposition 3.2 as a particular case. Note that ξ_t is non-negative (i.e. there is no short-selling) when f is a non decreasing function, e.g. in the case of European options we have $f(x) = (x - K)^+$.

3.6 Convergence of the CRR Model

In this section we consider the convergence of the discrete time model to the continuous-time Black Scholes model.

Continuous compounding - riskless asset

Consider the subdivision

$$\left[0, \frac{T}{N}, \frac{2T}{N}, \ldots, \frac{(N-1)T}{N}, T\right]$$

of the time interval $[0, T]$ into N time steps.

Note that

$$\lim_{N\to\infty} (1+r)^N = \infty,$$

thus we need to renormalize r so that the interest rate on each time interval becomes r_N, with $\lim_{N\to\infty} r_N = 0$.

It turns out that the correct renormalization is

$$r_N = r\frac{T}{N},$$

so that

$$\lim_{N\to\infty} (1+r_N)^N = \lim_{N\to\infty} \left(1 + r\frac{T}{N}\right)^N = e^{rT}, \qquad T \in \mathbb{R}_+. \tag{3.19}$$

Hence the price of the riskless asset satifies

$$A_t = A_0 e^{rt},$$

with the differential equation

$$\frac{dA_t}{dt} = rA_t,$$

also written as

$$dA_t = rA_t dt,$$

or

$$\frac{dA_t}{A_t} = rdt,$$

which means that the *return* of the riskless asset is rdt on the small time interval $[t, t+dt]$. Equivalently, one says that r is the instantaneous interest rate per unit of time.

The same equation rewrites in *integral form* as

$$A_T - A_0 = \int_0^T dA_t = r\int_0^T A_t dt.$$

Continuous compounding - risky asset

We need to apply a similar renormalization to the coefficients a and b of the CRR model. Let $\sigma > 0$ denote a positive parameter called the volatility and let a_N, b_N be defined from

$$\frac{1+a_N}{1+r_N} = e^{-\sigma\sqrt{\frac{T}{N}}} \qquad \text{and} \qquad \frac{1+b_N}{1+r_N} = e^{\sigma\sqrt{\frac{T}{N}}},$$

i.e.

$$a_N = (1+r_N)e^{-\sigma\sqrt{\frac{T}{N}}} - 1 \qquad \text{and} \qquad b_N = (1+r_N)e^{\sigma\sqrt{\frac{T}{N}}} - 1.$$

Consider the random return $R_k^{(N)} \in \{a_N, b_N\}$ and the price process defined as

$$S_t^{(N)} = S_0 \prod_{k=1}^{t} (1 + R_k^{(t)}), \qquad t = 1, \ldots, N.$$

Note that the risk-neutral probabilities are given by

$$\mathbb{P}^*(R_t = a_N) = \frac{b_N - r_N}{b_N - a_N} = \frac{e^{\sigma\sqrt{\frac{T}{N}}} - 1}{2\sinh\sqrt{\frac{\sigma^2 T}{N}}}, \qquad t = 1, \ldots, N,$$

and

$$\mathbb{P}^*(R_t = b_N) = \frac{r_N - a_N}{b_N - a_N} = \frac{1 - e^{-\sigma\sqrt{\frac{T}{N}}}}{2\sinh\sqrt{\frac{\sigma^2 T}{N}}}, \qquad t = 1, \ldots, N,$$

which both converge to $1/2$ as N goes to infinity.

Continuous-time limit

We have the following convergence result.

Proposition 3.6. *The price at time $t = 0$ of a contingent claim with payoff*

$$C = f\left(S_N^{(N)}\right)$$

converges as follows:

$$\lim_{N\to\infty} \frac{1}{(1+rT/N)^N}\, \mathbb{E}^*\left[f(S_N^{(N)})\right] = e^{-rT}\, \mathbb{E}\left[f(S_0 e^{\sigma\sqrt{T}X + rT - \sigma^2 T/2})\right] \tag{3.20}$$

where $X \simeq \mathcal{N}(0,1)$ is a standard Gaussian random variable.

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Proof. The convergence of the discount factor follows directly from (3.19). We refer to Theorem 5.53 and Corollary 5.56 of [21] for the stochastic part. □

Note that the expectation (3.20) can be written as a Gaussian integral:

$$e^{-rT}\,\mathbb{E}\left[f(S_0e^{\sigma\sqrt{T}X+rT-\sigma^2T/2})\right] = e^{-rT}\int_{-\infty}^{\infty} f(S_0e^{\sigma\sqrt{T}x+rT-\sigma^2T/2})\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx,$$

hence we have

$$\lim_{N\to\infty}\frac{1}{(1+rT/N)^N}\,\mathbb{E}^*\left[f(S_N^{(N)})\right] = e^{-rT}\int_{-\infty}^{\infty} f(S_0e^{\sigma\sqrt{T}x+rT-\sigma^2T/2})\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx.$$

It is a remarkable fact that in case $f(x) = (x-K)^+$, i.e. $C = (S_T - K)^+$ is the payoff of a European call option with strike K, the above integral can be computed according to the *Black-Scholes formula*:

$$e^{-rT}\,\mathbb{E}\left[(S_0e^{\sigma\sqrt{T}X+rT-\sigma^2T/2} - K)^+\right] = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

where

$$d_- = \frac{(r-\frac{1}{2}\sigma^2)T + \log\frac{S_0}{K}}{\sigma\sqrt{T}}, \qquad d_+ = d_- + \sigma\sqrt{T},$$

and

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-y^2/2}dy, \qquad x\in\mathbb{R},$$

is the Gaussian cumulative distribution function.

The Black-Scholes formula will be derived explicitly in the following chapters using both the PDE and probabilistic method. It can be considered as a building block for the pricing of financial derivatives, and its importance is not restricted to the pricing of options on stocks. Indeed, the complexity of the interest rate models makes it in general difficult to obtain closed form expressions, and in many situations one has to rely on the Black-Scholes framework in order to find pricing formulas, for example in the case of interest rate derivatives.

Our aim later will be to price and hedge options directly in continuous time using stochastic calculus, instead of applying the limit procedure described in the previous section.

Recall that the price $(A_t)_{t\in\mathbb{R}_+}$ of the riskless asset is given by

$$\frac{dA_t}{A_t} = r_t dt, \qquad A_0 = 1, \qquad t\in\mathbb{R}_+,$$

i.e.

$$A_t = A_0 e^{rt}, \qquad t \in \mathbb{R}_+.$$

We now need to construct a mathematical model for the price of the risly asset in continuous time.

The return at time $t \in \mathbb{R}_+$ of the risky asset $(S_t)_{t\in\mathbb{R}_+}$ will be defined as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where σdB_t is a "small" Gaussian random component, also called Brownian increment, parametrized by the volatility parameter $\sigma > 0$, that will be describer in the next chapter.

In the next section we will turn to the formal definition of the stochastic process $(B_t)_{t\in\mathbb{R}_+}$ which will be used for the modelling of risky assets in continuous time.

3.7 Exercises

Exercise 3.1. (Exercise 2.1 continued)

1. We consider a forward contract on S_N with strike K and payoff

$$C := S_N - K.$$

   Find a portfolio allocation $(\eta_N, \xi_N,)$ with price $V_N = \eta_N \pi_N + \xi_N S_N$ at time N, such that

$$V_N = C, \tag{3.21}$$

   by writing Condition (3.21) as a 2 system of equations.
2. Find a portfolio allocation (η_{N-1}, ξ_{N-1}) with price $V_{N-1} = \eta_{N-1}\pi_{N-1} + \xi_{N-1}S_{N-1}$ at time $N-1$, and verifying the self-financing condition

$$V_{N-1} = \eta_N \pi_{N-1} + \xi_N S_{N-1}.$$

   Next, at all times $t = 1, \ldots, N-1$, find a portfolio allocation (η_t, ξ_t) with price $V_t = \eta_t \pi_t + \xi_t S_t$ verifying (3.21) and the self-financing condition

$$V_t = \eta_{t+1}\pi_t + \xi_{t+1} S_t,$$

   where η_t, resp. ξ_t, represents the quantity of the riskless, resp. risky, asset in the portfolio over the time period $[t-1, t]$, $t = 1, \ldots, N$.
3. forward contract C, at time $t = 0, 1, \ldots, N$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

4. Check that the arbitrage price $\pi_t(C)$ satisfies the relation

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}}\,\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0, 1, \ldots, N.$$

Exercise 3.2. Consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$. The price S_t^0 of the riskless asset evolves as $S_t^0 = \pi^0(1+r)^t$, $t = 0, 1, \ldots, N$. The *return* of the risky asset, defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, \ldots, N,$$

is random and allowed to take only two values a and b, with $-1 < a < r < b$.

The discounted asset price is $X_t = S_t/(1+r)^t$, $t = 0, 1, \ldots, N$.

1. Show that this model admits a unique risk-neutral measure $\mathbb{P}^*$ and explicitly compute $P^*(R_t = a)$ and $P(R_t = b)$ for all $t = 1, \ldots, N$.
2. Does there exist arbitrage opportunities in this model ? Explain why.
3. Is this market model complete ? Explain why.
4. Consider a contingent claim with payoff[2]

$$C = (S_N)^2.$$

   Compute the discounted arbitrage price $\tilde{V}_t$, $t = 0, \ldots, N$, of a self-financing portfolio hedging the claim C, i.e. such that

$$\tilde{V}_N = \tilde{C} = \frac{(S_N)^2}{(1+r)^N}.$$

5. Compute the portfolio strategy

$$(\bar{\xi}_t)_{t=1,\ldots,N} = (\xi_t^0, \xi_t^1)_{t=1,\ldots,N}$$

   associated to $\tilde{V}_t$, i.e. such that

$$\tilde{V}_t = \bar{\xi}_t \cdot \bar{X}_t = \xi_t^0 X_t^0 + \xi_t^1 X_t^1, \qquad t = 1, \ldots, N.$$

6. Check that the above portfolio strategy is self-financing, i.e.

$$\bar{\xi}_{t+1} \cdot \bar{S}_t = \bar{\xi}_t \cdot \bar{S}_t, \qquad t = 1, \ldots, N-1.$$

Exercise 3.3. We consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$.

[2] This is the payoff of a power call option with strike $K = 0$.

The price π_t of the riskless asset evolves as $\pi_t = \pi_0(1+r)^t$, $t = 0, 1, \ldots, N$. The evolution of S_{t-1} to S_t is given by

$$S_t = \begin{cases} (1+b)S_{t-1} \\ \\ (1+a)S_{t-1} \end{cases}$$

with $-1 < a < r < b$. The *return* of the risky asset is defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, \ldots, N.$$

Let ξ_t, resp. η_t, denote the number of units of the risky, resp. riskless, asset held over the time period $[t-1, t]$ in the portfolio with value

$$V_t = \xi_t S_t + \eta_t \pi_t, \qquad t = 0, \ldots, N. \tag{3.22}$$

1. Show that

$$V_t = (1+R_t)\xi_t S_{t-1} + (1+r)\eta_t \pi_{t-1}, \qquad t = 1, \ldots, N. \tag{3.23}$$

2. Show that under the probability P^* defined by

$$P^*(R_t = a \mid \mathcal{F}_{t-1}) = \frac{b-r}{b-a}, \qquad P^*(R_t = b \mid \mathcal{F}_{t-1}) = \frac{r-a}{b-a},$$

where $\mathcal{F}_{t-1}$ represents the information generated by $\{R_1, \ldots, R_{t-1}\}$, we have

$$E^*[R_t | \mathcal{F}_{t-1}] = r.$$

3. Under the self-financing condition

$$V_{t-1} = \xi_t S_{t-1} + \eta_t \pi_{t-1} \qquad t = 1, \ldots, N, \tag{3.24}$$

show that

$$V_{t-1} = \frac{1}{1+r} E^*[V_t \mid \mathcal{F}_{t-1}],$$

using the result of Question 1.

4. Let $a = 5\%$, $b = 25\%$ and $r = 15\%$. Assume that the price V_t at time t of the portfolio is \$3 if $R_t = a$ and \$8 if $R_t = b$, and compute the price V_{t-1} of the portfolio at time $t-1$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Chapter 4
Brownian Motion and Stochastic Calculus

The modeling of random assets in finance is based on stochastic processes, which are families $(X_t)_{t\in I}$ of random variables indexed by a time interval I. In this chapter we present a description of Brownian motion and a construction of the associated Itô stochastic integral.

4.1 Brownian Motion

First of all we recall the definition of Brownian motion, which is a fundamental example of a stochastic process. Here we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathcal{C}_0(\mathbb{R}_+)$ is the space of continuous real-valued functions on $\mathbb{R}_+$ started at 0.

Definition 4.1. The standard Brownian motion is a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ such that

(i) $B_0 = 0$ almost surely,

(ii) The sample trajectories $t \mapsto B_t$ are continuous, with probability 1.

(iii) For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$

are independent.

(iv) For any times $0 \leq s < t$, $B_t - B_s$ is normally distributed with mean zero and variance $t - s$.

In particular, Condition (iv) above implies

$$\mathbb{E}[B_t - B_s] = 0 \quad \text{and} \quad \mathrm{Var}[B_t - B_s] = t - s, \qquad 0 \leq s \leq t.$$

In the sequel the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ will be generated by the Brownian paths up to time t, i.e.

$$\mathcal{F}_t = \sigma(B_s \ : \ 0 \leq s \leq t), \qquad t \geq 0. \tag{4.1}$$

A random variable F is said to be $\mathcal{F}_t$-*measurable* if the knowledge of F depends only on the information known up to time t. As an example, if t =today,

- the date of the previous MA5182 exam is $\mathcal{F}_t$-measurable, because it belongs to the past.

- the date of the next Chinese new year, although it refers to a future event, is also $\mathcal{F}_t$-measurable because it is known at time t.

- the date of the next typhoon is not $\mathcal{F}_t$-measurable since it is not known at time t.

- the maturity date T of a European option is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$, because it has been determined at time 0.

- the exercise date τ of an American option after time t (see Section 8.2) is not $\mathcal{F}_t$-measurable because it refers to a future random event.

Property (iii) above shows that $B_t - B_s$ is independent of all Brownian increments taken before time s, i.e.

$$(B_t - B_s) \perp\!\!\!\perp (B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}),$$

$0 \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq s \leq t$, hence $B_t - B_s$ is also independent of the whole Brownian history up to time s, hence $B_t - B_s$ is in fact independent of $\mathcal{F}_s$, $s \geq 0$.

We refer to Theorem 10.28 of [20] and to Chapter 1 of [44] for the proof of the existence of Brownian motion as a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ satisfying the above properties (i)-(iv).

For convenience we will informally regard Brownian motion as a random walk over infinitesimal time intervals of length Δt, with increments ΔB_t over the time interval $[t, t + \Delta t]$ given by

$$\Delta B_t = \pm\sqrt{\Delta t} \tag{4.2}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

with equal probabilities $(1/2, 1/2)$.

Note that the choice of the square root in (4.2) is not fortuitous. Indeed, any choice of $\pm(\Delta t)^\alpha$ with a power $\alpha > 1/2$ would lead to explosion of the process as dt tends to zero, whereas a power $\alpha \in (0, 1/2)$ would lead to a vanishing process.

Note that we have

$$\mathbb{E}[\Delta B_t] = \frac{1}{2}\sqrt{\Delta t} - \frac{1}{2}\sqrt{\Delta t} = 0,$$

and

$$\mathrm{Var}[\Delta B_t] = \mathbb{E}[(\Delta B_t)^2] = \frac{1}{2}\Delta t + \frac{1}{2}\Delta t = \Delta t.$$

According to this representation, the paths of Brownian motion are not differentiable, although they are continuous by Property (ii), as we have

$$\frac{dB_t}{dt} \simeq \frac{\pm\sqrt{dt}}{dt} = \pm\frac{1}{\sqrt{dt}} \simeq \pm\infty. \tag{4.3}$$

Recall that from the central limit theorem, given any sequence $(X_k)_{k\geq 1}$ of independent identically distributed centered random variables with variance σ^2, the normalized sum

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

converges (in distribution) to a centered Gaussian random variable $\mathcal{N}(0, \sigma^2)$ with variance σ^2 as n goes to infinity.

After splitting the interval $[0, T]$ into N intervals

$$\left(\frac{k-1}{N}T, \frac{k}{N}T\right], \qquad k = 1, \ldots, N,$$

of length $\Delta t = T/N$ with N "large" and letting

$$X_k = \pm\sqrt{T} = \pm\sqrt{N}\sqrt{\Delta t} = \sqrt{N}\Delta B_t$$

with probabilities $(1/2, 1/2)$, i.e. $\sigma^2 = Var(X_k) = T$ and

$$\Delta B_t = \frac{X_k}{\sqrt{N}} = \pm\sqrt{\Delta t}$$

is the increment of B_t over $((k-1)\Delta t, k\Delta t]$, and we get

$$B_T \simeq \sum_{0<t<T} \Delta B_t \simeq \frac{X_1 + \cdots + X_N}{\sqrt{N}}.$$

Hence by the central limit theorem we recover the fact that B_T has a centered Gaussian distribution with variance T, cf. point (iv) of the above definition of Brownian motion. Note that in the above description, ΔB_t could in fact be replaced by any centered random variable with variance Δt.

Note that there is no point in "computing" the value of B_t as it is a *random variable* for all $t > 0$, however we can generate samples of B_t, which are distributed according to the centered Gaussian distribution with variance t. Below we draw three sample paths of a standard Brownian motion obtained by computer simulation using (4.2).

Fig. 4.1: Sample paths of one-dimensional Brownian motion.

The n-dimensional Brownian motion can be constructed as $(B_t^1, \ldots, B_t^n)_{t\in\mathbb{R}_+}$ where $(B_t^1)_{t\in\mathbb{R}_+}$, $\ldots$,$(B_t^n)_{t\in\mathbb{R}_+}$ are independent copies of $(B_t)_{t\in\mathbb{R}_+}$.

Fig. 4.2: Two sample paths of a two-dimensional Brownian motion.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Next we turn to simulations of 2 dimensional and 3 dimensional Brownian motions. Recall that the movement of pollen particles originally observed by R. Brown in 1827 was indeed 2-dimensional.

Fig. 4.3: Sample paths of a three-dimensional Brownian motion.

4.2 Wiener Stochastic Integral

In this section we construct the Itô stochastic integral of square-integrable deterministic function with respect to Brownian motion.

Recall that Bachelier originally modeled the price S_t of a risky asset by $S_t = \sigma B_t$ where σ is a volatility parameter. The stochastic integral

$$\int_0^T f(t)dS_t = \sigma \int_0^T f(t)dB_t$$

can be used to represent the value of a portfolio as a sum of profits and losses $f(t)dS_t$ where dS_t represents the stock price variation and $f(t)$ is the quantity invested in the asset S_t over the short time interval $[t, t + dt]$.

A naive definition of the stochastic integral with respect to Brownian motion would consist in writing

$$\int_0^\infty f(t)dB_t = \int_0^\infty f(t)\frac{dB_t}{dt}dt,$$

and evaluating the above integral with respect to dt. However this definition fails because the paths of Brownian motion are not differentiable, cf. (4.3).

Next we present Itô's construction of the stochastic integral with respect to Brownian motion. Stochastic integrals will be first constructed as integrals of simple step functions of the form

$$f(t) = \sum_{i=1}^{n} a_i \mathbf{1}_{(t_{i-1},t_i]}(t), \qquad t \in \mathbb{R}_+, \tag{4.4}$$

i.e. the function f takes the value a_i on the interval $(t_{i-1}, t_i]$, $i = 1, \ldots, n$, with $0 \leq t_0 < \cdots < t_n$, as illustrated in Figure 4.1.

Fig. 4.1: Step function.

Recall that the classical integral of f given in (4.4) is interpreted as the area under the curve f and computed as

$$\int_0^\infty f(t)dt = \sum_{i=1}^{n} a_i(t_i - t_{i-1}).$$

In the next definition we adapt this construction to the setting of integration with respect to Brownian motion.

Definition 4.2. The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ of the simple step functions f of the form (4.4) is defined by

$$\int_0^\infty f(t)dB_t := \sum_{i=1}^{n} a_i(B_{t_i} - B_{t_{i-1}}). \tag{4.5}$$

In the next proposition we determine the probability distribution of $\displaystyle\int_0^\infty f(t)dB_t$ and we show that it is independent of the particular representation (4.4) chosen for $f(t)$.

In the sequel we will make use of the space $L^2(\mathbb{R}_+)$ of functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|f\|^2_{L^2(\mathbb{R}_+)} := \int_0^\infty |f(t)|^2 dt < \infty,$$

called square-integrable functions.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Proposition 4.1. *The definition of the stochastic integral* $\int_0^\infty f(t)dB_t$ *can be extended to any measurable function* $f \in L^2(\mathbb{R}_+)$*, i.e. to* f *such that*

$$\int_0^\infty |f(t)|^2 dt < \infty. \tag{4.6}$$

In this case, $\displaystyle\int_0^\infty f(t)dB_t$ *has a centered Gaussian distribution*

$$\int_0^\infty f(t)dB_t \simeq \mathcal{N}\left(0, \int_0^\infty |f(t)|^2 dt\right)$$

with variance $\displaystyle\int_0^\infty |f(t)|^2 dt$ *and we have the Itô isometry*

$$\mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right] = \int_0^\infty |f(t)|^2 dt. \tag{4.7}$$

Proof. Recall that if $X_1, \ldots, X_n$ are independent Gaussian random variables with probability laws $\mathcal{N}(m_1, \sigma_1^2), \ldots, \mathcal{N}(m_n, \sigma_n^2)$ then then sum $X_1 + \cdots + X_n$ is a Gaussian random variable with probability law $\mathcal{N}(m_1 + \cdots + m_n, \sigma_1^2 + \cdots + \sigma_n^2)$.

As a consequence, when f is the simple function

$$f(t) = \sum_{i=1}^n a_i \mathbf{1}_{(t_{i-1}, t_i]}(t), \qquad t \in \mathbb{R}_+,$$

the sum

$$\int_0^\infty f(t)dB_t = \sum_{k=1}^n a_k (B_{t_k} - B_{t_{k-1}})$$

has a centered Gaussian distribution with variance

$$\sum_{k=1}^n |a_k|^2 (t_k - t_{k-1}),$$

since

$$\mathrm{var}\left[a_k(B_{t_k} - B_{t_{k-1}})\right] = a_k^2 \mathrm{var}\left[B_{t_k} - B_{t_{k-1}}\right] = a_k^2 (t_k - t_{k-1}),$$

hence the stochastic integral

$$\int_0^\infty f(t)dB_t = \sum_{k=1}^n a_k (B_{t_k} - B_{t_{k-1}})$$

of the step function

$$f(t) = \sum_{k=1}^{n} a_k \mathbf{1}_{(t_{k-1},t_k]}(t)$$

has a centered Gaussian distribution with variance

$$\begin{aligned}
\mathrm{var}\left[\int_0^\infty f(t)dB_t\right] &= \sum_{k=1}^{n} |a_k|^2 (t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} |a_k|^2 \int_{t_{k-1}}^{t_k} dt \\
&= \int_0^\infty \sum_{k=1}^{n} |a_k|^2 1_{(t_{k-1},t_k]}(t)dt \\
&= \int_0^\infty |f(t)|^2 dt.
\end{aligned}$$

Finally we note that

$$\begin{aligned}
\mathrm{var}\left[\int_0^\infty f(t)dB_t\right] &= \mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right] - \left(\mathbb{E}\left[\int_0^\infty f(t)dB_t\right]\right)^2 \\
&= \mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right].
\end{aligned}$$

The extension of the stochastic integral to all functions satisfying (4.6) is obtained by density and a Cauchy sequence argument, applying the isometry (4.7). Namely, given f a function satisfying (4.6) and $(f_n)_{n\in\mathbb{N}}$ a sequence of simple functions converging to f for the norm

$$\|f - f_n\|_{L^2(\mathbb{R}_+)} := \left(\int_0^\infty |f(t) - f_n(t)|^2 dt\right)^{1/2}$$

i.e. in $L^2(\mathbb{R}_+)$, the isometry (4.7) shows that $\left(\int_0^\infty f_n(t)dB_t\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^2(\Omega)$ of square-integrable random variables $F : \Omega \to \mathbb{R}$ such that

$$\|F\|^2_{L^2(\Omega\times\mathbb{R}_+)} := \mathbb{E}[F^2] < \infty.$$

Indeed, we have

$$\begin{aligned}
&\left\|\int_0^\infty f_k(t)dB_t - \int_0^\infty f_n(t)dB_t\right\|_{L^2(\Omega)} \\
&\quad = \left(\mathbb{E}\left[\left(\int_0^\infty f_k(t)dB_t - \int_0^\infty f_n(t)dB_t\right)^2\right]\right)^{1/2} \\
&\quad = \|f_k - f_n\|_{L^2(\mathbb{R}_+)}
\end{aligned}$$

$$\leq \|f - f_k\|_{L^2(\mathbb{R}_+)} + \|f - f_n\|_{L^2(\mathbb{R}_+)},$$

which tends to 0 as k, n tend to infinity, hence $\left(\int_0^\infty f_n(t)dB_t\right)_{n\in\mathbb{N}}$ is it converges for the L^2-norm as $L^2(\Omega)$ is a complete space, cf. e.g. Chapter 4 of [16]. In this case we let

$$\int_0^\infty f(t)dB_t := \lim_{n\to\infty} \int_0^\infty f_n(t)dB_t$$

and the limit is unique from (4.7). □

For example, $\displaystyle\int_0^\infty e^{-t}dB_t$ has a centered Gaussian distribution with variance

$$\int_0^\infty e^{-2t}dt = \left[-\frac{1}{2}e^{-2t}\right]_0^\infty = \frac{1}{2}.$$

In the above proposition we made use of the fact that the space of simple step functions f of the form

$$f(t) = \sum_{i=1}^n a_i \mathbf{1}_{(t_{i-1},t_i]}(t), \qquad t \in \mathbb{R}_+, \tag{4.8}$$

forms a linear space which is dense in $L^2(\mathbb{R}_+)$ for the norm $\|\cdot\|_{L^2(\mathbb{R}_+)}$.

Again, the Wiener stochastic integral $\displaystyle\int_0^\infty f(s)dB_s$ is nothing but a Gaussian random variable and it can not be "computed" in the way standard integral are computed via the use of primitives.

However we have the following formula

$$\int_0^\infty f(t)dB_t = -\int_0^\infty f'(t)B_t dt,$$

provided $f \in L^2(\mathbb{R}_+)$ is $\mathcal{C}^1$ on $\mathbb{R}_+$, such that $\lim_{t\to\infty} t|f(t)|^2 = 0$ and

$$\lim_{t\to\infty} f(t)\int_0^t f(s)ds = 0,$$

cf. Remark 2.5.9 in [39].

4.3 Itô Stochastic Integral

In this section we extend the Wiener stochastic integral to square-integrable *adapted* processes. More precisely, a process $(X_t)_{t\in\mathbb{R}_+}$ is said to be $\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$, where the information flow $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ has been defined in (4.1).

In other words, a process $(X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted if the value of X_t at time t depends only on information known up to time t. Note that the value of X_t may still depend on to future data, for example a fixed future date in the calendar, such as a maturity time $T > t$, since its value is known at time t.

The extension of the stochastic integral to adapted random processes is actually necessary in order to compute a portfolio value when the portfolio process is no longer deterministic. This happens in particular when one needs to update the portfolio allocation based on random events occurring on the market.

Stochastic integrals will be first constructed as integrals of simple predictable processes $(u_t)_{t\in\mathbb{R}_+}$ of the form

$$u_t = \sum_{i=1}^{n} F_i \mathbf{1}_{(t_{i-1},t_i]}(t), \qquad t \in \mathbb{R}_+, \tag{4.9}$$

where F_i is an $\mathcal{F}_{t_{i-1}^n}$-measurable random variable for $i = 1, \ldots, n$. The notion of simple predictable process is natural in the context of portfolio investment, in which F_i will represent an investment allocation decided at time t_{i-1} and to remain unchanged over the time period $[t_{i-1}, t_i]$.

By convention, $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is denoted by $u_t(\omega)$, $t \in \mathbb{R}_+$, $\omega \in \Omega$, and the random outcome ω is often dropped for convenience of notation.

Definition 4.3. The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ of any simple predictable process $(u_t)_{t\in\mathbb{R}_+}$ of the form (4.9) is defined by

$$\int_0^\infty u_t dB_t := \sum_{i=1}^{n} F_i (B_{t_i} - B_{t_{i-1}}). \tag{4.10}$$

The next proposition gives the extension of the stochastic integral from simple predictable processes to square-integrable adapted processes.

In other words, $(X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted when the value of X_t at time t only depends on information contained in the Brownian path up to time t. This also means that knowing the future is not permitted in the definition of the Itô integral, for example a portfolio strategy that would allow the trader to "buy at the lowest" and "sell at the highest" is not possible.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Note that the difference between Relation (4.11) below and Relation (4.7) is the expectation on the right hand side.

Proposition 4.2. *The stochastic integral with respect to Brownian motion* $(B_t)_{t\in\mathbb{R}_+}$ *extends to all adapted processes* $(u_t)_{t\in\mathbb{R}_+}$ *such that*

$$\mathbb{E}\left[\int_0^\infty |u_t|^2 dt\right] < \infty,$$

with the Itô isometry

$$\mathbb{E}\left[\left(\int_0^\infty u_t dB_t\right)^2\right] = \mathbb{E}\left[\int_0^\infty |u_t|^2 dt\right]. \tag{4.11}$$

Proof. We start by showing that the Itô isometry (4.11) holds for the simple predictable process u of the form (4.9). We have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^\infty u_t dB_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n F_i(B_{t_i} - B_{t_{i-1}})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i,j=1}^n F_i F_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n |F_i|^2 (B_{t_i} - B_{t_{i-1}})^2\right] \\
&\quad +2\,\mathbb{E}\left[\sum_{1\le i<j\le n} F_i F_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] \\
&= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[|F_i|^2 (B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&\quad +2 \sum_{1\le i<j\le n} \mathbb{E}[\mathbb{E}[F_i F_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}]] \\
&= \sum_{i=1}^n \mathbb{E}[|F_i|^2\, \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_i}]] \\
&\quad +2 \sum_{1\le i<j\le n} \mathbb{E}[F_i F_j (B_{t_i} - B_{t_{i-1}})\, \mathbb{E}[(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}]] \\
&= \sum_{i=1}^n \mathbb{E}[|F_i|^2\, \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2]] \\
&\quad +2 \sum_{1\le i<j\le n} \mathbb{E}[F_i F_j (B_{t_i} - B_{t_{i-1}})\, \mathbb{E}[(B_{t_j} - B_{t_{j-1}})]]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \mathbb{E}[|F_i|^2(t_i - t_{i-1})] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} |F_i|^2(t_i - t_{i-1})\right] \\
&= \mathbb{E}\left[\int_0^\infty |u_t|^2 dt\right],
\end{aligned}
$$

where we used the tower property of conditional expectations and the facts that $B_{t_i} - B_{t_{i-1}}$ is independent of $\mathcal{F}_{t_{i-1}}$ and

$$
\mathbb{E}[B_{t_i} - B_{t_{i-1}}] = 0, \qquad \mathbb{E}\left[(B_{t_i} - B_{t_{i-1}})^2\right] = t_i - t_{i-1}, \qquad i = 1, \ldots, n.
$$

The extension of the stochastic integral to square-integrable adapted processes $(u_t)_{t\in\mathbb{R}_+}$ is obtained as in Proposition 4.1 by density and a Cauchy sequence argument using the isometry (4.11), in the same way as in the proof of Proposition 4.1. Let $L^2(\Omega \times \mathbb{R}_+)$ denote the space of square-integrable stochastic processes $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$
\|u\|^2_{L^2(\Omega\times\mathbb{R}_+)} := \mathbb{E}\left[\int_0^\infty |u_t|^2 dt\right] < \infty.
$$

By Lemma 1.1 of Ikeda and Watanabe [26], p. 22 and p. 46, or Proposition 2.5.3 of [39], the set of simple predictable processes forms a linear space which is dense in the subspace $L^2_{ad}(\Omega \times \mathbb{R}_+)$ of square-integrable adapted processes. In other words, given u a square-integrable adapted process and $(u^n)_{n\in\mathbb{N}}$ there exists a sequence of simple predictable processes converging to u in $L^2(\Omega \times \mathbb{R}_+)$, and the isometry (4.11) shows that $\left(\int u^n_t dB_t\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, hence it converges in the complete space $L^2(\Omega)$. In this case we let

$$
\int_0^\infty u_t dB_t := \lim_{n\to\infty} \int_0^\infty u^n_t dB_t
$$

and the limit is unique from (4.11). □

In addition, the Itô integral of an adapted process $(u_t)_{t\in\mathbb{R}_+}$ is always a centered random variable:

$$
\mathbb{E}\left[\int_0^\infty u_s dB_s\right] = 0. \tag{4.12}
$$

Note also that the Itô isometry (4.11) can also be written as

$$
\mathbb{E}\left[\int_0^\infty u_t dB_t \int_0^\infty v_t dB_t\right] = \mathbb{E}\left[\int_0^\infty u_t v_t dt\right],
$$

for all square-integrable adapted processes u, v.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

In addition, when the integrand $(u_t)_{t\in\mathbb{R}_+}$ is not a deterministic function, the random variable $\displaystyle\int_0^\infty u_s dB_s$ no longer has a Gaussian distribution, except in some exceptional cases.

The stochastic integral of u over the interval $[a,b]$ is defined as

$$\int_a^b u_t dB_t := \int_0^\infty \mathbf{1}_{[a,b]}(t) u_t dB_t.$$

In particular we have

$$\int_0^\infty \mathbf{1}_{[a,b]}(t) dB_t = B_b - B_a, \qquad 0 \le a \le b,$$

and

$$\int_a^b dB_t = B_b - B_a, \qquad 0 \le a \le b.$$

We also have the Chasles relation

$$\int_a^c u_t dB_t = \int_a^b u_t dB_t + \int_b^c u_t dB_t, \qquad 0 \le a \le b \le c,$$

and the stochastic integral has the following linearity property:

$$\int_0^\infty (u_t + v_t) dB_t = \int_0^\infty u_t dB_t + \int_0^\infty v_t dB_t, \qquad u, v \in L^2(\mathbb{R}_+).$$

In the sequel we wish to define the return at time $t \in \mathbb{R}_+$ of the risky asset $(S_t)_{t\in\mathbb{R}_+}$ as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t.$$

This equation can be formally rewritten in integral form as

$$S_T = S_0 + \mu \int_0^T S_t dt + \sigma \int_0^T S_t dB_t,$$

hence the need to define an integral with respect to dB_t, in addition to the usual integral with respect to dt.

In the previous chapter we have defined the stochastic integral of square-integrable processes with respect to Brownian motion, thus we have made sense of the equation

$$S_T = S_0 + \mu \int_0^T S_t dt + \sigma \int_0^T S_t dB_t,$$

which can be rewritten in differential form as

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

or

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{4.13}$$

This model will be used to represent the random price S_t of a risky asset at time t. Here the return dS_t/S_t of the asset is made of two components: a constant return μdt and a random return σdB_t parametrized by the coefficient σ, called the volatility.

Our goal is now to solve Equation (4.13) and for this we need to introduce Itô's calculus.

4.4 Deterministic Calculus

The *fundamental theorem of calculus* states that for any continuously differentiable (deterministic) function f we have

$$f(x) = f(0) + \int_0^x f'(y)dy.$$

In differential form this relation is written as the first order expansion

$$df(x) = f'(x)dx, \tag{4.14}$$

where dx is "small". Higher order expansions can be obtained from *Taylor's formula*, which, letting

$$df(x) = f(x + dx) - f(x),$$

states that

$$df(x) = f'(x)dx + \frac{1}{2}f''(x)(dx)^2 + \frac{1}{3!}f'''(x)(dx)^3 + \frac{1}{4!}f^{(4)}(x)(dx)^4 + \cdots.$$

Note that Relation (4.14) can be obtained by neglecting the terms of order larger than one in Taylor's formula, since $(dx)^n << dx$ when $n \geq 2$ and dx is "small".

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

4.5 Stochastic Calculus

Let us apply Taylor's formula to Brownian motion, taking

$$dB_t = B_{t+dt} - B_t,$$

and letting

$$df(B_t) = f(B_{t+dt}) - f(B_t),$$

we have

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 + \frac{1}{3!}f'''(B_t)(dB_t)^3 + \frac{1}{4!}f^{(4)}(B_t)(dB_t)^4 + \cdots .$$

From the construction of Brownian motion by its small increments $dB_t = \pm\sqrt{dt}$, it turns out that the terms in $(dt)^2$ and $dtdB_t = \pm(dt)^{3/2}$ can be neglected in Taylor's formula at the first order of approximation in dt. However, the term of order two

$$(dB_t)^2 = (\pm\sqrt{dt})^2 = dt$$

can no longer be neglected in front of dt.

Hence Taylor's formula written at the second order for Brownian motion reads

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt, \tag{4.15}$$

for "small" dt. Note that writing this formula as

$$\frac{df(B_t)}{dt} = f'(B_t)\frac{dB_t}{dt} + \frac{1}{2}f''(B_t)$$

does not make sense because the derivative

$$\frac{dB_t}{dt} \simeq \pm\frac{\sqrt{dt}}{dt} \simeq \pm\frac{1}{\sqrt{dt}} \simeq \pm\infty$$

does not exist.

Integrating (4.15) on both sides from the relation

$$f(B_t) - f(B_0) = \int_0^t df(B_s)$$

we get the integral form of Itô's formula for Brownian motion, i.e.

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

We now turn to the general expression of Itô's formula which applies to Itô processes of the form

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+, \tag{4.16}$$

or in differential notation

$$dX_t = v_t dt + u_t dB_t,$$

where $(u_t)_{t\in\mathbb{R}_+}$ and $(v_t)_{t\in\mathbb{R}_+}$ are square-integrable adapted processes.

Given $f(t,x)$ a smooth function of two variables, from now on we let $\dfrac{\partial f}{\partial x}$ denote partial differentiation with respect to the *second* variable, while $\dfrac{\partial f}{\partial s}$ denote partial differentiation with respect to the *first* (time) variable.

Theorem 4.1. (Itô formula for Itô processes). For any Itô process $(X_t)_{t\in\mathbb{R}_+}$ of the form (4.16) and any $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ we have

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t v_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t u_s \frac{\partial f}{\partial x}(s,X_s)dB_s \\ &+ \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \frac{1}{2}\int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds. \end{aligned} \tag{4.17}$$

Proof. cf. [43]. □

Using the relation

$$\int_0^t df(s,X_s) = f(t,X_t) - f(0,X_0),$$

we get

$$\begin{aligned} \int_0^t df(s,X_s) = &\int_0^t v_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t u_s \frac{\partial f}{\partial x}(s,X_s)dB_s \\ &+ \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \frac{1}{2}\int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds, \end{aligned}$$

which allows us to rewrite Itô's formula (4.17) in differential notation as

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + u_t \frac{\partial f}{\partial x}(t,X_t)dB_t + v_t \frac{\partial f}{\partial x}(t,X_t)dt + \frac{1}{2}|u_t|^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)dt, \tag{4.18}$$

or

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}|u_t|^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)dt.$$

Next, given two processes $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ written as

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,$$

and

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t a_s dB_s, \qquad t \in \mathbb{R}_+,$$

the Itô formula also shows that

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

where the product $dX_t dY_t$ is computed according to the *Itô rule*

$$(dt)^2 = 0, \qquad dt dB_t = 0, \qquad (dB_t)^2 = dt, \tag{4.19}$$

i.e.

$$\begin{aligned} dX_t dY_t &= (v_t dt + u_t dB_t)(b_t dt + a_t dB_t) \\ &= b_t v_t (dt)^2 + u_t b_t dt dB_t + v_t a_t dt dB_t + u_t a_t (dB_t)^2 \\ &= u_t a_t dt. \end{aligned}$$

Hence we have

$$\begin{aligned} (dX_t)^2 &= (v_t dt + u_t dB_t)^2 \\ &= (v_t)^2 (dt)^2 + (u_t)^2 (dB_t)^2 + 2 u_t v_t dt \cdot dB_t \\ &= (u_t)^2 dt, \end{aligned}$$

according to the Itô multiplication table

$\cdot$	dt	dB_t
dt	0	0
dB_t	0	dt

and (4.18) can also be rewriten as

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2.$$

Taking $u_t = 1$ and $v_t = 0$ in (4.16) yields $X_t = B_t$, in which case the Itô formula (4.17) reads

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial s}(s, B_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) ds,$$

i.e. in differential form:

$$df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t) dt + \frac{\partial f}{\partial x}(t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) dt. \tag{4.20}$$

As another example, applying Itô's formula (4.20) to B_t^2 with

$$B_t^2 = f(t, B_t) \quad \text{and} \quad f(t,x) = x^2,$$

we get

$$\begin{aligned} dB_t^2 &= df(B_t) \\ &= \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt \\ &= 2B_t dB_t + dt, \end{aligned}$$

since

$$\frac{\partial f}{\partial t}(t,x) = 0, \qquad \frac{\partial f}{\partial x}(t,x) = 2x, \qquad \text{and} \qquad \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 1,$$

hence by integration we find

$$\begin{aligned} B_T^2 &= B_0 + 2\int_0^T B_s dB_s + \int_0^T dt \\ &= 2\int_0^T B_s dB_s + T, \end{aligned}$$

and

$$\int_0^T B_s dB_s = \frac{B_T^2}{2} - \frac{T}{2}.$$

4.6 Geometric Brownian Motion

Our aim in this section is to solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{4.21}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation is rewritten in *integral form* as

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dB_s, \qquad t \in \mathbb{R}_+. \tag{4.22}$$

Proposition 4.3. *The solution of* (4.21) *is given by*

$$S_t = S_0 e^{\mu t + \sigma B_t - \sigma^2 t/2}, \qquad t \geq 0.$$

Proof. We decide to search for a solution of the form

$$S_t = f(t, B_t)$$

where $f(t,x)$ is a function to be determined.

By Itô's formula (4.20) we have

$$dS_t = df(t,B_t) = \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt.$$

Comparing this expression to (4.21) and identifying the terms in dB_t we get

$$\frac{\partial f}{\partial x}(t,B_t) = \sigma S_t,$$

and

$$\frac{\partial f}{\partial t}(t,B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t) = \mu S_t.$$

Using the relation $S_t = f(t,B_t)$ these two equations rewrite as

$$\frac{\partial f}{\partial x}(t,B_t) = \sigma f(t,B_t),$$

and

$$\frac{\partial f}{\partial t}(t,B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t) = \mu f(t,B_t).$$

Since B_t is a Gaussian random variable taking all possible values in $\mathbb{R}$, the equations should hold for all $x \in \mathbb{R}$, as follows:

$$\frac{\partial f}{\partial x}(t,x) = \sigma f(t,x), \tag{4.23}$$

and

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = \mu f(t,x). \tag{4.24}$$

Letting $g(t,x) = \log f(t,x)$, the first equation (4.23) shows that

$$\frac{\partial g}{\partial x}(t,x) = \frac{\partial \log f}{\partial x}(t,x) = \frac{1}{f(t,x)}\frac{\partial f}{\partial x}(t,x) = \sigma,$$

i.e.

$$\frac{\partial g}{\partial x}(t,x) = \sigma,$$

which is solved as

$$g(t,x) = g(t,0) + \sigma x,$$

hence

$$f(t,x) = e^{g(t,0)}e^{\sigma x} = f(t,0)e^{\sigma x}.$$

Plugging back this expression into the second equation (4.24) yields

$$e^{\sigma x}\frac{\partial f}{\partial t}(t,0)+\frac{1}{2}\sigma^2 e^{\sigma x}f(t,0)=\mu f(t,0)e^{\sigma x},$$

i.e. after division by $e^{\sigma x}$:

$$\frac{\partial f}{\partial t}(t,0)=\left(\mu-\sigma^2/2\right)f(t,0),$$

or

$$\frac{\partial g}{\partial t}(t,0)=\mu-\sigma^2/2,$$

i.e.

$$g(t,0)=g(0,0)+\left(\mu-\sigma^2/2\right)t,$$

and

$$\begin{aligned}f(t,x)&=e^{g(t,x)}\\&=e^{g(t,0)+\sigma x}\\&=e^{g(0,0)+\sigma x+\left(\mu-\sigma^2/2\right)t}\\&=f(0,0)e^{\sigma x+\left(\mu-\sigma^2/2\right)t},\end{aligned}$$

hence

$$S_t=f(t,B_t)=f(0,0)e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t},$$

and the solution to (4.21) is given by

$$S_t=S_0e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t}\qquad t\in\mathbb{R}_+.$$

□

Conversely, taking $S_t=f(t,B_t)$ with $f(t,x)=S_0e^{\sigma x-\sigma^2 t/2+\mu t}$ we may apply Itô's formula to check that

$$\begin{aligned}dS_t&=df(t,B_t)\\&=\frac{\partial f}{\partial t}(t,B_t)dt+\frac{\partial f}{\partial x}(t,B_t)dB_t+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt\\&=\left(\mu-\sigma^2/2\right)S_0e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t}dt+\sigma S_0e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t}dB_t\\&\quad+\frac{1}{2}\sigma^2S_0e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t}dt\\&=\mu S_0e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t}dt+\sigma S_0e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t}dB_t\\&=\mu S_tdt+\sigma S_tdB_t.\end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

4.7 Stochastic Differential Equations

In addition to geometric Brownian motion there exists a large family of stochastic differential equations that can be studied.

Let now

$$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$$

where $\mathbb{R}^d \otimes \mathbb{R}^n$ denotes the space of $d \times n$ matrices, and

$$b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$

satisfy the global Lipschitz condition

$$\|\sigma(t,x) - \sigma(t,y)\|^2 + \|b(t,x) - b(t,y)\|^2 \leq K^2 \|x - y\|^2,$$

$t \in \mathbb{R}_+$, $x, y \in \mathbb{R}^n$.

Then there exists a unique strong solution to the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a d-dimensional Brownian motion, cf. [43].

Examples

1. Consider the stochastic differential equation

   $$dX_t = -\alpha X_t dt + \sigma dB_t, \quad X_0 = x_0,$$

   with $\alpha > 0$ and $\sigma > 0$.

   Looking for a solution of the form

   $$X_t = a(t) \left(x_0 + \int_0^t b(s) dB_s \right)$$

   where $a(\cdot)$ and $b(\cdot)$ are deterministic functions, yields

   $$X_t = x_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s, \qquad t > 0,$$

   after applying Theorem 4.1 to the Itô process $x_0 + \int_0^t b(s) dB_s$ of the form (4.16) with $u_t = b(t)$ and $v(t) = 0$, and to the function $f(t,x) = a(t)x$.

Remark: the solution of this equation *cannot* be written as a function $f(t, B_t)$ of t and B_t as in the proof of Proposition 4.3.

2. Consider the stochastic differential equation

$$dX_t = tX_t dt + e^{t^2/2} dB_t, \qquad X_0 = x_0.$$

Looking for a solution of the form $X_t = a(t)\left(X_0 + \int_0^t b(s)dB_s\right)$, where $a(\cdot)$ and $b(\cdot)$ are deterministic functions we get $a'(t)/a(t) = t$ and $a(t)b(t) = e^{t^2/2}$, hence $a(t) = e^{t^2/2}$ and $b(t) = 1$, which yields $X_t = e^{t^2/2}(X_0 + B_t)$, $t \in \mathbb{R}_+$.

3. Consider the stochastic differential equation

$$dY_t = (2\mu Y_t + \sigma^2)dt + 2\sigma\sqrt{Y_t}dB_t,$$

where $\mu, \sigma > 0$.

Letting $X_t = \sqrt{Y_t}$ we have $dX_t = \mu X_t dt + \sigma dB_t$, hence

$$Y_t = \left(e^{\mu t}\sqrt{Y_0} + \sigma \int_0^t e^{\mu(t-s)} dB_s\right)^2.$$

4.8 Exercises

Exercise 4.1. Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion.

1. Let $c > 0$. Among the following processes, tell which is a standard Brownian motion and which is not. Justify your answer.
   a. $(B_{c+t} - B_c)_{t\in\mathbb{R}_+}$.
   b. $(cB_{t/c^2})_{t\in\mathbb{R}_+}$.
   c. $(B_{ct^2})_{t\in\mathbb{R}_+}$.
2. Compute the stochastic integrals

$$\int_0^T 2dB_t \qquad \text{and} \qquad \int_0^T (2 \times \mathbf{1}_{[0,T/2]}(t) + \mathbf{1}_{(T/2,T]}(t))dB_t$$

and determine their probability laws (including mean and variance).
3. Determine the probability law (including mean and variance) of the stochastic integral

$$\int_0^{2\pi} \sin(t)\, dB_t.$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

4. Compute $\mathbb{E}[B_t B_s]$ in terms of $s, t \geq 0$.
5. Let $T > 0$. Show that if f is a differentiable function with $f(0) = f(T) = 0$ we have
$$\int_0^T f(t)dB_t = -\int_0^T f'(t)B_t dt.$$
Hint: Apply Itô's calculus to $t \mapsto f(t)B_t$.

Exercise 4.2. Let $f \in L^2([0,T])$. Compute the conditional expectation
$$E\left[e^{\int_0^T f(s)dB_s}\middle|\mathcal{F}_t\right], \qquad 0 \leq t \leq T,$$
where $(\mathcal{F}_t)_{t\in[0,T]}$ denotes the filtration generated by $(B_t)_{t\in[0,T]}$.

Exercise 4.3. Compute the expectation
$$E\left[\exp\left(\beta\int_0^T B_t dB_t\right)\right]$$
for all $\beta < 1/T$. Hint: expand $(B_T)^2$ using Itô's formula.

Exercise 4.4. Given $T > 0$, let $(X_t^T)_{t\in[0,T]}$ denote the solution of the stochastic differential equation
$$dX_t^T = \sigma dB_t - \frac{X_t^T}{T-t}dt, \qquad t \in [0,T],$$
under the initial condition $X_0^T = 0$ and $\sigma > 0$.

1. Show that
$$X_t^T = \sigma(T-t)\int_0^t \frac{1}{T-s}dB_s, \qquad t \in [0,T].$$
Hint: Start by computing $d(X_t^T/(T-t))$ using Itô's calculus.
2. Show that $\mathbb{E}[X_t^T] = 0$ for all $t \in [0,T]$.
3. Show that $\mathrm{Var}[X_t^T] = \sigma^2 t(T-t)/T$ for all $t \in [0,T]$.
4. Show that $X_T^T = 0$. The process $(X_t^T)_{t\in[0,T]}$ is called a *Brownian bridge*.

Exercise 4.5. Exponential Vasicek model. Consider a short term rate interest rate proces $(r_t)_{t\in\mathbb{R}_+}$ in the exponential Vasicek model:
$$dr_t = r_t(\eta - a\log r_t)dt + \sigma r_t dB_t, \tag{4.25}$$
where η, a, σ are positive parameters.

1. Find the solution $(z_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation
$$dz_t = -az_t dt + \sigma dB_t$$
as a function of the initial condition z_0, where a and σ are positive parameters.
2. Find the solution $(y_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation
$$dy_t = (\theta - ay_t)dt + \sigma dB_t \tag{4.26}$$
as a function of the initial condition y_0. Hint: let $z_t = y_t - \theta/a$.
3. Let $x_t = e^{y_t}$, $t \in \mathbb{R}_+$. Determine the stochastic differential equation satisfied by $(x_t)_{t\in\mathbb{R}_+}$.
4. Find the solution $(r_t)_{t\in\mathbb{R}_+}$ of (4.25) in terms of the initial condition r_0.
5. Compute the mean[1] $E[r_t]$ of r_t, $t \geq 0$.
6. Compute the asymptotic mean $\lim_{t\to\infty} E[r_t]$.

Exercise 4.6. Cox-Ingerson-Ross model. Consider the equation
$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dB_t \tag{4.27}$$
modeling the variations of a short term interest rate process r_t, where α, β, σ and r_0 are positive parameters.

1. Write down the equation (4.27) in integral form.
2. Let $u(t) = E[r_t]$. Show, using the integral form of (4.27), that $u(t)$ satisfies the differential equation
$$u'(t) = \alpha - \beta u(t).$$
3. By an application of Itô's formula to r_t^2, show that
$$dr_t^2 = r_t(2\alpha + \sigma^2 - 2\beta r_t)dt + 2\sigma r_t^{3/2} dB_t. \tag{4.28}$$
4. Using the integral form of (4.28), find a differential equation satisfied by $v(t) = E[r_t^2]$.

Exercise 4.7. Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

1. Consider the Itô formula
$$f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s)dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s)ds + \frac{1}{2}\int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s)ds, \tag{4.29}$$

[1] You will need to use the formula $E[e^X] = e^{\alpha^2/2}$ for $X \simeq \mathcal{N}(0, \alpha^2)$.

where $X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds$.

Compute $S_t := e^{X_t}$ by the Itô formula (4.29) applied to $f(x) = e^x$ and $X_t = \sigma B_t + \nu t$, $\sigma > 0$, $\nu \in \mathbb{R}$.

2. Let $r > 0$. For which value of ν does $(S_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t \quad ?$$

3. Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0 e^{\sigma B_t + \nu t}$, $t \in \mathbb{R}_+$. Using the result of Exercise 16.1, show that the conditional probability $P(S_T > K \mid S_t = x)$ is given by

$$P(S_T > K \mid S_t = x) = \Phi\left(\frac{\log(x/K) + \nu\tau}{\sigma\sqrt{\tau}}\right),$$

where $\tau = T - t$. Hint: use the decomposition $S_T = S_t e^{\sigma(B_T - B_t) + \nu\tau}$.

4. Given $0 \le t \le T$ and $\sigma > 0$, let

$$X = \sigma(B_T - B_t) \qquad \text{and} \qquad \eta^2 = \mathrm{Var}[X], \qquad \eta > 0.$$

What is η equal to ?

Exercise 4.8. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion generating the information $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

1. Let $0 \le t \le T$. What is the probability law of $B_T - B_t$?
2. From the answer to Exercise 16.16-(2), show that

$$\mathbb{E}[(B_T)^+|\mathcal{F}_t] = \sqrt{\frac{\tau}{2\pi}} e^{-\frac{B_t^2}{2\tau}} + B_t \Phi\left(\frac{B_t}{\sqrt{\tau}}\right), \qquad 0 \le t \le T,$$

where $\tau = T - t$. Hint: write $B_T = B_T - B_t + B_t$.

3. Let $\sigma > 0$, $\nu \in \mathbb{R}$, and $X_t = \sigma B_t + \nu t$. Compute e^{X_t} using the Itô formula

$$f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s) dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s) ds + \frac{1}{2}\int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s) ds$$

stated here for a process $X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds$, $t \in \mathbb{R}_+$, and applied to $f(x) = e^x$.

4. Let $S_t = e^{X_t}$, $t \in \mathbb{R}_+$, and $r > 0$. For which value of ν does $(S_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t \quad ?$$

Exercise 4.9. From the answer to Exercise 16.2-(2), show that

$$\mathbb{E}[(\beta - B_T)^+ | \mathcal{F}_t] = \sqrt{\frac{\tau}{2\pi}} e^{-\frac{(\beta - B_t)^2}{2\tau}} + (\beta - B_t)\Phi\left(\frac{\beta - B_t}{\sqrt{\tau}}\right), \qquad 0 \leq t \leq T,$$

where $\tau = T - t$. Hint: write $B_T = B_T - B_t + B_t$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Chapter 5
The Black-Scholes PDE

In this section we review the notions of assets, self-financing portfolios, risk-neutral measures, and arbitrage in continuous time. We also derive the Black-Scholes PDE for self-financing portfolios, and we solve this equation using the heat kernel method.

5.1 Continuous-Time Market Model

Let $(A_t)_{t\in\mathbb{R}_+}$ be the riskless asset given by

$$\frac{dA_t}{A_t} = rdt, \qquad t \in \mathbb{R}_+,$$

i.e.

$$A_t = A_0 e^{rt}, \qquad t \in \mathbb{R}_+.$$

For $t > 0$, let $(S_t)_{t\in\mathbb{R}_+}$ be the price process defined as

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+.$$

We have

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \qquad t \in \mathbb{R}_+.$$

5.2 Self-Financing Portfolio Strategies

Let ξ_t and η_t be the numbers of units invested at time t, respectively in the assets $(S_t)_{t\in\mathbb{R}_+}$ and $(A_t)_{t\in\mathbb{R}_+}$, and let $\bar{\xi}_t = (\eta_t, \xi_t)$, $\bar{S}_t = (A_t, S_t)$, $t \in \mathbb{R}_+$, denote the associated portfolio and asset price processes. The value of the

portfolio V_t at time t is given by

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+. \tag{5.1}$$

The portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from (η_t, ξ_t) to $(\eta_{t+dt}, \xi_{t+dt})$, i.e.

$$\bar{\xi}_t \cdot \bar{S}_t = A_t\eta_t + S_t\xi_t = A_t\eta_{t+dt} + S_t\xi_{t+dt} = \bar{\xi}_t \cdot \bar{S}_{t+dt}, \tag{5.2}$$

which is the continuous-time equivalent of the self-financing condition already seen in the discrete case.

Portfolio value	$\bar{\xi}_t \cdot \bar{S}_{t-dt}$	$\longrightarrow$	$\bar{\xi}_t \cdot \bar{S}_t =$	$\bar{\xi}_{t+dt} \cdot \bar{S}_t$	$\longrightarrow$ $\bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}$
Asset value	S_{t-dt}		S_t	S_t	S_{t+dt}
Time scale	$t - dt$		t	t	$t + dt$
Portfolio allocation	ξ_t		ξ_t	ξ_{t+dt}	ξ_{t+dt}

Fig. 5.1: Illustration of the self-financing condition (5.2).

Equivalently, Condition (5.2) can be rewritten as

$$A_t d\eta_t + S_t d\xi_t = 0, \tag{5.3}$$

and this condition can be interpreted as

$$A_t d\eta_t = -S_t d\xi_t,$$

i.e. when one sells $-d\xi_t > 0$ units of the risky asset S_t over the time period $[t, t + dt]$ for a total amount $-S_t d\xi_t$, one should entirely use this income to buy $d\eta_t > 0$ units of the riskless asset for an amount $A_t d\eta_t > 0$.

Similarly, if one sells $-d\eta_t > 0$ units of the riskless asset A_t over the time period $[t, t + dt]$ for a total amount $-A_t d\eta_t$, one should entirely use this income to buy $d\xi_t > 0$ units of the risky asset for an amount $S_t d\xi_t > 0$, i.e.

$$S_t d\xi_t = -A_t d\eta_t.$$

Actually, the above analysis makes the implicit assumption that the asset prices A_t and S_t remain constant over time, which is not the case in continuous time.

Definition 5.1. The portfolio V_t is said to be self-financing if

$$dV_t = \eta_t dA_t + \xi_t dS_t. \tag{5.4}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Note that (5.4) reads

$$A_{t+dt}\eta_{t+dt} + S_{t+dt}\xi_{t+dt} - A_t\eta_t - S_t\xi_t = \eta_t(A_{t+dt} - A_t) + \xi_t(S_{t+dt} - S_t),$$

or

$$\eta_t A_{t+dt} + \xi_t S_{t+dt} = A_{t+dt}\eta_{t+dt} + S_{t+dt}\xi_{t+dt},$$

which differs from (5.2). On the other hand, from Itô's calculus we have

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t + A_t d\eta_t + S_t d\xi_t + d\eta_t \cdot dA_t + d\xi_t \cdot dS_t \\ &= \eta_t dA_t + \xi_t dS_t + A_t d\eta_t + S_t d\xi_t + d\xi_t \cdot dS_t, \end{aligned}$$

since $d\eta_t \cdot dA_t = rA_t dt \cdot d\eta_t = 0$, hence Condition (5.4) rewrites as

$$A_t d\eta_t + S_t d\xi_t + d\xi_t \cdot dS_t = 0,$$

which is equivalent to (5.2) and (5.3) provided we neglect the cross variation $d\xi_t \cdot dS_t$.

Let

$$\tilde{V}_t = e^{-rt}V_t \qquad \text{and} \qquad X_t = e^{-rt}S_t$$

respectively denote the discounted portfolio value and discounted risky asset prices at time $t \geq 0$. We have

$$\begin{aligned} dX_t &= d(e^{-rt}S_t) \\ &= -re^{-rt}S_t dt + e^{-rt}dS_t \\ &= -re^{-rt}S_t dt + \mu e^{-rt}S_t dt + \sigma e^{-rt}S_t dB_t \\ &= X_t((\mu - r)dt + \sigma dB_t). \end{aligned}$$

In the next lemma we show that when a portfolio is self-financing, its discounted value is a gain process given by the sum of discounted profits and losses (number of risky assets ξ_t times discounted price variation dX_t) over time.

Lemma 5.1. Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+.$$

The following statements are equivalent:

i) the portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

ii) we have

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u dX_u, \qquad t \in \mathbb{R}_+. \tag{5.5}$$

Proof. Assuming that (i) holds, the self-financing condition shows that

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t \qquad t \in \mathbb{R}_+, \end{aligned}$$

hence

$$\begin{aligned} d\tilde{V}_t &= d\left(e^{-rt}V_t\right) \\ &= -re^{-rt}V_t dt + e^{-rt}dV_t \\ &= (\mu - r)\xi_t X_t dt + \sigma\xi_t X_t dB_t \\ &= \xi_t dX_t, \qquad t \in \mathbb{R}_+, \end{aligned}$$

i.e. (5.5) holds by integrating on both sides as

$$\tilde{V}_t - \tilde{V}_0 = \int_0^t d\tilde{V}_u = \int_0^t \xi_u dX_u, \qquad t \in \mathbb{R}_+.$$

Conversely, if (5.5) is satisfied we have

$$\begin{aligned} dV_t &= d(e^{rt}\tilde{V}_t) \\ &= re^{rt}\tilde{V}_t dt + e^{rt}d\tilde{V}_t \\ &= re^{rt}\tilde{V}_t dt + e^{rt}\xi_t dX_t \\ &= rV_t dt + e^{rt}\xi_t dX_t \\ &= rV_t dt + e^{rt}\xi_t X_t((\mu - r)dt + \sigma dB_t) \\ &= rV_t dt + \xi_t S_t((\mu - r)dt + \sigma dB_t) \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= \eta_t dA_t + \xi_t dS_t, \end{aligned}$$

hence the portfolio is self-financing according to Definition 5.1. □

As a consequence of (5.5), the hedging problem of a claim C with maturiry T is reduced to that of finding the representation of the discounted claim $\tilde{C} = e^{-rT}C$ as a stochastic integral:

$$\tilde{C} = \tilde{V}_0 + \int_0^T \xi_u dX_u.$$

Note also that (5.5) shows that the value of a self-financing portfolio can also be written as

$$V_t = e^{rt}V_0 + (\mu - r)\int_0^t e^{r(t-u)}\xi_u S_u dt + \sigma\int_0^t e^{r(t-u)}\xi_u S_u dB_u, \qquad t \in \mathbb{R}_+. \tag{5.6}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

5.3 Arbitrage and Risk-Neutral Measures

In continuous-time, the definition of arbitrage follows the lines of its analogs in the discrete and two-step models.

Definition 5.2. A portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ constitutes an arbitrage opportunity if all *three* following conditions are satisfied:

i) $V_0 \leq 0$,
ii) $V_T \geq 0$,
iii) $\mathbb{P}(V_T > 0) > 0$.

Roughly speaking, (ii) means that the investor wants no loss, (iii) means that he wishes to sometimes make a strictly positive gain, and (i) means that he starts with zero capital or even with a debt.

Next we turn to the definition of risk-neutral measures in continuous time. Recall that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is generated by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, i.e.

$$\mathcal{F}_t = \sigma(B_u \ : \ 0 \leq u \leq t), \qquad t \in \mathbb{R}_+.$$

Definition 5.3. A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if it satisfies

$$\mathbb{E}^*[S_t|\mathcal{F}_u] = e^{r(t-u)}S_u, \qquad 0 \leq u \leq t, \tag{5.7}$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

From the relation

$$A_t = e^{r(t-u)}A_u, \qquad 0 \leq u \leq t,$$

we interpret (5.7) by saying that the expected return of the risky asset S_t under $\mathbb{P}^*$ equals the return of the riskless asset A_t. The discounted price X_t of the risky asset is defined by

$$X_t = e^{-rt}S_t = \frac{S_t}{A_t/A_0}, \qquad t \in \mathbb{R}_+,$$

i.e. A_t/A_0 plays the role of a *numeraire*.

Definition 5.4. A continuous time process $(Z_t)_{t\in\mathbb{R}_+}$ of integrable random variables is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s, \qquad 0 \leq s \leq t.$$

Note that when $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale, Z_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$.

As in the discrete case, the notion of martingale can be used to characterize risk-neutral measures.

Proposition 5.1. *The measure $\mathbb{P}^*$ is risk-neutral if and only if the discounted price process $(X_t)_{t\in\mathbb{R}_+}$ is a martingale* under $\mathbb{P}^*$.

Proof. We have

$$\begin{aligned}\mathbb{E}^*[X_t|\mathcal{F}_u] &= \mathbb{E}^*[e^{-rt}S_t|\mathcal{F}_u]\\ &= e^{-rt}\,\mathbb{E}^*[S_t|\mathcal{F}_u]\\ &= e^{-rt}e^{r(t-u)}S_u\\ &= e^{-ru}S_u\\ &= X_u, \qquad 0\le u\le t.\end{aligned}$$

□

As in the discrete time case, $\mathbb{P}^*$ would be called a risk-premium measure if it satisfied

$$\mathbb{E}^*[S_t|\mathcal{F}_u] > e^{r(t-u)}S_u, \qquad 0\le u\le t,$$

meaning that by taking risks in buying S_t, one could make an expected return higher than that of

$$A_t = e^{r(t-u)}A_u, \qquad 0\le u\le t.$$

Next we note that the first fundamental theorem of mathematical finance also holds in continuous time, and can be used to check for the existence of arbitrage opportunities.

Theorem 5.1. A market is *without* arbitrage opportunity if and only if it admits at least one risk-neutral measure.

Proof. cf. Chapter VII of [48]. □

5.4 Completeness

Definition 5.5. A contingent claim with payoff C is said to be attainable if there exists a portfolio strategy $(\eta_t,\xi_t)_{t\in[0,T]}$ such that

$$C = V_T.$$

In this case the price of the claim at time t will be equal to the value V_t of any self-financing portfolio hedging C.

Definition 5.6. A market model is said to be *complete* if every contingent claim C is attainable.

The next result is a continuous-time restatement of the second fundamental theorem of mathematical finance.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Theorem 5.2. A market model without arbitrage is complete if and only if it admits only one risk-neutral measure.

Proof. cf. Chapter VII of [48]. □

In the Black-Scholes model one can show the existence of a unique risk-neutral measure, hence the model is without arbitrage and complete.

5.5 The Black-Scholes PDE

We start by deriving the Black-Scholes partial differential equation (PDE) for the price of a self-financing portfolio.

Proposition 5.2. *Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy such that*

(i) $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing,
(ii) the value $V_t := \eta_t A_t + \xi_t S_t$, $t \in \mathbb{R}_+$, takes the form

$$V_t = g(t, S_t), \qquad t \in \mathbb{R}_+,$$

for some $g \in \mathcal{C}^2((0,\infty)\times(0,\infty))$.

Then the function $g(t,x)$ satisfies the Black-Scholes PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,x), \qquad t, x > 0,$$

and ξ_t is given by

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t), \qquad t \in \mathbb{R}_+.$$

Proof. First, note that the self-financing condition implies

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \end{aligned} \tag{5.8}$$

$t \in \mathbb{R}_+$.

We now rewrite (4.22) under the form of an Itô processes

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,$$

as in (4.16), by taking

$$u_t = \sigma S_t, \quad \text{and} \quad v_t = \mu S_t, \qquad t \in \mathbb{R}_+.$$

The application of Itô's formula Theorem 4.1 to $g(t,x)$ leads to

$$\begin{aligned} dg(t,S_t) &= g(0,S_0) + v_t \frac{\partial g}{\partial x}(t,S_t)dt + u_t \frac{\partial g}{\partial x}(t,S_t)dB_t \\ &\quad + \frac{\partial g}{\partial t}(t,S_t)dt + \frac{1}{2}|u_t|^2 \frac{\partial^2 g}{\partial x^2}(t,S_t)dt \\ &= \frac{\partial g}{\partial t}(t,S_t)dt + \mu S_t \frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}S_t^2\sigma^2 \frac{\partial^2 g}{\partial x^2}(t,S_t)dt + \sigma S_t \frac{\partial g}{\partial x}(t,S_t)dB_t. \end{aligned} \tag{5.9}$$

By respective identification of the terms in dB_t and dt in (5.8) and (5.9) we get

$$\begin{cases} r\eta_t A_t dt + \mu \xi_t S_t dt = \dfrac{\partial g}{\partial t}(t,S_t)dt + \mu S_t \dfrac{\partial g}{\partial x}(t,S_t)dt + \dfrac{1}{2}S_t^2\sigma^2 \dfrac{\partial^2 g}{\partial x^2}(t,S_t)dt, \\ \xi_t S_t \sigma dB_t = S_t \sigma \dfrac{\partial g}{\partial x}(t,S_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rV_t - r\xi_t S_t = \dfrac{\partial g}{\partial t}(t,S_t) + \dfrac{1}{2}S_t^2\sigma^2 \dfrac{\partial^2 g}{\partial x^2}(t,S_t), \\ \xi_t = \dfrac{\partial g}{\partial x}(t,S_t), \end{cases}$$

i.e.

$$\begin{cases} rg(t,S_t) = \dfrac{\partial g}{\partial t}(t,S_t) + rS_t \dfrac{\partial g}{\partial x}(t,S_t) + \dfrac{1}{2}S_t^2\sigma^2 \dfrac{\partial^2 g}{\partial x^2}(t,S_t), \\ \xi_t = \dfrac{\partial g}{\partial x}(t,S_t). \end{cases} \tag{5.10}$$

□

The process $(\xi_t)_{t\in\mathbb{R}_+}$ is called the Delta. The amount invested on the riskless asset is

$$\eta_t A_t = V_t - \xi_t S_t = g(t,S_t) - S_t \frac{\partial g}{\partial x}(t,S_t),$$

and η_t is given by

$$\begin{aligned} \eta_t &= \frac{V_t - \xi_t S_t}{A_t} \\ &= \frac{g(t,S_t) - S_t \dfrac{\partial g}{\partial x}(t,S_t)}{A_t} \end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$= \frac{g(t,S_t) - S_t \dfrac{\partial g}{\partial x}(t,S_t)}{A_0 e^{rt}}.$$

In the next proposition we add a boundary condition to the Black-Scholes PDE in order to hedge claim C of the form $C = f(S_T)$.

Proposition 5.3. *The price of any self-financing portfolio of the form $V_t = g(t,S_t)$ hedging an option with payoff $C = f(S_T)$ satisfies the Black-Scholes PDE*

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g}{\partial x^2}(t,x), \\ \\ g(T,x) = f(x), \end{cases}$$

with terminal condition $g(T,x) = f(x)$.

The Black-Scholes PDE admits an easy solution when $C = S_T - K$ is the (linear) payoff of a forward contract, i.e. $f(x) = x - K$. In this case we find

$$g(t,x) = x - Ke^{-r(T-t)}, \qquad t,x > 0,$$

and the Delta is given by

$$\xi_t = \frac{\partial g}{\partial x}(t,S_t) = 1, \qquad 0 \le t \le T.$$

Recall that in the case of a European call option with strike K the payoff function is given by $f(x) = (x-K)^+$ and the Black-Scholes PDE reads

$$\begin{cases} rg_c(t,x) = \dfrac{\partial g_c}{\partial t}(t,x) + rx\dfrac{\partial g_c}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g_c}{\partial x^2}(t,x) \\ \\ g_c(T,x) = (x-K)^+. \end{cases}$$

In the next sections we will prove that the solution of this PDE is given by the *Black-Scholes* formula

$$g_c(t,x) = \mathrm{Bl}(K,x,\sigma,r,T-t) = x\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \tag{5.11}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

denotes the Gaussian distribution function and

$$d_+ = \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_- = \frac{\log(x/K) + (r-\sigma_t^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

with

$$d_+ = d_- + \sigma\sqrt{T-t}.$$

One checks easily that when $t = T$,

$$d_+ = d_- = \begin{cases} +\infty, \, x > K, \\ \\ -\infty, \, x < K, \end{cases}$$

which allows one to recover the boundary condition

$$g_c(T, x) = \left\{ \begin{array}{ll} x\Phi(+\infty) - K\Phi(+\infty) = x - K, & x > K \\ \\ x\Phi(-\infty) - K\Phi(-\infty) = 0, & x < K \end{array} \right\} = (x - K)^+$$

at $t = T$.

Below is the graph of the solution

$$(t, x) \mapsto g_c(t, x) = x\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

of the Black-Scholes PDE for a call option.

Fig. 5.1: Graph of the Black-Scholes call price function.

Let us consider the stock price of HSBC Holdings (0005.HK) over one year:

Fig. 5.2: Graph of the stock price of HSBC Holdings.

Consider a call option issued by Societe Generale on 31 December 2008 with strike K=\$63.704, maturity T = October 05, 2009, and entitlement ratio 100. The next graph gives the time evolution of the Black-Scholes portfolio price

$$t \mapsto g_c(t, S_t)$$

driven by the market price $t \mapsto S_t$ of the underlying risky asset as given in Figure 5.2.

Fig. 5.3: Path of the Black-Scholes price for a call option on HSBC.

Note that in the case of a Black-Scholes call option the Delta is given by

$$\xi_t = \Phi(d_+).$$

This identity can be proved by direct differentiation of the Black-Scholes function, as follows:

$$\begin{aligned}
&\frac{\partial g}{\partial x}(x,t) = \\
&\frac{\partial}{\partial x}\left(x\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) - K\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= x\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad -K\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad +\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma\sqrt{T-t}}\exp\left(-\frac{1}{2}\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)^2\right) \\
&\quad -\frac{K}{\sqrt{2\pi}\sigma x\sqrt{T-t}}\exp\left(-\frac{1}{2}\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)^2\right) \\
&\quad +\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right),
\end{aligned}$$

and will be recovered later using a probabilistic argument in Proposition 6.7 below. Consequently, in the Black-Scholes model the amount invested on the risky asset is

$$S_t\Phi(d_+) = S_t\Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geq 0,$$

which is always positive, i.e. there is no short-selling, and the amount invested on the riskless asset is

$$\eta_t A_t = -Ke^{-r(T-t)}\Phi\left(\frac{\log(S_t/K)+(r-\sigma_t^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \leq 0,$$

which is always negative, i.e. we are constantly borrowing money.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Fig. 5.4: Time evolution of the hedging portfolio for a call option on HSBC.

Similarly, in the case of a European put option with strike K the payoff function is given by $f(x) = (K - x)^+$ and the Black-Scholes PDE reads

$$\begin{cases} rg_p(t,x) = \dfrac{\partial g_p}{\partial t}(t,x) + rx\dfrac{\partial g_p}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g_p}{\partial x^2}(t,x), \\ \\ g_p(T,x) = (K-x)^+, \end{cases}$$

with explicit solution

$$g_p(t,x) = Ke^{-r(T-t)}\Phi(-d_-) - x\Phi(-d_+),$$

as illustrated in the next graph.

Fig. 5.5: Graph of the Black-Scholes put price function.

Note that the call-put parity relation

$$g(t, S_t) = g_c(t, S_t) - g_p(t, S_t), \qquad 0 \leq t \leq T,$$

is satisfied here.

For one more example we consider a put option issued by BNP Paribas on 04 November 2008 with strike K=$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593.

