Statistics with Spa ows

Lecture 9

Julia Schroeder

Outline

- Linear models

Linear models

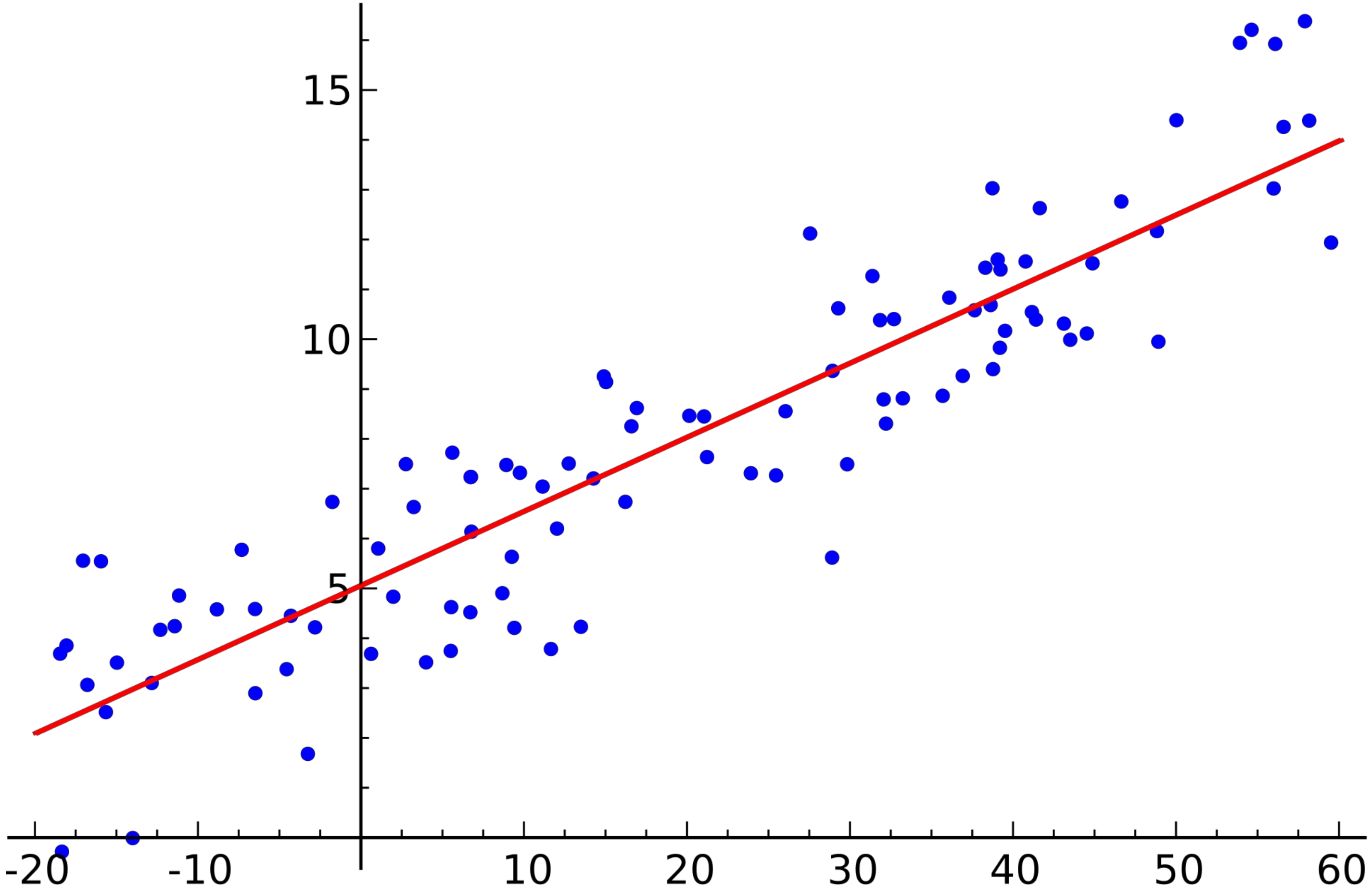
- Most important concepts
- Can model many questions (including previous t-test)
- If you understand linear models, everything else will be easy as a breeze!
- Aim to fit models to data

Fitting models to data

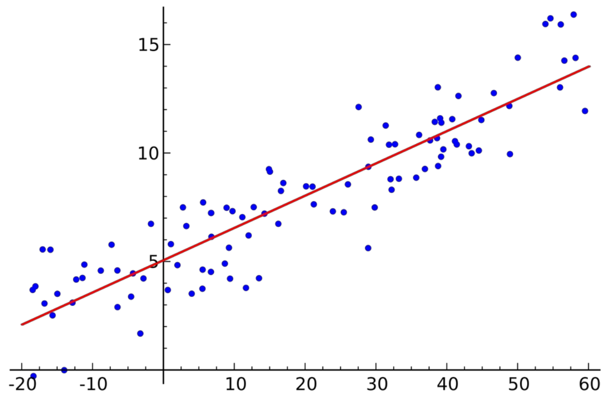
Fitting models to data

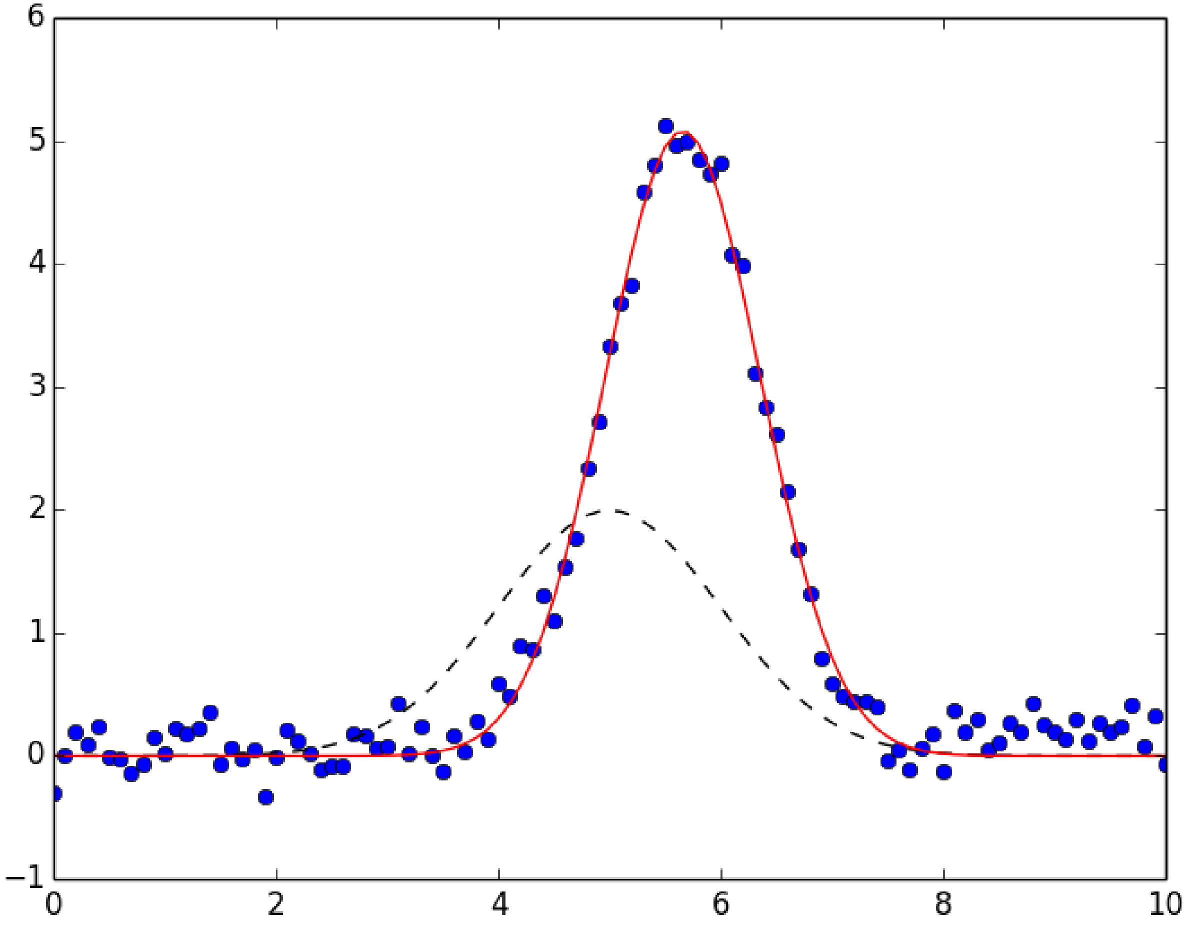
?

