A Circle Fitting Procedure and Its Error Analysis

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Abstract—**An efficient circle fitting procedure and its general random-error analysis are described. The first-order random errors of the center coordinates and the radius of the fitted circle are discussed in detail. The effect of data point distribution along the circle is investigated, and for an important microwave application (sliding termination measurements) the frequency dependence is also evaluated.**

The effects of the second-order error terms are also discussed and general formulas are given. Finally an estimation of data point error is provided.

I. Introduction

In a large number of measurement problems the useful information is obtained by fitting circles to the measured data points. Such fitting problems often occur in microwave engineering (e.g., [1]–[3]) and more generally in such problems where a complex bilinear transformation is involved.

This paper describes and analyzes an efficient [5] (i.e., easily implemented) least squares circle fitting procedure. In particular this procedure is examined for the effect that different data sampling schemes have on the estimators of the circle's parameters.

It is to be assumed that the only errors affecting the measured values are random and that the random errors are statistically independent.

The estimators of the hypothesized circle's radius and center coordinates as arrived at in this paper are not linear in the measured values, but one may fruitfully examine the effect of measurement errors on the estimators by means of a first-order approximation, i.e., the variances of the estimators are then linear in the variances of the measurement errors.

Adopting this approach—a reasonable one if the random errors are small compared to the radius of the circle—the estimators are considered for the effect that various data sampling schemes have on their variances. In this regard a typical microwave example—the sliding termination—is discussed; this discussion includes frequency-dependence of the errors, which is important in the broad-band measuring systems.

Finally some thought is given to a generalization using higher order error terms.

II. An Efficient Circle Fitting Method

Let (x_i,y_i) represent the x-y coordinates of the ith measured data point, N the number of the data points $(1 \le i \le N)$, (A,B) the coordinates of the center, and R the radius of the circle (Fig. 1).

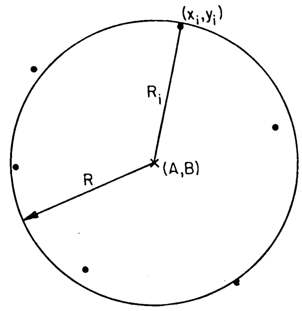

Fig. 1. The measured points and the parameters of the fitted circle.

A least squares error criterion for circle fitting is

$$\sum_{i=1}^{N} (R_i - R)^2 = \min \tag{1}$$

where

$$R_i = \sqrt{(x_i - A)^2 + (y_i - B)^2} \tag{2}$$

and $N \ge 3$.

Unfortunately, this expression is hard to handle analytically and, consequently, in accordance with Delonge [3] a modified least square error criterion will be used. This is

$$\sum_{i=1}^{N} (R_i^2 - R^2)^2 = \min \tag{3}$$

where R_i is given by (2). Note, that for small error (i.e., for $R_i \simeq R$), (3) gives a result close to (1) because of the factorization of (3),

$$\sum_{i=1}^{N} [(R_i + R)^2(R_i - R)^2] = \min \tag{4}$$

and the fact that $R_i + R \simeq 2R$ is nearly constant.

The criterion (3) is attractive because there are no analytical difficulties with it, and the measurement evaluation, as well as the error analysis, is simpler. Of course, the usefulness of this criterion should be supported by the analysis.

Equation (3) can be rewritten as

$$u = \sum_{i=1}^{N} [(x_i - A)^2 + (y_i - B)^2 - R^2]^2 = \min \tag{5}$$

where A, B, and R are to be determined. The sum u is a second-order function of A, B, and R, without constraints, hence the extremum can be simply evaluated

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by making the derivatives equal to zero

$$\frac{\partial u}{\partial A} = \frac{\partial u}{\partial B} = \frac{\partial u}{\partial R} = 0. \tag{6}$$

In this way

$$\frac{\partial u}{\partial R} = -4R \sum_{i=1}^{N} [(x_i - A)^2 + (y_i - B)^2 - R^2] = 0 \tag{7}$$

$$\frac{\partial u}{\partial A} = -4 \sum_{i=1}^{N} \{[(x_i - A)^2 + (y_i - B)^2 - R^2](x_i - A)\} = 0 \tag{8}$$

$$\frac{\partial u}{\partial B} = -4 \sum_{i=1}^{N} \{[(x_i - A)^2 + (y_i - B)^2 - R^2](y_i - B)\} = 0 \tag{9}$$