Fig. 5.6: Path of the Black-Scholes price for a put option on HSBC.

In the case of a Black-Scholes put option the Delta is given by

$$\xi_t = -\Phi(-d_+),$$

and the amount invested on the risky asset is

$$S_t\Phi(d_+) = -S_t\Phi\left(-\frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \leq 0,$$

i.e. there is always short-selling, and the amount invested on the riskless asset is

$$Ke^{-r(T-t)}\Phi\left(-\frac{\log(S_t/K) + (r-\sigma_t^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geq 0,$$

which is always positive, i.e. we are constantly investing on the riskless asset.

Fig. 5.7: Time evolution of the hedging portfolio for a put option on HSBC.

Finally let us consider a knock-out warrant with maturity T, strike K, barrier (or call price) B, and payoff

$$C = (S_T - K)^+ \mathbf{1}_{\left\{\min_{0\leq t\leq T} S_t > B\right\}} = \begin{cases} S_T - K & \text{if } \min_{0\leq t\leq T} S_t > B, \\ 0 & \text{if } \min_{0\leq t\leq T} S_t \leq B, \end{cases}$$

with $B \geq K$.

The price $g(t, S_t)$ at time t of this warrant satisfies the Black-Scholes PDE

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g}{\partial x^2}(t,x), \\ g(t,B) = 0, \quad t \in [0,T], \\ g(T,x) = (x-K)^+\mathbf{1}_{\{x>B\}}, \end{cases}$$

on the time-space domain $[0,T] \times [B,\infty)$ with terminal condition $g(T,x) = (x-K)^+\mathbf{1}_{\{x>B\}}$ and an additional boundary condition $g(t,B) = 0$ since the price of the claim at time t is 0 whenever $S_t = B$, cf. e.g. [19]. A closed-form solution for this PDE can be found in [49].

5.6 The Heat Equation

In this section we study the *heat equation* which is used to model the diffusion of heat in solids.

In the next section this equation will be shown to be equivalent to the Black-Scholes PDE after a change of variables. In particular this will lead to the explicit solution of the Black-Scholes PDE.

Proposition 5.4. *The heat equation*

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ g(0,y) = \psi(y) \end{cases} \tag{5.12}$$

with initial condition $\psi(y)$ has solution

$$g(t,y) = \int_{-\infty}^{\infty} \psi(z) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}. \tag{5.13}$$

Proof. We have

$$\begin{aligned} \frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \psi(z) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} \\ &= \int_{-\infty}^{\infty} \psi(z) \frac{\partial}{\partial t}\left(\frac{e^{-\frac{(y-z)^2}{2t}}}{\sqrt{2\pi t}}\right) dz \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z)\left(\frac{(y-z)^2}{t^2} - \frac{1}{t}\right) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial z^2} e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} \end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial y^2} e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} \psi(z) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y).
\end{aligned}
$$

On the other hand it can be checked that at time $t = 0$,

$$
\lim_{t\to 0}\int_{-\infty}^{\infty} \psi(z) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} = \lim_{t\to 0}\int_{-\infty}^{\infty} \psi(y+z) e^{-\frac{z^2}{2t}} \frac{dz}{\sqrt{2\pi t}} = \psi(y),
$$

$y \in \mathbb{R}$. □

Let us provide a second proof of Proposition 5.4 using stochastic calculus and Brownian motion. Note that under the change of variable $x = z - y$ we have

$$
\begin{aligned}
g(t,y) &= \int_{-\infty}^{\infty} \psi(z) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} \\
&= \int_{-\infty}^{\infty} \psi(y+x) e^{-\frac{x^2}{2t}} \frac{dx}{\sqrt{2\pi t}} \\
&= \mathbb{E}[\psi(y+B_t)],
\end{aligned}
$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Applying Itô's formula we have

$$
\begin{aligned}
\mathbb{E}[\psi(y+B_t)] &= \psi(y) + \mathbb{E}\left[\int_0^t \psi'(y+B_s)dB_s\right] + \frac{1}{2}\,\mathbb{E}\left[\int_0^t \psi''(y+B_s)ds\right] \\
&= \psi(y) + \frac{1}{2}\int_0^t \mathbb{E}\left[\psi''(y+B_s)\right] ds \\
&= \psi(y) + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2}\,\mathbb{E}\left[\psi(y+B_s)\right] ds,
\end{aligned}
$$

since the expectation of the stochastic integral is zero. Hence

$$
\begin{aligned}
\frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\,\mathbb{E}[\psi(y+B_t)] \\
&= \frac{1}{2}\frac{\partial^2}{\partial y^2}\,\mathbb{E}\left[\psi(y+B_s)\right] \\
&= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y).
\end{aligned}
$$

Concerning the initial condition we check that

$$
g(0,y) = \mathbb{E}[\psi(y+B_0)]
$$

$$= \mathbb{E}[\psi(y)]$$
$$= \psi(y).$$

5.7 Solution of the Black-Scholes PDE

Proposition 5.5. *Assume that $f(t,x)$ solves the Black-Scholes PDE*

$$\begin{cases} rf(t,x) = \dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 f}{\partial x^2}(t,x), \\ \\ f(T,x) = (x-K)^+, \end{cases} \tag{5.14}$$

with terminal condition $h(x) = (x-K)^+$. Then the function $g(t,y)$ defined by

$$g(t,y) = e^{rt} f(T-t, e^{\sigma y + (\frac{\sigma^2}{2} - r)t}) \tag{5.15}$$

solves the heat equation (5.12), i.e.

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ \\ g(0,y) = h(e^{\sigma y}), \end{cases}$$

with initial condition

$$g(0,y) = h(e^{\sigma y}). \tag{5.16}$$

Proof. Letting $s = T-t$ and $x = e^{\sigma y + (\frac{\sigma^2}{2} - r)t}$ we have

$$\begin{aligned} \frac{\partial g}{\partial t}(t,y) &= re^{rt} f(T-t, e^{\sigma y + (\frac{\sigma^2}{2} - r)t}) - e^{rt}\frac{\partial f}{\partial s}(T-t, e^{\sigma y + (\frac{\sigma^2}{2} - r)t}) \\ &\quad + \left(\frac{\sigma^2}{2} - r\right) e^{rt} e^{\sigma y + (\frac{\sigma^2}{2} - r)t}\frac{\partial f}{\partial x}(T-t, e^{\sigma y + (\frac{\sigma^2}{2} - r)t}) \\ &= re^{rt} f(T-t,x) - e^{rt}\frac{\partial f}{\partial s}(T-t,x) + \left(\frac{\sigma^2}{2} - r\right) e^{rt} x\frac{\partial f}{\partial x}(T-t,x) \\ &= \frac{1}{2}e^{rt}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(T-t,x) + \frac{\sigma^2}{2}e^{rt}x\frac{\partial f}{\partial x}(T-t,x), \end{aligned} \tag{5.17}$$

where on the last step we used the Black-Scholes PDE.

On the other hand we have

$$\frac{\partial g}{\partial y}(t,y) = \sigma e^{rt} e^{\sigma y + (\frac{\sigma^2}{2} - r)t}\frac{\partial f}{\partial x}(T-t, e^{\sigma y + (\frac{\sigma^2}{2} - r)t})$$

and

$$\begin{aligned}\frac{1}{2}\frac{\partial g^2}{\partial y^2}(t,y) &= \frac{\sigma^2}{2}e^{rt}e^{\sigma y+\frac{1}{2}(\frac{\sigma^2}{2}-r)t}\frac{\partial f}{\partial x}(T-t,e^{\sigma y+(\frac{\sigma^2}{2}-r)t})\\ &\quad+\frac{\sigma^2}{2}e^{rt}e^{2\sigma y+2(\frac{\sigma^2}{2}-r)t}\frac{\partial^2 f}{\partial x^2}(T-t,e^{\sigma y+(\frac{\sigma^2}{2}-r)t})\\ &= \frac{\sigma^2}{2}e^{rt}x\frac{\partial f}{\partial x}(T-t,x)+\frac{\sigma^2}{2}e^{rt}x^2\frac{\partial^2 f}{\partial x^2}(T-t,x),\end{aligned}$$

which, in view of (5.17), means that $g(t,x)$ satisfies the heat equation (5.12) with initial condition

$$g(0,y)=f(T,e^{\sigma y})=h(e^{\sigma y}).$$

□

In the next proposition we recover the Black-Scholes formula by solving the PDE (5.14).

Proposition 5.6. *When $h(x)=(x-K)^+$, the solution of the Black-Scholes PDE (5.14) is given by*

$$f(t,x)=x\Phi(d_+)-Ke^{-r(T-t)}\Phi(d_-),$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-y^2/2}dy,\qquad x\in\mathbb{R},$$

and

$$d_+=\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},\qquad d_-=\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Proof. By inversion of (5.15) with $s=T-t$ and $x=e^{\sigma y+(\frac{\sigma^2}{2}-r)t}$ we get

$$f(s,x)=e^{-r(T-s)}g\left(T-s,\frac{-(\frac{\sigma^2}{2}-r)(T-s)+\log x}{\sigma}\right).$$

Hence using the solution (11.14) and Relation (5.16) we get

$$\begin{aligned}f(t,x)&=e^{-r(T-t)}g\left(T-t,\frac{-(\frac{\sigma^2}{2}-r)(T-t)+\log x}{\sigma}\right)\\ &=e^{-r(T-t)}\int_{-\infty}^{\infty}\psi\left(\frac{-(\frac{\sigma^2}{2}-r)(T-t)+\log x}{\sigma}+z\right)e^{-\frac{z^2}{2(T-t)}}\frac{dz}{\sqrt{2\pi(T-t)}}\\ &=e^{-r(T-t)}\int_{-\infty}^{\infty}h\left(xe^{\sigma z-(\frac{\sigma^2}{2}-r)(T-t)}\right)e^{-\frac{z^2}{2(T-t)}}\frac{dz}{\sqrt{2\pi(T-t)}}\\ &=e^{-r(T-t)}\int_{-\infty}^{\infty}\left(xe^{\sigma z-(\frac{\sigma^2}{2}-r)(T-t)}-K\right)^+e^{-\frac{z^2}{2(T-t)}}\frac{dz}{\sqrt{2\pi(T-t)}}\end{aligned}$$

$$
\begin{aligned}
&= e^{-r(T-t)} \int_{\frac{(-r+\sigma^2/2)(T-t)+\log(K/x)}{\sigma}}^{\infty} \left(x e^{\sigma z - (\frac{\sigma^2}{2} - r)(T-t)} - K \right) e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&= x e^{-r(T-t)} \int_{-d_- \sqrt{T-t}}^{\infty} e^{\sigma z - (\frac{\sigma^2}{2} - r)(T-t)} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&\quad - K e^{-r(T-t)} \int_{-d_- \sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&= x \int_{-d_- \sqrt{T-t}}^{\infty} e^{\sigma z - \frac{\sigma^2}{2}(T-t) - \frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&\quad - K e^{-r(T-t)} \int_{-d_- \sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&= x \int_{-d_- \sqrt{T-t}}^{\infty} e^{-\frac{1}{2(T-t)}(z - \sigma(T-t))^2} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&\quad - K e^{-r(T-t)} \int_{-d_- \sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&= x \int_{-d_- \sqrt{T-t} - \sigma(T-t)}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&\quad - K e^{-r(T-t)} \int_{-d_- \sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} \\
&= x \int_{-d_- - \sigma\sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} - K e^{-r(T-t)} \int_{d_-}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \\
&= x \left(1 - \Phi\left(-d_+\right)\right) - K e^{-r(T-t)} \left(1 - \Phi\left(-d_-\right)\right) \\
&= x \Phi\left(d_+\right) - K e^{-r(T-t)} \Phi\left(d_-\right),
\end{aligned}
$$

where we used the relation

$$
1 - \Phi(a) = \Phi(-a), \qquad a \in \mathbb{R}.
$$

□

5.8 Exercises

Exercise 5.1.

1. Solve the stochastic differential equation

$$
dS_t = \alpha S_t dt + \sigma dB_t \tag{5.18}
$$

in terms of $\alpha, \sigma > 0$, and the initial condition S_0.

2. For which values α_M of α is the discounted price process $\tilde{S}_t = e^{-rt}S_t$, $t \in [0,T]$, a martingale under P ?
3. Compute the arbitrage price $C(t,S_t) = e^{-r(T-t)}\,\mathbb{E}[\exp(S_T)|\mathcal{F}_t]$ at time $t \in [0,T]$ of the contingent claim of $\exp(S_T)$, with $\alpha = \alpha_M$.
4. Explicitly compute the strategy $(\zeta_t, \eta_t)_{t\in[0,T]}$ that hedges the contingent claim $\exp(S_T)$.

Exercise 5.2. In the Black-Scholes model, the price at time t of a European claim on the underlying asset S_t, with strike price K, maturity T, interest rate r and volatility σ is given by the Black-Scholes formula as

$$f(t,S_t) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-),$$

where

$$d_- = \frac{(r-\frac{1}{2}\sigma^2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_+ = d_- + \sigma\sqrt{T-t}.$$

Recall that

$$\frac{\partial f}{\partial x}(t,S_t) = \Phi(d_+).$$

On December 18, 2007, a call warrant has been issued by Fortis Bank on the stock price S of the MTR Corporation with maturity $T = 23/12/2008$, Strike $K = 36.08$HK\$ and <u>Entitlement ratio=10</u>.

1. Using the values of the Gaussian cumulative distribution function, compute the Black-Scholes price of the corresponding call option at time t =November 07, 2008 with $S_t = 17.200$HK\$, assuming a volatility $\sigma = 90\% = 0.90$ and an *annual* risk-free interest rate $r = 4.377\% = 0.04377$, Hint: $\Phi(-0.08) = 0.4681$.
2. Still using the values of the Gaussian cumulative distribution function, compute the number of units of the risky asset required in your portfolio at time t =November 07, 2008 in order to hedge one such option at maturity $T = 23/12/2008$.
3. Figure 1 represents the Black-Scholes price of the call option as a function of $\sigma \in [0.5, 1.5] = [50\%, 150\%]$.

Fig. 5.1: Option price as a function of the volatility σ.

Knowing that the closing price of the warrant on November 07, 2008 was 0.023HK$, which value can you infer for the implied volatility σ at this date ?

Exercise 5.3. Recall that the price $\pi_t(C)$ of a claim $C = h(S_T)$ of maturity T can be written as $\pi_t(C) = g(t, S_t)$, where the function $g(t, x)$ satisfies the *Black-Scholes PDE*

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g}{\partial x^2}(t,x), \\ g(T,x) = h(x), \end{cases} \tag{1}$$

with terminal condition $g(T, x) = h(x)$.

1. Assume that C is a forward contract with payoff
$$C = S_T - K,$$
at time T. Find the function $h(x)$ in (1).
2. Find the solution $g(t, x)$ of the above PDE and compute the price $\pi_t(C)$ at time $t \in [0, T]$.
Hint: search for a solution of the form $g(t, x) = x - \alpha(t)$ where $\alpha(t)$ is a function of t to be determined.
3. Compute the quantity
$$\xi_t = \frac{\partial g}{\partial x}(t, S_t)$$
of risky assets in a self-financing portfolio hedging C.

This version: March 8, 2011
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Chapter 6
Martingale Approach to Pricing and Hedging

In this chapter we present the probabilistic *martingale approach* method to the pricing and hedging of options. In particular, this allows one to compute option prices as the expectations of the discounted option payoffs, and to determine the associated hedging portfolios.

6.1 Martingale Property of the Itô Integral

Recall that an integrable process $(X_t)_{t\in\mathbb{R}_+}$ is said to be a *martingale* with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s, \qquad 0 \le s \le t.$$

The following result shows that the indefinite Itô integral is a martingale with respect to the Brownian filtration.

Proposition 6.1. *The indefinite stochastic integral* $\left(\int_0^t u_s dB_s\right)_{t\in\mathbb{R}_+}$ *of* $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ *is a martingale, i.e.:*

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \Big| \mathcal{F}_s\right] = \int_0^s u_\tau dB_\tau, \quad 0 \le s \le t.$$

Proposition 6.1 is a consequence of Proposition 6.2 below.

Proposition 6.2. *For any* $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ *we have*

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right] = \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+.$$

In particular, $\int_0^t u_s dB_s$ *is* $\mathcal{F}_t$*-measurable,* $t \in \mathbb{R}_+$.

Proof. The statement is first proved in case u is a simple predictable process, and then extended to the general case, cf. e.g. Proposition 2.5.7 in [39]. □

In particular, since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, this recover the fact that the Itô integral is a centered random variable:

$$\mathbb{E}\left[\int_0^\infty u_s dB_s\right] = \mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_0\right] = \int_0^0 u_s dB_s = 0.$$

Examples

1. Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale because it has centered, independent increments:

$$\begin{aligned}\mathbb{E}[B_t|\mathcal{F}_s] &= \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s]\\ &= \mathbb{E}[B_t - B_s|\mathcal{F}_s] + \mathbb{E}[B_s|\mathcal{F}_s]\\ &= \mathbb{E}[B_t - B_s] + B_s\\ &= B_s, \qquad 0 \le s \le t.\end{aligned}$$

   This fact can be recovered from Proposition 6.1 since B_t can be written as

$$B_t = \int_0^t dB_s, \qquad t \in \mathbb{R}_+.$$

2. The discounted asset price

$$X_t = X_0 e^{(\mu - r)t + \sigma B_t - \sigma^2 t/2}$$

   is a martingale when $\mu = r$. Indeed we have

$$\begin{aligned}\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}[X_0 e^{\sigma B_t - \sigma^2 t/2}|\mathcal{F}_s]\\ &= X_0 e^{-\sigma^2 t/2}\, \mathbb{E}[e^{\sigma B_t}|\mathcal{F}_s]\\ &= X_0 e^{-\sigma^2 t/2}\, \mathbb{E}[e^{\sigma(B_t - B_s) + \sigma B_s}|\mathcal{F}_s]\\ &= X_0 e^{-\sigma^2 t/2 + \sigma B_s}\, \mathbb{E}[e^{\sigma(B_t - B_s)}|\mathcal{F}_s]\\ &= X_0 e^{-\sigma^2 t/2 + \sigma B_s}\, \mathbb{E}[e^{\sigma(B_t - B_s)}]\\ &= X_0 e^{-\sigma^2 t/2 + \sigma B_s} e^{\sigma^2 (t-s)/2}\\ &= X_0 e^{\sigma B_s - \sigma^2 s/2}\\ &= X_s, \qquad 0 \le s \le t.\end{aligned}$$

   This fact can also be recovered from Proposition 6.1 since X_t satisfies the equation

$$dX_t = \sigma X_t dB_t,$$

   i.e. it can be written as the Brownian stochastic integral

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$$X_t = X_0 + \sigma \int_0^t X_u dB_u, \qquad t \in \mathbb{R}_+.$$

3. The discounted value
$$\tilde{V}_t = e^{-rt} V_t$$
of a self-financing portfolio is given by
$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u dX_u, \qquad t \in \mathbb{R}_+,$$
is a martingale when $\mu = r$ because
$$\tilde{V}_t = \tilde{V}_0 + \sigma \int_0^t \xi_u X_u dB_u, \qquad t \in \mathbb{R}_+,$$
since
$$dX_t = X_t((\mu - r)dt + \sigma dB_t).$$

Since the Black-Scholes theory is in fact valid for any value of the parameter μ we will look forward to including the case $\mu \neq r$ in the sequel.

6.2 Risk-neutral Measures

Recall that by definition, a risk-neutral measure is a probability measure $\mathbb{P}^*$ under which the discounted asset price $(X_t)_{t\in\mathbb{R}_+}$ is a martingale. From the analysis of Section 6.1 it appears that when $\mu = r$, $(X_t)_{t\in\mathbb{R}_+}$ is a martingale and $\mathbb{P}^* = \mathbb{P}$ is risk-neutral.

In this section we address the construction of a risk-neutral measure in the general case $\mu \neq r$ and for this we will use the Girsanov theorem.

Note that the relation
$$dX_t = X_t((\mu - r)dt + \sigma dB_t)$$
can be written as
$$dX_t = \sigma X_t d\tilde{B}_t,$$
where
$$\tilde{B}_t := \frac{\mu - r}{\sigma} t + B_t, \qquad t \in \mathbb{R}_+.$$
Therefore the search for a risk-neutral measure can be replaced by the search for a probability measure $\mathbb{P}^*$ under which $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

Let us come back to the informal interpretation of Brownian motion via its infinitesimal increments:

$$\Delta B_t = \pm\sqrt{dt},$$

with

$$\mathbb{P}(\Delta B_t = +\sqrt{dt}) = \mathbb{P}(\Delta B_t = -\sqrt{dt}) = \frac{1}{2}.$$

Fig. 6.1: Drifted Brownian path.

Clearly, given $\nu \in \mathbb{R}$, the drifted process $\nu t + B_t$ is no longer a standard Brownian motion because it is not centered:

$$\mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t \neq 0,$$

cf. Figure 6.1. This identity can be formulated in terms of infinitesimal increments as

$$\mathbb{E}[\nu dt + dB_t] = \frac{1}{2}(\nu dt + \sqrt{dt}) + \frac{1}{2}(\nu dt - \sqrt{dt}) = \nu dt \neq 0.$$

In order to make $\nu t + B_t$ a centered process (i.e. a standard Brownian motion, since $\nu t + B_t$ conserves all the other properties (i)-(iii) in the definition of Brownian motion, one may change the probabilities of ups and downs, which have been fixed so far equal to $1/2$.

That is, the problem is now to find two numbers $p, q \in [0, 1]$ such that

$$\begin{cases} p(\nu dt + \sqrt{dt}) + q(\nu dt - \sqrt{dt}) = 0 \\ \\ p + q = 1. \end{cases}$$

The solution to this problem is given by

$$p = \frac{1}{2}(1 - \nu\sqrt{dt}) \quad \text{ and } q = \frac{1}{2}(1 + \nu\sqrt{dt}).$$

Still considering Brownian motion as a discrete random walk with independent increments $\pm\sqrt{dt}$, the corresponding probability density will be obtained by taking the product of the above probabilities divided by $1/2^N$, that is:

$$2^N \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{dt}\right)$$

where 2^N is a normalization factor and $N = T/dt$ is the (infinitely large) number of discrete time steps. Using elementary calculus, this density can be informally shown to converge as follows:

$$\begin{aligned} 2^N \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{dt}\right) &= \prod_{0<t<T} \left(1 \mp \nu\sqrt{dt}\right) \\ &= \exp\left(\log \prod_{0<t<T} \left(1 \mp \nu\sqrt{dt}\right)\right) \\ &= \exp\left(\sum_{0<t<T} \log\left(1 \mp \nu\sqrt{dt}\right)\right) \\ &\simeq \exp\left(\nu \sum_{0<t<T} \mp\sqrt{dt} - \frac{1}{2} \sum_{0<t<T} (\mp\nu\sqrt{dt})^2\right) \\ &= \exp\left(\nu \sum_{0<t<T} \mp\sqrt{dt} - \frac{1}{2}\nu^2 \sum_{0<t<T} dt\right) \\ &= \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right). \end{aligned}$$

The Girsanov theorem can be restated as follows in a more rigorous way. Recall that here, $\Omega = \mathcal{C}_0([0,T])$ is the Wiener space and $\omega \in \Omega$ is a continuous function on $[0,T]$ starting at 0 in $t = 0$. Consider the probability $\mathbb{Q}$ defined by

$$d\mathbb{Q}(\omega) = \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right) d\mathbb{P}(\omega).$$

Then the process $\nu t + B_t$ is a standard (centered) Brownian motion under $\mathbb{Q}$.

For example, the fact that $\nu T + B_T$ has a standard (centered) Gaussian law under $\mathbb{Q}$ can be recovered as follows:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[f(\nu T + B_T)] &= \int_\Omega f(\nu T + B_T) d\mathbb{Q} \\
&= \int_\Omega f(\nu T + B_T) \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right) d\mathbb{P} \\
&= \int_{-\infty}^{\infty} f(\nu T + x) \exp\left(-\nu x - \frac{1}{2}\nu^2 T\right) e^{-\frac{x^2}{2T}} \frac{dx}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{\infty} f(y) e^{-\frac{y^2}{2T}} \frac{dy}{\sqrt{2\pi T}} \\
&= \int_\Omega f(B_T) d\mathbb{P} \\
&= \mathbb{E}_{\mathbb{P}}[f(B_T)].
\end{aligned}$$

The Girsanov theorem can actually be extended to shifts by adapted processes as follows, cf. e.g. [43], Theorem III-42.

Theorem 6.1. Let $(\psi_t)_{t\in[0,T]}$ be an adapted process satisfying the Novikov integrability condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\psi_t|^2 dt\right)\right] < \infty, \tag{6.1}$$

and let $\mathbb{Q}$ denote the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T \psi_s^2 ds\right).$$

Then

$$\hat{B}_t := B_t + \int_0^t \psi_s ds, \qquad t \in [0,T],$$

is a standard Brownian motion under $\mathbb{Q}$.

When applied to

$$\psi_t := \frac{\mu - r}{\sigma},$$

the Girsanov theorem shows that

$$\tilde{B}_t := \frac{\mu - r}{\sigma} t + B_t, \qquad t \in \mathbb{R}_+,$$

is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} B_t - \frac{(\mu - r)^2}{2\sigma^2} t\right).$$

Hence the discounted price process given by

$$\frac{dX_t}{X_t} = (\mu - r)dt + \sigma dB_t = \sigma d\tilde{B}_t, \qquad t \in \mathbb{R}_+,$$

is a martingale under $\mathbb{P}^*$, hence $\mathbb{P}^*$ is a risk-neutral measure. We easily recover $\mathbb{P} = \mathbb{P}^*$ when $\mu = r$.

6.3 Pricing by the Martingale Method

In this section we give the expression of the Black-Scholes price using expectations of discounted payoffs.

Recall that from the first fundamental theorem of mathematical finance, a continuous market is without arbitrage opportunities if there exists (at least) a risk-neutral probability measure $\mathbb{P}^*$ under which the discounted price process

$$X_t := e^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

is a martingale under $\mathbb{P}^*$.

In addition, when the risk-neutral measure is unique, the market is *complete*.

In case the price process $(S_t)_{s\in[t,\infty)}$ satisfies the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+, \qquad S_0 > 0$$

we have

$$S_t = S_0 e^{\sigma B_t - \sigma^2 t/2 + \mu t}, \quad \text{and} \quad X_t = S_0 e^{(\mu - r)t + \sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

hence from Section 6.1 the discounted price process is a martingale under $\mathbb{P}^*$, and $\mathbb{P}^*$ is a martingale measure.

We have

$$dX_t = (\mu - r)X_t dt + \sigma X_t dB_t = \sigma X_t d\tilde{B}_t, \qquad t \in \mathbb{R}_+,$$

hence the discounted value $\tilde{V}_t$ of a self-financing portfolio is written as

$$\begin{aligned}\tilde{V}_t &= \tilde{V}_0 + \int_0^t \xi_u dX_u \\ &= \tilde{V}_0 + \sigma \int_0^t \xi_u X_u d\tilde{B}_u, \qquad t \in \mathbb{R}_+,\end{aligned}$$

and becomes a martingale under $\mathbb{P}^*$.

Proposition 6.3. *Let $(\xi_t, \eta_t)_{t\in[0,T]}$ be a portfolio strategy with price*

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in [0,T],$$

and let C be a contingent claim, such that

(i) $(\xi_t, \eta_t)_{t\in[0,T]}$ is a self-financing portfolio, and

(ii) $(\xi_t, \eta_t)_{t\in[0,T]}$ hedges the claim C, i.e. we have $V_T = C$.

Then we have

$$V_t = e^{-r(T-t)}\,\mathbb{E}^*[C|\mathcal{F}_t], \qquad 0 \le t \le T, \tag{6.2}$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral measure $\mathbb{P}^*$.

Proof. Since the portfolio strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ is self-financing we have

$$\tilde{V}_t = \tilde{V}_0 + \sigma \int_0^t \xi_u X_u d\tilde{B}_u, \qquad t \in \mathbb{R}_+,$$

which is a martingale from Proposition 6.1, hence

$$\begin{aligned}\tilde{V}_t &= \mathbb{E}^*[\tilde{V}_T \mid \mathcal{F}_t]\\ &= e^{-rT}\,\mathbb{E}^*[V_T \mid \mathcal{F}_t]\\ &= e^{-rT}\,\mathbb{E}^*[C \mid \mathcal{F}_t],\end{aligned}$$

which implies

$$V_t = e^{rt}\tilde{V}_t = e^{-r(T-t)}\,\mathbb{E}^*[C \mid \mathcal{F}_t].$$

□

In case the value V_t of the portfolio at time $t \in [0,T]$ is a function $C(t, S_t)$ of t and S_t, it can be computed from (6.2) as

$$V_t = C(t, S_t) = e^{-r(T-t)}\,\mathbb{E}^*[\phi(S_T)|\mathcal{F}_t], \qquad 0 \le t \le T,$$

and the function $C(t,x)$ solves the Black-Scholes PDE:

$$\begin{cases} rC(t,x) = \dfrac{\partial C}{\partial t}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 C}{\partial x^2}(t,x) + rx\dfrac{\partial C}{\partial x}(t,x)\\[2ex] C(T,x) = \phi(x).\end{cases}$$

In the case of European options with payoff function $\phi(x) = (x-K)^+$ we recover the Black-Scholes formula (5.11) by a probabilistic argument.

Proposition 6.4. *The price at time t of a European call option with strike K and maturity T is given by*

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http://www.ntu.edu.sg/home/nprivault/indext.html

$$C(t, S_t) = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-), \qquad t \in [0, T].$$

Proof. The proof of Proposition 6.4 is a consequence of (6.2) and the Lemma 6.1 below. Using the relation

$$S_T = S_t e^{r(T-t)+\sigma(\tilde{B}_T - \tilde{B}_t) - \sigma^2(T-t)/2}, \qquad t \in [0, T].$$

By Proposition 6.3 the price of the portfolio hedging C is given by

$$\begin{aligned}
V_t &= e^{-r(T-t)}\, \mathbb{E}^*[C|\mathcal{F}_t] \\
&= e^{-r(T-t)}\, \mathbb{E}^*[(S_T - K)^+|\mathcal{F}_t] \\
&= e^{-r(T-t)}\, \mathbb{E}^*[(S_t e^{r(T-t)+\sigma(\tilde{B}_T - \tilde{B}_t) - \sigma^2(T-t)/2} - K)^+|\mathcal{F}_t] \\
&= e^{-r(T-t)}\, \mathbb{E}^*[(x e^{r(T-t)+\sigma(\tilde{B}_T - \tilde{B}_t) - \sigma^2(T-t)/2} - K)^+]_{x=S_t} \\
&= e^{-r(T-t)}\, \mathbb{E}^*[(e^{m(x)+X} - K)^+]_{x=S_t}, \qquad 0 \le t \le T,
\end{aligned}$$

where

$$m(x) = r(T-t) - \sigma^2(T-t)/2 + \log x$$

and $X = \sigma(\tilde{B}_T - \tilde{B}_t)$ is a centered Gaussian random variable with variance

$$\mathrm{var}\,[X] = \mathrm{var}\,[\sigma(\tilde{B}_T - \tilde{B}_t)] = \sigma^2 \mathrm{var}\,[\tilde{B}_T - \tilde{B}_t] = \sigma^2(T-t)$$

under $\mathbb{P}^*$. Hence by Lemma 6.1 below we have

$$\begin{aligned}
V_t &= e^{-r(T-t)}\, \mathbb{E}^*[(e^{m(x)+X} - K)^+]_{x=S_t} \\
&= e^{-r(T-t)} e^{m(S_t)+\sigma^2(T-t)/2} \Phi(v + (m(S_t) - \log K)/v) \\
&\quad - K e^{-r(T-t)} \Phi((m(S_t) - \log K)/v) \\
&= S_t \Phi(v + (m(S_t) - \log K)/v) - K e^{-r(T-t)} \Phi((m(S_t) - \log K)/v) \\
&= S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-),
\end{aligned}$$

$0 \le t \le T$. □

Lemma 6.1. Let X be a centered Gaussian random variable with variance v^2. We have

$$\mathbb{E}[(e^{m+X} - K)^+] = e^{m+\frac{v^2}{2}} \Phi(v + (m - \log K)/v) - K\Phi((m - \log K)/v).$$

Proof. We have

$$\begin{aligned}
\mathbb{E}[(e^{m+X} - K)^+] &= \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+x} - K)^+ e^{-\frac{x^2}{2v^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} (e^{m+x} - K) e^{-\frac{x^2}{2v^2}}\, dx \\
&= \frac{e^m}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{x - \frac{x^2}{2v^2}}\, dx - \frac{K}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{-\frac{x^2}{2v^2}}\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{m+\frac{v^2}{2}}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{-\frac{(v^2-x)^2}{2v^2}} dx - \frac{K}{\sqrt{2\pi}} \int_{(-m+\log K)/v}^{\infty} e^{-x^2/2} dx \\
&= \frac{e^{m+\frac{v^2}{2}}}{\sqrt{2\pi v^2}} \int_{-v^2-m+\log K}^{\infty} e^{-\frac{x^2}{2v^2}} dx - K\Phi((m-\log K)/v) \\
&= e^{m+\frac{v^2}{2}} \Phi(v + (m-\log K)/v) - K\Phi((m-\log K)/v).
\end{aligned}
$$

□

Denoting by

$$P(t, S_t) = e^{-r(T-t)}\, \mathbb{E}^*[(K - S_T)^+ | \mathcal{F}_t]$$

the price of the put option with strike K and maturity T, we check from Proposition 6.3 that

$$
\begin{aligned}
C(t, S_t) - P(t, S_t) &= e^{-r(T-t)}\, \mathbb{E}^*[(S_T - K)^+ | \mathcal{F}_t] - e^{-r(T-t)}\, \mathbb{E}^*[(K - S_T)^+ | \mathcal{F}_t] \\
&= e^{-r(T-t)}\, \mathbb{E}^*[(S_T - K)^+ - (K - S_T)^+ | \mathcal{F}_t] \\
&= e^{-r(T-t)}\, \mathbb{E}^*[S_T - K | \mathcal{F}_t] \\
&= S_t - e^{-r(T-t)} K.
\end{aligned}
$$

This relation is called the *put-call parity*, and it shows that

$$
\begin{aligned}
P(t, S_t) &= C(t, S_t) - S_t + e^{-r(T-t)} K \\
&= S_t \Phi(d_+) + e^{-r(T-t)} K - S_t - e^{-r(T-t)} K \Phi(d_-) \\
&= -S_t(1 - \Phi(d_+)) + e^{-r(T-t)} K (1 - \Phi(d_-)) \\
&= -S_t \Phi(-d_+) + e^{-r(T-t)} K \Phi(-d_-).
\end{aligned}
$$

6.4 Hedging Strategies

In the next proposition we compute a self-financing hedging strategy leading to an arbitrary square-integrable random variable C admitting a stochastic integral representation formula of the form

$$C = \mathbb{E}[C] + \int_0^T \zeta_t dB_t, \tag{6.3}$$

where $(\zeta_t)_{t\in[0,t]}$ is a square-integrable adapted process. Consequently, the mathematical problem of finding the predictable representation (6.3) of a given random variable has important applications in finance. For example we have

$$B_T = T + 2\int_0^T B_t dB_t.$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Recall that the risky asset follows the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+, \qquad S_0 > 0,$$

and the discounted asset price satisfies

$$dX_t = \sigma X_t d\tilde{B}_t, \qquad t \in \mathbb{R}_+, \qquad X_0 = S_0 > 0,$$

where $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$.

The following proposition applies to arbitrary square-integrable payoff functions, i.e. it covers exotic and path-dependent options.

Proposition 6.5. *Consider a payoff $C \in L^2(\Omega)$ such that (6.3) holds, and let*

$$\xi_t = \frac{e^{-r(T-t)}}{\sigma S_t}\zeta_t, \tag{6.4}$$

$$\eta_t = \frac{e^{-r(T-t)}\,\mathbb{E}^*[C|\mathcal{F}_t] - \xi_t S_t}{A_t}, \qquad t \in [0,T]. \tag{6.5}$$

Then the portfolio $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing, and letting

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in [0,T], \tag{6.6}$$

we have

$$V_t = e^{-r(T-t)}\,\mathbb{E}^*[C|\mathcal{F}_t], \qquad t \in [0,T]. \tag{6.7}$$

In particular we have

$$V_T = C, \tag{6.8}$$

i.e. the portfolio $(\xi_t, \eta_t)_{t\in[0,T]}$ yields a hedging strategy leading to C, starting from the initial value

$$V_0 = e^{-rT}\,\mathbb{E}^*[C].$$

Proof. Relation (6.7) follows from (6.5) and (6.6), and it implies

$$V_0 = e^{-rT}\,\mathbb{E}^*[C] = \eta_0 A_0 + \xi_0 S_0$$

at $t = 0$, and (6.8) at $t = T$. It remains to show that the portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing. We have

$$\begin{aligned} V_t &= \eta_t A_t + \xi_t S_t \\ &= e^{-r(T-t)}\,\mathbb{E}^*[C|\mathcal{F}_t] \\ &= e^{-r(T-t)}\,\mathbb{E}^*\left[\mathbb{E}^*[C] + \int_0^T \zeta_u d\tilde{B}_u \Big| \mathcal{F}_t\right] \end{aligned}$$

$$
\begin{aligned}
&= e^{-r(T-t)} \left(\mathbb{E}^*[C] + \int_0^t \zeta_u d\tilde{B}_u \right) \\
&= e^{rt} V_0 + e^{-r(T-t)} \int_0^t \zeta_u d\tilde{B}_u \\
&= e^{rt} V_0 + \sigma \int_0^t \xi_u S_u e^{r(t-u)} d\tilde{B}_u \\
&= e^{rt} V_0 + \sigma \int_0^t \xi_u X_u e^{rt} d\tilde{B}_u \\
&= e^{rt} V_0 + e^{rt} \int_0^t \xi_u dX_u, \qquad t \in [0,T],
\end{aligned}
$$

which shows that the discounted portfolio value $\tilde{V}_t = e^{-rt} V_t$ satisfies

$$
V_t = V_0 + \int_0^t \xi_u dX_u, \qquad t \in [0,T],
$$

and this implies that $(\xi_t, \eta_t)_{t \in [0,T]}$ is self-financing. □

The above proposition shows that there always exists a hedging strategy starting from

$$
V_0 = \mathbb{E}^*[C] e^{-rT}.
$$

In addition, since there exists a hedging strategy leading to

$$
\tilde{V}_T = e^{-rT} C,
$$

then $(\tilde{V}_t)_{t \in [0,T]}$ is necessarily a martingale with

$$
\tilde{V}_t = \mathbb{E}^*[\tilde{V}_T | \mathcal{F}_t] = e^{-rT} \mathbb{E}^*[C | \mathcal{F}_t], \qquad t \in [0,T],
$$

and initial value

$$
\tilde{V}_0 = \mathbb{E}^*[\tilde{V}_T] = e^{-rT} \mathbb{E}^*[C].
$$

In practice, the hedging problem can now be reduced to the computation of the process $(\zeta_t)_{t \in [0,T]}$ appearing in (6.3). This computation, called the Delta hedging, can be performed by application of the Itô formula and the Markov property, see e.g. [42]. Consider the (non homogeneous) semi-group $(P_{s,t})_{0 \le s \le t \le T}$ associated to $(S_t)_{t \in [0,T]}$ and defined by

$$
P_{s,t} f(S_s) = \mathbb{E}[f(S_t) \mid \mathcal{F}_s] = \mathbb{E}[f(S_t) \mid S_s], \quad 0 \le s \le t,
$$

which acts on functions $f \in \mathcal{C}_b^2(\mathbb{R}^n)$, with

$$
P_{s,t} P_{t,u} = P_{s,u}, \qquad 0 \le s \le t \le u.
$$

Note that $(P_{t,T} f(S_t))_{t \in [0,T]}$ is an $\mathcal{F}_t$-martingale, i.e.:

$$\begin{aligned}\mathbb{E}[P_{t,T}f(S_t) \mid \mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[f(S_T) \mid \mathcal{F}_t] \mid \mathcal{F}_s] \\ &= \mathbb{E}[f(S_T) \mid \mathcal{F}_s] \\ &= P_{s,T}f(S_s), \end{aligned} \tag{6.9}$$

$0 \leq s \leq t \leq T$, and we have

$$P_{s,t}f(x) = \mathbb{E}[f(S_t) \mid S_s = x] = \mathbb{E}[f(xS_t/S_s)], \qquad 0 \leq s \leq t. \tag{6.10}$$

The next lemma allows us to compute the process $(\zeta_t)_{t\in[0,T]}$ in case the payoff C is of the form $C = \phi(S_T)$ for some function ϕ. In case $C \in L^2(\Omega)$ is the payoff of an exotic option, the process $(\zeta_t)_{t\in[0,T]}$ can be computed using the Malliavin gradient on the Wiener space, cf. [35], [39].

Lemma 6.2. Let $\phi \in \mathcal{C}_b^2(\mathbb{R}^n)$. The predictable representation

$$\phi(S_T) = \mathbb{E}[\phi(S_T)] + \int_0^T \zeta_t dB_t \tag{6.11}$$

is given by

$$\zeta_t = \sigma S_t \frac{\partial}{\partial x}(P_{t,T}\phi)(S_t), \qquad t \in [0,T]. \tag{6.12}$$

Proof. Since $P_{t,T}\phi$ is in $\mathcal{C}^2(\mathbb{R})$, we can apply the Itô formula to the process

$$t \mapsto P_{t,T}\phi(S_t) = \mathbb{E}[\phi(S_T) \mid \mathcal{F}_t],$$

which is a martingale from the tower property of conditional expectations as in (6.9). From the fact that the finite variation term in the Itô formula vanishes when $(P_{t,T}\phi(S_t))_{t\in[0,T]}$ is a martingale, (see e.g. Corollary 1, p. 72 of [43]), we obtain:

$$P_{t,T}\phi(S_t) = P_{0,T}\phi(S_0) + \sigma \int_0^t S_s \frac{\partial}{\partial x}(P_{s,T}\phi)(S_s)dB_s, \quad t \in [0,T], \tag{6.13}$$

with $P_{0,T}\phi(S_0) = \mathbb{E}[\phi(S_T)]$. Letting $t = T$, we obtain (6.12) by uniqueness of the predictable representation (6.11) of $C = \phi(S_T)$. □

By (6.10) we also have

$$\begin{aligned}\zeta_t &= \sigma S_t \frac{\partial}{\partial x}\, \mathbb{E}[\phi(S_T) \mid S_t = x]_{x=S_t} \\ &= \sigma S_t \frac{\partial}{\partial x}\, \mathbb{E}[\phi(xS_T/S_t)]_{x=S_t}, \qquad t \in [0,T],\end{aligned}$$

hence

$$\xi_t = \frac{1}{\sigma S_t} e^{-r(T-t)} \zeta_t$$

$$= e^{-r(T-t)} \frac{\partial}{\partial x} \mathbb{E}[\phi(xS_T/S_t)]_{x=S_t}, \qquad t \in [0,T],$$

which recovers the formula for the Delta of a European type option. As a consequence we have $\xi_t \geq 0$ and there is no short selling when the payoff function ϕ is non-decreasing.

In the case of European options, the process ζ can be computed via the next proposition.

Proposition 6.6. *Assume that $C = (S_T - K)^+$. Then for $0 \leq t \leq T$ we have*

$$\zeta_t = \sigma S_t \, \mathbb{E}\left[\frac{S_T}{S_t} \mathbf{1}_{[K,\infty)} \left(x \frac{S_T}{S_t} \right) \right]_{x=S_t}.$$

Proof. This result follows from Lemma 6.2 and the relation $P_{t,T} f(x) = \mathbb{E}[f(S^x_{t,T})]$, after approximation of $x \mapsto \phi(x) = (x-K)^+$ with $\mathcal{C}^2$ functions. □

From the above proposition we recover the formula for the Delta of a European call option in the Black-Scholes model.

Proposition 6.7. *The Delta of a European call option with payoff $f(x) = (x-K)^+$ is given by*

$$\xi_t = \Phi(d_+) = \Phi\left(\frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right), \qquad 0 \leq t \leq T.$$

Proof. By Propositions 6.5 and 6.6 we have

$$\begin{aligned}
\xi_t &= \frac{1}{\sigma S_t} e^{-r(T-t)} \zeta_t \\
&= e^{-r(T-t)} \, \mathbb{E}\left[\frac{S_T}{S_t} \mathbf{1}_{[K,\infty)} \left(x \frac{S_T}{S_t} \right) \right]_{x=S_t} \\
&= e^{-r(T-t)} \, \mathbb{E}\left[e^{\sigma(B_T - B_t) - \sigma^2(T-t)/2 + r(T-t)} \mathbf{1}_{[K,\infty)} \big(x e^{\sigma(B_T-B_t) - \sigma^2(T-t)/2 + r(T-t)} \big) \right]_{x=S_t} \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\sigma(T-t)/2 - r(T-t)/\sigma + \frac{1}{\sigma}\log\frac{K}{S_t}}^{\infty} e^{\sigma y - \sigma^2(T-t)/2 - y^2/(2(T-t))} dy \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-d_-/\sqrt{T-t}}^{\infty} e^{-\frac{1}{2(T-t)}(y - \sigma(T-t))^2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} e^{-\frac{1}{2}(y-\sigma(T-t))^2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-d_+}^{\infty} e^{-\frac{1}{2}y^2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_+} e^{-\frac{1}{2}y^2} dy
\end{aligned}$$

$= \Phi(d_+)$.

□

6.5 Pricing and Hedging of Some Exotic Options

We assume that the asset price $(S_t)_{t\in[0,T]}$ has the dynamics

$$dS_t = rS_t dt + \sigma S_t dB_t, \quad t \in [0,T],$$

under the risk-neutral probability measure $\mathbb{P}$. In particular we have

$$V_T e^{-rT} = V_0 + \sigma \int_0^T \xi_t S_t e^{-rt} dB_t, \quad t \in [0,T].$$

Asian options

The Asian option with payoff

$$\left(\frac{1}{T}\int_0^T S_u du - K\right)^+, \tag{6.14}$$

is a path-dependent option whose price at time $t \in [0,T]$ is given by

$$e^{-r(T-t)}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| \mathcal{F}_t\right]. \tag{6.15}$$

The next proposition gives a replicating hedging strategy for Asian options. Following [30], page 91, we define the auxiliary process

$$Y_t = \frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right), \qquad t \in \mathbb{R}_+.$$

Proposition 6.8. *The price* (6.15) *at time* t *of the Asian option with payoff* (6.14) *can be written as*

$$e^{r(T-t)} S_t \tilde{C}(t, Y_t), \qquad t \in [0,T],$$

where the measurable function $\tilde{C}$ *on* $\mathbb{R}_+ \times \mathbb{R}$ *is given by*

$$\tilde{C}(t,y) = \mathbb{E}\left[\left(y + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)^+\right], \tag{6.16}$$

and such that $\tilde{C}(t,\cdot)$ is $\mathcal{C}^1$, for all $t \in \mathbb{R}_+$. Moreover, the corresponding replicating portfolio is given by

$$\xi_t = e^{r(T-t)}\tilde{C}(t,Y_t) - e^{r(T-t)}Y_t\partial_2\tilde{C}(t,Y_t), \qquad t \in [0,T]. \tag{6.17}$$

Proof. With the above notation, the price of the Asian option at time t becomes

$$e^{r(T-t)}\,\mathbb{E}\left[S_T(Y_T)^+ \mid \mathcal{F}_t\right].$$

For $0 \le s \le t \le T$, we have

$$d\left(S_tY_t\right) = \frac{1}{T}d\left(\int_0^t S_u du - K\right) = \frac{S_t}{T}dt,$$

hence

$$\frac{S_tY_t}{S_s} = Y_s + \frac{1}{T}\int_s^t \frac{S_u}{S_s}du.$$

Since for any $t \in [0,T]$, S_t is positive and $\mathcal{F}_t$-measurable, and S_u/S_t is independent of $\mathcal{F}_t$, $u \ge t$, we have:

$$\begin{aligned}
\mathbb{E}\left[S_T(Y_T)^+ \mid \mathcal{F}_t\right] &= S_t\,\mathbb{E}\left[\left(\frac{S_T}{S_t}Y_T\right)^+ \Big|\mathcal{F}_t\right] \\
&= S_t\,\mathbb{E}\left[\left(Y_t + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)^+ \Big|\mathcal{F}_t\right] \\
&= S_t\,\mathbb{E}\left[\left(y + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)^+\right]_{y=Y_t} \\
&= S_t\tilde{C}(t,Y_t),
\end{aligned}$$

which proves (6.16). Next we proceed as in [45], which deals with the sum of a Brownian motion and a Poisson process. From the expression of $1/S_t$ we have

$$d\left(\frac{1}{S_t}\right) = \frac{1}{S_t}\left(\left(-r+\sigma^2\right)dt - \sigma dB_t\right),$$

hence

$$dY_t = Y_t\left(-r+\sigma^2\right)dt + \frac{1}{T}dt - Y_t\sigma dB_t.$$

Hence by Itô's calculus the process $(\zeta_t)_{t\in[0,T]}$ in (6.3) is given by

$$\zeta_t dB_t = d(S_t\tilde{C}(t,Y_t))$$

$$= \sigma S_t \tilde{C}(t, Y_t) dB_t - \sigma S_t Y_t \partial_2 \tilde{C}(t, Y_t) dB_t,$$

hence by Proposition 6.5 and Relation (6.4) we have

$$\xi_t = e^{r(T-t)} \tilde{C}(t, Y_t) - e^{r(T-t)} Y_t \partial_2 \tilde{C}(t, Y_t).$$

□

We also have

$$\begin{aligned}
\xi_t &= e^{r(T-t)} \sigma S_t \partial_2 C(t, S_t, Y_t) - \sigma Y_t \partial_3 C(t, S_t, Y_t) \\
&= e^{r(T-t)} \left(-Y_t \partial_2 \tilde{C}(t, Y_t) + \tilde{C}(t, Y_t) \right) \\
&= e^{r(T-t)} \left(S_t \frac{\partial}{\partial x} \tilde{C} \left(t, \frac{1}{x} \left(\frac{1}{T} \int_0^t S_u du - K \right) \right)_{|x=S_t} + \tilde{C}(t, Y_t) \right) \\
&= \frac{\partial}{\partial x} \left(x e^{r(T-t)} \tilde{C} \left(t, \frac{1}{x} \left(\frac{1}{T} \int_0^t S_u du - K \right) \right) \right)_{|x=S_t}, \quad t \in [0, T].
\end{aligned}$$

Note that the numerical computation of Asian option prices can be done using the probability density of $\int_0^T S_t dt$, cf. e.g. [8] and references therein.

Lookback Options

Let $m_s^t = \inf_{u \in [s,t]} S_u$, $M_s^t = \sup_{u \in [s,t]} S_u$, $0 \leq s \leq t \leq T$, and let $\mathcal{M}_s^t$ be either m_s^t or M_s^t. In the Lookback option case the payoff $H(S_T, \mathcal{M}_0^T)$ depends not only on the price of the underlying asset at maturity but also it depends on all prices of the underlying asset over the period which starts from the initial time and ends at maturity, and let Look_t be the price of the Lookback option given by

$$\mathrm{Look}_t = e^{-r(T-t)} \mathbb{E}[H(S_T, \mathcal{M}_0^T) | \mathcal{F}_t], \quad H \in \mathcal{C}_b^2(\mathbb{R}^2), \qquad t \in [0, T].$$

A standard Lookback call option is the right to buy the underlying asset at its historically lowest price. In this case the strike is m_0^T and the payoff is

$$G = S_T - m_0^T.$$

From [14], Proposition 4, page 271, the price Look_t at time t is given by

$$\begin{aligned}
\mathrm{Look}_t &= \mathbb{E}_Q[e^{-r(T-t)}(S_T - m_0^T) \Big| \mathcal{F}_t] \qquad (6.18) \\
&= S_t \Phi\left(d_t^T(m_0^t)\right) - e^{-r(T-t)} m_0^t \Phi\left(d_t^T(m_0^t) - \sigma\sqrt{T-t}\right) \\
&\quad + e^{-r(T-t)} \frac{S_t \sigma^2}{2r} \left(\left(\frac{S_t}{m_0^t} \right)^{\frac{-2r}{\sigma^2}} \Phi\left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma} \right) \right.
\end{aligned}$$

$$-e^{r(T-t)}\Phi\left(-d_t^T(m_0^t)\right)\Big).$$

Proposition 6.9. *The replicating portfolio of a Lookback option with payoff* $H(S_T, \mathcal{M}_0^T)$*, and price* $\mathrm{Look}_t = f(S_t, \mathcal{M}_0^t, t)$ *at time* t*, is given by*

$$\xi_t = e^{-r(T-t)}\partial_1 f(S_t, \mathcal{M}_0^t, t), \quad t \in [0,T], \tag{6.19}$$

where $f(x,y,t)$ *is given by*

$$f(S_t, \mathcal{M}_0^t, t) = \mathbb{E}[H(S_T, \mathcal{M}_0^T) \mid \mathcal{F}_t], \quad t \in [0,T].$$

Proof. It suffices to deal with the case $\mathcal{M}_s^t = m_s^t$. The existence of $f(x,y,t)$ follows from the Markov property, more precisely

$$f(x,y,t) = \mathbb{E}\left[H\left(S_{t,T}^x, y \wedge \mathcal{M}_t^T\right)\right].$$

Applying the change of variable formula, for $t \in [0,T]$, we have

$$\begin{aligned} df(S_t, \mathcal{M}_0^t, t) = &\left(\partial_3 f + rS_t\partial_1 f + \frac{1}{2}\sigma^2 S_t^2 \partial_1^2 f\right)(S_t, \mathcal{M}_0^t, t)dt \\ &+\partial_2 f(S_t, \mathcal{M}_0^t, t)d\mathcal{M}_0^t + \sigma S_t \partial_1 f(S_t, \mathcal{M}_0^t, t)dB_t. \end{aligned}$$

Since $\left(\mathbb{E}[H(S_T, \mathcal{M}_0^T)\Big|\mathcal{F}_t]\right)_{t\in[0,T]}$ is a P-martingale and $(\mathcal{M}_0^t)_{t\in[0,T]}$ has finite variation (it is in fact a non-increasing process), we have:

$$df(S_t, \mathcal{M}_0^t, t) = \sigma S_t \partial_1 f(S_t, \mathcal{M}_0^t, t)dB_t, \qquad t \in [0,T].$$

Then we have

$$e^{-rT}F = e^{-rT}\,\mathbb{E}[F] + \int_0^T e^{-rT}\sigma S_t \frac{\partial}{\partial x} f(x, \mathcal{M}_0^t, t)_{|x=S_t} dB_t, \qquad t \in [0,T],$$

which shows (6.19). □

See [2], § 2.6.1, page 29, for another approach to the following result using the Clark formula. Here we use Proposition 6.9 instead.

Proposition 6.10. *The hedging strategy for a standard Lookback call option is given by*

$$\begin{aligned} \xi_t = &\Phi(d_t^T(m_0^t)) - \frac{\sigma^2}{2r}\Phi(-d_t^T(m_0^t)) \\ &+e^{-r(T-t)}\left(\frac{S_t}{m_0^t}\right)^{\frac{-2r}{\sigma^2}}\left(\frac{\sigma^2}{2r} - 1\right)\Phi\left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma}\right). \end{aligned} \tag{6.20}$$

Proof. We need to compute the following derivatives:

$$
\begin{aligned}
&\frac{\partial}{\partial S_t}\left(\Phi\left(d_t^T(m_0^t)\right)\right)\\
&= \frac{\partial}{\partial S_t}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_t^T(m_0^t)} e^{-\frac{1}{2}u^2}du\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{\partial}{\partial S_t}\left(d_t^T(m_0^t)\right)\right)\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)\right)^2\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{\partial}{\partial S_t}\left(\frac{\log\frac{S_t}{m_0^t}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right)\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)\right)^2\right)\\
&= \frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)\right)^2\right),
\end{aligned}
$$

and

$$
\frac{\partial}{\partial S_t}\left(\Phi\left(d_t^T(m_0^t)-\sigma\sqrt{T-t}\right)\right) = \frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)-\sigma\sqrt{T-t}\right)^2\right).
$$

Similarly we have

$$
\begin{aligned}
&\frac{\partial}{\partial S_t}\left(\Phi\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)\right)\\
&= -\frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)^2\right),
\end{aligned}
$$

and

$$
\frac{\partial}{\partial S_t}\left(\Phi\left(-d_t^T(m_0^t)\right)\right) = -\frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)\right)^2\right).
$$

Finally,

$$
\frac{\partial}{\partial S_t}\left(\frac{S_t}{m_0^t}\right)^{-\frac{2r}{\sigma^2}} = \frac{-2r}{m_0^t\sigma^2}\left(\frac{S_t}{m_0^t}\right)^{-\frac{2r}{\sigma^2}-1}.
$$

The above expressions can be combined to compute the derivative of Look_t in (6.18), and to obtain:

$$
\begin{aligned}
&\xi_t = \Phi\left(d_t^T(m_0^t)\right) + \frac{1}{\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)\right)^2\right)\\
&-e^{-r(T-t)}m_0^t\frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)-\sigma\sqrt{T-t}\right)^2\right)\\
&+e^{-r(T-t)}\frac{\sigma^2}{2r}\left(\left(\frac{S_t}{m_0^t}\right)^{-\frac{2r}{\sigma^2}}\Phi\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&-e^{r(T-t)}\Phi\left(-d_t^T(m_0^t)\right)\Big) \\
&+e^{-r(T-t)}\frac{S_t\sigma^2}{2r}\left(\frac{-2r}{m_0^t\sigma^2}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2-1}\Phi\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)\right. \\
&-\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2}\frac{1}{\sigma S_t\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)^2\right) \\
&\left.+e^{r(T-t)}\frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)\right)^2\right)\right) \\
&=\Phi\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)\left(e^{-r(T-t)}\frac{\sigma^2}{2r}\left(\frac{S_t}{m_0^t}\right)^{-\frac{2r}{\sigma^2}}+e^{-r(T-t)}\frac{S_t\sigma^2}{2r}\right. \\
&\quad\left.\times\frac{-2r}{m_0^t\sigma^2}\left(\frac{S_t}{m_0^t}\right)^{-\frac{2r}{\sigma^2}-1}\right)+\Phi\left(d_t^T(m_0^t)\right) \\
&\quad-\Phi\left(-d_t^T(m_0^t)\right)\left(-e^{-r(T-t)}\frac{\sigma^2}{2r}e^{r(T-t)}\right) \\
&\quad+\frac{1}{\sigma\sqrt{2\pi(T-t)}}\left\{\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)\right)^2\right)\left(1+e^{-r(T-t)}\frac{S_t\sigma^2}{2r}\frac{e^{r(T-t)}}{S_t}\right)\right. \\
&\quad-e^{-r(T-t)}\left(\frac{m_0^t}{S_t}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)-\sigma\sqrt{T-t}\right)^2\right)\right. \\
&\quad\left.\left.+\frac{\sigma^2}{2r}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2}\exp\left(-\frac{1}{2}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)^2\right)\right)\right\} \\
&=e^{-r(T-t)}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2}\left(\frac{\sigma^2}{2r}-1\right)\Phi\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right) \\
&\quad+\Phi\left(d_t^T(m_0^t)\right)-\frac{\sigma^2}{2r}\Phi\left(-d_t^T(m_0^t)\right) \\
&\quad+\frac{1}{\sigma\sqrt{2\pi(T-t)}}\left\{\left(1+\frac{\sigma^2}{2r}\right)\exp\left(-\frac{1}{2}d_t^T(m_0^t)^2\right)\right. \\
&\quad-e^{-r(T-t)}\left(\frac{m_0^t}{S_t}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)-\sigma\sqrt{T-t}\right)^2\right)\right. \\
&\quad\left.\left.+\frac{\sigma^2}{2r}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2}\exp\left(-\frac{1}{2}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)^2\right)\right)\right\}.
\end{aligned}
$$

To obtain (6.20), it is sufficient to show that

$$
\begin{aligned}
0=&\left(1+\frac{\sigma^2}{2r}\right)\exp\left(-\frac{1}{2}d_t^T(m_0^t)^2\right) \\
&-e^{-r(T-t)}\left(\frac{m_0^t}{S_t}\exp\left(-\frac{1}{2}\left(d_t^T(m_0^t)-\sigma\sqrt{T-t}\right)^2\right)\right.
\end{aligned}
$$

$$+\frac{\sigma^2}{2r}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2}\exp\left(-\frac{1}{2}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)^2\right)\Bigg).$$

To see this, one can observe that

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\left(d_t^T(y)-\sigma\sqrt{T-t}\right)^2\right)\\
&=\exp\left(-\frac{1}{2}\left(\left(d_t^T(y)\right)^2+\sigma^2(T-t)-2d_t^T(y)\sigma\sqrt{T-t}\right)\right)\\
&=\exp\left(-\frac{1}{2}\left(d_t^T(y)\right)^2\right)\\
&\quad\times\exp\left(-\frac{1}{2}\left(\sigma^2(T-t)-2\left(\log\frac{S_t}{y}+(r+\frac{1}{2}\sigma^2)(T-t)\right)\right)\right)\\
&=\exp\left(-\frac{1}{2}\left(-2\log\frac{S_t}{y}-2r(T-t)\right)-\frac{1}{2}\left(d_t^T(y)\right)^2\right)\\
&=e^{r(T-t)}\frac{S_t}{y}\exp\left(-\frac{1}{2}\left(d_t^T(y)\right)^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\left(-d_t^T(y)+2r\sqrt{T-t}/\sigma\right)^2\right)\\
&=\exp\left(-\frac{1}{2}\left(d_t^T(y)\right)^2-\frac{1}{2}\left(\frac{4r^2}{\sigma^2}(T-t)-\frac{4r}{\sigma}d_t^T(y)\sqrt{T-t}\right)\right)\\
&=\exp\left(-\frac{1}{2}\left(d_t^T(y)\right)^2\right)\\
&\quad\times\exp\left(-\frac{1}{2}\left(\frac{4r^2}{\sigma^2}(T-t)-\frac{4r}{\sigma^2}\left(\log\frac{S_t}{y}+(r+\frac{1}{2}\sigma^2)(T-t)\right)\right)\right)\\
&=\exp\left(-\frac{1}{2}\left(d_t^T(y)\right)^2\right)\\
&\quad\times\exp\left(\frac{-2r^2}{\sigma^2}(T-t)+\frac{2r}{\sigma^2}\log\frac{S_t}{y}+\frac{2r^2}{\sigma^2}(T-t)+r(T-t)\right)\\
&=e^{r(T-t)}\left(\frac{S_t}{y}\right)^{\frac{2r}{\sigma^2}}\exp\left(-\frac{1}{2}\left(d_t^T(y)\right)^2\right).
\end{aligned}$$

□

Similar calculations using (6.19) are possible for other types of Lookback options, such as options on extrema and partial Lookback options, cf. [29].

6.6 Exercises

Exercise 6.1. In this problem, $(\eta_t, \xi_t)_{t\in[0,T]}$ denotes a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \le t \le T,$$

where S_t, resp. A_t, denotes the price at time t of a risky, resp. riskless, asset. Consider the price process $(S_t)_{t\in[0,T]}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

and a riskless asset of value $A_t = A_0 e^{rt}$, $t \in [0,T]$, with $r > 0$.

1. Compute the arbitrage price

$$C(t, S_t) = e^{-r(T-t)}\, \mathbb{E}_Q[|S_T|^2|\mathcal{F}_t],$$

   at time $t \in [0,T]$, of the contingent claim of payoff $|S_T|^2$.
2. Compute the portfolio strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ hedging the claim $|S_T|^2$.
3. Check that this strategy is self-financing.

Exercise 6.2. Again, $(\eta_t, \xi_t)_{t\in[0,T]}$ denotes a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \le t \le T,$$

where S_t, resp. A_t, denotes the price at time t of a risky, resp. riskless, asset.

1. Recall (without proof) the solution of the stochastic differential equation

$$dS_t = rS_t dt + \sigma dB_t. \tag{6.21}$$

2. Show that the discounted price process $\tilde{S}_t = e^{-rt} S_t$, $t \in [0,T]$, is a martingale under $\mathbb{P}$.
3. Compute the arbitrage price

$$C(t, S_t) = e^{-r(T-t)}\, \mathbb{E}[\exp(S_T)|\mathcal{F}_t]$$

   at time $t \in [0,T]$ of the contingent claim of $\exp(S_T)$.
4. Explicitly compute the portfolio strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ that hedges the contingent claim $\exp(S_T)$.
5. Check that this strategy is self-financing.

Exercise 6.3. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Recall that for $f \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R})$, Itô's formula for Brownian motion reads

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$f(t,B_t) = f(0,B_0) + \int_0^t \frac{\partial f}{\partial s}(s,B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)ds.$$

1. Let $r \in \mathbb{R}$, $\sigma > 0$, $f(x,t) = e^{rt+\sigma x-\sigma^2 t/2}$, and $S_t = f(t,B_t)$. Compute $df(t,B_t)$ by Itô's formula, and show that S_t solves the stochastic differential equation
$$dS_t = rS_t dt + \sigma S_t dB_t,$$
where $r > 0$ and $\sigma > 0$.
2. Show that
$$E[e^{\sigma B_T}|\mathcal{F}_t] = e^{\sigma B_t + \sigma^2(T-t)/2}, \qquad 0 \le t \le T.$$
Hint: use the independence of the increments of $B_T = (B_T - B_t) + (B_t - B_0)$ and the Laplace transform $E[e^{\alpha X}] = e^{\alpha^2\eta^2/2}$ when $X \simeq \mathcal{N}(0,\eta^2)$.
3. Show that the process $(S_t)_{t\in\mathbb{R}_+}$ satisfies
$$E[S_T|\mathcal{F}_t] = e^{r(T-t)}S_t, \qquad 0 \le t \le T.$$
4. Let $C = S_T - K$ denote the payoff of a forward contract with exercise price K and maturity T. Compute the discounted expected payoff
$$V_t := e^{-r(T-t)}E[C|\mathcal{F}_t].$$
5. Find a self-financing portfolio strategy $(\xi_t,\eta_t)_{t\in\mathbb{R}_+}$ such that
$$V_t = \xi_t S_t + \eta_t A_t, \qquad 0 \le t \le T,$$
where $A_t = A_0 e^{rt}$ is the price of a riskless asset with interest rate $r > 0$. Show that it recovers the result of Exercise 5.3-(3).
6. Show that the portfolio $(\xi_t,\eta_t)_{t\in[0,T]}$ found in Question 5 *hedges* the payoff $C = S_T - K$ at time T, i.e. show that $V_T = C$.

Exercise 6.4. Digital options. Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \qquad S_0 = 1,$$

under the risk-neutral measure $\mathbb{P}$.

The digital *call*, resp. *put*, option is a contract with maturity T, strike K, and payoff

$$C_d := \begin{cases} \$1 & \text{if } S_T \geq K, \\ 0 & \text{if } S_T < K, \end{cases} \qquad \text{resp.} \qquad P_d := \begin{cases} \$1 & \text{if } S_T \leq K, \\ 0 & \text{if } S_T > K. \end{cases}$$

Recall that the prices $\pi_t(C_d)$ and $\pi_t(P_d)$ at time t of the digital call and put options are given by the discounted expected payoffs

$$\pi_t(C_d) = e^{-r(T-t)}\,\mathbb{E}[C_d \mid \mathcal{F}_t] \qquad \text{and} \qquad \pi_t(P_d) = e^{-r(T-t)}\,\mathbb{E}[P_d \mid \mathcal{F}_t]. \tag{6.22}$$

1. Show that the payoffs C_d and P_d can be rewritten as

$$C_d = \mathbf{1}_{[K,\infty)}(S_T) \qquad \text{and} \qquad P_d = \mathbf{1}_{[0,K]}(S_T).$$

2. Using Relation (6.22) and Question 1, prove the call-put parity relation

$$\pi_t(C_d) + \pi_t(P_d) = e^{-r(T-t)}, \qquad 0 \leq t \leq T. \tag{6.23}$$

   If needed, you may use the fact that $\mathbb{P}(S_T = K) = 0$.
3. Using Relation (6.22), Question 1, and the relation

$$\mathbb{E}[\mathbf{1}_{[K,\infty)}(S_T) \mid S_t = x] = P(S_T \geq K \mid S_t = x),$$

   show that the price $\pi_t(C_d)$ is given by

$$\pi_t(C_d) = C_d(t, S_t),$$

   where $C_d(t,x)$ is the function defined by

$$C_d(t,x) = e^{-r(T-t)} P(S_T \geq K \mid S_t = x).$$

4. Using the results of Exercise 4.4-(3) and of Question 3, show that

$$C_d(t,x) = e^{-r(T-t)} \Phi\left(\frac{(r - \sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right),$$

   where $\tau = T - t$.
5. Show that the price $\pi_t(C_d)$ of the digital call option is given by

$$\pi_t(C_d) = e^{-r(T-t)} \Phi(d_-),$$

   where

$$d_- = \frac{(r - \sigma^2/2)\tau + \log(S_t/K)}{\sigma\sqrt{\tau}}.$$

6. Using the results of Questions 2 and 5, show that the price $\pi_t(P_d)$ of the digital put is given by

$$\pi_t(P_d) = e^{-r(T-t)} \Phi(-d_-).$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

7. Using the result of Question 5, compute the Delta

$$\xi_t := \frac{\partial C_d}{\partial x}(t, S_t)$$

of the digital call option. Does the Black-Scholes hedging strategy of such a call option involve short-selling ? Why ?

8. Using the result of Question 6, compute the Delta

$$\xi_t := \frac{\partial P_d}{\partial x}(t, S_t)$$

of the digital put option. Does the Black-Scholes hedging strategy of such a put option involve short-selling ? Why ?

Exercise 6.5.

1. Consider a market model made of a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ as in Exercise 4.8-(4) and a riskless asset with price $A_t = \$1 \times e^{rt}$ and riskless interest rate $r = \sigma^2/2$. From the answer to Exercise 4.8-(2), show that the arbitrage price

$$V_t = e^{-r(T-t)}\,\mathbb{E}[(\log S_T)^+|\mathcal{F}_t]$$

at time $t \in [0, T]$ of a log call option with payoff $(\log S_T)^+$ is equal to

$$V_t = \sigma e^{-r\tau}\sqrt{\frac{\tau}{2\pi}}e^{-\frac{B_t^2}{2\tau}} + \sigma e^{-r\tau}B_t\Phi\left(\frac{B_t}{\sqrt{\tau}}\right),$$

where $\tau = T - t$ denote the time to maturity.

2. Show that V_t can be written as

$$V_t = g(\tau, S_t),$$

where $g(\tau, x) = e^{-r\tau}f(\tau, \log x)$, and

$$f(\tau, y) = \sigma\sqrt{\frac{\tau}{2\pi}}e^{-\frac{y^2}{2\sigma^2\tau}} + y\Phi\left(\frac{y}{\sigma\sqrt{\tau}}\right).$$

3. Figure 6.1 represents the graph of $(\tau, x) \mapsto g(\tau, x)$, with $r = 0.05 = 5\%$ *per year* and $\sigma = 0.1$. Assume that the current underlying price is \$1 and there remains 700 days to maturity. What is the price of the option ?

Fig. 6.1: Option price as a function of the underlying and of time to maturity

4. Show[1] that the number $\Delta_t = \dfrac{\partial g}{\partial x}(\tau, S_t)$ of units of S_t at time t in a portfolio hedging the payoff $(\log S_T)^+$ is equal to

$$\Delta_t = e^{-r\tau}\frac{1}{S_t}\Phi\left(\frac{\log S_t}{\sigma\sqrt{\tau}}\right), \qquad 0 \le t \le T.$$

5. Figure 6.2 represents the graph of $(\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x)$. Assuming that the current underlying price is \$1 and that there remains 700 days to maturity, how many units of the risky asset should you hold in your portfolio in order to hedge one log option ?

Fig. 6.2: Delta as a function of the underlying and of time to maturity

[1] Recall the chain rule of derivation $\dfrac{\partial}{\partial x} f(\tau, \log x) = \dfrac{1}{x}\dfrac{\partial f}{\partial y}(\tau, y)_{|y=\log x}$.

6. Based on the framework and answers of Questions 3 and 5, should you borrow or lend the riskless asset $A_t = \$1 \times e^{rt}$, and for what amount ?
7. Show that the Gamma of the portfolio, defined as $\Gamma_t = \dfrac{\partial^2 g}{\partial x^2}(\tau, S_t)$, equals

$$\Gamma_t = e^{-r\tau}\frac{1}{S_t^2}\left(\frac{1}{\sigma\sqrt{2\pi\tau}}e^{\frac{-(\log S_t)^2}{2\sigma^2\tau}} - \Phi\left(\frac{\log S_t}{\sigma\sqrt{\tau}}\right)\right), \qquad 0 \le t \le T.$$

8. Figure 6.3 represents the graph of Gamma. Assume that there remains 60 days to maturity and that S_t, currently at \$1, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option ?

Fig. 6.3: Gamma as a function of the underlying and of time to maturity

9. Let now $\sigma = 1$. Show that the function $f(\tau, y)$ of Question 2 solves the *heat equation*

$$\begin{cases} \dfrac{\partial f}{\partial \tau}(\tau, y) = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial y^2}(\tau, y) \\ \\ f(0, y) = (y)^+. \end{cases}$$

Exercise 6.6.

1. Consider a market model made of a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ as in Exercise 4.7 and a riskless asset with price $A_t = \$1 \times e^{rt}$ and riskless interest rate $r = \sigma^2/2$. From the answer to Exercise 16.2-(2), show that the arbitrage price

$$V_t = e^{-r(T-t)}\,\mathbb{E}[(K - \log S_T)^+ | \mathcal{F}_t]$$

at time $t \in [0,T]$ of a log call option with strike K and payoff $(K - \log S_T)^+$ is equal to

$$V_t = \sigma e^{-r\tau}\sqrt{\frac{\tau}{2\pi}}e^{-\frac{(B_t - K/\sigma)^2}{2\tau}} + e^{-r\tau}(K - \sigma B_t)\Phi\left(\frac{K/\sigma - B_t}{\sqrt{\tau}}\right),$$

where $\tau = T - t$ denote the time to maturity.

2. Show that V_t can be written as

$$V_t = g(\tau, S_t),$$

where $g(\tau, x) = e^{-r\tau} f(\tau, \log x)$, and

$$f(\tau, y) = \sigma\sqrt{\frac{\tau}{2\pi}}e^{-\frac{(K-y)^2}{2\sigma^2\tau}} + (K - y)\Phi\left(\frac{K - y}{\sigma\sqrt{\tau}}\right).$$

3. Figure 6.4 represents the graph of $(\tau, x) \mapsto g(\tau, x)$, with $r = 0.125$ *per year* and $\sigma = 0.5$. Assume that the current underlying price is \$3 and there remains 700 days to maturity. What is the price of the option ?

Fig. 6.4: Option price as a function of the underlying and of time to maturity

4. Show[2] that the number $\Delta_t = \dfrac{\partial g}{\partial x}(\tau, S_t)$ of units of S_t at time t in a portfolio hedging the payoff $(K - \log S_T)^+$ is equal to

$$\Delta_t = -e^{-r\tau}\frac{1}{S_t}\Phi\left(\frac{K - \log S_t}{\sigma\sqrt{\tau}}\right), \qquad 0 \le t \le T.$$

5. Figure 6.5 represents the graph of $(\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x)$. Assuming that the current underlying price is \$3 and that there remains 700 days to matu-

[2] Recall the chain rule of derivation $\dfrac{\partial}{\partial x} f(\tau, \log x) = \dfrac{1}{x}\dfrac{\partial f}{\partial y}(\tau, y)_{|y=\log x}$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

rity, how many units of the risky asset should you hold in your portfolio in order to hedge one log option ?