Fitting models to data

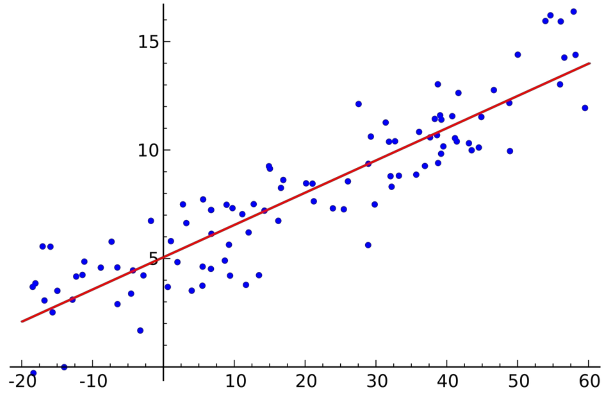


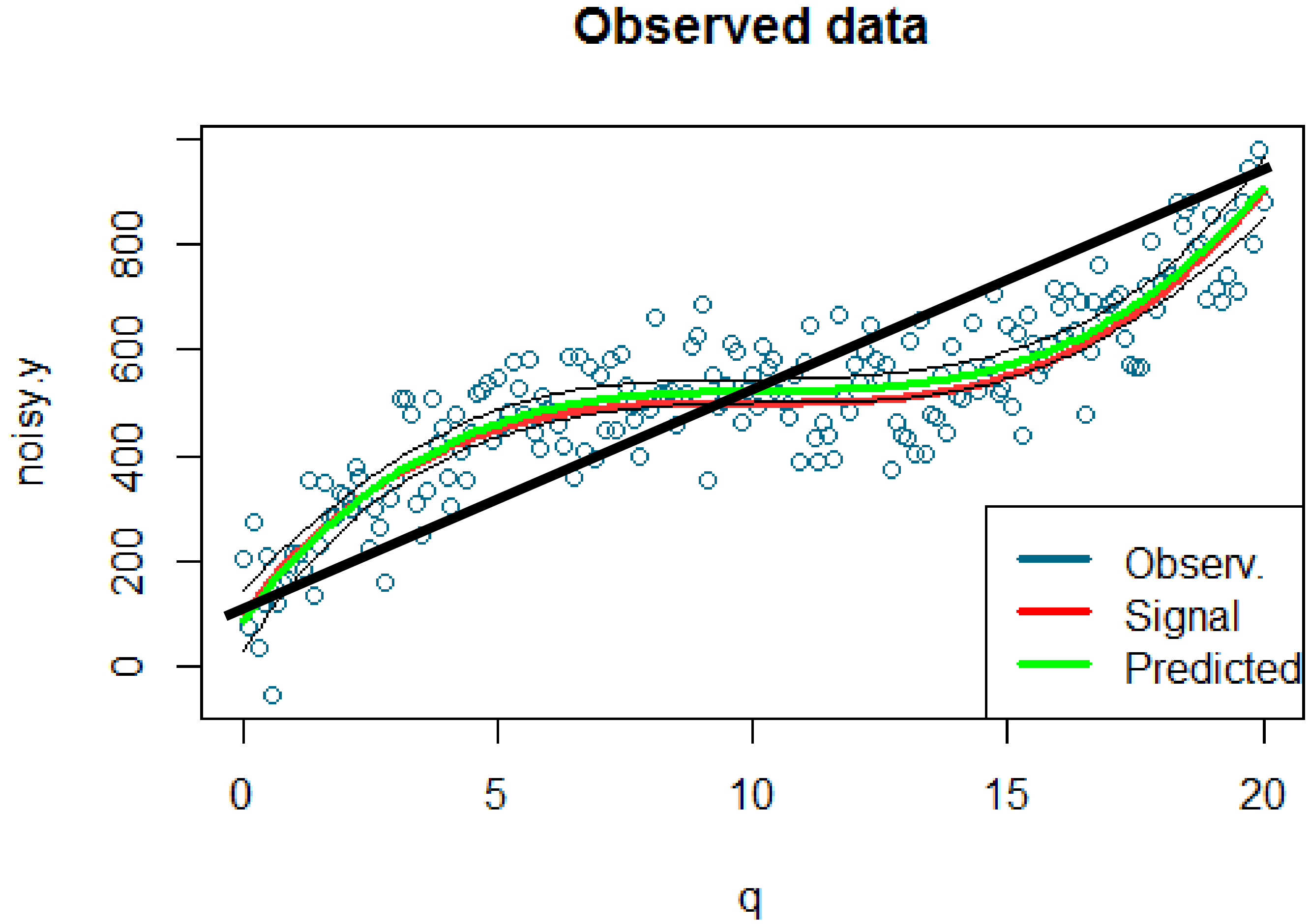
Fitting models to data

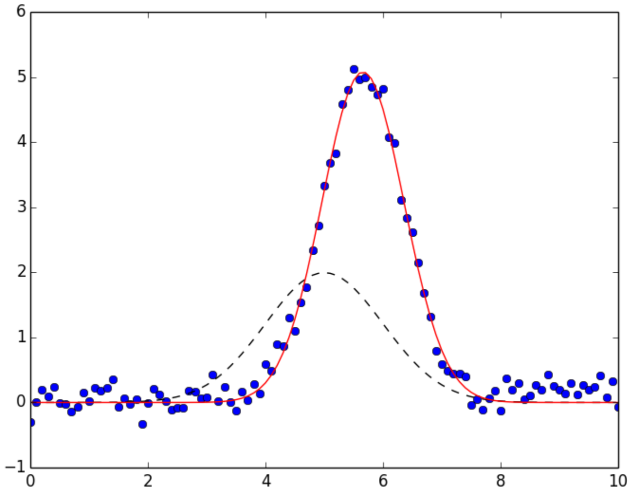




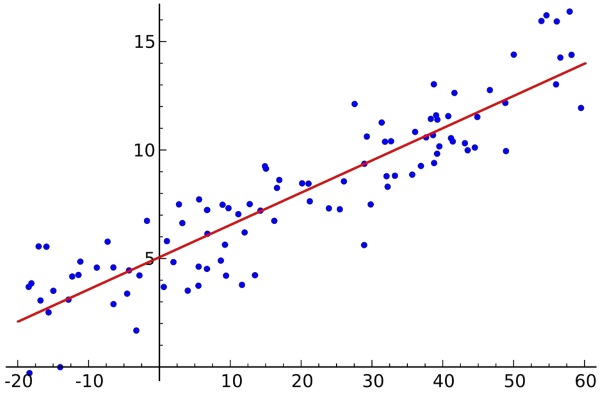
Fitting models to data

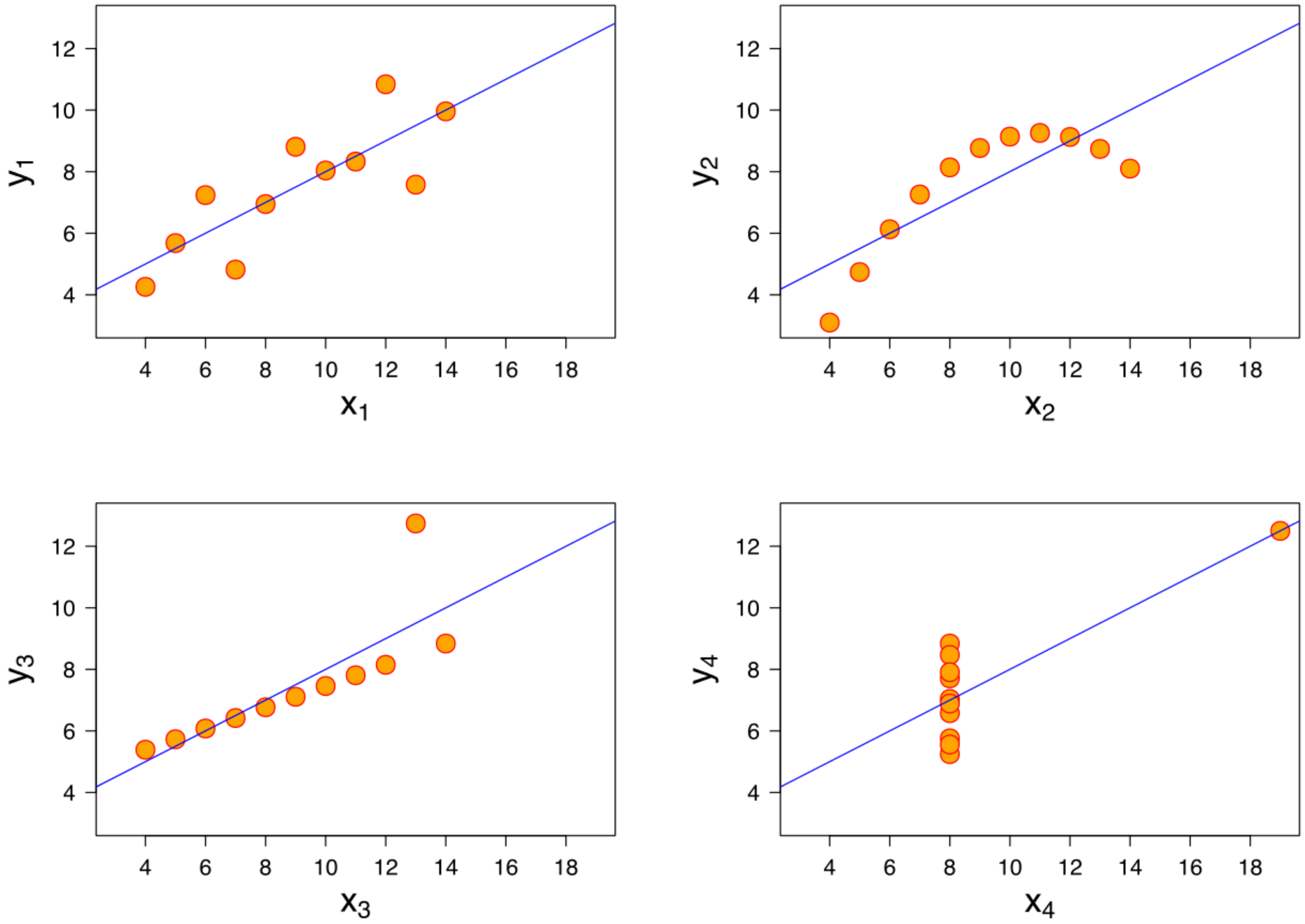






Fitting models to data





Linear models

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

Linear models

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y
5
3
6
10
4

Data. Response variable,
e.g. sparrow body mass.

Linear models

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

y	i
5	1
3	2
6	3
10	4
4	5

Data. Response variable. Observation 1, 2, 3, etc.
e.g. sparrow body mass.

Linear models

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

y	i
5	1
3	2
6	3
10	4
4	5

Data. Response variable. Observation 1, 2, 3, etc.
e.g. sparrow body mass.

x	i
3	1
1	2
4	3
8	4
2	5

Data. Explanatory variable.
e.g. sparrow tarsus length.

Linear models

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

	i
?	1
?	2
?	3
?	4
?	5

Linear models

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

$b_1 = ?$

	i
?	1
?	2
?	3
?	4
?	5

Linear models

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

$b_0 = ?$

$b_1 = ?$

	i
?	1
?	2
?	3
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?	5

Linear models

- Note difference in variable format
- Some are vectors, others are single values!

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

y	i
5	1
3	2
6	3
10	4
4	5

$b_0 = ?$

$b_1 = ?$

x	i
3	1
1	2
4	3
8	4
2	5

	i
?	1
?	2
?	3
?	4
?	5

Linear models

- Note difference in variable format
- Some are vectors, others are single values!
- We aim to estimate b_0 and b_1
- We will get from the results

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

Linear models

- Note difference in variable format
- Some are vectors, others are single values!
- We aim to estimate b_0 and b_1
- We will get from the results

Parameter estimates

Error, or residuals

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

Let's give this a try…

- Let's plot this

y	x
5	3
4	1
7	4
9	8
3	2

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

Let’s give this a try…

- Let’s plot this

y	x
5	3
4	1
7	4
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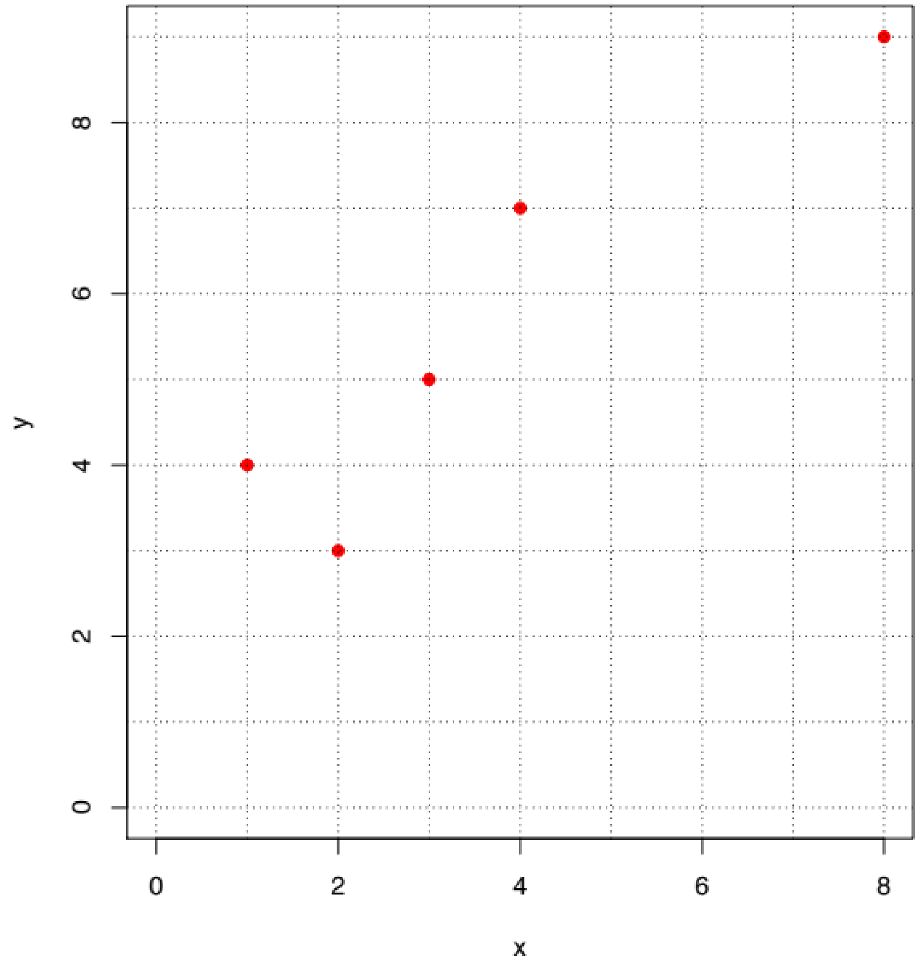

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

Let's give this a try…

- Let's plot this
- Now we "guesstimate" the line

y	x
5	3
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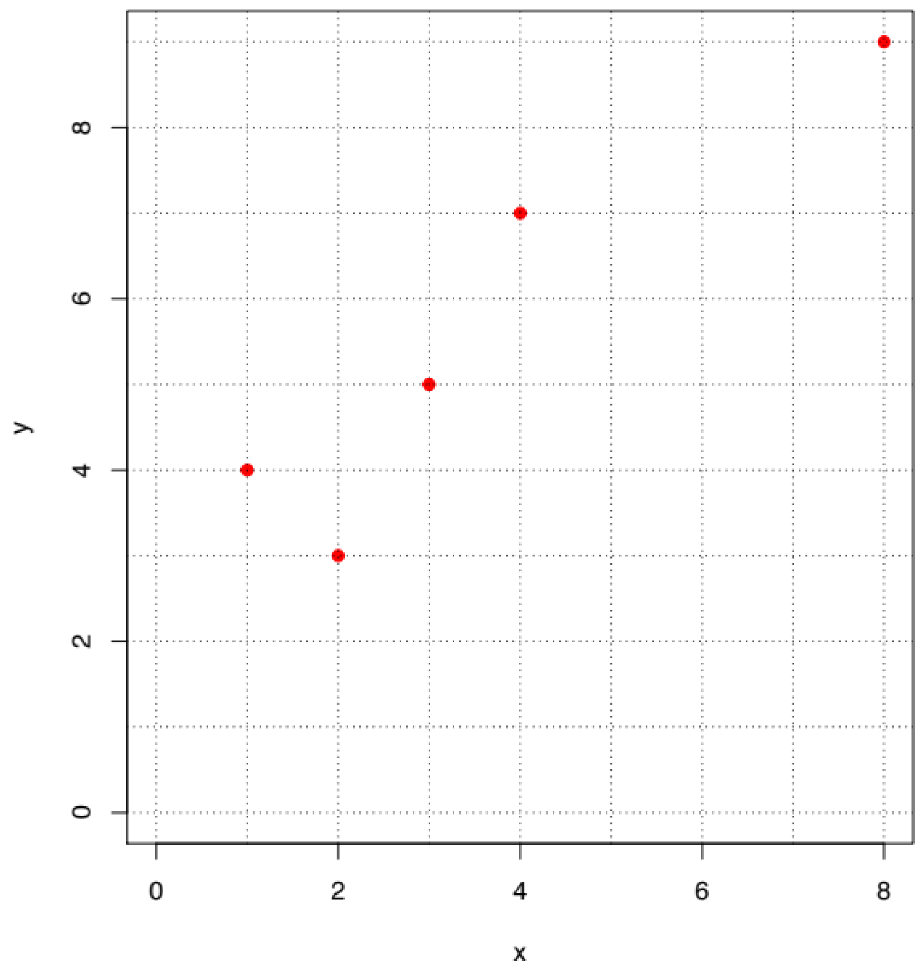