or after the possible simplifications

$$\sum_{i=1}^{N} [(x_i - A)^2 + (y_i - B)^2 - R^2] = 0 \tag{10}$$

$$\sum_{i=1}^{N} [(x_i - A)^2 x_i + (y_i - B)^2 x_i - R^2 x_i] = 0 \tag{11}$$

$$\sum_{i=1}^{N} [(x_i - A)^2 y_i + (y_i - B)^2 y_i - R^2 y_i] = 0. \tag{12}$$

Introducing the new variable C

$$C = R^2 - A^2 - B^2 \tag{13}$$

the equations (10)–(12) are rewritten as a *linear* equation system for A, B, and C, which are easily solved for A, B, and C, e.g., by using the classical Gauss method or some modified version of it.

$$A2\Sigma x_i + B2\Sigma y_i + CN = \Sigma(x_i^2 + y_i^2) \tag{14}$$

$$A2\Sigma x_i^2 + B2\Sigma x_i y_i + C\Sigma x_i = \Sigma(x_i^3 + x_i y_i^2) \tag{15}$$

$$A2\Sigma x_i y_i + B2\Sigma y_i^2 + C\Sigma y_i = \Sigma(x_i^2 y_i + y_i^3). \tag{16}$$

The summation of i is from 1 to N unless otherwise indicated.

Introducing the matrix D and the vectors E and Q, respectively,

$$D = \begin{pmatrix} 2\Sigma x_i & 2\Sigma y_i & N \\ 2\Sigma x_i^2 & 2\Sigma x_i y_i & \Sigma x_i \\ 2\Sigma x_i y_i & 2\Sigma y_i^2 & \Sigma y_i \end{pmatrix} \tag{17}$$

$$E = \begin{pmatrix} \Sigma(x_i^2 + y_i^2) \\ \Sigma(x_i^3 + x_i y_i^2) \\ \Sigma(x_i^2 y_i + y_i^3) \end{pmatrix} \tag{18}$$

and

$$Q = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{19}$$

formally one can write

$$E = DQ \tag{20}$$

and hence

$$Q = D^{-1}E. \tag{21}$$

The equations (14)–(16) or (20)–(21) are the basis of the further analysis. A, B, and C (or R) can be evaluated as continuous scalar functions of a $2N$-dimensional data set.

$$A = f_A(x_1 \cdots x_i \cdots x_N, y_1 \cdots y_i \cdots y_N) \tag{22}$$

$$B = f_B(\cdots x_i \cdots y_i \cdots) \tag{23}$$

$$C = f_C(\cdots x_i \cdots y_i \cdots) \tag{24}$$

or

$$R = f_R(\cdots x_i \cdots y_i \cdots). \tag{25}$$

III. Analysis of Random Errors

A. General

In context with the previous discussion, we assume that we are sampling from points on the circle

$$(X - A_0)^2 + (Y - B_0)^2 = R_0^2. \tag{26}$$

The N measured values (x_i, y_i), $i = 1, \cdots, N$ are such that

$$x_i = x_{i0} + \xi_i$$
$$y_i = y_{i0} + \eta_i \tag{27}$$

where (x_{i0}, y_{i0}) satisfy (26) and (ξ_i, η_i) is a purely random component. In this paper, we wish to investigate the effect of this random component on our estimators of A_0, B_0, and R_0 for both uniform and nonuniform schemes of sampling points from the hypothesized circle.

The term "purely random" is meant to denote that the means of the random components ξ_i and η_i, $i = 1, \cdots, N$ are zero, that is

$$E(\xi_i) = 0 \text{ and } E(\eta_i) = 0.$$

This in turn implies that

$$E(x_i) = x_{i0} \text{ and } E(y_i) = y_{i0}, \quad i = 1, \cdots, N.$$

Furthermore, it will be assumed that the variance–covariance matrix of

$$(\boldsymbol{\xi}, \boldsymbol{\eta})' = (\xi_1, \cdots, \xi_N, \eta_1, \cdots, \eta_N)$$

is $\sigma^2 I_{2N}$ where I_{2N} is the identity matrix of rank $2N$. That the covariance terms will be zero is true if all the measured values are statistically independent. In real measurement systems the statistical independence of (ξ_i, η_i) and (ξ_j, η_j), $i \neq j$ tends to be valid (which can be interpreted as the time intervals between the subsequent measurements are longer than the system memory time). However, the statistical dependence of ξ_i and η_i, $i = 1, \cdots, N$ is more likely to occur. Nevertheless, if the data of x and y are partly processed in different information channels and the errors arise in these different channels (i.e., the error processes are physically independent), one can rightfully assume that ξ_i and η_i are statistically independent. This is the case in a complex-ratio meter (network analyzer), which has two measurement channels and two detectors for the real and the imaginary parts, respectively, and the error is a consequence of the noise in these channels and detectors. If this is not the case, the covariance of ξ_i and η_i must be taken into consideration.