Fig. 6.5: Delta as a function of the underlying and of time to maturity

6. Based on the framework and answers of Questions 3 and 5, should you borrow or lend the riskless asset $A_t = \$1 \times e^{rt}$, and for what amount ?
7. Show that the Gamma of the portfolio, defined as $\Gamma_t = \dfrac{\partial^2 g}{\partial x^2}(\tau, S_t)$, equals

$$\Gamma_t = e^{-r\tau}\frac{1}{S_t^2}\left(\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(K-\log S_t)^2}{2\sigma^2\tau}} + \Phi\left(\frac{K-\log S_t}{\sigma\sqrt{\tau}}\right)\right), \qquad 0 \le t \le T.$$

8. Figure 6.6 represents the graph of Gamma. Assume that there remains 10 days to maturity and that S_t, currently at \$3, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option ?

Fig. 6.6: Gamma as a function of the underlying and of time to maturity

9. Show that the function $f(\tau, y)$ of Question 2 solves the *heat equation*

$$\begin{cases} \dfrac{\partial f}{\partial \tau}(\tau, y) = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 f}{\partial y^2}(\tau, y) \\ \\ f(0, y) = (K - y)^+. \end{cases}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Chapter 7
Estimation of Volatility

The values of the parameters r, t, S_t, T, and K used in the Black-Scholes formula for the computation of a call option can easily be obtained from market data. Estimating the volatility coefficient σ can be a more difficult task, and several estimation methods are considered in this section.

7.1 Historical Volatility

We consider the problem of estimating the parameters μ and σ from market data in the stock price model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t.$$

A natural estimator for μ can be written as

$$\hat{\mu}_N = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_{k+1} - t_k} \frac{S_{t_k+1} - S_{t_k}}{S_{t_k}},$$

where $(S_{t_k+1} - S_{t_k})/S_{t_k}$, $k = 0, \ldots, N-1$ is a family of log-returns observed at different times $t_0, \ldots, t_N$ on the market.

Similarly the parameter σ can be estimated as by the estimator $\hat{\sigma}_N$ built as

$$\hat{\sigma}_N^2 = \frac{1}{N-1} \sum_{k=1}^{N} \frac{1}{t_{k+1} - t_k} \left(\frac{S_{t_k+1} - S_{t_k}}{S_{t_k}} - \mu(t_{k+1} - t_k) \right)^2.$$

Parameter estimation based on historical data requires a lot of samples and it can only be valid on a given time interval, or as a moving average.

7.2 Implied Volatility

In contrast with the historical volatility, the computation of the implied volatility can be done at a fixed time and requires much less data.

Recall that when $h(x) = (x - K)^+$, the solution of the Black-Scholes PDE is given by

$$f(t, x, \sigma, r, T) = x\Phi(d_+) - Ke^{-(T-t)r}\Phi(d_-),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

and

$$d_+ = \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_- = \frac{\log(x/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}.$$

Equating

$$f(t, S_t, \sigma, r, T) = M$$

to the observed value M of a given market price, when t, S_t, r, T are known, allows one to infer a value for σ. This value is called the implied volatility and denoted here by $\sigma^{\text{imp}}(K, T)$.

Given two European call options with strikes K_1, resp. K_2 and maturities T_1, resp. T_2, on the same stock S, this procedure should yield two estimates $\sigma^{\text{imp}}(K_1, T_1)$ and $\sigma^{\text{imp}}(K_2, T_2)$ of implied volatilites.

Clearly there is no reason for $\sigma^{\text{imp}}(K_1, T_1)$ and $\sigma^{\text{imp}}(K_2, T_2)$ of implied volatilites.
σ_1 and σ_2 to be equal. However in the standard Black-Scholes model the value of the parameter σ is unique for a given stock S, and this is a reason for the development of stochastic volatility models.

Plotting the different values of σ as a function of K and T will yield a planar curve called the volatility smile.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

7.3 The Black-Scholes Formula vs Market Data

On July 28, 2009 a call warrant has been issued by Merrill Lynch on the stock price S of Cheung Kong Holdings (0001.HK) with Strike K=$109.99 and Maturity T = December 13, 2010.

Fig. 7.1: Graph of the stock price of Cheung Kong Holdings.

The market price of the option (17838.HK) on September 28 was $12.30, as obtained from http://www.hkex.com.hk/dwrc/search/listsearch.asp

The next graph shows the evolution of the market price of the option over time. One sees that the option price is much more volatile than the underlying stock price.

Fig. 7.2: Graph of the call option price on Cheung Kong Holdings.

We may compare this graph with that of the corresponding Black-Scholes price

$$t \mapsto g_c(t, S_t)$$

plotted in Figure 7.3 using the stock price data of Figure 7.1.

Fig. 7.3: Graph of the Black-Scholes call option price on Cheung Kong Holdings.

Another example

Let us consider the stock price of HSBC Holdings (0005.HK) over one year:

Fig. 7.4: Graph of the stock price of HSBC Holdings.

Next we consider the graph of the price of a call option issued by Societe Generale on 31 December 2008 with strike K=$63.704, maturity T = October 05, 2009, and entitlement ratio 100.

Fig. 7.5: Graph of the call option price on HSBC Holdings.

Again, we may compare this graph with the corresponding Black-Scholes price

$$t \mapsto g_c(t, S_t)$$

plotted in Figure 7.6 as a function of the stock price data of Figure 7.4.

Fig. 7.6: Graph of the Black-Scholes call option price on HSBC Holdings.

In this case we are in the money at maturity, and we also check that the option is worth $100 \times 0.2650 = \$26.650$ at that time which, by absence of arbitrage, is very close to the value \$90 - \$63.703 = \$26.296 of its payoff.

For one more example, consider the graph of the price of a put option issued by BNP Paribas on 04 November 2008 with strike K=\$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593.

Fig. 7.7: Graph of the put option price on HSBC Holdings.

One checks easily that at maturity, the put option is worth \$0.01 which almost equals its payoff \$0, by absence of arbitrage opportunities. We may compare this graph with the corresponding Black-Scholes put price

$$t \mapsto g_p(t, S_t)$$

plotted in Figure 7.8 as a function of the stock price data of Figure 7.6.

Fig. 7.8: Graph of the Black-Scholes put option price on HSBC Holdings.

7.4 Local Volatility

Since the constant volatility assumption in the Black-Scholes model appears to be not satisfying due to the existence of volatility smiles, it makes sense to consider models of the form

$$\frac{dS_t}{S_t} = rdt + \sigma_t dB_t$$

where σ_t is a random process. Such models are called stochastic volatility models.

A particular class of stochastic volatility models can be written as

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dB_t \tag{7.1}$$

where $\sigma(t, x)$ is a deterministic function of time and the stock price. Such models are called local volatility models. The corresponding Black-Scholes PDE can be written as

$$\begin{cases} rg(t,x,K) = \dfrac{\partial g}{\partial t}(t,x,K) + rx\dfrac{\partial g}{\partial x}(t,x,K) + \dfrac{1}{2}x^2\sigma^2(t,x)\dfrac{\partial^2 g}{\partial x^2}(t,x,K), \\ \\ g(T,x,K) = (x-K)^+, \end{cases} \tag{7.2}$$

with terminal condition $g(T, x, K) = (x - K)^+$, i.e. we consider European call options.

Note that the Black-Scholes PDE would allow one to recover the value of $\sigma(t, x)$ as a function of the option price $g(t, x, K)$, as

$$\sigma(t,x) = \sqrt{\frac{2rg(t,x,K) - 2\dfrac{\partial g}{\partial t}(t,x,K) - 2rx\dfrac{\partial g}{\partial x}(t,x,K)}{x^2\dfrac{\partial^2 g}{\partial x^2}(t,x,K)}}, \qquad x, t > 0,$$

however this formula requires the knowledge of the option price for different values of the underlying x.

The Dupire formula brings a solution to the local volatility calibration problem by providing as estimator of $\sigma(t, x)$ based on different values of the strike price K.

Proposition 7.1. *Assume that a family $(C(T, K))_{T,K>0}$ of market call option prices with maturities T and strikes K is given at time t with $S_t = x$, while the values of r and x are fixed.*

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

The Dupire formula states that defining the volatility function $\sigma(t,y)$ by

$$\sigma(t,y) = \sqrt{\frac{2\dfrac{\partial C}{\partial t}(t,y) + 2ry\dfrac{\partial C}{\partial y}(t,y)}{y^2\dfrac{\partial^2 C}{\partial y^2}(t,y)}} \tag{7.3}$$

the prices $g(t,x,K)$ computed from the Black-Scholes PDE (7.2) will match the option prices $C(T,K)$ in the sense that

$$g(T,x,K) = C(T,K), \qquad T,K > 0. \tag{7.4}$$

Proof. We use the probabilistic approach that allows us to write $g(T,x,K)$ as

$$g(T,x,K) = e^{-rT}\,\mathbb{E}[(S_T - K)^+ \mid S_t = x], \tag{7.5}$$

where $(S_t)_{t\in\mathbb{R}_+}$ is defined by (7.1). Hence the condition (7.4) can be written as

$$C(T,K) = e^{-rT}\int_{-\infty}^{\infty}(y-K)^+\varphi_T(y)dy,$$

where $\varphi_T(y)$ is the probability density of S_T. After differentiating both sides twice with respect to K one gets

$$\frac{\partial^2 C}{\partial K^2}(T,K) = e^{-rT}\varphi_T(K). \tag{7.6}$$

On the other hand, for any sufficiently smooth function f, using the Itô formula we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\varphi_T(y)f(y)dy = \mathbb{E}[f(S_T)]\\
&= \mathbb{E}\left[f(S_0) + \int_0^T f'(S_t)dS_t + \frac{1}{2}\int_0^T f''(S_t)\sigma^2(t,S_t)dt\right]\\
&= f(S_0) + \mathbb{E}\left[r\int_0^T f'(S_t)S_t dt + \frac{1}{2}\int_0^T f''(S_t)S_t^2\sigma^2(t,S_t)dt\right]\\
&= f(S_0) + r\int_{-\infty}^{\infty}\int_0^T yf'(y)dt\varphi_t(y)dy + \frac{1}{2}\int_{-\infty}^{\infty}\int_0^T y^2 f''(y)\sigma^2(t,y)dt\varphi_t(y)dy,
\end{aligned}$$

hence after differentiation with respect to T,

$$\int_{-\infty}^{\infty}\frac{\partial\varphi_T}{\partial T}(y)f(y)dy = r\int_{-\infty}^{\infty}yf'(y)\varphi_T(y)dy + \frac{1}{2}\int_{-\infty}^{\infty}y^2 f''(y)\sigma^2(T,y)\varphi_T(y)dy.$$

Integrating by parts in the above relation yields

$$\int_{-\infty}^{\infty} \frac{\partial \varphi_T}{\partial T}(y) f(y) dy$$
$$= -r \int_{-\infty}^{\infty} f(y) \frac{\partial}{\partial y}(y\varphi_T(y)) dy + \frac{1}{2} \int_{-\infty}^{\infty} f(y) \frac{\partial^2}{\partial y^2}(y^2\sigma^2(T,y)\varphi_T(y)) dy,$$

for all smooth functions $f(y)$ with compact support, hence

$$\frac{\partial \varphi_T}{\partial T}(y) = -r \frac{\partial}{\partial y}(y\varphi_T(y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2}(y^2\sigma^2(T,y)\varphi_T(y)), \qquad y \in \mathbb{R}.$$

Making use of (7.6) we get

$$-r \frac{\partial^2 C}{\partial y^2}(T,y) - \frac{\partial}{\partial T} \frac{\partial^2 C}{\partial y^2}(T,y)$$
$$= r \frac{\partial}{\partial y} \left(y \frac{\partial^2 C}{\partial y^2}(T,y) \right) - \frac{1}{2} \frac{\partial^2}{\partial y^2} \left(y^2 \sigma^2(T,y) \frac{\partial^2 C}{\partial y^2}(T,y) \right), \qquad y \in \mathbb{R}.$$

After a first integration with respect to y under the limiting condition $\lim_{K \to +\infty} C(T,K) = 0$, we obtain

$$-r \frac{\partial C}{\partial y}(T,y) - \frac{\partial}{\partial T} \frac{\partial C}{\partial y}(T,y) = ry \frac{\partial^2 C}{\partial y^2}(T,y) - \frac{1}{2} \frac{\partial}{\partial y} \left(y^2 \sigma^2(T,y) \frac{\partial^2 C}{\partial y^2}(T,y) \right),$$

i.e.

$$-r \frac{\partial C}{\partial y}(T,y) - \frac{\partial}{\partial T} \frac{\partial C}{\partial y}(T,y)$$
$$= r \frac{\partial}{\partial y} \left(y \frac{\partial C}{\partial y}(T,y) \right) - r \frac{\partial C}{\partial y}(T,y) - \frac{1}{2} \frac{\partial}{\partial y} \left(y^2 \sigma^2(T,y) \frac{\partial^2 C}{\partial y^2}(T,y) \right),$$

or

$$-\frac{\partial}{\partial T} \frac{\partial C}{\partial y}(T,y) = r \frac{\partial}{\partial y} \left(y \frac{\partial C}{\partial y}(T,y) \right) - \frac{1}{2} \frac{\partial}{\partial y} \left(y^2 \sigma^2(T,y) \frac{\partial^2 C}{\partial y^2}(T,y) \right).$$

Integrating one more time with respect to y yields

$$-\frac{\partial C}{\partial T}(T,y) = ry \frac{\partial C}{\partial y}(T,y) - \frac{1}{2} y^2 \sigma^2(T,y) \frac{\partial^2 C}{\partial y^2}(T,y), \qquad y \in \mathbb{R},$$

which conducts to (7.3) and is called the Dupire [18] PDE. □

Note that the local volatility can be estimated given a value of implied volatility σ, after computing $C(T,K)$ from the Black-Scholes formula using σ. See [1] for numerical methods applied to volatility estimation in this framework.

Chapter 8
American Options

8.1 Stopping Times and Martingales

We consider the *filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, i.e. an increasing family of σ-algebras, generated by a stochastic process $(X_t)_{t\in\mathbb{R}_+}$, in which $\mathcal{F}_t$ denotes all events possibly generated by $\{X_s \ : \ 0 \leq s \leq t\}$. Examples of such events include the event

$$\{X_{t_0} \leq a_0,\ X_{t_1} \leq a_1,\ \ldots,\ X_{t_n} \leq a_n\}$$

for $a_0, a_1, \ldots, a_n$ a given fixed sequence of real numbers and $0 \leq t_1 < \cdots < t_n < t$. Note that we have the inclusion $\mathcal{F}_s \subset \mathcal{F}_t$, $0 \leq s \leq t$.

One refers to $\mathcal{F}_t$ as the *information* generated by $(X_s)_{s\in\mathbb{R}_+}$ up to time t.

Let us recall the definition of *martingale* and introduce the definition of *super*martingale and *sub*martingale.

Definition 8.1. An integrable stochastic process $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale (resp. a *super*martingale, resp. a *sub*martingale) with respect to $(\mathcal{F}_t)_{t\imath\mathbb{R}_+}$ if it satisfies the property

$$Z_s = \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leq s \leq t,$$

resp.

$$Z_s \geq \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leq s \leq t,$$

resp.

$$Z_s \leq \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leq s \leq t.$$

A particular property of martingales is that their expectation is constant.

Proposition 8.1. *Let $(Z_t)_{t\in\mathbb{R}_+}$ be a martingale. Then we have*

$$\mathbb{E}[Z_t] = \mathbb{E}[Z_s], \qquad 0 \leq s \leq t.$$

Similarly, a *super*martingale has a *decreasing* expectation, while a *sub*martingale has a *increasing* expectation.

Proposition 8.2. *Let* $(Z_t)_{t\in\mathbb{R}_+}$ *be a* super*martingale, resp. a* sub*martingale. Then we have*

$$\mathbb{E}[Z_t] \leq \mathbb{E}[Z_s], \qquad 0 \leq s \leq t,$$

resp.

$$\mathbb{E}[Z_t] \geq \mathbb{E}[Z_s], \qquad 0 \leq s \leq t.$$

Proof. From the tower property of conditional expectations we have:

$$\mathbb{E}[Z_t] = \mathbb{E}[\mathbb{E}[Z_t \mid \mathcal{F}_s]] \leq \mathbb{E}[Z_s], \qquad 0 \leq s \leq t.$$

The proof is similar in the *sub*martingale case. □

Independent increments processes whose increments have negative expectation give examples of *super*martingales. For example, if $(X_t)_{t\in\mathbb{R}_+}$ is such a process then we have

$$\begin{aligned}
\mathbb{E}[X_t \mid \mathcal{F}_s] &= \mathbb{E}[X_s \mid \mathcal{F}_s] + \mathbb{E}\left[X_t - X_s \mid \mathcal{F}_s\right] \\
&= \mathbb{E}[X_s \mid \mathcal{F}_s] + \mathbb{E}[X_t - X_s] \\
&\leq \mathbb{E}[X_s \mid \mathcal{F}_s] \\
&= X_s, \qquad 0 \leq s \leq t.
\end{aligned}$$

Similarly, a process with independent increments which have positive expectation will be a *sub*martingales. Brownian motion $B_t + \mu t$ with positive drift $\mu > 0$ is such an example, as in Figure 8.1 below.

Fig. 8.1: Drifted Brownian path.

This version: March 8, 2011
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The following example comes from gambling.

Fig. 8.2: Evolution of the fortune of a poker player according to the number of games played.

A natural way to construct *sub*martingales is to take convex functions of martingales. Indeed, if $(M_t)_{t\in\mathbb{R}_+}$ is a martingale and ϕ is a convex function, Jensen's inequality states that

$$\phi(\mathbb{E}[M_t \mid \mathcal{F}_s]) \leq \mathbb{E}[\phi(M_t) \mid \mathcal{F}_s], \qquad 0 \leq s \leq t, \tag{8.1}$$

which shows that

$$\phi(M_s) = \phi(\mathbb{E}[M_t \mid \mathcal{F}_s]) \leq \mathbb{E}[\phi(M_t) \mid \mathcal{F}_s], \qquad 0 \leq s \leq t,$$

i.e. $(\phi(M_t))_{t\in\mathbb{R}_+}$ is a *sub*martingale. More generally, the above shows that $\phi(M_t)_{t\in\mathbb{R}_+}$ remains a *sub*martingale when ϕ is convex non-decreasing and $(M_t)_{\in\mathbb{R}_+}$ is a *sub*martingale. Similarly, $(\phi(M_t))_{t\in\mathbb{R}_+}$ will be *super*martingale when $(M_t)_{\in\mathbb{R}_+}$ is a martingale and the function ϕ is *concave*.

Other examples of (super, sub)-martingales include the geometric Brownian motion

$$S_t = S_0 e^{rt+\sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

which is a martingale for $r = 0$, a *super*martingale for $r \leq 0$, and a *sub*martingale for $r \geq 0$.

Next we turn to the definition of *stopping time*.

Definition 8.2. A stopping time is a random variable $\tau : \Omega \to \mathbb{R}_+$ such that

$$\{\tau > t\} \in \mathcal{F}_t, \qquad t \in \mathbb{R}_+. \tag{8.2}$$

The meaning of Relation (8.2) is that the knowledge of the event $\{\tau > t\}$ depends only on the information present in $\mathcal{F}_t$ up to time t, i.e. on the knowledge of $(X_s)_{0 \leq s \leq t}$.

Note that a constant is always a stopping time, and if τ and θ are stopping times then $\tau \wedge \theta$ is also a stopping time, since

$$\{\tau \wedge \theta > t\} = \{\tau > t \quad \text{and} \quad \theta > t\} = \{\tau > t\} \cap \{\theta > t\} \in \mathcal{F}_t, \qquad t \in \mathbb{R}_+.$$

Hitting times provide natural examples of stopping times.

For example, the hitting time

$$\tau_x = \inf\{t \in \mathbb{R}_+ 0 \ : \ X_t = x\}$$

of $x \in \mathbb{R}$ is a stopping time, as we have (here in discrete time)

$$\begin{aligned}\{\tau_x > t\} &= \{X_s \neq x, \ 0 \leq s \leq t\} \\ &= \{X_0 \neq x\} \cap \{X_1 \neq x\} \cap \cdots \cap \{X_t \neq x\} \in \mathcal{F}_t, \qquad t \in \mathbb{N}.\end{aligned}$$

However, not every $\mathbb{R}_+$-valued random variable is a stopping time. For example the random time

$$\tau = \inf\left\{t \in [0,T] \ : \ X_t = \sup_{s \in [0,T]} X_s\right\},$$

which represents the first time the process $(X_t)_{t \in [0,T]}$ reaches its maximum over $[0,T]$, is not a stopping time with respect to the filtration generated by $(X_t)_{t \in [0,T]}$. Indeed, the information known at time $t \in (0,T)$ is not sufficient to determine whether $\{\tau > t\}$.

The following Theorem 8.1 is called the *stopping time* (or *optional sampling*, or *optional stopping*) theorem, it is due to the mathematician J.L. Doob (1910-2004).

Given $(Z_t)_{t \in \mathbb{R}_+}$ a stochastic process and $\tau : \Omega \to \mathbb{N}$ a stopping time, the *stopped process* $(Z_{t \wedge \tau})_{t \in \mathbb{R}_+}$ is defined as

$$Z_{t \wedge \tau} = \begin{cases} Z_\tau \text{ if } \tau \leq t, \\ \\ Z_t \text{ if } \tau > t, \end{cases}$$

Using indicator functions we may also write

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$Z_{t\wedge\tau} = Z_\tau \mathbf{1}_{\{\tau\le t\}} + Z_t \mathbf{1}_{\{\tau>t\}}, \qquad t\in\mathbb{R}_+.$$

The following figure is an illustration of the path of a stopped process.

Fig. 8.3: Stopped process

Theorem 8.1. Assume that $(M_t)_{t\in\mathbb{R}_+}$ is a martingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Then the *stopped process* $(M_{t\wedge\tau})_{t\in\mathbb{R}_+}$ is also a martingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

Proof. We only give the proof in discrete time.[1] Writing

$$M_n = M_0 + \sum_{l=1}^{n}(M_l - M_{l-1}) = M_0 + \sum_{l=1}^{\infty}\mathbf{1}_{\{l\le n\}}(M_l - M_{l-1}),$$

we have

$$M_{\tau\wedge n} = M_0 + \sum_{l=1}^{\tau\wedge n}(M_l - M_{l-1}) = M_0 + \sum_{l=1}^{\infty}\mathbf{1}_{\{l\le \tau\wedge n\}}(M_l - M_{l-1}),$$

and for $k\le n$,

$$\begin{aligned}\mathbb{E}[M_{\tau\wedge n}\mid\mathcal{F}_k] &= M_0 + \sum_{l=1}^{\infty}\mathbb{E}[\mathbf{1}_{\{l\le\tau\wedge n\}}(M_l - M_{l-1})\mid\mathcal{F}_k]\\ &= M_0 + \sum_{l=1}^{k}\mathbb{E}[\mathbf{1}_{\{l\le\tau\wedge n\}}(M_l - M_{l-1})\mid\mathcal{F}_k]\end{aligned}$$

[1] This proof can be skipped.

$$
\begin{aligned}
&+ \sum_{l=k+1}^{\infty} \mathbb{E}[\mathbf{1}_{\{l \leq \tau \wedge n\}}(M_l - M_{l-1}) \mid \mathcal{F}_k] \\
&= M_0 + \sum_{l=1}^{k} (M_l - M_{l-1})\, \mathbb{E}[\mathbf{1}_{\{l \leq \tau \wedge n\}} \mid \mathcal{F}_k] \\
&\quad + \sum_{l=k+1}^{\infty} \mathbb{E}[\mathbb{E}[(M_l - M_{l-1})\mathbf{1}_{\{l \leq \tau \wedge n\}} \mid \mathcal{F}_{l-1}] \mid \mathcal{F}_k] \\
&= M_0 + \sum_{l=1}^{k} (M_l - M_{l-1})\mathbf{1}_{\{l \leq \tau \wedge n\}} \\
&\quad + \sum_{l=k+1}^{\infty} \mathbb{E}[\mathbf{1}_{\{l \leq \tau \wedge n\}}\, \mathbb{E}[(M_l - M_{l-1}) \mid \mathcal{F}_{l-1}] \mid \mathcal{F}_k] \\
&= M_0 + \sum_{l=1}^{\tau \wedge n \wedge k} (M_l - M_{l-1})\mathbf{1}_{\{l \leq \tau \wedge n\}} \\
&= M_0 + \sum_{l=1}^{\tau \wedge k} (M_l - M_{l-1})\mathbf{1}_{\{l \leq \tau \wedge n\}} \\
&= M_{\tau \wedge k}, \qquad 0 \leq k \leq n.
\end{aligned}
$$

□

By Theorem 8.1 we know that the stopped process $(M_{\tau \wedge t})_{t \in \mathbb{R}_+}$ is a martingale. Since the stopped process $(M_{\tau \wedge t})_{t \in \mathbb{R}_+}$ is a martingale by Theorem 8.1 we find that its expectation is constant by Proposition 8.1. More generally, if $(M_t)_{t \in \mathbb{R}_+}$ is a *super*martingale with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, then the *stopped process* $(M_{t \wedge \tau})_{t \in \mathbb{R}_+}$ remains a *super*martingale with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$.

As a consequence, if τ is a stopping time bounded by $T > 0$ we have

$$
\mathbb{E}[M_\tau] = \mathbb{E}[M_{\tau \wedge T}] = \mathbb{E}[M_{\tau \wedge 0}] = \mathbb{E}[M_0], \qquad 0 \leq t \leq T.
$$

In case τ is not bounded we may also write

$$
\mathbb{E}[M_\tau] = \mathbb{E}[\lim_{t \to \infty} M_{\tau \wedge t}] = \lim_{t \to \infty} \mathbb{E}[M_{\tau \wedge t}] = \mathbb{E}[M_0], \tag{8.3}
$$

provided the limit and expectation signs can be exchanged, however this may not be always the case.[2] Dealing with unbounded stopping times can be necessary in the case of hitting times.

In addition, if τ and ν are two *bounded* stopping times such that $\tau \leq \nu$, a.s., we have

[2] In some situations the exchange of limit and expectation signs can be difficult to justify.

$$\mathbb{E}[M_\tau] \geq \mathbb{E}[M_\nu] \tag{8.4}$$

if $(M_t)_{t\in\mathbb{R}_+}$ is a *super*martingale, and

$$\mathbb{E}[M_\tau] \leq \mathbb{E}[M_\nu] \tag{8.5}$$

if $(M_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale, cf. Exercise 8 below for a proof in discrete time. As a consequence of (8.4) and (8.5), if τ and ν are two *bounded* stopping times such that $\tau \leq \nu$, a.s., we have

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_\nu] \tag{8.6}$$

if $(M_t)_{t\in\mathbb{R}_+}$ is a martingale.

Relations (8.4), (8.5) and (8.6) can be extended to general stopping times along the same lines and conditions as (8.3).

Nevertheless, for all stopping times τ (bounded or unbounded) we may write

$$\mathbb{E}[M_\tau] = \mathbb{E}[\lim_{t\to\infty} M_{\tau\wedge t}] \leq \lim_{t\to\infty} \mathbb{E}[M_{\tau\wedge t}] \leq \lim_{t\to\infty} \mathbb{E}[M_0] = \mathbb{E}[M_0], \tag{8.7}$$

provided $(M_t)_{t\in\mathbb{R}_+}$ is a nonnegative *super*martingale, where we used Fatou's lemma.[3]

As counterexample to (8.6), the random time

$$\tau = \inf\left\{ t \in [0,T] \ : \ M_t = \sup_{s\in[0,T]} M_s \right\},$$

which is not a stopping time, will satisfy

$$\mathbb{E}[M_\tau] > \mathbb{E}[M_T],$$

although $\tau \leq T$ almost surely.

In the table below we summarize some of the results of this section.

[3] $\mathbb{E}[\lim_{n\to\infty} F_n] \leq \lim_{n\to\infty} \mathbb{E}[F_n]$ for all sequences $(F_n)_{n\in\mathbb{N}}$ of nonnegative random variables.

		stopping times $\tau \leq \nu$
M_t	*super*martingale	$\mathbb{E}[M_\tau] \geq \mathbb{E}[M_\nu]$
	martingale	$\mathbb{E}[M_\tau] = \mathbb{E}[M_\nu]$
	*sub*martingale	$\mathbb{E}[M_\tau] \leq \mathbb{E}[M_\nu]$

Examples of application

In this section we will show that, as an application of the stopping time theorem, a number of expectations can be computed in a simple and elegant way.

Brownian motion hitting a barrier

Given $a, b \in \mathbb{R}$, $a < b$, let the hitting[4] time $\tau_{a,b} : \Omega \to \mathbb{R}_+$ be defined by

$$\tau_{a,b} = \inf\{t \geq 0 \ : \ B_t = a \text{ or } B_t = b\},$$

which is the hitting time of the boundary $\{a, b\}$ of Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, $a, b \in \mathbb{R}$, $a < b$.

Recall that Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale since it has independent increments, and those increments are centered:

$$\mathbb{E}[B_t - B_s] = 0, \qquad 0 \leq s \leq t.$$

Consequently, $(B_{\tau_{a,b}\wedge t})_{t\in\mathbb{R}_+}$ is still a martingale and by (8.3) we have

$$\mathbb{E}[B_{\tau_{a,b}} \mid B_0 = x] = \mathbb{E}[B_0 \mid B_0 = x] = x,$$

hence

$$x = \mathbb{E}[B_{\tau_{a,b}} \mid B_0 = x] = a \times \mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) + b \times \mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x),$$

under the additional condition

$$\mathbb{P}\left(X_{\tau_{a,b}} = a \mid X_0 = x\right) + \mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = 1.$$

which yields

[4] A hitting time is a stopping time

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) = \frac{x-a}{b-a}, \qquad a \le x \le b,$$

which also shows that

$$\mathbb{P}(B_{\tau_{a,b}} = 0 \mid B_0 = x) = \frac{b-x}{b-a}, \qquad a \le x \le b.$$

Note that the above result actually applies to any continuous martingale, and not only to Brownian motion.

Drifted Brownian motion hitting a barrier

Next, let us turn to the case of drifted Brownian motion

$$X_t = x + B_t + \mu t, \qquad t \in \mathbb{R}_+.$$

In this case the process $(X_t)_{t\in\mathbb{R}_+}$ is no longer a martingale and in order to use Theorem 8.1 we need to construct a martingale of a different type. Here we note that the process

$$M_t := e^{\sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

is a martingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Indeed, we have

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}\left[e^{\sigma B_t - \sigma^2 t/2}\middle|\mathcal{F}_s\right] = e^{\sigma B_s - \sigma^2 s/2}, \qquad 0 \le s \le t.$$

By Theorem 8.1 we know that the stopped process $(M_{\tau_{a,b}\wedge t})_{t\in\mathbb{R}_+}$ is a martingale, hence its expectation is constant by Proposition 8.1, and (8.3) gives

$$1 = \mathbb{E}[M_0] = \mathbb{E}[M_{\tau_{a,b}}].$$

Next we note that letting $\sigma = -2\mu$ we have

$$e^{\sigma X_t} = e^{\sigma x + \sigma B_t + \sigma\mu t} = e^{\sigma x + \sigma B_t - \sigma^2 t/2} = e^{\sigma x} M_t,$$

hence

$$\begin{aligned} 1 &= \mathbb{E}[M_{\tau_{a,b}}] \\ &= e^{-\sigma x}\,\mathbb{E}[e^{\sigma X_{\tau_{a,b}}}] \\ &= e^{\sigma(a-x)}\mathbb{P}\left(X_{\tau_{a,b}} = a \mid X_0 = x\right) + e^{\sigma(b-x)}\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x), \end{aligned}$$

under the additional condition

$$\mathbb{P}\left(X_{\tau_{a,b}} = a \mid X_0 = x\right) + \mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = 1.$$

Finally this gives

$$\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) = \frac{e^{\sigma x} - e^{\sigma b}}{e^{\sigma a} - e^{\sigma b}} = \frac{e^{-2\mu x} - e^{-2\mu b}}{e^{-2\mu a} - e^{-2\mu b}}, \tag{8.8}$$

$a \leq x \leq b$, and

$$\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = \frac{e^{-2\mu a} - e^{-2\mu x}}{e^{-2\mu a} - e^{-2\mu b}}, \qquad a \leq x \leq b.$$

Mean hitting time for Brownian motion

The martingale method also allows us to compute the expectation $\mathbb{E}[B_{\tau_{a,b}}]$, after checking that $(B_t^2 - t)_{t\in\mathbb{R}_+}$ is also a martingale. Indeed we have

$$\begin{aligned}
&\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] = \mathbb{E}[(B_s + (B_t - B_s))^2 - t \mid \mathcal{F}_s] \\
&= \mathbb{E}[B_s^2 + (B_t - B_s)^2 + 2B_s(B_t - B_s) - t \mid \mathcal{F}_s] \\
&= \mathbb{E}[B_s^2 - s \mid \mathcal{F}_s] - (t-s) + \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + 2\,\mathbb{E}[B_s(B_t - B_s) \mid \mathcal{F}_s] \\
&= B_s^2 - s - (t-s) + \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + 2B_s\,\mathbb{E}[B_t - B_s \mid \mathcal{F}_s] \\
&= B_s^2 - s - (t-s) + \mathbb{E}[(B_t - B_s)^2] + 2B_s\,\mathbb{E}[B_t - B_s] \\
&= B_s^2 - s, \qquad 0 \leq s \leq t.
\end{aligned}$$

Consequently the stopped process $(B^2_{\tau_{a,b}\wedge t} - \tau_{a,b} \wedge t)_{t\in\mathbb{R}_+}$ is still a martingale by Theorem 8.1 hence the expectation $\mathbb{E}[B^2_{\tau_{a,b}\wedge t} - \tau_{a,b} \wedge t]$ is constant in $t \in \mathbb{R}_+$, hence by (8.3) we get

$$\begin{aligned}
x^2 &= \mathbb{E}[B^2_{\tau_{a,b}} - \tau_{a,b} \mid B_0 = x] \\
&= \mathbb{E}[B^2_{\tau_{a,b}} \mid B_0 = x] - \mathbb{E}[\tau_{a,b} \mid B_0 = x] \\
&= b^2\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) + a^2\mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) - \mathbb{E}[\tau_{a,b} \mid B_0 = x],
\end{aligned}$$

i.e.

$$\begin{aligned}
\mathbb{E}[\tau_{a,b} \mid B_0 = x] &= b^2\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) + a^2\mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) - x^2 \\
&= b^2\frac{x-a}{b-a} + a^2\frac{b-x}{b-a} - x^2 \\
&= (x-a)(b-x), \qquad a \leq x \leq b.
\end{aligned}$$

Mean hitting time for drifted Brownian motion

Finally we show how to recover the value of the mean hitting time of drifted Brownian motion $X_t = B_t + \mu t$. As above, the process $X_t - \mu t$ is a martingale the stopped process $(X_{\tau_{a,b}\wedge t} - \mu(\tau_{a,b} \wedge t))_{t\in\mathbb{R}_+}$ is still a martingale by The-

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

orem 8.1. Hence the expectation $\mathbb{E}[X_{\tau_{a,b}\wedge t} - \mu(\tau_{a,b}\wedge t)]$ is constant in $t \in \mathbb{R}_+$.

Since the stopped process $(X_{\tau_{a,b}\wedge t} - \mu t)_{t\in\mathbb{R}_+}$ is a martingale, we have

$$x = \mathbb{E}[X_{\tau_{a,b}} - \mu\tau_{a,b} \mid X_0 = x],$$

which gives

$$\begin{aligned} x &= \mathbb{E}[X_{\tau_{a,b}} - \mu\tau_{a,b} \mid X_0 = x] \\ &= \mathbb{E}[X_{\tau_{a,b}} \mid X_0 = x] - \mu\,\mathbb{E}[\tau_{a,b} \mid X_0 = x] \\ &= b\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) + a\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) - \mu\,\mathbb{E}[\tau_{a,b} \mid X_0 = x], \end{aligned}$$

i.e.

$$\begin{aligned} \mu\,\mathbb{E}[\tau_{a,b} \mid X_0 = x] &= b\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) + a\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) - x \\ &= b\frac{e^{2\mu a} - e^{2\mu x}}{e^{2\mu a} - e^{2\mu b}} + a\frac{e^{2\mu x} - e^{2\mu b}}{e^{2\mu a} - e^{2\mu b}} - x \\ &= \frac{b(e^{2\mu a} - e^{2\mu x}) + a(e^{2\mu x} - e^{2\mu b}) - x(e^{2\mu a} - e^{2\mu b})}{e^{2\mu a} - e^{2\mu b}}, \end{aligned}$$

from (8.8), hence

$$\mathbb{E}[\tau_{a,b} \mid X_0 = x] = \frac{b(e^{2\mu a} - e^{2\mu x}) + a(e^{2\mu x} - e^{2\mu b}) - x(e^{2\mu a} - e^{2\mu b})}{\mu(e^{2\mu a} - e^{2\mu b})},$$

$a \le x \le b$.

8.2 Perpetual American Options

The price of an American put option expiring at time $T > 0$ with strike K can be expressed as the expected value of its discounted payoff:

$$f(t, S_t) = \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)^+ \middle| S_t\right],$$

under the risk-neutral probability measure $\mathbb{P}^*$, where the supremum is taken over stopping times between t and a fixed maturity T. Similarly, the price of a finite expiration American call option with strike K is expressed as

$$f(t, S_t) = \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)^+ \middle| S_t\right].$$

In this section we take $T = +\infty$, in which case we refer to these options as *perpetual* options, and the corresponding put and call are respectively priced

as

$$f(t,S_t) = \sup_{\substack{\tau \geq t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau-t)}(K-S_\tau)^+ \middle| S_t \right],$$

and

$$f(t,S_t) = \sup_{\substack{\tau \geq t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau-t)}(S_\tau-K)^+ \middle| S_t \right].$$

Two-choice optimal stopping at a fixed price level for perpetual put options

In this section we consider the pricing of perpetual put options. Given $L \in (0,K)$ a fixed price, consider the following choices for the exercise of a *put* option with strike K:

1. If $S_t \leq L$, then exercise at time t.
2. Otherwise, wait until the first hitting time

$$\tau_L = \inf\{u \geq t \ : \ S_u = L\}$$

and exercise the option at time τ_L.

In case $S_t \leq L$, the payoff will be

$$(K-S_t)^+ = K - S_t$$

since $K > L \geq S_t$, however in this case one would buy the option at price $K - S_t$ only to exercise it immediately for the same amount.

In case $S_t > L$, the price of the option will be

$$\begin{aligned} f_L(t,S_t) &= \mathbb{E}^* \left[e^{-r(\tau_L-t)}(K-S_{\tau_L})^+ \middle| S_t \right] \\ &= \mathbb{E}^* \left[e^{-r(\tau_L-t)}(K-L)^+ \middle| S_t \right] \\ &= (K-L)\, \mathbb{E}^* \left[e^{-r(\tau_L-t)} \middle| S_t \right]. \end{aligned} \tag{8.9}$$

We note that the starting date t does not matter when pricing perpetual options, hence $f_L(t,x) = f_L(x)$ is actually independent of $t \in \mathbb{R}_+$, and we take $t = 0$ in the remaining of this section. Recall that the underlying asset price is written as

$$S_t = S_0 e^{rt+\sigma \tilde{B}_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+, \tag{8.10}$$

where $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, r is the risk-free interest rate, and $\sigma > 0$ is the volatility coefficient.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Proposition 8.3. *Assume that $r \geq 0$. We have*

$$\begin{aligned} f_L(x) &= \mathbb{E}^* \left[e^{-r(\tau_L - t)} (K - S_{\tau_L})^+ \middle| S_t = x \right] \\ &= \begin{cases} K - x, & 0 < x \leq L, \\ (K - L) \left(\dfrac{x}{L} \right)^{-2r/\sigma^2}, & x \geq L. \end{cases} \end{aligned} \tag{8.11}$$

Proof. The result is obvious for $x \leq L$, so that we focus on the case $x \geq L$. We note that from (8.23), for all $\lambda \in \mathbb{R}$ the process $(Z_t)_{t \in \mathbb{R}_+}$ defined as

$$Z_t := \left(\frac{S_t}{S_0} \right)^\lambda e^{-r\lambda t + \lambda \sigma^2 t/2 - \lambda^2 \sigma^2 t/2} = e^{\lambda \sigma \tilde{B}_t - \lambda^2 \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, hence we have[5]

$$\mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*[Z_0] = 1,$$

which yields

$$\mathbb{E}^* \left[\left(\frac{S_{\tau_L}}{S_0} \right)^\lambda e^{-(r\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2)\tau_L} \right] = 1,$$

i.e.

$$\mathbb{E}^* \left[e^{-r\tau_L} \right] = \left(\frac{S_0}{L} \right)^\lambda, \tag{8.12}$$

provided we choose λ such that

$$-(r\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2) = -r,$$

i.e.

$$0 = \lambda^2\sigma^2/2 + \lambda(r - \sigma^2/2) - r = \frac{\sigma^2}{2}(\lambda + 2r/\sigma^2)(\lambda - 1). \tag{8.13}$$

This equation admits two solutions and we choose the negative solution $\lambda = -2r/\sigma^2$ since $S_0 = x > L$ and the expectation $\mathbb{E}^*[e^{-r\tau_L}] > 1$ in (8.12) is lower than 1 as $r \geq 0$. □

We note that taking $L = K$ would yield a payoff always equal to 0 for the option holder, hence the value of L should be strictly lower than K. On the other hand, if $L = 0$ the value of τ_L will be infinite almost surely, hence the option price will be 0 when $r \geq 0$ from (17.3). Therefore there should be an optimal value L^*, which should be strictly comprised between 0 and K.

Figure 8.1 shows for $K = 100$ that there exists an optimal value $L^* = 85.71$ which maximizes the option price for all values of the underlying.

[5] Here the exchange of limit and expectation can be justified by monotone convergence, cf. p. 347 of [49].

Fig. 8.1: Graphs of the option price by exercising at τ_L for several values of L.

In order to compute L^* we observe that, geometrically, the slope of $f_L(x)$ at $x = L^*$ is equal to -1, i.e.

$$f'_{L^*}(L^*) = -\frac{2r}{\sigma^2}(K - L^*)\frac{(L^*)^{-2r/\sigma^2-1}}{(L^*)^{-2r/\sigma^2}} = -1,$$

i.e.

$$\frac{2r}{\sigma^2}(K - L^*) = L^*,$$

or

$$L^* = \frac{2r}{2r + \sigma^2}K < K,$$

cf. [49] page 351 for another derivation.

The next figure is a 2-dimensional animation that also shows the optimal value L^* of L.

Fig. 8.2: Animated graph of the option price depending on the values of L.

The next figure gives another view of the put option prices according to different values of L, with the optimal value L^*.

Fig. 8.3: Graph of the option price as a function of L and of the underlying.

In Figure 8.4 which is based on the stock price of HSBC Holdings (0005.HK) over year 2009, the optimal exercise strategy for an American put option with strike K=\$77.67 would have been to exercise whenever the underlying price goes above $L^* = \$62$, i.e. at approximately 54 days, for a payoff of \$38. Note that waiting a longer time, e.g. until 85 days, would have yielded a higher payoff of at least \$65. This is due to the fact that, here, optimization

is done based on the past information only and makes sense in expectation (or average) over all possible future paths.

Fig. 8.4: Path of the American put option price on the HSBC stock.

We can check by hand that

$$f_{L^*}(x) = \begin{cases} K - x, & 0 < x \leq L^* = \dfrac{2r}{2r+\sigma^2}K, \\ \\ \dfrac{K\sigma^2}{2r+\sigma^2}\left(\dfrac{2r+\sigma^2}{2r}\dfrac{x}{K}\right)^{-2r/\sigma^2}, & x \geq L^* = \dfrac{2r}{2r+\sigma^2}K, \end{cases}$$

satisfies the PDE

$$rf_{L^*}(x) - rxf'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) = \begin{cases} rK, \, 0 < x \leq L^* < K, \\ \\ 0, \quad x > L^*. \end{cases} \tag{8.14}$$

in addition to the condition

$$\begin{cases} f_{L^*}(x) = K - x, \qquad 0 < x < L^* < K, \\ \\ f_{L^*}(x) > (K-x)^+, \, x \geq L^*. \end{cases}$$

This can be summarized in the following variational PDE:

$$\begin{cases} f_{L^*}(x) \geq (K-x)^+, & (8.15a) \\ rxf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) \leq rf_{L^*}(x), & (8.15b) \\ \left(rf_{L^*}(x) - rxf'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x)\right)(f_{L^*}(x) - (K-x)^+) = 0, & (8.15c) \end{cases}$$

which admits an interpretation in terms of absence of arbitrage, as shown below.

By Itô's formula the discounted portfolio price $\tilde{f}_{L^*}(S_t) = e^{-rt} f_{L^*}(S_t)$ satisfies

$$\begin{aligned} d(\tilde{f}_{L^*}(S_t)) = &\left(-rf_{L^*}(S_t) + rS_t f'_{L^*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 f''_{L^*}(S_t)\right) e^{-rt} dt \\ &+ e^{-rt}\sigma S_t f'_{L^*}(S_t) d\tilde{B}_t, \end{aligned} \tag{8.16}$$

hence from Equation (8.15c), when $f_{L^*}(S_t) > (K - S_t)^+$, i.e. $S_t > L^*$, the expected return of the option equals that of the risk-free asset since the investor prefers to wait.

On the other hand if $f_{L^*}(S_t) = (K - S_t)^+$, i.e. $0 < S_t < L^*$, it is not worth waiting as (8.15b) and (8.15c) show that the return of the option is lower than that of the risk-free asset, i.e.:

$$-rf_{L^*}(S_t) + rS_t f'_{L^*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 f''_{L^*}(S_t) < 0,$$

and exercise becomes immediate.

In view of the above derivation it should make sense to assert that $f_{L^*}(S_t)$ is the price at time t of the perpetual American put option. The next proposition shows that this is indeed the case, and that the optimal exercise time is $\tau^* = \tau_{L^*}$.

Proposition 8.4. *The price of the perpetual American put option is given for all $t \geq 0$ by*

$$\begin{aligned} f_{L^*}(S_t) &= \sup_{\substack{\tau \geq t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau - t)} (K - S_\tau)^+ \middle| S_t \right] \\ &= \mathbb{E}^* \left[e^{-r(\tau_{L^*} - t)} (K - S_{\tau_{L^*}})^+ \middle| S_t \right] \end{aligned}$$

$$= \begin{cases} K - S_t, \quad 0 < S_t \leq L^*, \\ \\ \dfrac{K\sigma^2}{2r+\sigma^2}\left(\dfrac{2r+\sigma^2}{2r}\dfrac{S_t}{K}\right)^{-2r/\sigma^2}, \qquad S_t \geq L^*. \end{cases}$$

Proof. By Itô's formula (8.16) one checks that the discounted portfolio price

$$t \mapsto e^{-rt} f_{L^*}(S_t)$$

is a supermartingale. As a consequence, for all stopping times τ we have, by (8.7),

$$\begin{aligned} e^{-rt} f_{L^*}(S_t) &\geq \mathbb{E}^* \left[e^{-r\tau} f_{L^*}(S_\tau) \middle| S_t \right] \\ &\geq \mathbb{E}^* \left[e^{-r\tau} (K - S_\tau)^+ \middle| S_t \right], \end{aligned}$$

from (8.15a), which implies

$$e^{-rt} f_{L^*}(S_t) \geq \sup_{\substack{\tau \geq t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r\tau} (K - S_\tau)^+ \middle| S_t \right]. \tag{8.17}$$

On the other hand, taking $\tau = \tau_{L^*}$ we note that the process

$$t \mapsto e^{-rt \wedge \tau_{L^*}} f_{L^*}(S_{t \wedge \tau_{L^*}})$$

is not only a supermartingale, it is also a martingale since $S_{T \wedge \tau_{L^*}} \geq L^*$ and from (8.14), hence after letting T go to infinity we obtain

$$\begin{aligned} e^{-rt} f_{L^*}(S_t) &= \mathbb{E}^* \left[e^{-r\tau_{L^*}} f_{L^*}(S_{\tau_{L^*}}) \middle| S_t \right] \\ &= \mathbb{E}^* \left[e^{-r\tau_{L^*}} f_{L^*}(L^*) \middle| S_t \right] \\ &= \mathbb{E}^* \left[e^{-r\tau_L} (K - S_\tau)^+ \middle| S_t \right], \end{aligned}$$

which shows that

$$e^{-rt} f_{L^*}(S_t) \leq \sup_{\substack{\tau \geq t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r\tau} (K - S_\tau)^+ \middle| S_t \right], \qquad t \geq 0,$$

and the conclusion follows from (8.17). □

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Two-choice optimal stopping at a fixed price level for perpetual call options

In this section we consider the pricing of perpetual call options. Given $L > K$ a fixed price, consider the following choices for the exercise of a *call* option with strike K:

1. If $S_t \geq L$, then exercise at time t.
2. Otherwise, wait until the first hitting time

$$\tau_L = \inf\{u \geq t \ : \ S_u = L\}$$

   and exercise the option and time τ_L.

In case $S_t \geq L$, the payoff will be

$$(S_t - K)^+ = S_t - K$$

since $K < L \leq S_t$.

In case $S_t < L$, the price of the option will be

$$\begin{aligned} f_L(t, S_t) &= \mathbb{E}^* \left[e^{-r(\tau_L - t)} (S_{\tau_L} - K)^+ \middle| S_t \right] \\ &= \mathbb{E}^* \left[e^{-r(\tau_L - t)} (L - K)^+ \middle| S_t \right] \\ &= (L - K)\, \mathbb{E}^* \left[e^{-r(\tau_L - t)} \middle| S_t \right]. \end{aligned}$$

Proposition 8.5. *We have*

$$f_L(x) = \begin{cases} x - K, & x \geq L > K, \\ (L - K)\dfrac{x}{L}, & 0 < x \leq L. \end{cases} \tag{8.18}$$

Proof. We note that for all $\lambda \in \mathbb{R}$,

$$Z_t := \left(\frac{S_t}{S_0} \right)^\lambda e^{-r\lambda t + \lambda\sigma^2 t/2 - \lambda^2\sigma^2 t/2} = e^{\lambda\sigma\tilde{B}_t - \lambda^2\sigma^2 t/2}$$

is a martingale under the risk-neutral probability measure $\tilde{\mathbb{P}}$. Hence the stopped process $(Z_{t\wedge\tau_L})_{t\in\mathbb{R}_+}$ is a martingale and it has constant expectation. Hence we have

$$\mathbb{E}^*[Z_{t\wedge\tau_L}] = \mathbb{E}^*[Z_0] = 1,$$

and by letting t go to infinity we get

$$\mathbb{E}^*\left[\left(\frac{S_{\tau_L}}{S_0}\right)^{\lambda} e^{-(r\lambda-\lambda\sigma^2/2+\lambda^2\sigma^2/2)\tau_L}\right] = 1,$$

which yields

$$\mathbb{E}^*\left[e^{-(r\lambda-\lambda\sigma^2/2+\lambda^2\sigma^2/2)\tau_L}\right] = \left(\frac{S_0}{L}\right)^{\lambda},$$

i.e.

$$\mathbb{E}^*\left[e^{-r\tau_L}\right] = \left(\frac{S_0}{L}\right)^{\lambda},$$

provided that λ is chosen such that

$$-(r\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2) = -r,$$

i.e.

$$0 = \lambda^2\sigma^2/2 + \lambda(r - \sigma^2/2) - r = \frac{\sigma^2}{2}(\lambda + 2r/\sigma^2)(\lambda - 1).$$

Here we choose the positive solution $\lambda = 1$ since $S_0 = x > L$ and the expectation is lower than 1. □

One sees from Figure 8.5 that the situation completely differs from the perpetual put option case, as there do not exist an optimal value L^* that would maximize the option price for all values of the underlying.

Fig. 8.5: Graphs of the option price by exercising at τ_L for several values of L.

The intuition behind this picture is that there is no upper limit above which one should exercise the option, and in order to price the American perpetual call option we have to take $L \to \infty$ and the "optimal" exercise strategy is to wait indefinitely.

Fig. 8.6: Graphs of the option prices parametrized by different values of L.

We check from (8.18) that

$$\lim_{L\to infty} f_L(x) = x - \lim_{L\to\infty} K\frac{x}{L} = x, \qquad x > 0. \tag{8.19}$$

As a consequence we have the following proposition.

Proposition 8.6. *The price of the perpetual American call option is given by*

$$\sup_{\substack{\tau\geq t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)^+\Big|S_t\right] = S_t, \qquad t\in\mathbb{R}_+. \tag{8.20}$$

Proof. For all $L > K$ we have

$$\begin{aligned} f_L(S_t) &= \mathbb{E}^*\left[e^{-r(\tau_L-t)}(S_{\tau_L} - K)^+\Big|S_t\right] \\ &\leq \sup_{\substack{\tau\geq t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)^+\Big|S_t\right], \qquad t\geq 0, \end{aligned}$$

hence taking the limit as $L\to\infty$ yields

$$S_t \leq \sup_{\substack{\tau\geq t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)^+\Big|S_t\right] \tag{8.21}$$

from (8.19). On the other hand, for all stopping times $\tau\geq t$ we have, by (8.7),

$$\mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)^+\Big|S_t\right] \leq \mathbb{E}^*\left[e^{-r(\tau-t)}S_\tau\Big|S_t\right] \leq S_t, \qquad t\geq 0,$$

since $u\mapsto e^{-r(u-t)}S_u$ is a martingale, hence

$$\sup_{\substack{\tau \geq t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau-t)} (S_\tau - K)^+ \middle| S_t \right] \leq S_t, \qquad t \geq 0,$$

which shows (8.20) by (8.21). □

We may also check that since $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, the process $t \mapsto (e^{-rt}S_t - K)^+$ is a *sub*martingale since the function $x \mapsto (x-K)^+$ is convex, hence for all bounded stopping times $\tau \geq t$ we have

$$(S_t - K)^+ \leq \mathbb{E}^* \left[(e^{-r(\tau-t)} S_\tau - K)^+ \middle| S_t \right] \leq \mathbb{E}^* \left[e^{-r(\tau-t)} (S_\tau - K)^+ \middle| S_t \right],$$

$t \geq 0$, showing that it is always better to wait than to exercise at time t, and the optimal exercise time is $\tau^* = +\infty$.

8.3 Finite Expiration American Options

In this section we consider finite expirations American put and call options with strike K, whose prices can be expressed as

$$f(t, S_t) = \sup_{\substack{t \leq \tau \leq T \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau-t)} (K - S_\tau)^+ \middle| S_t \right],$$

and

$$f(t, S_t) = \sup_{\substack{t \leq \tau \leq T \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau-t)} (S_\tau - K)^+ \middle| S_t \right].$$

Two-choice optimal stopping at fixed times with finite expiration

We start by considering the optimal stopping problem in a simplified setting where $\tau \in \{t, T\}$ is allowed at time t to take only *two* values t (which corresponds to immediate exercise) and T (wait until maturity).

Since $x \mapsto (x-K)^+$ is a non-decreasing convex function and the $(S_t)_{t\in\mathbb{R}_+}$ is a submartingale under $\mathbb{P}^*$, the process $t \mapsto (S_t - K)^+$ remains a submartingale, hence we have

$$(S_t - K)^+ \leq e^{-r(T-t)} \, \mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t],$$

i.e.

$$(x - K)^+ \leq e^{-r(T-t)} \, \mathbb{E}^*[(S_T - K)^+ | S_t = x], \qquad x, t > 0,$$

as illustrated in Figure 8.1 using the Black-Scholes formula for European call options.

This version: March 8, 2011
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In other words, taking $x = S_t$, the payoff $(S_t - K)^+$ of immediate exercise at time t is always lower than the expected payoff $e^{-r(T-t)}\,\mathbb{E}^*[(S_T - K)^+ | S_t = x]$ given by exercise at maturity T. As a consequence, the optimal strategy for the investor is to wait until time T to exercise an American call option, rather than exercising earlier at time t.

Fig. 8.1: Expected Black-Scholes European call price vs payoff function $(x, t) \mapsto (x - K)^+$.

More generally it can be in fact shown that the price of an American call option equals the price of the corresponding European call option with maturity T, i.e.

$$f(t, S_t) = e^{-r(T-t)}\,\mathbb{E}^*\left[(S_T - K)^+ \middle| S_t\right],$$

i.e. T is the optimal exercise date, cf. e.g. §14.4 of [50] for a proof.

For put options the situation is entirely different. The Black-Scholes formula for European put options shows that the inequality

$$(K - x)^+ \leq e^{-r(T-t)}\,\mathbb{E}^*[(K - S_T)^+ | S_t = x],$$

does not always hold, as illustrated in Figure 8.2.

Fig. 8.2: Comparison of the Black-Scholes put price function with the payoff $(x,t) \mapsto (K-x)^+$.

As a consequence, the optimal exercise decision for a put option depends on whether $(K-S_t)^+ \leq e^{-r(T-t)}\,\mathbb{E}^*[(K-S_T)^+|S_t]$ (in which case one chooses to exercise at time T) or $(K-S_t)^+ > e^{-r(T-t)}\,\mathbb{E}^*[(K-S_T)^+|S_t]$ (in which case one chooses to exercise at time t).

A view from above of the graph of Figure 8.2 shows the existence of an optimal frontier depending on time to maturity and on the value of the underlying asset, cf. Figure 8.3:

Fig. 8.3: Optimal frontier for exercise of a put option.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

At a given time t, one will choose to exercise immediately if $(S_t, T-t)$ belongs to the blue area on the right, and to wait until maturity if $(S_t, T-t)$ belongs to the red area on the left.

PDE characterization of the finite expiration American put price

Let us describe the PDE associated to American put options. After discretization $\{0 = t_0, t_1, \ldots, t_N = T\}$ of the time interval $[0, T]$, the optimal exercise strategy for an American put option can be described as follow at each time step:

If $f(t, S_t) > (K - S_t)^+$, wait.
If $f(t, S_t) = (K - S_t)^+$, exercise the option at time t.

Note that we cannot have $f(t, S_t) < (K - S_t)^+$.

If $f(t, S_t) > (K - S_t)^+$ the expected return of the option equals that of the risk-free asset. This means that $f(t, S_t)$ follows the Black-Scholes PDE

$$rf(t, S_t) = \frac{\partial f}{\partial t}(t, S_t) + rS_t \frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t),$$

whereas if $f(t, S_t) = (K - S_t)^+$ it is not worth waiting as the return of the option is lower than that of the risk-free asset:

$$rf(t, S_t) \geq \frac{\partial f}{\partial t}(t, S_t) + rS_t \frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t).$$

As a consequence, $f(t, x)$ should solve the following variational PDE:

$$\begin{cases} f(t,x) \geq (K-x)^+, & \text{(8.22a)} \\[2ex] \dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 f}{\partial x^2}(t,x) \leq rf(t,x), & \text{(8.22b)} \\[2ex] \left(\dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 f}{\partial x^2}(t,x) - rf(t,x)\right) & \text{(8.22c)} \\ \qquad\qquad \times \left(f(t,x) - (K-x)^+\right) = 0, & \end{cases}$$

subject to the terminal condition $f(T, x) = (K - x)^+$. In other words, equality holds either in (8.22a) or in (8.22b) due to the presence of the term $(f(t, x) - (K - x)^+)$ in (8.22c).

The optimal exercise strategy consists in exercising the put option as soon as the equality $f(u, S_u) = (K - S_u)^+$ holds, i.e. at

$$\tau^* = \inf\{u \geq t \ : \ f(u, S_u) = (K - S_u)^+\}.$$

A simple procedure to compute numerically the price of an American put option is to use a finite difference scheme while simply enforcing the condition $f(t,x) \geq (K-x)^+$ at iteration, adding the condition

$$f(t_i, x_j) := \max(f(t_i, x_j), (K - x_j)^+)$$

right after the computation of $f(t_i, x_j)$.

The next figure shows a numerical resolution of the variational PDE (8.22a)-(8.22c) using the above simplified (implicit) finite difference scheme. In comparison with Figure 8.4, one can check that the PDE solution becomes time-dependent in the finite expiration case.

Fig. 8.4: Numerical values of the finite expiration American put price.

In general, one will choose to exercise the put option when

$$f(t, S_t) = (K - S_t)^+,$$

i.e. within the blue area in Figure (8.4). We check that the optimal threshold $L^* = 90.64$ of the corresponding perpetual put option is within the exercise region, which is consistent since the perpetual optimal strategy should allow one to wait longer than in the finite expiration case.

The numerical computation of the put price

$$f(t, S_t) = \sup_{\substack{t \le \tau \le T \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau - t)} (K - S_\tau)^+ \middle| S_t \right]$$

can also be done by dynamic programming and backward optimization using the Longstaff-Schwartz (or Least Square Monte Carlo, LSM) algorithm [32], as in Figure 8.5.

Fig. 8.5: Longstaff-Schwartz algorithm applied to the finite expiration American put price.

In Figure 8.5 above and Figure 8.6 below the optimal threshold of the corresponding perpetual put option is again $L^* = 90.64$ and falls within the exercise region. Also, the optimal threshold is closer to L^* for large time to maturities, which shows that the perpetual option approximates the finite expiration option in that situation. In the next Figure 8.6 we compare the numerical computation of the American put price by the finite difference and Longstaff-Schwartz methods.

Fig. 8.6: Longstaff-Schwartz algorithm applied to the finite expiration American put price.

It turns out that, although both results are very close, the Longstaff-Schwartz method performs better in the critical area close to exercise at it yields the expected continuously differentiable solution, and the simple numerical PDE solution tends to underestimate the optimal threshold. Also, a small error in the values of the solution translates into a large error on the value of the optimal exercise threshold.

The finite expiration American call option

In the next proposition we compute the price of a finite expiration American call option with an arbitrary convex payoff function ϕ.

Proposition 8.7. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function such that $\phi(0) = 0$. The price of the finite expiration American call option with payoff function ϕ on the underlying asset $(S_t)_{t\in\mathbb{R}_+}$ is given by*

$$f(t, S_t) = \sup_{\substack{t\leq\tau\leq T \\ \tau \ \text{stopping time}}} \mathbb{E}^* \left[e^{-r(\tau-t)}\phi(S_\tau) \middle| S_t \right] = e^{-r(T-t)}\, \mathbb{E}^* \left[\phi(S_T) \middle| S_t \right],$$

i.e. the optimal strategy is to wait until the maturity time T to exercise the option, and $\tau^* = T$.

Proof. Since the function ϕ is convex and $\phi(0) = 0$ we have

$$\phi(px) = \phi((1-p) \times 0 + px) \leq (1-p) \times \phi(0) + p\phi(x) \leq p\phi(x),$$

for all $p \in [0, 1]$ and $x \geq 0$. Hence the process $s \mapsto e^{-rs}\phi(S_{t+s})$ is a *sub*martingale since taking $p = e^{-r(\tau-t)}$ we have

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\begin{aligned}\mathbb{E}^*\left[e^{-rs}\phi\left(S_{t+s}\right) \mid \mathcal{F}_t\right] &\geq e^{-rs}\phi\left(\mathbb{E}^*\left[S_{t+s} \mid \mathcal{F}_t\right]\right)\\ &\geq \phi\left(\mathbb{E}^*\left[e^{-rs}S_{t+s} \mid \mathcal{F}_t\right]\right)\\ &= \phi(S_t),\end{aligned}$$

where we used Jensen's inequality (8.1) applied to the convex function ϕ. Hence by the optional sampling theorem for *sub*martingales, cf (8.5), for all (bounded) stopping times τ comprised between t and T we have,

$$\mathbb{E}^*[e^{-r(\tau-t)}\phi(S_\tau) \mid \mathcal{F}_t] \leq e^{-r(T-t)}\,\mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t],$$

i.e. it is always better to wait until time T than to exercise at time $\tau \in [t, T]$, and this yields

$$\sup_{\substack{t\leq\tau\leq T\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}\phi(S_\tau)\Big|S_t\right] \leq e^{-r(T-t)}\,\mathbb{E}^*\left[\phi(S_T)\Big|S_t\right].$$

The converse inequality

$$e^{-r(T-t)}\,\mathbb{E}^*\left[\phi(S_T)\Big|S_t\right] \leq \sup_{\substack{t\leq\tau\leq T\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}\phi(S_\tau)\Big|S_t\right],$$

being obvious because T is a stopping time. □

As a consequence of Proposition 8.7 applied to the convex function $\phi(x) = (x-K)^+$, the price of the finite expiration American call option is given by

$$f(t,S_t) = \sup_{\substack{t\leq\tau\leq T\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau-K)^+\Big|S_t\right] = e^{-r(T-t)}\,\mathbb{E}^*\left[(S_T-K)^+\Big|S_t\right],$$

i.e. the optimal strategy is to wait until the maturity time T to exercise the option.

In the following table we summarize some results on the pricing of American options.

option type	perpetual	finite expiration
put option	$\begin{cases} K-S_t, & 0<S_t\leq L^*,\\ (K-L^*)\left(\dfrac{S_t}{L^*}\right)^{-2r/\sigma^2}, & S_t\geq L^*.\end{cases}$ $\tau^* = \tau_{L^*}$	solve the PDE for $f(t,S_t)$ or use Longstaff-Schwartz [32] $\tau^* = \inf\{u\geq t : f(u,S_u) = (K-S_u)^+\}$
call option	S_t $\tau^* = +\infty$	$e^{-r(T-t)}\,\mathbb{E}^*[(S_T-K)^+ \mid S_t]$, $\tau^* = T$

8.4 Exercises

Exercise 8.1. American digital options. An American digital call (resp. put) option with maturity $T > 0$ can be exercised at any time $t \in [0, T]$, at the choice of the option holder.

The call (resp. put) option exercised at time t yields the payoff $\mathbf{1}_{[K,\infty)}(S_t)$ (resp. $\mathbf{1}_{[0,K]}(S_t)$), and the option holder wants to find an exercise strategy that will maximize his payoff.

1. Consider the following possible situations at time t:

   (i) $S_t \geq K$,
   (ii) $S_t < K$.

   In each case (i) and (ii), tell whether you would choose to exercise the <u>call</u> option immediately or to wait.
2. Consider the following possible situations at time t:

   (i) $S_t > K$,
   (ii) $S_t \leq K$.

   In each case (i) and (ii), tell whether you would choose to exercise the <u>put</u> option immediately or to wait.
3. The price $C_d^{\mathrm{Am}}(t, S_t)$ of an American digital call option is known to satisfy the Black-Scholes PDE

   $$rC_d^{\mathrm{Am}}(t,x) = \frac{\partial}{\partial t} C_d^{\mathrm{Am}}(t,x) + rx\frac{\partial}{\partial x} C_d^{\mathrm{Am}}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial}{\partial x} C_d^{\mathrm{Am}}(t,x).$$

   Based on your answers to Question 1, how would you set the boundary conditions $C_d^{\mathrm{Am}}(t, K)$, $0 \leq t < T$, and $C_d^{\mathrm{Am}}(T, x)$, $0 \leq x < K$?
4. The price $P_d^{\mathrm{Am}}(t, S_t)$ of an American digital put option is known to satisfy the same Black-Scholes PDE

   $$rP_d^{\mathrm{Am}}(t,x) = \frac{\partial}{\partial t} P_d^{\mathrm{Am}}(t,x) + rx\frac{\partial}{\partial x} P_d^{\mathrm{Am}}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial}{\partial x} P_d^{\mathrm{Am}}(t,x).$$

   Based on your answers to Question 2, how would you set the boundary conditions $P_d^{\mathrm{Am}}(t, K)$, $0 \leq t < T$, and $P_d^{\mathrm{Am}}(T, x)$, $x > K$?
5. Show that the optimal exercise strategy for the American digital call option with strike K is to exercise as soon as the underlying reaches the level K, at the time

   $$\tau_K = \inf\{u \geq t \ : \ S_u = K\},$$

   starting from any level $S_t \leq K$, and that the price $C_d^{\mathrm{Am}}(t, S_t)$ of the American digital call is given by

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$C_d^{\mathrm{Am}}(t,x) = \mathbb{E}[e^{-r(\tau_K - t)}\mathbf{1}_{\{\tau_K < T\}} \mid S_t = x].$$

6. The price $C_d^{\mathrm{Am}}(t,S_t)$ of the American digital call can be shown to be equal to

$$\begin{aligned} C_d^{\mathrm{Am}}(t,x) = {} & \frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ & + \left(\frac{x}{K}\right)^{2r/\sigma^2}\Phi\left(\frac{-(r+\sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right), \qquad 0 \le x \le K, \end{aligned}$$

   where $\tau = T - t$. Show that this formula is consistent with your answer to Question 3.
7. Show that the optimal exercise strategy for the American digital put option with strike K is to exercise as soon as the underlying reaches the level K, at the time

$$\tau_K = \inf\{u \ge t \ : \ S_u = K\},$$

   starting from any level $S_t \ge K$, and that the price $P_d^{\mathrm{Am}}(t,S_t)$ of the American digital put is

$$P_d^{\mathrm{Am}}(t,x) = \mathbb{E}[e^{-r(\tau_K - t)}\mathbf{1}_{\{\tau_K < T\}} \mid S_t = x].$$

8. The price $P_d^{\mathrm{Am}}(t,S_t)$ of the American digital put can be shown to be equal to

$$\begin{aligned} P_d^{\mathrm{Am}}(t,x) = {} & \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ & + \left(\frac{x}{K}\right)^{2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right), \qquad x \ge K. \end{aligned}$$

   Show that this formula is consistent with your answer to Question 4.
9. Does the call-put parity (6.23) hold for American digital options ?

Exercise 8.2. American forward contracts. Consider $(S_t)_{t\in\mathbb{R}_+}$ an asset price process given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

1. Compute the price

$$f(t,S_t) = \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)\Big| S_t\right],$$

and optimal exercise strategy of a finite expiration American type put forward contract with strike K on the underlying asset $(S_t)_{t\in\mathbb{R}_+}$, with payoff $K - S_T$.

2. Compute the price

$$f(t, S_t) = \sup_{\substack{t\leq\tau\leq T \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau-t)}(S_\tau - K) \middle| S_t \right],$$

and optimal exercise strategy of a finite expiration American type call forward contract with strike K on the underlying asset $(S_t)_{t\in\mathbb{R}_+}$, with payoff $S_T - K$.

3. How are the answers to Questions 1 and 2 modified in the case of perpetual options ?

Exercise 8.3. Let $p \geq 1$ and consider a power put option with payoff

$$((\kappa - S_T)^+)^p = \begin{cases} (\kappa - S_T)^p & \text{if } S_T \leq \kappa, \\ 0 & \text{if } S_T > \kappa, \end{cases}$$

on an underlying asset whose price is written as

$$S_t = S_0 e^{rt+\sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+, \tag{8.23}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, $r \geq 0$ is the risk-free interest rate, and $\sigma > 0$ is the volatility coefficient.

Given $L \in (0, \kappa)$ a fixed price, consider the following choices for the exercise of a *put* option with strike κ:

- If $S_t \leq L$, then exercise at time t.
- Otherwise, wait until the first hitting time $\tau_L := \inf\{u \geq t \ : \ S_u = L\}$, and exercise the option at time τ_L.

(i) Under the above strategy, what is the option payoff equal to if $S_t \leq L$?

(ii) Show that in case $S_t > L$, the price of the option is equal to

$$f_L(t, S_t) = (\kappa - L)^p \mathbb{E}^* \left[e^{-r(\tau_L - t)} \middle| S_t \right].$$

(iii) Compute the price $f_L(S_t)$ of the option at time t.

Hint. Recall that $\mathbb{E}^*[e^{-r(\tau_L-t)} \mid S_t = x] = (x/L)^{-2r/\sigma^2}$, $x \geq L$.

(iv) Compute the optimal value L^* that maximizes $L \mapsto f_L(x)$ for all fixed $x > 0$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Hint. Observe that, geometrically, the slope of $x \mapsto f_L(x)$ at $x = L^*$ is equal to $-p(\kappa - L^*)^{p-1}$.

(v) How would you compute the American option price

$$f(t, S_t) = \sup_{\substack{\tau \geq t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[e^{-r(\tau - t)} ((\kappa - S_\tau)^+)^p \middle| S_t \right] \quad ?$$

Exercise 8.4. Let $(M_n)_{n \in \mathbb{N}}$ be a discrete-time martingale with respect to a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$, with $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$.

1. Show that there exists two processes $(N_n)_{n \in \mathbb{N}}$ and $(A_n)_{n \in \mathbb{N}}$ such that

   (i) $(N_n)_{n \in \mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$,
   (ii) $(A_n)_{n \in \mathbb{N}}$ is non-decreasing, i.e. $A_n \leq A_{n+1}$ a.s., $n \in \mathbb{N}$,
   (iii) $(A_n)_{n \in \mathbb{N}}$ is predictable in the sense that A_n is $\mathcal{F}_{n-1}$-measurable, $n \in \mathbb{N}$, and
   (iv) $M_n = N_n + A_n$, $n \in \mathbb{N}$.

   Hint: Let $A_0 = 0$,

   $$A_{n+1} = A_n + E[M_{n+1} - M_n \mid \mathcal{F}_n], \quad n \geq 0,$$

   and define $(N_n)_{n \in \mathbb{N}}$ in such a way that it satisfies the four required properties.
2. Let $(M_n)_{n \in \mathbb{N}}$ be a *sub*martingale. Show that for all bounded stopping times σ and τ such that $\sigma \leq \tau$ a.s., we have

   $$E[M_\sigma] \leq E[M_\tau].$$

   Hint: Use the stopping time Theorem 8.1 for martingales and the definition of conditional expectation.

Chapter 9
Change of Numéraire and Forward Measures

9.1 Notion of Numéraire

A *numéraire* is any strictly positive $\mathcal{F}_t$-adapted stochastic process $(N_t)_{t\in\mathbb{R}_+}$ that can be taken as a unit of reference when pricing an asset or a claim.

In general the price S_t of an asset in the numéraire N_t is given by

$$\hat{S}_t = \frac{S_t}{N_t}, \qquad t \in \mathbb{R}_+.$$

Deterministic numéraires transformations are easy to handle as a change of numéraire by a deterministic factor is a formal algebraic transformation that does not involve any risk. As an example, the exchange rate from Euro to French Franc is 6.55957 and has been fixed on January 1st, 1999.

On the other hand, a random numéraire may involve risk and also arbitrage opportunities.

Examples of numéraire include:

- the money market account

$$N_t = \exp\left(\int_0^t r_s ds\right),$$

where $(r_t)_{t\in\mathbb{R}_+}$ is a possibly random and time-dependent risk-free interest rate.

In this case,

$$\hat{S}_t = \frac{S_t}{N_t} = e^{-\int_0^t r_s ds} S_t, \qquad t \in \mathbb{R}_+,$$

represents the discounted price of the asset at time 0.

- an exchange rate $N_t = R_t$ with respect to a foreign currency.