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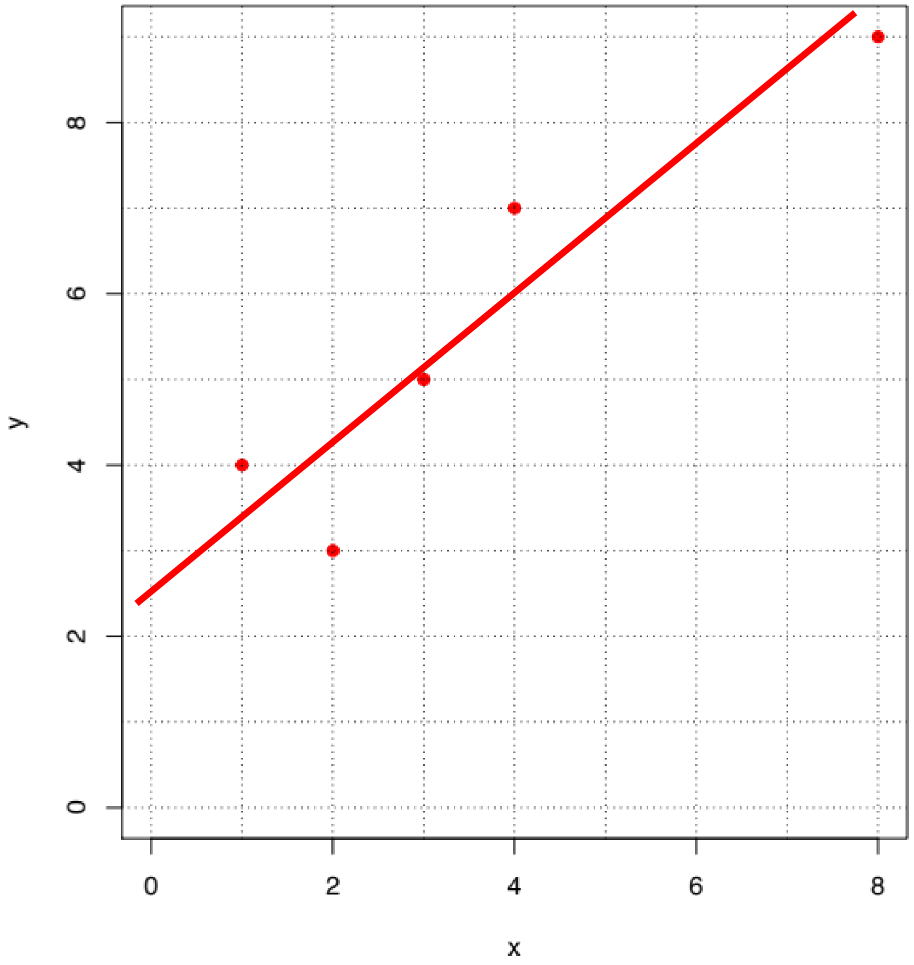
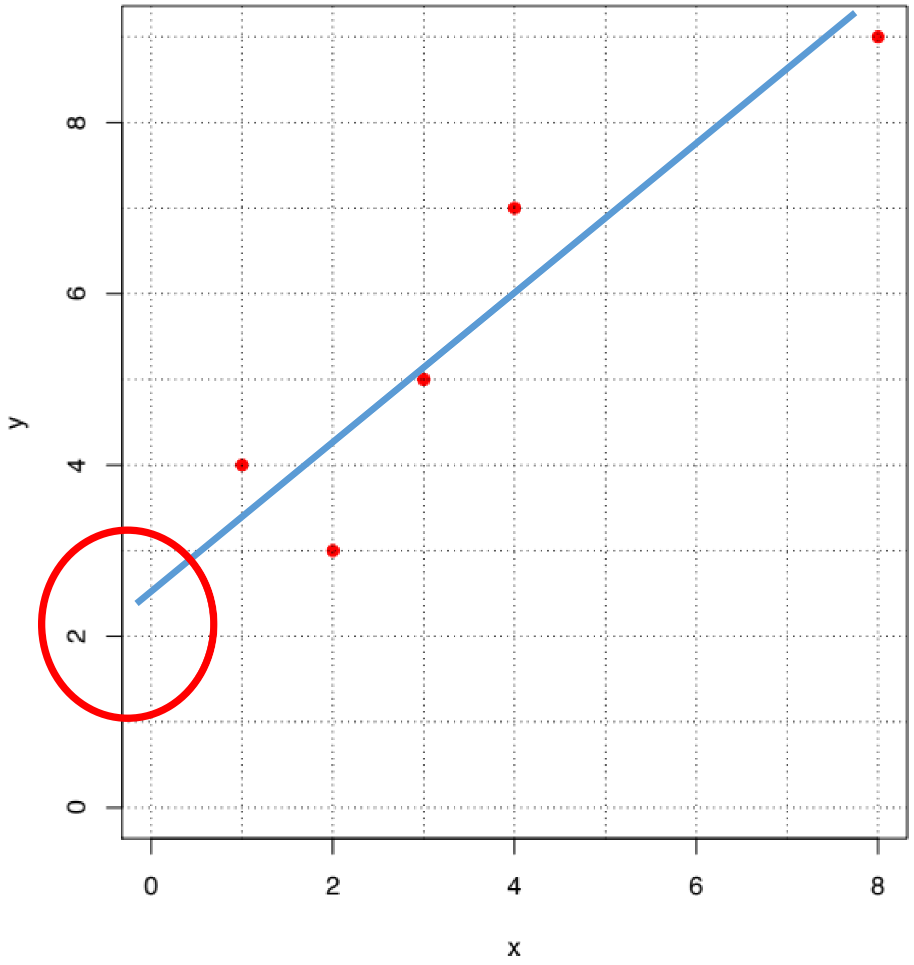


$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

Let’s give this a try…

- Let’s plot this
- Now we “guesstimate” the line
- Now we “guesstimate” b_0 and b_1:
- Intercept:



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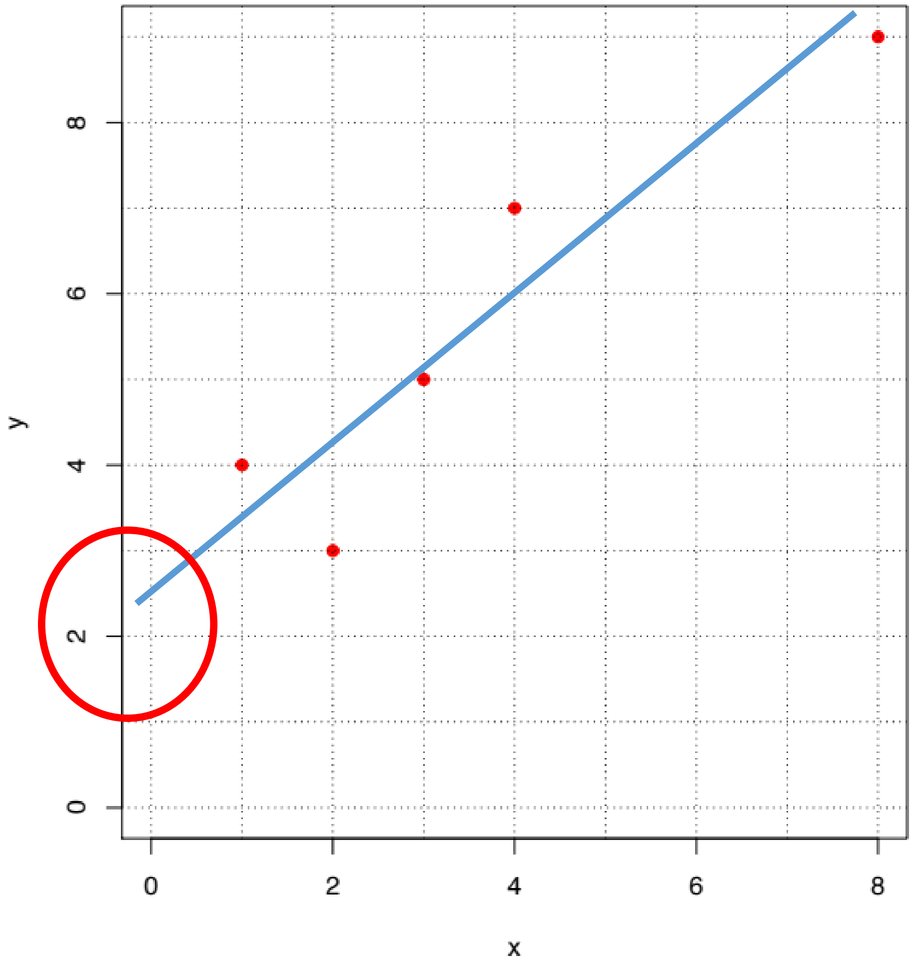
- Now we "guesstimate" the line
- Now we "guesstimate" b_0 and b_1:
- Intercept: something 2.2



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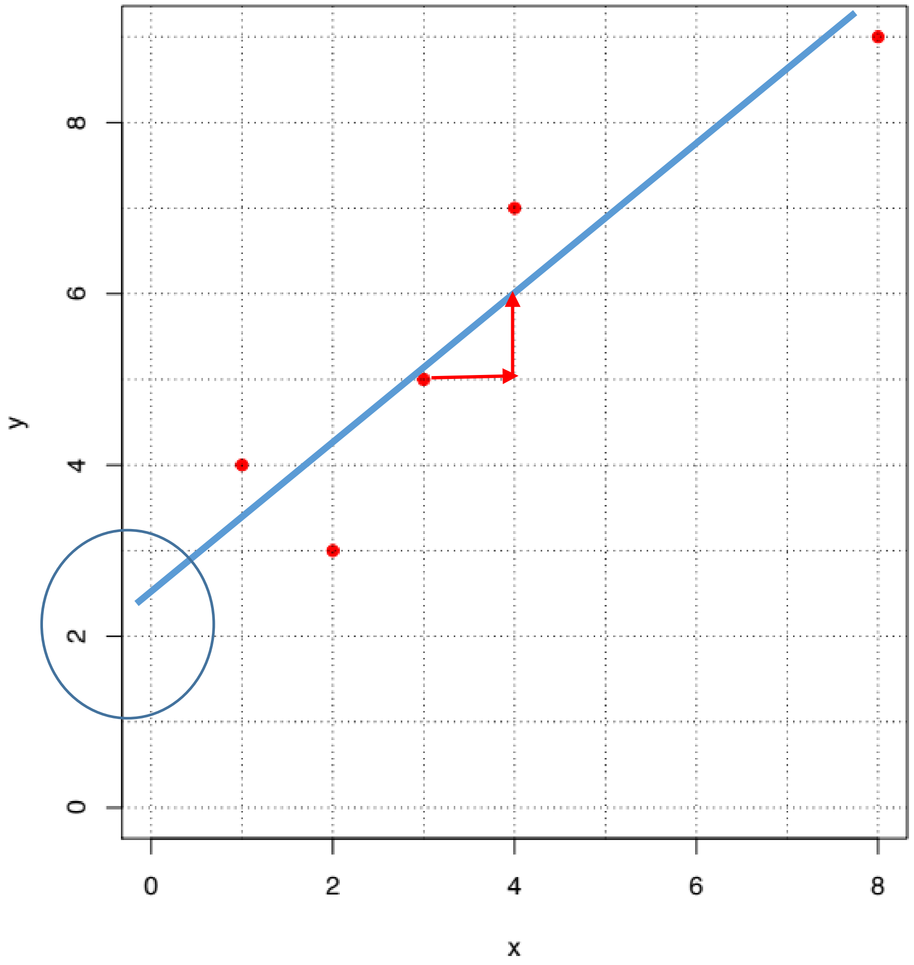
- Now we "guesstimate" b_0 and b_1:
- Intercept: something 2.2
- Slope: close enough to 1