On the basis of stationary features of a measurement system (which itself needs experimental confirmation) and identical character of the two channels one can assume that the variances of all the error variables are equal, that is

$$\sigma_{xi}^2 = \sigma_{yi}^2 = \sigma^2, \quad i = 1, \cdots, N. \tag{28}$$

In the general case the functions f_A, f_B, f_C, and f_R are nonlinear ones and the means $E(f)$ as well as the variances σ_f^2, of them can be determined (at least in principle) by the general formulas of probability theory:

$$E(f) = \int_{R^{(2N)}} f(x_{i0}, y_{i0}, \xi_i, \eta_i) p(\xi_i, \eta_i) \, d\epsilon \, d\eta \tag{29}$$

and

$$\sigma_f^2 = E(f - E(f))^2 = E(f^2) - [E(f)]^2 \tag{30}$$

where the integration is taken on the whole $2N$-dimensional Euclidean space of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ (or at least in a sufficiently large $2N$-dimensional sphere).

It may be appreciated that the equations (29) and (30) are, in general, very hard to evaluate. For relatively small errors an easier way is provided by serial expansion and first-order approximation.

B. First-Order Approximation

If the errors (ξ_i, η_i), $i = 1, \cdots, N$ are small (compared to the radius of the circle), a serial expansion of f_A, f_B, f_C, or f_R around the point $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ in $2N$ space proves to be useful with the first-order approximation giving good estimators.

In this case, the errors of A, B, C, and R can be written in serial form, e.g., for A the error α:

$$\alpha = A - A_0 = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial x_{i0}} \xi_i + \frac{\partial A}{\partial y_{i0}} \eta_i \right) + \frac{1}{2} \sum_{i=1}^{N} \left(\frac{\partial^2 A}{\partial x_{i0}^2} \xi_i^2 + 2 \frac{\partial^2 A}{\partial x_{i0} y_{i0}} \xi_i \eta_i + \frac{\partial^2 A}{\partial y_{i0}^2} \eta_i^2 \right) + \cdots. \tag{31}$$

Similarly the first-order error terms for A, B, C, and R are, respectively, α, β, γ, and ρ:

$$\alpha = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial x_{i0}} \xi_i + \frac{\partial A}{\partial y_{i0}} \eta_i \right) \tag{32}$$

$$\beta = \sum_{i=1}^{N} \left(\frac{\partial B}{\partial x_{i0}} \xi_i + \frac{\partial B}{\partial y_{i0}} \eta_i \right) \tag{33}$$

$$\gamma = \sum_{i=1}^{N} \left(\frac{\partial C}{\partial x_{i0}} \xi_i + \frac{\partial C}{\partial y_{i0}} \eta_i \right) \tag{34}$$

and

$$\rho = \sum_{i=1}^{N} \left(\frac{\partial R}{\partial x_{i0}} \xi_i + \frac{\partial R}{\partial y_{i0}} \eta_i \right) \tag{35}$$

where

$$\frac{\partial R}{\partial z} = \frac{1}{2R_0} \left(2A_0 \frac{\partial A}{\partial z} + 2B_0 \frac{\partial B}{\partial z} + \frac{\partial C}{\partial z} \right). \tag{36}$$

As can be seen, to a first-order approximation

$$E(\alpha) = E(\beta) = E(\gamma) = E(\rho) = 0$$

and

$$\sigma_A^2 = \sigma^2 \sum_{i=1}^{N} \left(\left(\frac{\partial A}{\partial x_{i0}} \right)^2 + \left(\frac{\partial A}{\partial y_{i0}} \right)^2 \right) \tag{37}$$

$$\sigma_B^2 = \sigma^2 \sum_{i=1}^{N} \left(\left(\frac{\partial B}{\partial x_{i0}} \right)^2 + \left(\frac{\partial B}{\partial y_{i0}} \right)^2 \right) \tag{38}$$