In this case,

$$\hat{S}_t = \frac{S_t}{R_t}, \qquad t \in \mathbb{R}_+,$$

represents the price of the asset in units of the foreign currency. For example, if $R_t = 1.7$ is the exchange rate from Euro to Singapore dollar and $S_t = S\$1$, then $\hat{S}_t = S_t/R_t =$€ 0.59.
- *forward numéraire*: the price

$$P(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right], \qquad 0 \le t \le T,$$

of a bond paying $P(T,T) = \$1$ at maturity T, in this case $R_t = P(t,T)$, $0 \le t \le T$. We check that

$$t \mapsto e^{-\int_0^t r_s ds} P(t,T) = \mathbb{E}^* \left[e^{-\int_0^T r_s ds} \middle| \mathcal{F}_t \right], \qquad 0 \le t \le T,$$

is a martingale.
- an *annuity numéraire* of the form

$$R_t = \sum_{k=1}^{n} (T_k - T_{k-1}) P(t, T_k)$$

where $P(t,T_1), \ldots, P(t,T_n)$ are bond prices with maturities $T_1 < \cdots < T_n$ arranged according to a *tenor structure*.
- combinations of the above, for example a foreign money market account $e^{\int_0^t r_s^f ds} R_t$, expressed in local currency, where $(r_t^f)_{t \in \mathbb{R}_+}$ represents a short term interest rate on the foreign market.

When the numéraire is a random process, the pricing of a claim whose value has been transformed under change of numéraire, e.g. under a change of currency, has to take into account the risks existing on the foreign market.

In particular, in order to perform a fair pricing, one has to determine a probability measure (for example on the foreign market), under which the transformed process $\hat{S}_t = S_t/N_t$ will be a martingale.

For example in case $N_t = e^{\int_0^t r_s ds}$, the risk-neutral measure $\mathbb{P}^*$ is a measure under which the discounted price process

$$\hat{S}_t = \frac{S_t}{N_t} = e^{-\int_0^t r_s ds} S_t, \qquad t \in \mathbb{R}_+,$$

is a martingale.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

In the next section we will see that this property can be extended to any kind of numéraire.

9.2 Change of Numéraire

In this section we review the option pricing by change of measure associated to a numéraire N_t, cf. e.g. [22] and references therein.

We assume that under $\mathbb{P}^*$, the discounted price process

$$t \mapsto e^{-\int_0^t r_s ds} N_t$$

is an $\mathcal{F}_t$-martingale.

Taking the process $(N_t)_{t\in[0,T]}$ as a numéraire, we define the *forward measure* $\hat{\mathbb{P}}$ by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} = e^{-\int_0^T r_s ds} \frac{N_T}{N_0}. \tag{9.1}$$

The above relation rewrites as

$$d\hat{\mathbb{P}} = e^{-\int_0^T r_s ds} \frac{N_T}{N_0} d\mathbb{P}^*,$$

which is equivalent to stating that

$$\int_\Omega \xi(\omega) d\hat{\mathbb{P}}(\omega) = \int_\Omega e^{-\int_0^T r_s ds} \frac{N_T}{N_0} \xi(\omega) d\mathbb{P}^*(\omega),$$

or, under a different notation,

$$\hat{\mathbb{E}}[\xi] = \mathbb{E}^* \left[e^{-\int_0^T r_s ds} \frac{N_T}{N_0} \xi \right],$$

for all integrable $\mathcal{F}_T$-measurable random variable ξ.

More generally it can be shown in addition to (9.1) that

$$\mathbb{E}^* \left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} \Big| \mathcal{F}_t \right] = e^{-\int_0^t r_s ds} \frac{N_t}{N_0}, \qquad 0 \le t \le T, \tag{9.2}$$

for all integrable $\mathcal{F}_t$-measurable random variables ξ, due to the fact that

$$t \mapsto e^{-\int_0^t r_s ds} N_t$$

is a martingale under $\mathbb{P}^*$. Indeed we have

$$\mathbb{E}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}\Big|\mathcal{F}_t\right] = \mathbb{E}\left[\frac{N_T}{N_0}e^{-\int_0^T r_s ds}\Big|\mathcal{F}_t\right] = \frac{N_t}{N_0}e^{-\int_0^t r_s ds}, \qquad 0 \le t \le T. \quad (9.3)$$

We note that in case $N_t = e^{\int_0^t r_s ds}$ we simply have $\hat{\mathbb{P}} = \mathbb{P}^*$.

In the sequel we let $(X_t)_{t\in\mathbb{R}_+}$ denote a continuous $\mathcal{F}_t$-adapted asset price process such that

$$t \mapsto e^{-\int_0^t r_s ds}X_t$$

is a martingale under $\mathbb{P}^*$, and we let

$$\hat{X}_t := \frac{X_t}{N_t}, \qquad 0 \le t \le T,$$

denote the value at time t of X_t, expressed in units of the numéraire N_t.

Proposition 9.1. *Under change of numéraire, the forward price process $\hat{X}_t := X_t/N_t$, $0 \le t \le T$, is a martingale under $\hat{\mathbb{P}}$, provided it is integrable under $\hat{\mathbb{P}}$.*

Proof. For all bounded $\mathcal{F}_s$-measurable random variables G we have

$$\begin{aligned}\hat{\mathbb{E}}\left[G\frac{X_t}{N_t}\right] &= \mathbb{E}^*\left[Ge^{-\int_0^t r_u du}\frac{X_t}{N_0}\right]\\ &= \mathbb{E}^*\left[Ge^{-\int_0^s r_u du}\frac{X_s}{N_0}\right]\\ &= \hat{\mathbb{E}}\left[G\frac{X_s}{N_s}\right], \qquad 0 \le s \le t,\end{aligned}$$

because

$$t \mapsto e^{-\int_0^t r_s ds}X_t$$

is a martingale. This shows that

$$\hat{\mathbb{E}}\left[\frac{X_t}{N_t}\Big|\mathcal{F}_s\right] = \frac{X_s}{N_s}, \qquad 0 \le s \le t,$$

by a standard characterization of conditional expectation. □

Letting

$$\Phi_t = \mathbb{E}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}\Big|\mathcal{F}_t\right], \qquad t \in [0,T],$$

Relation (9.3) shows that

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\begin{aligned}\Phi_t &= \mathbb{E}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}\Big|\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\frac{N_T}{N_0}e^{-\int_0^T r_s ds}\Big|\mathcal{F}_t\right]\\ &= \frac{N_t}{N_0}e^{-\int_0^t r_s ds}, \qquad 0 \le t \le T,\end{aligned}$$

hence

$$d\Phi_t = -\Phi_t r_t dt + \frac{\Phi_t}{N_t}dN_t,$$

and by the Girsanov theorem,[1] it is known that the process $(\hat{W}_t)_{t\in\mathbb{R}_+}$ defined by

$$d\hat{W}_t = dW_t - \frac{1}{\Phi_t}d\Phi_t \cdot dW_t = dW_t - \frac{1}{N_t}dN_t \cdot dW_t, \qquad t \in \mathbb{R}_+, \tag{9.4}$$

is a standard Brownian motion under $\hat{\mathbb{P}}$.

The next proposition confirms the statement of Proposition 9.1.

Proposition 9.2. *Assume that $(X_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equations*

$$dX_t = r_t X_t dt + \sigma_t^X X_t dW_t, \quad \textit{and} \quad dN_t = r_t N_t dt + \sigma_t^N N_t dW_t.$$

Then we have

$$d\hat{X}_t = (\sigma_t^X - \sigma_t^N)\hat{X}_t d\hat{W}_t. \tag{9.5}$$

Proof. By Itô's calculus we have

$$\begin{aligned}d\hat{X}_t &= d\left(\frac{X_t}{N_t}\right)\\ &= \frac{dX_t}{N_t} - \frac{X_t}{N_t^2}dN_t - \frac{1}{N_t^2}dN_t \cdot dX_t + X_t\frac{(dN_t)^2}{N_t^3}\\ &= \frac{1}{N_t}(r_t X_t dt + \sigma_t^X X_t dW_t) - \frac{X_t}{N_t^2}(r_t N_t dt + \sigma_t^N N_t dW_t)\\ &\quad -\frac{X_t N_t}{N_t^2}\sigma_t^X\sigma_t^N dt + X_t\frac{(\sigma_t^N)^2 N_t^2}{N_t^3}dt\\ &= \frac{X_t}{N_t}\sigma_t^X dW_t - \frac{X_t}{N_t}\sigma_t^N dW_t - \frac{X_t}{N_t}\sigma_t^X\sigma_t^N dt + X_t\frac{(\sigma_t^N)^2}{N_t}dt\\ &= \frac{X_t}{N_t}\left(\sigma_t^X dW_t - \sigma_t^N dW_t - \sigma_t^X\sigma_t^N dt + (\sigma_t^N)^2 dt\right)\\ &= \hat{X}_t(\sigma_t^X - \sigma_t^N)dW_t - \hat{X}_t(\sigma_t^X - \sigma_t^N)\sigma_t^N dt\end{aligned}$$

[1] See e.g. Theorem 40 in [41].

$$= \hat{X}_t(\sigma_t^X - \sigma_t^N)d\hat{W}_t,$$

with

$$d\hat{W}_t = dW_t - \sigma_t^N dt, \qquad t \in \mathbb{R}_+,$$

from (9.4). □

Inverse Change of Measure

In the next proposition we compute a conditional inverse density.

Proposition 9.1. *We have*

$$\hat{\mathbb{E}}\left[\frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\Big|\mathcal{F}_t\right] = \frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right) \qquad 0 \le t \le T, \tag{9.6}$$

and the process

$$t \mapsto \frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right), \qquad 0 \le t \le T,$$

is an $\mathcal{F}_t$-martingale under $\hat{\mathbb{P}}$.

Proof. For all bounded and $\mathcal{F}_t$-measurable random variables F we have,

$$\begin{aligned}
\hat{\mathbb{E}}\left[F\frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\right] &= \mathbb{E}^*\left[F\right] \\
&= \mathbb{E}^*\left[F\frac{N_t}{N_t}\right] \\
&= \mathbb{E}^*\left[F\frac{N_T}{N_t}\exp\left(-\int_t^T r_s ds\right)\right] \\
&= \hat{\mathbb{E}}\left[F\frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right)\right].
\end{aligned}$$

□

By Itô's calculus we also have

$$\begin{aligned}
d\left(\frac{1}{N_t}\right) &= -\frac{1}{N_t^2}dN_t + \frac{1}{N_t^3}(dN_t)^2 \\
&= -\frac{1}{N_t^2}(r_t N_t dt + \sigma_t^N N_t dW_t) + \frac{(\sigma_t^N)^2}{N_t}dt \\
&= -\frac{1}{N_t^2}(r_t N_t dt + \sigma_t^N N_t(d\hat{W}_t + \sigma_t^N dt)) + \frac{(\sigma_t^N)^2}{N_t}dt
\end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$= -\frac{1}{N_t}(r_t dt + \sigma_t^N d\hat{W}_t),$$

and

$$d\left(\frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)\right) = -\frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)\sigma_t^N d\hat{W}_t,$$

which recovers the second part of Proposition 9.1, i.e. the martingale property of

$$t \mapsto \frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)$$

under $\hat{\mathbb{P}}$.

9.3 Foreign Exchange

Let R_t denote the foreign exchange rate, i.e. R_t is the number of units of local currency that correspond to one unit of foreign currency, and take

$$N_t = e^{r^f t} R_t, \qquad t \in \mathbb{R}_+,$$

as numéraire, i.e. the process

$$t \mapsto e^{-rt} N_t = e^{-(r-r^f)t} R_t$$

is a martingale under $\mathbb{P}^*$.

Assume that the foreign exchange rate R_t satisfies a stochastic differential equation of the form

$$dR_t = \mu R_t dt + \sigma R_t dW_t,$$

where $(W)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, with solution

$$R_t = R_0 e^{\mu t + \sigma W_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+.$$

The discounted value of the foreign money market account on the local market is

$$e^{-rt+r^f t} R_t = R_0 e^{-rt+r^f t+\mu t+\sigma W_t-\sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

Hence $e^{-(r-r^f)t} R_t$ is a martingale under $\mathbb{P}^*$ provided $r^f - r + \mu = 0$, i.e.

$$\mu = r - r^f. \tag{9.7}$$

Consequently, under absence of arbitrage, a local investor who buys a unit of foreign in the hope of a higher return $r^f >> r$ will have to face a lower (or even more negative) drift μ in his exchange rate R_t, and we have

$$dR_t = (r - r^f)R_t dt + \sigma R_t dW_t. \tag{9.8}$$

under $\mathbb{P}^*$.

Let now $X_t = e^{rt}$ denote the local money market account. The discounted value of the local money market account on the foreign market is

$$t \mapsto e^{(r-r^f)t}\frac{1}{R_t} = \frac{X_t}{N_t} = R_0 e^{(r-r^f)t-\mu t-\sigma W_t+\sigma^2 t/2} = R_0 e^{(r-r^f)t-\mu t-\sigma \hat{W}_t-\sigma^2 t/2}, \tag{9.9}$$

where

$$d\hat{W}_t = dW_t - \frac{1}{N_t}dN_t \cdot dW_t = dW_t - \frac{1}{R_t}dR_t \cdot dW_t = dW_t - \sigma dt, \qquad t \in \mathbb{R}_+,$$

is a standard Brownian motion under $\hat{\mathbb{P}}$ by (9.4). Since (9.9) is a martingale under $\hat{\mathbb{P}}$ by Proposition 9.1, we get $-r^f + r - \mu = 0$, i.e.

$$\mu = r - r^f. \tag{9.10}$$

Consequently, under absence of arbitrage, a foreign investor who buys a unit of the local currency in the hope of a higher return $r >> r^f$ will have to face a lower (or even more negative) drift $-\mu$ in his exchange rate $1/R_t$ written by Itô's calculus as

$$\begin{aligned} d\left(\frac{1}{R_t}\right) &= -\frac{1}{R_t^2}(\mu R_t dt + \sigma R_t dW_t) + \frac{1}{R_t^3}\sigma^2 R_t^2 dt \\ &= -(\mu - \sigma^2)\frac{1}{R_t}dt - \sigma\frac{1}{R_t}dW_t \\ &= -\mu\frac{1}{R_t}dt - \sigma\frac{1}{R_t}d\hat{W}_t \\ &= (r^f - r)\frac{1}{R_t}dt - \sigma\frac{1}{R_t}d\hat{W}_t, \end{aligned} \tag{9.11}$$

under $\hat{\mathbb{P}}$.

Note that the symmetry between (9.8) and (9.11) holds provided one uses two different Brownian motions W_t and $\hat{W}_t$.

As a consequence of (9.8) rewritten as

$$dN_t = d(e^{r^f t}R_t) = re^{r^f t}R_t dt + \sigma e^{r^f t}R_t dW_t = rN_t dt + \sigma N_t dW_t,$$

a foreign exchange option with payoff $(R_T - \kappa)^+$ can be priced under $\mathbb{P}^*$ by the Black-Scholes formula, as

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\begin{aligned} e^{-(T-t)r}\,\mathbb{E}^*[(R_T-\kappa)^+ \mid R_t] &= e^{-(T-t)r}\,\mathbb{E}^*[(e^{-r^fT}N_T-\kappa)^+ \mid R_t] \\ &= e^{-(T-t)r}e^{-r^fT}\,\mathbb{E}^*[(N_T-\kappa e^{r^fT})^+ \mid R_t] \\ &= e^{-(T-t)r^f}R_t\Phi_+(t,R_t)-\kappa e^{-(T-t)r}\Phi_-(t,R_t), \end{aligned}$$

where

$$\Phi_+(t,x)=\Phi\left(\frac{\log(x/\kappa)+(r-r^f+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x)=\Phi\left(\frac{\log(x/\kappa)+(r-r^f-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Similarly, from (9.11) rewritten as

$$d\left(\frac{e^rt}{R_t}\right)=r^f\frac{e^rt}{R_t}dt-\sigma\frac{e^rt}{R_t}d\hat{W}_t,$$

a foreign exchange option with payoff $(1/R_T-\kappa)^+$ can be priced under $\hat{\mathbb{P}}$ as a put option by the Black-Scholes formula, as

$$e^{-(T-t)r^f}\hat{\mathbb{E}}\left[\left(\frac{1}{R_T}-\kappa\right)^+\Big|R_t\right] \tag{9.12}$$

$$= e^{-(T-t)r^f}e^{-rT}\hat{\mathbb{E}}\left[\left(\frac{e^{rT}}{R_T}-\kappa e^{rT}\right)^+\Big|R_t\right] \tag{9.13}$$

$$= \frac{e^{-r(T-t)}}{R_t}\Phi_+\left(t,\frac{1}{R_t}\right)-\kappa e^{-(T-t)r^f}\Phi_-\left(t,\frac{1}{R_t}\right),$$

where

$$\Phi_+(t,x)=\Phi\left(\frac{\log(x/\kappa)+(r^f-r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x)=\Phi\left(\frac{\log(x/\kappa)+(r^f-r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).$$

9.4 Pricing under Change of Numéraire

The change of numéraire technique is specially useful for pricing under random interest rates.

The next proposition is the basic result of this section.

Proposition 9.3. *An option with payoff ξ is priced at time t as*

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds}\xi \,\Big|\mathcal{F}_t\right] = N_t\hat{\mathbb{E}}\left[\frac{\xi}{N_T}\Big|\mathcal{F}_t\right], \qquad 0 \le t \le T, \tag{9.14}$$

provided $\xi/N_T \in L^1(\hat{\mathbb{P}}, \mathcal{F}_T)$.

Proof. This proposition relies on the relation

$$\frac{d\hat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} = e^{-\int_t^T r_s ds}\frac{N_T}{N_t}, \qquad 0 \le t \le T, \tag{9.15}$$

which can be proved as follows. For all integrable $\mathcal{F}_T$-measurable random variable $\hat{\xi}$ we have

$$\begin{aligned}
\hat{\mathbb{E}}[\hat{\mathbb{E}}[\hat{\xi}|\mathcal{F}_t]] &= \hat{\mathbb{E}}[\hat{\xi}] \\
&= \mathbb{E}^*\left[\hat{\xi}e^{-\int_0^T r_s ds}\frac{N_T}{N_0}\right] \\
&= \mathbb{E}^*\left[\frac{N_t}{N_0}e^{-\int_0^t r_s ds}\,\mathbb{E}^*\left[\hat{\xi}e^{-\int_t^T r_s ds}\frac{N_T}{N_t}\Big|\mathcal{F}_t\right]\right] \\
&= \hat{\mathbb{E}}\left[\mathbb{E}^*\left[\hat{\xi}e^{-\int_t^T r_s ds}\frac{N_T}{N_t}\Big|\mathcal{F}_t\right]\right],
\end{aligned}$$

which yields (9.15). □

Examples:

- N_t is the money market account $N_t = e^{\int_0^t r_s ds}$, where $(r_t)_{t\in\mathbb{R}_+}$ is a possibly random and time-dependent risk-free interest rate.

In this case we have $\hat{\mathbb{P}} = \mathbb{P}$ and (9.14) yields no particular information.

- $N_t = e^{r^f t}R_t$ where R_t is an exchange rate with respect to a foreign currency and r_f is the foreign market interest rate.

From Proposition 9.3 and (9.14) we have

$$e^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{\kappa} - R_T\right)^+\Big|R_t\right] = N_t\hat{\mathbb{E}}\left[\frac{1}{e^{r^f T}R_T}\left(\frac{1}{\kappa} - R_T\right)^+\Big|R_t\right],$$

and this yields we have the call/put duality

$$\begin{aligned}
e^{-(T-t)r^f}\hat{\mathbb{E}}\left[\left(\frac{1}{R_T} - \kappa\right)^+\Big|R_t\right] &= e^{-(T-t)r^f}\hat{\mathbb{E}}\left[\frac{\kappa}{R_T}\left(\frac{1}{\kappa} - R_T\right)^+\Big|R_t\right] \\
&= e^{r^f t}\hat{\mathbb{E}}\left[\frac{\kappa}{e^{r^f T}R_T}\left(\frac{1}{\kappa} - R_T\right)^+\Big|R_t\right]
\end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$= \frac{\kappa}{N_t} e^{r^f t-(T-t)r}\, \mathbb{E}^* \left[\left(\frac{1}{\kappa} - R_T \right)^+ \Big| R_t \right]$$
$$= \frac{\kappa}{R_t} e^{-(T-t)r}\, \mathbb{E}^* \left[\left(\frac{1}{\kappa} - R_T \right)^+ \Big| R_t \right],$$

between a call option with strike κ and a κ/R_t units of put option with strike $1/\kappa$.

In the Black-Scholes case this duality can be directly checked by verifying that (9.12) coincides with

$$\begin{aligned}
&\frac{\kappa}{R_t} e^{-(T-t)r}\, \mathbb{E}^* \left[\left(\frac{1}{\kappa} - R_T \right)^+ \Big| R_t \right] \\
&\quad = \frac{\kappa}{R_t} e^{-(T-t)r} e^{-r^f T}\, \mathbb{E}^* \left[\left(\frac{e^{r^f T}}{\kappa} - e^{r^f T} R_T \right)^+ \Big| R_t \right] \\
&\quad = \frac{\kappa}{R_t} e^{-(T-t)r} e^{-r^f T}\, \mathbb{E}^* \left[\left(\frac{e^{r^f T}}{\kappa} - N_T \right)^+ \Big| R_t \right] \\
&\quad = \frac{\kappa}{R_t} \left(\frac{e^{-(T-t)r}}{\kappa} \Phi^p_-(t, R_t) - e^{-r^f (T-t)} R_t \Phi^p_+(t, R_t) \right) \\
&\quad = \frac{e^{-(T-t)r}}{R_t} \Phi^p_-(t, R_t) - \kappa e^{-r^f (T-t)} \Phi^p_+(t, R_t),
\end{aligned}$$

where

$$\Phi^p_-(t,x) = \Phi\left(-\frac{\log(x\kappa) + (r - r^f - \sigma_t^2/2)(T-t)}{\sigma\sqrt{T-t}} \right),$$

and

$$\Phi^p_+(t,x) = \Phi\left(-\frac{\log(x\kappa) + (r - r^f + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right).$$

- *forward numéraire*: we take $N_t = P(t,T)$ the price $P(t,T)$ of a bond maturing at T, $0 \le t \le T$. In this case, (9.14) shows that

$$\mathbb{E}^* \left[e^{-\int_t^T r_s ds} \xi \,\Big|\, \mathcal{F}_t \right] = P(t,T) \hat{\mathbb{E}} \left[\xi \Big| \mathcal{F}_t \right], \qquad 0 \le t \le T,$$

since $P(T,T) = 1$.
- an *annuity numéraire* of the form

$$N_t = \sum_{k=1}^{n} (T_k - T_{k-1}) P(t, T_k)$$

where $P(t,T_1),\ldots,P(t,T_n)$ are bond prices with maturities $T_1<\cdots<T_n$. Here, (9.14) shows that

$$\begin{aligned}\mathbb{E}^*\left[e^{-\int_t^T r_s ds}(P(T,T_n)-P(T,T_0)-\kappa R_T)^+\Big|\mathcal{F}_t\right]\\ = N_t\hat{\mathbb{E}}\left[\left(\frac{P(T,T_n)-P(T,T_0)}{R_T}-\kappa\right)^+\Big|\mathcal{F}_t\right],\end{aligned}$$

$0\le t\le T$, where $(P(T,T_n)-P(T,T_0))/R_T$ is a *swap rate*.

9.5 Pricing of Exchange Options

Consider a continuous martingale $\hat{X}_t$ under $\hat{\mathbb{P}}$, written as

$$d\hat{X}_t=\hat{\sigma}_t d\hat{W}_t,\qquad t\in\mathbb{R}_+,\tag{9.16}$$

where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}$ and $(\hat{\sigma}_t)_{t\in\mathbb{R}_+}$ is an adapted process.

In addition, assume that $(\hat{X}_t)_{t\in\mathbb{R}_+}$ has the Markov property, and the dynamics

$$d\hat{X}_t=\hat{\sigma}_t(\hat{X}_t)d\hat{W}_t,\tag{9.17}$$

where $x\mapsto\hat{\sigma}_t(x)$ is a Lipschitz function, uniformly in $t\in\mathbb{R}_+$, the option with payoff $\xi=N_T\hat{g}(\hat{X}_T)$ is priced at time t as

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds}N_T\hat{g}(\hat{X}_T)\Big|\mathcal{F}_t\right]=N_t\hat{\mathbb{E}}\left[\hat{g}(\hat{X}_T)\Big|\mathcal{F}_t\right]=N_t\hat{C}(t,\hat{X}_t),\tag{9.18}$$

for some measurable function $\hat{C}(t,x)$ on $\mathbb{R}_+$.

In case $\hat{\sigma}_t(\hat{X}_t)=\hat{\sigma}(t)\hat{X}_t$, i.e. the martingale $(\hat{X}_t)_{t\in[0,T]}$ is a geometric Brownian motion under $\hat{\mathbb{P}}$ with deterministic variance $(\hat{\sigma}(t))_{t\in[0,T]}$ and for the call option with payoff

$$(X_T-\kappa N_T)^+=N_T(\hat{X}_T-\kappa)^+,$$

(9.18) shows that

$$\begin{aligned}\mathbb{E}^*\left[e^{-\int_t^T r_s ds}(X_T-\kappa N_T)^+\Big|\mathcal{F}_t\right]&=\mathbb{E}^*\left[e^{-\int_t^T r_s ds}N_T(\hat{X}_T-\kappa)^+\Big|\mathcal{F}_t\right]\\&=N_t\hat{\mathbb{E}}\left[(\hat{X}_T-\kappa)^+\Big|\mathcal{F}_t\right]\\&=N_t\hat{C}(t,\hat{X}_t),\end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

where the function $\hat{C}(t,\hat{X}_t)$ is given by the Black-Scholes formula

$$\hat{C}(t,x) = x\Phi_+^0(t,x) - \kappa\Phi_-^0(t,x),$$

with zero interest rate, where

$$\Phi_+^0(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} + \frac{v(t,T)}{2}\right),$$

and

$$\Phi_-^0(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} - \frac{v(t,T)}{2}\right),$$

and

$$v^2(t,T) = \int_t^T \hat{\sigma}^2(s)ds,$$

and the corresponding exchange option is priced by the Margrabe [33] formula

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds}\left(X_T - \kappa N_T\right)^+ \Big| \mathcal{F}_t\right] = X_t\Phi_+^0(t,\hat{X}_t) - \kappa N_t\Phi_-^0(t,\hat{X}_t), \qquad t \in \mathbb{R}_+.$$

In particular, from Proposition 9.2 and (9.5), we can take $\hat{\sigma}(t) = \sigma_t^X - \sigma_t^N$ when $(\sigma_t^X)_{t\in\mathbb{R}_+}$ and $(\sigma_t^N)_{t\in\mathbb{R}_+}$ are deterministic.

Examples:

- when the short rate process $(r(t))_{t\in[0,T]}$ is a *deterministic* function and $N_t = e^{-\int_t^T r(s)ds}$, $0 \le t \le T$, this yields Merton's [34] "zero interest rate" version of the Black-Scholes formula

$$\begin{aligned} &e^{-\int_t^T r(s)ds}\,\mathbb{E}^*\left[\left(X_T - \kappa\right)^+ \Big| \mathcal{F}_t\right] \\ &\qquad = X_t\Phi_+^0\left(t, e^{\int_t^T r(s)ds}X_t\right) - \kappa e^{-\int_t^T r(s)ds}\Phi_-^0\left(t, e^{\int_t^T r(s)ds}X_t\right), \end{aligned}$$

where $(X_t)_{t\in\mathbb{R}_+}$ satisfies the equation

$$\frac{dX_t}{X_t} = r(t)dt + \hat{\sigma}(t)dW_t, \qquad 0 \le t \le T.$$

- in the case of pricing under a *forward numéraire*, i.e. $N_t = P(t,T)$, $0 \le t \le T$, we get

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds}\left(X_T - \kappa\right)^+ \Big| \mathcal{F}_t\right] = X_t\Phi_+(t,\hat{X}_t) - \kappa P(t,T)\Phi_-(t,\hat{X}_t),$$

$t \in \mathbb{R}_+$, since $P(T,T) = 1$. In particular, when $X_t = P(t,S)$, this allows us to price a bond call option on $P(T,S)$ as

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds}\left(P(T,S) - \kappa\right)^+ \Big| \mathcal{F}_t\right] = P(t,S)\Phi_+(t,\hat{X}_t) - \kappa P(t,T)\Phi_-(t,\hat{X}_t),$$

$0 \leq t \leq T$, provided $\hat{X}_t = P(t,S)/P(t,T)$ is given by a geometric Brownian motion.

9.6 Self-Financing Hedging by Change of Numéraire

In this section we use the stochastic integral representation of the forward payoff ξ/N_T and change of numéraire to compute self-financing portfolio strategies. Our hedging portfolios will be built on the assets (X_t, N_t), and not on X_t and the money market account $B_t = e^{\int_0^t r_s ds}$, extending the classical hedging portfolios that are available in from the Black-Scholes formula, using a technique from [27], cf. also [40].

Assume that the forward claim $\xi/N_T \in L^2(\Omega)$ has the stochastic integral representation

$$\frac{\xi}{N_T} = \hat{\mathbb{E}}\left[\frac{\xi}{N_T}\right] + \int_0^T \hat{\phi}_t d\hat{X}_t, \tag{9.19}$$

where $(\hat{X}_t)_{t\in[0,T]}$ is given by (9.16) and $(\hat{\phi}_t)_{t\in[0,T]}$ is a square-integrable adapted process under $\hat{\mathbb{P}}$, from which it follows that the forward claim price

$$\hat{V}_t := \hat{\mathbb{E}}\left[\frac{\xi}{N_T}\Big|\mathcal{F}_t\right], \qquad 0 \leq t \leq T,$$

is a martingale that can be decomposed as

$$\hat{V}_t = \hat{\mathbb{E}}\left[\frac{\xi}{N_T}\right] + \int_0^t \hat{\phi}_s d\hat{X}_s, \qquad 0 \leq t \leq T. \tag{9.20}$$

The next proposition extends the argument of [27] to the general framework of pricing under change of numéraire. Note that this result differs from the standard formula that uses the money market account $B_t = e^{\int_0^t r_s ds}$ for hedging instead of N_t, cf. e.g. [22] pages 453-454.

Proposition 9.4. *Letting $\hat{\eta}_t = \hat{V}_t - \hat{X}_t\hat{\phi}_t$, $0 \leq t \leq T$, the portfolio $(\hat{\phi}_t, \hat{\eta}_t)_{t\in[0,T]}$ with value*

$$V_t = \hat{\phi}_t X_t + \hat{\eta}_t N_t, \qquad 0 \leq t \leq T,$$

is self-financing and hedges the claim ξ.

Proof. In order to check that the portfolio $(\hat{\phi}_t, \hat{\eta}_t)_{t\in[0,T]}$ hedges the claim ξ it suffices to recall that the price V_t at time $t \in [0,T]$ of the hedging portfolio is given from (9.14) by

$$V_t = N_t\hat{V}_t = \mathbb{E}^*\left[e^{-\int_t^T r_s ds}\xi \,\Big|\mathcal{F}_t\right], \qquad 0 \leq t \leq T.$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Next, we show that the portfolio $(\hat{\phi}_t, \hat{\eta}_t)_{t\in[0,T]}$ is self-financing. We have

$$\begin{aligned}
dV_t &= d(N_t\hat{V}_t)\\
&= \hat{V}_t dN_t + N_t d\hat{V}_t + dN_t \cdot d\hat{V}_t\\
&= \hat{V}_t dN_t + N_t\hat{\phi}_t d\hat{X}_t + \hat{\phi}_t dN_t \cdot d\hat{X}_t\\
&= \hat{\phi}_t\hat{X}_t dN_t + N_t\hat{\phi}_t d\hat{X}_t + \hat{\phi}_t dN_t \cdot d\hat{X}_t + (\hat{V}_t - \hat{\phi}_t\hat{X}_t)dN_t\\
&= \hat{\phi}_t d(N_t\hat{X}_t) + (\hat{V}_t - \hat{\phi}_t\hat{X}_t)dN_t\\
&= \hat{\phi}_t dX_t + \hat{\eta}_t dN_t.
\end{aligned}$$

□

We now consider an application to the forward Delta hedging of European type options with payoff $\xi = \hat{g}(\hat{X}_T)$ where $\hat{g} : \mathbb{R} \to \mathbb{R}$ and $(\hat{X}_t)_{t\in\mathbb{R}_+}$ has the Markov property as in (9.17), where $\hat{\sigma} : \mathbb{R}_+ \times \mathbb{R}$. Assuming that the function $\hat{C}(t, x)$ defined by

$$\hat{V}_t := \hat{\mathbb{E}}\left[g(\hat{X}_T)\middle|\mathcal{F}_t\right] = \hat{C}(t, \hat{X}_t)$$

is $\mathcal{C}^2$ on $\mathbb{R}_+$, we have the following corollary of Proposition 9.4.

Corollary 9.1. Letting $\hat{\eta}_t = \hat{C}(t, \hat{X}_t) - \hat{X}_t\dfrac{\partial\hat{C}}{\partial x}(t, \hat{X}_t)$, $0 \le t \le T$, the portfolio $\left(\dfrac{\partial\hat{C}}{\partial x}(t, \hat{X}_t), \hat{\eta}_t\right)_{t\in[0,T]}$ with value

$$V_t = \hat{\eta}_t N_t + X_t\frac{\partial\hat{C}}{\partial x}(t, \hat{X}_t), \qquad t \in \mathbb{R}_+,$$

is self-financing and hedges the claim $\xi = N_T\hat{g}(\hat{X}_T)$.

Proof. This result follows directly from Proposition 9.4 by noting that by Itô's formula, and the martingale property of $\hat{V}_t$ under $\hat{\mathbb{P}}$ the stochastic integral representation (9.20) is given by

$$\hat{\phi}_t = \frac{\partial\hat{C}}{\partial x}(t, \hat{X}_t), \qquad 0 \le t \le T.$$

□

In the case of a call option with payoff function $\xi = (X_T - \kappa N_T)^+ = N_T(\hat{X}_T - \kappa)^+$ on the geometric Brownian motion $(\hat{X}_t)_{t\in[0,T]}$ under $\hat{\mathbb{P}}$ with

$$\hat{\sigma}_t(\hat{X}_t) = \hat{\sigma}(t)\hat{X}_t,$$

where $(\hat{\sigma}(t))_{t\in[0,T]}$ is a deterministic function, we have the following corollary on the hedging of exchange options.

Corollary 9.2. The decomposition

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds}\left(X_T-\kappa N_T\right)^+\Big|\mathcal{F}_t\right]=X_t\Phi_+(t,\hat{X}_t)-\kappa N_t\Phi_-(t,\hat{X}_t)$$

yields a self-financing portfolio $(\Phi_+(t,\hat{X}_t),-\kappa\Phi_-(t,\hat{X}_t))_{t\in[0,T]}$ in (X_t,N_t) that hedges the claim $\xi=(X_T-\kappa N_T)^+$.

Proof. We apply Corollary 9.1 and the classical relation

$$\frac{\partial\hat{C}}{\partial x}(t,x)=\Phi_+(t,x),\qquad x\in\mathbb{R},$$

applied to the function $\hat{C}(t,x)=x\Phi_+(t,x)-\kappa\Phi_-(t,x)$. □

Note that the Delta hedging method requires the computation of the function $\hat{C}(t,x)$ and that of the associated finite differences, and may not apply to path-dependent claims.

Examples:

- when the short rate process $(r(t))_{t\in[0,T]}$ is a *deterministic* function and $N_t=e^{\int_t^T r(s)ds}$, Corollary 9.2 yields the usual Black-Scholes hedging strategy.
- in case $N_t=P(t,T)$ and $X_t=P(t,S)$, $0\le t\le T<S$, Corollary 9.2 shows that a bond call option with payoff $(P(T,S)-\kappa)^+$ can be hedged as

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds}\left(P(T,S)-\kappa\right)^+\Big|\mathcal{F}_t\right]=P(t,S)\Phi_+(t,\hat{X}_t)-\kappa P(t,T)\Phi_-(t,\hat{X}_t)$$

by the self-financing portfolio $(\Phi_+(t,\hat{X}_t),-\kappa\Phi_-(t,\hat{X}_t))_{t\in[0,T]}$ in $(P(t,S),P(t,T))$, i.e. one needs to hold $\Phi_+(t,\hat{X}_t)$ units of the bond maturing at S, and to short $\kappa\Phi_-(t,\hat{X}_t)$ units of the bond maturing at T.

9.7 Exercises

Exercise 9.1. Bond options. Consider two bonds with maturities T and S, with prices $P(t,T)$ and $P(t,S)$ given by

$$\frac{dP(t,T)}{P(t,T)}=r_t dt+\zeta_t^T dW_t,$$

and

$$\frac{dP(t,S)}{P(t,S)}=r_t dt+\zeta_t^S dW_t,$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

where $(\zeta^T(s))_{s\in[0,T]}$ and $(\zeta^S(s))_{s\in[0,S]}$ are deterministic functions.

1. Show, using Itô's formula, that

$$d\left(\frac{P(t,S)}{P(t,T)}\right) = \frac{P(t,S)}{P(t,T)}(\zeta^S(t) - \zeta^T(t))d\hat{W}_t,$$

   where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}$.
2. Show that

$$P(T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(\int_t^T (\zeta^S(s) - \zeta^T(s))d\hat{W}_s - \frac{1}{2}\int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds\right).$$

   Let $\hat{\mathbb{P}}$ denote the forward measure associated to the numéraire

$$N_t := P(t,T), \qquad 0 \le t \le T.$$

3. Show that for all $S, T > 0$ the price at time t

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S) - \kappa)^+ \middle| \mathcal{F}_t\right]$$

   of a bond call option on $P(T,S)$ with payoff $(P(T,S) - \kappa)^+$ is equal to

$$\begin{aligned} &\mathbb{E}^*\left[e^{-\int_t^T r_s ds}(P(T,S) - \kappa)^+ \middle| \mathcal{F}_t\right] \qquad (9.21)\\ &= P(t,S)\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{\kappa P(t,T)}\right) - \kappa P(t,T)\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{\kappa P(t,T)}\right), \end{aligned}$$

   where

$$v^2 = \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds.$$

4. Compute the self-financing hedging strategy that hedges the bond option using a portfolio based on the assets $P(t,T)$ and $P(t,S)$.

Exercise 9.2. Extension of Proposition 9.2 to correlated Brownian motions. Assume that $(S_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equations

$$dS_t = r_t S_t dt + \sigma_t^S S_t dW_t^S, \quad \text{and} \quad dN_t = \eta_t N_t dt + \sigma_t^N N_t dW_t^N,$$

where $(W_t^S)_{t\in\mathbb{R}_+}$ and $(W_t^N)_{t\in\mathbb{R}_+}$ have the correlation

$$dW_t^S \cdot dW_t^N = \rho dt,$$

where $\rho \in [-1,1]$.

1. Show that $(W_t^N)_{t\in\mathbb{R}_+}$ can be written as

$$W_t^N = \rho W_t^S + \sqrt{1-\rho^2} W_t, \qquad t \in \mathbb{R}_+,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, independent of $(W_t^S)_{t\in\mathbb{R}_+}$.

2. Letting $X_t = S_t/N_t$, show that

$$dX_t = (r_t - \eta_t + (\sigma_t^N)^2 - \rho\sigma_t^N\sigma_t^S)X_t dt + \hat{\sigma}_t X_t dW_t^X,$$

where

$$\hat{\sigma}_t = \sqrt{(\sigma_t^S)^2 - 2\rho\sigma_t^R\sigma_t^S + (\sigma_t^R)^2}$$

and

$$W_t^X = \frac{\sigma_t^S - \rho\sigma_t^N}{\hat{\sigma}_t} W_t^S - \sqrt{1-\rho^2}\frac{\sigma_t^N}{\hat{\sigma}_t} W_t, \qquad t \in \mathbb{R}_+,$$

is a standard Brownian motion under $\mathbb{P}^*$.

Exercise 9.3. (Exercise 9.5 in [49]). Quanto options. Consider an asset priced S_t at time t, with

$$dS_t = rS_t dt + \sigma^S R_t dW_t^S,$$

and an exchange rate $(R_t)_{t\in\mathbb{R}_+}$ given by

$$dR_t = (r - r^f)R_t dt + \sigma^R R_t dW_t^R,$$

where $(W_t^R)_{t\in\mathbb{R}_+}$ is written as

$$W_t^R = \rho W_t^S + \sqrt{1-\rho^2} W_t, \qquad t \in \mathbb{R}_+,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, independent of $(W_t^S)_{t\in\mathbb{R}_+}$, i.e. we have

$$dW_t^R \cdot dW_t^S = \rho dt,$$

where ρ is a correlation coefficient.

1. Let

$$a = r - r^f + \rho\sigma^R\sigma^S - (\sigma^R)^2$$

and $X_t = e^{at} S_t/R_t$, $t \in \mathbb{R}_+$, and show that, by Exercise 9.2,

$$dX_t = rX_t dt + \hat{\sigma} X_t dW_t^X,$$

where $\hat{\sigma} = \sqrt{(\sigma^S)^2 - 2\rho\sigma^R\sigma^S + (\sigma^R)^2}$, and $(W_t^X)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$.

2. Show that the price

$$\mathbb{E}^*\left[\left(\frac{S_T}{R_T} - \kappa\right)^+ \Big| \mathcal{F}_t\right]$$

at time t of a quanto option is given by

$$\mathbb{E}^*\left[\left(\frac{S_T}{R_T}-\kappa\right)^+\Big|\mathcal{F}_t\right] = e^{-aT}\,\mathbb{E}^*\left[\left(X_T - e^{aT}\kappa\right)^+\Big|\mathcal{F}_t\right]$$
$$= \frac{S_t}{R_t}e^{(r-a)(T-t)}\Phi\left(\frac{(r-a+\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} + \frac{1}{\hat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right)$$
$$-\kappa e^{-r(T-t)}\Phi\left(\frac{(r-a-\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} + \frac{1}{\hat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right).$$

Chapter 10
Forward Rate Modeling

10.1 Short Term Models

Mean Reversion

The first model to capture the mean reversion property of interest rates, a property not possessed by geometric Brownian motion, is the Vasicek [51] model, which is based on the Ornstein-Uhlenbeck process. Here, the short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ solves the equation

$$dr_t = b(a - r_t)dt + \sigma dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with solution

$$r_t = r_0 e^{-bt} + a(1 - e^{-bt}) + \sigma \int_0^t e^{-b(t-s)} dB_s, \qquad t \in \mathbb{R}_+. \tag{10.1}$$

The law of r_t is Gaussian at all times t, with mean $r_0 e^{-bt} + a(1 - e^{-bt})$ and variance

$$\sigma^2 \int_0^t (e^{-b(t-s)})^2 ds = \sigma^2 \int_0^t e^{-2bs} ds = \frac{\sigma^2}{2b}.$$

This model has the interesting properties of being statistically stationary in time in the long run, and to admit a Gaussian $\mathcal{N}(a, \sigma^2/(2b))$ invariant distribution when $b > 0$, however its drawback is to allow for negative values of r_t.

The Cox-Ingersoll-Ross (CIR) [13] model brings a solution to the positivity problem encountered with the Vasicek model, by the use the nonlinear equation

$$dr_t = \beta(\alpha - r_t)dt + r_t^{1/2} \sigma dB_t.$$

Other classical mean reverting models include the Courtadon (1982) model

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dB_t$$

where α, β, σ are nonnegative, and the exponential-Vasicek model

$$dr_t = r_t(\eta - a \log r_t)dt + \sigma r_t dB_t,$$

where a, η, σ are nonnegative.

Constant Elasticity of Variance (CEV)

Constant Elasticity of Variance models are designed to take into account non-constant volatilities that can vary as a power of the underlying asset. The Marsh-Rosenfeld (1983) model

$$dr_t = (\beta r_t^{-(1-\gamma)} + \alpha r_t)dt + \sigma r_t^{\gamma/2} dB_t$$

where α, β, σ, γ are nonnegative constants, covers most of the CEV models. For $\gamma = 1$ this is the CIR model, and for $\beta = 0$ we get the standard CEV model

$$dr_t = \alpha r_t dt + \sigma r_t^{\gamma/2} dB_t.$$

If $\gamma = 2$ this yields the Dothan [15], [36] model

$$dr_t = \alpha r_t dt + \sigma r_t dB_t.$$

Affine Models

The class of short rate interest rate models admits a number of generalizations that can be found in the references quoted in the introduction of this chapter, among which is the class of affine models of the form

$$dr_t = (\eta(t) + \lambda(t) r_t)dt + \sqrt{\delta(t) + \gamma(t) r_t} dB_t. \tag{10.2}$$

Such models are called affine because the associated zero-coupon bonds can be priced using an affine PDE as will be seen in Proposition 10.1.

They also include the Ho-Lee model

$$dr_t = \theta(t)dt + \sigma dB_t,$$

where $\theta(t)$ is a deterministic function of time, and the Hull-White model

$$dr_t = (\theta(t) - \alpha(t) r_t)dt + \sigma(t) dB_t$$

which is a time-dependent extension of the Vasicek model.

10.2 Zero-Coupon Bonds

A zero-coupon bond is a contract priced $P_0(t,T)$ at time $t < T$ to deliver $P_0(T,T) = 1\$$ at time T. The computation of the arbitrage price $P_0(t,T)$ of a zero-coupon bond based on an underlying short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ is a basic and important issue in interest rate modeling.

We postulate that

$$P(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] \tag{10.3}$$

under some risk-neutral measure $\mathbb{P}^*$. It is natural to write $P(t,T)$ as a conditional expectation under a martingale measure, as the use of conditional expectation helps to "filter out" the future information past time t contained in $\displaystyle\int_t^T r_s ds$. Expression (10.3) makes sense as the "best possible estimate" of the future quantity $e^{-\int_t^T r_s ds}$ given information known up to time t.

Pricing bonds with non-zero coupon is not difficult in the case of a deterministic continuous-time coupon yield at rate $c > 0$. In this case the price $P_c(t,T)$ of the coupon bound is given by

$$P_c(t,T) = e^{c(T-t)} P_0(t,T), \qquad 0 \le t \le T,$$

In the sequel we will only consider zero-coupon bonds, and let $P(t,T) = P_0(t,T)$, $0 \le t \le T$.

The following proposition shows that the bond price process $t \mapsto P(t,T)$ can be taken as a numéraire.

Proposition 10.1. *The discounted bond price process*

$$t \mapsto e^{-\int_0^t r_s ds} P(t,T)$$

is a martingale under $\mathbb{P}^*$.

Proof. We have

$$\begin{aligned} e^{-\int_0^t r_s ds} P(t,T) &= e^{-\int_0^t r_s ds}\, \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^* \left[e^{-\int_0^t r_s ds} e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^* \left[e^{-\int_0^T r_s ds} \middle| \mathcal{F}_t \right] \end{aligned}$$

and this suffices to conclude since by the tower property, of conditional expectations, any process of the form $t \mapsto \mathbb{E}^*[F \mid \mathcal{F}_t]$, $F \in L^1(\Omega)$, is a martingale. □

Bond pricing PDE

We assume from now on that the underlying short rate process is solution to the stochastic differential equation

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dB_t \tag{10.4}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$.

Since all solutions of stochastic differential equations such as (10.4) have the Markov property, cf e.g. Theorem V-32 of [43], the arbitrage price $P(t, T)$ can be rewritten as a function $F(t, r_t)$ of r_t, i.e.

$$P(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \middle| r_t \right] = F(t, r_t),$$

and depends on r_t only instead of depending on all information available in $\mathcal{F}_t$ up to time t, meaning that the pricing problem can now be formulated as a search for the function $F(t, x)$.

Proposition 10.1. *The bond pricing PDE for $P(t, T) = F(t, r_t)$ is written as*

$$xF(t, x) = \mu(t, x)\frac{\partial F}{\partial x}(t, x) + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 F}{\partial x^2}(t, x) + \frac{\partial F}{\partial t}(t, x), \tag{10.5}$$

$t \in \mathbb{R}_+$, $x \in \mathbb{R}$, subject to the terminal condition

$$F(T, x) = 1, \qquad x \in \mathbb{R}. \tag{10.6}$$

Proof. From Itô's formula we have

$$\begin{aligned}
&d\left(e^{-\int_0^t r_s ds} P(t, T)\right) = -r_t e^{-\int_0^t r_s ds} P(t, T)dt + e^{-\int_0^t r_s ds} dP(t, T) \\
&= -r_t e^{-\int_0^t r_s ds} F(t, r_t)dt + e^{-\int_0^t r_s ds} dF(t, r_t) \\
&= -r_t e^{-\int_0^t r_s ds} F(t, r_t)dt + e^{-\int_0^t r_s ds} \frac{\partial F}{\partial x}(t, r_t)(\mu(t, r_t)dt + \sigma(t, r_t)dB_t) \\
&\quad + e^{-\int_0^t r_s ds} \left(\frac{1}{2}\sigma^2(t, r_t)\frac{\partial^2 F}{\partial x^2}(t, r_t)dt + \frac{\partial F}{\partial t}(t, r_t)dt \right) \\
&= e^{-\int_0^t r_s ds} \sigma(t, r_t)\frac{\partial F}{\partial x}(t, r_t)dB_t \\
&\quad + e^{-\int_0^t r_s ds} \left(-r_t F(t, r_t) + \mu(t, r_t)\frac{\partial F}{\partial x}(t, r_t) \right. \\
&\quad \left. + \frac{1}{2}\sigma^2(t, r_t)\frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \right) dt.
\end{aligned} \tag{10.7}$$

Given that $t \mapsto e^{-\int_0^t r_s ds} P(t, T)$ is a martingale, the above expression (10.7) should only contain terms in dB_t (cf. Corollary 1, p. 72 of [43]), and all terms

This version: March 8, 2011
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in dt should vanish inside (10.7). This leads to the identity

$$-r_t F(t,r_t) + \mu(t,r_t)\frac{\partial F}{\partial x}(t,r_t) + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial x^2}(t,r_t) + \frac{\partial F}{\partial t}(t,r_t) = 0.$$

Condition (10.6) is due to the fact that $P(T,T) = \$1$. □

The above proposition also shows that $t \mapsto e^{-\int_0^t r_s ds}P(t,T)$ satisfies the stochastic differential equations

$$d\left(e^{-\int_0^t r_s ds}P(t,T)\right) = e^{-\int_0^t r_s ds}\sigma(t,r_t)\frac{\partial F}{\partial x}(t,r_t)dB_t.$$

Consequently we have

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \frac{\sigma(t,r_t)}{P(t,T)}\frac{\partial F}{\partial x}(t,r_t)dB_t = r_t dt + \sigma(t,r_t)\frac{\partial \log F}{\partial x}(t,r_t)dB_t.$$

In the Vasicek case

$$dr_t = (a - br_t)dt + \sigma dW_t,$$

we get

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \frac{\sigma}{b}(1 - e^{-b(T-t)})dW_t.$$

Note that more generally, all affine short rate models as defined in Relation (10.2), including the Vasicek model, will yield a bond pricing formula of the form

$$P(t,T) = e^{A(T-t)+C(T-t)r_t},$$

cf. e.g. § 3.2.4. of [7].

Next we solve the PDE (10.5) by direct computation of the conditional expectation

$$P(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right], \tag{10.8}$$

in the Vasicek [51] model

$$dr_t = (a - br_t)dt + \sigma dB_t,$$

i.e. when the short rate $(r_t)_{t\in\mathbb{R}_+}$ has the expression

$$r_t = g(t) + \int_0^t h(t,s)dB_s = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma\int_0^t e^{-b(t-s)}dB_s,$$

where $g(t)$ and $h(t,s)$ are deterministic functions.

Letting $u \vee t = \max(u,t)$, using the fact that Wiener integrals are Gaussian random variables and the Gaussian characteristic function, we have

$$
\begin{aligned}
P(t,T) &= \mathbb{E}^*\left[e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[e^{-\int_t^T (g(s)+\int_0^s h(s,u)dB_u)ds}\middle|\mathcal{F}_t\right] \\
&= e^{-\int_t^T g(s)ds}\,\mathbb{E}^*\left[e^{-\int_t^T \int_0^s h(s,u)dB_u ds}\middle|\mathcal{F}_t\right] \\
&= e^{-\int_t^T g(s)ds}\,\mathbb{E}^*\left[e^{-\int_0^T \int_{u\vee t}^T h(s,u)dsdB_u}\middle|\mathcal{F}_t\right] \\
&= e^{-\int_t^T g(s)ds} e^{-\int_0^t \int_{u\vee t}^T h(s,u)dsdB_u}\,\mathbb{E}^*\left[e^{-\int_t^T \int_{u\vee t}^T h(s,u)dsdB_u}\middle|\mathcal{F}_t\right] \\
&= e^{-\int_t^T g(s)ds} e^{-\int_0^t \int_t^T h(s,u)dsdB_u}\,\mathbb{E}^*\left[e^{-\int_t^T \int_u^T h(s,u)dsdB_u}\middle|\mathcal{F}_t\right] \\
&= e^{-\int_t^T g(s)ds} e^{-\int_0^t \int_t^T h(s,u)dsdB_u}\,\mathbb{E}^*\left[e^{-\int_t^T \int_u^T h(s,u)dsdB_u}\right] \\
&= e^{-\int_t^T g(s)ds} e^{-\int_0^t \int_t^T h(s,u)dsdB_u} e^{\frac{1}{2}\int_t^T \left(\int_u^T h(s,u)ds\right)^2 du} \\
&= e^{-\int_t^T (r_0 e^{-bs}+\frac{a}{b}(1-e^{-bs}))ds} e^{-\sigma\int_0^t \int_t^T e^{-b(s-u)}dsdB_u} \\
&\quad \times e^{\frac{\sigma^2}{2}\int_t^T \left(\int_u^T e^{-b(s-u)}ds\right)^2 du} \\
&= e^{-\int_t^T (r_0 e^{-bs}+\frac{a}{b}(1-e^{-bs}))ds} e^{-\frac{\sigma}{b}(1-e^{-b(T-t)})\int_0^t e^{-b(t-u)}dB_u} \\
&\quad \times e^{\frac{\sigma^2}{2}\int_t^T e^{2bu}\left(\frac{e^{-bu}-e^{-bT}}{b}\right)^2 du} \\
&= e^{-\frac{r_t}{b}(1-e^{-b(T-t)})+\frac{1}{b}(1-e^{-b(T-t)})(r_0 e^{-bt}+\frac{a}{b}(1-e^{-bt}))} \\
&\quad \times e^{-\int_t^T (r_0 e^{-bs}+\frac{a}{b}(1-e^{-bs}))ds+\frac{\sigma^2}{2}\int_t^T e^{2bu}\left(\frac{e^{-bu}-e^{-bT}}{b}\right)^2 du} \\
&= e^{C(T-t)r_t+A(T-t)},
\end{aligned}
$$

where

$$C(T-t) = -\frac{1}{b}(1-e^{-b(T-t)}),$$

and

$$A(T-t) = \frac{4ab-3\sigma^2}{4b^3} + \frac{\sigma^2-2ab}{2b^2}(T-t) + \frac{\sigma^2-ab}{b^3}e^{-b(T-t)} - \frac{\sigma^2}{4b^3}e^{-2b(T-t)}.$$

In order to solve the PDE (10.5) analytically we may look for a solution of the form

$$F(t,x) = e^{A(T-t)+xC(T-t)}, \tag{10.9}$$

where A and C are functions to be determined under the conditions $A(0)=0$ and $C(0)=0$. Plugging (10.9) into the PDE (10.5) yields the system of Riccati and linear differential equations

$$\begin{cases} -A'(s) = -aC(s) - \dfrac{\sigma^2}{2}C^2(s) \\ \\ -C'(s) = bC(s)+1, \end{cases}$$

which can be solved to recover the above value of $P(t,T)$.

10.3 Forward Rates

A forward interest rate contract gives its holder a loan decided at present time t and to be delivered over a future period of time $[T, S]$ at a rate denoted by $f(t, T, S)$, $t \leq T \leq S$, and called a forward rate.

Let us determine the arbitrage or "fair" value of this rate using the instruments available in a bond market, which are bonds priced at $P(t, T)$ for various maturity dates $T > t$.

The loan can be realized using the bonds available on the market by proceeding in two steps:

1) borrow \$1 at time t, at the price $P(t, S)$. The amount \$1 is to be repaid at time S.
2) since one only needs the money at time T, it makes sense to invest the amount $P(t, S)$ over the period $[t, T]$ in a bond with maturity T, that will yield $P(t, S)/P(t, T)$ at time T.

As a consequence the investor will receive $P(t, S)/P(t, T)$ at time T and repay a unit amount at time S.

The corresponding forward rate $f(t, T, S)$, $0 \leq t \leq T \leq S$, is then given by the relation

$$\exp\left((S - T) f(t, T, S)\right) = \left(\frac{P(t, S)}{P(t, T)}\right)^{-1}, \tag{10.10}$$

where we used exponential compounding, which leads to the following definition.

Definition 10.1. The forward rate $f(t, T, S)$ at time t for a loan on $[T, S]$ is given by

$$f(t, T, S) = -\frac{\log P(t, S) - \log P(t, T)}{S - T}.$$

The *spot* forward rate $F(t, T)$ is given by

$$F(t, T) := f(t, t, T) = -\frac{\log P(t, T)}{T - t}.$$

Figure 10.1 presents a typical forward rate curve on the LIBOR (London Interbank Offered Rate) market with t =07 may 2003, $\delta =$ six months.

TimeSerieNb	505
AsOfDate	7-mai-03
2D	2,55
1W	2,53
1M	2,56
2M	2,52
3M	2,48
1Y	2,34
2Y	2,49
3Y	2,79
4Y	3,07
5Y	3,31
6Y	3,52
7Y	3,71
8Y	3,88
9Y	4,02
10Y	4,14
11Y	4,23
12Y	4,33
13Y	4,4
14Y	4,47
15Y	4,54
20Y	4,74
25Y	4,83
30Y	4,86

Fig. 10.1: Graph of $T \mapsto f(t, T, T + \delta)$.

The instantaneous forward rate $f(t, T)$ is defined by taking the limit of $f(t, T, S)$ as $S \searrow T$, i.e.

$$\begin{aligned} f(t,T) :&= -\lim_{S \searrow T} \frac{\log P(t,S) - \log P(t,T)}{S - T} \\ &= -\lim_{\varepsilon \searrow 0} \frac{\log P(t, T+\varepsilon) - \log P(t,T)}{\varepsilon} \\ &= -\frac{\partial \log P(t,T)}{\partial T} \\ &= -\frac{1}{P(t,T)} \frac{\partial P(t,T)}{\partial T}. \end{aligned}$$

The above equation can be viewed as a differential equation to be solved for $\log P(t, T)$ under the initial condition $P(T, T) = 1$, which yields the following proposition.

Proposition 10.2. *We have*

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right), \qquad 0 \le t \le T. \tag{10.11}$$

Proof. We check that

$$\log P(t,T) = \log P(t,T) - \log P(t,t) = \int_t^T \frac{\partial \log P(t,s)}{\partial s} ds = -\int_t^T f(t,s)ds.$$

□

As a consequence of (10.10) and (10.11) the forward rate $f(t,T,S)$ can be recovered from the instantaneous forward rate $f(t,s)$, as:

$$f(t,T,S) = \frac{1}{S-T}\int_T^S f(t,s)ds, \qquad 0 \le t \le T < S. \tag{10.12}$$

Swap Rates

An interest rate swap makes it possible to exchange a variable forward rate $f(t,T,S)$ against a fixed rate κ. Such an exchange will generate a cash flow valued *at time* t as

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(f(t,T_k,T_{k+1})-\kappa),$$

in which we use linear interest rate compounding and the bond prices as discount factors.

The swap rate $S(t,T_i,T_j)$ is by definition the fair value of κ that cancels this cash flow:

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(f(t,T_k,T_{k+1})-S(t,T_i,T_j)) = 0, \tag{10.13}$$

and it satisfies

$$S(t,T_i,T_j) = \frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})f(t,T_k,T_{k+1}), \tag{10.14}$$

where $P(t,T_i,T_j)$ is the annuity numeraire.

10.4 The HJM Model

In this section we consider the problem of how to model the instantaneous forward rate $f(t,T)$. The graph given in Figure 10.1 presents a possible random evolution of a forward interest rate curve using the Musiela convention, i.e. we will write

$$g(x) = f(t,t+x) = f(t,T),$$

under the substitution $x = T-t$, $x \ge 0$, and represent a sample of the instantaneous forward curve $x \mapsto f(t,t+x)$ for each $t \in \mathbb{R}_+$. In the HJM

Fig. 10.1: Stochastic process of forward curves.

model, the instantaneous forward rate $f(t,T)$ is modeled by a stochastic differential equation of the form

$$d_t f(t,T) = \alpha(t,T)dt + \sigma(t,T)dB_t, \tag{10.15}$$

where $t \mapsto \alpha(t,T)$ and $t \mapsto \sigma(t,T)$, $0 \leq t \leq T$, are allowed to be random (adapted) processes. In the above equation, the date T is fixed and the differential d_t is with respect to t.

Next, we determine the dynamics of the spot forward rate

$$f(t,t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds.$$

The [25] Hull-White model

$$dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dB_t,$$

cf. Section 10.1, with explicit solution

$$r_t = r_s e^{-\int_s^t b(\tau)d\tau} + \int_s^t e^{-\int_u^t b(\tau)d\tau} a(u)du + \int_s^t \sigma(u)e^{-\int_u^t b(\tau)d\tau} dB_u,$$

$0 \leq s \leq t$, can be obtained from a HJM model with deterministic coefficients $\alpha(t,T)$ and $\sigma(t,T)$, by assuming that the short rate has the Markov property, cf. § 6.6 of [37].

The HJM Condition

An important question is to determine how absence of arbitrage can be "encoded" in the defining equation (10.15). Recall that under absence of arbitrage, the bond price $P(t,T)$ has been defined as

$$P(t,T) = \mathbb{E}^* \left[\exp\left(-\int_t^T r_s ds \right) \Big| \mathcal{F}_t \right], \tag{10.16}$$

and that the discounted bond price process

$$\begin{aligned} t \mapsto \exp\left(-\int_0^t r_s ds \right) P(t,T) &= \exp\left(-\int_0^t r_s ds - \int_t^T f(t,s) ds \right) \\ &= \exp\left(-\int_0^t r_s ds - X_t \right) \end{aligned} \tag{10.17}$$

is a martingale by Proposition 10.1 and Relation (10.11).

Proposition 10.3. *The fact that* (10.17) *is a martingale implies that*

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s) ds, \tag{10.18}$$

which is known as the HJM absence of arbitrage condition*, cf. [24].*

Proof. Writing

$$X_t = \int_t^T f(t,s) ds = -\log P(t,T), \qquad 0 \le t \le T,$$

we have

$$f(t,t,T) = \frac{1}{T-t} \int_t^T f(t,s) ds = \frac{X_t}{T-t}, \qquad 0 \le t \le T, \tag{10.19}$$

where the dynamics of $t \mapsto f(t,s)$ is given by (10.15). We have

$$\begin{aligned} d_t X_t &= -f(t,t)dt + \int_t^T d_t f(t,s) ds \\ &= -f(t,t)dt + \int_t^T \alpha(t,s) ds dt + \int_t^T \sigma(t,s) ds dB_t \\ &= -r_t dt + \left(\int_t^T \alpha(t,s) ds \right) dt + \left(\int_t^T \sigma(t,s) ds \right) dB_t, \end{aligned}$$

hence

$$|d_t X_t|^2 = \left(\int_t^T \sigma(t,s) ds \right)^2 dt.$$

Hence by Itô's calculus we have

$$\begin{aligned}
d_t e^{-X_t} &= -e^{-X_t} d_t X_t + \frac{1}{2} e^{-X_t} (d_t X_t)^2 \\
&= -e^{-X_t} d_t X_t + \frac{1}{2} e^{-X_t} \left(\int_t^T \sigma(t,s) ds \right)^2 dt \\
&= -e^{-X_t} \left(-r_t dt + \int_t^T \alpha(t,s) ds dt + \int_t^T \sigma(t,s) ds dB_t \right) \\
&\quad + \frac{1}{2} e^{-X_t} \left(\int_t^T \sigma(t,s) ds \right)^2 dt,
\end{aligned}$$

and

$$\begin{aligned}
& d_t \exp\left(-\int_0^t r_s ds - \int_t^T f(t,s) ds \right) = d_t \exp\left(-\int_0^t r_s ds - X_t \right) \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t \right) dt + \exp\left(-\int_0^t r_s ds \right) d_t e^{-X_t} \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t \right) dt - \exp\left(-\int_0^t r_s ds - X_t \right) d_t X_t \\
&\quad + \frac{1}{2} \exp\left(-\int_0^t r_s ds - X_t \right) \left(\int_t^T \sigma(t,s) ds \right)^2 dt \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t \right) dt \\
&\quad - \exp\left(-\int_0^t r_s ds - X_t \right) \left(-r_t dt + \int_t^T \alpha(t,s) ds dt + \int_t^T \sigma(t,s) ds dB_t \right) \\
&\quad + \frac{1}{2} \exp\left(-\int_0^t r_s ds - X_t \right) \left(\int_t^T \sigma(t,s) ds \right)^2 dt \\
&= -\exp\left(-\int_0^t r_s ds - X_t \right) \int_t^T \sigma(t,s) ds dB_t \\
&\quad - \exp\left(-\int_0^t r_s ds - X_t \right) \left(\int_t^T \alpha(t,s) ds dt - \frac{1}{2} \left(\int_t^T \sigma(t,s) ds \right)^2 \right) dt.
\end{aligned}$$

Thus the martingale property of the above process implies that

$$\int_t^T \alpha(t,s)ds - \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 = 0, \qquad 0 \le t \le T. \tag{10.20}$$

Differentiating the above relation with respect to T, we get

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds,$$

which is in fact equivalent to (10.20). □

10.5 Forward Vasicek Rates

In this section we consider the Vasicek model, in whcih the short rate process is solution of

$$dr_t = (a - br_t)dt + \sigma dB_t, \tag{10.21}$$

with

$$r_t = e^{-bt}r_0 + \frac{a}{b}(1 - e^{-bt}) + \sigma\int_0^t e^{-b(t-s)}dB_s, \qquad t \in \mathbb{R}_+.$$

Figure 10.1 presents a random simulation of $t \mapsto r_t$ in the Vasicek model with $r_0 = a/b = 5\%$, i.e. the reverting property of the process is with respect to its initial value $r_0 = 5\%$. Note that the interest rate in Figure 10.1 becomes negative for a short period of time, which is unusual for interest rates but may nevertheless happen.

Fig. 10.1: Graph of $t \mapsto r_t$.

Figure 10.2 presents a random simulation of $t \mapsto P(t,T)$ in the same Vasicek model. The graph of the corresponding deterministic bond price obtained for $a = b = \sigma = 0$ is also shown on the same Figure 10.2.

Fig. 10.2: Graphs of $t \mapsto P(t,T)$ and $t \mapsto e^{-r_0(T-t)}$.

Figure 10.3 presents a random simulation of $t \mapsto P(t,T)$ for a coupon bond with price $P_c(t,T) = e^{c(T-t)}P(t,T)$, $0 \leq t \leq T$.

Fig. 10.3: Graph of $t \mapsto P(t,T)$ for a bond with a 2.3% coupon.

The above simulation can be compared to the actual market data of a coupon bond in Figure 10.2 below.

Fig. 10.4: Graph of a bond price with maturity Jan 18, 2008 and coupon rate 6.25%.

Concerning the forward rate we find

$$
\begin{aligned}
f(t,T,S) &= -\frac{\log P(t,S) - \log P(t,T)}{S-T} \\
&= -\frac{r_t(C(S-t) - C(T-t)) + A(S-t) - A(T-t))}{S-T} \\
&= -\frac{\sigma^2 - 2ab}{2b^2} \\
&-\frac{1}{S-T}\left(\left(\frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3}\right)(e^{-b(S-t)} - e^{-b(T-t)}) - \frac{\sigma^2}{4b^3}(e^{-2b(S-t)} - e^{-2b(T-t)})\right).
\end{aligned}
$$

In this model the forward rate $t \mapsto f(t,T,S)$ can be represented as in Figure 10.5, with here $b/a > r_0$.

Fig. 10.5: Forward rate process $t \mapsto f(t,T,S)$.

Note that the forward rate cure $t \mapsto f(t,T,S)$ is flat for small values of t.

The instantaneous short rate is given by

$$\begin{aligned} f(t,T) :&= -\frac{\partial \log P(t,T)}{\partial T} \\ &= r_t e^{-b(T-t)} + \frac{a}{b}(1-e^{-b(T-t)}) - \frac{\sigma^2}{2b^2}(1-e^{-b(T-t)})^2, \end{aligned} \tag{10.22}$$

and the relation $\lim_{T \searrow t} f(t,T) = r_t$ is easy to recover from this formula.

The instantaneous forward rate $t \mapsto f(t,T)$ can be represented as in Figure 10.6, with here $t = 0$ and $b/a > r_0$:

Fig. 10.6: Instantaneous forward rate process $t \mapsto f(t,T)$.

The HJM coefficients in the Vasicek model are in fact deterministic and taking $a = 0$ we have

$$d_t f(t,T) = \sigma^2 e^{-b(T-t)} \int_t^T e^{b(t-s)} ds dt + \sigma e^{-b(T-t)} dB_t,$$

i.e.

$$\alpha(t,T) = \sigma^2 e^{-b(T-t)} \int_t^T e^{-b(T-s)} ds, \qquad \text{and} \qquad \sigma(t,T) = \sigma e^{-b(T-t)},$$

and the HJM condition reads

$$\alpha(t,T) = \sigma^2 e^{-b(T-t)} \int_t^T e^{-b(s-t)} ds = \sigma(t,T) \int_t^T \sigma(t,s) ds. \tag{10.23}$$

Random simulation of the Vasicek instantaneous forward rates are given in Figures 10.7 and 10.8.

Fig. 10.7: Forward instantaneous curve $(t, x) \mapsto f(t, t + x)$ in the Vasicek model.

Fig. 10.8: Forward instantaneous curve $x \mapsto f(0, x)$ in the Vasicek model.

For $x = 0$ the first "slice" of this surface is actually the short rate Vasicek process $r_t = f(t,t) = f(t,t+0)$ which is represented in Figure 10.9 using another discretization.

Fig. 10.9: Short term interest rate curve $t \mapsto r_t$ in the Vasicek model.

10.6 Modeling Issues

Parametrization of Forward Rates

In the Nelson-Siegel parametrization the forward interest rate curves are parametrized by 4 coefficients z_1, z_2, z_3, z_4, as

$$g(x) = z_1 + (z_2 + z_3 x)e^{-xz_4}, \qquad x \geq 0.$$

An example of a graph obtained by the Nelson-Siegel parametrization is given in Figure 10.1, for $z_1 = 1$, $z_2 = -10$, $z_3 = 100$, $z_4 = 10$.

Fig. 10.1: Graph of $x \mapsto g(x)$ in the Nelson-Siegel model.

The Svensson parametrization has the advantage to reproduce two humps instead of one, the location and height of which can be chosen via 6 parameters z_1, z_2, z_3, z_4, z_5, z_6 as

$$g(x) = z_1 + (z_2 + z_3 x)e^{-xz_4} + z_5 x e^{-xz_6}, \qquad x \geq 0.$$

A typical graph of a Svensson parametrization is given in Figure 10.2, for $z_1 = 7$, $z_2 = -5$, $z_3 = -100$, $z_4 = 10$, $z_5 = -1/2$, $z_6 = 1$.

Fig. 10.2: Graph of $x \mapsto g(x)$ in the Svensson model.

Figure 10.3 presents a fit of the market data of Figure 10.1 using a Svensson curve.

Fig. 10.3: Comparison of market data *vs* a Svensson curve.

One may think of constructing an instantaneous rate process taking values in the Svensson space, however this type of modelization is not consistent with absence of arbitrage, and it is proved that the HJM curves can not live in the Nelson-Siegel or Svensson spaces, cf. §3.5 of [5].

For example it can be easily shown that the forward curves of the Vasicek model are included neither in the Nelson-Siegel space, nor in the Svensson space, cf. [37] and §3.5 of [5]. In addition, such curves do not appear to correctly model the market forward curves considered above, cf. e.g. Figure 10.1. In the Vasicek model we have

$$\frac{\partial f}{\partial T}(t,T) = \left(-br_t + a - \frac{\sigma^2}{b} + \frac{\sigma^2}{b}e^{-b(T-t)}\right)e^{-b(T-t)},$$

and one can check that the sign of the derivatives of f can only change once at most. As a consequence, the possible forward curves in the Vasicek model are limited to one change of "regime" per curve, as illustrated in Figure 10.4 for various values of r_t, and in Figure 10.5.

Fig. 10.4: Graphs of forward rates.

Fig. 10.5: Forward instantaneous curve $(t, x) \mapsto f(t, t + x)$ in the Vasicek model.

Another way to deal with the curve fitting problem is to use deterministic shifts for the fitting of one forward curve, such as the initial curve at $t = 0$, cf. e.g. § 8.2. of [37].

The Correlation Problem and a Two-Factor Model

The correlation problem is another issue of concern when using the affine models considered so far. Let us consider the following bond price simulation with maturity $T_3 = 30$, cf. Figure 10.6.

Fig. 10.6: Graph of $t \mapsto P(t, T_3)$.

Next, the simulation of another bond with maturity $T_2 = 20$ is given in Figure 10.7.

Fig. 10.7: Graph of $t \mapsto P(t, T_2)$.

Let us compare these graphs together with a simulation of a bond with maturity $T_1 = 10$ as given in Figure 10.8. Clearly, the bond prices $P(t, T_1)$ and $P(t, T_2)$ with maturities T_1 and T_2 are linked by the relation

$$P(t, T_2) = P(t, T_1) \exp(A(t, T_2) - A(t, T_1) + r_t(C(t, T_2) - C(t, T_1)),$$

meaning that bond prices with different maturities could be deduced from each other, which is unrealistic.

Fig. 10.8: Graph of $t \mapsto P(t, T_1)$.

For affine models of short rates we have the perfect correlation

$$\mathrm{Cor}(\log P(t, T_1), \log P(t, T_2)) = 1,$$

cf. § 8.3 of [37], since $\log P(t, T_1)$ and $\log P(t, T_2)$ two linear combinations of a same random variable Z. A solution to the correlation problem is to consider two control processes $(X_t)_{t\in\mathbb{R}_+}$, $(Y_t)_{t\in\mathbb{R}_+}$ which are solution of

$$\begin{cases} dX_t = \mu_1(t, X_t)dt + \sigma_1(t, X_t)dB_t^{(1)}, \\ \\ dY_t = \mu_2(t, Y_t)dt + \sigma_2(t, Y_t)dB_t^{(2)}, \end{cases} \tag{10.24}$$

where $(B_t^{(1)})_{t\in\mathbb{R}_+}$, $(B_t^{(2)})_{t\in\mathbb{R}_+}$ have correlated Brownian motion with

$$\mathrm{Cov}(B_s^{(1)}, B_t^{(2)}) = \rho \min(s, t), \qquad s, t \in \mathbb{R}_+, \tag{10.25}$$

and

$$dB_t^{(1)} dB_t^{(2)} = \rho dt, \tag{10.26}$$

for some $\rho \in [-1, 1]$. In practice, $(B^{(1)})_{t\in\mathbb{R}_+}$ and $(B^{(2)})_{t\in\mathbb{R}_+}$ can be constructed from two independent Brownian motions $(W^{(1)})_{t\in\mathbb{R}_+}$ and $(W^{(2)})_{t\in\mathbb{R}_+}$, by letting

$$\begin{cases} B_t^{(1)} = W_t^{(1)}, \\ B_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}, \qquad t \in \mathbb{R}_+, \end{cases}$$

and Relations (10.25) and (10.26) are easily satisfied from this construction.