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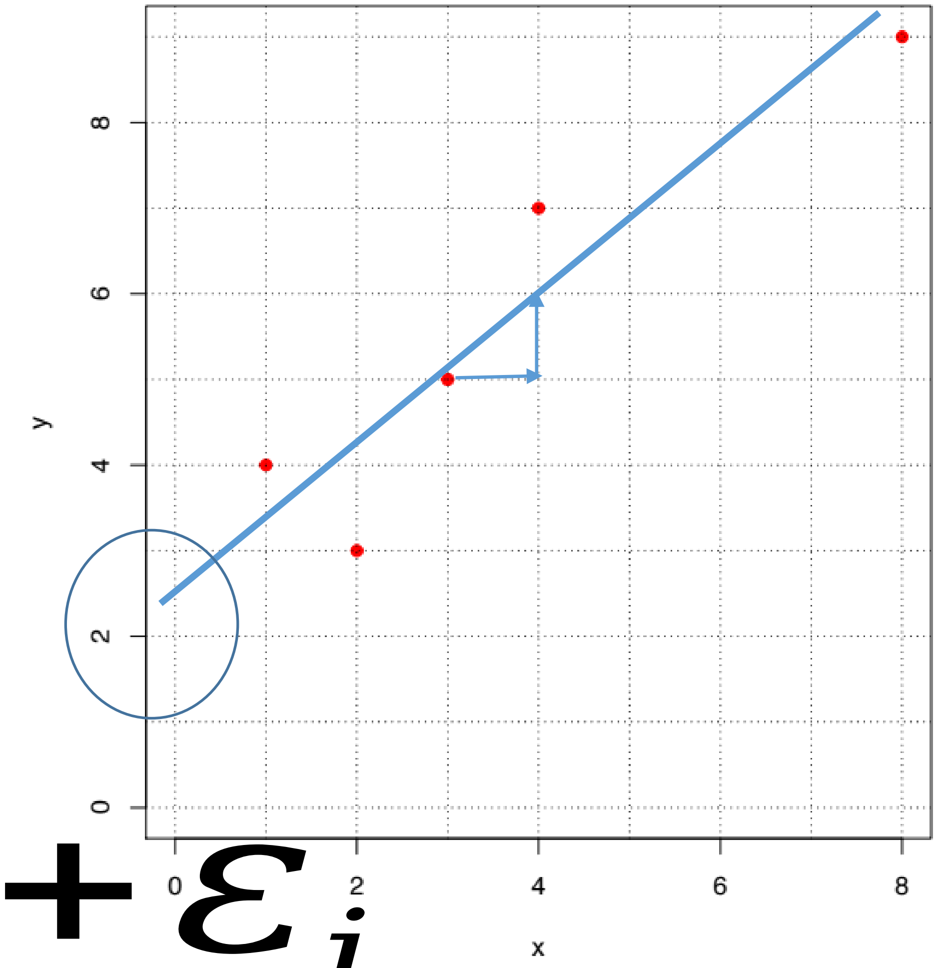
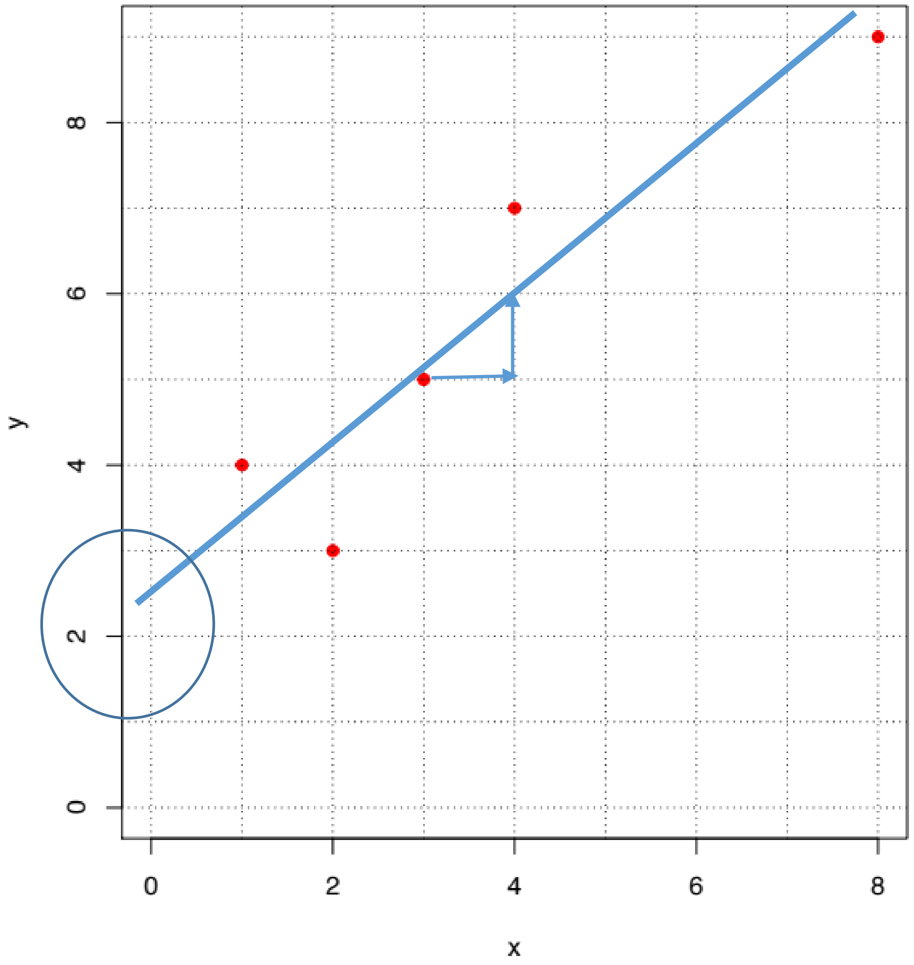


$$y_i = 2.2 + 1\, x_i + \varepsilon_i$$

$$y_i = b_0 + b_1\, x_i + \varepsilon_i$$

Let’s give this a try…

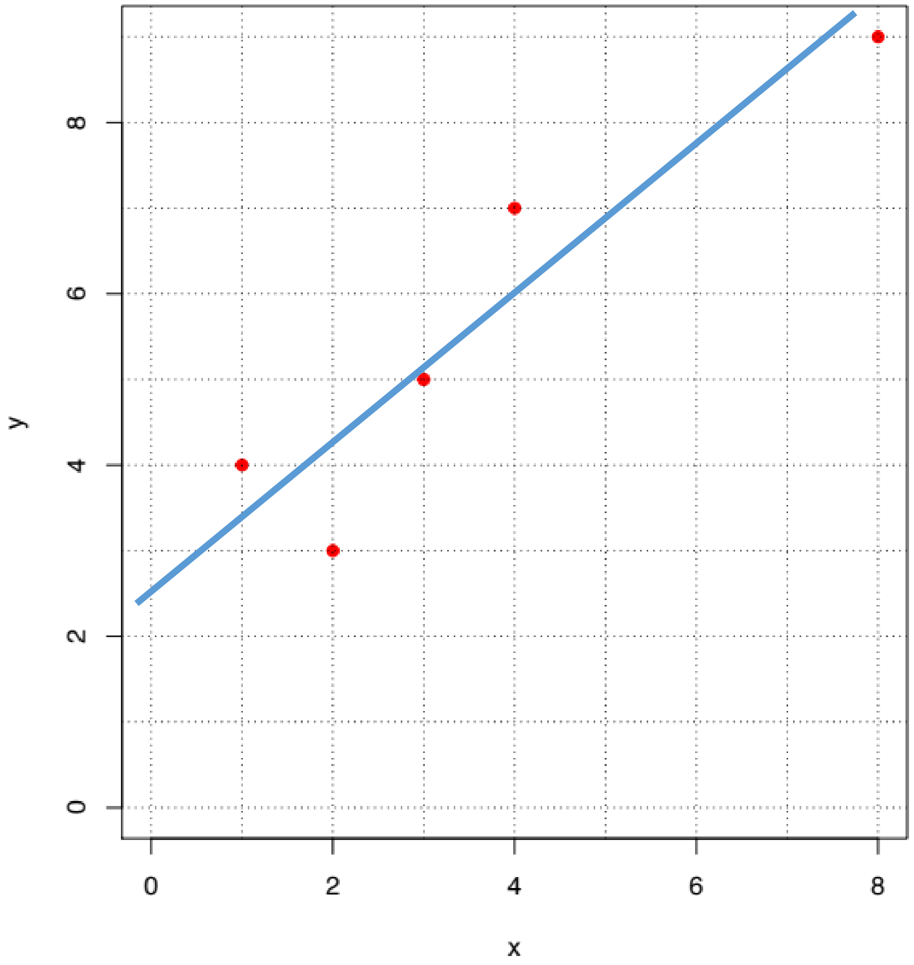
- Let’s plot this
- Now we “guesstimate” the line
- Now we “guesstimate” b_0 and b_1:
- Intercept: something 2.2
- Slope: close enough to 1
- But what’s with ?



$$y_i = 2.2 + 1\,x_i + \varepsilon_i$$

Let's give this a try…

- But what's with ?
- The residuals are the "error" of the model
- We get them by plotting the vertical (y) distance:



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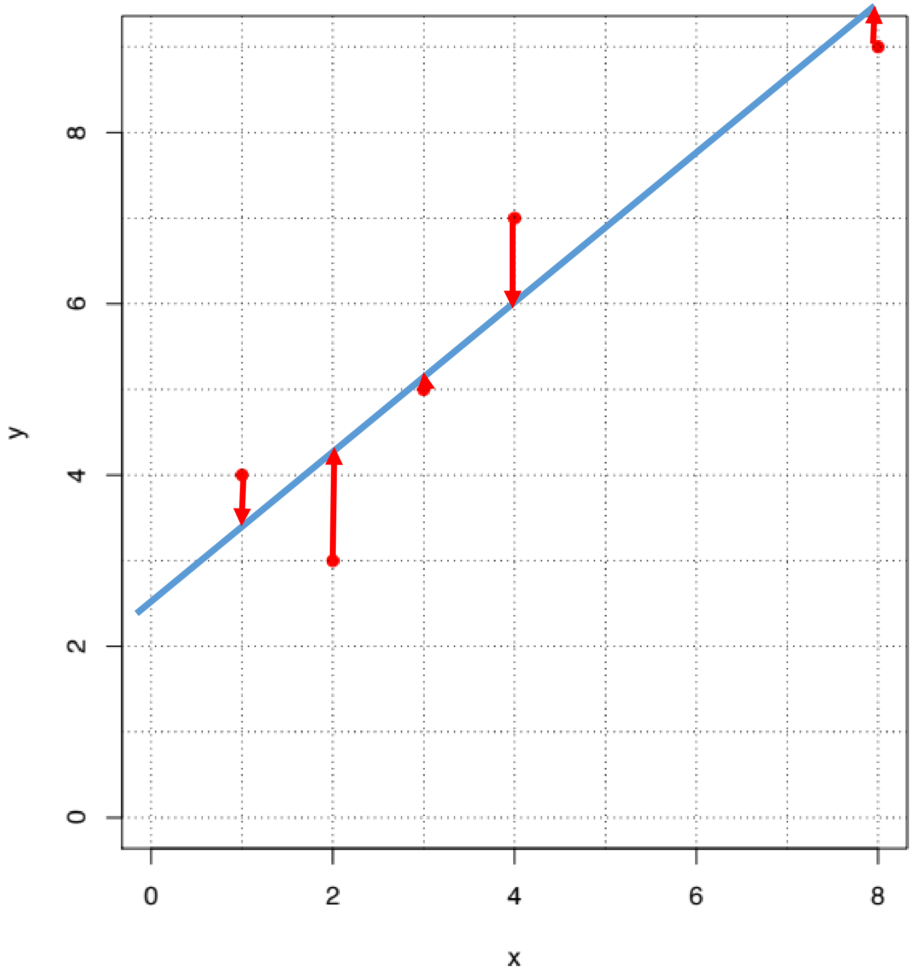