$$\sigma_{AB}^2 = \sigma^2 \sum_{i=1}^{N} \left(\frac{\partial A}{\partial x_{i0}} \cdot \frac{\partial B}{\partial x_{i0}} + \frac{\partial A}{\partial y_{i0}} \cdot \frac{\partial B}{\partial y_{i0}} \right) \tag{39}$$

$$\sigma_C^2 = \sigma^2 \sum_{i=1}^{N} \left(\left(\frac{\partial C}{\partial x_{i0}} \right)^2 + \left(\frac{\partial C}{\partial y_{i0}} \right)^2 \right) \tag{40}$$

$$\sigma_R^2 = \sigma^2 \sum_{i=1}^{N} \left(\left(\frac{\partial R}{\partial x_{i0}} \right)^2 + \left(\frac{\partial R}{y_{i0}} \right)^2 \right) \tag{41}$$

where $\sigma^2 = E(\xi_i^2) = E(\eta_i^2)$ and the covariance

$$\sigma_{\xi\eta} = E(\xi_i \eta_i) = 0. \tag{42}$$

The covariance of α and β (i.e., σ_{AB}) will, in general, be nonzero even for the first-order approximation.

The characteristic roots, σ_1^2 and σ_2^2, of the covariance matrix of A and B are expressed by (43) and (44):

$$\sigma_1^2 = \frac{\sigma_A^2 + \sigma_B^2 + \sqrt{(\sigma_A^2 - \sigma_B^2)^2 + 4\sigma_{AB}^2}}{2} \tag{43}$$

$$\sigma_2^2 = \frac{\sigma_A^2 + \sigma_B^2 - \sqrt{(\sigma_A^2 - \sigma_B^2)^2 + 4\sigma_{AB}^2}}{2}. \tag{44}$$

The larger value σ_1^2 is the least upper bound of the variances of normalized linear combinations of A and B, i.e., if

$$L = aA + bB$$

such that $a^2 + b^2 = 1$ then $\sigma_L^2 \leq \sigma_1^2$. As can be seen, σ_1^2 provides a criterion for choosing among schemes for sampling points from the circle.

Since all first-order terms are proportional to the scale factor σ, it will be convenient to normalize with respect to this factor. Therefore, we define the following (relative) error sensitivities:

$$S_A = \frac{\sigma_A}{\sigma} \qquad S_1 = \frac{\sigma_1}{\sigma}$$

$$S_B = \frac{\sigma_B}{\sigma} \qquad S_2 = \frac{\sigma_2}{\sigma}$$

$$S_{AB} = \frac{\sigma_{AB}^{1/2}}{\sigma}$$

$$S_C = \frac{\sigma_C}{\sigma}$$

$$S_R = \frac{\sigma_R}{\sigma}. \tag{45}$$

Now, in order to calculate the above, certain first-order partial derivatives must be evaluated. Differentiate equation (20). Then

$$\dot{D}Q + D\dot{Q} = \dot{E} \tag{46}$$

where the dot denotes the derivatives, and $\dot{Q}$ contains the derivatives to be evaluated

$$\dot{Q} = \begin{pmatrix} \dot{A} \\ \dot{B} \\ \dot{C} \end{pmatrix}. \tag{47}$$

Hence

$$\dot{Q} = D^{-1}(\dot{E} - \dot{D}Q) \tag{48}$$

and the explicit derivatives with respect to x_i and y_i are, respectively,

$$\frac{\partial E}{\partial x_i} = \begin{pmatrix} 2x_i \\ 3x_i{}^2 + y_i{}^2 \\ 2x_i y_i \end{pmatrix} \tag{49}$$

$$\frac{\partial E}{\partial y_i} = \begin{pmatrix} 2y_i \\ 2x_i y_i \\ x_i{}^2 + 3y_i{}^2 \end{pmatrix} \tag{50}$$

and

$$\frac{\partial D}{\partial x_i} = \begin{pmatrix} 2 & 0 & 0 \\ 4x_i & 2y_i & 1 \\ 2y_i & 0 & 0 \end{pmatrix} \tag{51}$$

$$\frac{\partial D}{\partial y_i} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2x_i & 0 \\ 2x_i & 4y_i & 1 \end{pmatrix} \tag{52}$$

The equations (48)–(52) provide a simple way to evaluate the derivatives.

C. The Effect of the Number of Points Used and Their Placement

The purpose of this analysis is to evaluate the error sensitivities as functions of how the points are chosen on the circle.