In two-factor models one chooses to build the short term interest rate r_t via

$$r_t = X_t + Y_t, \qquad t \in \mathbb{R}_+.$$

By the previous standard arbitrage arguments we define the price of a bond with maturity T as

$$\begin{aligned} P(t,T) :&= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| X_t,\ Y_t\right] \\ &= F(t, X_t, Y_t), \end{aligned} \tag{10.27}$$

since the couple $(X_t, Y_t)_{t\in\mathbb{R}_+}$ is Markovian. Using the Itô formula with two variables and the fact that

$$t \mapsto e^{-\int_0^t r_s ds} P(t,T) = e^{-\int_0^t r_s ds}\, \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right]$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$ we can derive a PDE

$$\begin{aligned} &-(x+y)F(t,x,y) + \mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y) + \mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y) \\ &+\frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y) \\ &+\rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x \partial y}(t,x,y) + \frac{\partial F}{\partial t}(t,X_t,Y_t)dt = 0, \end{aligned} \tag{10.28}$$

on $\mathbb{R}^2$ for the bond price $P(t,T)$. In the Vasicek model

$$\begin{cases} dX_t = -aX_t dt + \sigma dB_t^{(1)}, \\ dY_t = -bY_t dt + \eta dB_t^{(2)}, \end{cases} \tag{10.29}$$

this yields

$$P(t,T) = F_1(t,X_t)F_2(t,Y_t)\exp\left(U(t,T)\right), \tag{10.30}$$

where $F_1(t, X_t)$ and $F_2(t, Y_t)$ are the bond prices associated to X_t and Y_t in the Vasicek model, and

$$U(t,T) = \rho \frac{\sigma\eta}{ab} \left(T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a+b} \right)$$

is a correlation term which vanishes when $(B_t^{(1)})_{t\in\mathbb{R}_+}$ and $(B_t^{(2)})_{t\in\mathbb{R}_+}$ are independent, i.e. when $\rho = 0$, cf [7], Chapter 4, Appendix A, and [37].

Partial differentiation of $\log P(t,T)$ with respect to T leads to the instantaneous forward rate

$$f(t,T) = f_1(t,T) + f_2(t,T) - \rho \frac{\sigma\eta}{ab}(1 - e^{-a(T-t)})(1 - e^{-b(T-t)}), \qquad (10.31)$$

where $f_1(t,T)$, $f_2(t,T)$ are the instantaneous forward rates corresponding to X_t and Y_t respectively, cf. § 8.4 of [37].

An example of a forward rate curve obtained in this way is given in Figure 10.9.

Fig. 10.9: Graph of forward rates in a two-factor model.

Next in Figure 10.10 we present a graph of the evolution of forward curve in a two factor model.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Fig. 10.10: Random evolution of forward rates in a two-factor model.

10.7 The LIBOR Model

Recall that the forward rate $f(t,T,S)$, $0 \le t \le T \le S$, is defined using exponential compounding, from the relation

$$f(t,T,S) = -\frac{\log P(t,S) - \log P(t,T)}{S-T}. \tag{10.32}$$

In order to compute swaption prices one prefers to use forward rates as defined on the London InterBank Offered Rates (LIBOR) market instead of the standard forward rates given by (10.32).

The forward LIBOR $L(t,T,S)$ for a loan on $[T,S]$ is defined using linear compounding, i.e. by replacing (10.32) with the relation

$$1 + (S-T)L(t,T,S) = \frac{P(t,T)}{P(t,S)},$$

which yields the following definition.

Definition 10.2. The forward LIBOR rate $L(t,T,S)$ at time t for a loan on $[T,S]$ is given by

$$L(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right), \qquad 0 \le t \le T < S.$$

Swap Rates on the LIBOR Market

From (10.13) the forward swap rate $S(t, T_i, T_j)$ on the LIBOR market satisfies

$$\sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t, T_{k+1})(L(t, T_k, T_{k+1}) - S(t, T_i, T_j)) = 0.$$

Proposition 10.2. *We have*

$$S(t, T_i, T_j) = \frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)}, \qquad 0 \le t \le T_i, \quad 1 \le i < j \le n.$$

Proof. By definition of the forward LIBOR $L(t, T, S)$ we have

$$P(t, T_k) - P(t, T_{k+1}) - (T_{k+1} - T_k)P(t, T_{k+1})L(t, T_k, T_{k+1}) = 0,$$

hence by summation on $k = i, \ldots, j-1$ we get

$$P(t, T_i) - P(t, T_j) - \sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t, T_{k+1})L(t, T_k, T_{k+1}) = 0.$$

Finally, (10.14) yields

$$\begin{aligned} S(t, T_i, T_j) &= \frac{1}{P(t, T_i, T_j)} \sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t, T_{k+1})L(t, T_k, T_{k+1}) \\ &= \frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)}. \end{aligned} \tag{10.33}$$

□

Clearly, a simple expression for the swap rate such as that of Proposition 10.2 cannot be obtained using the standard (i.e. non LIBOR) rates defined in (10.32).

When $j = i + 1$, the swap rate $S(t, T_i, T_{i+1})$ coincides with the forward rate $L(t, T_i, T_{i+1})$:

$$S(t, T_i, T_{i+1}) = L(t, T_i, T_{i+1}), \qquad 1 \le i \le n-1, \tag{10.34}$$

and the bond prices $P(t, T_i)$ can be recovered from the forward swap rates $S(t, T_i, T_j)$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

The BGM Model

The models (HJM, affine, etc.) considered in the previous chapter suffer from various drawbacks such as non-positivity of interest rates in Vasicek model, and lack of explicit analytical formulas in more complex models, The BGM [6] model has the advantage of yielding positive interest rates, and to permit to derive explicit formulas for the computation of prices for interest rate derivatives such as caps and swaptions on the LIBOR market.

In the BGM model the forward LIBOR rate $L(t, T_i, T_{i+1})$ is a geometric Brownian motion under $\mathbb{P}_{i+1}$, i.e.

$$\frac{dL(t, T_i, T_{i+1})}{L(t, T_i, T_{i+1})} = \gamma_i(t) dB_t^{i+1}, \tag{10.35}$$

$0 \leq t \leq T_i$, $i = 1, \ldots, n-1$, for some deterministic function $\gamma_i(t)$, $i = 1, \ldots, n-1$, with solution

$$L(u, T_i, T_{i+1}) = L(t, T_i, T_{i+1}) \exp\left(\int_t^u \gamma_i(s) dB_s^{i+1} - \frac{1}{2} \int_t^u |\gamma_i|^2(s) ds \right),$$

i.e. for $u = T_i$,

$$L(T_i, T_i, T_{i+1}) = L(t, T_i, T_{i+1}) \exp\left(\int_t^{T_i} \gamma_i(s) dB_s^{i+1} - \frac{1}{2} \int_t^{T_i} |\gamma_i|^2(s) ds \right).$$

Since $L(t, T_i, T_{i+1})$ is a geometric Brownian motion under $\mathbb{P}_{i+1}$, $i = 0, \ldots, n-1$, standard caplets can be priced at time $t \in [0, T_i]$ from the Black-Scholes formula.

10.8 Exercises

Exercise 10.1. We consider a short term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ in a centered Ho-Lee model with constant deterministic volatility:

$$dr_t = \sigma dW_t, \qquad \sigma > 0,$$

and let $P(t, T)$ will denote the arbitrage price of a zero-coupon bond in this model:

$$P(t, T) = \mathbb{E}^* \left[\exp\left(- \int_t^T r_s ds \right) \Big| \mathcal{F}_t \right], \qquad 0 \leq t \leq T. \tag{10.36}$$

1. State the bond pricing PDE satisfied by the function $F(t,x)$ defined via

$$F(t,x) = \mathbb{E}^* \left[\exp\left(-\int_t^T r_s ds \right) \Big| r_t = x \right], \qquad 0 \le t \le T.$$

2. Compute the arbitrage price $F(t,r_t) = P(t,T)$ from its expression (10.36) as a conditional expectation.

   Hint. One may use the *integration by parts* relation

$$\begin{aligned} \int_t^T B_s ds &= TB_T - tB_t - \int_t^T s dB_s \\ &= (T-t)B_t + T(B_T - B_t) - \int_t^T s dB_s \\ &= (T-t)B_t + \int_t^T (T-s) dB_s, \end{aligned}$$

   and the Laplace transform identity $\mathbb{E}[e^{\lambda X}] = e^{\lambda^2 \eta^2}$ for $X \simeq \mathcal{N}(0,\eta^2)$.
3. Check that the function $F(t,x)$ computed in question 2 does satisfy the PDE derived in question 1.
4. Compute the forward rate $f(t,T,S)$ in this model.
5. Compute the instantaneous forward rate $f(t,T)$ in this model.
6. Derive the stochastic equation satisfied by the instantaneous forward rate $f(t,T)$.
7. Check that the HJM absence of arbitrage condition is satisfied in this equation.

Exercise 10.2. Let $(r_t)_{t\in\mathbb{R}_+}$ denote a short term interest rate process. For any $T > 0$, let $P(t,T)$ denote the price at time $t \in [0,T]$ of a zero coupon bond defined by the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma_t^T dB_t, \qquad 0 \le t \le T, \tag{10.37}$$

under the terminal condition $P(T,T) = 1$, where $(\sigma_t^T)_{t\in[0,T]}$ is an adapted process. Let the forward measure $\mathbb{P}_T$ be defined by

$$\mathbb{E}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}} \Big| \mathcal{F}_t \right] = \frac{P(t,T)}{P(0,T)} e^{-\int_0^t r_s ds}, \qquad 0 \le t \le T.$$

Recall that

$$B_t^T := B_t - \int_0^t \sigma_s^T ds, \qquad 0 \le t \le T,$$

is a standard Brownian motion under $\mathbb{P}_T$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

1. Solve the stochastic differential equation (10.37).
2. Derive the stochastic differential equation satisfied by the discounted bond price process

$$t \mapsto e^{-\int_0^t r_s ds} P(t,T), \qquad 0 \le t \le T,$$

and show that it is a martingale.
3. Show that

$$\mathbb{E}\left[e^{-\int_0^T r_s ds} \middle| \mathcal{F}_t \right] = e^{-\int_0^t r_s ds} P(t,T), \qquad 0 \le t \le T.$$

4. Show that

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right], \qquad 0 \le t \le T.$$

5. Compute $P(t,S)/P(t,T)$, $0 \le t \le T$, show that it is a martingale under $\mathbb{P}_T$ and that

$$P(T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(\int_t^T (\sigma_s^S - \sigma_s^T) dB_s^T - \frac{1}{2} \int_t^T (\sigma_s^S - \sigma_s^T)^2 ds \right).$$

6. Assuming that $(\sigma_t^T)_{t\in[0,T]}$ and $(\sigma_t^S)_{t\in[0,S]}$ are deterministic functions, compute the price

$$\mathbb{E}\left[e^{-\int_t^T r_s ds} \left(P(T,S) - \kappa\right)^+ \middle| \mathcal{F}_t \right] = P(t,T)\, \mathbb{E}_T\left[\left(P(T,S) - \kappa\right)^+ \middle| \mathcal{F}_t \right]$$

of a bond option with strike κ.

Recall that if X is a centered Gaussian random variable with mean m_t and variance v_t^2 given $\mathcal{F}_t$, we have

$$\begin{aligned} \mathbb{E}[(e^X - K)^+ \mid \mathcal{F}_t] = {} & e^{m_t + v_t^2/2} \Phi\left(\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log K) \right) \\ & - K\Phi\left(-\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log K) \right) \end{aligned}$$

where $\Phi(x)$, $x \in \mathbb{R}$, denotes the Gaussian distribution function.

Exercise 10.3. (Exercise 4.4 continued). Assume that the price $P(t,T)$ of a zero coupon bond is modeled as

$$P(t,T) = e^{-\mu(T-t) + X_t^T}, \qquad t \in [0,T],$$

where $\mu > 0$. Show that the terminal condition $P(T,T) = 1$ is satisfied.

Exercise 10.4. (Exercise 4.6 continued). Write down the bond pricing PDE for the function

$$F(t,x) = E\left[e^{-\int_t^T r_s ds}\Big| r_t = x\right]$$

and show that in case $\alpha = 0$ the corresponding bond price $P(t,T)$ equals

$$P(t,T) = e^{-B(T-t)r_t}, \qquad 0 \le t \le T,$$

where

$$B(x) = \frac{2(e^{\gamma x}-1)}{2\gamma + (\beta+\gamma)(e^{\gamma x}-1)},$$

with $\gamma = \sqrt{\beta^2 + 2\sigma^2}$.

Exercise 10.5. (Exercise 4.4 continued).

1. Compute the forward rate

$$f(t,T,S) = -\frac{1}{S-T}(\log P(t,S) - \log P(t,T)).$$

2. Compute the instantaneous forward rate

$$f(t,T) = -\lim_{S \searrow T} \frac{1}{S-T}(\log P(t,S) - \log P(t,T)).$$

3. Show that the limit $\lim\limits_{T \searrow t} f(t,T)$ does not exist in $L^2(\Omega)$.
4. Show that $P(t,T)$ satisfies the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = \sigma dB_t + \frac{1}{2}\sigma^2 dt - \frac{\log P(t,T)}{T-t}dt, \qquad t \in [0,T].$$

5. Show, using the results of Exercise 10.2-(4), that

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s^T ds}\Big| \mathcal{F}_t\right],$$

   where $(r_t^T)_{t\in[0,T]}$ is a process to be determined.
6. Compute the conditional density

$$\mathbb{E}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}}\Big| \mathcal{F}_t\right] = \frac{P(t,T)}{P(0,T)} e^{-\int_0^t r_s^T ds}$$

   of the forward measure $\mathbb{P}_T$ with respect to $\mathbb{P}$.
7. Show that the process

$$\tilde{B}_t := B_t - \sigma t, \qquad 0 \le t \le T,$$

   is a standard Brownian motion under $\mathbb{P}_T$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

8. Compute the dynamics of X_t^S and $P(t,S)$ under $\mathbb{P}_T$.
   Hint: Show that

$$-\mu(S-T)+\sigma(S-T)\int_0^t \frac{1}{S-s}dB_s = \frac{S-T}{S-t}\log P(t,S).$$

9. Compute the bond option price

$$\mathbb{E}\left[e^{-\int_t^T r_s^T ds}(P(T,S)-K)^+\Big|\mathcal{F}_t\right] = P(t,T)\,\mathbb{E}_T\left[(P(T,S)-K)^+\Big|\mathcal{F}_t\right],$$

   $0 \le t < T < S$.

Chapter 11
Pricing of Interest Rate Derivatives

11.1 Forward Measures and Tenor Structure

The maturity dates are arranged according to a discrete *tenor structure*

$$\{0 = T_0 < T_1 < T_2 < \cdots < T_n\}.$$

An example of forward interest rate curve data is given in the table of Figure 11.1, which contains the values of $(T_1, T_2, \ldots, T_{23})$ and of $\{f(t, t+T_i, t+T_i+\delta)\}_{i=1,\ldots,23}$, with $t = 07/05/2003$ and $\delta =$ six months.

TimeSerieNb	505
AsOfDate	7-mai-03
2D	2,55
1W	2,53
1M	2,56
2M	2,52
3M	2,48
1Y	2,34
2Y	2,49
3Y	2,79
4Y	3,07
5Y	3,31
6Y	3,52
7Y	3,71
8Y	3,88
9Y	4,02
10Y	4,14
11Y	4,23
12Y	4,33
13Y	4,4
14Y	4,47
15Y	4,54
20Y	4,74
25Y	4,83
30Y	4,86

Fig. 11.1: Forward rates according to a tenor structure.

Recall that by definition of $P(t, T_i)$ and absence of arbitrage the process

$$t \mapsto e^{-\int_0^t r_s ds} P(t, T_i), \qquad 0 \le t \le T_i, \quad i = 1, \ldots, n,$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}$, and as a consequence $(P(t,T))_{t\in[0,T]}$ can be taken as numéraire in the definition

$$\frac{d\hat{\mathbb{P}}_i}{d\mathbb{P}} = \frac{1}{P(0,T_i)} e^{-\int_0^{T_i} r_s ds}$$

of the *forward measure* $\hat{\mathbb{P}}_i$. The following proposition will allow us to price contingent claims using the forward measure $\hat{\mathbb{P}}_i$.

Proposition 11.1. *For all sufficiently integrable random variables F we have*

$$\mathbb{E}\left[F e^{-\int_t^{T_i} r_s ds} \middle| \mathcal{F}_t \right] = P(t,T_i)\hat{\mathbb{E}}_i[F \mid \mathcal{F}_t], \qquad 0 \le t \le T, \quad i = 1,\ldots,n. \tag{11.1}$$

Recall that for all $T_i, T_j \ge 0$, the process

$$t \mapsto \frac{P(t,T_j)}{P(t,T_i)}, \qquad 0 \le t \le T_i \wedge T_j,$$

is an $\mathcal{F}_t$-martingale under $\hat{\mathbb{P}}_i$.

Dynamics under the forward measure

In order to apply Proposition 11.1 and to compute the price

$$\mathbb{E}\left[e^{-\int_t^{T} r_s ds} F \middle| \mathcal{F}_t \right] = P(t,T_i)\hat{\mathbb{E}}_i[F \mid \mathcal{F}_t],$$

it can be useful to determine the dynamics of the underlying processes r_t, $f(t,T,S)$, and $P(t,T)$ under the forward measure $\hat{\mathbb{P}}_i$.

Let us assume that the dynamics of the bond price $P(t,T_i)$ is given by

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_i(t) dW_t, \tag{11.2}$$

for $i = 1,\ldots,n$, where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$ and $(r_t)_{t\in\mathbb{R}_+}$ and $(\zeta_i(t))_{t\in\mathbb{R}_+}$ are adapted processes with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(W_t)_{t\in\mathbb{R}_+}$.

By the Girsanov theorem,

$$W_t^i := W_t - \int_0^t \zeta_i(s) ds, \qquad 0 \le t \le T_i, \tag{11.3}$$

is a standard Brownian motion under $\hat{\mathbb{P}}_i$ for all $i = 1,\ldots,n$, and we have

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$dW_t^j = dW_t^i - (\zeta_j(t) - \zeta_i(t))dt.$$

Hence the dynamics of $t \mapsto P(t, T_i)$ under $\hat{\mathbb{P}}_i$ is given by

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + |\zeta_i(t)|^2 dt + \zeta_t d\hat{W}_t^i, \tag{11.4}$$

where $(\hat{W}_t^i)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}_i$.

In case the short rate process $(r_t)_{t\in\mathbb{R}_+}$ is Markovian and solution of

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t,$$

its dynamics will be given under $\hat{\mathbb{P}}_i$ by

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)\zeta_t dt + \sigma(t, r_t)d\hat{W}_t^i.$$

In the Vasicek case

$$dr_t = (a - br_t)dt + \sigma dW_t,$$

we get

$$dr_t = (a - br_t)dt - \frac{\sigma^2}{b}(1 - e^{-b(T-t)})dt + \sigma d\hat{W}_t^i \tag{11.5}$$

and

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \frac{\sigma^2}{b^2}(1 - e^{-b(T-t)})^2 dt - \frac{\sigma}{b}(1 - e^{-b(T-t)})d\hat{W}_t^i.$$

The price

$$\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}(P(T_i,T_j) - \kappa)^+ \middle| \mathcal{F}_t\right] = P(t,T_i)\hat{\mathbb{E}}_i\left[(P(T_i,T_j) - \kappa)^+ \middle| \mathcal{F}_t\right]$$

of a bond call option on $P(T_i, T_j)$ with payoff $F = (P(T_i, T_j) - \kappa)^+$ can be computed as

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}(P(T_i,T_j) - \kappa)^+ \middle| \mathcal{F}_t\right] \\
&= P(t,T_j)\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,T_j)}{\kappa P(t,T_i)}\right) - \kappa P(t,T_i)\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,T_j)}{\kappa P(t,T_i)}\right),
\end{aligned}$$

with

$$v^2 = \int_t^T (\zeta^j(s) - \zeta^i(s))^2 ds,$$

from the Black-Scholes formula and the martingale property of $P(t,T_j)/P(t,T_i)$, written as

$$P(T_i,T_j)=\frac{P(t,T_j)}{P(t,T_i)}\exp\left(\int_t^{T_i}(\zeta^j(s)-\zeta^i(s))d\hat{W}_s^i-\frac{1}{2}\int_t^{T_i}|\zeta^j(s)-\zeta^i(s)|^2ds\right),$$

when $(\zeta^i(s))_{s\in[0,T_i]}$ and $(\zeta^j(s))_{s\in[0,T_j]}$ in (11.4) are deterministic functions.

In the Vasicek case the above bond option price could also be computed from the joint law of $\left(r_T,\int_t^T r_s ds\right)$, which is Gaussian, or from the dynamics (11.4)-(11.5) of $P(t,T)$ and r_t under $\hat{\mathbb{P}}^i$, cf § 7.3 of [37].

11.2 Caplets and Caps

The caplet on the spot rate $f(T,T,T_i)$ with strike κ is a contract with payoff

$$(f(T,T,T_i)-\kappa)^+,$$

priced at time $t\in[0,T]$ under the forward measure $\hat{\mathbb{P}}_i$ as

$$\mathbb{E}\left[e^{-\int_t^{T_i}r_s ds}(f(T,T,T_i)-\kappa)^+\Big|\mathcal{F}_t\right]=P(t,T_i)\hat{\mathbb{E}}_i\left[(f(T,T,T_i)-\kappa)^+\mid\mathcal{F}_t\right]. \tag{11.6}$$

Similarly, a floorlet on $f(T,T,T_i)$ with strike κ is a contract with payoff $(\kappa-f(T,T,T_i))^+$, priced at time $t\in[0,T]$ as

$$\mathbb{E}\left[e^{-\int_t^{T_i}r_s ds}(\kappa-f(T,T,T_i))^+\Big|\mathcal{F}_t\right]=P(t,T_i)\hat{\mathbb{E}}_i\left[(\kappa-f(T,T,T_i))^+\mid\mathcal{F}_t\right].$$

Caplet Pricing

Consider the caplet with payoff

$$(L(T_i,T_i,T_{i+1})-\kappa)^+$$

where the LIBOR rate

$$t\mapsto L(t,T_i,T_{i+1})=\frac{1}{T_{i+1}-T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})}-1\right),\qquad 0\le t\le T_i<T_{i+1},$$

is a martingale under $\hat{\mathbb{P}}_i$ which is modeled in the BGM model by

$$\frac{dL(t,T_i,T_{i+1})}{L(t,T_i,T_{i+1})}=\gamma_i(t)dB_t^{i+1},$$

$0 \leq t \leq T_i$, $i = 1, \ldots, n-1$, for some deterministic function $\gamma_i(t)$, $i = 1, \ldots, n-1$.

As a consequence of (11.6) and using the Black-Scholes formula, this caplet can be priced as time $t \in [0, T_i]$ as

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^{T_{i+1}} r_s ds}(L(T_i, T_i, T_{i+1}) - \kappa)^+ \middle| \mathcal{F}_t\right] \\
&= P(t, T_{i+1})\, \mathbb{E}_{i+1}\left[(L(T_i, T_i, T_{i+1}) - \kappa)^+ \mid \mathcal{F}_t\right] \\
&= P(t, T_{i+1}) \mathrm{Bl}(\kappa, L(t, T_i, T_{i+1}), \sigma_i(t), 0, T_i - t),
\end{aligned}$$

where

$$\mathrm{Bl}(\kappa, x, \sigma, r, \tau) = x\Phi(d_+) - \kappa e^{-r\tau}\Phi(d_-)$$

is the Black-Scholes function with

$$|\sigma_i(t)|^2 = \frac{1}{T_i - t}\int_t^{T_i} |\gamma_i|^2(s)ds, \tag{11.7}$$

and this formula is known as the *Black caplet formula*. Floorlets can be similarly priced by the call/put parity in the Black-Scholes formula.

Precisely we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^{T_{i+1}} r_s ds}(L(T_i, T_i, T_{i+1}) - \kappa)^+ \middle| \mathcal{F}_t\right] \\
&\quad = P(t, T_{i+1}) L(t, T_i, T_{i+1})\Phi(d_+) - \kappa P(t, T_{i+1})\Phi(d_-),
\end{aligned}$$

where

$$d_+ = \frac{\log(L(t, T_i, T_{i+1})/\kappa) + \sigma_i^2(t)(T_i - t)/2}{\sigma_i(t)\sqrt{T_i - t}},$$

and

$$d_- = \frac{\log(L(t, T_i, T_{i+1})/\kappa) - \sigma_i^2(t)(T_i - t)/2}{\sigma_i(t)\sqrt{T_i - t}}.$$

We may also write

$$\begin{aligned}
&(T_{i+1} - T_i)\, \mathbb{E}\left[e^{-\int_t^{T_{i+1}} r_s ds}(L(T_i, T_i, T_{i+1}) - \kappa)^+ \middle| \mathcal{F}_t\right] \\
&= P(t, T_{i+1})\left(\frac{P(t, T_i)}{P(t, T_{i+1})} - 1\right)\Phi(d_+) - \kappa(T_{i+1} - T_i)P(t, T_{i+1})\Phi(d_-) \\
&= (P(t, T_i) - P(t, T_{i+1}))\Phi(d_+) - \kappa(T_{i+1} - T_i)P(t, T_{i+1})\Phi(d_-) \\
&= P(t, T_i)\Phi(d_+) - (1 + \kappa(T_{i+1} - T_i))\, P(t, T_{i+1})\Phi(d_-),
\end{aligned}$$

and this yields a self-financing hedging strategy $(\Phi(d_+), -(1 + \kappa(T_{i+1} - T_i))\,\Phi(d_-))$ in the bonds $(P(t, T_i), P(t, T_{i+1}))$ with maturities T_i and T_{i+1}, cf. [40]. The

above decomposition can also be computed by taking $P(t,S)$ as a numéraire and letting $\hat{X}_t = P(t,T_i)/P(t,T_{i+1}) = 1 + (T_{i+1} - T_i)L(T_i,T_i,T_{i+1})$.

Cap Pricing

More generally one can consider caps that are relative to a given tenor structure $\{T_i,\ldots,T_j\}$, $1 \le i < j \le n$, with discounted payoff

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)e^{-\int_t^{T_{k+1}} r_s ds}(f(T_k,T_k,T_{k+1})-\kappa)^+.$$

Pricing formulas for caps are easily deduced from analog formulas for caplets, since the payoff of a cap can be decomposed into a sum of caplet payoffs. Thus the price of a cap at time $t \in [0,T_i]$ is given by

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=i}^{j-1}(T_{k+1}-T_k)e^{-\int_t^{T_{k+1}} r_s ds}(f(T_k,T_k,T_{k+1})-\kappa)^+\Big|\mathcal{F}_t\right]\\
&=\sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathbb{E}\left[e^{-\int_t^{T_{k+1}} r_s ds}(f(T_k,T_k,T_{k+1})-\kappa)^+\Big|\mathcal{F}_t\right]\\
&=\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\hat{\mathbb{E}}_{k+1}\left[(f(T_k,T_k,T_{k+1})-\kappa)^+\Big|\mathcal{F}_t\right].
\end{aligned}$$

In the above BGM model, the cap with payoff

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)(L(T_k,T_k,T_{k+1})-\kappa)^+$$

can be priced at time $t \in [0,T_i]$ as

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\mathrm{Bl}(\kappa,L(t,T_k,T_{k+1}),\sigma_k(t),0,T_k-t).$$

11.3 Forward Swap Measures

In this section we introduce the forward measures to be used for the pricing of swaptions, and we study their properties. We start with the definition of the *annuity numeraire*

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$P(t,T_i,T_j)=\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}),\qquad 0\le t\le T_i, \tag{11.8}$$

$1\le i<j\le n$, with in particular

$$P(t,T_i,T_{i+1})=(T_{i+1}-T_i)P(t,T_{i+1}),\qquad 0\le t\le T_i.$$

$1\le i<n$. The annuity numéraire satisfies the following martingale property, which can be proved by linearity and the fact that $t\mapsto e^{-\int_0^t r_s ds}P(t,T_k)$ is a martingale for all $k=i,\dots,j$.

Proposition 11.2. *The discounted annuity numeraire*

$$t\mapsto e^{-\int_0^t r_s ds}P(t,T_i,T_j),\qquad 0\le t\le T_i,$$

is a martingale under $\mathbb{P}$, $1\le i<j\le n$.

The forward swap measure $\hat{\mathbb{P}}_{i,j}$ is defined by

$$\frac{d\hat{\mathbb{P}}_{i,j}}{d\mathbb{P}}=e^{-\int_0^{T_i} r_s ds}\frac{P(T_i,T_i,T_j)}{P(0,T_i,T_j)}, \tag{11.9}$$

$1\le i<j\le n$. We have

$$\begin{aligned}\mathbb{E}\left[\frac{d\hat{\mathbb{P}}_{i,j}}{d\mathbb{P}}\Big|\mathcal{F}_t\right]&=\frac{1}{P(0,T_i,T_j)}\,\mathbb{E}\left[e^{-\int_0^{T_i} r_s ds}P(T_i,T_i,T_j)\Big|\mathcal{F}_t\right]\\&=\frac{P(t,T_i,T_j)}{P(0,T_i,T_j)}e^{-\int_0^t r_s ds},\end{aligned}$$

$0\le t\le T_i$, by Proposition 11.2, and

$$\frac{d\hat{\mathbb{P}}_{i,j|\mathcal{F}_t}}{d\mathbb{P}_{|\mathcal{F}_t}}=e^{-\int_t^{T_i} r_s ds}\frac{P(T_i,T_i,T_j)}{P(t,T_i,T_j)},\qquad 0\le t\le T_{i+1}. \tag{11.10}$$

We also know that the process

$$t\mapsto v_k^{i,j}(t):=\frac{P(t,T_k)}{P(t,T_i,T_j)}$$

is an $\mathcal{F}_t$-martingale under $\hat{\mathbb{P}}_{i,j}$. It follows that the swap rate

$$S(t,T_i,T_j):=\frac{P(t,T_i)-P(t,T_j)}{P(t,T_i,T_j)}=v_i^{i,j}(t)-v_j^{i,j}(t),$$

$0\le t\le T_i$, is also a martingale under $\hat{\mathbb{P}}_{i,j}$

11.4 Swaption Pricing on the LIBOR

A swaption is a contract to protect oneself against a risk based on an interest rate swap, and has payoff

$$\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)e^{-\int_{T_i}^{T_{k+1}} r_s ds}(f(T_i,T_k,T_{k+1})-\kappa)\right)^+.$$

This swaption can be priced at time $t \in [0,T_i]$ as

$$\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)e^{-\int_{T_i}^{T_{k+1}} r_s ds}(f(T_i,T_k,T_{k+1})-\kappa)\right)^+\Big|\mathcal{F}_t\right]. \tag{11.11}$$

In the sequel and in practice the price (11.11) of the swaption will be evaluated as

$$\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(f(T_i,T_k,T_{k+1})-\kappa)\right)^+\Big|\mathcal{F}_t\right], \tag{11.12}$$

meaning that we approximate the discount factor $e^{-\int_{T_i}^{T_{k+1}} r_s ds}$ by its conditional expectation $P(T_i,T_{k+1})$ given $\mathcal{F}_{T_i}$.

Note that when $j=i+1$, the swaption price (11.12) coincides with the price at time t of a caplet on $[T_i,T_{i+1}]$ up to a factor $\delta_i := T_{i+1}-T_i$ since

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}\left((T_{i+1}-T_i)P(T_i,T_{i+1})(f(T_i,T_i,T_{i+1})-\kappa)\right)^+\Big|\mathcal{F}_t\right]\\
&=(T_{i+1}-T_i)\,\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}P(T_i,T_{i+1})\left((f(T_i,T_i,T_{i+1})-\kappa)\right)^+\Big|\mathcal{F}_t\right]\\
&=\delta_i\,\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}\,\mathbb{E}\left[e^{-\int_{T_i}^{T_{i+1}} r_s ds}\Big|\mathcal{F}_{T_i}\right]\left((f(T_i,T_i,T_{i+1})-\kappa)\right)^+\Big|\mathcal{F}_t\right]\\
&=\delta_i\,\mathbb{E}\left[\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}e^{-\int_{T_i}^{T_{i+1}} r_s ds}\left((f(T_i,T_i,T_{i+1})-\kappa)\right)^+\Big|\mathcal{F}_{T_i}\right]\Big|\mathcal{F}_t\right]\\
&=\delta_i\,\mathbb{E}\left[e^{-\int_t^{T_{i+1}} r_s ds}\left((f(T_i,T_i,T_{i+1})-\kappa)\right)^+\Big|\mathcal{F}_t\right],
\end{aligned}$$

$0 \le t \le T_i$.

Proposition 11.1. *The swaption with payoff*

$$\left(\sum_{k=i}^{j-1}\delta_k P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

on the LIBOR market can be priced under the forward swap measure $\hat{\mathbb{P}}_{i,j}$ *as*

$$P(t,T_i,T_j)\hat{\mathbb{E}}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+\,\middle|\mathcal{F}_t\right], \qquad 0\le t\le T_i.$$

Proof. The relation

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(L(t,T_k,T_{k+1})-S(t,T_i,T_j))=0$$

that defines the forward swap rate $S(t,T_i,T_j)$ shows that

$$\begin{aligned}\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1}) &= S(t,T_i,T_j)\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\\ &= P(t,T_i,T_j)S(t,T_i,T_j)\\ &= P(t,T_i)-P(t,T_j),\end{aligned}$$

by the definition (11.8) of $P(t,T_i,T_j)$, hence

$$\begin{aligned}&\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(L(t,T_k,T_{k+1})-\kappa)\\ &\qquad = P(t,T_i,T_j)\left(S(t,T_i,T_j)-\kappa\right),\end{aligned}$$

and for $t=T_i$,

$$\begin{aligned}&\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+\\ &\qquad = (P(T_i,T_i)-P(T_i,T_j)-\kappa P(T_i,T_i,T_j))^+\\ &\qquad = P(T_i,T_i,T_j)\left(S(T_i,T_i,T_j)-\kappa\right)^+.\end{aligned}$$

As a consequence, from (11.12) we have

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_t^{T_i}r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+\,\middle|\mathcal{F}_t\right]\\ &\quad = \mathbb{E}\left[e^{-\int_t^{T_i}r_s ds}\left(P(T_i,T_i)-P(T_i,T_j)-\kappa P(T_i,T_i,T_j)\right)^+\,\middle|\mathcal{F}_t\right]\\ &\quad = \mathbb{E}\left[e^{-\int_t^{T_i}r_s ds}P(T_i,T_i,T_j)\left(S(T_i,T_i,T_j)-\kappa\right)^+\,\middle|\mathcal{F}_t\right]\\ &\quad = P(t,T_i,T_j)\hat{\mathbb{E}}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+\,\middle|\mathcal{F}_t\right],\end{aligned} \tag{11.13}$$

after applying (11.9) on the last line. □

Assume now the swap rate is a geometric Brownian motion under $\mathbb{P}_{i,j}$, i.e.

$$dS(t,T_i,T_j) = S(t,T_i,T_j)\hat{\sigma}(t)dW_t^{i,j}.$$

In this case the swaption with payoff

$$(P_T(T_i) - P_T(T_j) - \kappa P(T_i,T_i,T_j))^+ = P(T_i,T_i,T_j)\left(S(T_i,T_i,T_j) - \kappa\right)^+,$$

can be priced using the Black-Scholes formula as

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds} P(T_i,T_i,T_j)\left(S(T_i,T_i,T_j) - \kappa\right)^+ \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{-\int_t^{T} r_s ds} (P_T(T_i) - P_T(T_j) - \kappa P(T_i,T_i,T_j))^+ \Big| \mathcal{F}_t\right] \\
&= P(t,T_i,T_j)\hat{\mathbb{E}}_{i,j}\left[\left(S(T_i,T_i,T_j) - \kappa\right)^+ \Big| \mathcal{F}_t\right] \\
&= P(t,T_i,T_j)\mathrm{Bl}(\kappa, S(T_i,T_i,T_j), \sigma_{i,j}(t), 0, T_i - t),
\end{aligned}$$

where

$$|\sigma_{i,j}(t)|^2 = \frac{1}{T_i - t}\int_t^{T_i} |\hat{\sigma}|^2(s)ds.$$

Precisely we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds} P(T_i,T_i,T_j)\left(S(T_i,T_i,T_j) - \kappa\right)^+ \Big| \mathcal{F}_t\right] \\
&= P(t,T_i,T_j)\left(S(t,T_i,T_j)\Phi_+(t,S(t,T_i,T_j)) - \kappa\Phi_-(t,S(t,T_i,T_j))\right) \\
&= (P_t(T_i) - P_t(T_j))\Phi_+(t,S(t,T_i,T_j)) - \kappa P(T_i,T_i,T_j)\Phi_-(t,S(t,T_i,T_j)) \\
&= (P_t(T_i) - P_t(T_j))\Phi_+(t,S(t,T_i,T_j)) - \kappa\Phi_-(t,S(t,T_i,T_j))\sum_{k=i+1}^{j-1}\tau_{k-1}P(t,T_k),
\end{aligned}$$

where

$$d_+ = \frac{\log(S(t,T_i,T_{i+1})/\kappa) + \sigma_{i,j}^2(t)(T_i - t)/2}{\sigma_{i,j}(t)\sqrt{T_i - t}},$$

and

$$d_- = \frac{\log(S(t,T_i,T_{i+1})/\kappa) - \sigma_{i,j}^2(t)(T_i - t)/2}{\sigma_{i,j}(t)\sqrt{T_i - t}},$$

and the hedging strategy

$$\begin{aligned}
&(\Phi_+(t,S(t,T_i,T_j)), -\kappa\Phi_-(t,S(t,T_i,T_j))\tau_i, \ldots \\
&\quad \ldots, -\kappa\Phi_-(t,S(t,T_i,T_j))\tau_{j-2}, -\Phi_+(t,S(t,T_i,T_j)))
\end{aligned}$$

in $(P(t,T_i),\ldots,P(t,T_j))$ is self-financing, cf. [40].

Swaption prices can also be computed by an approximation formula, from the exact dynamics of the swap rate $S(t,T_i,T_j)$ under $\hat{\mathbb{P}}_{i,j}$, based on the bond price dynamics of the form (11.2), cf. [47], page 17.

11.5 Exercises

Exercise 11.1. We work in the short rate model

$$dr_t = \sigma dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$, and $\mathbb{P}_2$ is the forward measure defined by

$$\frac{d\mathbb{P}_2}{d\mathbb{P}} = \frac{1}{P(0,T_2)} e^{-\int_0^{T_2} r_s ds}.$$

1. Recall the expressions of ζ_t^1 and ζ_t^2 in

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_t^i dB_t, \qquad i = 1,2,$$

   and the dynamics of the $P(t,T_1)/P(t,T_2)$ under $\mathbb{P}_2$, where $P(t,T_1)$ and $P(t,T_2)$ are bond prices with maturities T_1 and T_2.
2. Recall the expression of the forward rate $f(t,T_1,T_2)$.
3. Compute the dynamics of $f(t,T_1,T_2)$ under the forward measure $\mathbb{P}_2$ with

$$\frac{d\mathbb{P}_2}{d\mathbb{P}} = \frac{1}{P(0,T_2)} e^{-\int_0^{T_2} r_s ds}.$$

4. Compute the price

$$(T_2 - T_1)\,\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(f(T_1,T_1,T_2) - \kappa)^+ \middle| \mathcal{F}_t\right]$$

   of a cap at time $t \in [0,T_1]$, using the expectation under the forward measure $\mathbb{P}_2$.
5. Compute the dynamics of the swap rate process

$$S(t,T_1,T_2) = \frac{P(t,T_1) - P(t,T_2)}{(T_2 - T_1)P(t,T_2)}, \qquad t \in [0,T_1],$$

   under $\mathbb{P}_2$.
6. Compute the swaption price

$$(T_2 - T_1)\,\mathbb{E}\left[e^{-\int_t^{T_1} r_s ds} P(T_1,T_2)(S(T_1,T_1,T_2) - \kappa)^+ \middle| \mathcal{F}_t\right]$$

   on the swap rate $S(T_1,T_1,T_2)$ using the expectation under the forward swap measure $\mathbb{P}_{1,2}$.

Exercise 11.2. Consider three zero-coupon bonds $P(t,T_1)$, $P(t,T_2)$ and $P(t,T_3)$ with maturities $T_1 = \delta$, $T_2 = 2\delta$ and $T_3 = 3\delta$ respectively, and the forward LIBOR $L(t,T_1,T_2)$ and $L(t,T_2,T_3)$ defined by

$$L(t,T_i,T_{i+1}) = \frac{1}{\delta}\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1\right), \qquad i = 1,2.$$

Assume that $L(t,T_1,T_2)$ and $L(t,T_2,T_3)$ are modeled in the BGM model by

$$\frac{dL(t,T_1,T_2)}{L(t,T_1,T_2)} = e^{-at}dW_t^2, \qquad 0 \le t \le T_1, \tag{11.14}$$

and $L(t,T_2,T_3) = b$, $0 \le t \le T_2$, for some constants $a, b > 0$, where W_t^2 is a standard Brownian motion under the forward rate measure $\mathbb{P}_2$ defined by

$$\frac{d\mathbb{P}_2}{d\mathbb{P}} = \frac{e^{-\int_0^{T_2} r_s ds}}{P(0,T_2)}.$$

(i) Compute $L(t,T_1,T_2)$, $0 \le t \le T_2$ by solving Equation (11.14).

(ii) Compute the price at time t:

$$\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(L(T_1,T_1,T_2)-\kappa)^+\Big|\mathcal{F}_t\right] = P(t,T_2)\hat{\mathbb{E}}_2\left[(L(T_1,T_1,T_2)-\kappa)^+ \mid \mathcal{F}_t\right],$$

of the caplet with strike κ, where $\hat{\mathbb{E}}_2$ denotes the expectation under the forward measure $\mathbb{P}_2$.

(iii) Compute

$$\frac{P(t,T_1)}{P(t,T_1,T_3)}, \quad 0 \le t \le T_1, \quad \text{and} \quad \frac{P(t,T_3)}{P(t,T_1,T_3)}, \qquad 0 \le t \le T_2,,$$

in terms of b and $L(t,T_1,T_2)$, where $P(t,T_1,T_3)$ is the annuity numeraire

$$P(t,T_1,T_3) = \delta P(t,T_2) + \delta P(t,T_3), \qquad 0 \le t \le T_2.$$

(iv) Compute the dynamics of the swap rate

$$t \mapsto S(t,T_1,T_3) = \frac{P(t,T_1) - P(t,T_3)}{P(t,T_1,T_3)}, \qquad 0 \le t \le T_1,$$

i.e. show that we have

$$dS(t,T_1,T_3) = \sigma_{1.3}(t)S(t,T_1,T_3)dW_t^2,$$

where $\sigma_{1,3}(t)$ is a process to be determined.

(v) Using the Black-Scholes formula, compute an approximation of the swaption price

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_t^{T_1} r_s ds}P(T_1,T_1,T_3)(S(T_1,T_1,T_3)-\kappa)^+\Big|\mathcal{F}_t\right]\\ &= P(t,T_1,T_3)\hat{\mathbb{E}}_2\left[(S(T_1,T_1,T_3)-\kappa)^+ \mid \mathcal{F}_t\right],\end{aligned}$$

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http://www.ntu.edu.sg/home/nprivault/indext.html

at time $t \in [0, T_1]$. You will need to approximate $\sigma_{1,3}(s)$, $s \geq t$, by "freezing" all random terms at time t.

Hint. Given X a centered Gaussian random variable with variance v^2 we have

$$\mathbb{E}[(e^{m+X} - \kappa)^+] = e^{m+\frac{v^2}{2}} \Phi(v + (m - \log \kappa)/v) - \kappa \Phi((m - \log \kappa)/v),$$

where Φ denotes the Gaussian cumulative distribution function.

Exercise 11.3. Consider a portfolio $(\xi_t^T, \xi_t^S)_{t \in [0,T]}$ made of two bonds with maturities T, S, and value

$$V_t = \xi_t^T P(t,T) + \xi_t^S P(t,S), \qquad 0 \leq t \leq T,$$

at time t. We assume that the portfolio is self-financing, i.e.

$$dV_t = \xi_t^T dP(t,T) + \xi_t^S dP(t,S), \qquad 0 \leq t \leq T, \tag{11.15}$$

and that it *hedges* the claim $(P(T,S) - \kappa)^+$, so that

$$\begin{aligned} V_t &= \mathbb{E}\left[e^{-\int_t^T r_s ds} (P(T,S) - \kappa)^+ \Big| \mathcal{F}_t \right] \\ &= P(t,T)\, \mathbb{E}_T \left[(P(T,S) - \kappa)^+ \Big| \mathcal{F}_t \right], \qquad 0 \leq t \leq T. \end{aligned}$$

1. Show that

$$\begin{aligned} &\mathbb{E}\left[e^{-\int_t^T r_s ds} (P(T,S) - K)^+ \Big| \mathcal{F}_t \right] \\ &= P(0,T)\, \mathbb{E}_T \left[(P(T,S) - K)^+ \right] + \int_0^t \xi_s^T dP(s,T) + \int_0^t \xi_s^S dP(s,S). \end{aligned}$$

2. Show that under the self-financing condition (11.15), the discounted portfolio price $\tilde{V}_t = e^{-\int_0^t r_s ds} V_t$ satisfies

$$d\tilde{V}_t = \xi_t^T d\tilde{P}(t,T) + \xi_t^S d\tilde{P}(t,S),$$

where $\tilde{P}(t,T) = e^{-\int_0^t r_s ds} P(t,T)$ and $\tilde{P}(t,S) = e^{-\int_0^t r_s ds} P(t,S)$ denote the discounted bond prices.

3. Show that

$$\begin{aligned} \mathbb{E}_T \left[(P(T,S) - K)^+ | \mathcal{F}_t \right] &= \mathbb{E}_T \left[(P(T,S) - K)^+ \right] \\ &\quad + \int_0^t \frac{\partial C}{\partial x}(X_u, T-u, v(u,T)) dX_u. \end{aligned}$$

Hint: use the martingale property and the Itô formula.

4. Show that the discounted portfolio price $\hat{V}_t = V_t/P(t,T)$ satisfies

$$\begin{aligned} d\hat{V}_t &= \frac{\partial C}{\partial x}(X_t, T-t, v(t,T))dX_t \\ &= \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t^T. \end{aligned}$$

5. Show that

$$dV_t = P(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \hat{V}_t dP(t,T).$$

6. Show that

$$d\tilde{V}_t = \tilde{P}(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \hat{V}_t d\tilde{P}(t,T).$$

7. Compute the hedging strategy $(\xi_t^T, \xi_t^S)_{t\in[0,T]}$ of the bond option.
8. Show that

$$\frac{\partial C}{\partial x}(x,\tau,v) = \Phi\left(\frac{\log(x/K) + \tau v^2/2}{\sqrt{\tau}v}\right).$$

Exercise 11.4. Consider a bond market with tenor structure $\{T_i, \ldots, T_j\}$ and bonds with maturities $T_i, \ldots, T_j$, whose prices $P(t,T_i), \ldots P(t,T_j)$ at time t are given by

$$\frac{dP(t,T_k)}{P(t,T_k)} = r_t dt + \zeta_k(t)dB_t, \qquad k = i, \ldots, j,$$

where $(r_t)_{t\in\mathbb{R}_+}$ is a short term interest rate process and $(B_t)_{t\in\mathbb{R}_+}$ denotes a standard Brownian motion generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and $\zeta_i(t), \ldots, \zeta_j(t)$ are volatility processes.

The swap rate $S(t,T_i,T_j)$ is defined by

$$S(t,T_i,T_j) = \frac{P(t,T_i) - P(t,T_j)}{P(t,T_i,T_j)},$$

where

$$P(t,T_i,T_j) = \sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t,T_{k+1})$$

is the annuity numeraire. Recall that a swaption on the LIBOR market can be priced at time $t \in [0,T_i]$ as

$$\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1} - T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1}) - \kappa)\right)^+ \Big| \mathcal{F}_t\right]$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$= P(t, T_i, T_j)\, \mathbb{E}_{i,j}\left[(S(T_i, T_i, T_j) - \kappa)^+ \,\middle|\, \mathcal{F}_t\right], \tag{11.16}$$

under the forward swap measure $\mathbb{P}_{i,j}$ defined by

$$\frac{d\mathbb{P}_{i,j}}{d\mathbb{P}} = e^{-\int_0^{T_i} r_s ds} \frac{P(T_i, T_i, T_j)}{P(0, T_i, T_j)}, \qquad 1 \le i < j \le n,$$

under which

$$B_t^{i,j} := B_t - \sum_{k=i}^{j-1} (T_{k+1} - T_k) \frac{P(t, T_{k+1})}{P(t, T_i, T_j)} \zeta_{k+1}(t) dt \tag{11.17}$$

is a standard Brownian motion. We assume that the swap rate is modeled as a geometric Brownian motion

$$dS(t, T_i, T_j) = S(t, T_i, T_j) \sigma_{i,j}(t) dB_t^{i,j}, \qquad 0 \le t \le T_i, \tag{11.18}$$

where the swap rate volatility is a deterministic function $\sigma_{i,j}(t)$. In the sequel we denote $S_t = S(t, T_i, T_j)$ for simplicity of notation.

1. Solve the equation (11.18) on the interval $[t, T_i]$, and compute $S(T_i, T_i, T_j)$ from the initial condition $S(t, T_i, T_j)$.
2. Show that the price (11.13) of the swaption can be written as

   $$P(t, T_i, T_j) C(S_t, v(t, T_i)),$$

   where

   $$v^2(t, T_i) = \int_t^{T_i} |\sigma_{i,j}(s)|^2 ds,$$

   and $C(x, v)$ is a function to be specified using the Black-Scholes formula

   $$\begin{aligned} \mathrm{Bl}(K, x, \sigma, r, \tau) &= \mathbb{E}[(xe^{m+X} - K)^+] \\ &= \Phi(v + (m + \log(x/K))/v) - K\Phi((m + \log(x/K))/v), \end{aligned}$$

   where X is a centered Gaussian random variable with mean $m = r\tau - v^2/2$ and variance v^2.
3. Consider a portfolio $(\xi_t^i, \ldots, \xi_t^j)_{t \in [0, T_i]}$ made of bonds with maturities $T_i, \ldots, T_j$ and value

   $$V_t = \sum_{k=i}^{j} \xi_t^k P(t, T_k),$$

   at time $t \in [0, T_i]$. We assume that the portfolio is self-financing, i.e.

   $$dV_t = \sum_{k=i}^{j} \xi_t^k dP(t, T_k), \qquad 0 \le t \le T_i, \tag{11.19}$$

and that it *hedges* the claim $(S(T_i,T_i,T_j)-\kappa)^+$, so that

$$V_t = \mathbb{E}\left[e^{-\int_t^{T_i} r_s ds} \left(\sum_{k=i}^{j-1} (T_{k+1}-T_k) P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa) \right)^+ \Big| \mathcal{F}_t \right]$$
$$= P(t,T_i,T_j)\, \mathbb{E}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+ \Big| \mathcal{F}_t \right],$$

$0 \le t \le T_i$. Show that

$$\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds} \left(\sum_{k=i}^{j-1} (T_{k+1}-T_k) P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa) \right)^+ \Big| \mathcal{F}_t \right]$$
$$= P(0,T_i,T_j)\, \mathbb{E}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+ \right] + \sum_{k=i}^{j} \int_0^t \xi_s^k dP(s,T_i),$$

$0 \le t \le T_i$.

4. Show that under the self-financing condition (11.19), the discounted portfolio price $\tilde{V}_t = e^{-\int_0^t r_s ds} V_t$ satisfies

$$d\tilde{V}_t = \sum_{k=i}^{j} \xi_t^k d\tilde{P}(t,T_k),$$

where $\tilde{P}(t,T_k) = e^{-\int_0^t r_s ds} P(t,T_k)$, $k=i,\ldots,j$, denote the discounted bond prices.

5. Show that

$$\mathbb{E}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+ \Big| \mathcal{F}_t \right]$$
$$= \mathbb{E}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+ \right] + \int_0^t \frac{\partial C}{\partial x}(S_u, v(u,T_i)) dS_u.$$

Hint: use the martingale property and the Itô formula.

6. Show that the discounted portfolio price $\hat{V}_t = V_t / P(t,T_i,T_j)$ satisfies

$$d\hat{V}_t = \frac{\partial C}{\partial x}(S_t, v(t,T_i)) dS_t = S_t \frac{\partial C}{\partial x}(S_t, v(t,T_i)) \sigma_t^{i,j} dB_t^{i,j}.$$

7. Show that

$$dV_t = (P(t,T_i) - P(t,T_j)) \frac{\partial C}{\partial x}(S_t, v(t,T_i)) \sigma_t^{i,j} dB_t + \hat{V}_t dP(t,T_i,T_j).$$

8. Show that

$$
\begin{aligned}
dV_t = {} & S_t\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})dB_t \\
& +(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i)))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\zeta_{k+1}(t)dB_t \\
& +\frac{\partial C}{\partial x}(S_t, v(t,T_i))P(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t.
\end{aligned}
$$

9. Show that

$$
\begin{aligned}
d\tilde{V}_t = {} & \frac{\partial C}{\partial x}(S_t, v(t,T_i))d(\tilde{P}(t,T_i)-\tilde{P}(t,T_j)) \\
& +(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i)))d\tilde{P}(t,T_i,T_j).
\end{aligned}
$$

10. Show that

$$
\frac{\partial C}{\partial x}(x, v(t,T_i)) = \Phi\left(\frac{\log(x/K)}{v(t,T_i)} + \frac{v(t,T_i)}{2}\right).
$$

11. Show that we have

$$
\begin{aligned}
d\tilde{V}_t = {} & \Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} + \frac{v(t,T_i)}{2}\right)d(\tilde{P}(t,T_i)-\tilde{P}(t,T_j)) \\
& -\kappa\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} - \frac{v(t,T_i)}{2}\right)d\tilde{P}(t,T_i,T_j).
\end{aligned}
$$

12. Show that the hedging strategy is given by

$$
\xi_t^i = \Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} + \frac{v(t,T_i)}{2}\right),
$$

$$
\xi_t^j = -\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} + \frac{v(t,T_i)}{2}\right) - \kappa(T_{j+1}-T_j)\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} - \frac{v(t,T_i)}{2}\right),
$$

and

$$
\xi_t^k = -\kappa(T_{k+1}-T_k)\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} - \frac{v(t,T_i)}{2}\right), \qquad i+1 \le k \le j-1.
$$

Chapter 12
Credit Default

The bond models studied in the previous chapters rely on the hypothesis that the bond payoff at maturity is always equal to \$1, thus excluding the possibility of default. In this chapter we study a bond model in which default can occur at a random time τ. We also consider the associated options (credit default swaps) that are designed as a protection against default. We refer to [3] for a complete account of credit risk modeling.

12.1 Exponential Distribution

A random variable τ has the exponential distribution with parameter $\lambda > 0$ if

$$\mathbb{P}(\tau > t) = e^{-\lambda t}, \qquad t \geq 0.$$

Given $(\tau_n)_{n\geq 1}$ a sequence of i.i.d. exponentially distributed random variables, letting

$$T_n = \tau_1 + \cdots + \tau_n, \qquad n \geq 1,$$

and

$$N_t = \sum_{n\geq 1} \mathbf{1}_{[T_n,\infty)}(t)$$

defines a standard Poisson process with intensity $\lambda > 0$ and

$$\mathbb{P}(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}, \qquad k \geq 0.$$

The intensity of the process can in fact made time-dependent (e.g. by a time change), i.e.

$$\mathbb{P}(N_t - N_s = k) = e^{-\int_s^t \lambda(u)du} \frac{\left(\int_s^t \lambda(u)du\right)^k}{k!}, \qquad k \geq 0.$$

In particular,

$$\mathbb{P}(N_{t+dt} - N_t = k) = \begin{cases} e^{-\lambda(t)dt} \simeq 1 - \lambda(t)dt, & k = 0, \\ \lambda(t)e^{-\lambda(t)dt}dt \simeq \lambda(t)dt, & k = 1, \\ o(dt), & k \geq 2. \end{cases}$$

The intensity process $\lambda(s)$ can also be made random in the case of Cox processes.

12.2 Survival Probabilities

Let τ denote the lifetime of an entity, and let $\mathbb{P}(\tau \geq t)$, $t > 0$, denote its probability of surviving at least t years.

The probability of surviving up to time T, given that the entity has survived up to time t, $T \geq t$, is (p. 50)

$$\begin{aligned} \mathbb{P}(\tau > T \mid \tau > t) &= \frac{\mathbb{P}(\tau > T \text{ and } \tau > t)}{\mathbb{P}(\tau > t)} \\ &= \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}. \end{aligned}$$

Letting

$$\begin{aligned} f(t) &= \frac{\mathbb{P}(\tau < t + dt \mid \tau > t)}{dt} \\ &= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(\tau < t + dt \text{ and } \tau > t)}{dt} \\ &= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > t + dt)}{dt} \\ &= -\frac{d}{dt} \log \mathbb{P}(\tau > t) \\ &= -\frac{1}{\mathbb{P}(\tau > t)} \frac{d}{dt} \mathbb{P}(\tau > t), \end{aligned}$$

we get

$$\mathbb{P}(\tau > t) = \exp\left(- \int_0^t f(u)du \right),$$

and

$$\mathbb{P}(\tau > T \mid \tau > t) = \exp\left(- \int_t^T f(u)du \right).$$

Given survival up to time t, survival over the time period $[t, t+dt]$ occurs with probability

$$\mathbb{P}(\tau > t + dt \mid \tau > t) = e^{-\lambda(t)dt} \simeq 1 - \lambda(t)dt,$$

and default occurs on $[t, t+dt]$ with probability

$$\mathbb{P}(\tau < t + dt \mid \tau > t) = 1 - e^{-\lambda dt} \simeq \lambda(t)dt,$$

which coincide respectively with $P(N_{t+dt} - N_t = 0)$ and $P(N_{t+dt} - N_t = 1)$ above.

The above formulas can be recovered informally as

$$\begin{aligned}\mathbb{P}(\tau > T) &= \prod_{0<t<T} \mathbb{P}(\tau > t + dt \mid \tau > t) \\ &= \prod_{0<t<T} \exp\left(-\lambda(t)dt\right) \\ &= \exp\left(-\int_0^T \lambda(t)dt\right),\end{aligned}$$

i.e. in the limit

$$\mathbb{P}(\tau > t) = e^{-\int_0^t \lambda(s)ds}.$$

12.3 Stochastic Default

We now assume that the default rate $\lambda(t)$ is a random process adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

The random time τ is called a *stopping time* with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ we have

$$\{\tau > t\} \in \mathcal{F}_t, \qquad t \in \mathbb{R}_+,$$

i.e. the knowledge of whether default already occurred at time t is contained in $\mathcal{F}_t$, $t \in \mathbb{R}_+$. In this case we have

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \mathbb{E}[\mathbf{1}_{\{\tau>t\}} \mid \mathcal{F}_t] = \mathbf{1}_{\{\tau>t\}}.$$

In the sequel we will work in a framework where τ is not a stopping time with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and we will assume that $\mathbb{P}(\tau > t \mid \mathcal{F}_t)$ takes the form

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = e^{-\int_0^t \lambda(u)du}.$$

This condition holds in particular when $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the filtration generated by a process $(X_t)_{t\in\mathbb{R}_+}$, $\lambda(u)$ has the form $\lambda(u) = h(X_u)$, and

$$\tau = \inf\left\{t \in \mathbb{R}_+ \ : \ \int_0^t \lambda(u)du \geq L\right\},$$

cf. [31].

We let $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ be the filtration defined by

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\} \ : \ 0 \leq u \leq t), \qquad t \in \mathbb{R}_+,$$

i.e. $\mathcal{G}_t$ contains the additional information whether default at time τ has occurred or not before time t.

Lemma 12.1. ([23]) For any $\mathcal{F}_T$-measurable integrable random variable F we have

$$\mathbb{E}\left[F\mathbf{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] = \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[Fe^{-\int_t^T \lambda(u)du}\middle|\mathcal{F}_t\right].$$

Proof. Using the fact that

$$\begin{aligned}\frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} &= \frac{e^{-\int_0^T \lambda(u)du}}{e^{-\int_0^t \lambda(u)du}} \\ &= e^{-\int_t^T \lambda(u)du},\end{aligned}$$

we get

$$\begin{aligned}\mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[F\frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\middle|\mathcal{F}_t\right] &= \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\,\mathbb{E}\left[F\,\mathbb{E}[\mathbf{1}_{\{\tau>T\}} \mid \mathcal{F}_T]\middle|\mathcal{F}_t\right] \\ &= \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\,\mathbb{E}\left[\mathbb{E}[F\mathbf{1}_{\{\tau>T\}} \mid \mathcal{F}_T]\middle|\mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}\frac{E\left[F\mathbf{1}_{\{\tau>T\}}\middle|\mathcal{F}_t\right]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \\ &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[F\mathbf{1}_{\{\tau>T\}}\middle|\mathcal{F}_t \vee \{\tau > t\}\right] \\ &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[F\mathbf{1}_{\{\tau>T\}}\middle|\mathcal{G}_t\right].\end{aligned}$$

□

Taking $F = 1$ in Lemma 12.1 we get that the probability of survival up to time T, given information known up to time t, is

$$\begin{aligned}\mathbb{P}(\tau > T \mid \mathcal{G}_t) &= \mathbb{E}\left[\mathbf{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[e^{-\int_t^T \lambda(u)du}\middle|\mathcal{F}_t\right].\end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

The process $\lambda(t)$ can be chosen among the classical mean-reverting diffusion processes, and may contain jumps.

The computation of $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ is then similar to bond pricing. In addition, HJM type default intensities can also be introduce to compute

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{E}\left[\exp\left(-\int_t^T f(t,u)du\right)\Big|\mathcal{F}_t\right],$$

where the process $f(t,T)$ can be modeled in the HJM framework.

12.4 Defaultable Bonds

The price of a bond with maturity T and (random) default time τ is given by

$$\begin{aligned} P(t,T) &= \mathbb{E}\left[\mathbf{1}_{\{\tau>T\}}\exp\left(-\int_t^T r_u du\right)\Big|\mathcal{G}_t\right] \\ &\quad + \mathbb{E}\left[\xi\mathbf{1}_{\{\tau\le T\}}\exp\left(-\int_t^T r_u du\right)\Big|\mathcal{G}_t\right], \end{aligned}$$

where ξ is the recovery rate.

Taking $F = \exp\left(-\int_t^T r_u du\right)$ in Lemma 12.1 we get

$$\mathbb{E}\left[\exp\left(-\int_t^T r_u du\right)\mathbf{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\right] = \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T (r_u+\lambda(u))du\right)\Big|\mathcal{F}_t\right],$$

cf. [31], [23], [17].

In the case of complete default we have $\xi = 0$ (zero-recovery) and

$$\begin{aligned} P(t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\mathbf{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda(s))ds\right)\Big|\mathcal{F}_t\right]. \end{aligned} \tag{12.1}$$

The interest in this expression yields in the fact that the effect of the presence of a default time τ is to decrease the bond price that can be viewed as an

increase $\lambda(u)$ in the short rate.

This treatment has some similarity with that of coupon bonds which can be priced as

$$P(t,T) = e^{-c(T-t)}\,\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{G}_t\right],$$

where $c > 0$ is a continuous time coupon rate.

Finally, (12.1) can also be expressed under the forward measure $\tilde{\mathbb{P}}$ as

$$\begin{aligned}
P(t,T) &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda(s))ds\right)\Big|\mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right]\mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(-\int_t^T \lambda(s)ds\right)\Big|\mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right]Q(t,T),
\end{aligned}$$

where

$$Q(t,T) = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(-\int_t^T \lambda(s)ds\right)\Big|\mathcal{F}_t\right]$$

denotes the survival probability under the forward measure, cf. [10], [9].

12.5 Credit Default Swaps

We work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$.

A Credit Default Swap (CDS) is a contract consisting in

- a premium leg: the buyer is buying protection at time t against default at time T_k, $k = i+1,\ldots,j$, and has to make a fixed payment S_t at regular times $T_{i+1},\ldots,T_j$ between times t and T.

  The discounted value at time t of the premium leg is

$$\begin{aligned}
V(t,T) &= \mathbb{E}\left[\sum_{k=i}^{j-1} S_t\delta_k \mathbf{1}_{\{\tau>T_{k+1}\}} e^{-\int_t^{T_{k+1}} r_s ds}\Big|\mathcal{G}_t\right] \\
&= \sum_{k=i}^{j-1} S_t\delta_k\,\mathbb{E}\left[\mathbf{1}_{\{\tau>T_{k+1}\}} e^{-\int_t^{T_{k+1}} r_s ds}\Big|\mathcal{G}_t\right]
\end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$= S_t \sum_{k=i}^{j-1} \delta_k P(t, T_{k+1})$$
$$= S_t P(t, T_i, T_j),$$

where $\delta_k = T_{k+1} - T_k$, $P(t, T_i, T_j)$ is the annuity numeraire, and

$$P(t, T_k) = \mathbf{1}_{\{\tau > t\}} \, \mathbb{E}\left[\exp\left(-\int_t^{T_k} (r_s + \lambda(s)) ds \right) \Big| \mathcal{F}_t \right],$$

$k = 1, \ldots, n$. In the above value of the premium leg we have ignored a possible accrual interest term on the time period $[T_k, \tau]$ when $\tau \in [T_k, T_{k+1}]$.

- a protection leg: the seller makes a payment to the buyer if default occurs.

The value at time t of the protection leg is

$$\mathbb{E}\left[\sum_{k=i}^{j-1} \mathbf{1}_{[T_k, T_{k+1}]}(\tau)(1 - \xi_{k+1}) e^{-\int_t^{T_{k+1}} r_s ds} \Big| \mathcal{G}_t \right],$$

where ξ_{k+1} is the recovery rate with maturity T_{k+1}.

In the case of a constant recovery rate ξ_k the value of the protection leg is

$$\sum_{k=i}^{j-1} (1 - \xi_{k+1}) \, \mathbb{E}\left[\mathbf{1}_{[T_k, T_{k+1}]}(\tau) e^{-\int_t^{T_{k+1}} r_s ds} \Big| \mathcal{G}_t \right].$$

The spread S_t is computed by equating the protection leg and the premium leg, i.e. from the relation

$$V(t, T) = S_t P(t, T_i, T_j) = \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbf{1}_{[T_k, T_{k+1}]}(\tau)(1 - \xi_{k+1}) e^{-\int_t^{T_{k+1}} r_s ds} \Big| \mathcal{G}_t \right],$$

i.e.

$$S_t = \frac{1}{P(t, T_i, T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbf{1}_{[T_k, T_{k+1}]}(\tau)(1 - \xi_{k+1}) e^{-\int_t^{T_{k+1}} r_s ds} \Big| \mathcal{G}_t \right].$$

In the case of a constant recovery rate ξ we have

$$S_t = \frac{1 - \xi}{P(t, T_i, T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbf{1}_{[T_k, T_{k+1}]}(\tau) e^{-\int_t^{T_{k+1}} r_s ds} \Big| \mathcal{G}_t \right],$$

and if τ is constrained to take values in the tenor structure we get

$$S_t = \frac{1-\xi}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbf{1}_{[t,T]}(\tau) e^{-\int_t^\tau r_s ds} \Big| \mathcal{G}_t \right].$$

12.6 Exercises

Exercise 12.1. Defaultable bonds. Consider a (random) default time τ with law

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = e^{-\int_0^t \lambda_u du},$$

where λ_t is a (random) default rate process which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Recall that the probability of survival up to time T, given information known up to time t, is given by

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbf{1}_{\{\tau>t\}} E\left[e^{-\int_t^T \lambda_u du} \Big| \mathcal{F}_t \right],$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\} \ : \ 0 \le u \le t)$, $t \in \mathbb{R}_+$, is the filtration defined by adding the default time information to the history $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. In this framework, the price $P(t,T)$ of defaultable bond with maturity T, short term interest rate r_t and (random) default time τ is given by

$$\begin{aligned} P(t,T) &= E\left[\mathbf{1}_{\{\tau>T\}} \exp\left(-\int_t^T r_u du \right) \Big| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau>t\}} E\left[\exp\left(-\int_t^T (r_u + \lambda_u) du \right) \Big| \mathcal{F}_t \right]. \end{aligned} \tag{12.2}$$

In the sequel we assume that the processes $(r_t)_{t\in\mathbb{R}_+}$ and $(\lambda_t)_{t\in\mathbb{R}_+}$ are modeled according to

$$\begin{cases} dr_t = -a r_t dt + \sigma dB_t^1, \\ d\lambda_t = -b\lambda_t dt + \eta dB_t^2, \end{cases} \tag{12.3}$$

where $(B_t^1)_{t\in\mathbb{R}_+}$ and $(B_t^2)_{t\in\mathbb{R}_+}$ are two standard $\mathcal{F}_t$-Brownian motions with correlation $\rho \in [-1,1]$, and $dB_t^1 dB_t^2 = \rho dt$.

1. Give a justification for the fact that

$$E\left[\exp\left(-\int_t^T (r_u + \lambda_u) du \right) \Big| \mathcal{F}_t \right]$$

   can be written as a function $F(t, r_t, \lambda_t)$ of t, r_t and λ_t, $t \in [0,T]$.
2. Show that

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$t \mapsto e^{-\int_0^t (r_s+\lambda_s)ds} E\left[\exp\left(-\int_t^T (r_u+\lambda_u)du\right)\Big|\mathcal{F}_t\right]$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}$.

3. Use the Itô formula with two variables to derive a PDE on $\mathbb{R}^2$ for the function $F(t,x,y)$.
4. Show that we have

$$\int_t^T r_s ds = C(a,t,T)r_t + \sigma \int_t^T C(a,s,T)dB_s^1,$$

and

$$\int_t^T \lambda_s ds = C(b,t,T)\lambda_t + \eta \int_t^T C(b,s,T)dB_s^2,$$

where

$$C(a,t,T) = -\frac{1}{a}(e^{-a(T-t)} - 1).$$

5. Show that the random variable

$$\int_t^T r_s ds + \int_t^T \lambda_s ds$$

is Gaussian and compute its mean

$$\mathbb{E}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds\Big|\mathcal{F}_t\right]$$

and variance

$$\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds\Big|\mathcal{F}_t\right],$$

conditionally to $\mathcal{F}_t$.

6. Compute $P(t,T)$ from its expression (12.2) as a conditional expectation.
7. Show that the solution $F(t,x,y)$ to the 2-dimensional PDE of Question 3 is

$$F(t,x,y) = \exp\left(-C(a,t,T)x - C(b,t,T)y\right)$$
$$\times \exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds + \rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right).$$

8. Show that the defaultable bond price $P(t,T)$ can also be written as

$$P(t,T) = e^{U(t,T)}\mathbb{P}(\tau > T \mid \mathcal{G}_t)\,\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right],$$

where

$$U(t,T) = \rho \frac{\sigma\eta}{ab} \left(T - t - C(a,t,T) - C(b,t,T) + C(a+b,t,T) \right).$$

9. By partial differentiation of $\log P(t,T)$ with respect to T, compute the corresponding instantaneous short rate $f(t,T) = -\dfrac{\partial \log P(t,T)}{\partial T}$.
10. Show that $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ can be written using an HJM type default rate as
$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbf{1}_{\{\tau > t\}} e^{-\int_t^T f_2(t,u)du},$$
where
$$f_2(t,u) = \lambda_t e^{-b(u-t)} - \frac{\eta^2}{2} C^2(b,t,u).$$
11. Show how the result of Question 8 can be simplified when $(B_t^1)_{t\in\mathbb{R}_+}$ and $(B_t^2)_{t\in\mathbb{R}_+}$ are independent.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Chapter 13
Stochastic Calculus for Jump Processes

The modelling of risky asset by stochastic processes with continuous paths, based on Brownian motions, suffers from several defects. First, the path continuity assumption does not seem reasonable in view of the possibility of sudden price variations (jumps) resulting of market crashes. Secondly, the modeling of risky asset prices by Brownian motion relies on the use of the Gaussian distribution which tends to underestimate the probabilities of extreme events.

A solution is to use stochastic processes with jumps, that will account for sudden variations of the asset prices. On the other hand, such jump models are generally based on the Poisson distribution which has a slower tail decay than the Gaussian distribution. This allows one to assign higher probabilities to extreme events, resulting in a more realistic modeling of asset prices.

13.1 The Poisson Process

The most elementary and useful jump process is the *standard Poisson process* which is a stochastic process $(N_t)_{t\in\mathbb{R}_+}$ with jumps of size $+1$ only, and whose paths are constant in between two jumps, i.e. at time t, the value N_t of the process is given by

$$N_t = \sum_{k=1}^{\infty} \mathbf{1}_{[T_k,\infty)}(t), \qquad t \in \mathbb{R}_+, \tag{13.1}$$

where

$$\mathbf{1}_{[T_k,\infty)}(t) = \begin{cases} 1 \text{ if } t \geq T_k, \\ \\ 0 \text{ if } 0 \leq t < T_k, \end{cases}$$

$k \geq 1$, and $(T_k)_{k\geq 1}$ is the increasing family of jump times of $(N_t)_{t\in\mathbb{R}_+}$ such that

$$\lim_{k\to\infty} T_k = +\infty.$$

In addition, $(N_t)_{t\in\mathbb{R}_+}$ satisfies the following conditions:

1. Independence of increments: for all $0 \leq t_0 < t_1 < \cdots < t_n$ and $n \geq 1$ the random variables
$$N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}},$$
are independent.
2. Stationarity of increments: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leq s \leq t$.

The meaning of the above stationarity condition is that for all fixed $k \in \mathbb{N}$ we have

$$\mathbb{P}(N_{t+h} - N_{s+h} = k) = \mathbb{P}(N_t - N_s = k),$$

for all $h > 0$ and $0 \leq s \leq t$.

The stationarity of increments means that for all $k \in \mathbb{N}$, the probability $\mathbb{P}(N_{t+h} - N_{s+h} = k)$ does not depend on $h > 0$.

The next figure represents a sample path of a Poisson process.

Fig. 13.1: Sample path of a Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Based on the above assumption, a natural question arises:

what is the distribution of N_t at time t ?

We already know that N_t takes values in $\mathbb{N}$ and therefore it has a discrete distribution for all $t \in \mathbb{R}_+$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

It is a remarkable fact that the distribution of the increments of $(N_t)_{t\in\mathbb{R}_+}$, can be completely determined from the above conditions, as shown in the following theorem.

As seen in the next result, $N_t - N_s$ has the Poisson distribution with parameter $\lambda(t-s)$.

Theorem 13.1. Assume that the counting process $(N_t)_{t\in\mathbb{R}_+}$ satisfies the above Conditions 1 and 2. Then we have

$$P(N_t - N_s = k) = e^{-\lambda(t-s)}\frac{(\lambda(t-s))^k}{k!}, \qquad k \in \mathbb{N}, \quad 0 \le s \le t, \tag{13.2}$$

for some constant $\lambda > 0$.

The proof of the above result is skipped here, as we could as well include the distribution property (13.2) among the hypotheses that define the Poisson process.

Precisely, the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ is a process defined by (13.1), which has independent increments distributed according to the Poisson law, in the sense that for all $0 \le t_0 \le t_1 < \cdots < t_n$,

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

is a vector of independent Poisson random variables with respective parameters

$$(\lambda(t_1 - t_0), \ldots, \lambda(t_n - t_{n-1})).$$

The parameter $\lambda > 0$ is called the *intensity* of the process and it is given by

$$\lambda := \lim_{h\to 0}\frac{1}{h}\mathbb{P}(N_h = 1). \tag{13.3}$$

In particular, N_t has a Poisson distribution with parameter λt:

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}, \qquad t > 0.$$

From (13.3) above we see that[1]

$$\mathbb{P}(N_h = 1) = h\lambda e^{-h\lambda} \simeq h\lambda, \qquad h \to 0,$$

and

$$\mathbb{P}(N_h = 0) = e^{-h\lambda} \simeq 1 - h\lambda, \qquad h \to 0,$$

and more generally that

$$\mathbb{P}(N_{t+h} - N_t = 1) = h\lambda e^{-h\lambda} \simeq h\lambda, \qquad h \to 0,$$

[1] We use the notation $f(h) \simeq h^k$ to mean that $\lim_{h\to 0} f(h)/h^k = 1$.

and

$$\mathbb{P}(N_{t+h} - N_t = 0) = e^{-h\lambda} \simeq 1 - h\lambda, \qquad h \to 0,$$

for all $t > 0$. This means that within a "short" interval $[t, t+h]$ of length h, the increment $N_{t+h} - N_t$ behaves like a Bernoulli random variable with parameter λh. This fact can be used for the random simulation of Poisson process paths.