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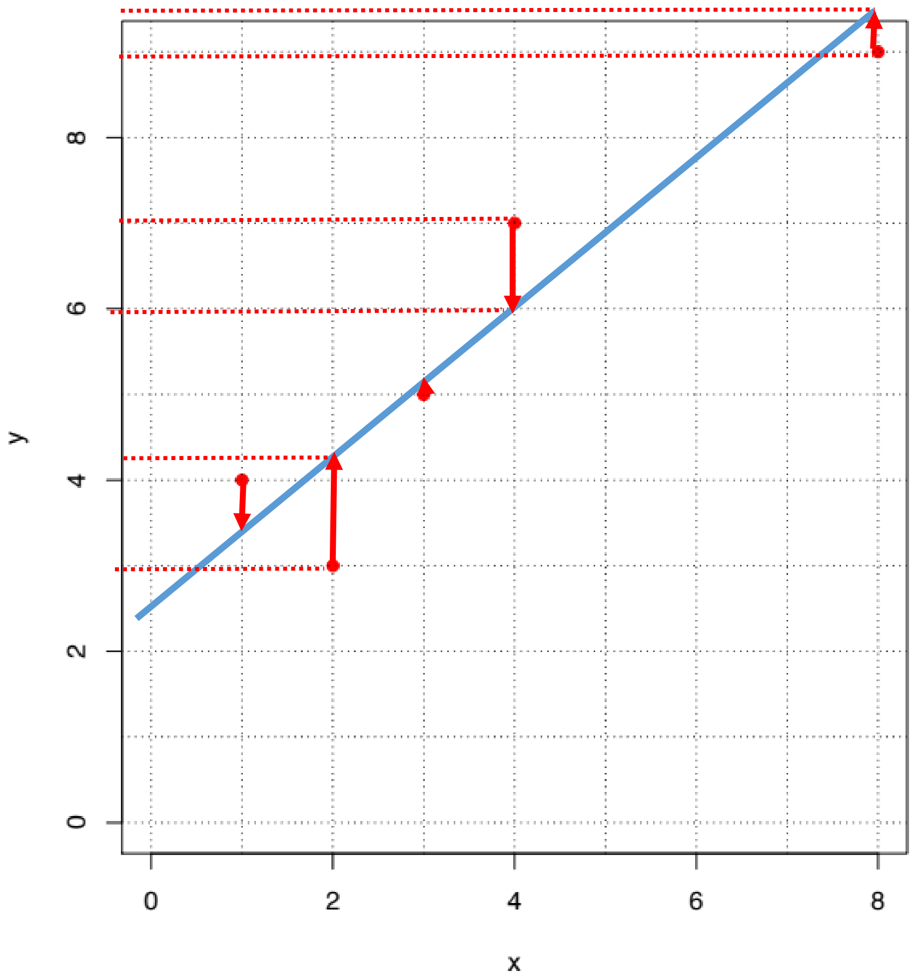
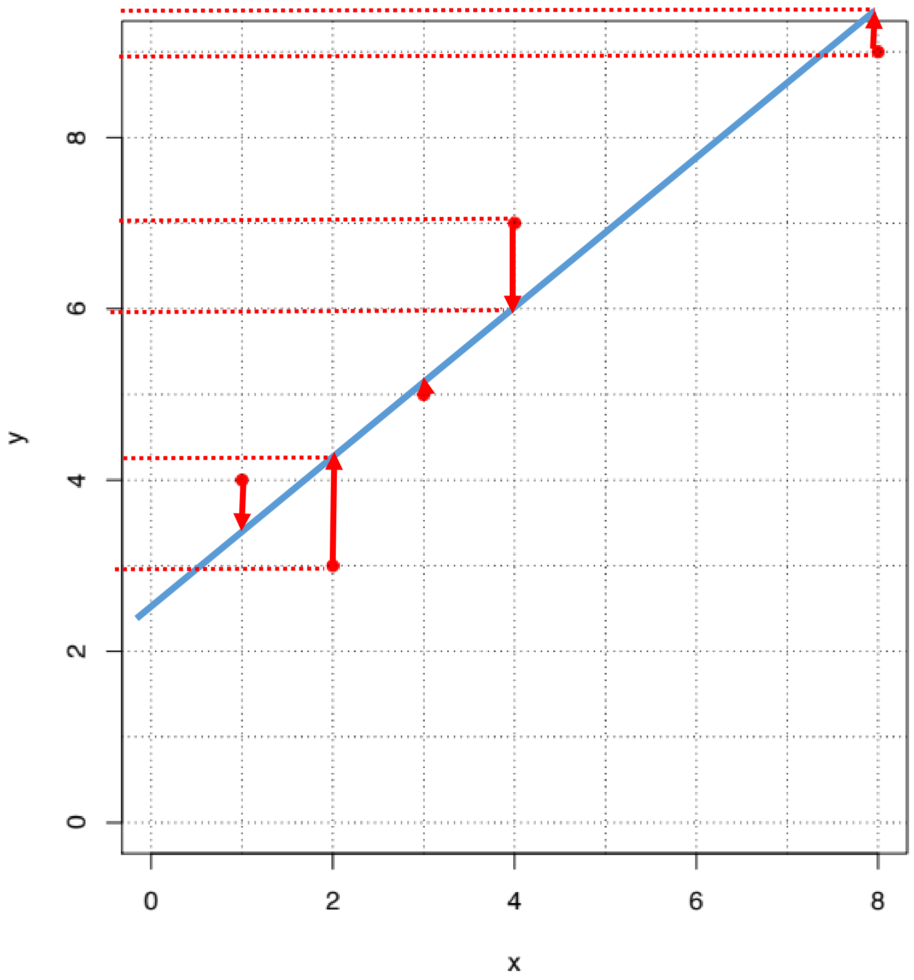


-0.7
1.2
0.1
1
0.3

$$y_i = 2.2 + 1\,x_i + \varepsilon_i$$

Let's give this a try...

- But what's with ?
- The residuals are the "error" of the model
- We get them by plotting the vertical (y) distance
- Just that R does this all for us
- But, HOW?

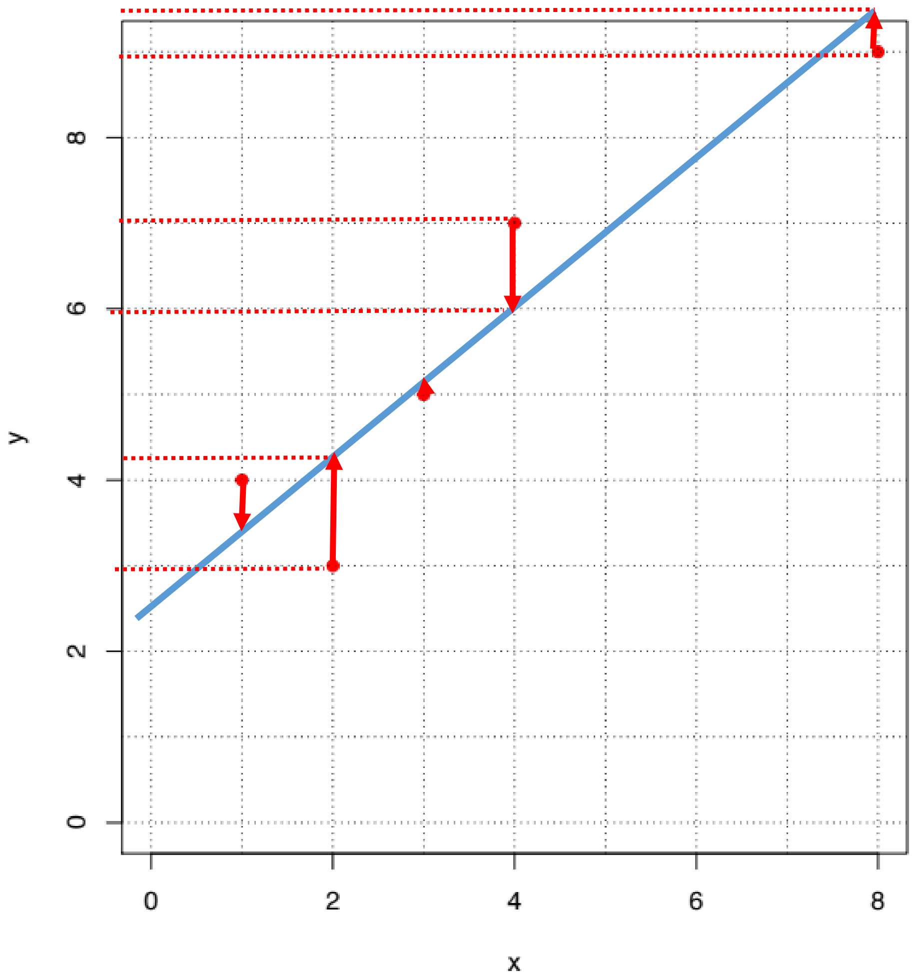


$$y_i = 2.2 + 1\,x_i + \varepsilon_i$$

-0.7
1.2
0.1
1
0.3

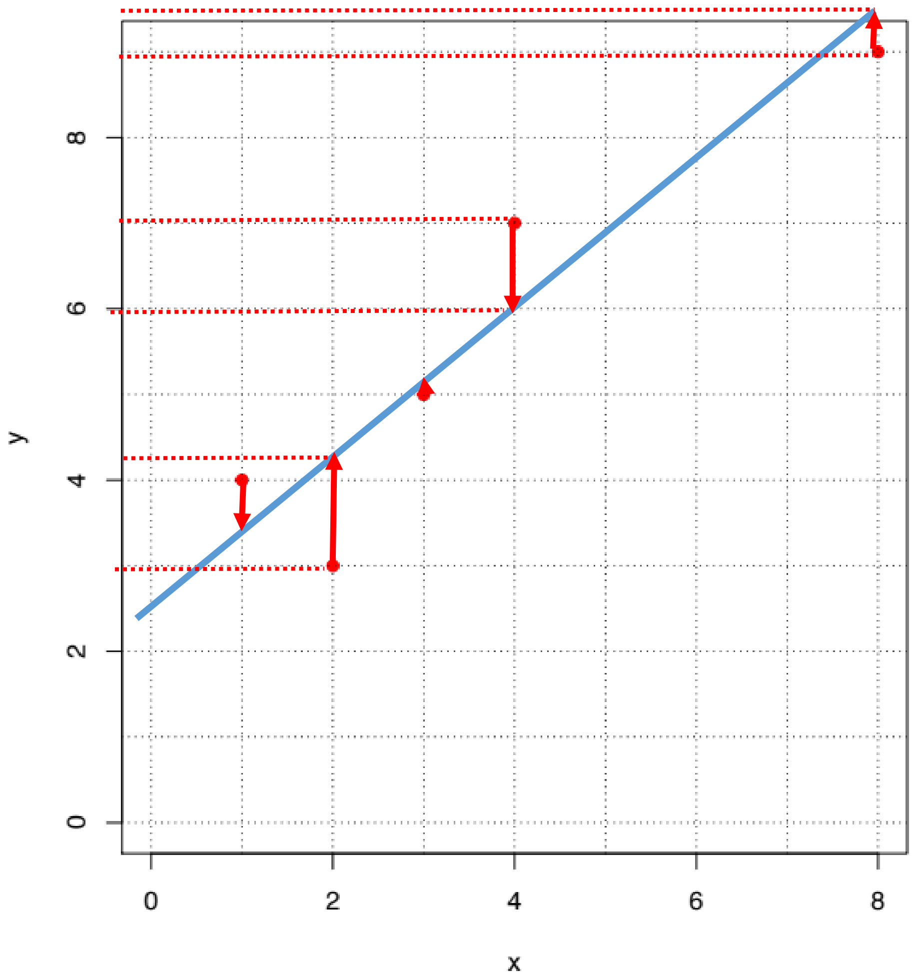
No more guesstimates

- How can we make this process scientific and mathematically tractable?



No more guesstimates

- How can we make this process scientific and mathematically tractable?
- Idea: line with smallest residuals wins!