If the measurement error is independent of the position of the circle, then without any loss of generality, we may choose

$$A_0 = 0$$

$$B_0 = 0.$$

Also x_{i0} and y_{i0} can be written as

$$x_{i0} = R_0 \cos \phi_i$$

$$y_{i0} = R_0 \sin \phi_i$$

where R_0 is the radius of the circle and ϕ_i is the angle which the ith point makes with the x axis.

Likewise,

$$x_i = R_0 \cos \phi_i + \xi_i \tag{53}$$

$$y_i = R_0 \sin \phi_i + \eta_i. \tag{54}$$

Prior to investigating a general case it is interesting to evaluate a specific example of practical importance. This example is the regular equidistant placement of N data points along a circle. In this case

$$\phi_i = i\frac{2\pi}{N}, \qquad i = 1, \cdots, N \tag{55}$$

and as first-order approximations

$$D = N \begin{pmatrix} 0 & 0 & 1 \\ R_0{}^2 & 0 & 0 \\ 0 & R_0{}^2 & 0 \end{pmatrix} \tag{56}$$

and, hence,

$$D^{-1} = \frac{1}{N} \begin{pmatrix} 0 & R_0{}^{-2} & 0 \\ 0 & 0 & R_0{}^{-2} \\ 1 & 0 & 0 \end{pmatrix} \tag{57}$$

Consequently

$$\frac{\partial Q}{\partial x_i} = \frac{2}{N} \begin{pmatrix} \cos^2 \phi_i \\ \cos \phi_i \sin \phi_i \\ R_0 \cos \phi_i \end{pmatrix} \tag{58}$$

$$\frac{\partial Q}{\partial y_i} = \begin{pmatrix} \cos \phi_i \sin \phi_i \\ \frac{2}{N} \sin^2 \phi_i \\ R_0 \sin \phi_i \end{pmatrix} \tag{59}$$

Hence

$$S_A{}^2 = \left(\frac{2}{N}\right)^2 \sum_{i=1}^{N} (\cos^4 \phi_i + \cos^2 \phi_i \sin^2 \phi_i) = \frac{2}{N} \tag{60}$$

and

$$S_A = \sqrt{\frac{2}{N}}. \tag{61}$$

Similarly

$$S_B = \sqrt{\frac{2}{N}} \tag{62}$$

$$S_{AB} = 0 \tag{63}$$

$$S_C = \frac{2R_0}{\sqrt{N}} \tag{64}$$

$$S_R = \frac{1}{\sqrt{N}}. \tag{65}$$

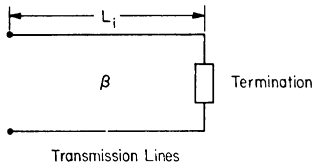

Fig. 2. Schematic of the sliding termination.

It is clear that the accuracy can be improved by increasing the number of data points N. This, of course, is valid also for nonuniform data point distributions.

In sliding termination applications, the angle of the data point is determined by the phase shift due to the position of termination (see Fig. 2), i.e.,

$$\phi_i = 2\beta L_i \tag{66}$$

where β is the phase coefficient. It is frequency-dependent and for TEM-mode transmission lines it is linearly proportional to the frequency.

For a given set of L_i's the ϕ_i angles and therefore the random errors will vary with the frequency. Although the ϕ_i values vary, their ratio one to another does not (if the L_i's are fixed). Introducing the ratio s_i of the ith to the Nth data point

$$s_i = \frac{\phi_i}{\phi_N} \tag{67}$$

which can be rewritten as

$$\phi_i = s_i \cdot \phi_N.$$

If the random-error sensitivities (for fixed s_i's) as functions of the point angle ϕ_N are evaluated and an accuracy requirement is prescribed, the relative frequency range of measurement procedure can be determined.

Note that generally the exact position of data points cannot be given prior to the actual measurement, but in many cases (in calibration measurements) the distribution of termination positions provides a good approximation to the actual data point distribution. If this is not the case, an *a posteriori* error evaluation always can be accomplished by the given analysis.

Another consequence of practical importance, that a minimum usable length (deviation) of a sliding termination can be determined (for a given frequency).

On the basis of these considerations a number of error evaluations were carried out (for different N and ϕ_i ratios). Some of the results are plotted on Figs. 3–6. These figures present normalized error sensitivities σ_1, σ_2, and σ_r. To relate these curves to practical cases, a criteria for the particular case must be established, for example a value of one (1). Then, when the curve of interest falls below this value the criteria has been met.