We also find that

$$\mathbb{P}(N_{t+h} - N_t = 2) \simeq h^2 \frac{\lambda^2}{2}, \qquad h \to 0, \qquad t > 0,$$

and more generally

$$\mathbb{P}(N_{t+h} - N_t = k) \simeq h^k \frac{\lambda^k}{k!}, \qquad h \to 0, \qquad t > 0.$$

In order to prove the next proposition we note that we have the equivalence

$$\{T_1 > t\} \Longleftrightarrow \{N_t = 0\},$$

and more generally

$$\{T_n > t\} \Longleftrightarrow \{N_t \leq n-1\}, \qquad n \geq 1.$$

Proposition 13.1. *The law of T_n has the density $t \mapsto \lambda^n \mathrm{e}^{-\lambda t} \frac{t^{n-1}}{(n-1)!}$ on $\mathbb{R}_+$, $n \geq 1$.*

Proof. We have

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \in \mathbb{R}_+,$$

and by induction, assuming that

$$\mathbb{P}(T_{n-1} > t) = \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds, \qquad n \geq 2,$$

we obtain

$$\begin{aligned}
\mathbb{P}(T_n > t) &= \mathbb{P}(T_n > t \geq T_{n-1}) + \mathbb{P}(T_{n-1} > t) \\
&= \mathbb{P}(N_t = n-1) + \mathbb{P}(T_{n-1} > t) \\
&= \mathrm{e}^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds \\
&= \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

where we applied an integration by parts to derive the last line. □

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

In particular, for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(N_t = n) = p_n(t) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

i.e. $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $n \geq 1$, is the density function of T_n.

Similarly we could show that the time

$$\tau_k := T_{k+1} - T_k$$

spent in state $k \in \mathbb{N}$, with $T_0 = 0$, forms a sequence of independent identically distributed random variables having the exponential distribution with parameter $\lambda > 0$, i.e.

$$\mathbb{P}(\tau_0 > t_0, \ldots, \tau_n > t_n) = e^{-\lambda(t_1 + \cdots + t_n)}, \qquad t_1, \ldots, t_n \in \mathbb{R}_+.$$

Since the expectation of the exponentially distributed random variable τ_k with parameter $\lambda > 0$ is given by

$$\mathbb{E}[\tau_k] = \frac{1}{\lambda},$$

we can check that the higher the intensity λ (i.e. the higher the probability of having a jump within a small interval), the smaller is the time spent in each state $k \in \mathbb{N}$ on average.

In addition, given that $\{N_T = n\}$, the n jump times on $[0, T]$ of the Poisson process $(N_t)_{t \in \mathbb{R}_+}$ are independent uniformly distributed random variables on $[0, T]^n$.

Since

$$\mathbb{E}[N_t] = \lambda t,$$

it can be shown that the compensated Poisson process

$$(N_t - \lambda t)_{t \in \mathbb{R}_+}$$

is a *martingale*.

Extensions of the Poisson process include Poisson processes with time-dependent intensity, and with random time-dependent intensity (Cox processes). Renewal processes are counting processes

$$N_t = \sum_{n \geq 1} \mathbf{1}_{[T_n, \infty)}(t), \qquad t \in \mathbb{R}_+,$$

in which $\tau_k = T_{k+1} - T_k$, $k \in \mathbb{N}$, is a sequence of independent identically distributed random variables. In particular, Poisson processes are renewal processes.

13.2 Compound Poisson Processes

The Poisson process itself appears to be too limited to develop realistic price models as its jumps are of constant size. Therefore there is some interest in considered jump processes having random jump sizes.

Let $(Z_k)_{k\geq 1}$ denote an i.i.d. sequence of random variables with probability distribution $\nu(dy)$ on $\mathbb{R}$, independent of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Definition 13.1. The process

$$Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+, \tag{13.4}$$

is called a compound Poisson process.

The next figure represents a sample path of a compound Poisson process, with here $Z_1 = 0.9$, $Z_2 = -0.7$, $Z_3 = 1.4$, $Z_4 = 0.6$, $Z_5 = -2.5$, $Z_6 = 1.5$, $Z_7 = -1.2$.

Fig. 13.1: Sample path of a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$.

Given that $\{N_T = n\}$, the n jump sizes of $(Y_t)_{t\in\mathbb{R}_+}$ on $[0, T]$ are independent random variables which are distributed on $\mathbb{R}$ according to $\nu(dx)$. Based on this fact, the next proposition gives the characteristic function of the increment $Y_T - Y_t$.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Proposition 13.2. *For any $t \in [0,T]$ we have*

$$\mathbb{E}\left[\exp\left(i\alpha(Y_T - Y_t)\right)\right] = \exp\left(\lambda(T-t)\int_{-\infty}^{\infty}(\mathrm{e}^{iy\alpha}-1)\nu(dy)\right),$$

$\alpha \in \mathbb{R}$.

Proof. Since N_t has a Poisson distribution with parameter $t > 0$ and is independent of $(Z_k)_{k\geq 1}$, for all $\alpha \in \mathbb{R}$ we have by conditioning:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\alpha(Y_T - Y_t)\right)\right] &= \sum_{n=0}^{\infty}\mathbb{E}\left[\exp\left(i\alpha\sum_{k=1}^{n}Z_k\right)\right]\mathbb{P}(N_T - N_t = n)\\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(T-t)^n\,\mathbb{E}\left[\exp\left(i\alpha\sum_{k=1}^{n}Z_k\right)\right]\\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(T-t)^n\left(\mathbb{E}\left[\exp\left(i\alpha Z_1\right)\right]\right)^n\\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\lambda(T-t)\int_{-\infty}^{\infty}\mathrm{e}^{i\alpha y}\nu(dy)\right)^n\\
&= \exp\left(\lambda(T-t)\int_{-\infty}^{\infty}(\mathrm{e}^{i\alpha y}-1)\nu(dy)\right),
\end{aligned}$$

since $\nu(dy)$ is the probability distribution of Z_1 and $\int_{-\infty}^{\infty}\nu(dy) = 1$. □

In particular we have

$$\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{k=1}^{N_t}Z_k\Big|N_t\right]\right]\\
&= \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}\,\mathbb{E}\left[\sum_{k=1}^{n}Z_k\Big|N_t = n\right]\\
&= \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}\,\mathbb{E}\left[\sum_{k=1}^{n}Z_k\right]\\
&= \lambda t\mathrm{e}^{-\lambda t}\,\mathbb{E}[Z_1]\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}\\
&= \lambda t\,\mathbb{E}[Z_1].
\end{aligned}$$

This relation can be recovered from the characteristic function of Y_t, as

$$\begin{aligned}
\mathbb{E}[Y_t] &= -i\frac{d}{d\alpha}\,\mathbb{E}[\mathrm{e}^{i\alpha Y_t}]_{|\alpha=0}\\
&= \lambda t\int_{-\infty}^{\infty}y\mu(dy)
\end{aligned}$$

$$= \lambda t \, \mathbb{E}[Z_1].$$

More generally one can show that for all $0 \leq t_0 \leq t_1 \cdots \leq t_n$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ we have

$$\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^{n} \mathrm{e}^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right] &= \exp\left(\lambda \sum_{k=1}^{n} (t_k - t_{k-1}) \int_{-\infty}^{\infty} (\mathrm{e}^{i\alpha_k y} - 1)\nu(dy)\right) \\
&= \prod_{k=1}^{n} \exp\left(\lambda (t_k - t_{k-1}) \int_{-\infty}^{\infty} (\mathrm{e}^{i\alpha_k y} - 1)\nu(dy)\right) \\
&= \prod_{k=1}^{n} \mathbb{E}\left[\mathrm{e}^{i\alpha (Y_{t_k} - Y_{t_{k-1}})}\right],
\end{aligned}$$

hence the compound Poisson process $(Y_t)_{t \in \mathbb{R}_+}$ has independent increments. As a consequence we have the following proposition.

Proposition 13.3. *The compensated compound Poisson process process*

$$M_t := Y_t - \lambda t \, \mathbb{E}[Z_1], \qquad t \in \mathbb{R}_+,$$

is a martingale.

Compound Poisson processes belong to the more general family of Lévy processes which may have an infinite number of jumps on any finite interval, cf. [11].

13.3 Stochastic Integrals with Jumps

Given $(\phi_t)_{t \in \mathbb{R}_+}$ a stochastic process we let the stochastic integral of $(\phi_t)_{t \in \mathbb{R}_+}$ with respect to $(Y_t)_{t \in \mathbb{R}_+}$ be defined by

$$\int_0^T \phi_t dY_t := \sum_{k=1}^{N_T} \phi_{T_k} Z_k.$$

The stochastic integral $\displaystyle\int_0^T \phi_t dY_t$ can be interpreted as the value at time T of a portfolio containing ϕ_t units of a risky asset at time t, whose price evolves according to random returns Z_k at random times T_k.

Next, given $(W_t)_{t \in \mathbb{R}_+}$ a standard Brownian motion independent of $(Y_t)_{t \in \mathbb{R}_+}$ and $(X_t)_{t \in \mathbb{R}_+}$ a jump-diffusion process of the form

$$X_t = \int_0^t u_s dW_s + \int_0^t v_s ds + Y_t, \qquad t \in \mathbb{R}_+,$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

where $(\phi_t)_{t\in\mathbb{R}_+}$ is a process adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(W_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$, and such that

$$\mathbb{E}\left[\int_0^\infty \phi_s^2 u_s^2 ds\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^\infty |\phi_s v_s| ds\right] < \infty,$$

we let the stochastic integral of $(\phi_s)_{s\in\mathbb{R}_+}$ with respect to $(X_s)_{s\in\mathbb{R}_+}$ be defined by

$$\int_0^T \phi_s dX_s := \int_0^T \phi_s u_s dW_s + \int_0^T \phi_s v_s ds + \sum_{k=1}^{N_T} \phi_{T_k} Z_k, \qquad T > 0.$$

When

$$X_t = \int_0^t u_s dW_s + Y_t - \lambda t\, \mathbb{E}[Z_1], \qquad t \in \mathbb{R}_+,$$

is a compensated martingale, the stochastic integral can be shown to satisfy the Itô isometry

$$\mathbb{E}\left[\left(\int_0^T \phi_s dX_s\right)^2\right] = \mathbb{E}\left[\int_0^T \phi_s^2 u_s^2 ds\right] + \lambda\, \mathbb{E}[Z_1^2]\, \mathbb{E}\left[\int_0^T \phi_s^2 ds\right].$$

When $(X_t)_{t\in\mathbb{R}_+}$ takes the form

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds + \int_0^t \eta_s dY_t, \qquad t \in \mathbb{R}_+,$$

the stochastic integral of $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(X_t)_{t\in\mathbb{R}_+}$ satisfies

$$\begin{aligned}\int_0^T \phi_s dX_s &:= \int_0^T \phi_s u_s dW_s + \int_0^T \phi_s v_s ds + \int_0^T \eta_s \phi_s dY_s \\ &= \int_0^T \phi_s u_s dW_s + \int_0^T \phi_s v_s ds + \sum_{k=1}^{N_T} \phi_{T_k} \eta_{T_k} Z_k, \qquad T > 0.\end{aligned}$$

13.4 Itô Formula with Jumps

Let us first consider the case of a simple Poisson process $(N_t)_{t\in\mathbb{R}_+}$. We have

$$\begin{aligned}f(N_t) &= f(0) + \sum_{k=1}^{N_t} f(k) - f(k-1) \\ &= f(0) + \int_0^t (f(1+N_{s^-}) - f(N_{s^-}))dN_s\end{aligned}$$

$$= f(0) + \int_0^t (f(N_s) - f(N_{s^-}))dN_s.$$

Here, N_{s^-} denotes the left limit of the Poisson process at time s, i.e.

$$N_{s^-} = \lim_{h \searrow 0} N_{s-h}.$$

In particular we have

$$k = N_{T_k} = 1 + N_{T_k^-}, \qquad k \geq 1.$$

By the same argument we find, in the case of the compound Poisson process $(Y_t)_{t \in \mathbb{R}_+}$,

$$\begin{aligned} f(Y_t) &= f(0) + \sum_{k=1}^{N_T} (f(Y_{T_k^-} + Z_k) - f(Y_{T_k^-})) \\ &= f(0) + \int_0^t (f(Z_{N_s} + Y_{s^-}) - f(Y_{s^-}))dN_s \\ &= f(0) + \int_0^t (f(Y_s) - f(Y_{s^-}))dN_s. \end{aligned}$$

More generally, for a process of the form

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds + \int_0^t \eta_s dY_s, \qquad t \in \mathbb{R}_+,$$

we find, by combining the Itô formula for Brownian motion with the above argument,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t u_s f'(X_s)dW_s + \frac{1}{2}\int_0^t f''(X_s)u_s^2 ds \\ &\quad + \int_0^t v_s f'(X_s)ds + \sum_{k=1}^{N_T} (f(X_{T_k^-} + \eta_{T_k} Z_k) - f(X_{T_k^-})) \\ &= f(X_0) + \int_0^t u_s f'(X_s)dW_s + \frac{1}{2}\int_0^t f''(X_s)u_s^2 ds + \int_0^t v_s f'(X_s)ds \\ &\quad + \int_0^t (f(X_{s^-} + \eta_s Z_{N_s}) - f(X_{s^-}))dN_s \\ &= f(X_0) + \int_0^t u_s f'(X_s)dW_s + \frac{1}{2}\int_0^t f''(X_s)u_s^2 ds + \int_0^t v_s f'(X_s)ds \\ &\quad + \int_0^t (f(X_s) - f(X_{s^-}))dN_s, \qquad t \in \mathbb{R}_+. \end{aligned}$$

For example, in case

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$X_t = \int_0^t u_s dW_s + \int_0^t v_s ds + \int_0^t \eta_s dN_s, \qquad t \in \mathbb{R}_+,$$

we get

$$\begin{aligned} f(X_t) &= f(0) + \int_0^t u_s f'(X_s) dW_s + \frac{1}{2}\int_0^t u_s^2 f''(X_s) dW_s \\ &\quad + \int_0^t v_s f'(X_s) ds + \int_0^t (f(X_{s^-} + \eta_s) - f(X_{s^-})) dN_s \\ &= f(0) + \int_0^t u_s f'(X_s) dW_s + \frac{1}{2}\int_0^t u_s^2 f''(X_s) dW_s \\ &\quad + \int_0^t v_s f'(X_s) ds + \int_0^t (f(X_s) - f(X_{s^-})) dN_s. \end{aligned}$$

For a process of the form

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t \eta_s dY_t, \qquad t \in \mathbb{R}_+,$$

the Itô formula can be rewritten as

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_0^t u_s f'(X_{s^-}) dW_s + \frac{1}{2}\int_0^t f''(X_s) u_s^2 ds + \int_0^t \eta_s f'(X_{s^-}) dY_s \\ &+ \int_0^t \left(f(X_s) - f(X_{s^-}) - \Delta X_s f'(X_{s^-})\right) d(N_s - s) \\ &+ \int_0^t \left(f(X_s) - f(X_{s^-}) - \Delta X_s f'(X_{s^-})\right) ds, \qquad t \in \mathbb{R}_+, \end{aligned}$$

and this formulation is at the basis of the extension of Itô's formula to Lévy processes with an infinite number of jumps using the bound

$$|f(x+y) - f(x) - y f'(x)| \le C y^2,$$

for f a $\mathcal{C}_b^2(\mathbb{R})$ function. Such processes, also called "infinite activity Lévy processes" [11] are also useful in financial modeling and include the gamma process, stable processes, variance gamma processes, etc.

13.5 Stochastic Differential Equations with Jumps

Let us consider the simplest example

$$dS_t = \eta S_{t^-} dN_t, \tag{13.5}$$

of a stochastic differential equation with respect to the standard Poisson process, with constant coefficient $\eta \in \mathbb{R}$. When $\Delta N_t = 1$, i.e. the Poisson process has a jump at time t, the equation (13.5) reads

$$dS_t = S_t - S_{t^-} = \eta S_{t^-}, \qquad t > 0.$$

which yields

$$S_t = (1+\eta) S_{t^-}, \qquad t > 0.$$

Applying the same procedure at each jump time gives us the solution

$$S_t = S_0 (1+\eta)^{N_t}, \qquad t \in \mathbb{R}_+.$$

Next, consider the case where η becomes time-dependent, i.e.

$$dS_t = \eta_t S_{t^-} dN_t. \tag{13.6}$$

At each jump time T_k, Relation (13.6) reads

$$dS_{T_k} = S_{T_k} - S_{T_k^-} = \eta_{T_k} S_{T_k^-},$$

i.e.

$$S_{T_k} = (1+\eta_{T_k}) S_{T_k^-},$$

and repeating this argument at each jump time yields the product solution

$$S_t = S_0 \prod_{k=1}^{N_t} (1+\eta_{T_k}) = S_0 \prod_{\substack{\Delta N_s = 1 \\ 0 \le s \le t}} (1+\eta_s), \qquad t \in \mathbb{R}_+.$$

On the other hand,

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-} (dN_t - \lambda dt), \tag{13.7}$$

is solved as

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda \int_0^t \eta_s ds \right) \prod_{k=1}^{N_t} (1+\eta_{T_k}) \qquad t \in \mathbb{R}_+.$$

A random simulation of the numerical solution of the above equation (13.7) is given in Figure 13.1.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Fig. 13.1: Geometric Poisson process.

A random simulation of the geometric compound Poisson process

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda\, \mathbb{E}[Z_1] \int_0^t \eta_s ds\right) \prod_{k=1}^{N_t} (1 + \eta_{T_k} Z_k) \qquad t \in \mathbb{R}_+,$$

solution of

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dY_t - \lambda\, \mathbb{E}[Z_1]dt),$$

is given in Figure 13.2.

Fig. 13.2: Geometric compound Poisson process.

In the case of a jump-diffusion stochastic differential equation of the form

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dY_t - \lambda\,\mathbb{E}[Z_1]dt) + \sigma_t S_t dW_t,$$

we get

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda\,\mathbb{E}[Z_1]\int_0^t \eta_s ds + \int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t |\sigma_s|^2 ds\right)\prod_{k=1}^{N_t}(1+\eta_{T_k}Z_k),$$

$t \in \mathbb{R}_+$. A random simulation of the geometric Brownian motion with compound Poisson jumps is given in Figure 13.3.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Fig. 13.3: Geometric Brownian motion with compound Poisson jumps.

By rewriting S_t as

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds + \int_0^t \eta_s (dY_s - \lambda\, \mathbb{E}[Z_1] ds) + \int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t |\eta_s|^2 ds\right)$$
$$\times \prod_{k=1}^{N_t} (e^{-\eta_{T_k}}(1 + \eta_{T_k} Z_k)),$$

$t \in \mathbb{R}_+$, one can extend the model to processes with an infinite number of jumps.

13.6 Girsanov Theorem for Jump Processes

Recall that in its simplest form, the Girsanov theorem for Brownian motion reads

$$\begin{aligned}
\mathbb{E}[f(W_T - \mu T)] &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(x - \mu T) e^{-x^2/(2T)} dx \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(x) e^{-(x+\mu T)^2/(2T)} dx \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(x) e^{-\mu x - \mu^2 T/2} e^{-x^2/(2T)} dx
\end{aligned}$$

$$= \mathbb{E}[f(W_T)e^{-\mu W_T - \mu^2 T/2}],$$

for any bounded measurable function f on $\mathbb{R}$, hence W_T has mean $-\mu T$ under the probability measure

$$e^{-\mu W_T - \mu^2 T/2} d\mathbb{P},$$

and $W_T + \mu T$ is centered Gaussian under that measure. When Brownian motion is replaced with a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, the above space shift may not be used because $N_t + \mu t$ cannot be a Poisson process, whatever the change of probability applied, due to the drift μt and by definition of the standard Poisson process.

The correct way to proceed is to replace the space shift with a time contraction or dilation by a factor $1 + c$ with $c > -1$, as follows:

$$\begin{aligned}
\mathbb{E}[f(N_{T(1+c)})] &= e^{-\lambda(1+c)T} \sum_{k=0}^{\infty} f(k) \frac{(\lambda T(1+c))^k}{k!} \\
&= e^{-\lambda T} e^{-\lambda c T} \sum_{k=0}^{\infty} f(k)(1+c)^k \frac{(\lambda T)^k}{k!} \\
&= e^{-\lambda c T} \sum_{k=0}^{\infty} f(k)(1+c)^k \mathbb{P}(N_T = k) \\
&= e^{-\lambda c T}\, \mathbb{E}[f(N_T)(1+c)^{N_T}] \\
&= e^{-\lambda c T} \int_\Omega (1+c)^{N_T} f(N_T) d\mathbb{P},
\end{aligned}$$

for f any bounded function on $\mathbb{N}$.

Hence, under the probability measure

$$e^{-\lambda c T}(1+c)^{N_T} d\mathbb{P},$$

the random variable N_T has same law as $N_{T(1+c)}$ under $\mathbb{P}$, i.e. it is a Poisson random variable with intensity $\lambda T(1+c)$. Given $\tilde{\lambda} > 0$ and letting $c = -1 + \tilde{\lambda}/\lambda$, we find that N_T is a Poisson random variable with intensity $\tilde{\lambda} T = \lambda(1+c)T$ under the probability measure

$$d\tilde{\mathbb{P}} = e^{-(\tilde{\lambda}-\lambda)T} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T} d\mathbb{P}.$$

Consequently the compensated Poisson process

$$N_t - \tilde{\lambda} t$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

is a martingale under $\tilde{\mathbb{P}}$.

In the case of compound Poisson processes we have:

Theorem 13.2. Let $(Y_t)_{t\geq 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump distribution $\nu(dx)$, and consider another jump distribution $\tilde{\nu}(dx)$, and let

$$\phi(x) = \frac{\tilde{\lambda}}{\lambda}\frac{d\tilde{\nu}}{d\nu}(x) - 1, \qquad x \in \mathbb{R}.$$

Then, under the probability measure

$$d\mathbb{P}_{\tilde{\lambda},\tilde{\nu}} = e^{-(\tilde{\lambda}-\lambda)T}\prod_{k=1}^{N_T}(1+\phi(Z_k))d\mathbb{P}_{\lambda,\nu},$$

the process $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process with intensity $\tilde{\lambda} > 0$ and jump distribution $\tilde{\nu}$, i.e. we have

$$d\mathbb{P}_{\tilde{\lambda},\tilde{\nu}} = e^{-(\tilde{\lambda}-\lambda)T}\prod_{k=1}^{N_T}(1+\phi(Z_k))d\mathbb{P}_{\lambda,\nu}.$$

Proof. For any bounded measurable function f on $\mathbb{R}$ we have

$$\begin{aligned}
&e^{-(\tilde{\lambda}-\lambda)T}\,\mathbb{E}_{\lambda,\nu}\left[f(Y_T)\prod_{i=1}^{N_T}(1+\phi(Z_i))\right]\\
&= \sum_{k=0}^{\infty}\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}\left[f(Y_T)\prod_{i=1}^{k}(1+\phi(Z_i))\Big|N_T=k\right]\mathbb{P}(N_T=k)\\
&= e^{-\lambda T}\sum_{k=0}^{\infty}\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}\left[f(Y_T)\prod_{i=1}^{k}(1+\phi(Z_i))\right]\frac{(\lambda T)^k}{k!}\\
&= e^{-\tilde{\lambda}T}\sum_{k=0}^{\infty}\frac{(\lambda T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\prod_{i=1}^{k}(1+\phi(z_i))\nu(dz_1)\cdots\nu(dz_k)\\
&= e^{-\tilde{\lambda}T}\sum_{k=0}^{\infty}\frac{(\tilde{\lambda}T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\left(\prod_{i=1}^{k}\frac{d\tilde{\nu}}{d\nu}(x_i)\right)\nu(dz_1)\cdots\nu(dz_k)\\
&= e^{-\tilde{\lambda}T}\sum_{k=0}^{\infty}\frac{(\tilde{\lambda}T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\tilde{\nu}(dz_1)\cdots\tilde{\nu}(dz_k)\\
&= \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}\left[f(Y_T)\right],
\end{aligned}$$

for f any bounded function on $\mathbb{N}$. See Proposition 9.6 of [11] for the independence of increments of $(Y_t)_{t\in\mathbb{R}_+}$ under $\mathbb{P}_{\tilde{\lambda},\tilde{\nu}}$. □

As a consequence, under the probability measure

$$d\mathbb{P}_{\tilde{\lambda},\tilde{\nu}} = e^{-(\tilde{\lambda}-\lambda)T} \prod_{k=1}^{N_T} (1+\phi(Z_k)) d\mathbb{P}_{\lambda,\nu},$$

the compensated process

$$Y_t - \tilde{\lambda} t \, \mathbb{E}_{\tilde{\nu}}[Z_1]$$

is a martingale.

More generally we have the following result, in which $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, independent of the compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ under $\mathbb{P}$.

Theorem 13.3. Let $(Y_t)_{t\geq 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump distribution $\nu(dx)$. Consider another jump distribution $\tilde{\nu}(dx)$, let

$$\phi(x) = \frac{\tilde{\lambda}}{\lambda} \frac{d\tilde{\nu}}{d\nu}(x) - 1, \qquad x \in \mathbb{R},$$

and let $(u_t)_{t\in\mathbb{R}_+}$ be a bounded adapted process. Then the process

$$\left(W_t + \int_0^t u_s ds + Y_t - \tilde{\lambda} \, \mathbb{E}_{\tilde{\nu}}[Z_1] t \right)_{t\in\mathbb{R}_+}$$

is a martingale under the probability measure

$$d\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}} = \exp\left(-(\tilde{\lambda}-\lambda)T - \int_0^T u_s dW_s - \frac{1}{2}\int_0^T u_s^2 ds \right) \prod_{k=1}^{N_T} (1+\phi(Z_k)) d\mathbb{P}_{\lambda,\nu}.$$

Consequently, if

$$W_t + \int_0^t v_s ds + Y_t$$

is not a martingale under $\mathbb{P}_{\lambda,\nu}$, it will become a martingale under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$ provided $u, \tilde{\lambda}$ and $\tilde{\nu}$ are chosen in such a way that

$$v_s = u_s - \tilde{\lambda} \, \mathbb{E}_{\tilde{\nu}}[Z_1], \qquad s \in \mathbb{R}. \tag{13.8}$$

Moreover, under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$, the process $W_t + \int_0^t u_s ds$ is a standard Brownian motion, and $Y_t - \tilde{\lambda} \, \mathbb{E}_{\tilde{\nu}}[Z_1]t$ is a martingale.

When $\tilde{\lambda} = \lambda = 0$, Theorem 13.3 coincides with the usual Girsanov theorem for Brownian motion, in which case (13.8) admits only one solution $u = v$ and there is uniqueness of $\tilde{P}_{u,0,0}$. Note that uniqueness occurs also when $u = 0$ in the absence of Brownian motion with Poisson jumps of fixed size a (i.e. $\tilde{\nu}(dx) = \nu(dx) = \delta_a(dx)$) since in this case (13.8) also admits only one solution $\tilde{\lambda} = v$ and there is uniqueness of $\tilde{P}_{0,\tilde{\lambda},\delta_a}$.

13.7 Exercises

Exercise 13.1. Show, by direct computation or using the characteristic function, that the variance of the compound Poisson process Y_t with intensity $\lambda > 0$ satisfies

$$\operatorname{var}[Y_t] = \lambda t\, \mathbb{E}[|Z_1|^2] = \lambda t \int_{-\infty}^{\infty} x^2 \nu(dx).$$

Exercise 13.2. Consider an exponential compound Poisson process of the form

$$S_t = S_0 e^{\mu t + \sigma W_t + Y_t}, \qquad t \in \mathbb{R}_+,$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process of the form (13.4).

1. Derive the stochastic differential equation with jumps satisfied by $(S_t)_{t\in\mathbb{R}_+}$.
2. Let $r > 0$. Find a family $(\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}})$ of probability measures under which the discounted asset price $e^{-rt}S_t$ is a martingale.

Chapter 14
Pricing and Hedging in Jump Models

14.1 Risk-Neutral Measures

Consider an asset price modeled by the equation,

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_{t^-} dY_t,$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process, with

$$S_t = S_0 \exp\left(\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t\right) \prod_{k=1}^{N_t} (1 + Z_k),$$

$t \in \mathbb{R}_+$. An important issue for non-abitrage pricing is to determine a risk-neutral probability measure under which the discounted process $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, and this goal can be achieved using the Girsanov theorem for jump processes.

We have

$$\begin{aligned} d(e^{-rt}S_t) &= (\mu - r)e^{-rt}S_t dt + \sigma e^{-rt}S_t dW_t + e^{-rt}S_{t^-} dY_t \\ &= (\mu - r + \lambda\, \mathbb{E}_\nu[Z_1])e^{-rt}S_t dt + \sigma e^{-rt}S_t dW_t + e^{-rt}S_{t^-}(dY_t - \lambda\, \mathbb{E}_\nu[Z_1]dt), \end{aligned}$$

which yields a martingale under $\mathbb{P}$ provided

$$\mu - r + \lambda\, \mathbb{E}_\nu[Z_1] = 0,$$

however that condition may not be satisfied by the market parameters.

In order for the discounted process $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ to be a martingale, we choose $u \in \mathbb{R}$, $\tilde{\lambda} > 0$, and the measure $\tilde{\nu}$ such that

$$\mu - r = \sigma u - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[Z_1], \tag{14.1}$$

and the Girsanov theorem for jump processes shows that

$$dW_t + udt + dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]dt$$

is a martingale under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$, hence the discounted asset price

$$\begin{aligned} d(e^{-rt}S_t) &= (\mu - r)e^{-rt}S_t dt + \sigma e^{-rt}S_t dW_t + e^{-rt}S_{t^-}dY_t \\ &= \sigma e^{-rt}S_t(dW_t + udt) + e^{-rt}S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]dt), \end{aligned}$$

is a martingale under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$.

In this setting the non-uniqueness of the risk neutral measure is apparent since more degrees of freedom are involved in the choices of u, λ, and the measure $\tilde{\nu}$, where as in the continuous case the choice of $u = (\mu - r)/\sigma$ is unique.

14.2 Pricing in Jump Models

Recall that a market is without arbitrage if and only it admits at least one risk-neutral measure.

Consider a probability measure $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$ under which the discounted asset price

$$d(e^{-rt}S_t) = e^{-rt}S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\nu}[Z_1]dt) + \sigma e^{-rt}S_t d\tilde{W}_t,$$

is a martingale, where $\tilde{W}_t = W_t + udt$ is a standard Brownian motion under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$.

Then the price of a claim with payoff C is given by

$$e^{-r(T-t)}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[C \mid \mathcal{F}_t] \tag{14.2}$$

under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$.

Clearly the price (14.2) of C is no longer unique in the presence of jumps due to the infinity of choices satisfying the martingale condition (14.1), and such a market is not complete, except if either $\tilde{\lambda} = \lambda = 0$, or ($\sigma = 0$ and $\tilde{\nu} = \nu = \delta_1$).

We have

$$\begin{aligned} &e^{-r(T-t)}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t] \\ &= e^{-r(T-t)}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(S_0 \exp\left(\mu T + \sigma W_T - \frac{1}{2}\sigma^2 T\right)\prod_{k=1}^{N_T}(1+Z_k)\right)\Big|\mathcal{F}_t\right] \end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$= e^{-r(T-t)}\,\mathbb{E}_{u,\tilde\lambda,\tilde\nu}\left[\phi\left(S_t\exp\left(\mu(T-t)+\sigma(W_T-W_t)-\frac{1}{2}\sigma^2(T-t)\right)\prod_{k=N_t}^{N_T}(1+Z_k)\right)\Big|\mathcal{F}_t\right]$$

$$= e^{-r(T-t)}\,\mathbb{E}_{u,\tilde\lambda,\tilde\nu}\left[\phi\left(x\exp\left(\mu(T-t)+\sigma(W_T-W_t)-\frac{1}{2}\sigma^2(T-t)\right)\prod_{k=N_t}^{N_T}(1+Z_k)\right)\right]_{x=S_t}$$

$$= e^{-r(T-t)}\sum_{n=0}^{\infty}\mathbb{P}_{u,\tilde\lambda,\tilde\nu}(N_T-N_t=n)$$

$$\mathbb{E}_{u,\tilde\lambda,\tilde\nu}\left[\phi\left(xe^{\mu(T-t)+\sigma(W_T-W_t)-\frac{1}{2}\sigma^2(T-t)}\prod_{k=N_t}^{N_T}(1+Z_k)\right)\Big|N_T-N_t=n\right]_{x=S_t}$$

$$= e^{-r(T-t)-\tilde\lambda(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde\lambda(T-t))^n}{n!}\,\mathbb{E}_{u,\tilde\lambda,\tilde\nu}\left[\phi\left(xe^{\mu(T-t)+\sigma(W_T-W_t)-\frac{1}{2}\sigma^2(T-t)}\prod_{k=1}^{n}(1+Z_k)\right)\right]_{x=S_t}$$

$$= e^{-(r+\tilde\lambda)(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde\lambda(T-t))^n}{n!}$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathbb{E}_{u,\tilde\lambda,\tilde\nu}\left[\phi\left(xe^{\mu(T-t)+\sigma(W_T-W_t)-\frac{1}{2}\sigma^2(T-t)}\prod_{k=1}^{n}(1+z_k)\right)\right]_{x=S_t}\tilde\nu(dz_1)\cdots\tilde\nu(dz_n),$$

hence

$$e^{-r(T-t)}\,\mathbb{E}_{0,\tilde\lambda,\tilde\nu}[\phi(S_T)\mid\mathcal{F}_t]$$

$$= \frac{1}{\sqrt{2\pi(T-t)}}e^{-(r+\tilde\lambda)(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde\lambda(T-t))^n}{n!}$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi\left(S_te^{\mu(T-t)+\sigma(x-u)-\frac{1}{2}\sigma^2(T-t)}\prod_{k=1}^{n}(1+z_k)\right)e^{-\frac{x^2}{2(T-t)}}\tilde\nu(dz_1)\cdots\tilde\nu(dz_n)dx.$$

14.3 Black-Scholes PDE with Jumps

By the Markov property we have

$$e^{-r(T-t)}\,\mathbb{E}_{u,\tilde\lambda,\tilde\nu}[\phi(S_T)\mid\mathcal{F}_t]=f(t,S_t),$$

where $f(t,x)$ is a function such that $f(T,x)=\phi(x)$, and

$$t\longmapsto e^{r(T-t)}f(t,S_t)$$

is a martingale under $\mathbb{P}_{u,\tilde\lambda,\tilde\nu}$.

We have

$$dS_t = rS_tdt+\sigma S_td\tilde W_t+S_{t^-}(dY_t-\tilde\lambda\,\mathbb{E}_{\tilde\nu}[Z_1]dt),$$

where $\tilde{W}_t = W_t + ut$ is a standard Brownian motion under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$.

Hence the Itô formula with jumps shows that

$$\begin{aligned}
df(t,S_t) &= \frac{\partial f}{\partial t}(t,S_t)dt + rS_t\frac{\partial f}{\partial x}(t,S_t)dt + \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\tilde{W}_t + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)dt \\
&\quad -\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]S_t\frac{\partial f}{\partial x}(t,S_t)dt + (f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t \\
&= \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\tilde{W}_t \\
&\quad +(f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z_1)) - f(t,x))]_{x=S_{t^-}}dt \\
&\quad +\frac{\partial f}{\partial t}(t,S_t)dt + rS_t\frac{\partial f}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)dt \\
&\quad +\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z_1)) - f(t,x))]_{x=S_{t^-}}dt - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]S_t\frac{\partial f}{\partial x}(t,S_t)dt,
\end{aligned}$$

and since $e^{r(T-t)}f(t,S_t)$ is a martingale under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$ we have

$$\begin{aligned}
&-rf(t,S_t) + \frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t) \\
&\quad +\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z_1)) - f(t,x))]_{x=S_{t^-}} - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]S_t\frac{\partial f}{\partial x}(t,S_t) = 0,
\end{aligned}$$

or

$$\begin{aligned}
&-rf(t,x) + \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2}(t,x) \\
&\quad +\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z_1)) - f(t,x))] - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]x\frac{\partial f}{\partial x}(t,x) = 0,
\end{aligned}$$

which leads to the *integro-differential* PDE

$$\begin{aligned}
rf(t,x) &= \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2}(t,x) \\
&\quad +\tilde{\lambda}\int_{-\infty}^{\infty}\left(f(t,x(1+y)) - f(t,x) - yx\frac{\partial f}{\partial x}(t,x)\right)\tilde{\nu}(dy),
\end{aligned}$$

under the terminal condition $f(T,x) = \phi(x)$, and we find that

$$\begin{aligned}
df(t,S_t) &= \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\tilde{W}_t + rf(t,S_t)dt \\
&\quad +(f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z_1)) - f(t,x))]_{x=S_{t^-}}dt.
\end{aligned} \tag{14.3}$$

In the case of Poisson jumps with fixed size a, i.e. $Y_t = aN_t$, $\nu(dx) = \delta_a(dx)$, the PDE reads

$$
\begin{aligned}
rf(t,x) = {} & \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x) \\
& + \tilde{\lambda}\left(f(t,x(1+a)) - f(t,x) - ax\frac{\partial f}{\partial x}(t,x) \right),
\end{aligned}
$$

and we have

$$
\begin{aligned}
df(t,S_t) = {} & \sigma S_t \frac{\partial f}{\partial x}(t,S_t)d\tilde{W}_t + rf(t,S_t)dt \\
& + (f(t,S_{t^-}(1+a)) - f(t,S_{t^-}))dN_t - \tilde{\lambda}(f(t,x(1+a)) - f(t,x))dt.
\end{aligned}
$$

14.4 Exponential models

Consider an exponential price process of the form

$$
S_t = S_0 e^{\mu t + \sigma W_t + Y_t} = S_0 \exp\left(\mu t + \sigma W_t + \sum_{k=1}^{N_t} Z_k \right), \qquad t \in \mathbb{R}+,
$$

and choose a risk-neutral measure $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$ under which $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale. Then

$$
e^{-r(T-t)}\, \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t]
$$

is a (non-unique) arbitrage price at time $t \in [0,T]$ for the contingent claim with payoff $\phi(S_T)$.

We have

$$
\begin{aligned}
& e^{-r(T-t)}\, \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t] = e^{-r(T-t)}\, \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_0 e^{\mu T + \sigma W_T + Y_T}) \mid \mathcal{F}_t] \\
& = e^{-r(T-t)}\, \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_t e^{\mu(T-t) + \sigma(W_T - W_t) + Y_T - Y_t}) \mid \mathcal{F}_t] \\
& = e^{-r(T-t)}\, \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(x e^{\mu(T-t) + \sigma(W_T - W_t) + Y_T - Y_t})]_{x=S_t} \\
& = e^{-r(T-t) - \tilde{\lambda}(T-t)} \sum_{n=0}^{\infty} \frac{(\tilde{\lambda}T)^n}{n!}\, \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}} \left[\phi\left(x e^{\mu(T-t) + \sigma(W_T - W_t)} \exp\left(\sum_{k=1}^{n} Z_k \right) \right) \right]_{x=S_t}.
\end{aligned}
$$

In the exponential model

$$
S_t = S_0 e^{\mu t + \sigma W_t + Y_t}
$$

the process S_t has jumps of size $S_{T_k^-}(e^{Z_k} - 1)$, $k \geq 1$, hence (14.1) reads

$$
\mu + \frac{1}{2}\sigma^2 - r = \sigma u - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[e^{Z_1} - 1].
$$

The Merton model

We assume that $(Z_k)_{k\geq 1}$ is a family of independent identically distributed Gaussian $\mathcal{N}(\delta,\eta^2)$ random variables under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$ with

$$\mu+\frac{1}{2}\sigma^2-r=\sigma u-\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[e^{Z_1}-1]=\sigma u-\tilde{\lambda}(e^{\delta+\eta^2/2}-1),$$

from (14.1), hence is a standard Brownian motion under $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$. For simplicity we choose $u=0$, i.e.

$$\mu=r-\frac{1}{2}\sigma^2-\tilde{\lambda}(e^{\delta+\eta^2/2}-1),$$

Hence we have

$$\begin{aligned}
&e^{-r(T-t)}\,\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(S_T)\mid\mathcal{F}_t]\\
&=e^{-r(T-t)-\tilde{\lambda}(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}(T-t))^n}{n!}\,\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}\left[\phi\left(xe^{\mu(T-t)+\sigma(W_T-W_t)}\exp\left(\sum_{k=1}^{n}Z_k\right)\right)\right]_{x=S_t}\\
&=e^{-r(T-t)-\tilde{\lambda}(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}(T-t))^n}{n!}\,\mathbb{E}\left[\phi(xe^{\mu(T-t)+n\delta+X})\right]_{x=S_t},
\end{aligned}$$

where $X\simeq\mathcal{N}(0,\sigma^2(T-t)+n\eta^2)$.

Hence when $\phi(x)=(x-\kappa)^+$ we get

$$\begin{aligned}
&e^{-r(T-t)-\tilde{\lambda}(T-t)}\,\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[(S_T-\kappa)^+\mid\mathcal{F}_t]\\
&=e^{-r(T-t)-\tilde{\lambda}(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}(T-t))^n}{n!}\,\mathbb{E}\left[(xe^{\mu(T-t)+n\delta+X}-\kappa)^+\right]_{x=S_t}\\
&=e^{-r(T-t)-\tilde{\lambda}(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}(T-t))^n}{n!}\,\mathbb{E}\left[(xe^{(r-\frac{1}{2}\sigma^2-\tilde{\lambda}(e^{\delta+\eta^2/2}-1))(T-t)+n\delta+X}-\kappa)^+\right]_{x=S_t}\\
&=e^{-\tilde{\lambda}(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}(T-t))^n}{n!}\mathrm{Bl}(S_te^{n\delta+\frac{1}{2}n\eta^2-\tilde{\lambda}(e^{\delta+\eta^2/2}-1)(T-t)},\sigma^2+n\eta^2/(T-t),r,\kappa).
\end{aligned}$$

14.5 Self-Financing Hedging with Jumps

Consider a self-financing portfolio with value

$$V_t=\eta_te^{rt}+\xi_tS_t$$

and

$$dV_t = r\eta_t e^{rt}dt + \xi_t dS_t.$$

If the portfolio hedges the claim $\phi(S_T)$ we must have $V_t = f(t, S_t)$ and

$$\begin{aligned}
dV_t &= df(t, S_t) \\
&= r\eta_t e^{rt}dt + \xi_t dS_t \\
&= r\eta_t e^{rt}dt + \xi_t(rS_t dt + \sigma S_t d\tilde{W}_t + S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]dt)) \\
&= rV_t dt + \sigma\xi_t S_t d\tilde{W}_t + \xi_t S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]dt) \\
&= rf(t, S_t)dt + \sigma\xi_t S_t d\tilde{W}_t + \xi_t S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]dt), \qquad (14.4)
\end{aligned}$$

has to match

$$\begin{aligned}
df(t, S_t) &= rf(t, S_t)dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t)d\tilde{W}_t \qquad (14.5) \\
&+(f(t, S_{t^-}(1 + Z_{N_t})) - f(t, S_{t^-}))dN_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t, x(1 + Z_1)) - f(t, x))]_{x=S_{t^-}}dt,
\end{aligned}$$

which is obtained from (14.3).

Exact replication is possible in only two situations:

(i) Continuous market, $\lambda = \tilde{\lambda} = 0$. In this case we find the usual Black-Scholes Delta:

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t). \qquad (14.6)$$

(ii) Poisson jump market, $\sigma = 0$ and $Y_t = aN_t$, $\nu(dx) = \delta_a(dx)$. In this case we find

$$\xi_t = \frac{1}{aS_{t^-}}(f(t, S_{t^-}(1 + a)) - f(t, S_{t^-})). \qquad (14.7)$$

Note that in the limit $a \to 0$ this expression recovers the Black-Scholes Delta formula (14.6).

In the general case we find an hedging error given from (14.4) and (14.5) by

$$\begin{aligned}
V_T - \phi(S_T) &= V_T - f(T, S_T) \\
&= V_0 - f(0, S_0) + \int_0^T dV_t - \int_0^T df(t, S_t) \\
&= V_0 - f(0, S_0) + \sigma \int_0^T S_t \left(\xi_t - \frac{\partial f}{\partial x}(t, S_t)\right) d\tilde{W}_t \\
&\quad + \int_0^T \xi_t S_{t^-}(Z_{N_t}dN_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z_1]dt) \\
&\quad - \int_0^T (f(t, S_{t^-}(1 + Z_{N_t})) - f(t, S_{t^-}))dN_t \\
&\quad + \tilde{\lambda} \int_0^T \mathbb{E}_{\tilde{\nu}}[(f(t, x(1 + Z_1)) - f(t, x))]_{x=S_{t^-}}dt.
\end{aligned}$$

Assuming for simplicity that $Y_t = aN_t$, i.e. $\nu(dx) = \delta_a(dx)$, we get

$$V_T - f(T, S_T) = V_0 - f(0, S_0) + \sigma \int_0^T S_t \left(\xi_t - \frac{\partial f}{\partial x}(t, S_t) \right) d\tilde{W}_t$$
$$- \int_0^T (f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})(dN_t - \tilde{\lambda} dt),$$

hence the mean square hedging error is given by

$$\begin{aligned}
&\mathbb{E}_{u,\tilde{\lambda}}[(V_T - f(T, S_T))^2] \\
&= (V_0 - f(0, S_0))^2 + \sigma^2 \, \mathbb{E}_{u,\tilde{\lambda}} \left[\left(\int_0^T S_{t^-} \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-}) \right) d\tilde{W}_t \right)^2 \right] \\
&\quad + \mathbb{E}_{u,\tilde{\lambda}} \left[\left(\int_0^T (f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})(dN_t - \tilde{\lambda} dt) \right)^2 \right] \\
&= (V_0 - f(0, S_0))^2 + \sigma^2 \, \mathbb{E}_{u,\tilde{\lambda}} \left[\int_0^T S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-}) \right)^2 dt \right] \\
&\quad + \tilde{\lambda} \, \mathbb{E}_{u,\tilde{\lambda}} \left[\int_0^T \left((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-}) \right)^2 dt \right].
\end{aligned}$$

Clearly, the initial value that minimizes the above quantity is

$$V_0 = f(0, S_0) = e^{-rT} \, \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T)].$$

When hedging only the risk generated by the Brownian part we let

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t)$$

as in the Black-Scholes model, and in this case the hedging error becomes

$$\mathbb{E}_{u,\tilde{\lambda}}[(V_T - f(T, S_T))^2] = \tilde{\lambda} \, \mathbb{E}_{u,\tilde{\lambda}} \left[\int_0^T \left((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-}) \right)^2 dt \right].$$

Next, let us find the optimal strategy $(\xi_t)_{t \in \mathbb{R}_+}$ that minimizes the remaining hedging error

$$\mathbb{E}_{u,\tilde{\lambda}} \left[\int_0^T \left(\sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-}) \right)^2 + \tilde{\lambda} \left((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-}) \right)^2 \right) dt \right].$$

For all $t \in [0, T]$, the almost-sure minimum of

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\xi_t \longmapsto \sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-})\right)^2 + \tilde{\lambda}\left((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})\right)^2$$

is given by differentiation with respect to ξ_t, as the solution of

$$\sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-})\right) - a\tilde{\lambda} S_{t^-}\left((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})\right) = 0,$$

i.e.

$$\xi_t = \frac{\sigma^2 \dfrac{\partial f}{\partial x}(t, S_{t^-}) + \dfrac{a\tilde{\lambda}}{S_{t^-}}(f(t, S_{t^-}(1+a)) - f(t, S_{t^-}))}{\sigma^2 + a^2\tilde{\lambda}}, \qquad t \in [0, T]. \tag{14.8}$$

We note that the optimal strategy (14.8) is a weighted average of the Brownian and jump hedging strategies (14.6) and (14.7) according to the respective variance parameters σ^2 and $a^2\tilde{\lambda}$ of the continuous and jump components.

Clearly, if $a\tilde{\lambda} = 0$ we get

$$\xi_t = \frac{\partial f}{\partial x}(t, S_{t^-}), \qquad t \in [0, T],$$

which is the Black-Scholes perfect replication strategy, and when $\sigma = 0$ we recover

$$\xi_t = \frac{f(t, S_{t^-}(1+a)) - f(t, S_{t^-})}{aS_{t^-}}, \qquad t \in [0, T].$$

which is (14.7).

Note that the fact that perfect replication is not possible in a jump-diffusion model can be interpreted as a more realistic feature of the model, as perfect replication is not possible in the real world.

14.6 Exercises

Exercise 14.1. Consider a forward call option with payoff $S_T - K$ on a jump diffusion risky asset $(S_t)_{t\in\mathbb{R}_+}$ given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_{t^-} dY_t.$$

1. Show that the forward claim admits a unique arbitrage price to be computed in a market with risk-free rate $r > 0$.
2. Show that the forward claim admits an exact replicating portfolio strategy based on the two assets S_t and e^{rt}.

3. Show that the portfolio strategy of Question 2 coincides with the optimal portfolio strategy (14.8).

Exercise 14.2. Consider $(W_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion and $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$, independent of $(W_t)_{t\in\mathbb{R}_+}$, under a probability measure $\mathbb{P}^*$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + \eta S_{t^-} dN_t + \sigma S_t dW_t. \tag{14.9}$$

(i) Solve the equation (14.9).
(ii) We assume that μ, η and the risk-free rate $r > 0$ are chosen such that the discounted process $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$. What relation does this impose on μ, η and r ?
(iii) Under the relation of Question (ii), compute the price at time t of a European call option on S_T with strike κ and maturity T, using a series expansion of Black-Scholes functions.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Chapter 15
Basic Numerical Methods

This chapter is an elementary introduction to finite difference methods for the resolution of PDEs and stochastic differential equations. We cover the explicit and implicit finite difference schemes for the heat equations and the Black-Scholes PDE, as well as the Euler and Milshtein schemes for SDEs.

15.1 The heat equation

Consider the heat equation

$$\frac{\partial \phi}{\partial t}(t,x) = \frac{\partial^2 \phi}{\partial x^2}(t,x) \tag{15.1}$$

with initial condition

$$\phi(0,x) = f(x)$$

on a compact interval $[0,T] \times [0,X]$ divided into the grid

$$(t_i, x_j) = (i\Delta t, j\Delta x), \qquad i = 0,\dots,N, \quad j = 0,\dots,M,$$

with $\Delta t = T/N$ and $\Delta x = X/M$. Our goal is to obtain a discrete approximation

$$(\phi(t_i,x_j))_{0\le i\le N,\ 0\le j\le M}$$

of the solution to (15.1), by approximation of the derivative using finite differences.

Explicit method

Using the *forward* time difference approximation of (15.1) we get

$$\frac{\phi(t_{i+1},x_j)-\phi(t_i,x_j)}{\Delta t}=\frac{\phi(t_i,x_{j+1})+\phi(t_i,x_{j-1})-2\phi(t_i,x_j)}{(\Delta x)^2}$$

and letting $\rho=(\Delta t)/(\Delta x)^2$ this yields

$$\phi(t_{i+1},x_j)=\rho\phi(t_i,x_{j+1})+(1-2\rho)\phi(t_i,x_j)+\rho\phi(t_i,x_{j-1}),$$

$1\leq j\leq M-1$, i.e

$$\Phi_{i+1}=A\Phi_i+\rho\begin{bmatrix}\phi(t_i,x_0)\\0\\\vdots\\0\\\phi(t_i,x_M)\end{bmatrix},\qquad i=0,1,\ldots,N-1,$$

with

$$\Phi_i=\begin{bmatrix}\phi(t_i,x_1)\\\vdots\\\phi(t_i,x_{M-1})\end{bmatrix},\qquad i=0,1,\ldots,N,$$

and

$$A=\begin{bmatrix}1-2\rho&\rho&0&\cdots&0&0&0\\\rho&1-2\rho&\rho&\cdots&0&0&0\\0&\rho&1-2\rho&\cdots&0&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&0&\cdots&1-2\rho&\rho&0\\0&0&0&\cdots&\rho&1-2\rho&\rho\\0&0&0&\cdots&0&\rho&1-2\rho\end{bmatrix}.$$

The vector

$$\begin{bmatrix}\phi(t_i,x_0)\\0\\\vdots\\0\\\phi(t_i,x_M)\end{bmatrix},\qquad i=0,\ldots,N,$$

can be given by the lateral boundary conditions $\phi(t,0)$ and $\phi(t,X)$.

Implicit method

Using the *backward* time difference approximation

$$\frac{\phi(t_i,x_j)-\phi(t_{i-1},x_j)}{\Delta t}=\frac{\phi(t_i,x_{j+1})+\phi(t_i,x_{j-1})-2\phi(t_i,x_j)}{(\Delta x)^2}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

and letting $\rho = (\Delta t)/(\Delta x)^2$ we get

$$\phi(t_{i-1}, x_j) = -\rho\phi(t_i, x_{j+1}) + (1+2\rho)\phi(t_i, x_j) - \rho\phi(t_i, x_{j-1}),$$

$1 \le j \le M-1$, i.e.

$$\Phi_{i-1} = B\Phi_i + \rho \begin{bmatrix} \phi(t_i, x_0) \\ 0 \\ \vdots \\ 0 \\ \phi(t_i, x_M) \end{bmatrix}, \qquad i = 1, 2, \ldots, N,$$

with

$$B = \begin{bmatrix} 1+2\rho & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+2\rho & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+2\rho & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\rho & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+2\rho & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1+2\rho \end{bmatrix}.$$

By inversion of the matrix B, Φ_i is given in terms of Φ_{i-1} as

$$\Phi_i = B^{-1}\Phi_{i-1} - \rho B^{-1} \begin{bmatrix} \phi(t_i, x_0) \\ 0 \\ \vdots \\ 0 \\ \phi(t_i, x_M) \end{bmatrix}, \qquad i = 1, \ldots, N.$$

15.2 The Black-Scholes PDE

Consider the Black-Scholes PDE

$$r\phi(t,x) = \frac{\partial \phi}{\partial t}(t,x) + rx\frac{\partial \phi}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 \phi}{\partial x^2}(t,x), \tag{15.2}$$

under the terminal condition $\phi(T,x) = (x-K)^+$, resp. $\phi(T,x) = (K-x)^+$, for a European call, resp. put, option.

Note that here time runs *backwards* as we start from a terminal condition at time T. Thus here the explicit method uses *backward* differences while the implicit method uses *forward* differences.

Explicit method

Using the *backward* time difference approximation of (15.2) we get

$$r\phi(t_i, x_j) = \frac{\phi(t_i, x_j) - \phi(t_{i-1}, x_j)}{\Delta t} + r x_j \frac{\phi(t_i, x_{j+1}) - \phi(t_i, x_{j-1})}{2\Delta x} + \frac{1}{2} x_j^2 \sigma^2 \frac{\phi(t_i, x_{j+1}) + \phi(t_i, x_{j-1}) - 2\phi(t_i, x_j)}{(\Delta x)^2},$$

$1 \leq j \leq M-1$, i.e.

$$\phi(t_{i-1}, x_j) = \frac{1}{2}\Delta t(\sigma^2 j^2 - rj)\phi(t_i, x_{j-1}) + (1 - \Delta t(\sigma^2 j^2 + r))\phi(t_i, x_j) + \frac{1}{2}\Delta t(\sigma^2 j^2 + rj)\phi(t_i, x_{j+1}),$$

$1 \leq j \leq M-1$, where the boundary conditions $\phi(t_i, x_0)$ and $\phi(t_i, x_M)$ may be provided as

$$\phi(t_i, x_0) = 0, \qquad \phi(t_i, x_M) = x_M - Ke^{-r(T-t_i)}, \qquad 0 \leq i \leq N,$$

for a European call option, and

$$\phi(t_i, x_0) = Ke^{-r(T-t_i)}, \qquad \phi(t_i, x_M) = 0 \qquad 0 \leq i \leq N,$$

for a European put option.

The explicit finite difference method is known to have a divergent behaviour when time runs backwards, as illustrated in Figure 15.1.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Fig. 15.1: Divergence of the explicit finite difference method.

Implicit method

Using the *forward* time difference approximation of (15.2) we get

$$r\phi(t_i,x_j) = \frac{\phi(t_{i+1},x_j)-\phi(t_i,x_j)}{\Delta t} + rx_j\frac{\phi(t_i,x_{j+1})-\phi(t_i,x_{j-1})}{2\Delta x} + \frac{1}{2}x_j^2\sigma^2\frac{\phi(t_i,x_{j+1})+\phi(t_i,x_{j-1})-2\phi(t_i,x_j)}{(\Delta x)^2},$$

$1 \le j \le M-1$, i.e.

$$\phi(t_{i+1},x_j) = -\frac{1}{2}\Delta t(\sigma^2 j^2 - rj)\phi(t_i,x_{j-1}) + (1+\Delta t(\sigma^2 j^2 + r))\phi(t_i,x_j) - \frac{1}{2}\Delta t(\sigma^2 j^2 + rj)\phi(t_i,x_{j+1}),$$

$1 \le j \le M-1$, i.e.

$$\Phi_{i+1} = B\Phi_i + \begin{bmatrix} \frac{1}{2}\Delta t\left(r-\sigma^2\right)\phi(t_i,x_0) \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}\Delta t\left(r(M-1)+\sigma^2(M-1)^2\right)\phi(t_i,x_M) \end{bmatrix}, \qquad i=0,1,\ldots,N-1,$$

with

$$B_{j,j-1} = \frac{1}{2}\Delta t\left(rj-\sigma^2 j^2\right), \quad B_{j,j} = 1+\sigma^2 j^2\Delta t + r\Delta t, \quad \text{and} \quad B_{j,j+1} = -\frac{1}{2}\Delta t\left(rj+\sigma^2 j^2\right),$$

for $j = 1, \ldots, M-1$, and $B(i,j) = 0$ otherwise.

By inversion of the matrix B, Φ_i is given in terms of Φ_{i+1} as

$$\Phi_i = B^{-1}\Phi_{i+1} - B^{-1}\begin{bmatrix} \frac{1}{2}\Delta t\left(r-\sigma^2\right)\phi(t_i,x_0) \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}\Delta t\left(r(M-1)+\sigma^2(M-1)^2\right)\phi(t_i,x_M) \end{bmatrix},$$

$i = 0, 1, \ldots, N-1$, where the boundary conditions $\phi(t_i, x_0)$ and $\phi(t_i, x_M)$ can be provided as in the case of the explicit method.

Note that for all $j = 1, \ldots, M-1$ we have

$$B_{j,j-1} + B_{j,j} + B_{j,j+1} = 1 + r\Delta t,$$

hence when the terminal condition is a constant $\phi(T,x) = c > 0$ we get

$$\phi(t_i, x) = c(1 + r\Delta t)^{-(N-i)} = c(1 + rT/N)^{-(N-i)}, \qquad i = 0, \ldots, N,$$

hence for all $s \in [0, T]$,

$$\begin{aligned} \phi(s,x) &= \lim_{N\to\infty} \phi(t_{[Ns/T]}, x) \\ &= c \lim_{N\to\infty} (1 + rT/N)^{-(N-[Ns/T])} \\ &= c \lim_{N\to\infty} (1 + rT/N)^{-[N(T-s)/T]} \\ &= c \lim_{N\to\infty} (1 + rT/N)^{-(T-s)/T} \\ &= ce^{-r(T-s)}, \end{aligned}$$

as expected, where $[x]$ denotes the integer part of $x \in \mathbb{R}$. The implicit finite difference method is known to be more stable than the explicit method, as illustrated in Figure 15.2, in which the discretization parameters have been taken to be the same as in Figure 15.1.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Fig. 15.2: Stability of the implicit finite difference method.

15.3 Euler discretization

In order to apply the Monte Carlo method for the numerical computation of expectations and for option pricing, we need to generate random samples of solutions to stochastic differential equations. This can be done by discretizing the equation.

The Euler discretization scheme for the stochastic differential equation

$$dX_t = b(X_t)dt + a(X_t)dW_t$$

is given by

$$\begin{aligned}\hat{X}^N_{t_{k+1}} &= \hat{X}^N_{t_k} + \int_{t_k}^{t_{k+1}} b(X_s)ds + \int_{t_k}^{t_{k+1}} a(X_s)dW_s \\ &\simeq \hat{X}^N_{t_k} + b(\hat{X}^N_{t_k})(t_{k+1} - t_k) + a(\hat{X}^N_{t_k})(W_{t_{k+1}} - W_{t_k}).\end{aligned}$$

In particular, when X_t is the geometric Brownian motion given by

$$dX_t = rX_t dt + \sigma X_t dW_t$$

we get

$$\hat{X}^N_{t_{k+1}} = \hat{X}^N_{t_k} + r\hat{X}^N_{t_k}(t_{k+1} - t_k) + \sigma\hat{X}^N_{t_k}(W_{t_{k+1}} - W_{t_k}),$$

which can be computed as

$$\hat{X}_{t_k}^N = \hat{X}_{t_0}^N \prod_{i=1}^{k} \left(1 + r(t_i - t_{i-1}) + \sigma(W_{t_i} - W_{t_{i-1}})\right).$$

15.4 Milshtein discretization

In the Milshtein scheme we expand $a(X_s)$ as

$$a(X_s) \simeq a(X_{t_k}) + a'(X_{t_k})b(X_{t_k})(s - t_k) + a'(X_{t_k})a(X_{t_k})(W_s - W_{t_k}).$$

As a consequence we get

$$\begin{aligned}
\hat{X}_{t_{k+1}}^N &= \hat{X}_{t_k}^N + \int_{t_k}^{t_{k+1}} b(X_s)ds + \int_{t_k}^{t_{k+1}} a(X_s)dW_s \\
&\simeq \hat{X}_{t_k}^N + \int_{t_k}^{t_{k+1}} b(X_s)ds + a(X_{t_k})(W_{t_{k+1}} - W_{t_k}) \\
&\quad + a'(X_{t_k})b(X_{t_k}) \int_{t_k}^{t_{k+1}} (s - t_k) + a'(X_{t_k})a(X_{t_k}) \int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s \\
&\simeq \hat{X}_{t_k}^N + \int_{t_k}^{t_{k+1}} b(X_s)ds + a(X_{t_k})(W_{t_{k+1}} - W_{t_k}) \\
&\quad + \frac{1}{2}a'(X_{t_k})b(X_{t_k})(t_{k+1} - t_k)^2 + a'(X_{t_k})a(X_{t_k}) \int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s \\
&\simeq \hat{X}_{t_k}^N + \int_{t_k}^{t_{k+1}} b(X_s)ds + a(X_{t_k})(W_{t_{k+1}} - W_{t_k}) \\
&\quad + a'(X_{t_k})a(X_{t_k}) \int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s.
\end{aligned}$$

Next using Itô's formula we note that

$$(W_{t_{k+1}} - W_{t_k})^2 = 2\int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s + \int_{t_k}^{t_{k+1}} ds,$$

hence

$$\int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s = \frac{1}{2}((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)),$$

and

$$\hat{X}_{t_{k+1}}^N \simeq \hat{X}_{t_k}^N + \int_{t_k}^{t_{k+1}} b(X_s)ds + a(X_{t_k})(W_{t_{k+1}} - W_{t_k})$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$
\begin{aligned}
&+\frac{1}{2}a'(X_{t_k})a(X_{t_k})((W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k))\\
&=\hat{X}^N_{t_k}+\int_{t_k}^{t_{k+1}} b(X_s)ds+a(X_{t_k})(W_{t_{k+1}}-W_{t_k})\\
&+\frac{1}{2}a'(X_{t_k})a(X_{t_k})((W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k))\\
&=\hat{X}^N_{t_k}+b(X_{t_k})(t_{k+1}-t_k)+a(X_{t_k})(W_{t_{k+1}}-W_{t_k})\\
&+\frac{1}{2}a'(X_{t_k})a(X_{t_k})((W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k)).
\end{aligned}
$$

As a consequence the Milshtein scheme is written as

$$
\begin{aligned}
\hat{X}^N_{t_{k+1}} \simeq\ & \hat{X}^N_{t_k}+b(\hat{X}^N_{t_k})(t_{k+1}-t_k)+a(\hat{X}^N_{t_k})(W_{t_{k+1}}-W_{t_k})\\
&+\frac{1}{2}a'(\hat{X}^N_{t_k})a(\hat{X}^N_{t_k})((W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k)),
\end{aligned}
$$

i.e. in the Milshtein scheme we take into account the "small" difference

$$(W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k)$$

existing between $(\Delta W_t)^2$ and Δt. Taking $(\Delta W_t)^2$ equal to Δt brings us back to the Euler scheme.

When X_t is the geometric Brownian motion given by

$$dX_t = rX_t dt + \sigma X_t dW_t$$

we get

$$\hat{X}^N_{t_{k+1}}=\hat{X}^N_{t_k}+(r-\sigma^2/2)\hat{X}^N_{t_k}(t_{k+1}-t_k)+\sigma\hat{X}^N_{t_k}(W_{t_{k+1}}-W_{t_k})+\frac{1}{2}\sigma^2\hat{X}^N_{t_k}(W_{t_{k+1}}-W_{t_k})^2,$$

which can be computed as

$$\hat{X}^N_{t_k}=\hat{X}^N_{t_0}\prod_{i=1}^{k}\left(1+(r-\sigma^2/2)(t_i-t_{i-1})+\sigma(W_{t_i}-W_{t_{i-1}})+\frac{1}{2}\sigma^2(W_{t_i}-W_{t_{i-1}})^2\right).$$

Chapter 16
Appendix: Background on Probability Theory

In this appendix we review a number of basic probabilistic tools that are needed in option pricing and hedging.

16.1 Probability Spaces

We will need the following notation coming from set theory. Given A and B to abstract sets, "$A \subset B$" means that A is contained in B. The property that ω belongs to the set A is denoted by "$\omega \in A$". The finite set made of n elements $\omega_1, \ldots, \omega_n$ is denoted by $\{\omega_1, \ldots, \omega_n\}$. Note that the element ω is in fact distinct from the set $\{\omega\}$.

A probability space is an abstract set Ω made of the possible outcomes of a random experiment.

Examples:

- Coin tossing: $\Omega = \{H, T\}$.
- Rolling one die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Picking on card at random in a pack of 52: $\Omega = \{1, 2, 3, , \ldots, 52\}$.
- An integer-valued random outcome: $\Omega = \mathbb{N}$.

  In this case the outcome $\omega \in \mathbb{N}$ can be the random number of trials needed until some event occurs.
- A non-negative, real-valued outcome: $\Omega = \mathbb{R}_+$.

In this case the outcome $\omega \in \mathbb{R}_+$ may represent the (non-negative) value of a stock price, or a continuous random time.

- A random continuous parameter (such as time, weather, price, temperature, ...): $\Omega = \mathbb{R}$.

- Random choice of a continuous path in the space $\Omega = \mathcal{C}(\mathbb{R}_+)$ of all continuous functions on $\mathbb{R}_+$.

  In this case, $\omega \in \Omega$ is a function $\omega : \mathbb{R}_+ \to \mathbb{R}$ and a typical example is the graph $t \mapsto \omega(t)$ of a stock price over time.

Product spaces:

Probability spaces can be built as product spaces and used for the modeling of repeated random experiments.

- Rolling two dice: $\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$.

  In this case a typical element of Ω is written as $\omega = (k, l)$ with $k, l \in \{1, 2, 3, 4, 5, 6\}$.

- A finite number n of real-valued samples: $\Omega = \mathbb{R}^n$.

  In this case the outcome ω is a vector $\omega = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with n components.

Note that to some extent, the more complex Ω is, the better it fits a practical and useful situation, e.g. $\Omega = \{H, T\}$ corresponds to a simple coin tossing experiment while $\Omega = \mathcal{C}(\mathbb{R}_+)$ can be applied to the modeling of stock markets.

On the other hand, in many situations and especially in the most complex situations, we will *not* attempt to specify Ω explicitly.

16.2 Events

An event is a collection of outcomes, which is represented by a subset of Ω. The collection of all events will be denoted by $\mathcal{F}$. In the context of stochastic processes, $\mathcal{F}$ will be used to represent a given quantity of information.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Note that in the set theoretic notation, an event A is a subset of Ω, i.e. $A \subset \Omega$, while it is an element of $\mathcal{F}$, i.e. $A \in \mathcal{F}$.

Example: $\Omega = \{1, 2, 3, 4, 5, 6\}$. The event $A = \{2, 4, 6\}$ corresponds to

"the result of the experiment is an even number".

This formalism is helpful to describe events in a short and precise way.

The collection of all events in Ω will be generally denoted by $\mathcal{F}$.

Example:

Take

$$\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H.T), (T, H), (T, T)\}.$$

In this case, the collection $\mathcal{F}$ of all possible events is given by

$$\begin{aligned}\mathcal{F} = \{&\emptyset, \{(H, H)\}, \{(T, T)\}, \{(H, T)\}, \{(T, H)\},\\ &\{(T, T), (H, H)\}, \{(H, T), (T, H)\}, \{(H, T), (T, T)\},\\ &\{(T, H), (T, T)\}, \{(H, T), (H, H)\}, \{(T, H), (H, H)\},\\ &\{(H, H), (T, T), (T, H)\}, \{(H, H), (T, T), (H, T)\},\\ &\{(H, T), (T, H), (H, H)\}, \{(H, T), (T, H), (T, T)\}, \Omega\}.\end{aligned}$$

The empty set $\emptyset$ and the full space Ω are considered as events but they are of less importance because Ω corresponds to "any outcomet can occur" while $\emptyset$ corresponds to an absence of outcome, or no experiment.

Note that taking $\underline{n = 4}$, the set above $\mathcal{F}$ of all events has altogether

$$\begin{aligned}
1 &= \binom{n}{0} \text{ event of cardinal 0,}\\
4 &= \binom{n}{1} \text{ events of cardinal 1,}\\
6 &= \binom{n}{2} \text{ events of cardinal 2,}\\
4 &= \binom{n}{3} \text{ events of cardinal 3,}\\
1 &= \binom{n}{1} \text{ event of cardinal 1,}
\end{aligned}$$

for a total of

$$16 = 2^n = \sum_{k=0}^{n} \binom{n}{k} = 1 + 4 + 6 + 4 + 1$$

events.

Exercise: Write down the set of all events of $\Omega = \{H, T\}$.

Note also that (H, T) is different from (T, H), whereas $\{(H, T), (T, H)\}$ is equal to $\{(T, H), (H, T)\}$.

In addition we will usually make a distinction between the *outcome* $\omega \in \Omega$ from its associated *event* $\{\omega\} \in \mathcal{F}$, which satisfies $\{\omega\} \subset \Omega$.

16.3 Probability Measures

A probability measure is a mapping $\mathbb{P} : \mathcal{F} \to [0, 1]$ that assigns a probability $\mathbb{P}(A) \in [0, 1]$ to any event A, with the properties

a) $\mathbb{P}(\Omega) = 1$ and
b) $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$, whenever $A_k \cap A_l = \emptyset$, $k \neq l$.

In particular we have

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n)$$

when the subsets $A_1, \ldots, A_n$ of Ω are disjoints, and

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

if $A \cap B = \emptyset$. In the general case we can write

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

The triple

$$(\Omega, \mathcal{F}, \mathbb{P}) \tag{16.1}$$

was introduced by A.N. Kolmogorov (1903-1987), and is generally referred to as the Kolmogorov framework. A property is said to hold $\mathbb{P}$-almost surely (also written $\mathbb{P}$-a.s.) if it holds with probability equal to one.
In addition we have the following convergence properties.

1. Let $(A_n)_{n\in\mathbb{N}}$ be an *increasing* sequence of events, i.e. $A_n \subset A_{n+1}$, $n \in \mathbb{N}$. Then we have
$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$
2. Let $(A_n)_{n\in\mathbb{N}}$ be a *decreasing* sequence of events, i.e. $A_{n+1} \subset A_n$, $n \in \mathbb{N}$. Then we have

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

16.4 Conditional Probabilities and Independence

We start with an example.

Consider a population $\Omega = M \cup W$ made of a set M of men and a set W of women. After polling the population, e.g. for a market survey, it turns out that a proportion $p \in [0,1]$ of the population declares to like apples, while a proportion $1-p$ declares to dislike apples. Let $L \subset \Omega$ denote the subset of individuals who like apples, while $D \subset \Omega$ denotes the subset individuals who dislike apples, with

$$p = \mathbb{P}(L) \quad \text{and} \quad 1-p = \mathbb{P}(D).$$

Suppose now that $p = 60\%$ of the population likes apples. It may be interesting to get a more precise information and to determine

- the relative proportion of women who like apples, and
- the relative proportion of men who like apples.

Given that $M \cap L$ (resp. $W \cap L$) denotes the set of men (resp. women) who like apples, these relative proportions are respectively given by

$$\frac{\mathbb{P}(L\cap M)}{\mathbb{P}(M)} \quad \text{and} \quad \frac{\mathbb{P}(L\cap W)}{\mathbb{P}(W)}.$$

Here, $\mathbb{P}(L\cap W)/\mathbb{P}(W)$ represents the probability that a woman picked at random in W likes apples, and $\mathbb{P}(L\cap M)/\mathbb{P}(M)$ represents the probability that a man picked at random in M likes apples. Thus these two proportions are also interpreted as *conditional probabilities*, i.e. for example $\mathbb{P}(L\cap M)/\mathbb{P}(M)$ denotes the probability that an individual likes apples *given that* he is a man.

More generally, given any two events $A, B \subset \Omega$ with $\mathbb{P}(B) \neq 0$, we call

$$\mathbb{P}(A \mid B) := \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

the probability of A *given* B, or *conditionally to* B.

The events A and B are said to be *independent* if

$$\mathbb{P}(A \mid B) = \mathbb{P}(A),$$

i.e. if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

We also recall the following property:

$$\begin{aligned}\mathbb{P}\left(B \cap \bigcup_{n=1}^{\infty} A_n\right) &= \sum_{n=1}^{\infty} \mathbb{P}(B \cap A_n) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(A_n \mid B)\mathbb{P}(B) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(B \mid A_n)\mathbb{P}(A_n),\end{aligned}$$

provided $A_i \cap A_j = \emptyset$, $i \neq j$, and $\mathbb{P}(A_i) > 0$, $i \geq 1$.

Note that conditional probability measures are probability measures, in the sense that if $\mathbb{P}(B) > 0$ we have

a) $\mathbb{P}(\Omega \mid B) = 1$, and

b) $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n \middle| B\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n \mid B)$, whenever $A_k \cap A_l = \emptyset$, $k \neq l$.

However we have in general

$$\mathbb{P}\left(A \middle| \bigcup_{n=1}^{\infty} B_n\right) \neq \sum_{n=1}^{\infty} \mathbb{P}(A \mid B_n),$$

even when $B_k \cap B_l = \emptyset$, $k \neq l$. Taking for example $A = \Omega = B_1 \cup B_2$ with $B_1 \cap B_2 = \emptyset$ and $\mathbb{P}(B_1) = \mathbb{P}(B_2) = 1/2$, we have

$$1 = \mathbb{P}(\Omega \mid B_1 \cup B_2) \neq \mathbb{P}(\Omega \mid B_1) + \mathbb{P}(\Omega \mid B_2) = 2.$$

16.5 Random Variables

A random variable is a mapping

$$\begin{aligned} X : \Omega &\to \mathbb{R} \\ \omega &\mapsto X(\omega) \end{aligned}$$

where Ω is a probability space.