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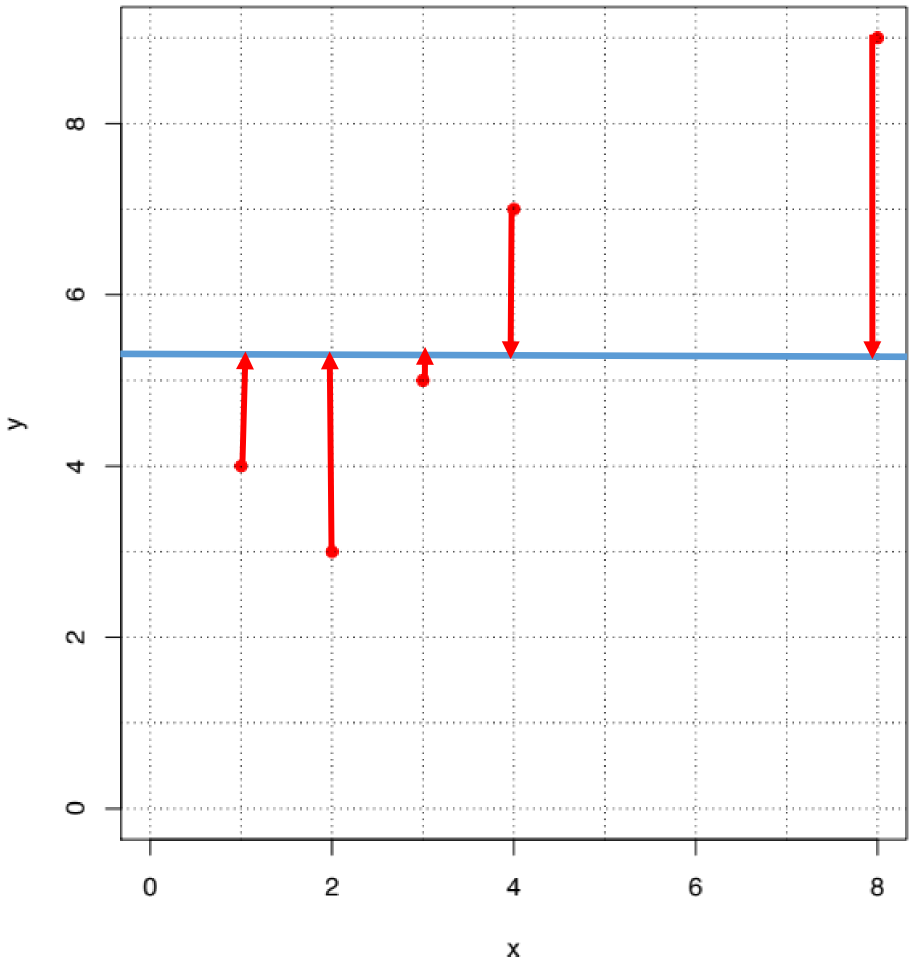
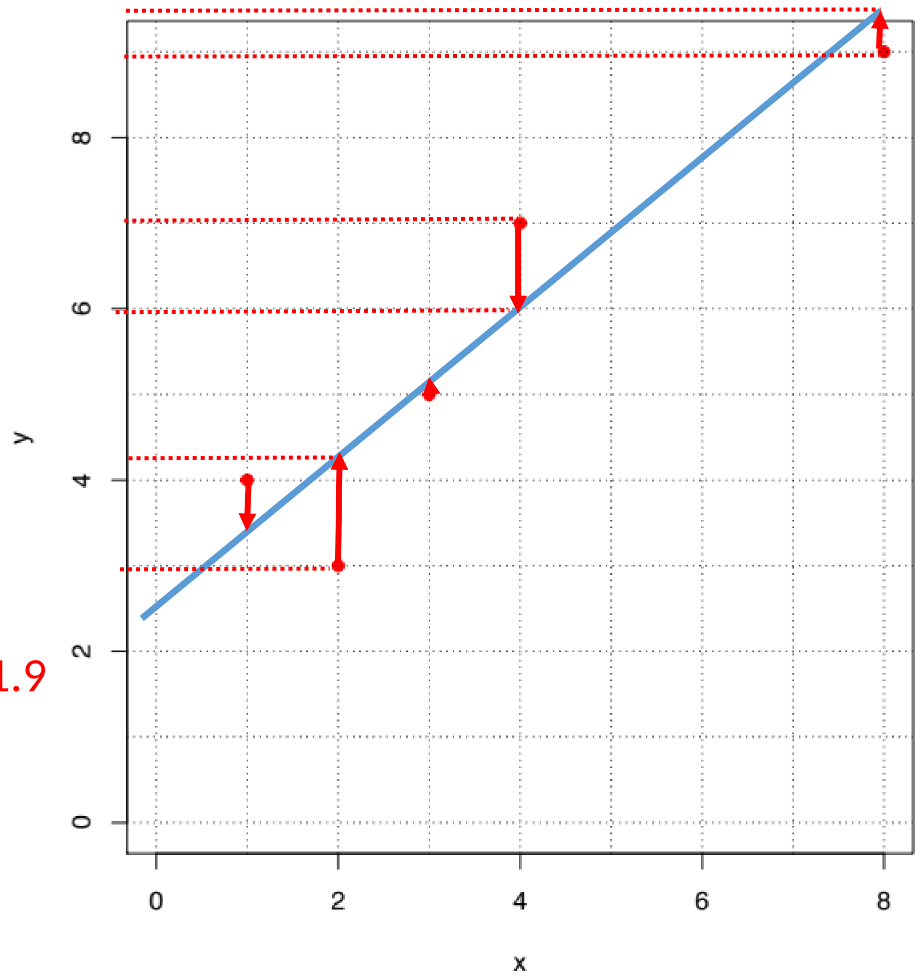
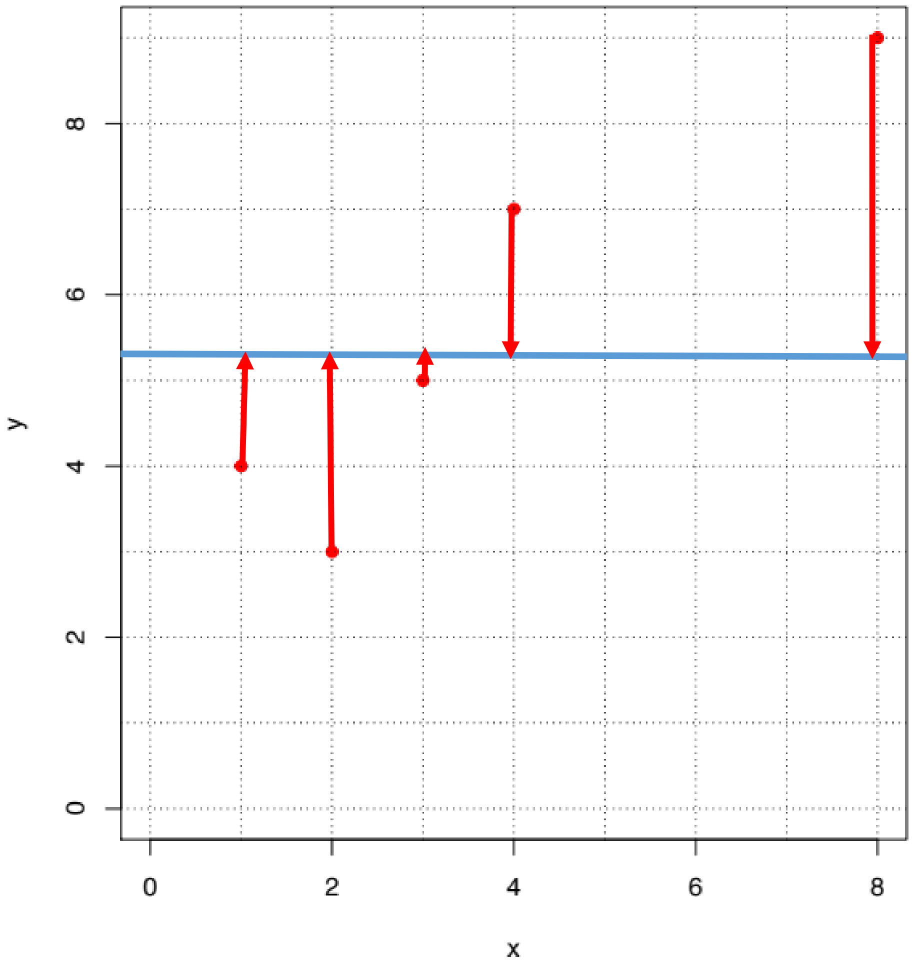