An essential result of these diagrams is that the number of data points is important, not because of the error level but because of the broad-banding of the measurement process. The error drastically increases when the data points are crowded around two points and eventually this limits the high-frequency usability of the measurements.

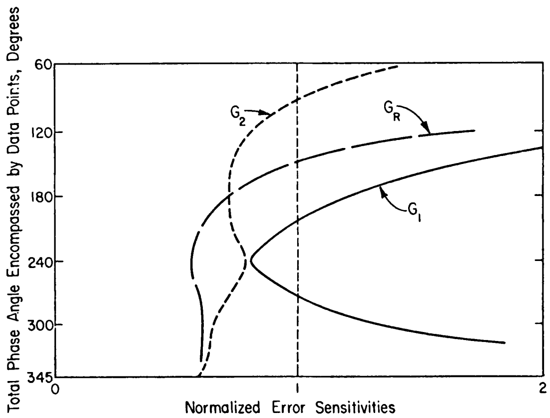

Fig. 3. Output of the computer program. $N = 3$, uniform data point distribution: G_1 and G_2 here signify the larger and the smaller characteristic error sensitivities of the center A, B, and G_R gives the error sensitivity of the radius R.

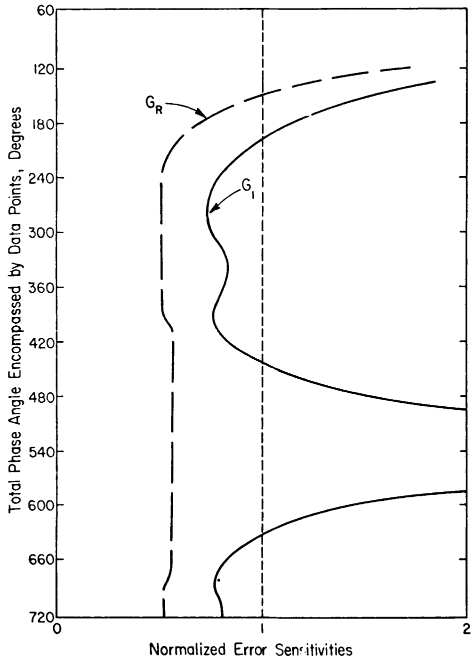

Fig. 4. Error sensitivity of the center (solid line) and the radius (dotted line). $N = 4$, uniform data point distribution.

A further interesting result is that a nonuniform data distribution gives better broad-band performances than a uniform one. This is because by proper choice of data points it can be assured that even in a wide-frequency range there are at least three different, relatively distant, points for evaluation.

Unfortunately, it seems to be very difficult to determine an exact optimum distribution (with maximum frequency range) for a prescribed error, but heuristic considerations give reasonable approximate results.

It can be seen that the $N = 3$ is hopelessly narrow-band situation, but for $N > 3$ the bandwidth can be considerably improved by nonuniform data distribution.

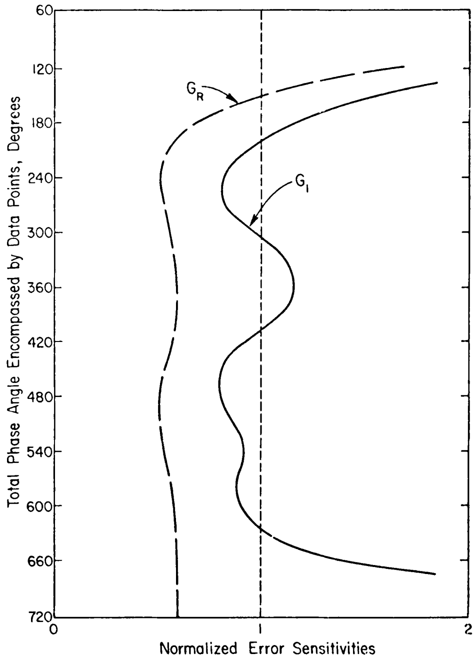

Fig. 5. Error sensitivity of the center (solid line) and the radius (dotted line). N = 4, relative data point distribution: 0, 0.25, 0.5, 1.