Given $X : \Omega \to \mathbb{R}$ a random variable and A a measurable subset of $\mathbb{R}$, we denote by $\{X \in A\}$ the event

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\{X \in A\} = \{\omega \in \Omega \ : \ X(\omega) \in A\}.$$

Examples:

- Let $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$, and consider the mapping

$$\begin{aligned} X : \Omega &\longrightarrow \mathbb{R} \\ (k,l) &\mapsto k+l. \end{aligned}$$

Then X is a random variable giving the sum of the numbers appearing on each die.

- the time needed everyday to travel from home to work or school is a random variable, as the precise value of this time may change from day to day under unexpected circumstances.

- the price of a risky asset is a random variable.

In the sequel we will often use the notion of *indicator function* $\mathbf{1}_A$ of a an event A. The indicator function $\mathbf{1}_A$ is the random variable The indicator function $\mathbf{1}_A$ is the random variable

$$\begin{aligned} \mathbf{1}_A : \Omega &\longrightarrow \{0,1\} \\ \omega &\mapsto \ \mathbf{1}_A(\omega) \end{aligned}$$

defined by

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

For example if $\Omega = \mathbb{N}$ and $A = \{k\}$, for all $l \in \mathbb{N}$ we have

$$\mathbf{1}_{\{k\}}(l) = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

If X is a random variable we also let

$$\mathbf{1}_{\{X=n\}} = \begin{cases} 1 & \text{if } X = n, \\ 0 & \text{if } X \neq n, \end{cases}$$

and

$$\mathbf{1}_{\{X<n\}} = \begin{cases} 1 & \text{if } X < n, \\ 0 & \text{if } X \geq n. \end{cases}$$

16.6 Probability Distributions

In practice we will be interested in the *probability distribution* of X, which is the collection

$$\{\mathbb{P}(X \in A) \ : \ A \text{ measurable subset of } \mathbb{R}\}.$$

In fact the distributions of X can be reduced to the knowledge of either

$$\{\mathbb{P}(a < X \leq b) \ : \ a, b \in \mathbb{R}\},$$

or

$$\{\mathbb{P}(X \leq a) \ : \ a \in \mathbb{R}\},$$

or

$$\{\mathbb{P}(X \geq a) \ : \ a \in \mathbb{R}\}.$$

Two random variables X and Y are said to be independent under the probability $\mathbb{P}$ if their probability distributions satisfy

$$\mathbb{P}(X \in A \ , \ Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

for all measurable[1] subsets A and B of $\mathbb{R}$.

Distributions Admitting a Density

The distribution of X is given by

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$$

where the function $f : \mathbb{R} \to \mathbb{R}_+$ is called the *density* of the distribution of X. In this case we also say that the distribution of X is continuous, or that X is a continuous random variable. This, however, does *not* imply that the density function $f : \mathbb{R} \to \mathbb{R}_+$ is continuous.

In particular we always have

$$\int_{-\infty}^{\infty} f(x)dx = \mathbb{P}(-\infty \leq X \leq \infty) = 1$$

for all probability density functions $f : \mathbb{R} \to \mathbb{R}_+$.

The density f_X can be recovered from the distribution functions

[1] This concept will not be defined in this course.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$x \mapsto \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(s)ds,$$

and

$$x \mapsto \mathbb{P}(X \geq x) = \int_{x}^{\infty} f_X(s)ds,$$

as

$$f_X(x) = \frac{\partial}{\partial x}\int_{-\infty}^{x} f_X(s)ds = -\frac{\partial}{\partial x}\int_{x}^{\infty} f_{(X,Y)}(s)ds,$$

$x \in \mathbb{R}$.

Examples:

- The Gaussian distribution.

  The density of the standard normal distribution is given by

  $$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

  More generally, X has a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ (in this case we write $X \simeq \mathcal{N}(\mu, \sigma^2)$) if

  $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/(2\sigma^2)}.$$

- The exponential distribution with parameter $\lambda > 0$.

  In this case,

  $$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

  We also have

  $$\mathbb{P}(X > t) = e^{-\lambda t}, \qquad t \in \mathbb{R}_+.$$

  In addition, if $X_1, \ldots, X_n$ are independent exponentially distributed random variables with parameters $\lambda_1, \ldots, \lambda_n$ we have

  $$\begin{aligned} \mathbb{P}(\min(X_1, \ldots, X_n) > t) &= \mathbb{P}(X_1 > t, \ldots, X_n > t) \\ &= \mathbb{P}(X_1 > t) \cdots \mathbb{P}(X_n > t) \\ &= e^{-t(\lambda_1 + \cdots + \lambda_n)}, \qquad t \in \mathbb{R}_+, \end{aligned} \tag{16.2}$$

  hence $\min(X_1, \ldots, X_n)$ is an exponentially distributed random variable with parameter $\lambda_1 + \cdots + \lambda_n$.
- The gamma distribution.

  In this case,

$$f(x) = \begin{cases} \dfrac{a^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-ax}, & x \geq 0 \\ \\ 0, & x < 0, \end{cases}$$

where $a > 0$ and $\lambda > 0$ are parameters and

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} e^{-x} dx, \qquad \lambda > 0,$$

is the Gamma function.

- The Cauchy distribution.

  In this case we have

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}, \qquad x \in \mathbb{R}.$$

- The lognormal distribution.

  In this case,

$$f(x) = \begin{cases} \dfrac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\mu - \log x)^2}{2\sigma^2}}, & x > 0 \\ \\ 0, & x \leq 0. \end{cases}$$

Exercise: For each of the above probability density functions, check that the condition

$$\int_{-\infty}^\infty f(x) dx = 1$$

is satisfied.

Remark 16.1. Note that if the distribution of X admits a density then for all $a \in \mathbb{R}$, we have

$$\mathbb{P}(X = a) = \int_a^a f(x) dx = 0, \tag{16.3}$$

and this is not a contradiction.

In particular this shows that

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(X = a) + \mathbb{P}(a < X \leq b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a < X < b),$$

for $a \leq b$.

In practice, Property (16.3) appears for example in the framework of lottery games with a large number of participants, in which a given number "a"

chosen in advance has a very low (almost zero) probability to be chosen.

Given two continuous random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ we can form the $\mathbb{R}^2$-valued random variable (X, Y) defined by

$$\begin{aligned}(X, Y) : \Omega &\longrightarrow \mathbb{R}^2 \\ \omega &\mapsto (X(\omega), Y(\omega)).\end{aligned}$$

We say that (X, Y) admits a joint probability density

$$f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}_+$$

when

$$\mathbb{P}((X, Y) \in A \times B = \int_A \int_B f_{(X,Y)}(x, y)dxdy$$

for all measurable subsets A, B of $\mathbb{R}$. The density $f_{(X,Y)}$ can be recovered from the distribution functions

$$(x, y) \mapsto \mathbb{P}(X \leq x,\ Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{(X,Y)}(s, t)dsdt,$$

and

$$(x, y) \mapsto \mathbb{P}(X \geq x,\ Y \geq y) = \int_{x}^{\infty} \int_{y}^{\infty} f_{(X,Y)}(s, t)dsdt,$$

as

$$f_{(X,Y)}(x, y) = \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^{x} \int_{-\infty}^{y} f_{(X,Y)}(s, t)dsdt = \frac{\partial^2}{\partial x \partial y} \int_{x}^{\infty} \int_{y}^{\infty} f_{(X,Y)}(s, t)dsdt,$$

$x, y \in \mathbb{R}$.

The probability densities $f_X : \mathbb{R} \to \mathbb{R}_+$ and $f_Y : \mathbb{R} \to \mathbb{R}_+$ of $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are called the *marginal* densities of (X, Y) and are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x, y)dy, \qquad x \in \mathbb{R},$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{(X,Y)}(x, y)dx, \qquad y \in \mathbb{R}.$$

The conditional density $f_{X|Y=y} : \mathbb{R} \to \mathbb{R}_+$ X given $Y = y$ is given by

$$f_{X|Y=y}(x) = \frac{f_{(X,Y)}(x, y)}{f_Y(y)}, \qquad x \in \mathbb{R},$$

provided $f_Y(y) > 0$.

Discrete Distributions

We only consider integer-valued random variables, i.e. the distribution of X is given by the values of $\mathbb{P}(X = k)$, $k \in \mathbb{N}$.

Examples:

- The Bernoulli distribution.

  We have
  $$\mathbb{P}(X = 1) = p \qquad \text{and} \qquad \mathbb{P}(X = 0) = 1 - p, \tag{16.4}$$
  where $p \in [0, 1]$ is a parameter.

- The binomial distribution.

  We have
  $$\mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \qquad k = 0, 1, \ldots, n,$$
  where $n \geq 1$ and $p \in [0, 1]$ are parameters.

- The geometric distribution.

  We have
  $$\mathbb{P}(X = k) = (1 - p)p^k, \qquad k \in \mathbb{N}, \tag{16.5}$$
  where $p \in (0, 1)$ is a parameter.

  Note that if $(X_k)_{k\in\mathbb{N}}$ is a sequence of independent Bernoulli random variables with distribution (16.4), then the random variable
  $$X := \inf\{k \in \mathbb{N} \ : \ X_k = 1\}$$
  has the geometric distribution (16.5).

- The negative binomial distribution (or Pascal distribution).

  We have
  $$\mathbb{P}(X = k) = \binom{k + r - 1}{r - 1} (1 - p)^r p^k, \qquad k \in \mathbb{N},$$
  where $p \in (0, 1)$ and $k \geq 1$ are parameters. Note that the negative binomial distribution recovers the geometric distribution when $r = 1$.

- The Poisson distribution.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

We have

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k \in \mathbb{N},$$

where $\lambda > 0$ is a parameter.

Remark 16.2. The distribution of a discrete random variable can not admit a density. If this were the case, by Remark 16.1 we would have $\mathbb{P}(X = k) = 0$ for all $k \in \mathbb{N}$ and

$$1 = \mathbb{P}(X \in \mathbb{R}) = \mathbb{P}(X \in \mathbb{N}) = \sum_{k=0}^{\infty} \mathbb{P}(X = k) = 0,$$

which is a contradiction.

Given two discrete random variables X and Y, the conditional distribution of X given $Y = k$ is given by

$$\mathbb{P}(X = n \mid Y = k) = \frac{\mathbb{P}(X = n \text{ and } Y = k)}{\mathbb{P}(Y = k)}, \qquad n \in \mathbb{N}.$$

16.7 Expectation of a Random Variable

The *expectation* of a random variable X is the mean, or average value, of X. In practice, expectations can be even more useful than probabilities. For example, knowing that a given equipment (such as a bridge) has a failure probability of 1.78493 out of a billion can be of less practical use than knowing the expected lifetime (e.g. 200000 years) of that equipment.

For example, the time $T(\omega)$ to travel from home to work/school can be a random variable with a new outcome and value every day, however we usually refer to its expectation $\mathbb{E}[T]$ rather than to its sample values that may change from day to day.

The notion of expectation takes its full meaning under conditioning. For example, the expected return of a random asset usually depends on information such as economic data, location, etc. In this case, replacing the expectation by a conditional expectation will provide a better estimate of the expected value.

For example, life expectancy is a natural example of a conditional expectation since it typically depends on location, gender, and other parameters.

Discrete Distributions

In general, the expectation of the indicator function $\mathbf{1}_A$ is defined as

$$\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A),$$

for any event A.

Next, let $X : \Omega \to \mathbb{N}$ be a discrete random variable. We have

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\mathbb{P}(X = k),$$

and more generally,

$$\mathbb{E}[\phi(X)] = \sum_{k=0}^{\infty} \phi(k)\mathbb{P}(X = k),$$

for all sufficiently summable functions $\phi : \mathbb{N} \to \mathbb{R}$.

Note that the expectation of the indicator function $\mathbf{1}_A : \Omega \to \{0, 1\}$ is simply given by

$$\mathbb{E}[\mathbf{1}_A] = 0 \times \mathbb{P}(\Omega \setminus A) + 1 \times \mathbb{P}(A) = \mathbb{P}(A).$$

Note that the expectation is *linear*, i.e. we have

$$\mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y], \qquad a, b \in \mathbb{R}, \tag{16.6}$$

provided

$$\mathbb{E}[|X|] + \mathbb{E}[|Y|] < \infty.$$

The conditional expectation of $X : \Omega \to \mathbb{N}$ given an event A is defined by

$$\mathbb{E}[X \mid A] = \sum_{k=0}^{\infty} k\mathbb{P}(X = k \mid A),$$

with

$$\begin{aligned}
\mathbb{E}[X \mid A] &= \frac{1}{\mathbb{P}(A)} \sum_{k=0}^{\infty} k\mathbb{P}(\{X = k\} \cap A) \\
&= \frac{1}{\mathbb{P}(A)} \sum_{k=0}^{\infty} k\,\mathbb{E}\left[\mathbf{1}_{\{X=k\}\cap A}\right] \\
&= \frac{1}{\mathbb{P}(A)}\,\mathbb{E}\left[\mathbf{1}_A \sum_{k=0}^{\infty} k\mathbf{1}_{\{X=k\}}\right]
\end{aligned}$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$= \frac{1}{\mathbb{P}(A)} \mathbb{E}\left[X \mathbf{1}_A\right]. \tag{16.7}$$

In particular we have

$$\mathbb{E}[\mathbf{1}_A] = 0 \times \mathbb{P}(\Omega \setminus A) + 1 \times \mathbb{P}(A) = \mathbb{P}(A),$$

and

$$\begin{aligned}\mathbb{E}[\mathbf{1}_B \mid A] &= 0 \times \mathbb{P}(X = 0 \mid A) + 1 \times \mathbb{P}(X = 1 \mid A) \\ &= \mathbb{P}(X = 1 \mid A) \\ &= \mathbb{P}(B \mid A).\end{aligned}$$

One can also define the conditional expectation of X given that $Y = k$, as

$$\mathbb{E}[X \mid Y = k] = \sum_{n=0}^{\infty} n\mathbb{P}(X = n \mid Y = k).$$

In general we have

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]], \tag{16.8}$$

since

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X \mid Y]] &= \sum_{k=0}^{\infty} \mathbb{E}[X \mid Y = k]\mathbb{P}(Y = k) \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} n\mathbb{P}(X = n \mid Y = k)\mathbb{P}(Y = k) \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} n\mathbb{P}(X = n \text{ and } \mid Y = k) \\ &= \sum_{n=0}^{\infty} n\mathbb{P}(X = n) \\ &= \mathbb{E}[X].\end{aligned}$$

Relation 16.8 is sometimes referred to as the *tower property*.

Example:

The life expectancy in Singapore is $\mathbb{E}[T] = 80$ years overall. Let $G \in \{M, W\}$ denote the gender of an individual chosen at random. The statistics show that

$$\mathbb{E}[T \mid G = W] = 78 \quad \text{and} \quad \mathbb{E}[T \mid G = M] = 81.9,$$

and we have

$$\begin{aligned}
80 &= \mathbb{E}[T] \\
&= \mathbb{E}[\mathbb{E}[T|G]] \\
&= \mathbb{P}(G = W)\,\mathbb{E}[T \mid G = W] + \mathbb{P}(G = M)\,\mathbb{E}[T \mid G = M] \\
&= 81.9 \times \mathbb{P}(G = W) + 78 \times \mathbb{P}(G = M) \\
&= 81.9 \times (1 - \mathbb{P}(G = M)) + 78 \times \mathbb{P}(G = M),
\end{aligned}$$

showing that

$$80 = 81.9 \times (1 - \mathbb{P}(G = M)) + 78 \times \mathbb{P}(G = M),$$

i.e.

$$\mathbb{P}(G = M) = \frac{81.9 - 80}{81.9 - 78} = \frac{1.9}{3.9} = 0.487.$$

Distributions Admitting a Density

Given a random variable X whose distribution admits a density $f : \mathbb{R} \to \mathbb{R}_+$ we have

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx,$$

and more generally,

$$\mathbb{E}[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) f(x) dx,$$

for all sufficiently integrable function ϕ on $\mathbb{R}$.

For example, if X has a standard normal distribution we have

$$\mathbb{E}[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}.$$

In case X has a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ we get

$$\mathbb{E}[\phi(X)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \phi(x) e^{-(x-\mu)^2/(2\sigma^2)} dx.$$

The expectation of a continuous random variable satisfies the same linearity property (16.6) as in the discrete case.

Exercise: In case X has a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$, check that

$$\mu = \mathbb{E}[X] \qquad \text{and} \qquad \sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

The conditional expectation of a continuous random variable can be defined as

$$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y=y}(x)dx$$

with the relation

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$$

as in the discrete case, since

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X \mid Y]] &= \int_{-\infty}^{\infty} \mathbb{E}[X \mid Y = y] f_Y(y)dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X|Y=y}(x) f_Y(y)dxdy \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)dydx \\
&= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \mathbb{E}[X].
\end{aligned}$$

16.8 Characteristic Functions and Laplace Transforms

The characteristic function of a random variable X is the function $\Psi_X : \mathbb{R} \to \mathbb{C}$ defined by

$$\Psi_X(t) = \mathbb{E}[e^{itX}], \qquad t \in \mathbb{R}.$$

The Laplace transform of a random variable X is the function $\Phi_X : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi_X(t) = \mathbb{E}[e^{tX}], \qquad t \in \mathbb{R},$$

provided the expectation is finite.

In particular we have

$$E[X^n] = \frac{\partial^n}{\partial t} \Phi_X(0), \qquad n \geq 1.$$

The Laplace transform Φ_X of a random variable X with density $f : \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx, \qquad t \in \mathbb{R}.$$

Note that in probability we are using the *bilateral* Laplace transform.

The characteristic function Ψ_X of a random variable X with density $f : \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\Psi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx, \qquad t \in \mathbb{R}.$$

On the other hand, if $X : \Omega \to \mathbb{N}$ is a discrete random variable we have

$$\Psi_X(t) = \sum_{n=0}^{\infty} e^{itn} \mathbb{P}(X = n), \qquad t \in \mathbb{R}.$$

The main applications of characteristic functions lie in the following theorems:

Theorem 16.1. Two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ have same distribution if and only if

$$\Psi_X(t) = \Psi_Y(t), \qquad t \in \mathbb{R}.$$

Theorem 16.1 is used to identify or to determine the probability distribution of a random variable X, by comparison with the characteristic function Ψ_Y of a random variable Y whose distribution is known.

The characteristic function of a random vector (X, Y) is the function $\Psi_{X,Y} : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\Psi_{X,Y}(s,t) = \mathbb{E}[e^{isX+itY}], \qquad s, t \in \mathbb{R}.$$

Theorem 16.2. Given two *independent* random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are independent if and only if

$$\Psi_{X,Y}(s,t) = \Psi_X(s)\Psi_Y(t), \qquad s, t \in \mathbb{R}.$$

A random variable X is Gaussian with mean μ and variance σ^2 if and only if its characteristic function satisfies

$$\mathbb{E}[e^{i\alpha X}] = e^{i\alpha\mu - \alpha^2\sigma^2/2}, \qquad \alpha \in \mathbb{R}. \tag{16.9}$$

In terms of Laplace transforms we have, replacing $i\alpha$ by α,

$$\mathbb{E}[e^{\alpha X}] = e^{\alpha\mu + \alpha^2\sigma^2/2}, \qquad \alpha \in \mathbb{R}. \tag{16.10}$$

From Theorems 16.1 and 16.2 we deduce the following proposition.

Proposition 16.1. *Let $X \simeq \mathcal{N}(\mu, \sigma_X^2)$ and $Y \simeq \mathcal{N}(\nu, \sigma_Y^2)$ be independent Gaussian random variables. Then $X + Y$ also has a Gaussian distribution*

$$X + Y \simeq \mathcal{N}(\mu + \nu, \sigma_X^2 + \sigma_Y^2).$$

Proof. Since X and Y are independent, by Theorem 16.2 the characteristic function Ψ_{X+Y} of $X + Y$ is given by

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\begin{aligned}\Phi_{X+Y}(t) &= \Phi_X(t)\Phi_Y(t)\\ &= e^{it\mu - t^2\sigma_X^2/2} e^{it\nu - t^2\sigma_Y^2/2}\\ &= e^{it(\mu+\nu) - t^2(\sigma_X^2+\sigma_Y^2)/2},\end{aligned}$$

where we used (16.9). Consequently, the characteristic function of $X+Y$ is that of a Gaussian random variable with mean $\mu+\nu$ and variance $\sigma_X^2+\sigma_Y^2$ and we conclude by Theorem 16.1. □

The generating function of a *discrete* random variable X is the function

$$G_X : [-1,1] \to \mathbb{R}$$

defined by

$$G_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k \mathbb{P}(X=k), \qquad s \in [-1,1]. \tag{16.11}$$

Note that since $0^0 = 1$ we have

$$G_X(0) = \mathbb{E}[0^X] = 0^0\mathbb{P}(X=0) = \mathbb{P}(X=0).$$

On the other hand we have[2]

$$G'_X(1^-) = \mathbb{E}[X] = \sum_{k=0}^{\infty} k\mathbb{P}(X=k),$$

provided $\mathbb{E}[|X|] < \infty$, and

$$\mathrm{Var}[X] = G''(1^-) + G'(1^-) - (G'(1^-))^2,$$

provided $\mathbb{E}[|X|^2] < \infty$.

Given two independent variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ we have

$$G_{X+Y}(t) = G_X(t)G_Y(t), \qquad t \in [-1,1].$$

Example: Consider a random variable X with generating function

$$G(s) = e^{s(1-p)}, \qquad s \in [-1,1],$$

for some $p \in [0,1]$. What is the distribution of X ?

We have

[2] In case $G_X(s)$ is not defined at $s=1$, then $G'_X(1^-)$ denotes the left limit of the derivative G'_X at the point 1.

$$G(s) = e^{s(1-p)} = e^{-p} \sum_{n=0}^{\infty} s^n \frac{p^n}{n!}, \qquad s \in [-1, 1],$$

hence by identification with (16.11) we find

$$\mathbb{P}(X = n) = e^{-p} \frac{p^n}{n!}, \qquad n \in \mathbb{N},$$

i.e. X has the Poisson distribution with parameter p.

Exercises

Exercise 16.1. Let X denote a centered Gaussian random variable with variance η^2, $\eta > 0$. Show that the probability $P(e^X > c)$ is given by

$$P(e^X > c) = \Phi(-(\log c)/\eta),$$

where $\log = \ln$ denotes the natural logarithm and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

denotes the Gaussian cumulative distribution function.

Exercise 16.2. Let X be a centered Gaussian random variable with variance $\alpha^2 > 0$ and density $x \mapsto \frac{1}{\sqrt{2\pi\alpha^2}} e^{-x^2/(2\alpha^2)}$ and let $\beta \in \mathbb{R}$.

1. Write down $\mathbb{E}[(\beta - X)^+]$ as an integral. Hint: $(\beta - x)^+$ is zero when $x > \beta$.
2. Compute this integral to show that

$$\mathbb{E}[(\beta - X)^+] = \frac{\alpha}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2\alpha^2}} + \beta \Phi(\beta/\alpha),$$

where

$$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}, \qquad x \in \mathbb{R}.$$

Exercise 16.3. Let X be a centered Gaussian random variable with variance $\alpha^2 > 0$ and density $x \mapsto \frac{1}{\sqrt{2\pi\alpha^2}} e^{-x^2/(2\alpha^2)}$ and let $\beta \in \mathbb{R}$.

1. Write down $\mathbb{E}[(\beta + X)^+]$ as an integral. Hint: $(\beta + x)^+$ is zero when $x < -\beta$.
2. Compute this integral to show that

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\mathbb{E}[(\beta + X)^+] = \frac{\alpha}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2\alpha^2}} + \beta \Phi(\beta/\alpha),$$

where

$$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}, \qquad x \in \mathbb{R}.$$

Exercise 16.4. Let X be a centered Gaussian random variable with variance v^2.

1. Compute

$$\mathbb{E}\left[e^{\sigma X} \mathbf{1}_{[K,\infty[}(xe^{\sigma X})\right] = \frac{1}{\sqrt{2\pi v^2}} \int_{\frac{1}{\sigma}\log\frac{K}{x}}^{\infty} e^{\sigma y - y^2/(2v^2)} dy.$$

   Hint: use the decomposition

$$\sigma y - \frac{y^2}{v^2} = \frac{v^2\sigma^2}{4} - \left(\frac{y}{v} - \frac{v\sigma}{2}\right)^2.$$

2. Compute

$$\mathbb{E}[(e^{m+X} - K)^+] = \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+x} - K)^+ e^{-\frac{x^2}{2v^2}} dx.$$

3. Compute the expectation (16.10) above.

Chapter 17
Exercise Solutions

Chapter 1

Exercise 1.1.

1. The possible values of R are a and b.
2. We have

$$\begin{aligned}\mathbb{E}^*[R] &= a\mathbb{P}^*(R=a) + b\mathbb{P}^*(R=b)\\ &= a\frac{b-r}{b-a} + b\frac{r-a}{b-a}\\ &= r.\end{aligned}$$

3. By Theorem 1.1, there do not exist arbitrage opportunities in this market since there exists a risk-neutral measure $\mathbb{P}^*$ from Question 2.
4. The risk-neutral measure is unique hence the market model is complete by Theorem 1.2.
5. Taking

$$\eta = \frac{\alpha(1+b) - \beta(1+a)}{\pi_1(b-a)} \quad \text{and} \quad \xi = \frac{\alpha-\beta}{S_0(a-b)},$$

we check that

$$\begin{cases}\eta\pi_1 + \xi S_0(1+a) = \alpha\\ \\ \eta\pi_1 + \xi S_0(1+b) = \beta,\end{cases}$$

which shows that

$$\eta\pi_1 + \xi S_1 = C.$$

6. We have

$$\begin{aligned}\pi(C) &= \eta\pi_0 + \xi S_0\\ &= \frac{\alpha(1+b) - \beta(1+a)}{(1+r)(b-a)} + \frac{\alpha-\beta}{a-b}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha(1+b) - \beta(1+a) - (1+r)(\alpha-\beta)}{(1+r)(b-a)} \\
&= \frac{\alpha b - \beta a - r(\alpha-\beta)}{(1+r)(b-a)}. \qquad (17.1)
\end{aligned}
$$

7. We have

$$
\begin{aligned}
\mathbb{E}^*[C] &= \alpha \mathbb{P}^*(R=a) + \beta \mathbb{P}^*(R=b) \\
&= \alpha \frac{b-r}{b-a} + \beta \frac{r-a}{b-a}. \qquad (17.2)
\end{aligned}
$$

8. Comparing (17.1) and (17.2) above we get

$$
\pi(C) = \frac{1}{1+r}\, \mathbb{E}^*[C]
$$

9. The initial value $\pi(C)$ of the portfolio is interpreted as the arbitrage price of the option contract and it equals the expected value of the discounted payoff.
10. We have

$$
C = (K - S^1)^+ = (11 - S^1)^+ = \begin{cases} C(\omega^-) = 11 - S^1 \text{ if } K > S^1, \\ C(\omega^+) = 0 \text{ if } K \leq S^1. \end{cases}
$$

11. We have

$$
\xi^1 = \frac{C(\omega^+) - C(\omega^-)}{b-a} = -\frac{2}{3}, \qquad \xi^0 = \frac{bC(\omega^-) - aC(\omega^+)}{(1+r)(b-a)} = \frac{8}{1.05}.
$$

12. The arbitrage price $\pi(C)$ of the contingent claim C is

$$
\pi(C) = \xi^0 \pi^0 + \xi^1 \pi^1 = 0.952.
$$

Chapter 2

Exercise 2.1.

1. The possible values of R_t are a and b.
2. We have

$$
\begin{aligned}
\mathbb{E}^*[R_{t+1} \mid \mathcal{F}_t] &= a\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + b\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) \\
&= a\frac{b-r}{b-a} + b\frac{r-a}{b-a} = r.
\end{aligned}
$$

3. We have

$$\begin{aligned}
\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_t] &= \sum_{i=0}^{k} \left(\frac{r-a}{b-a}\right)^i \left(\frac{b-r}{b-a}\right)^{k-i} \binom{k}{i} (1+b)^i (1+a)^{k-i} S_t \\
&= S_t \sum_{i=0}^{k} \binom{k}{i} \left(\frac{r-a}{b-a}(1+b)\right)^i \left(\frac{b-r}{b-a}(1+a)\right)^{k-i} \\
&= S_t \left(\frac{r-a}{b-a}(1+b) + \frac{b-r}{b-a}(1+a)\right)^k \\
&= (1+r)^k S_t.
\end{aligned}$$

Assuming that the formula holds for $k = 1$, its extension to $k \geq 2$ can also be proved recursively from the "tower property" (16.8) of conditional expectations, as follows:

$$\begin{aligned}
\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_t] &= \mathbb{E}^*[\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_{t+k-1}] \mid \mathcal{F}_t] \\
&= (1+r)\, \mathbb{E}^*[S_{t+k-1} \mid \mathcal{F}_t] \\
&= (1+r)\, \mathbb{E}^*[\mathbb{E}^*[S_{t+k-1} \mid \mathcal{F}_{t+k-2}] \mid \mathcal{F}_t] \\
&= (1+r)^2\, \mathbb{E}^*[S_{t+k-2} \mid \mathcal{F}_t] \\
&= (1+r)^2\, \mathbb{E}^*[\mathbb{E}^*[S_{t+k-2} \mid \mathcal{F}_{t+k-3}] \mid \mathcal{F}_t] \\
&= (1+r)^3\, \mathbb{E}^*[S_{t+k-3} \mid \mathcal{F}_t] \\
&= \cdots \\
&= (1+r)^{k-2}\, \mathbb{E}^*[S_{t+2} \mid \mathcal{F}_t] \\
&= (1+r)^{k-2}\, \mathbb{E}^*[\mathbb{E}^*[S_{t+2} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \\
&= (1+r)^{k-1}\, \mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t] \\
&= (1+r)^k S_t.
\end{aligned}$$

Chapter 3

Exercise 3.1.

1. The condition $V_N = C$ reads

$$\begin{cases} \eta_N \pi_N + \xi_N (1+a) S_{N-1} = (1+a) S_{N-1} - K \\ \\ \eta_N \pi_N + \xi_N (1+b) S_{N-1} = (1+b) S_{N-1} - K \end{cases}$$

from which we deduce $\xi_N = 1$ and $\eta_N = -K(1+r)^{-N}/\pi_0$.

2. We have

$$\begin{cases} \eta_{N-1}\pi_{N-1} + \xi_{N-1}(1+a)S_{N-1} = \eta_N\pi_{N-1} + \xi_N(1+a)S_{N-1} \\ \\ \eta_{N-1}\pi_{N-1} + \xi_{N-1}(1+b)S_{N-1} = \eta_N\pi_{N-1} + \xi_N(1+b)S_{N-1}, \end{cases}$$

which yields $\xi_{N-1} = \xi_N = 1$ and $\eta_{N-1} = \eta_N = -K(1+r)^{-N}/\pi_0$. Similarly, solving the self-financing condition

$$\begin{cases} \eta_t\pi_t + \xi_t(1+a)S_t = \eta_{t+1}\pi_t + \xi_{t+1}(1+a)S_t \\ \\ \eta_t\pi_t + \xi_t(1+b)S_t = \eta_{t+1}\pi_t + \xi_{t+1}(1+b)S_t, \end{cases}$$

at time t yields $\xi_t = 1$ and $\eta_t = -K(1+r)^{-N}/\pi_0$, $t = 1, 2, \ldots, N$.

3. We have

$$\pi_t(C) = V_t = \eta_t\pi_t + \xi_t S_t = S_t - K(1+r)^{-N}\pi_t/\pi_0 = S_t - K(1+r)^{-(N-t)}.$$

4. For all $t = 0, 1, \ldots, N$ we have

$$\begin{aligned} (1+r)^{-(N-t)}\,\mathbb{E}^*[C \mid \mathcal{F}_t] &= (1+r)^{-(N-t)}\,\mathbb{E}^*[S_N - K \mid \mathcal{F}_t], \\ &= (1+r)^{-(N-t)}\,\mathbb{E}^*[S_N \mid \mathcal{F}_t] - (1+r)^{-(N-t)}\,\mathbb{E}^*[K \mid \mathcal{F}_t] \\ &= (1+r)^{-(N-t)}(1+r)^{N-t}S_t - K(1+r)^{-(N-t)} \\ &= S_t - K(1+r)^{-(N-t)} \\ &= V_t = \pi_t(C). \end{aligned}$$

Exercise 3.2.

1. This model admits a unique risk-neutral measure $\mathbb{P}^*$ because we have $a < r < b$. We have

$$\mathbb{P}^*(R_t = a) = \frac{b-r}{b-a} = \frac{0.07 - 0.05}{0.07 - (-0.02)},$$

and

$$\mathbb{P}(R_t = b) = \frac{r-a}{b-a} = \frac{0.05 - (-0.02)}{0.07 - (-0.02)},$$

$t = 1, \ldots, N$.

2. There are no arbitrage opportunities in this model, due to the existence of a risk-neutral measure.
3. This market model is complete because the risk-neutral measure is unique.
4. We have

$$C = (S_N)^2,$$

hence

$$H = (S_N)^2/(1+r)^N = h(X_N),$$

with

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http://www.ntu.edu.sg/home/nprivault/indext.html

$$h(x) = x^2(1+r)^N.$$

Now we have

$$V_t = v_t(X_t),$$

where the function $v_t(x)$ is given by

$$\begin{aligned}
v_t(x) &= \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \left(\frac{r-a}{b-a}\right)^k \left(\frac{b-r}{b-a}\right)^{N-t-k} h\left(x\left(\frac{1+b}{1+r}\right)^k \left(\frac{1+a}{1+r}\right)^{N-t-k}\right) \\
&= \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \left(\frac{r-a}{b-a}\right)^k \left(\frac{b-r}{b-a}\right)^{N-t-k} x^2(1+r)^N \left(\frac{1+b}{1+r}\right)^{2k} \left(\frac{1+a}{1+r}\right)^{2(N-t-k)} \\
&= x^2(1+r)^N \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \left(\frac{(r-a)(1+b)^2}{(b-a)(1+r)^2}\right)^k \left(\frac{(b-r)(1+a)^2}{(b-a)(1+r)^2}\right)^{N-t-k} \\
&= x^2(1+r)^N \left(\frac{(r-a)(1+b)^2}{(b-a)(1+r)^2} + \frac{(b-r)(1+a)^2}{(b-a)(1+r)^2}\right)^{N-t} \\
&= \frac{x^2\left((r-a)(1+b)^2 + (b-r)(1+a)^2\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= \frac{x^2\left((r-a)(1+2b+b^2) + (b-r)(1+2a+a^2)\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= \frac{x^2\left(r(1+2b+b^2) - a(1+2b+b^2) + b(1+2a+a^2) - r(1+2a+a^2)\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= x^2 \frac{(1+r(2+a+b)-ab)^{N-t}}{(1+r)^{N-2t}}.
\end{aligned}$$

5. We have

$$\begin{aligned}
\xi_t^1 &= \frac{v_t\left(\frac{1+b}{1+r}X_{t-1}\right) - v_t\left(\frac{1+a}{1+r}X_{t-1}\right)}{X_{t-1}(b-a)/(1+r)} \\
&= X_{t-1} \frac{\left(\frac{1+b}{1+r}\right)^2 - \left(\frac{1+a}{1+r}\right)^2}{(b-a)/(1+r)} \frac{(1+r(2+a+b)-ab)^{N-t}}{(1+r)^{N-2t}} \\
&= S_{t-1}(2+b+a)\frac{(1+r(2+a+b)-ab)^{N-t}}{(1+r)^{N-t}}, \qquad t = 1, \ldots, N,
\end{aligned}$$

representing the number of units of the risky asset to be present in the portfolio at time t. On the other hand we have

$$\begin{aligned}
\xi_t^0 &= \frac{V_t - \xi_t^1 X_t}{X_t^0} \\
&= \frac{V_t - \xi_t^1 X_t}{\pi_0}
\end{aligned}$$

$$
\begin{aligned}
&= X_t(1+r(2+a+b)-ab)^{N-t}\frac{X_t - X_{t-1}(2+b+a)/(1+r)}{\pi_0(1+r)^{N-2t}} \\
&= S_t(1+r(2+a+b)-ab)^{N-t}\frac{S_t - S_{t-1}(2+b+a)}{\pi_0(1+r)^{N}} \\
&= -(S_{t-1})^2(1+r(2+a+b)-ab)^{N-t}\frac{(1+a)(1+b)}{\pi_0(1+r)^{N}}, \qquad t=1,\ldots,N.
\end{aligned}
$$

6. Let us check that the portfolio is self-financing. We have

$$
\begin{aligned}
&\bar{\xi}_{t+1}\cdot\bar{S}_t = \xi^0_{t+1}S^0_t + \xi^1_{t+1}S^1_t \\
&= -(S_t)^2(1+r(2+a+b)-ab)^{N-t-1}\frac{(1+a)(1+b)}{\pi_0(1+r)^{N}}S^0_t \\
&\quad +(S_t)^2(2+b+a)\frac{(1+r(2+a+b)-ab)^{N-t-1}}{(1+r)^{N-t-1}} \\
&= (S_t)^2(1+r(2+a+b)-ab)^{N-t-1}\frac{1}{(1+r)^{N-t}}\left((2+b+a)(1+r)-(1+a)(1+b)\right) \\
&= (X_t)^2(1+r(2+a+b)-ab)^{N-t}\frac{1}{(1+r)^{N-3t}} \\
&= (1+r)^t V_t \\
&= \bar{\xi}_t\cdot\bar{S}_t, \qquad t=1,\ldots,N.
\end{aligned}
$$

Exercise 3.3.

1. We have

$$
\begin{aligned}
V_t &= \xi_t S_t + \eta_t\pi_t \\
&= \xi_t(1+R_t)S_{t-1} + \eta_t(1+r)\pi_{t-1}.
\end{aligned}
$$

2. We have

$$
\begin{aligned}
\mathbb{E}^*[R_t|\mathcal{F}_{t-1}] &= a\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + b\mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
&= a\frac{b-r}{b-a} + b\frac{r-a}{b-a} \\
&= b\frac{r}{b-a} - a\frac{r}{b-a} \\
&= r.
\end{aligned}
$$

3. By the result of Question 1 we have

$$
\begin{aligned}
\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] &= \mathbb{E}^*[\xi_t(1+R_t)S_{t-1} \mid \mathcal{F}_{t-1}] + \mathbb{E}^*[\eta_t(1+r)\pi_{t-1} \mid \mathcal{F}_{t-1}] \\
&= \xi_t S_{t-1}\,\mathbb{E}^*[1+R_t \mid \mathcal{F}_{t-1}] + (1+r)\,\mathbb{E}^*[\eta_t\pi_{t-1} \mid \mathcal{F}_{t-1}] \\
&= (1+r)\xi_t S_{t-1} + (1+r)\eta_t\pi_{t-1} \\
&= (1+r)\xi_t S_t + (1+r)\eta_t\pi_t
\end{aligned}
$$

$$= (1+r)V_{t-1},$$

where we used the self-financing condition.

4. We have

$$\begin{aligned}
V_{t-1} &= \frac{1}{1+r}\,\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] \\
&= \frac{3}{1+r}\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + \frac{8}{1+r}\mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
&= \frac{1}{1+0.15}\left(3\frac{0.25-0.15}{0.25-0.05} + 8\frac{0.15-0.05}{0.25-0.05}\right) \\
&= \frac{1}{1.15}\left(\frac{3}{2}+\frac{8}{2}\right) \\
&= 4.78.
\end{aligned}$$

Chapter 4

Exercise 4.1.

1. We need to check whether the four properties of the definition of Brownian motion are satisfied. Checking Conditions (i) to (iii) does not pose any particular problem since the time changes $t \mapsto c+t$, $t \mapsto t/c^2$ and $t \mapsto ct^2$ are deterministic, continuous, and increasing. As for Condition (iv), $B_{c+t} - B_{c+s}$ clearly has a centered Gaussian distribution with variance t, and the same property holds for cB_{t/c^2} since

$$\mathrm{var}\,(c(B_{t/c^2} - B_{s/c^2})) = c^2\mathrm{var}\,(B_{t/c^2} - B_{s/c^2}) = c^2(t-s)/c^2 = t-s.$$

As a consequence, (a) and (b) are standard Brownian motions.

Concerning (c), we note that B_{ct^2} is a centered Gaussian random variable with variance ct^2 - not t, hence $(B_{ct^2})_{t\in\mathbb{R}_+}$ is not a standard Brownian motion.

2. We have $\displaystyle\int_0^T 2dB_t = 2(B_T - B_0) = 2B_T$, which has a Gaussian law with mean 0 and variance $4T$. On the other hand,

$$\int_0^T (2\times\mathbf{1}_{[0,T/2]}(t)+\mathbf{1}_{(T/2,T]}(t))dB_t = 2(B_{T/2}-B_0)+(B_T-B_{T/2}) = B_T+B_{T/2},$$

which has a Gaussian law with mean 0 and variance $4(T/2)+T/2 = 5T/2$.

3. The stochastic integral $\displaystyle\int_0^{2\pi} \sin(t)\, dB_t$ has a Gaussian distribution with mean 0 and variance
$$\int_0^{2\pi} \sin^2(t)dt = \int_0^{2\pi} \frac{1-\cos(2t)}{2} dt = \pi.$$
4. If $0 \leq s \leq t$ we have
$$\mathbb{E}[B_t B_s] = \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}[B_s^2] = \mathbb{E}[(B_t - B_s)]\, \mathbb{E}[B_s] + \mathbb{E}[B_s^2] = 0 + s = s,$$
and similarly we obtain $\mathbb{E}[B_t B_s] = t$ when $0 \leq t \leq s$, hence in general we have $\mathbb{E}[B_t B_s] = \min(s,t)$, $s, t \geq 0$.
5. We have
$$\begin{aligned} d(f(t)B_t) &= f(t)dB_t + B_t df(t) + df(t) \cdot dB_t \\ &= f(t)dB_t + B_t f'(t)dt + f'(t)dt \cdot dB_t \\ &= f(t)dB_t + B_t f'(t)dt, \end{aligned}$$
and by integration on both sides we get
$$\begin{aligned} 0 &= f(T)B_T - f(0)B_0 \\ &= \int_0^T d(f(t)B_t) \\ &= \int_0^T f(t)dB_t + \int_0^T B_t f'(t)dt, \end{aligned}$$
hence the conclusion.

Exercise 4.2.Let $f \in L^2([0,T])$. We have
$$E\left[e^{\int_0^T f(s)dB_s} \middle| \mathcal{F}_t \right] = \exp\left(\int_0^t f(s)dB_s + \frac{1}{2}\int_0^T |f(s)|^2 ds \right), \qquad 0 \leq t \leq T.$$

Exercise 4.3.
We have
$$\begin{aligned} E\left[\exp\left(\beta \int_0^T B_t dB_t \right) \right] &= E\left[\exp\left(\beta(B_T^2 - T)/2 \right) \right] \\ &= e^{-\beta T/2} E\left[\exp\left(e^{\beta (B_T)^2/2} \right) \right] \\ &= \frac{e^{-\beta T/2}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{(\beta - \frac{1}{T})\frac{x^2}{2}} dx \end{aligned}$$

$$= \frac{e^{-\beta T/2}}{\sqrt{1-\beta T}}.$$

for all $\beta < 1/T$.

Exercise 4.4.

1. We have
$$d(X_t^T/(T-t)) = \frac{dX_t^T}{T-t} + \frac{X_t^T}{(T-t)^2}dt = \sigma\frac{dB_t}{T-t},$$
hence by integration using the initial condition $X_0 = 0$ we have
$$\frac{X_t^T}{T-t} = \sigma\int_0^t \frac{1}{T-s}dB_s, \qquad t \in [0,T].$$
2. We have
$$\mathbb{E}[X_t^T] = \sigma(T-t)\,\mathbb{E}\left[\int_0^t \frac{1}{T-s}dB_s\right] = 0.$$
3. Using the Itô isometry we have
$$\begin{aligned}\mathrm{Var}[X_t^T] &= \sigma^2(T-t)^2\,\mathrm{Var}\left[\int_0^t \frac{1}{T-s}dB_s\right]\\ &= \sigma^2(T-t)^2\int_0^t \frac{1}{(T-s)^2}ds\\ &= \sigma^2(T-t)^2\left(\frac{1}{T-t} - \frac{1}{T}\right)\\ &= \sigma^2(1-t/T).\end{aligned}$$
4. We have $\mathrm{Var}[X_T^T] = 0$ hence $X_T^T = \mathbb{E}[X_T^T] = 0$ by Question 2.

Exercise 4.5.Exponential Vasicek model.

1. We have $z_t = e^{-at}z_0 + \sigma\int_0^t e^{-a(t-s)}dB_s$.
2. We have $y_t = e^{-at}y_0 + \frac{\theta}{a}(1-e^{-at}) + \sigma\int_0^t e^{-a(t-s)}dB_s$.
3. We have $dx_t = x_t\left(\theta + \frac{\sigma^2}{2} - a\log x_t\right)dt + \sigma x_t dB_t$.
4. We have $r_t = \exp\left(e^{-at}\log r_0 + \frac{\theta}{a}(1-e^{-at}) + \sigma\int_0^t e^{-a(t-s)}dB_s\right)$, with $\eta = \theta + \sigma^2/2$.
5. We have

$$\mathbb{E}[r_t] = \exp\left(e^{-at}\log r_0 + \frac{\theta}{a}(1-e^{-at}) + \frac{\sigma^2}{4a}(1-e^{-2at})\right).$$

6. We have $\lim_{t\to\infty} \mathbb{E}[r_t] = \exp\left(\frac{\theta}{a} + \frac{\sigma^2}{4a}\right)$.

Exercise 4.6. Cox-Ingersoll-Ross (CIR) model.

1. We have $r_t = r_0 + \int_0^t (\alpha - \beta r_s)ds + \sigma \int_0^t \sqrt{r_s}dB_s$.
2. Using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, we get, taking expectations on both sides of the above integral equation: $u'(t) = \alpha - \beta u(t)$.
3. Apply Itô's formula to
$$r_t^2 = f\left(r_0 + \int_0^t (\alpha - \beta r_s)ds + \sigma \int_0^t \sqrt{r_s}dB_s\right),$$
with $f(x) = x^2$, to obtain $d(r_t)^2 = r_t(\sigma^2 + 2\alpha - 2\beta r_t)dt + 2r_t\sigma\sqrt{r_t}dB_t$.
4. Taking again the expectation on both sides of the above equation we get
$$v_t' = (\sigma^2 + 2\alpha)u(t) - 2\beta v(t).$$

Exercise 4.7.

1. We have
$$\begin{aligned} S_t &= e^{X_t} \\ &= e^{X_0} + \int_0^t u_s e^{X_s}dB_s + \int_0^t v_s e^{X_s}ds + \frac{1}{2}\int_0^t u_s^2 e^{X_s}ds \\ &= e^{X_0} + \nu\int_0^t e^{X_s}dB_s + \nu\int_0^t e^{X_s}ds + \frac{1}{2}\sigma^2\int_0^t e^{X_s}ds \\ &= S_0 + \nu\int_0^t S_s dB_s + \nu\int_0^t S_s ds + \frac{1}{2}\sigma^2\int_0^t S_s ds. \end{aligned}$$
2. Let $r > 0$. The process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation
$$dS_t = rS_t dt + \sigma S_t dB_t$$
when $r = \nu + \sigma^2/2$.
3. We have
$$\eta^2 = \mathrm{Var}[X] = \mathrm{Var}[\sigma(B_T - B_t)] = \sigma^2\,\mathrm{Var}[B_T - B_t] = \sigma^2(T-t),$$
hence $\eta = \sigma\sqrt{T-t}$.

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4. Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0 e^{\sigma B_t + \nu t}$, $t \in \mathbb{R}_+$. Using the decomposition $S_T = S_t e^{\sigma(B_T - B_t)+\nu\tau}$, we have

$$\begin{aligned}
\mathbb{P}(S_T > K \mid S_t = x) &= \mathbb{P}(S_t e^{\sigma(B_T-B_t)+\nu(T-t)} > K \mid S_t = x) \\
&= \mathbb{P}(x e^{\sigma(B_T-B_t)+\nu(T-t)} > K) \\
&= \mathbb{P}(e^{\sigma(B_T-B_t)} > K e^{-\nu(T-t)}/x) \\
&= \Phi\left(\frac{-\log(K e^{-\nu(T-t)}/x)}{\sigma\sqrt{\tau}}\right) \\
&= \Phi\left(\frac{\log(x/K)+\nu\tau}{\sigma\sqrt{\tau}}\right),
\end{aligned}$$

where $\tau = T - t$.

Chapter 5

Exercise 5.1.

1. We have
$$S_t = S_0 e^{\alpha t} + \sigma \int_0^t e^{\alpha(t-s)} dB_s.$$
2. We have $\alpha_M = r$.
3. After computing the conditional expectation
$$C(t,x) = e^{-r(T-t)} \exp\left(x e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)} - 1)\right).$$
4. Here we need to note that the usual Black-Scholes argument applies and yields $\zeta_t = \partial C(t, S_t)/\partial x$, that is
$$\zeta_t = \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)} - 1)\right).$$

Exercise 5.2.

1. We have, counting approximately 46 days to maturity,
$$\begin{aligned}
d_- &= \frac{(r - \frac{1}{2}\sigma^2)(T-t) + \log\frac{S_t}{K}}{\sigma\sqrt{T-t}} \\
&= \frac{(0.04377 - \frac{1}{2}(0.9)^2)(46/365) + \log\frac{17.2}{36.08}}{0.9\sqrt{46/365}} \\
&= -2.46,
\end{aligned}$$

and

$$d_+ = d_- + 0.9\sqrt{46/365} = -2.14.$$

From the attached table we get

$$\Phi(d_+) = \Phi(-2.14) = 0.0162$$

and

$$\Phi(d_-) = \Phi(-2.46) = 0.0069,$$

hence

$$\begin{aligned} f(t, S_t) &= S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) \\ &= 17.2 \times 0.0162 - 36.08 * e^{-0.04377\times 46/365} \times 0.0069 \\ &= 0.031 \text{ HK\$}. \end{aligned}$$

2. We have

$$\eta_t = \frac{\partial f}{\partial x}(t, S_t) = \Phi(d_+) = \Phi(-2.14) = 0.0162,$$

hence one should only hold 16.2 units of the risky asset in order to hedge 1000 such call options when $\sigma = 0.90$.

3. From the curve it turns out that when $f(t, S_t) = 10 \times 0.023 = 0.23$ HK\$, the volatility σ is approximately equal to $\sigma = 122\%$.

   This approximate value of implied volatility can be found under the column "Implied Volatility (IV.)" on this set of market data from the Hong Kong Stock Exchange:

Updated: 6 November 2008

	Basic Data														Market Data		
DW Code ▽	Issuer △	UL	Call /Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y) △	Strike △	Entitle-ment Ratio^	Total Issue Size	O/S (%) △	Delta (%) △	IV. (%) △	Day High ($)	Day Low ($)	Closing Price # ($)	T/O ('000)	UL Price ($)
01897	FB	00066	Call	Standard	18-12-2007	23-12-2008	36.08	10	138,000,000	16.43	0.780	125.375	0.000	0.000	0.023	0	17.200

Remark: a typical value for the volatility in standard market conditions would be around 20%. The observed volatility value $\sigma = 1.22$ per year is actually quite high.

Exercise 5.3.

1. We find $h(x) = x - K$.
2. Letting $g(t, x)$, the PDE rewrites as

$$r(x - \alpha(t)) = -\alpha'(t) + rx,$$

hence $\alpha(t) = \alpha(0)e^{rt}$ and $g(t, x) = x - \alpha(0)e^{-rt}$. The final condition

$$g(T,x) = h(x) = x - K$$

yields $\alpha(0) = Ke^{rT}$ and $g(t,x) = x - Ke^{-r(T-t)}$.

3. We have
$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1.$$

Chapter 6

Exercise 6.1.

1. For all $t \in [0,T]$ we have
$$\begin{aligned} C(t,S_t) &= e^{-r(T-t)} S_t^2 \, \mathbb{E}\left[\frac{S_T^2}{S_t^2}\right] \\ &= e^{-r(T-t)} S_t^2 \, \mathbb{E}\left[e^{2\sigma(B_T - B_t) - \sigma^2(T-t) + 2r(T-t)}\right] \\ &= S_t^2 e^{(r+\sigma^2)(T-t)}. \end{aligned}$$

2. For all $t \in [0,T]$ we have
$$\xi_t = \frac{\partial C}{\partial x}(t,x)_{|x=S_t} = 2S_t e^{(r+\sigma^2)(T-t)},$$
and
$$\begin{aligned} \eta_t = \frac{C(t,S_t) - \xi_t S_t}{A_t} &= \frac{e^{-rt}}{A_0}(S_t^2 e^{(r+\sigma^2)(T-t)} - 2S_t^2 e^{(r+\sigma^2)(T-t)}) \\ &= -\frac{S_t^2}{A_0} e^{\sigma^2(T-t) + r(T-2t)}. \end{aligned}$$

3. We have
$$\begin{aligned} dC(t,S_t) &= d(S_t^2 e^{(r+\sigma^2)(T-t)}) \\ &= -(r+\sigma^2) e^{(r+\sigma^2)(T-t)} S_t^2 dt + e^{(r+\sigma^2)(T-t)} d(S_t^2) \\ &= -(r+\sigma^2) e^{(r+\sigma^2)(T-t)} S_t^2 dt + e^{(r+\sigma^2)(T-t)}(2S_t dS_t + \sigma^2 S_t^2 dt) \\ &= -r e^{(r+\sigma^2)(T-t)} S_t^2 dt + 2S_t e^{(r+\sigma^2)(T-t)} dS_t, \end{aligned}$$
and
$$\begin{aligned} \xi_t dS_t + \eta_t dA_t &= 2S_t e^{(r+\sigma^2)(T-t)} dS_t - r\frac{S_t^2}{A_0} e^{\sigma^2(T-t) + r(T-2t)} A_t dt \\ &= 2S_t e^{(r+\sigma^2)(T-t)} dS_t - r S_t^2 e^{\sigma^2(T-t) + r(T-t)} dt, \end{aligned}$$

hence we can check that the strategy is self-financing since $dC(t, S_t) = \xi_t dS_t + \eta_t dA_t$.

Exercise 6.2.

1. We have
$$S_t = S_0 e^{rt} + \sigma \int_0^t e^{r(t-s)} dB_s.$$
2. We have
$$\tilde{S}_t = S_0 + \sigma \int_0^t e^{-rs} dB_s,$$
which is a martingale, being a stochastic integral with respect to Brownian motion.

   This fact can also be proved directly by computing the conditional expectation $E[\tilde{S}_t \mid \mathcal{F}_s]$ and showing it is equal to $\tilde{S}_s$:
$$\begin{aligned}
E[\tilde{S}_t \mid \mathcal{F}_s] &= E\left[S_0 + \sigma \int_0^t e^{-ru} dB_u \mid \mathcal{F}_s\right] \\
&= E[S_0] + \sigma E\left[\int_0^t e^{-ru} dB_u \mid \mathcal{F}_s\right] \\
&= S_0 + \sigma E\left[\int_0^s e^{-ru} dB_u \mid \mathcal{F}_s\right] + \sigma E\left[\int_s^t e^{-ru} dB_u \mid \mathcal{F}_s\right] \\
&= S_0 + \sigma \int_0^s e^{-ru} dB_u + \sigma E\left[\int_s^t e^{-ru} dB_u\right] \\
&= S_0 + \sigma \int_0^s e^{-ru} dB_u \\
&= \tilde{S}_s.
\end{aligned}$$
3. We have
$$\begin{aligned}
&C(t, S_t) = e^{-r(T-t)} E[\exp(S_T)|\mathcal{F}_t] \\
&= e^{-r(T-t)} E\left[\exp\left(e^{rT} S_0 + \sigma \int_0^T e^{r(T-u)} dB_u\right) \Big| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} E\left[\exp\left(e^{rT} S_0 + \sigma \int_0^t e^{r(T-u)} dB_u + \sigma \int_t^T e^{r(T-u)} dB_u\right) \Big| \mathcal{F}_t\right] \\
&= \exp\left(-r(T-t) + e^{r(T-t)} S_t\right) E\left[\exp\left(\sigma \int_t^T e^{r(T-u)} dB_u\right) \Big| \mathcal{F}_t\right] \\
&= \exp\left(-r(T-t) + e^{r(T-t)} S_t\right) E\left[\exp\left(\sigma \int_t^T e^{r(T-u)} dB_u\right)\right]
\end{aligned}$$

$$= \exp\left(-r(T-t) + e^{r(T-t)}S_t\right) \exp\left(\frac{\sigma^2}{2}\int_t^T (e^{r(T-u)})^2 du\right)$$
$$= \exp\left(-r(T-t) + e^{r(T-t)}S_t + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right).$$

4. We have

$$\xi_t = \frac{\partial C}{\partial x}(t, S_t) = \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)$$

and

$$\eta_t = \frac{C(t,S_t) - \xi_t S_t}{A_t}$$
$$= \frac{e^{-r(T-t)}\exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right) - S_t \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)}{A_t}.$$

5. We have

$$\begin{aligned}
dC(t,S_t) &= re^{-r(T-t)}\exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad -rS_t \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad -\frac{\sigma^2}{2}e^{r(T-t)}\exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad +\exp\left(S_t + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dS_t \\
&\quad +\frac{1}{2}e^{r(T-t)}\exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)\sigma^2 dt \\
&= re^{-r(T-t)}\exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad -rS_t \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad +\xi_t dS_t.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\xi_t dS_t + \eta_t dA_t = \xi_t dS_t \\
&\quad +re^{-r(T-t)}\exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad -rS_t \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt,
\end{aligned}$$

showing that

$$dC(t,S_t) = \xi_t dS_t + \eta_t dA_t,$$

and confirming that the strategy $(\xi_t,\eta_t)_{t\in\mathbb{R}_+}$ is self-financing.

Exercise 6.3.

1. We have

$$\frac{\partial f}{\partial t}(t,x) = (r-\sigma^2/2)f(t,x), \qquad \frac{\partial f}{\partial x}(t,x) = \sigma f(t,x), \qquad \frac{\partial^2 f}{\partial x^2}(t,x) = \sigma^2 f(t,x),$$

hence

$$\begin{aligned} dS_t &= df(t,B_t) \\ &= \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt \\ &= \left(r - \frac{1}{2}\sigma^2\right) f(t,B_t)dt + \sigma f(t,B_t)dB_t + \frac{1}{2}\sigma^2 f(t,B_t)dt \\ &= rf(t,B_t)dt + \sigma f(t,B_t)dB_t \\ &= rS_t dt + \sigma S_t dB_t. \end{aligned}$$

2. We have

$$\begin{aligned} E[e^{\sigma B_T}|\mathcal{F}_t] &= E[e^{\sigma(B_T-B_t+B_t)}|\mathcal{F}_t] \\ &= e^{\sigma B_t}E[e^{\sigma(B_T-B_t)}|\mathcal{F}_t] \\ &= e^{\sigma B_t}E[e^{\sigma(B_T-B_t)}] \\ &= e^{\sigma B_t+\sigma^2(T-t)/2}. \end{aligned}$$

3. We have

$$\begin{aligned} E[S_T|\mathcal{F}_t] &= E[e^{\sigma B_T+rT-\sigma^2 T/2}|\mathcal{F}_t] \\ &= e^{rT-\sigma^2 T/2}E[e^{\sigma B_T}|\mathcal{F}_t] \\ &= e^{rT-\sigma^2 T/2}e^{\sigma B_t+\sigma^2(T-t)/2} \\ &= e^{rT+\sigma B_t-\sigma^2 t/2} \\ &= e^{r(T-t)+\sigma B_t+rt-\sigma^2 t/2} \\ &= e^{r(T-t)}S_t. \end{aligned}$$

4. We have

$$\begin{aligned} V_t &= e^{-r(T-t)}E[C|\mathcal{F}_t] \\ &= e^{-r(T-t)}E[S_T-K|\mathcal{F}_t] \\ &= e^{-r(T-t)}E[S_T|\mathcal{F}_t] - e^{-r(T-t)}E[K|\mathcal{F}_t] \\ &= S_t - e^{-r(T-t)}K. \end{aligned}$$

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http://www.ntu.edu.sg/home/nprivault/indext.html

5. We take $\xi_t = 1$ and $\eta_t = -Ke^{-rT}/A_0$, $t \in [0,T]$.
6. We have
$$V_T = E[C \mid \mathcal{F}_{\mathcal{T}}] = C.$$

Exercise 6.4.Digital options.

1. By definition of the indicator functions $\mathbf{1}_{[K,\infty)}$ and $\mathbf{1}_{[0,K]}$ we have
$$\mathbf{1}_{[K,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq K, \\ 0 & \text{if } x < K, \end{cases} \qquad \text{resp.} \qquad \mathbf{1}_{[0,K]}(x) = \begin{cases} 1 & \text{if } x \leq K, \\ 0 & \text{if } x > K, \end{cases}$$
which shows the claimed result by the definition of C_d and P_d.
2. We have
$$\begin{aligned} \pi_t(C_d) + \pi_t(P_d) &= e^{-r(T-t)}\,\mathbb{E}[C_d \mid \mathcal{F}_t] + e^{-r(T-t)}\,\mathbb{E}[P_d \mid \mathcal{F}_t] \\ &= e^{-r(T-t)}\,\mathbb{E}[C_d + P_d \mid \mathcal{F}_t] \\ &= e^{-r(T-t)}\,\mathbb{E}[\mathbf{1}_{[K,\infty)}(S_T) + \mathbf{1}_{[0,K]}(S_T) \mid \mathcal{F}_t] \\ &= e^{-r(T-t)}\,\mathbb{E}[\mathbf{1}_{[0,\infty)}(S_T) \mid \mathcal{F}_t] \\ &= e^{-r(T-t)}\,\mathbb{E}[1 \mid \mathcal{F}_t] \\ &= e^{-r(T-t)}, \qquad 0 \leq t \leq T, \end{aligned}$$
since $\mathbb{P}(S_T = K) = 0$.
3. We have
$$\begin{aligned} \pi_t(C_d) &= e^{-r(T-t)}\,\mathbb{E}[C_d \mid \mathcal{F}_t] \\ &= e^{-r(T-t)}\,\mathbb{E}[\mathbf{1}_{[K,\infty)}(S_T) \mid S_t] \\ &= e^{-r(T-t)} P(S_T \geq K \mid S_t) \\ &= C_d(t, S_t). \end{aligned}$$
4. We have
$$\begin{aligned} C_d(t,x) &= e^{-r(T-t)} P(S_T > K \mid S_t = x) \\ &= e^{-r(T-t)} \Phi\left(\frac{r\tau - \sigma^2\tau/2 + \log(x/K)}{\sigma\sqrt{\tau}}\right), \end{aligned}$$
where $\tau = T - t$.
5. We have
$$\begin{aligned} \pi_t(C_d) &= C_d(t, S_t) \\ &= e^{-r(T-t)} \Phi\left(\frac{r\tau - \sigma^2\tau/2 + \log(S_t/K)}{\sigma\sqrt{\tau}}\right) \\ &= e^{-r(T-t)} \Phi\left(d_-\right), \end{aligned}$$

where

$$d_- = \frac{(r - \sigma^2/2)\tau + \log(S_t/K)}{\sigma\sqrt{\tau}}.$$

6. We have

$$\begin{aligned}
\pi_t(P_d) &= e^{-r(T-t)} - \pi_t(C_d) \\
&= e^{-r(T-t)} - e^{-r(T-t)}\Phi\left(\frac{r\tau - \sigma^2\tau/2 + \log(x/K)}{\sigma\sqrt{\tau}}\right) \\
&= e^{-r(T-t)}(1 - \Phi(d_-)) \\
&= e^{-r(T-t)}\Phi(-d_-).
\end{aligned}$$

7. We have

$$\begin{aligned}
\xi_t &= \frac{\partial C_d}{\partial x}(t, S_t) \\
&= e^{-r(T-t)}\frac{\partial}{\partial x}\Phi\left(\frac{r\tau - \sigma^2\tau/2 + \log(x/K)}{\sigma\sqrt{\tau}}\right)_{x=S_t} \\
&= e^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi\tau}S_t}e^{-(d_-)^2/2} \\
&> 0.
\end{aligned}$$

The Black-Scholes hedging strategy of such a call option does not involve short-selling because $\xi_t > 0$ for all t.

8. Here we have

$$\begin{aligned}
\xi_t &= \frac{\partial P_d}{\partial x}(t, S_t) \\
&= e^{-r(T-t)}\frac{\partial}{\partial x}\Phi\left(-\frac{r\tau - \sigma^2\tau/2 + \log(x/K)}{\sigma\sqrt{\tau}}\right)_{x=S_t} \\
&= -e^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi\tau}S_t}e^{-(d_-)^2/2} \\
&< 0.
\end{aligned}$$

The Black-Scholes hedging strategy of such a call option does involve short-selling because $\xi_t < 0$ for all t.

Chapter 8

Exercise 8.1. American digital options.

1. The optimal strategy is as follows:

(i) if $S_t \geq K$, then exercise immediately.
(ii) if $S_t < K$, then wait.

2. The optimal strategy is as follows:

   (i) if $S_t > K$, then wait.
   (ii) if $S_t \leq K$, exercise immediately.

3. Based on the answers to Question 1 we set

$$C_d^{\mathrm{Am}}(t, K) = 1, \qquad 0 \leq t < T,$$

and

$$C_d^{\mathrm{Am}}(T, x) = 0, \qquad 0 \leq x < K.$$

4. Based on the answers to Question 2, we set

$$P_d^{\mathrm{Am}}(t, K) = 1, \qquad 0 \leq t < T,$$

and

$$P_d^{\mathrm{Am}}(T, x) = 0, \qquad x > K.$$

5. Starting from $S_t \leq K$, the maximum possible payoff is clearly reached as soon as S_t hits the level K before the expiration date T, hence the discounted optimal payoff of the option is $e^{-r(\tau_K - t)}\mathbf{1}_{\{\tau_K < T\}}$.
6. We have

$$\begin{aligned} C_d^{\mathrm{Am}}(t, K) &= \Phi\left(\frac{(r + \sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right) + \Phi\left(\frac{-(r + \sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} C_d^{\mathrm{Am}}(T, x) &= \frac{x}{K}\Phi(-\infty) + \left(\frac{x}{K}\right)^{2r/\sigma^2}\Phi(-\infty) \\ &= 0, \end{aligned}$$

since $x < K$ and $\tau = 0$, which is consistent with the answers to Question 3.
7. Starting from $S_t \geq K$, the maximum possible payoff is clearly reached as soon as S_t hits the level K before the expiration date T, hence the discounted optimal payoff of the option is $e^{-r(\tau_K - t)}\mathbf{1}_{\{\tau_K < T\}}$.
8. We have

$$\begin{aligned} P_d^{\mathrm{Am}}(t, K) &= \Phi\left(\frac{-(r + \sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right) + \Phi\left(\frac{(r + \sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} P_d^{\mathrm{Am}}(T,x) &= \frac{x}{K}\Phi(-\infty) + \left(\frac{x}{K}\right)^{2r/\sigma^2}\Phi(-\infty) \\ &= 0, \end{aligned}$$

since $x < K$ and $\tau = 0$, which is consistent with the answers to Question 3.

9. The call-put parity does not hold for American digital options since

$$\begin{aligned} C_d^{\mathrm{Am}}(t,x) + P_d^{\mathrm{Am}}(t,x) &= \frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ &\quad + \left(\frac{x}{K}\right)^{2r/\sigma^2}\Phi\left(\frac{-(r+\sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ &\quad + \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ &\quad + \left(\frac{x}{K}\right)^{2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ &= \frac{x}{K} + \left(\frac{x}{K}\right)^{2r/\sigma^2}. \end{aligned}$$

Exercise 8.2. American forward contracts. Consider $(S_t)_{t\in\mathbb{R}_+}$ an asset price process given by

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

1. For all stopping times τ such that $t \le \tau \le T$ we have

$$\begin{aligned} \mathbb{E}^*\left[e^{-r(\tau-t)}(K-S_\tau)\middle|S_t\right] &= K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\middle|S_t\right] - \mathbb{E}^*\left[e^{-r(\tau-t)}S_\tau\middle|S_t\right] \\ &= e^{-r(\tau-t)}K - S_t, \end{aligned}$$

since $\tau \in [t,T]$ is bounded and $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, and the above quantity is clearly maximized by taking $\tau = t$. Hence we have

$$f(t,S_t) = \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K-S_\tau)\middle|S_t\right] = K - S_t,$$

and the optimal strategy is to exercise immediately at time t.

2. Similarly we have

$$\begin{aligned} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau-K)\middle|S_t\right] &= \mathbb{E}^*\left[e^{-r(\tau-t)}S_\tau\middle|S_t\right] - K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\middle|S_t\right] \\ &= S_t - K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\middle|S_t\right], \end{aligned}$$

since $\tau \in [t,T]$ is bounded and $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, and the above quantity is clearly maximized by taking $\tau = T$. Hence we have

$$f(t,S_t) = \ sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] = S_t - e^{-r(T-t)}K,$$

and the optimal strategy is to exercise at time T.

3. Concerning the perpetual American call forward contract, since $u \mapsto e^{-r(u-t)}S_u$ is a martingale, for all stopping times τ we have[1]

$$\begin{aligned}\mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] &= \mathbb{E}^*\left[e^{-r(\tau-t)}S_\tau\Big|S_t\right] - K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\Big|S_t\right] \\ &\le S_t - K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\Big|S_t\right] \\ &\le S_t, \qquad t \ge 0.\end{aligned}$$

On the other hand, for all fixed $T > 0$ we have

$$\begin{aligned}\mathbb{E}^*\left[e^{-r(T-t)}(S_T - K)\Big|S_t\right] &= \mathbb{E}^*\left[e^{-r(T-t)}S_T\Big|S_t\right] - K\,\mathbb{E}^*\left[e^{-r(T-t)}\Big|S_t\right] \\ &= S_t - e^{-r(T-t)}K, \qquad t \ge 0,\end{aligned}$$

hence letting $T \to \infty$ we get

$$\begin{aligned}\sup_{\substack{\tau\ge t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)\Big|S_t\right] &\ge \lim_{T\to\infty}\left(S_t - e^{-r(T-t)}K\right) \\ &= S_t,\end{aligned}$$

hence we have

$$f(t,S_t) = \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] = S_t,$$

and the optimal strategy $\tau^* = +\infty$ is to wait indefinitely.

Concerning the perpetual American put forward contract we have

$$\begin{aligned}f(t,S_t) &= \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)\Big|S_t\right] \\ &\le \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)^+\Big|S_t\right] \\ &= f_{L^*}(S_t).\end{aligned}$$

On the other hand, for $\tau = \tau_{L^*}$ we have

$$(K - S_{\tau_{L^*}}) = (K - L^*) = (K - L^*)^*$$

[1] by Fatou's Lemma.

since $0 < L^* = 2Kr/(2r+\sigma^2) < K$, hence

$$\begin{aligned} f_{L^*}(S_t) &= \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_{\tau_{L^*}})^+\Big|S_t\right] \\ &= \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_{\tau_{L^*}})\Big|S_t\right] \\ &\leq \sup_{\substack{t\leq\tau\leq T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)\Big|S_t\right] \\ &= f(t,S_t), \end{aligned}$$

which shows that

$$f(t,S_t) = f_{L^*}(S_t),$$

i.e. the perpetual American put forward contract has same price and exercise strategy as the perpetual American put option.

Exercise 8.3.

(i) The option payoff equals $(\kappa - S_t)^p$ if $S_t \leq L$.

(ii) We have

$$\begin{aligned} f_L(t,S_t) &= \mathbb{E}^*\left[e^{-r(\tau_L-t)}((\kappa - S_{\tau_L})^+)^p\Big|S_t\right] \\ &= \mathbb{E}^*\left[e^{-r(\tau_L-t)}((\kappa - L)^+)^p\Big|S_t\right] \\ &= (\kappa - L)^p\mathbb{E}^*\left[e^{-r(\tau_L-t)}\Big|S_t\right]. \end{aligned} \tag{17.3}$$

(iii) We have

$$\begin{aligned} f_L(x) &= \mathbb{E}^*\left[e^{-r(\tau_L-t)}(\kappa - S_{\tau_L})^+\Big|S_t = x\right] \\ &= \begin{cases} (\kappa - x)^p, & 0 < x \leq L, \\ (\kappa - L)^p\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geq L. \end{cases} \end{aligned}$$

(iv) We have

$$f'_{L^*}(L^*) = -\frac{2r}{\sigma^2}(\kappa - L^*)^p\frac{(L^*)^{-2r/\sigma^2-1}}{(L^*)^{-2r/\sigma^2}} = -p(\kappa - L^*)^{p-1},$$

i.e.

$$\frac{2r}{\sigma^2}(\kappa - L^*) = pL^*,$$

or

$$L^* = \frac{2r}{2r+p\sigma^2}\kappa < \kappa.$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

(v) The price can be computed as

$$f(t,S_t)=f_{L^*}(S_t)=\begin{cases}(\kappa-S_t)^p, & 0<S_t\le L^*,\\ \left(\dfrac{p\sigma^2\kappa}{2r+p\sigma^2}\right)^p\left(\dfrac{2r+p\sigma^2}{2r}\dfrac{S_t}{\kappa}\right)^{-2r/\sigma^2}, & S_t\ge L^*.\end{cases}$$

Chapter 9

Exercise 9.1.Bond options.

1. Itô's formula yields

$$\begin{aligned}d\left(\frac{P(t,S)}{P(t,T)}\right)&=\frac{P(t,S)}{P(t,T)}(\zeta^S(t)-\zeta^T(t))(dW_t-\zeta^T(t)dt)\\&=\frac{P(t,S)}{P(t,T)}(\zeta^S(t)-\zeta^T(t))d\hat{W}_t,\end{aligned}\tag{17.4}$$

   where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}$ by the Girsanov theorem.
2. From (17.4) we have

$$\frac{P(t,S)}{P(t,T)}=\frac{P(0,S)}{P(0,T)}\exp\left(\int_0^t(\zeta^S(s)-\zeta^T(s))d\hat{W}_s-\frac{1}{2}\int_0^t|\zeta^S(s)-\zeta^T(s)|^2ds\right),$$

   hence

$$\frac{P(u,S)}{P(u,T)}=\frac{P(t,S)}{P(t,T)}\exp\left(\int_t^u(\zeta^S(s)-\zeta^T(s))d\hat{W}_s-\frac{1}{2}\int_t^u|\zeta^S(s)-\zeta^T(s)|^2ds\right),$$

   $u\ge t$, and for $u=T$ this yields

$$P(T,S)=\frac{P(t,S)}{P(t,T)}\exp\left(\int_t^T(\zeta^S(s)-\zeta^T(s))d\hat{W}_s-\frac{1}{2}\int_t^T|\zeta^S(s)-\zeta^T(s)|^2ds\right),$$

   since $P(T,T)=1$. Let $\hat{\mathbb{P}}$ denote the forward measure associated to the numéraire

$$N_t:=P(t,T),\qquad 0\le t\le T.$$

3. For all $S\ge T>0$ we have

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S)-K)^+\middle|\mathcal{F}_t\right]$$

$$= P(t,T)\hat{\mathbb{E}}\left[\left(\frac{P(t,S)}{P(t,T)}\exp\left(X - \frac{1}{2}\int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds\right) - K\right)^+ \Big| \mathcal{F}_t\right]$$
$$= P(t,T)\hat{\mathbb{E}}\left[\left(e^{X+m(t,T,S)} - K\right)^+ \Big| \mathcal{F}_t\right],$$

where X is a centered Gaussian random variable with variance

$$v^2(t,T,S) = \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds$$

given $\mathcal{F}_t$, and

$$m(t,T,S) = -\frac{1}{2}v^2(t,T,S) + \log\frac{P(t,S)}{P(t,T)}.$$

Recall that when X is a centered Gaussian random variable with variance v^2, the expectation of $(e^{m+X} - K)^+$ is given, as in the standard Black-Scholes formula, by

$$\mathbb{E}[(e^{m+X} - K)^+] = e^{m+\frac{v^2}{2}}\Phi(v + (m - \log K)/v) - K\Phi((m - \log K)/v),$$

where

$$\Phi(z) = \int_{-\infty}^z e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}, \qquad z \in \mathbb{R},$$

denotes the Gaussian cumulative distribution function and for simplicity of notation we dropped the indices t, T, S in $m(t,T,S)$ and $v^2(t,T,S)$.