-0.7
1.2
0.1
1
0.3

Sum: 1.9



No more guesstimates

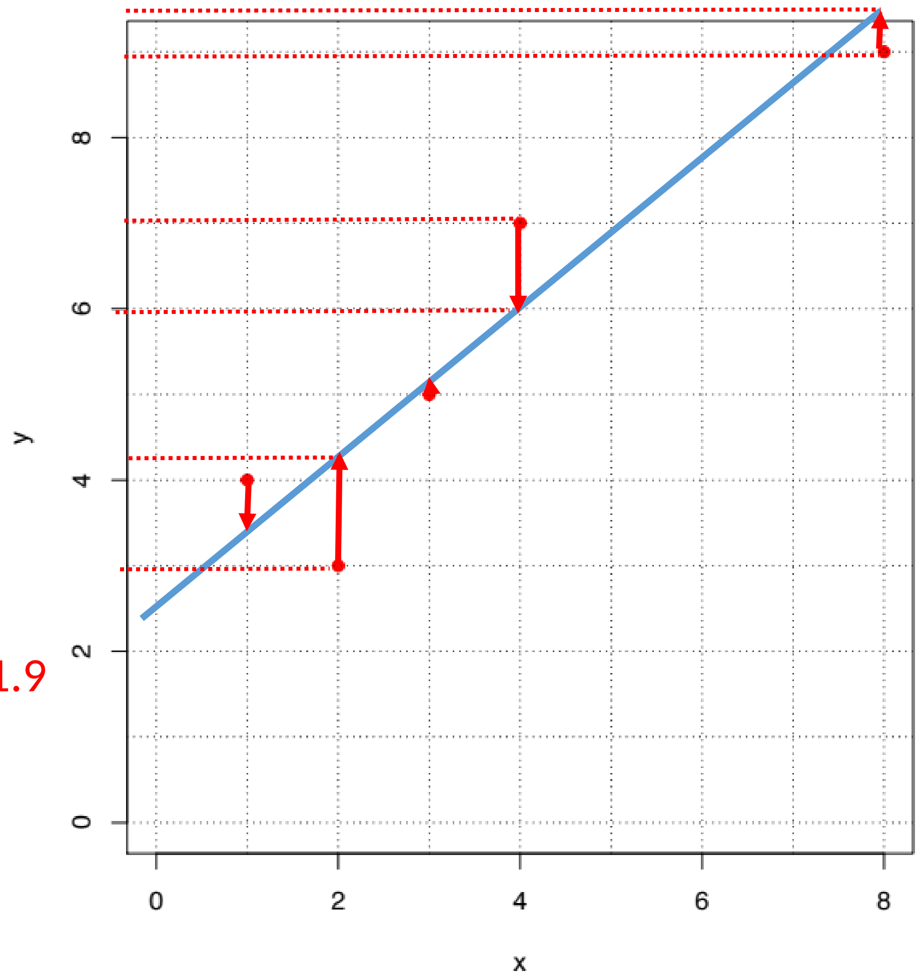


1.2
2.2
0.2
-1.8
-3.4

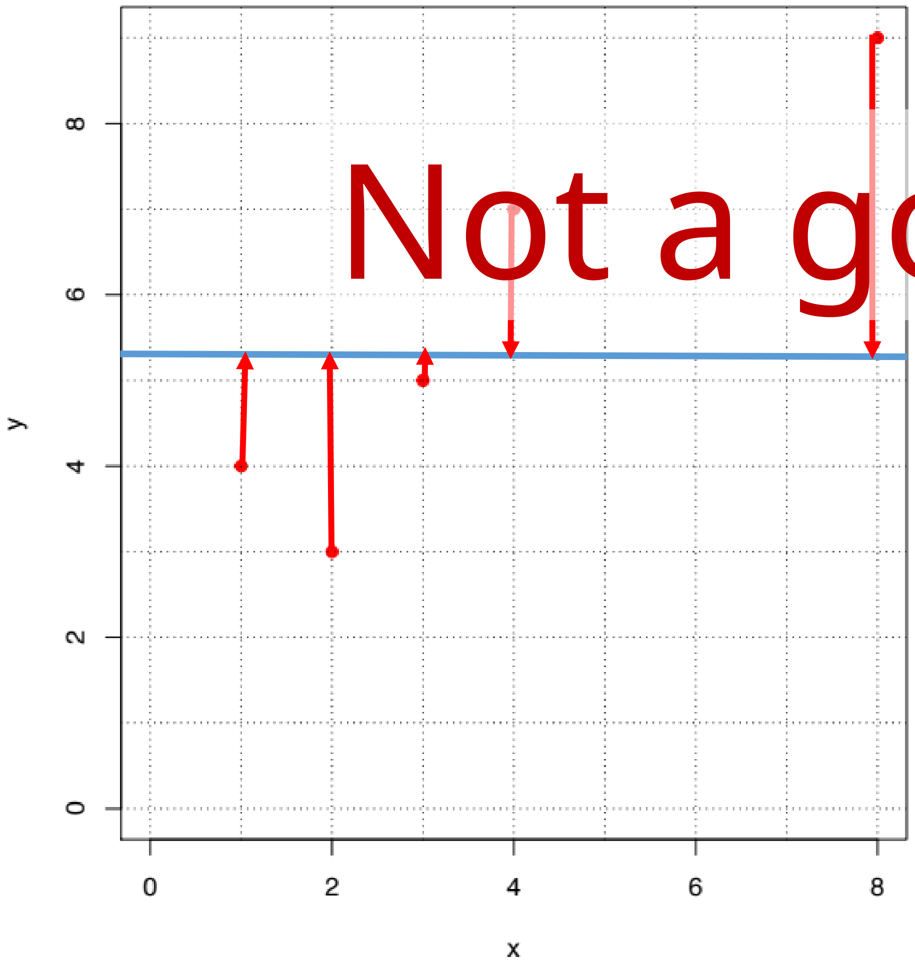
Sum: -1.6

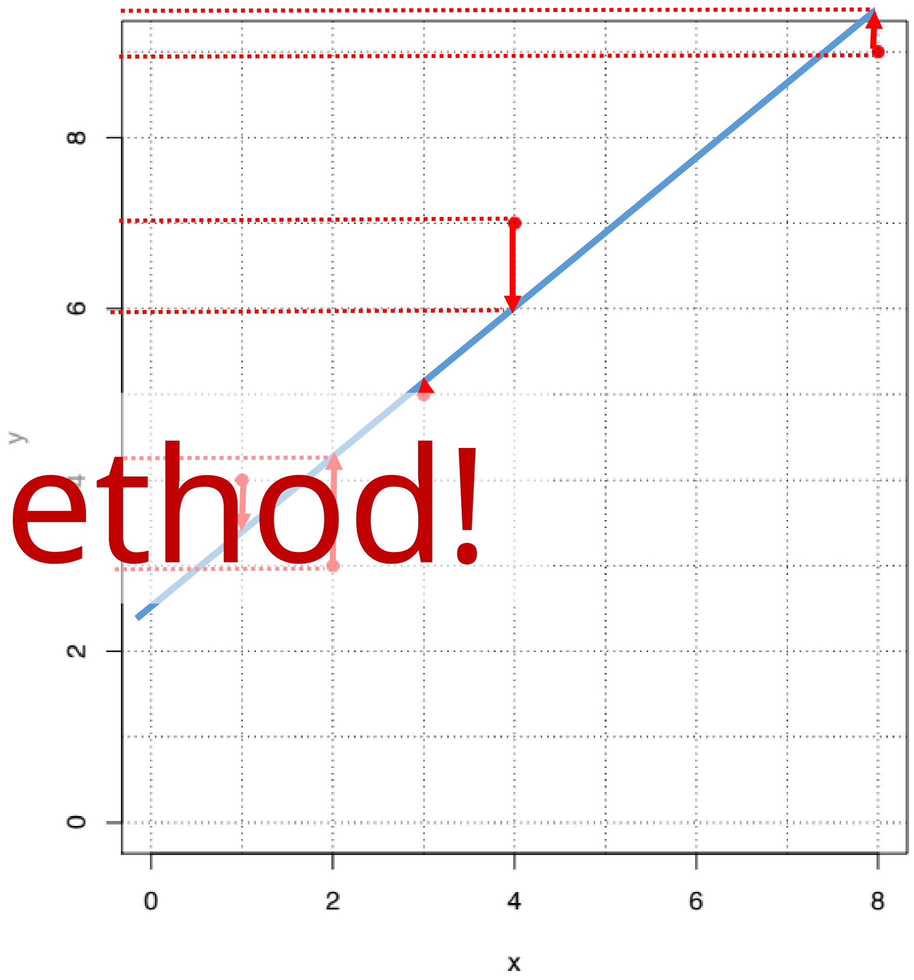
-0.7
1.2
0.1
1
0.3

Sum: 1.9



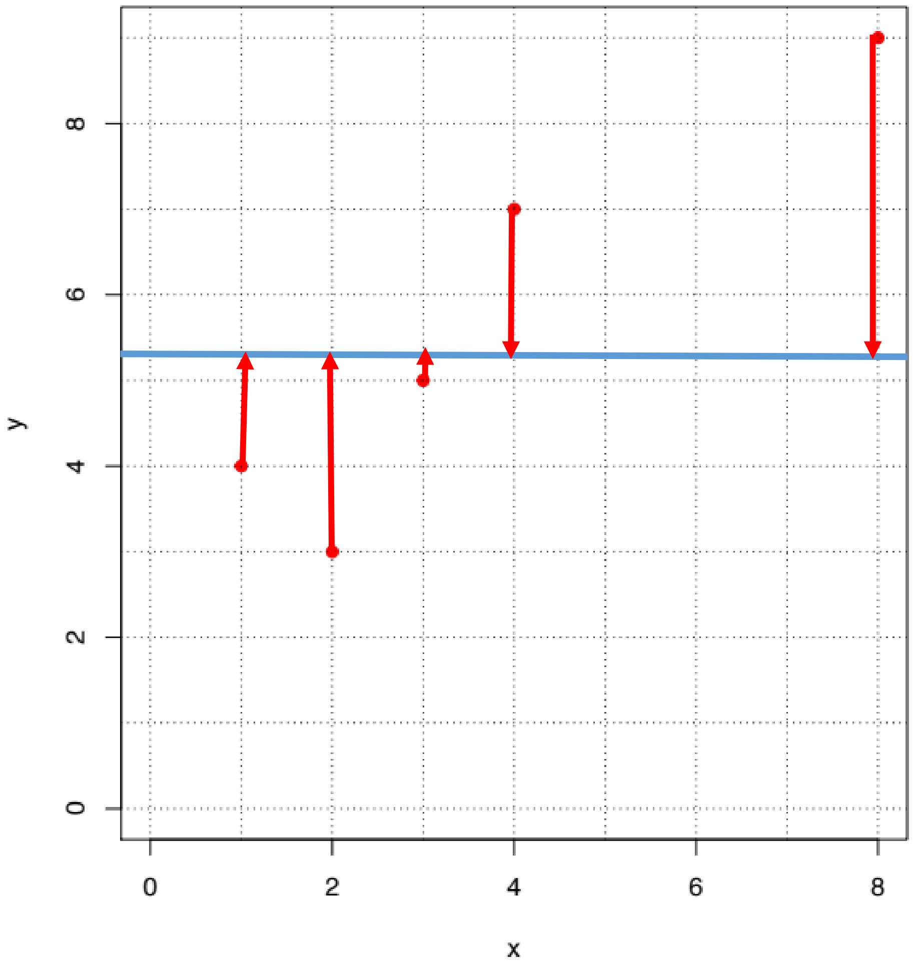
No more guesstimates



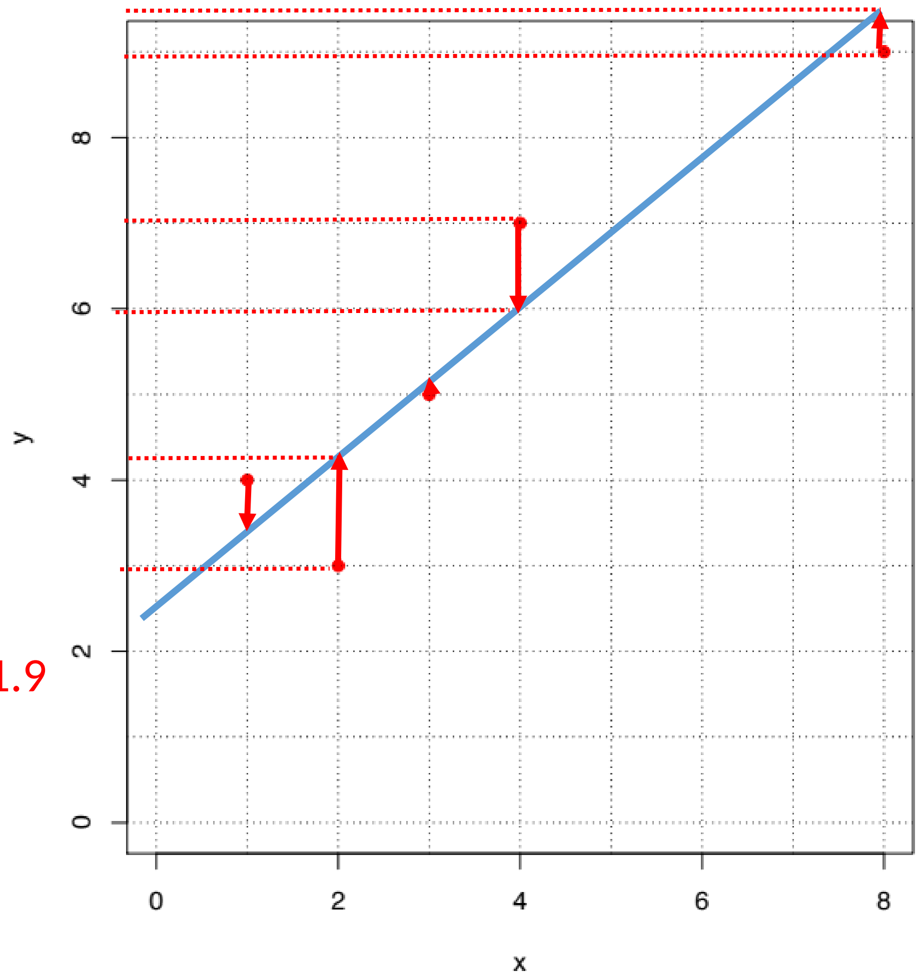


Not a good method!

No more guesstimates



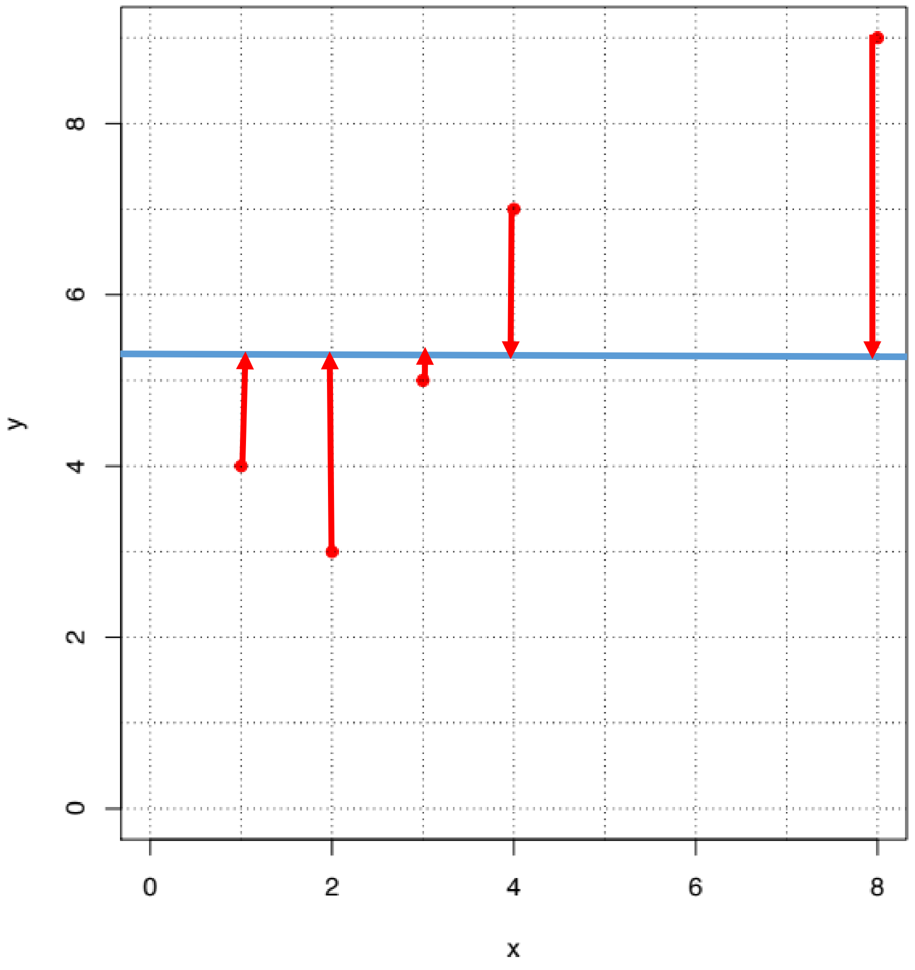
1.2	-0.7
2.2	1.2
0.2	0.1
-1.8	1
-3.4	0.3
Sum: -1.6	Sum: 1.9



Need to ensure residuals are evaluated as absolute value

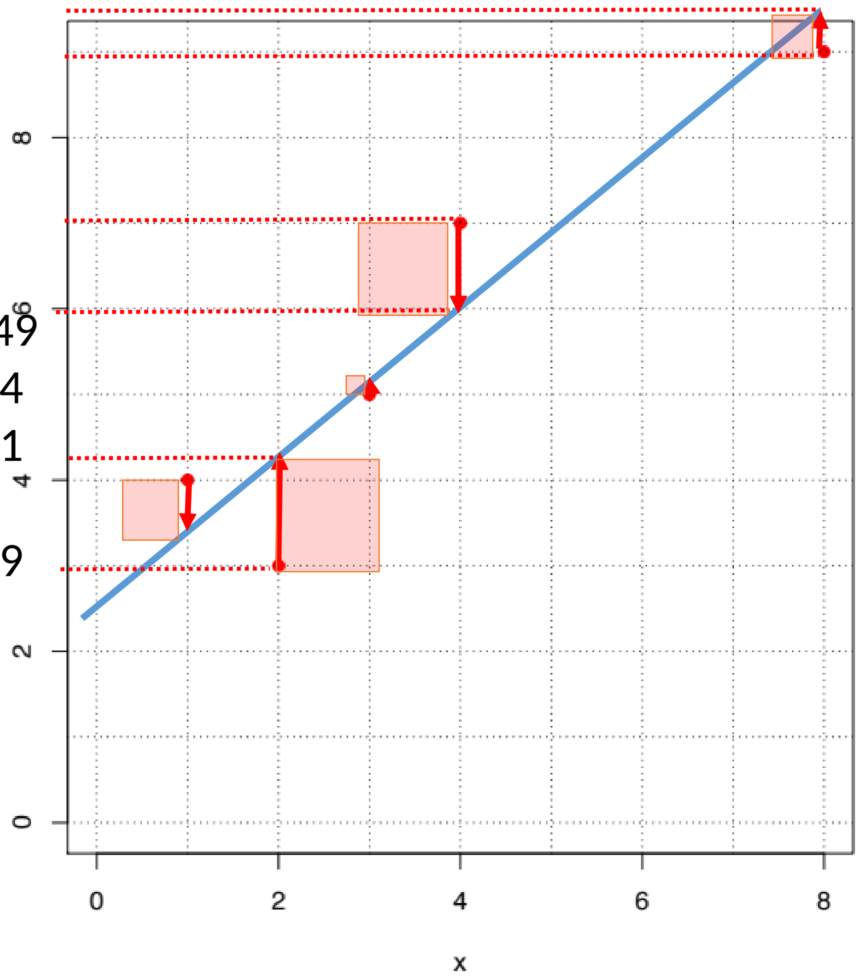
Square all of the residuals!