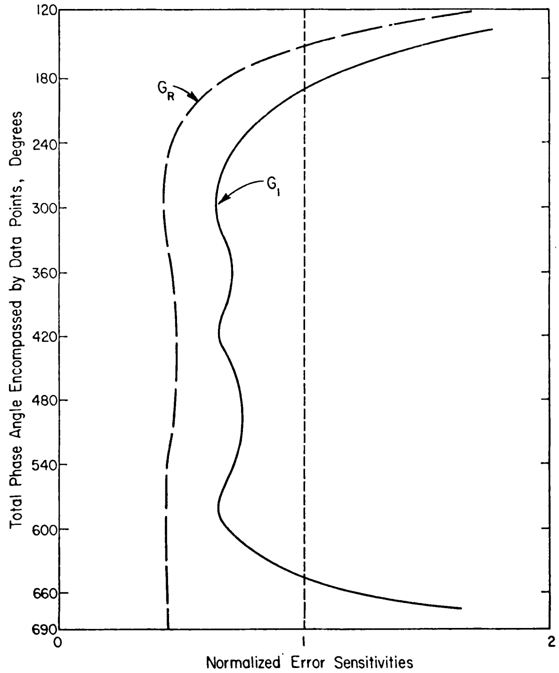

Fig. 6. Error sensitivity of the center (solid line) and the radius (dotted line). N = 5, uniform data point distribution.

For practical purposes $N = 4$ or $N = 5$ give good results with a reasonable number of data points.

D. Higher Order Terms

The estimation of the effect of random errors is more accurate if the higher order terms of the serial expansion (31) are taken into account.

Essentially the higher order terms describe two kinds of errors:

a. the bias of the results, i.e., the mean (or expectation) of the result will differ from the exact value.
b. random errors due to higher level of the data point errors. Especially important are the error terms due to simultaneous errors of more than one data point.

The bias of A considering the second-order terms is

$$E(\alpha) = \frac{\sigma^2}{2}\sum_{i=1}^{N}\left(\frac{\partial^2 A}{\partial x_i^2} + \frac{\partial^2 A}{\partial y_i^2}\right) \tag{68}$$

and the variance $\sigma_A{}^2$, using the definition (30) and the expansion (31), is

$$\sigma_A{}^2 = \sum_{i=1}^{N}\left\{\sigma^2\left(\left(\frac{\partial A}{\partial x_i}\right)^2 + \left(\frac{\partial A}{\partial y_i}\right)^2\right) + \frac{1}{4}\left(\frac{\partial^2 A}{\partial x^2}\right)^2 E(\xi_i{}^4) + \left(\frac{\partial^2 A}{\partial x_i \partial y_i}\right)^2 E(\xi_i{}^2\eta_i{}^2) + \frac{1}{4}\left(\frac{\partial^2 A}{\partial y_i{}^2}\right)^2 E(\eta_i{}^4) + \frac{1}{4}\frac{\partial^2 A}{\partial x_i{}^2} \cdot \frac{\partial^2 A}{\partial y_i{}^2} E(\xi_i{}^2\eta_i{}^2) - [E(\alpha)]^2\right\} \tag{69}$$

where

$$E(\xi_i{}^2\eta_i{}^2) = \sigma^4$$

and if one assumes that the ξ_i and η_i are Gaussian distributed (see [4]),

$$E(\xi_i{}^4) = E(\eta_i{}^4) = 3\sigma^4.$$

In this way, the second-order approximation of variance is obtained;

$$\sigma_A{}^2 = \sigma^2\sum_{i=1}^{N}\left(\left(\frac{\partial A}{\partial x_i}\right)^2 + \left(\frac{\partial A}{\partial y_i}\right)^2\right) + \frac{\sigma^4}{2}\sum_{i=1}^{N}\left(\left(\frac{\partial^2 A}{\partial x_i{}^2}\right) + \left(\frac{\partial^2 A}{\partial y_i{}^2}\right)^2\right) + \sigma^4\sum_{i=1}^{N}\left(\left(\frac{\partial^2 A}{\partial x_i \partial y_i}\right)^2 - \frac{1}{4}\frac{\partial^2 A}{\partial x_i{}^2} \cdot \frac{\partial^2 A}{\partial y_i{}^2}\right). \tag{70}$$

In order to evaluate these formulas, it is necessary to calculate the second derivatives. For this purpose let us differentiate (47)

$$\ddot{D}_{ps}Q + \dot{D}_p\dot{Q}_s + \dot{D}_s\dot{Q}_p + D\ddot{Q}_{ps} = \ddot{E}_{ps} \tag{71}$$

where the dots denote derivatives by the variables p and s. Hence

$$\ddot{Q}_{ps} = D^{-1}(\ddot{E}_{ps} - \ddot{D}_{ps}Q - \dot{D}_p\dot{Q}_s - \dot{D}_s\dot{Q}_p). \tag{72}$$

It can be shown that for $A_0 = 0$, $B_0 = 0$

$$\ddot{D}_{ps}Q = 0. \tag{73}$$

$\dot{Q}$ can be calculated from (48).