Consequently we have

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S) - K)^+ \Big| \mathcal{F}_t\right] \tag{17.5}$$
$$= P(t,S)\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right) - KP(t,T)\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right).$$

4. The self-financing hedging strategy that hedges the bond option is obtained by holding

$$\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right)$$

units of the bond with maturity S, and shorting

$$K\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right)$$

units of the bond with maturity T.

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

Chapter 10

Exercise 10.1.

1. We have $r_t = r_0 + \sigma W_t$ and
$$F(t, r_t) = F(t, r_0 + \sigma W_t),$$
hence by standard arbitrage arguments the PDE satisfied by $F(t, x)$ is
$$-xF(t,x) + \frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0, \tag{17.6}$$
with terminal condition $F(T, x) = 1$.
2. We have
$$\begin{aligned}
F(t, r_t) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\exp\left(-r_0(T-t) - \sigma\int_t^T B_s ds\right)\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\exp\left(-r_0(T-t) - \sigma(T-t)B_t - \sigma\int_t^T (T-s)dB_s\right)\Big|\mathcal{F}_t\right] \\
&= \exp\left(-r_0(T-t) - \sigma(T-t)B_t\right)\mathbb{E}\left[\exp\left(-\sigma\int_t^T (T-s)dB_s\right)\Big|\mathcal{F}_t\right] \\
&= \exp\left(-r_0(T-t) - \sigma(T-t)B_t\right)\mathbb{E}\left[\exp\left(-\sigma\int_t^T (T-s)dB_s\right)\right] \\
&= \exp\left(-(T-t)r_t + \frac{\sigma^2}{2}\int_t^T (T-s)^2 ds\right) \\
&= \exp\left(-(T-t)r_t + \frac{\sigma^2}{2}\int_0^{T-t} (T-t-s)^2 ds\right) \\
&= \exp\left(-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right).
\end{aligned}$$
Note that the PDE (17.6) can also be solved by looking for a solution of the form $F(t,x) = e^{A(T-t)+xC(T-t)}$, in which case one would find $A(s) = \sigma^2 s^3/6$ and $C(s) = -s$.
3. The function $F(t, x)$ is easily shown to satisfy the PDE (17.6) and the terminal condition $F(0, x) = 1$.
4. We have

$$
\begin{aligned}
f(t,T,S) &= \frac{1}{S-T}(\log P(t,T) - \log P(t,S)) \\
&= \frac{1}{S-T}\left(\left(-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right) - \left(-(S-t)r_t + \frac{\sigma^2}{6}(S-t)^3\right)\right) \\
&= r_t + \frac{1}{S-T}\frac{\sigma^2}{6}((T-t)^3 - (S-t)^3).
\end{aligned}
$$

5. We have
$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T) = r_t - \frac{\sigma^2}{2}(T-t)^2.$$
6. We have
$$d_t f(t,T) = \sigma^2(T-t)dt + \sigma dW_t.$$
7. The HJM condition is satisfied since the drift of $d_t f(t,T)$ equals $\sigma\int_t^T \sigma ds$.

Exercise 10.2.

1. We have
$$P(t,T) = P(s,T)\exp\left(\int_s^t r_u du + \int_s^t \sigma_u^T dB_u - \frac{1}{2}\int_s^t |\sigma_u^T|^2 du\right),$$
$0 \le s \le t \le T$.
2. We have
$$d\left(e^{-\int_0^t r_s ds}P(t,T)\right) = e^{-\int_0^t r_s ds}P(t,T)\sigma_t^T dB_t,$$
which gives a martingale after integration, from the properties of the Itô integral.
3. By the martingale property of the previous question we have
$$
\begin{aligned}
\mathbb{E}\left[e^{-\int_0^T r_s ds}\middle|\mathcal{F}_t\right] &= \mathbb{E}\left[P(T,T)e^{-\int_0^T r_s ds}\middle|\mathcal{F}_t\right] \\
&= P(t,T)e^{-\int_0^t r_s ds}, \qquad 0 \le t \le T.
\end{aligned}
$$
4. By the previous question we have
$$
\begin{aligned}
P(t,T) &= e^{\int_0^t r_s ds}\,\mathbb{E}\left[e^{-\int_0^T r_s ds}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{\int_0^t r_s ds}e^{-\int_0^T r_s ds}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right], \qquad 0 \le t \le T,
\end{aligned}
$$
since $e^{-\int_0^t r_s ds}$ is an $\mathcal{F}_t$-measurable random variable.
5. We have

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$\begin{aligned}\frac{P(t,S)}{P(t,T)} &= \frac{P(s,S)}{P(s,T)}\exp\left(\int_s^t(\sigma_u^S-\sigma_u^T)dB_u-\frac{1}{2}\int_s^t(|\sigma_u^S|^2-|\sigma_u^T|^2)du\right)\\ &= \frac{P(s,S)}{P(s,T)}\exp\left(\int_s^t(\sigma_u^S-\sigma_u^T)dB_u^T-\frac{1}{2}\int_s^t(\sigma_u^S-\sigma_u^T)^2du\right),\end{aligned}$$

$0\le t\le T$, hence letting $s=t$ and $t=T$ in the above expression we have

$$P(T,S)=\frac{P(t,S)}{P(t,T)}\exp\left(\int_t^T(\sigma_s^S-\sigma_s^T)dB_s^T-\frac{1}{2}\int_t^T(\sigma_s^S-\sigma_s^T)^2ds\right).$$

6. We have

$$\begin{aligned}&P(t,T)\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+\right]\\ &= P(t,T)\,\mathbb{E}_T\left[\left(\frac{P(t,S)}{P(t,T)}\exp\left(\int_t^T(\sigma_s^S-\sigma_s^T)dB_s^T-\frac{1}{2}\int_t^T(\sigma_s^S-\sigma_s^T)^2ds\right)-\kappa\right)^+\right]\\ &= P(t,T)\,\mathbb{E}[(e^X-\kappa)^+\mid\mathcal{F}_t]\\ &= P(t,T)e^{m_t+v_t^2/2}\Phi\left(\frac{v_t}{2}+\frac{1}{v_t}(m_t+v_t^2/2-\log\kappa)\right)\\ &\quad-\kappa P(t,T)\Phi\left(-\frac{v_t}{2}+\frac{1}{v_t}(m_t+v_t^2/2-\log\kappa)\right),\end{aligned}$$

with

$$m_t=\log(P(t,S)/P(t,T))-\frac{1}{2}\int_t^T(\sigma_s^S-\sigma_s^T)^2ds$$

and

$$v_t^2=\int_t^T(\sigma_s^S-\sigma_s^T)^2ds,$$

i.e.

$$\begin{aligned}&P(t,T)\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+\right]\\ &= P(t,S)\Phi\left(\frac{v_t}{2}+\frac{1}{v_t}\log\frac{P(t,S)}{\kappa P(t,T)}\right)-\kappa P(t,T)\Phi\left(-\frac{v_t}{2}+\frac{1}{v_t}\log\frac{P(t,S)}{\kappa P(t,T)}\right).\end{aligned}$$

Exercise 10.3. We check that $P(T,T)=e^{X_T^T}=1$.

Exercise 10.4. From Proposition 10.1 the bond pricing PDE is

$$\begin{cases}\dfrac{\partial F}{\partial t}(t,x)=xF(t,x)-(\alpha-\beta x)\dfrac{\partial F}{\partial x}(t,x)-\dfrac{1}{2}\sigma^2x^2\dfrac{\partial^2F}{\partial x^2}(t,x)\\ F(T,x)=1.\end{cases}$$

Let us search for a solution of the form

$$F(t,x) = e^{A(T-t)-xB(T-t)},$$

with $A(0) = B(0) = 0$, which implies

$$\begin{cases} A'(s) = 0 \\ B'(s) + \beta B(s) + \frac{1}{2}\sigma^2 B^2(s) = 1. \end{cases}$$

hence in particular $A(s) = 0$, $s \in \mathbb{R}$, and $B(s)$ solves a Riccatti equation, whose solution is easily checked to be

$$B(s) = \frac{2(e^{\gamma s} - 1)}{2\gamma + (\beta + \gamma)(e^{\gamma s} - 1)},$$

with $\gamma = \sqrt{\beta^2 + 2\sigma^2}$.

Exercise 10.5.

1. We have

$$\begin{aligned} f(t,T,S) &= -\frac{1}{S-T}\left(X_t^S - X_t^T - \mu(S-T)\right) \\ &= \mu - \sigma\frac{1}{S-T}\left((S-t)\int_0^t \frac{1}{S-s}dB_s - (T-t)\int_0^t \frac{1}{T-s}dB_s\right) \\ &= \mu - \sigma\frac{1}{S-T}\int_0^t \left(\frac{S-t}{S-s} - \frac{T-t}{T-s}\right)dB_s \\ &= \mu - \sigma\frac{1}{S-T}\int_0^t \frac{(T-s)(S-t)-(T-t)(S-s)}{(S-s)(T-s)}dB_s \\ &= \mu + \frac{\sigma}{S-T}\int_0^t \frac{(s-t)(S-T)}{(S-s)(T-s)}dB_s. \end{aligned}$$

2. We have

$$f(t,T) = \mu - \sigma\int_0^t \frac{t-s}{(T-s)^2}dB_s.$$

3. We note that

$$\lim_{T\searrow t} f(t,T) = \mu - \sigma\int_0^t \frac{1}{t-s}dB_s$$

does not exist in $L^2(\Omega)$.

4. By Itô's calculus we have

$$\frac{dP(t,T)}{P(t,T)} = \sigma dB_t + \frac{1}{2}\sigma^2 dt + \mu dt - \frac{X_t^T}{T-t}dt$$

This version: March 8, 2011
http://www.ntu.edu.sg/home/nprivault/indext.html

$$= \sigma dB_t + \frac{1}{2}\sigma^2 dt - \frac{\log P(t,T)}{T-t}dt, \qquad t \in [0,T].$$

5. Let

$$\begin{aligned} r_t^S &= \mu + \frac{1}{2}\sigma^2 - \frac{X_t^S}{S-t} \\ &= \mu + \frac{1}{2}\sigma^2 - \sigma \int_0^t \frac{1}{S-s} dB_s, \end{aligned}$$

and apply the result of Exercise 10.2-(4).

6. We have

$$\mathbb{E}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}}\Big|\mathcal{F}_t\right] = e^{\sigma B_t - \sigma^2 t/2}.$$

7. By the Girsanov theorem, the process $\tilde{B}_t := B_t - \sigma t$ is a standard Brownian motion under $\mathbb{P}_T$.
8. We have

$$\begin{aligned} &\log P(T,S) = -\mu(S-T) + \sigma(S-T)\int_0^T \frac{1}{S-s} dB_s \\ &= -\mu(S-T) + \sigma(S-T)\int_0^t \frac{1}{S-s} dB_s + \sigma(S-T)\int_t^T \frac{1}{S-s} dB_s \\ &= \frac{S-T}{S-t}\log P(t,S) + \sigma(S-T)\int_t^T \frac{1}{S-s} dB_s \\ &= \frac{S-T}{S-t}\log P(t,S) + \sigma(S-T)\int_t^T \frac{1}{S-s} d\tilde{B}_s + \sigma^2(S-T)\int_t^T \frac{1}{S-s} ds \\ &= \frac{S-T}{S-t}\log P(t,S) + \sigma(S-T)\int_t^T \frac{1}{S-s} d\tilde{B}_s + \sigma^2(S-T)\log\frac{S-t}{S-T}, \end{aligned}$$

$0 < T < S$.

9. We have

$$\begin{aligned} &P(t,T)\,\mathbb{E}_T\left[(P(T,S)-K)^+\Big|\mathcal{F}_t\right] \\ &= P(t,T)\,\mathbb{E}[(e^X - \kappa)^+ \mid \mathcal{F}_t] \\ &= P(t,T)e^{m_t + v_t^2/2}\Phi\left(\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log\kappa)\right) \\ &\quad -\kappa P(t,T)\Phi\left(-\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log\kappa)\right) \\ &= P(t,T)e^{m_t + v_t^2/2}\Phi\left(v_t + \frac{1}{v_t}(m_t - \log\kappa)\right) - \kappa P(t,T)\Phi\left(\frac{1}{v_t}(m_t - \log\kappa)\right), \end{aligned}$$

with

$$m_t = \frac{S-T}{S-t}\log P(t,S) + \sigma^2(S-T)\log\frac{S-t}{S-T}$$

and

$$\begin{aligned} v_t^2 &= \sigma^2 (S-T)^2 \int_t^T \frac{1}{(S-s)^2} ds \\ &= \sigma^2 (S-T)^2 \left(\frac{1}{S-T} - \frac{1}{S-t} \right) \\ &= \sigma^2 (S-T) \frac{(T-t)}{(S-t)}, \end{aligned}$$

hence

$$\begin{aligned} & P(t,T)\, \mathbb{E}_T \left[(P(T,S)-K)^+ \middle| \mathcal{F}_t \right] \\ & = P(t,T) \left(P(t,S) \right)^{(S-T)(S-t)} \left(\frac{S-t}{S-T} \right)^{\sigma^2 (S-T)} e^{v_t^2/2} \\ & \times \Phi \left(v_t + \frac{1}{v_t} \log \left(\frac{(P(t,S))^{(S-T)(S-t)}}{\kappa} \left(\frac{S-t}{S-T} \right)^{\sigma^2 (S-T)} \right) \right) \\ & - \kappa P(t,T) \Phi \left(\frac{1}{v_t} \log \left(\frac{(P(t,S))^{(S-T)(S-t)}}{\kappa} \left(\frac{S-t}{S-T} \right)^{\sigma^2 (S-T)} \right) \right). \end{aligned}$$

Chapter 11

Exercise 11.1.

1. We have

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_t^i dB_t, \qquad i = 1,2,$$

and

$$P(T,T_i) = P(t,T_i) \exp \left(\int_t^T r_s ds + \int_t^T \zeta_s^i dB_s - \frac{1}{2} \int_t^T (\zeta_s^i)^2 ds \right), \qquad 0 \le t \le T \le T_i, \quad i = 1,2,$$

hence

$$\log P(T,T_i) = \log P(t,T_i) + \int_t^T r_s ds + \int_t^T \zeta_s^i dB_s - \frac{1}{2} \int_t^T (\zeta_s^i)^2 ds, \qquad 0 \le t \le T \le T_i, \quad i = 1,2,$$

and

$$d \log P(t,T_i) = r_t dt + \zeta_t^i dB_t - \frac{1}{2} (\zeta_t^i)^2 dt, \qquad i = 1,2.$$

In the present model

$$dr_t = \sigma dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$, we have

$$\zeta_t^i = -\sigma(T_i - t), \qquad 0 \le t \le T_i, \quad i = 1, 2.$$

Letting

$$dB_t^i = dB_t - \zeta_t^i dt,$$

defines a standard Brownian motion under $\mathbb{P}_i$, $i = 1, 2$, and we have

$$\begin{aligned}\frac{P(T,T_1)}{P(T,T_2)} &= \frac{P(t,T_1)}{P(t,T_2)} \exp\left(\int_t^T (\zeta_s^1 - \zeta_s^2) dB_s - \frac{1}{2}\int_t^T ((\zeta_s^1)^2 - (\zeta_s^2)^2) ds\right) \\ &= \frac{P(t,T_1)}{P(t,T_2)} \exp\left(\int_t^T (\zeta_s^1 - \zeta_s^2) dB_s^2 - \frac{1}{2}\int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds\right),\end{aligned}$$

which is an $\mathcal{F}_t$-martingale under $\mathbb{P}_2$ and under $\mathbb{P}_{1,2}$, and

$$\frac{P(T,T_2)}{P(T,T_1)} = \frac{P(t,T_2)}{P(t,T_1)} \exp\left(-\int_t^T (\zeta_s^1 - \zeta_s^2) dB_s^1 - \frac{1}{2}\int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds\right),$$

which is an $\mathcal{F}_t$-martingale under $\mathbb{P}_1$.

2. We have

$$\begin{aligned}f(t,T_1,T_2) &= -\frac{1}{T_2 - T_1}(\log P(t,T_2) - \log P(t,T_1)) \\ &= r_t + \frac{1}{T_2 - T_1}\frac{\sigma^2}{6}((T_1 - t)^3 - (T_2 - t)^3).\end{aligned}$$

3. We have

$$\begin{aligned}df(t,T_1,T_2) &= -\frac{1}{T_2 - T_1} d\log\left(P(t,T_2)/P(t,T_1)\right) \\ &= -\frac{1}{T_2 - T_1}\left((\zeta_t^2 - \zeta_t^1) dB_t - \frac{1}{2}((\zeta_t^2)^2 - (\zeta_t^1)^2) dt\right) \\ &= -\frac{1}{T_2 - T_1}\left((\zeta_t^2 - \zeta_t^1)(dB_t^2 + \zeta_t^2 dt) - \frac{1}{2}((\zeta_t^2)^2 - (\zeta_t^1)^2) dt\right) \\ &= -\frac{1}{T_2 - T_1}\left((\zeta_t^2 - \zeta_t^1) dB_t^2 - \frac{1}{2}(\zeta_t^2 - \zeta_t^1)^2 dt\right).\end{aligned}$$

4. We have

$$f(T,T_1,T_2) = -\frac{1}{T_2 - T_1}\log\left(P(T,T_2)/P(T,T_1)\right)$$

$$
\begin{aligned}
&= f(t,T_1,T_2) - \frac{1}{T_2-T_1}\left(\int_t^T (\zeta_s^2-\zeta_s^1)dB_s - \frac{1}{2}((\zeta_s^2)^2-(\zeta_s^1)^2)ds\right)\\
&= f(t,T_1,T_2) - \frac{1}{T_2-T_1}\left(\int_t^T (\zeta_s^2-\zeta_s^1)dB_s^2 - \frac{1}{2}\int_t^T (\zeta_s^2-\zeta_s^1)^2ds\right)\\
&= f(t,T_1,T_2) - \frac{1}{T_2-T_1}\left(\int_t^T (\zeta_s^2-\zeta_s^1)dB_s^1 + \frac{1}{2}\int_t^T (\zeta_s^2-\zeta_s^1)^2ds\right).
\end{aligned}
$$

Hence $f(T,T_1,T_2)$ has a Gaussian distribution given $\mathcal{F}_t$ with conditional mean

$$
m = f(t,T_1,T_2) + \frac{1}{2}\int_t^T (\zeta_s^2-\zeta_s^1)^2ds
$$

under $\mathbb{P}_2$, resp.

$$
m = f(t,T_1,T_2) - \frac{1}{2}\int_t^T (\zeta_s^2-\zeta_s^1)^2ds)
$$

under $\mathbb{P}_1$, and variance

$$
v^2 = \frac{1}{(T_2-T_1)^2}\int_t^T (\zeta_s^2-\zeta_s^1)^2ds.
$$

Hence

$$
\begin{aligned}
&(T_2-T_1)\,\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(f(T_1,T_1,T_2)-\kappa)^+\middle|\mathcal{F}_t\right]\\
&= (T_2-T_1)P(t,T_2)\,\mathbb{E}_2\left[(f(T_1,T_1,T_2)-\kappa)^+\middle|\mathcal{F}_t\right]\\
&= (T_2-T_1)P(t,T_2)\,\mathbb{E}_2\left[(m+X-\kappa)^+\middle|\mathcal{F}_t\right]\\
&= (T_2-T_1)P(t,T_2)\left(\frac{v}{\sqrt{2\pi}}e^{-\frac{(\kappa-m)^2}{2v^2}} + (m-\kappa)\Phi((m-\kappa)/v)\right).
\end{aligned}
$$

5. We have

$$
\begin{aligned}
&L(T,T_1,T_2) = S(T,T_1,T_2)\\
&= \frac{1}{T_2-T_1}\left(\frac{P(T,T_1)}{P(T,T_2)}-1\right)\\
&= \frac{1}{T_2-T_1}\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T (\zeta_s^1-\zeta_s^2)dB_s - \frac{1}{2}\int_t^T ((\zeta_s^1)^2-(\zeta_s^2)^2)ds\right)-1\right)\\
&= \frac{1}{T_2-T_1}\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T (\zeta_s^1-\zeta_s^2)dB_s^2 - \frac{1}{2}\int_t^T (\zeta_s^1-\zeta_s^2)^2ds\right)-1\right)
\end{aligned}
$$

$$= \frac{1}{T_2 - T_1}\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T (\zeta_s^1 - \zeta_s^2)dB_s^1 + \frac{1}{2}\int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds\right) - 1\right),$$

and by Itô calculus,

$$\begin{aligned}
dS(t,T_1,T_2) &= \frac{1}{T_2 - T_1} d\left(\frac{P(t,T_1)}{P(t,T_2)}\right) \\
&= \frac{1}{T_2 - T_1}\frac{P(t,T_1)}{P(t,T_2)}\left((\zeta_t^1 - \zeta_t^2)dB_t + \frac{1}{2}(\zeta_t^1 - \zeta_t^2)^2 dt - \frac{1}{2}((\zeta_t^1)^2 - (\zeta_t^2)^2)dt\right) \\
&= \left(\frac{1}{T_2 - T_1} + S(t,T_1,T_2)\right)\left((\zeta_t^1 - \zeta_t^2)dB_t + \zeta_t^2(\zeta_t^2 - \zeta_t^1)dt)dt\right) \\
&= \left(\frac{1}{T_2 - T_1} + S(t,T_1,T_2)\right)\left((\zeta_t^1 - \zeta_t^2)dB_t^1 + ((\zeta_t^2)^2 - (\zeta_t^1)^2)dt\right) \\
&= \left(\frac{1}{T_2 - T_1} + S(t,T_1,T_2)\right)(\zeta_t^1 - \zeta_t^2)dB_t^2, \qquad t \in [0,T_1],
\end{aligned}$$

hence $\frac{1}{T_2 - T_1} + S(t,T_1,T_2)$ is a geometric Brownian motion, with

$$\begin{aligned}
&\frac{1}{T_2 - T_1} + S(T,T_1,T_2) \\
&= \left(\frac{1}{T_2 - T_1} + S(t,T_1,T_2)\right)\exp\left(\int_t^T (\zeta_s^1 - \zeta_s^2)dB_s^2 - \frac{1}{2}\int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds\right),
\end{aligned}$$

$0 \le t \le T \le T_1$.

6. We have

$$\begin{aligned}
&(T_2 - T_1)\,\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(L(T_1,T_1,T_2) - \kappa)^+ \middle| \mathcal{F}_t\right] \\
&= (T_2 - T_1)\,\mathbb{E}\left[e^{-\int_t^{T_1} r_s ds}P(T_1,T_2)(L(T_1,T_1,T_2) - \kappa)^+ \middle| \mathcal{F}_t\right] \\
&= P(t,T_1,T_2)\,\mathbb{E}_{1,2}\left[(S(T_1,T_1,T_2) - \kappa)^+ \middle| \mathcal{F}_t\right].
\end{aligned}$$

The forward measure $\mathbb{P}_2$ is defined by

$$\mathbb{E}\left[\frac{d\mathbb{P}_2}{d\mathbb{P}}\middle|\mathcal{F}_t\right] = \frac{P(t,T_2)}{P(0,T_2)}e^{-\int_0^t r_s ds}, \qquad 0 \le t \le T_2,$$

and the forward swap measure is defined by

$$\mathbb{E}\left[\frac{d\mathbb{P}_{1,2}}{d\mathbb{P}}\middle|\mathcal{F}_t\right] = \frac{P(t,T_2)}{P(0,T_2)}e^{-\int_0^t r_s ds}, \qquad 0 \le t \le T_1,$$

hence $\mathbb{P}_2$ and $\mathbb{P}_{1,2}$ coincide up to time T_1 and $(B_t^2)_{t\in[0,T_1]}$ is a standard Brownian motion until time T_1 under $\mathbb{P}_2$ and under $\mathbb{P}_{1,2}$, consequently under $\mathbb{P}_{1,2}$,

$$
\begin{aligned}
&L(T,T_1,T_2) = S(T,T_1,T_2) \\
&= -\frac{1}{T_2-T_1} + \left(\frac{1}{T_2-T_1} + S(t,T_1,T_2)\right) \exp\left(\int_t^T (\zeta_s^1-\zeta_s^2)dB_s^2 - \frac{1}{2}\int_t^T (\zeta_s^1-\zeta_s^2)^2 ds\right),
\end{aligned}
$$

has same law as

$$
\frac{1}{T_2-T_1}\left(\frac{P(t,T_1)}{P(t,T_2)} e^{X-\frac{1}{2}\mathrm{var}\,[X]} - 1\right),
$$

where X is a centered Gaussian random variable with variance

$$
\int_t^{T_1} (\zeta_s^1-\zeta_s^2)^2 ds
$$

given $\mathcal{F}_t$. Hence

$$
\begin{aligned}
&(T_2-T_1)\,\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(L(T_1,T_1,T_2)-\kappa)^+ \middle| \mathcal{F}_t\right] \\
&= P(t,T_1,T_2)\mathrm{Bl}\left(\frac{1}{T_2-T_1} + S(t,T_1,T_2), \frac{\int_t^{T_1}(\zeta_s^1-\zeta_s^2)^2 ds}{T_1-t}, \kappa + \frac{1}{T_2-T_1}, T_1-t\right).
\end{aligned}
$$

Exercise 11.2.

(i) We have

$$
L(t,T_1,T_2) = L(0,T_1,T_2) e^{\int_0^t \gamma_1(s)dW_s^2 - \frac{1}{2}\int_0^T |\gamma_1(s)|^2 ds}, \qquad 0 \le t \le T_1,
$$

and $L(t,T_2,T_3) = b$. Note that we have $P(t,T_2)/P(t,T_3) = 1+\delta b$ hence $\mathbb{P}_2 = \mathbb{P}_3 = \mathbb{P}_{1,2}$ up to time T_1.

(ii) We have

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(L(T_1,T_1,T_2)-\kappa)^+ \middle| \mathcal{F}_t\right] = P(t,T_1)\hat{\mathbb{E}}_2\left[(L(T_1,T_1,T_2)-\kappa)^+ \mid \mathcal{F}_t\right] \\
&= P(t,T_1)\hat{\mathbb{E}}_2\left[(L(t,T_1,T_2)e^{\int_0^t \gamma_1(s)dW_s^2 - \frac{1}{2}\int_0^T |\gamma_1(s)|^2 ds} - \kappa)^+ \mid \mathcal{F}_t\right] \\
&= P(t,T_1)\mathrm{Bl}(\kappa, L(t,T_1,T_2), \sigma_1(t), 0, T_1-t),
\end{aligned}
$$

where

$$
\sigma_1^2(t) = \frac{1}{T_1-t}\int_t^{T_1} |\gamma_1(s)|^2 ds.
$$

(iii) We have

$$
\begin{aligned}
\frac{P(t,T_1)}{P(t,T_1,T_3)} &= \frac{P(t,T_1)}{\delta P(t,T_2) + \delta P(t,T_3)} \\
&= \frac{P(t,T_1)}{\delta P(t,T_2)} \frac{1}{1+P(t,T_3)/P(t,T_2)}
\end{aligned}
$$

$$= \frac{1+\delta b}{\delta(\delta b+2)}(1+\delta L(t,T_1,T_2)), \qquad 0 \le t \le T_1,$$

and

$$\begin{aligned}\frac{P(t,T_3)}{P(t,T_1,T_3)} &= \frac{P(t,T_3)}{P(t,T_2)+P(t,T_3)} \\ &= \frac{1}{1+P(t,T_2)/P(t,T_3)} \\ &= \frac{1}{\delta}\frac{1}{2+\delta b}, \qquad 0 \le t \le T_2. \end{aligned} \tag{17.7}$$

(iv) We have

$$\begin{aligned}S(t,T_1,T_3) &= \frac{P(t,T_1)}{P(t,T_1,T_3)} - \frac{P(t,T_3)}{P(t,T_1,T_3)} \\ &= \frac{1+\delta b}{\delta(2+\delta b)}(1+\delta L(t,T_1,T_2)) - \frac{1}{\delta(2+\delta b)} \\ &= \frac{1}{2+\delta b}(b+(1+\delta b)L(t,T_1,T_2)), \qquad 0 \le t \le T_2.\end{aligned}$$

We have

$$\begin{aligned}dS(t,T_1,T_3) &= \frac{1+\delta b}{2+\delta b}L(t,T_1,T_2)\gamma_1(t)dW_t^2 \\ &= \left(S(t,T_1,T_3) - \frac{b}{2+\delta b}\right)\gamma_1(t)dW_t^2 \\ &= S(t,T_1,T_3)\sigma_{1,3}(t)dW_t^2, \qquad 0 \le t \le T_2,\end{aligned}$$

with

$$\begin{aligned}\sigma_{1,3}(t) &= \left(1 - \frac{b}{S(t,T_1,T_3)(2+\delta b)}\right)\gamma_1(t) \\ &= \left(1 - \frac{b}{b+(1+\delta b)L(t,T_1,T_2)}\right)\gamma_1(t) \\ &= \frac{(1+\delta b)L(t,T_1,T_2)}{b+(1+\delta b)L(t,T_1,T_2)}\gamma_1(t) \\ &= \frac{(1+\delta b)L(t,T_1,T_2)}{(2+\delta b)S(t,T_1,T_3)}\gamma_1(t).\end{aligned}$$

(v) The process $(W^2)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}_2$ and

$$\begin{aligned}&P(t,T_1,T_3)\hat{\mathbb{E}}_{1,3}\left[(S(T_1,T_1,T_3)-\kappa)^+ \mid \mathcal{F}_t\right] \\ &\qquad = P(t,T_2)\mathrm{Bl}(\kappa, S(t,T_1,T_2), \tilde{\sigma}_{1,3}(t), 0, T_1-t),\end{aligned}$$

where $|\tilde{\sigma}_{1,3}(t)|^2$ is the approximation of the volatility

$$\frac{1}{T_1-t}\int_t^{T_1}|\sigma_{1,3}(s)|^2ds=\frac{1}{T_1-t}\int_t^{T_1}\left(\frac{(1+\delta b)L(s,T_1,T_2)}{(2+\delta b)S(s,T_1,T_3)}\right)^2\gamma_1(s)ds$$

obtained by freezing the random component of $\sigma_{1,3}(s)$ at time t, i.e.

$$\tilde{\sigma}_{1,3}^2(t)=\frac{1}{T_1-t}\left(\frac{(1+\delta b)L(t,T_1,T_2)}{(2+\delta b)S(t,T_1,T_3)}\right)^2\int_t^{T_1}|\gamma_1(s)|^2ds.$$

Exercise 11.3.

1. We have

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_t^T r_s ds}\left(P(T,S)-\kappa\right)^+\Big|\mathcal{F}_t\right]=V_T=V_0+\int_0^T dV_t\\&=P(0,T)\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+\right]+\int_0^t\xi_s^T dP(s,T)+\int_0^t\xi_s^S dP(s,S).\end{aligned}$$

2. We have

$$\begin{aligned}d\tilde{V}_t&=d\left(e^{-\int_0^t r_s ds}V_t\right)\\&=-r_te^{-\int_0^t r_s ds}V_tdt+e^{-\int_0^t r_s ds}dV_t\\&=-r_te^{-\int_0^t r_s ds}(\xi_t^TP(t,T)\\&\quad+\xi_t^SP(t,S))dt+e^{-\int_0^t r_s ds}\xi_t^TdP(t,T)+e^{-\int_0^t r_s ds}\xi_t^SdP(t,S)\\&=\xi_t^Td\tilde{P}(t,T)+\xi_t^Sd\tilde{P}(t,S).\end{aligned}$$

3. By Itô's formula we have

$$\begin{aligned}&\mathbb{E}_T\left[(P(T,S)-\kappa)^+|\mathcal{F}_t\right]=C(X_T,0,0)\\&=C(X_0,T,v(0,T))+\int_0^t\frac{\partial C}{\partial x}(X_s,T-s,v(s,T))dX_s\\&=\mathbb{E}_T\left[(P(T,S)-\kappa)^+\right]+\int_0^t\frac{\partial C}{\partial x}(X_s,T-s,v(s,T))dX_s,\end{aligned}$$

since the process

$$t\mapsto\mathbb{E}_T\left[(P(T,S)-\kappa)^+|\mathcal{F}_t\right]$$

is a martingale under $\tilde{\mathbb{P}}$.

4. We have

$$\begin{aligned}d\hat{V}_t&=d(V_t/P(t,T))\\&=d\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+|\mathcal{F}_t\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial C}{\partial x}(X_t, T-t, v(t,T))dX_t \\
&= \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t^T.
\end{aligned}
$$

5. We have

$$
\begin{aligned}
dV_t &= d(P(t,T)\hat{V}_t) \\
&= P(t,T)d\hat{V}_t + \hat{V}_t dP(t,T) + d\hat{V}_t \cdot dP(t,T) \\
&= P(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t^T + \hat{V}_t dP(t,T) \\
&\quad + P(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)\sigma_t^T dt \\
&= P(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \hat{V}_t dP(t,T).
\end{aligned}
$$

6. We have

$$
\begin{aligned}
d\tilde{V}_t &= d(e^{-\int_0^t r_s ds}V_t) \\
&= -r_t e^{-\int_0^t r_s ds}V_t dt + e^{-\int_0^t r_s ds}dV_t \\
&= \tilde{P}(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \hat{V}_t d\tilde{P}(t,T).
\end{aligned}
$$

7. We have

$$
\begin{aligned}
d\tilde{V}_t &= \tilde{P}(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \hat{V}_t d\tilde{P}(t,T) \\
&= \frac{\partial C}{\partial x}(X_t, T-t, v(t,T))d\tilde{P}(t,S) \\
&\quad - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))d\tilde{P}(t,T) + \hat{V}_t d\tilde{P}(t,T) \\
&= \left(\hat{V}_t - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))\right)d\tilde{P}(t,T) \\
&\quad + \frac{\partial C}{\partial x}(X_t, T-t, v(t,T))d\tilde{P}(t,S),
\end{aligned}
$$

hence the hedging strategy $(\xi_t^T, \xi_t^S)_{t\in[0,T]}$ of the bond option is given by

$$
\begin{aligned}
\xi_t^T &= \hat{V}_t - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T)) \\
&= C(X_t, T-t, v(t,T)) - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T)),
\end{aligned}
$$

and

$$
\xi_t^S = \frac{\partial C}{\partial x}(X_t, T-t, v(t,T)),
$$

$t \in [0,T]$.

8. We have

$$\begin{aligned}
\frac{\partial C}{\partial x}(x,\tau,v) &= \frac{\partial}{\partial x}\left[x\Phi\left(\frac{v\sqrt{\tau}}{2}+\frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)-\kappa\Phi\left(-\frac{v\sqrt{\tau}}{2}+\frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)\right]\\
&= x\frac{\partial}{\partial x}\Phi\left(\frac{v\sqrt{\tau}}{2}+\frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)-\kappa\frac{\partial}{\partial x}\Phi\left(-\frac{v\sqrt{\tau}}{2}+\frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)\\
&\quad+\Phi\left(\frac{v\sqrt{\tau}}{2}+\frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)\\
&= x\frac{e^{-\frac{1}{2}\left(\frac{v\sqrt{\tau}}{2}+\frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)^2}}{\sqrt{2\pi}}\left(\frac{1}{v\sqrt{\tau}x}\right)-\kappa\frac{e^{-\frac{1}{2}\left(-\frac{v\sqrt{\tau}}{2}+\frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)^2}}{\sqrt{2\pi}}\left(\frac{1}{v\sqrt{\tau}x}\right)\\
&\quad+\Phi\left(\frac{v\sqrt{\tau}}{2}+\frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)\\
&= \Phi\left(\frac{\log(x/\kappa)+\tau v^2/2}{\sqrt{\tau}v}\right).
\end{aligned}$$

As a consequence we get

$$\begin{aligned}
\xi_t^T &= C(X_t,T-t,v(t,T))-\frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t,T-t,v(t,T))\\
&= \frac{P(t,S)}{P(t,T)}\Phi\left(\frac{(T-t)v^2(t,T)/2+\log X_t}{\sqrt{T-t}v(t,T)}\right)-\kappa\Phi\left(-\frac{v(t,T)}{2}+\frac{1}{v(t,T)}\log\frac{P(t,S)}{\kappa P(t,T)}\right)\\
&\quad-\frac{P(t,S)}{P(t,T)}\Phi\left(\frac{\log(X_t/\kappa)+(T-t)v^2(t,T)/2}{\sqrt{T-t}v(t,T)}\right)\\
&= -\kappa\Phi\left(\frac{\log(X_t/\kappa)-(T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right),
\end{aligned}$$

and

$$\xi_t^S = \frac{\partial C}{\partial x}(X_t,T-t,v(t,T)) = \Phi\left(\frac{\log(X_t/\kappa)+(T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right),$$

$t \in [0,T]$, and the hedging strategy is given by

$$\begin{aligned}
V_T &= \mathbb{E}\left[e^{-\int_t^T r_s ds}\left(P(T,S)-\kappa\right)^+\Big|\mathcal{F}_t\right]\\
&= V_0+\int_0^t \xi_s^T dP(s,T)+\int_0^t \xi_s^S dP(s,S)\\
&= V_0-\kappa\int_0^t \Phi\left(\frac{\log(X_t/\kappa)-(T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right)dP(s,T)\\
&\quad+\int_0^t \Phi\left(\frac{\log(X_t/\kappa)+(T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right)dP(s,S).
\end{aligned}$$

Consequently the bond option can be hedged by shortselling a bond with maturity T for the amount

$$\kappa\Phi\left(\frac{\log(X_t/\kappa)-(T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right),$$

and by buying a bond with maturity S for the amount

$$\Phi\left(\frac{\log(X_t/\kappa)+(T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right).$$

Exercise 11.4.

1. We have

$$S(T_i,T_i,T_j)=S(t,T_i,T_j)\exp\left(\int_t^{T_i}\sigma_{i,j}(s)dB_s^{i,j}-\frac{1}{2}\int_t^{T_i}|\sigma_{i,j}|^2(s)ds\right).$$

2. We have

$$\begin{aligned}
&P(t,T_i,T_j)\,\mathbb{E}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+\,\middle|\mathcal{F}_t\right]\\
&=P(t,T_i,T_j)\,\mathbb{E}_{i,j}\left[\left(S(t,T_i,T_j)\exp\left(\int_t^{T_i}\sigma_{i,j}(s)dB_s^{i,j}-\frac{1}{2}\int_t^{T_i}|\sigma_{i,j}|^2(s)d_s\right)-\kappa\right)^+\,\middle|\mathcal{F}_t\right]\\
&=P(t,T_i,T_j)\mathrm{B}l(\kappa,v(t,T_i)/\sqrt{T_i-t},0,T_i-t)\\
&=P(t,T_i,T_j)\left(S(t,T_i,T_j)\Phi\left(\frac{\log(x/K)}{v(t,T_i)}+\frac{v(t,T_i)}{2}\right)-\kappa\Phi\left(\frac{\log(x/K)}{v(t,T_i)}-\frac{v(t,T_i)}{2}\right)\right),
\end{aligned}$$

where

$$v^2(t,T_i)=\int_t^{T_i}|\sigma_{i,j}|^2(s)ds.$$

3. Integrate the self-financing condition (11.19) between 0 and t.
4. We have

$$\begin{aligned}
d\tilde{V}_t&=d\left(e^{-\int_0^t r_s ds}V_t\right)\\
&=-r_te^{-\int_0^t r_s ds V_t}dt+e^{-\int_0^t r_s ds}dV_t\\
&=-r_te^{-\int_0^t r_s ds}\sum_{k=i}^{j}\xi_t^kP(t,T_k),dt+e^{-\int_0^t r_s ds}\sum_{k=i}^{j}\xi_t^kdP(t,T_k)\\
&=\sum_{k=i}^{j}\xi_t^kd\tilde{P}(t,T_k),\qquad 0\le t\le T_i.
\end{aligned}$$

since

$$\frac{d\tilde{P}(t,T_k)}{\tilde{P}(t,T_k)} = \zeta_k(t)dt, \qquad k = i,\ldots,j.$$

5. We apply the Itô formula and the fact that

$$t \mapsto \mathbb{E}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+ \,\middle|\, \mathcal{F}_t\right]$$

and $(S_t)_{t\in\mathbb{R}_+}$ are both martingales under $\mathbb{P}_{i,j}$.

6. Use the fact that

$$\hat{V}_t = \mathbb{E}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+ \,\middle|\, \mathcal{F}_t\right],$$

and apply the result of Question 5.

7. Apply the Itô rule to $V_t = P(t,T_i,T_j)\hat{V}_t$ using Relation (11.17) and the result of Question 6.

8. We have

$$\begin{aligned}
dV_t &= S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i)) \\
&\quad \times \left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(\zeta_i(t)-\zeta_{k+1}(t)) + P(t,T_j)(\zeta_i(t)-\zeta_j(t))\right) dB_t \\
&\quad + \hat{V}_t dP(t,T_i,T_j) \\
&= S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i)) \\
&\quad \times \left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(\zeta_i(t)-\zeta_{k+1}(t)) + P(t,T_j)(\zeta_i(t)-\zeta_j(t))\right) dB_t \\
&\quad + \hat{V}_t\sum_{k=i}^{j-1}(T_{k+1}-T_k)\zeta_{k+1}(t)P(t,T_{k+1})dB_t \\
&= S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(\zeta_i(t)-\zeta_{k+1}(t))dB_t \\
&\quad + \frac{\partial C}{\partial x}(S_t, v(t,T_i))P(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t \\
&\quad + \hat{V}_t\sum_{k=i}^{j-1}(T_{k+1}-T_k)\zeta_{k+1}(t)P(t,T_{k+1})dB_t \\
&= S_t\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})dB_t \\
&\quad - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\zeta_{k+1}(t)dB_t
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial C}{\partial x}(S_t, v(t,T_i))P(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t \\
&+\hat{V}_t\sum_{k=i}^{j-1}(T_{k+1}-T_k)\zeta_{k+1}(t)P(t,T_{k+1})dB_t \\
=\ & S_t\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})dB_t \\
&+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\zeta_{k+1}(t)dB_t \\
&+\frac{\partial C}{\partial x}(S_t, v(t,T_i))P(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t.
\end{aligned}
$$

9. We have

$$
\begin{aligned}
d\tilde{V}_t &= S_t\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)\tilde{P}(t,T_{k+1})dB_t \\
&\quad+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)\sum_{k=i}^{j-1}(T_{k+1}-T_k)\tilde{P}(t,T_{k+1})\zeta_{k+1}(t)dB_t \\
&\quad+\frac{\partial C}{\partial x}(S_t, v(t,T_i))\tilde{P}(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t \\
&= (\tilde{P}(t,T_i)-\tilde{P}(t,T_j))\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))dB_t \\
&\quad+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)d\tilde{P}(t,T_i,T_j) \\
&\quad+\frac{\partial C}{\partial x}(S_t, v(t,T_i))\tilde{P}(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t \\
&= (\zeta_i(t)\tilde{P}(t,T_i)-\zeta_j(t)\tilde{P}(t,T_j))\frac{\partial C}{\partial x}(S_t, v(t,T_i))dB_t \\
&\quad+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)d\tilde{P}(t,T_i,T_j) \\
&= \frac{\partial C}{\partial x}(S_t, v(t,T_i))d(\tilde{P}(t,T_i)-\tilde{P}(t,T_j)) \\
&\quad+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)d\tilde{P}(t,T_i,T_j).
\end{aligned}
$$

10. We have

$$
\begin{aligned}
&\frac{\partial C}{\partial x}(x,\tau,v) = \frac{\partial}{\partial x}\left[x\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)-\kappa\Phi\left(-\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)\right] \\
&= x\frac{\partial}{\partial x}\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)-\kappa\frac{\partial}{\partial x}\Phi\left(-\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)+\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)
\end{aligned}
$$

$$= x\frac{e^{-\frac{1}{2}\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)^2}}{\sqrt{2\pi}}\left(\frac{1}{vx}\right) - \kappa\frac{e^{-\frac{1}{2}\left(-\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)^2}}{\sqrt{2\pi}}\left(\frac{1}{vx}\right) + \Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)$$
$$= \Phi\left(\frac{\log(x/\kappa)}{v}+\frac{v}{2}\right).$$

11. We have

$$\begin{aligned} d\tilde{V}_t &= \frac{\partial C}{\partial x}(S_t, v(t,T_i))d(\tilde{P}(t,T_i)-\tilde{P}(t,T_j)) \\ &\quad + \left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)d\tilde{P}(t,T_i,T_j) \\ &= \Phi\left(\frac{\log(S_t/K)}{v(t,T_i)}+\frac{v(t,T_i)}{2}\right)d(\tilde{P}(t,T_i)-\tilde{P}(t,T_j)) \\ &\quad -\kappa\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)}-\frac{v(t,T_i)}{2}\right)d\tilde{P}(t,T_i,T_j). \end{aligned}$$

12. We compare the results of Questions 11 and 4.

Chapter 12

Exercise 12.1.Defaultable bonds.

1. Use the fact that $(r_t,\lambda_t)_{t\in[0,T]}$ is a Markov process.
2. Use the tower property for the conditional expectation under $\mathcal{F}_t$.
3. We have

$$\begin{aligned} & d\left(e^{-\int_0^t(r_s+\lambda_s)ds}P(t,T)\right) \qquad (17.8) \\ &= -(r_t+\lambda_t)e^{-\int_0^t(r_s+\lambda_s)ds}P(t,T)dt + e^{-\int_0^t(r_s+\lambda_s)ds}dP(t,T) \\ &= -(r_t+\lambda_t)e^{-\int_0^t(r_s+\lambda_s)ds}P(t,T)dt + e^{-\int_0^t(r_s+\lambda_s)ds}dF(t,r_t,\lambda_t) \\ &= -(r_t+\lambda_t)e^{-\int_0^t(r_s+\lambda_s)ds}P(t,T)dt + e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial x}(t,r_t,\lambda_t)dr_t \\ &\quad + e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial y}(t,r_t,\lambda_t)d\lambda_t + \frac{1}{2}e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t)dt \\ &\quad + \frac{1}{2}e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t)dt \\ &\quad + e^{-\int_0^t(r_s+\lambda_s)ds}\rho\frac{\partial^2 F}{\partial x\partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t)dt + e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial t}(t,r_t,\lambda_t)dt \\ &= e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial x}(t,r_t,\lambda_t)\sigma_1(t,r_t)dB_t^1 + e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial y}(t,r_t,\lambda_t)\sigma_2(t,\lambda_t)dB_t^2 \end{aligned}$$

$$e^{-\int_0^t (r_s+\lambda_s)ds}\left(-(r_t+\lambda_t)P(t,T)+\frac{\partial F}{\partial x}(t,r_t,\lambda_t)\mu_1(t,r_t)+\frac{\partial F}{\partial y}(t,r_t,\lambda_t)\mu_2(t,\lambda_t)\right.$$
$$+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t)+\frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t)$$
$$\left.+\rho\frac{\partial^2 F}{\partial x\partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t)+\frac{\partial F}{\partial t}(t,r_t,\lambda_t)\right)dt,$$

hence the bond pricing PDE is

$$\begin{aligned}&-(x+y)F(t,x,y)+\mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y)\\&+\mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y)+\frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y)\\&+\frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y)+\rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x\partial y}(t,x,y)+\frac{\partial F}{\partial t}(t,r_t,\lambda_t)=0\end{aligned}\tag{119}$$

4. We have

$$\begin{aligned}\int_0^t r_s ds &= \frac{1}{a}\left(\sigma B_t^1 - r_t\right)\\&=\frac{\sigma}{a}\left(B_t^1-\int_0^t e^{-a(t-s)}dB_s^1\right)\\&=\frac{\sigma}{a}\int_0^t(1-e^{-a(t-s)})dB_s^1,\end{aligned}$$

hence

$$\begin{aligned}&\int_t^T r_s ds=\int_0^T r_s ds-\int_0^t r_s ds\\&=\frac{\sigma}{a}\int_0^T(1-e^{-a(T-s)})dB_s^1-\frac{\sigma}{a}\int_0^t(1-e^{-a(t-s)})dB_s^1\\&=-\frac{\sigma}{a}\left(\int_0^t(e^{-a(T-s)}-e^{-a(t-s)})dB_s^1+\int_t^T(e^{-a(T-s)}-1)dB_s^1\right)\\&=-\frac{\sigma}{a}(e^{-a(T-t)}-1)\int_0^t e^{-a(t-s)}dB_s^1-\frac{\sigma}{a}\int_t^T(e^{-a(T-s)}-1)dB_s^1\\&=-\frac{1}{a}(e^{-a(T-t)}-1)r_t-\frac{\sigma}{a}\int_t^T(e^{-a(T-s)}-1)dB_s^1.\end{aligned}$$

The answer for λ_t is similar.

5. As a consequence of the previous question we have

$$\mathbb{E}\left[\int_t^T r_s ds+\int_t^T\lambda_s ds\middle|\mathcal{F}_t\right]=C(a,t,T)r_t+C(b,t,T)\lambda_t,$$

and

$$\begin{aligned}
&\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \Big| \mathcal{F}_t\right] = \\
&= \mathrm{Var}\left[\int_t^T r_s ds \Big| \mathcal{F}_t\right] + \mathrm{Var}\left[\int_t^T \lambda_s ds \Big| \mathcal{F}_t\right] + 2\,\mathrm{Cov}\left(\int_t^T X_s ds, \int_t^T Y_s ds \Big| \mathcal{F}_t\right) \\
&= \frac{\sigma^2}{a^2}\int_t^T (e^{-a(T-s)}-1)^2 ds + 2\rho\frac{\sigma\eta}{ab}\int_t^T (e^{-a(T-s)}-1)(e^{-b(T-s)}-1)ds \\
&\quad + \frac{\eta^2}{b^2}\int_t^T (e^{-b(T-s)}-1)^2 ds \\
&= \sigma^2\int_t^T C^2(a,s,T)ds + 2\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds + \eta^2\int_t^T C^2(b,sT)ds,
\end{aligned}$$

from the Itô isometry.

6. We have

$$\begin{aligned}
P(t,T) &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T r_s ds - \int_t^T \lambda_s ds\right)\Big|\mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}\exp\left(-\mathbb{E}\left[\int_t^T r_s ds \Big| \mathcal{F}_t\right] - \mathbb{E}\left[\int_t^T \lambda_s ds \Big| \mathcal{F}_t\right]\right) \\
&\quad \times \exp\left(\frac{1}{2}\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \Big| \mathcal{F}_t\right]\right) \\
&= \mathbf{1}_{\{\tau>t\}}\exp\left(-C(a,t,T)r_t - C(b,t,T)\lambda_t\right) \\
&\quad \times \exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)e^{-b(T-s)}ds\right) \\
&\quad \times \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right).
\end{aligned}$$

7. This is a direct consequence of the answers to Questions 3 and 6.
8. The above analysis shows that

$$\begin{aligned}
\mathbb{P}(\tau > T \mid \mathcal{G}_t) &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right)\Big|\mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}\exp\left(-C(b,t,T)\lambda_t + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right),
\end{aligned}$$

for $a = 0$ and

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right] = \exp\left(-C(a,t,T)r_t + \frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds\right),$$

for $b = 0$, and this implies

$$\begin{aligned} U(t,T) &= \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right) \\ &= \exp\left(\rho\frac{\sigma\eta}{ab}\left(T - t - C(a,t,T) - C(b,t,T) + C(a+b,t,T)\right)\right). \end{aligned}$$

9. We have

$$\begin{aligned} f(t,T) &= -\mathbf{1}_{\{\tau>t\}}\frac{\partial \log P(t,T)}{\partial T} \\ &= \mathbf{1}_{\{\tau>t\}}\left(r_t e^{-a(T-t)} - \frac{\sigma^2}{2}C^2(a,t,T) + \lambda_t e^{-b(T-t)} - \frac{\eta^2}{2}C^2(b,t,T) - \rho\sigma\eta C(a,t,T)C(b,t,T)\right). \end{aligned}$$

10. We use the relation

$$\begin{aligned} \mathbb{P}(\tau > T \mid \mathcal{G}_t) &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right)\Big|\mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}\exp\left(-C(b,t,T)\lambda_t + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right) \\ &= \mathbf{1}_{\{\tau>t\}}e^{-\int_t^T f_2(t,u)du}, \end{aligned}$$

where $f_2(t,T)$ is the Vasiçek forward rate corresponding to λ_t, i.e.

$$f_2(t,u) = \lambda_t e^{-b(u-t)} - \frac{\eta^2}{2}C^2(b,t,u).$$

11. In this case we have $\rho = 0$ and

$$P(t,T) = \mathbb{P}(\tau > T \mid \mathcal{G}_t)\,\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right],$$

since $U(t,T) = 0$.

Chapter 13

Exercise 13.1.We have

$$
\begin{aligned}
\operatorname{var}[Y_T] &= \mathbb{E}\left[\left(\sum_{k=1}^{N_T} Z_k - \mathbb{E}[Y_T]\right)^2\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[\left(\sum_{k=1}^{N_T} Z_k - \lambda t\,\mathbb{E}[Z_1]\right)^2 \Big| N_T = k\right] \mathbb{P}(N_T = k) \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!}\, \mathbb{E}\left[\left(\sum_{k=1}^{n} Z_k - \lambda t\,\mathbb{E}[Z_1]\right)^2\right] \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!}\, \mathbb{E}\left[2 \sum_{1\leq k<l\leq n} Z_k Z_l + \sum_{k=1}^{n} |Z_k|^2 - 2\lambda t\,\mathbb{E}[Z_1] \sum_{k=1}^{n} Z_k + \lambda^2 t^2 (\mathbb{E}[Z_1])^2\right] \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} (n(n-1)(\mathbb{E}[Z_1])^2 + n\,\mathbb{E}[|Z_1|^2] - 2n\lambda t(\mathbb{E}[Z_1])^2 + \lambda^2 t^2 (\mathbb{E}[Z_1])^2) \\
&= \mathrm{e}^{-\lambda t} (\mathbb{E}[Z_1])^2 \sum_{n=2}^{\infty} \frac{\lambda^n t^n}{(n-2)!} + \mathrm{e}^{-\lambda t}\,\mathbb{E}[|Z_1|^2] \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{(n-1)!} \\
&\quad -2\mathrm{e}^{-\lambda t} \lambda t (\mathbb{E}[Z_1])^2 \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{(n-1)!} + \lambda^2 t^2 (\mathbb{E}[Z_1])^2) \\
&= \lambda t\,\mathbb{E}[|Z_1|^2],
\end{aligned}
$$

or, using the characteristic function,

$$
\begin{aligned}
\operatorname{var}[Y_T] &= \mathbb{E}[|Y_T|^2] - (\mathbb{E}[Y_T])^2 \\
&= -\frac{d^2}{d\alpha^2}\,\mathbb{E}[\mathrm{e}^{i\alpha Y_T}]_{|\alpha=0} - t^2 (\mathbb{E}[Z_1])^2 \\
&= \lambda t \int_{-\infty}^{\infty} |y|^2 \mu(dy) \\
&= \lambda t\,\mathbb{E}[|Z_1|^2].
\end{aligned}
$$

Exercise 13.2.

1. We have

$$
\begin{aligned}
dS_t &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_t - S_{t^-}) dN_t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 e^{\mu t + \sigma W_t + Y_t} - S_0 e^{\mu t + \sigma W_t + Y_{t^-}}) dN_t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 e^{\mu t + \sigma W_t + Y_{t^-} + Z_{N_t}} - e^{\mu t + \sigma W_t + Y_{t^-}}) dN_t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + S_{t^-} (e^{Z_{N_t}} - 1) dN_t,
\end{aligned}
$$

hence the jumps of S_t are given by the sequence $(e^{Z_k}-1)_{k\geq 1}$.

2. The discounted process $e^{-rt}S_t$ satisfies

$$d(e^{-rt}S_t) = e^{-rt}\left(\mu - r + \frac{1}{2}\sigma^2\right)S_t dt + \sigma e^{-rt}S_t dW_t + e^{-rt}S_{t^-}(e^{Z_{N_t}}-1)dN_t.$$

Hence by the Girsanov theorem, choosing u, $\tilde{\lambda}$, $\tilde{\nu}$ such that

$$\mu - r + \frac{1}{2}\sigma^2 = u - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[e^{Z_1}-1],$$

shows that

$$d(e^{-rt}S_t) = \sigma e^{-rt}S_t(dW_t + u dt) + e^{-rt}S_{t^-}((e^{Z_{N_t}}-1)dN_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[e^{Z_1}-1]dt)$$

is a martingale under $(\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}})$.

Chapter 14

Exercise 14.1. Consider a forward call option with payoff $S_T - K$ on a jump diffusion risky asset $(S_t)_{t\in\mathbb{R}_+}$ given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_{t^-}dY_t.$$

1. Independently of the choice of a risk-neutral measure $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$ we have

$$\begin{aligned} e^{-r(T-t)}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[S_T - K \mid \mathcal{F}_t] &= e^{rt}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[e^{-rT}S_T \mid \mathcal{F}_t] - Ke^{-r(T-t)} \\ &= e^{rt}e^{-rt}S_t - Ke^{-r(T-t)} \\ &= S_t - Ke^{-r(T-t)} \\ &= f(t, S_t), \end{aligned}$$

for

$$f(t,x) = x - Ke^{-r(T-t)}, \qquad t, x > 0.$$

2. Clearly, holding one unit of the risky asset and shorting Ke^{-rT} units of the riskless asset will hedge the payoff $S_T - K$, and this hedging strategy is self-financing because it is constant in time.
3. Since $\dfrac{\partial f}{\partial x}(t,x) = 1$ we have

$$\xi_t = \frac{\sigma^2\dfrac{\partial f}{\partial x}(t, S_{t^-}) + \dfrac{a\tilde{\lambda}}{S_{t^-}}(f(t, S_{t^-}(1+a)) - f(t, S_{t^-}))}{\sigma^2 + a^2\tilde{\lambda}}$$

$$
\begin{aligned}
&= \frac{\sigma^2 + \dfrac{a\tilde{\lambda}}{S_{t^-}}(S_{t^-}(1+a) - S_{t^-})}{\sigma^2 + a^2\tilde{\lambda}} \\
&= 1, \qquad t \in [0,T],
\end{aligned}
$$

which coincides with the result of Question 2.

Exercise 14.2.

(i) We have

$$S_t = S_0 \exp\left(\mu t + \sigma B_t - \frac{1}{2}\sigma^2 t\right)(1+\eta)^{N_t}.$$

(ii) We need to take

$$\mu - r + \lambda\eta = 0.$$

(iii) We have

$$
\begin{aligned}
&e^{-r(T-t)}\mathbb{E}^*[(S_T-\kappa)^+ \mid S_t] \\
&= e^{-r(T-t)}\mathbb{E}^*\left[\left(S_0 \exp\left(\mu T + \sigma B_T - \frac{1}{2}\sigma^2 T\right)(1+\eta)^{N_T} - \kappa\right)^+ \Big| S_t\right] \\
&= e^{-r(T-t)}\mathbb{E}^*\left[\left(S_t e^{\mu(T-t)+\sigma(B_T-B_t)-\frac{1}{2}\sigma^2(T-t)}(1+\eta)^{N_T-N_t} - \kappa\right)^+ \Big| S_t\right] \\
&= e^{-r(T-t)}\sum_{n=0}^{\infty}\mathbb{E}^*\left[\left(S_t e^{\mu(T-t)+\sigma(B_T-B_t)-\frac{1}{2}\sigma^2(T-t)}(1+\eta)^{n} - \kappa\right)^+ \Big| S_t\right]\mathbb{P}(N_T-N_t=n) \\
&= e^{-(r+\lambda)(T-t)}\sum_{n=0}^{\infty}\mathbb{E}^*\left[\left(S_t e^{\mu(T-t)+\sigma(B_T-B_t)-\frac{1}{2}\sigma^2(T-t)}(1+\eta)^{n} - \kappa\right)^+ \Big| S_t\right]\frac{(\lambda(T-t))^n}{n!} \\
&= e^{-(r+\lambda)(T-t)}\sum_{n=0}^{\infty}\mathrm{Bl}(S_t(1+\eta)^n,\mu,\sigma^2,\kappa)\frac{(\lambda(T-t))^n}{n!} \\
&= e^{-(r+\lambda)(T-t)}\sum_{n=0}^{\infty}\left(S_t(1+\eta)^n\Phi(d_+) - \kappa e^{-\mu(T-t)}\Phi(d_-)\right)\frac{(\lambda(T-t))^n}{n!},
\end{aligned}
$$

with

$$
\begin{aligned}
d_+ &= \frac{\log(S_t(1+\eta)^n/\kappa) + (\mu+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_- &= \frac{\log(S_t(1+\eta)^n/\kappa) + (\mu-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},
\end{aligned}
$$

and $\mu = r - \lambda\eta$.

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Chapter 16

Exercise 15.1.

1. We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{\infty} k\mathbb{P}(X=k) \\
&= (1-p)\lim_{n\to\infty}\sum_{k=0}^{n} kp^k \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} kp^k - \lim_{n\to\infty}\sum_{k=0}^{n} kp^{k+1} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} kp^k - \lim_{n\to\infty}\sum_{k=0}^{n}(k+1)p^{k+1} + \lim_{n\to\infty}\sum_{k=0}^{n} p^{k+1} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} kp^k - \lim_{n\to\infty}\sum_{k=1}^{n+1} kp^k - 1 + \lim_{n\to\infty}\sum_{k=0}^{n+1} p^k \\
&= -1 + \lim_{n\to\infty}\sum_{k=0}^{n+1} p^k \\
&= -1 + \sum_{k=0}^{\infty} p^k \\
&= -1 + \frac{1}{1-p}\mathbb{P}(X\in\mathbb{N}) \\
&= -1 + \frac{1}{1-p} \\
&= \frac{p}{1-p}.
\end{aligned}$$

2. We have

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=0}^{\infty} k^2\mathbb{P}(X=k) \\
&= (1-p)\lim_{n\to\infty}\sum_{k=0}^{n} k^2p^k \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} k^2p^k - \lim_{n\to\infty}\sum_{k=0}^{n} k^2p^{k+1} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} k^2p^k - \lim_{n\to\infty}\sum_{k=0}^{n}(k+1)^2p^{k+1} + 2\lim_{n\to\infty}\sum_{k=0}^{n} kp^{k+1} + \lim_{n\to\infty}\sum_{k=0}^{n} p^{k+1}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{k=0}^{n} k^2 p^k - \lim_{n\to\infty} \sum_{k=1}^{n+1} k^2 p^k + \frac{2p}{1-p}\,\mathbb{E}[X] + \frac{p}{1-p}\mathbb{P}(X \in \mathbb{N}) \\
&= \frac{2p^2}{(1-p)^2} + \frac{p}{1-p} \\
&= \frac{p+p^2}{(1-p)^2}.
\end{aligned}
$$

From this we deduce

$$
\operatorname{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{p}{(1-p)^2}.
$$

Exercise 15.2.

1. We have

$$
\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \lim_{A\to+\infty} \int_{-A}^{A} y e^{-\frac{y^2}{2}} dy \\
&= 0,
\end{aligned}
$$

by symmetry of the function $y \mapsto y e^{-\frac{y^2}{2}}$. Note that we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} |y| e^{-\frac{y^2}{2}} dy &= \lim_{A\to+\infty} \int_{-A}^{A} |y| e^{-\frac{y^2}{2}} dy \\
&= 2 \lim_{A\to+\infty} \int_{0}^{A} y e^{-\frac{y^2}{2}} dy \\
&= -2 \lim_{A\to+\infty} \left[e^{-\frac{y^2}{2}} \right]_0^A \\
&= 2 \lim_{A\to+\infty} (1 - e^{-\frac{A^2}{2}}) \\
&= 2 \\
&< \infty,
\end{aligned}
$$

hence the function $y \mapsto y e^{-\frac{y^2}{2}}$ is integrable on $\mathbb{R}$ and the above computation of $\mathbb{E}[X]$ is valid.

2. We have

$$
\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-m)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (y+m) e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy + \frac{m}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \frac{m}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= m \int_{-\infty}^{\infty} f(y) dy \\
&= m\mathbb{P}(X \in \mathbb{R}) \\
&= m.
\end{aligned}
$$

Exercise 15.3. We have

$$
\begin{aligned}
\mathbb{P}(e^X > c) &= \mathbb{P}(X > \log c) \\
&= \int_{\log c}^{\infty} e^{-y^2/(2\eta^2)} \frac{dy}{\sqrt{2\pi\eta^2}} \\
&= \int_{(\log c)/\eta}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} \\
&= 1 - \Phi((\log c)/\eta) \\
&= \Phi(-(\log c)/\eta).
\end{aligned}
$$

References

1. Y. Achdou and O. Pironneau. *Computational methods for option pricing*, volume 30 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.
2. H. Bermin. Essays on Lookback options: a Malliavin calculus approach. PhD thesis, Lund University, 1998.
3. T.R. Bielecki and M. Rutkowski. *Credit risk: modelling, valuation and hedging*. Springer Finance. Springer-Verlag, Berlin, 2002.
4. T. Bjork. *Arbitrage Theory in Continuous Time*, volume 121 of *Oxford Finance*. Oxford University Press, 2004.
5. T. Björk. On the geometry of interest rate models. In *Paris-Princeton Lectures on Mathematical Finance 2003*, volume 1847 of *Lecture Notes in Math.*, pages 133–215. Springer, Berlin, 2004.
6. A. Brace, D. Gatarek, and M. Musiela. The market model of interest rate dynamics. *Math. Finance*, 7(2):127–155, 1997.
7. D. Brigo and F. Mercurio. *Interest rate models—theory and practice*. Springer Finance. Springer-Verlag, Berlin, second edition, 2006.
8. P. Carr and M. Schröder. Bessel processes, the integral of geometric Brownian motion, and Asian options. *Theory Probab. Appl.*, 48(3):400–425, 2004.
9. R.R. Chen, X. Cheng, F.J. Fabozzi, and B. Liu. An explicit, multi-factor credit default swap pricing model with correlated factors. *Journal of Financial and Quantitative Analysis*, 43(1):123–160, 2008.
10. R.R. Chen and J.Z. Huang. Credit spread bounds and their implications for credit risk modeling. Working paper, Rutgers University and Penn State University, 2001.
11. R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.
12. J. Cox, S. Ross, and M. Rubenstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7:87–106, 1979.
13. J C Cox, J E Ingersoll, and S A Ross. A theory of the term structure of interest rates. *Econometrica*, 53:385–407, 1985.
14. R.-A. Dana and M. Jeanblanc. *Financial markets in continuous time*. Springer Finance. Springer-Verlag, Berlin, 2003.
15. L.U. Dothan. On the term structure of interest rates. *Jour. of Fin. Ec.*, 6:59–69, 1978.
16. R.M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

17. D. Duffie and K.J. Singleton. *Credit risk*. Princeton series in finance. Princeton University Press, Princeton, NJ, 2003.
18. B. Dupire. Pricing with a smile. *Risk*, 7(1):18–20, 1994.
19. J. Eriksson and J. Persson. Pricing turbo warrants. Preprint, submitted to Journal of Economic Dynamics and Control, 2006.
20. G. B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1999.
21. H. Föllmer and A. Schied. *Stochastic finance*, volume 27 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2004.
22. H. Geman, N. El Karoui, and J.-C. Rochet. Changes of numéraire, changes of probability measure and option pricing. *J. Appl. Probab.*, 32(2):443–458, 1995.
23. X. Guo, R. Jarrow, and C. Menn. A note on Lando's formula and conditional independence. Preprint, 2007.
24. D. Heath, R. Jarrow, and A. Morton. Bond pricing and the term structure of interest rates: a new methodology. *Econometrica*, 60:77–105, 1992.
25. J. Hull and A. White. Pricing interest rate derivative securities. *The Review of Financial Studies*, 3:537–592, 1990.
26. N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. North-Holland, 1989.
27. F. Jamshidian. Sorting out swaptions. *Risk*, 9(3):59–60, 1996.
28. M. Jeanblanc, M. Yor, and M. Chesney. *Mathematical methods for financial markets*. Springer Finance. Springer-Verlag London Ltd., London, 2009.
29. Y. El Khatib. Contributions to the study of discontinuous markets via the Malliavin calculus. Thèse, Université de La Rochelle, 2002.
30. D. Lamberton and B. Lapeyre. *Introduction to stochastic calculus applied to finance*. Chapman & Hall, London, 1996.
31. D. Lando. On Cox processes and credit risky securities. *Review of Derivative Research*, 2:99–120, 1998.
32. F.A. Longstaff and E.S. Schwartz. Valuing american options by simulation: a simple least-squares approach. *Review of Financial Studies*, 14:113–147, 2001.
33. W. Margrabe. The value of an option to exchange one asset for another. *The Journal of Finance*, XXXIII(1):177–186, 1978.
34. R. Merton. Theory of rational option pricing. *Bell Journal of Economics*, 4(1):141–183, 1973.
35. G. Di Nunno, B. Øksendal, and F. Proske. *Malliavin Calculus for Lévy Processes with Applications to Finance*. Universitext. Springer-Verlag, Berlin, 2009.
36. C. Pintoux and N. Privault. The Dothan pricing model revisited. *Math. Finance*, 21:355–363, 2011.
37. N. Privault. *An Elementary Introduction to Stochastic Interest Rate Modeling*. Advanced Series on Statistical Science & Applied Probability, 12. World Scientific Publishing Co., Singapore, 2008.
38. N. Privault. Stochastic analysis of Bernoulli processes. *Probab. Surv.*, 5:435–483 (electronic), 2008.
39. N. Privault. *Stochastic Analysis in Discrete and Continuous Settings*, volume 1982 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
40. N. Privault and T.R. Teng. Forward hedging of swaptions and related derivatives in bond markets. Preprint, 2010.
41. Ph. Protter. *Stochastic integration and differential equations. A new approach*. Springer-Verlag, Berlin, 1990.
42. Ph. Protter. A partial introduction to financial asset pricing theory. *Stochastic Process. Appl.*, 91(2):169–203, 2001.
43. Ph. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2005.
44. D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer-Verlag, 1994.

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45. L.C.G. Rogers and Z. Shi. The value of an Asian option. *J. Appl. Probab.*, 32(4):1077–1088, 1995.
46. J. Ruiz de Chávez. Predictable representation of the binomial process and application to options in finance. In *XXXIII National Congress of the Mexican Mathematical Society (Spanish) (Saltillo, 2000)*, volume 29 of *Aportaciones Mat. Comun.*, pages 223–230. Soc. Mat. Mexicana, México, 2001.
47. J. Schoenmakers. *Robust LIBOR modelling and pricing of derivative products*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2005.
48. A.N. Shiryaev. *Probability*, volume 95 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.
49. S.E. Shreve. *Stochastic calculus for finance. II*. Springer Finance. Springer-Verlag, New York, 2004. Continuous-time models.
50. J.M. Steele. *Stochastic calculus and financial applications*, volume 45 of *Applications of Mathematics*. Springer-Verlag, New York, 2001.
51. O. Vašiček. An equilibrium characterisation of the term structure. *Journal of Financial Economics*, 5:177–188, 1977.
52. D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.
53. H.Y. Wong and C.M. Chan. Turbo warrants under stochastic volatility. *Quant. Finance*, 8(7):739–751, 2008.

Index

absence of arbitrage, 205
adapted process, 70
affine model, 204
American option
  finite expiration, 170
  perpetual, 159
American options perpetual options, 159
annuity numeraire, 242
arbitrage, 11
  continuous time, 91
  discrete time, 29
Asian options, 123

BGM model, 231
Black-Scholes
  formula, 95, 231
  PDE, 93, 95, 101, 104, 108, 297
Black-Scholes PDE
  with jumps, 287
bond pricing PDE, 206, 227, 369
Brownian motion, 61

call option, 4
cap, 242
cap pricing, 242
caplet, 240
caplet pricing, 240
CEV model, 204
characteristic function, 321
CIR model, 203
completeness
  continuous time, 92
  discrete time, 35
compound Poisson process, 270
conditional expectation, 31
contingent claim, 14, 29
continuous-time model, 87
correlation problem, 224
credit default swap, 260
CRR model, 36

defaultable bond, 259

Euler discretization, 301
event, 306
exotic options
  continuous time, 123
  discrete time, 50
expectation, 317
exponential distribution, 255
exponential model, 289

finite differences
  explicit method, 295, 298
  implicit method, 296, 299
floorlet, 240
forward
  measure, 238
  rate, 209
forward rate, 209

Gaussian
  distribution, 95
  random variable, 322
geometric Brownian motion, 78
Girsanov theorem, 113
Girsanov theorem for jump processes, 279

heat equation, 102, 295
hedging, 43, 47, 118
hedging strategy, 119
hedging with jumps, 290
hitting probability, 156

hitting time, 152
HJM
  condition, 213
  model, 212
Hull-White model, 212

instantaneous forward rate, 210
Itô
  formula, 130
  integral, 109
  isometry, 71
  stochastic integral, 66, 70, 71
Itô formula
  with jumps, 273

LIBOR
  model, 229
lookback options, 125

market completeness, 16
martingale, 31, 149
  method, 115
  submartingale, 149
  supermartingale, 149
mean hitting time, 158
mean reversion, 203
Merton model, 290
Milshtein discretization, 302
Musiela notation, 211

Nelson-Siegel, 221

optimal stopping, 170

payoff function, 5
Poisson
  compound martingale, 270
Poisson process, 265
portfolio, 10, 26, 88
predictable representation, 118, 121
pricing, 39, 44
pricing with jumps, 286
probability
  distribution, 312
  measure, 308
  space, 305
put option, 4, 5

random variable, 310
risk-neutral measure, 13, 285
  continuous time, 111
risk-neutral measures
  continuous time, 91
  discrete time, 34
riskless asset, 57, 87

self-financing portfolio
  continuous time, 87, 88
  discrete time, 27
short-selling, 10
stochastic calculus, 75
stochastic differential equations, 81
stochastic integral, 65, 70
stochastic integral with jump, 272
stopped process, 153
stopping time, 149, 257
stopping time theorem, 153
survival probability, 256
Svensson parametrization, 222
swap, 211
  measure, 243, 251
  rate, 211, 230
swaption, 244

tenor structure, 237
two-factor model, 227

variational PDE, 173
Vasicek model, 203
volatility
  historical, 139
  implied, 140
  local, 146

zero-coupon bond, 205

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N. Privault

This document is an introduction to the stochastic calculus and PDE approaches to the pricing and hedging of financial derivatives, including vanilla options and some exotic options. The presentation is done both in discrete and continuous-time financial models, with an emphasis on the complementarity between algebraic and probabilistic methods. It also covers some interest rate derivatives and stochastic models with jumps.

386

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