An interesting consequence of (71) that even if only one of the first derivatives became large, all the second-order error terms increase.

IV. Estimation of σ^2

For the objective function

$$u = \sum_{i=1}^{N}[(x_i - A)^2 + (y_i - B)^2 - R^2]^2 \tag{74}$$

if $A = A_0$, $B = B_0$, and $R = R_0$, then it can be shown that, assuming η_i, ξ_i are Gaussian distributed,

$$E\{u\} = 4N\sigma^2(R_0^2 + 2\sigma^2) \tag{75}$$

or to a first-order approximation

$$E\{u\} \simeq 4R_0^2N\sigma^2. \tag{76}$$

This is irrespective of the relative positions of the points about the circle. For $N \gg 3$ and σ^2 small, one might estimate σ^2 as

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{N} [(x_i - \hat{A})^2 + (y_i - \hat{B})^2 - \hat{R}^2]^2}{4\hat{R}^2N} \tag{77}$$

where $\hat{A}$, $\hat{B}$, and $\hat{R}$ are the values of A, B, and R chosen to minimize u.

Now, if we define

$$\hat{A} = A_0 + \alpha$$
$$\hat{B} = B_0 + \beta$$
$$\hat{R} = R_0 + \rho$$

then

$$\min u = \sum_{i=1}^{N} \{(x_{i0} + \xi_i - \alpha - A_0)^2 + (y_{i0} + \eta_i - \beta - B_0)^2 - (R_0 - \rho)^2\}^2. \tag{78}$$

Considering the special equidistant placement case described in Section III-C with $A_0 = B_0 = 0$, to a first-order approximation

$$\min u = \sum_{i=1}^{N} \{2x_{i0}(\xi_i - \alpha) + 2y_{i0}(\eta_i - \beta) - 2R_0\rho\}^2. \tag{79}$$

Let $\min u = u_0$. Then to first order

$$E(u_0) = \sigma^2 \sum_{i=1}^{N} \left(\left(2x_{i0}\left(1 - \frac{\partial A}{\partial x_i}\right) - 2y_{i0}\frac{\partial B}{\partial x_i} - \frac{\partial C}{\partial x_i}\right)^2 + \left(2y_{i0}\left(1 - \frac{\partial B}{\partial y_i}\right) - 2x_{i0} - \frac{\partial A}{\partial x_i} - \frac{\partial C}{\partial y_i}\right)^2\right) = \sigma^2 4R_0^2N\left(1 - \frac{3}{N}\right)^2. \tag{80}$$

See (36), (58), and (59). The estimate for σ^2 would then be

$$\hat{\sigma}^2 = \frac{u_0}{4\hat{R}_0^2N\left(1 - \frac{3}{N}\right)^2}. \tag{81}$$

V. Summary

In this paper, an efficient circle fitting procedure was described and analyzed. A general analysis was given for the first-order random errors and the variance of the center and radius of the fitted circle were evaluated. For the analysis computer programs were written.

The error terms were investigated as a function of the data point number and data point distribution and it was shown that for $N > 3$ the nonuniform data distribution provides better broad-band performance. For $N = 4$ and $N = 5$ successive division of sections into halves gives reasonably good performance. In order to illustrate the effects of data distribution, several examples are calculated for different N's and distributions.

A systematic way to extend the results to higher-order terms is also shown and general formulas for the second-order bias and the second-order random-error variances are provided.

Finally, an *a posteriori* estimation of data point error was presented.

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References

[1] G. A. Deschamps, "Determination of reflection coefficients and inversion loss of a waveguide junction," *Appl. Phys.*, vol. 24, no. 8, pp. 1046–1050, 1950.
[2] M. Sucher and F. Fox, *Handbook of Microwave Measurements*, vol. 1. Brooklyn, NY: Polytechnic Press.
[3] P. Delonge, "Computer optimization of Deschamps' method and error cancellation in reflectometry," *Proc. IMEKO-Symp. Microwave Measurement* (Budapest), pp. 117–123, 1972.
[4] W. B. Davenport and W. L. Root, *Random Signals and Noise*. New York: McGraw-Hill, 1958.
[5] L. Miller, Engineering Dimensional Metrology. London, England: Edward Arnold Pub. Ltd, 1962, pp. 69–75.