Sums of squares:

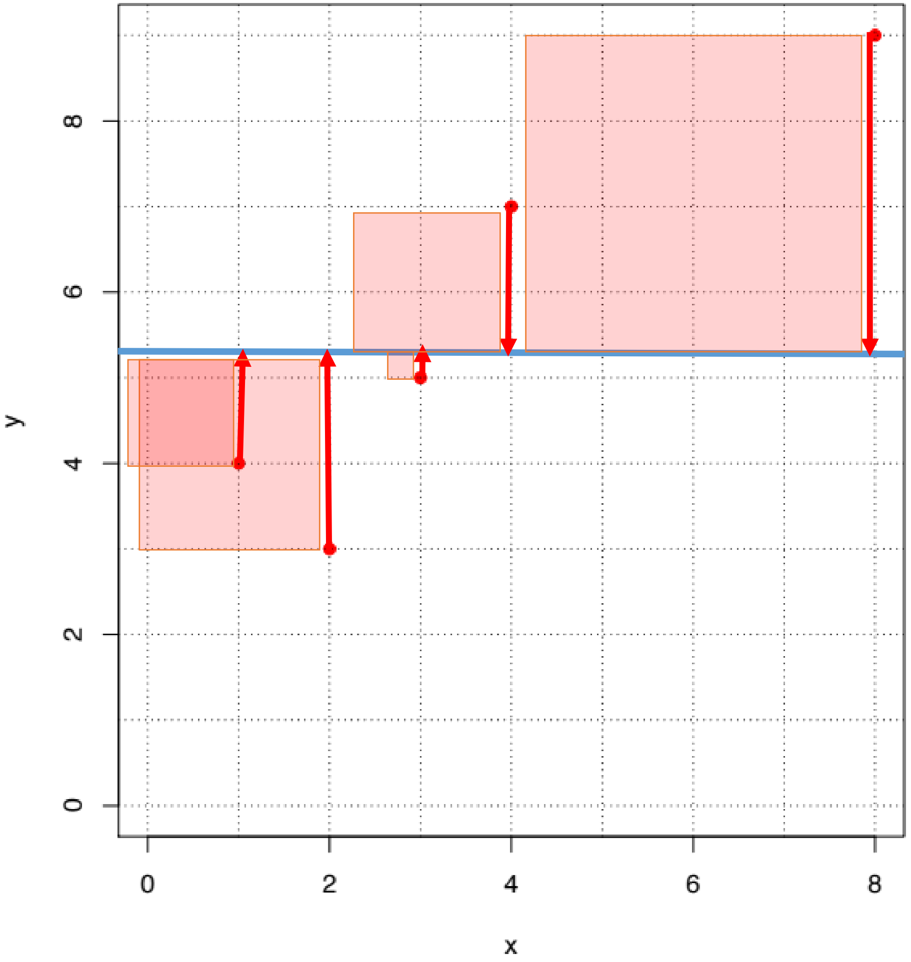


1.2
2.2
0.2
-1.8
-3.4

-0.7 -> 0.49
1.2 -> 1.44
0.1 -> 0.01
1 -> 1
0.3 -> 0.09



Sums of squares:



1.2 -> 1.44
2.2 -> 4.84
0.2 -> 0.04
-1.8 -> 3.24
-3.4 -> 11.56

-0.7 -> 0.49
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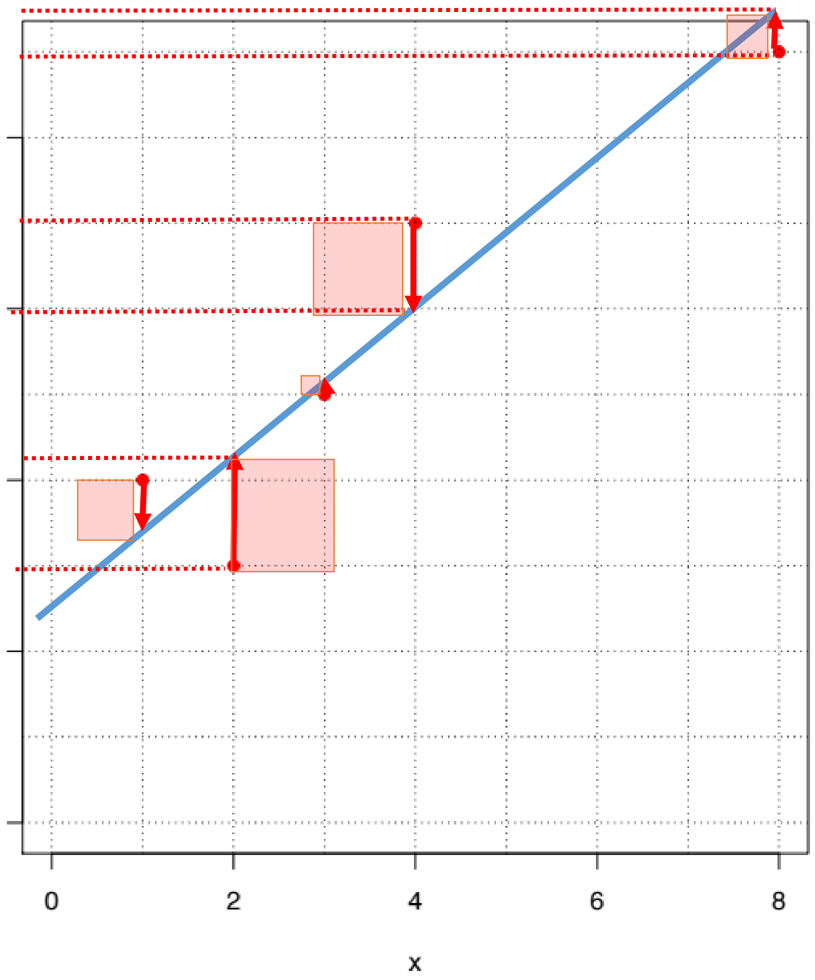
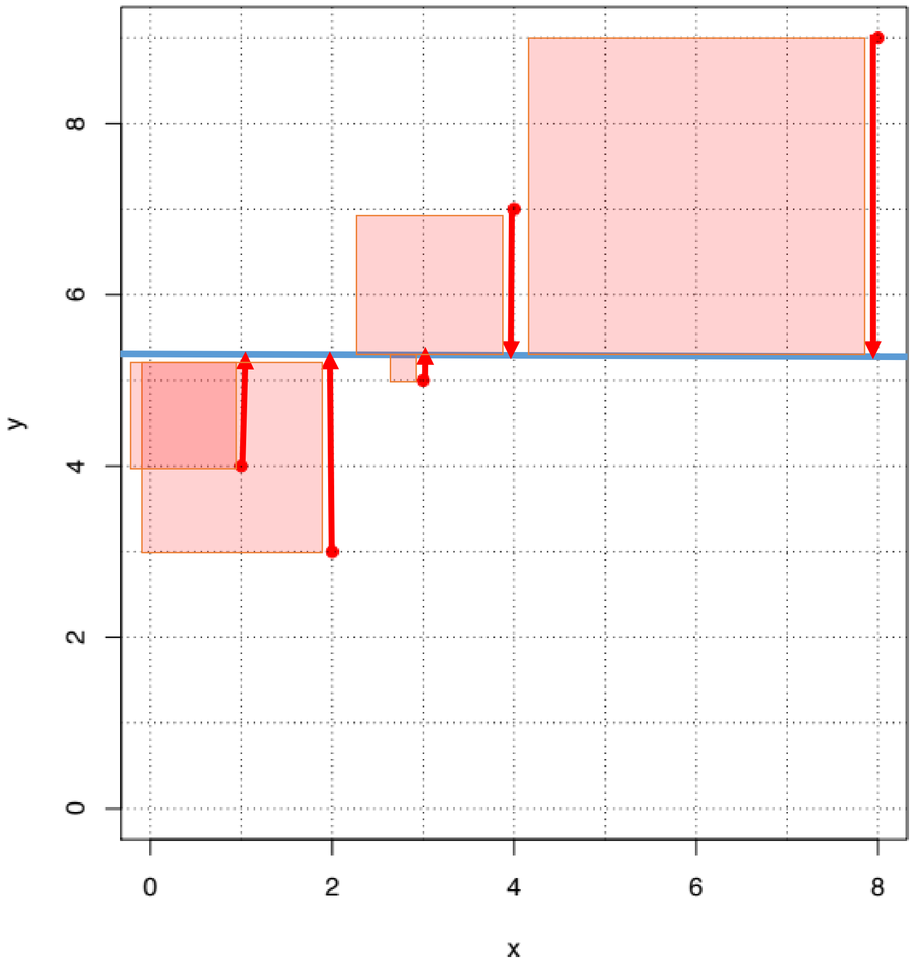


Sums of squares:



1.2 -> 1.44
2.2 -> 4.84
0.2 -> 0.04
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-3.4 -> 11.56

SS: 21.12

-0.7 -> 0.49
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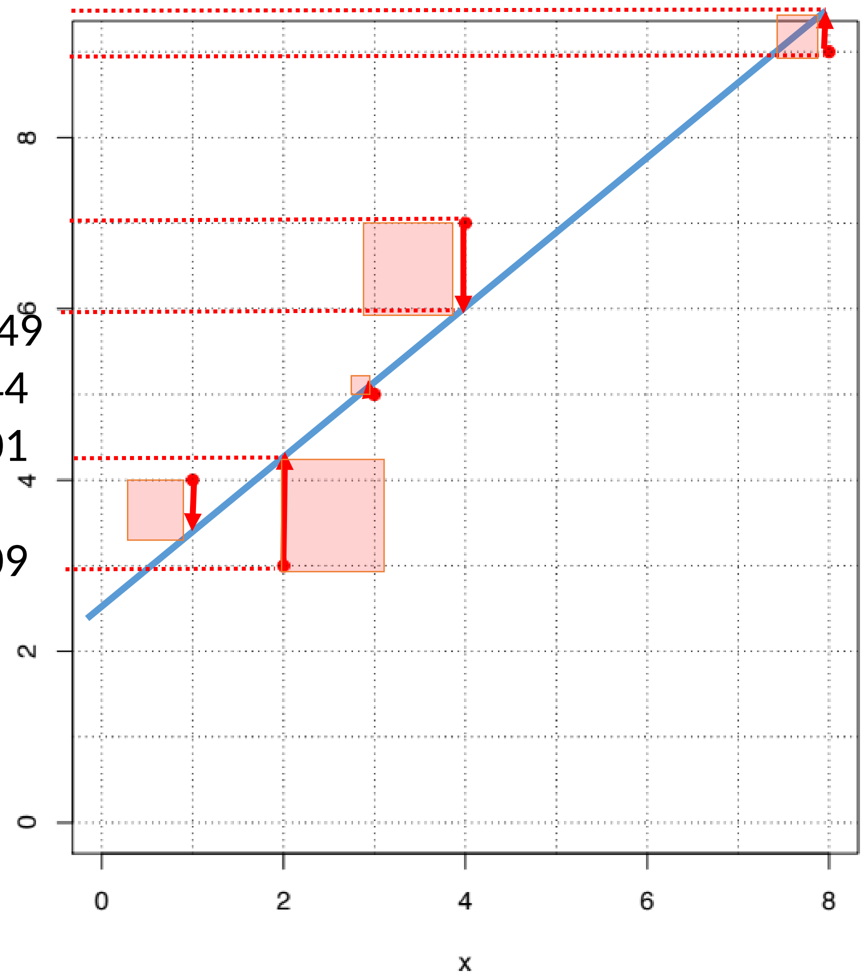
SS: 3.03



Use Sums of Squares to assess goodness of fit

- Find b_0 and b_1 of a line that is positioned so that it minimizes the sum of the squared residuals for the data y_i and x_i

Use Sums of Squares to assess goodness of fit

- Find b_0 and b_1 of a line that is positioned so that it minimizes the sum of the squared residuals for the data y_i and x_i

- Solve for b_1, and b_0

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ALGEBRA

- Solve for b_1, and b_0

Use Sums of Squares to assess goodness of fit

- Find b_0 and b_1 of a line that is positioned so that it minimizes the sum of the squared residuals for the data y_i and x_i

ALGEBRA

- Solve for b_1, and b_0

$$\overline{y} = b_0 + b_1 \overline{x}$$

$$b_0 = \overline{y} - b_1 \overline{x}$$

Use Sums of Squares to assess goodness of fit

- Find b_0 and b_1 of a line that is positioned so that it minimizes the sum of the squared residuals for the data y_i and x_i

ALGEBRA

- Solve for b_1, and b_0

$$b_0 = \bar{y} - b_1 \bar{x} \qquad b_1 = \frac{\sum x_i y_i - \frac{1}{n}\sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2}$$

Use Sums of Squares to assess goodness of fit

- Find b_0 and b_1 of a line that is positioned so that it minimizes the sum of the squared residuals for the data y_i and x_i

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Use Sums of Squares to assess goodness of fit

- Find b_0 and b_1 of a line that is positioned so that it minimizes the sum of the squared residuals for the data y_i and x_i

ALGEBRA

- Solve for b_1, and b_0

Covariance between x and y

$$b_0 = \overline{y} - b_1 \overline{x}$$

$$b_1 = \frac{\sum x_i y_i - \frac{1}{n} \sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n} \left(\sum x_i\right)^2}$$

$$¿\, \frac{Cov[x, y]}{\sigma_x^2}$$

Use Sums of Squares to assess goodness of fit

- Find b_0 and b_1 of a line that is positioned so that it minimizes the sum of the squared residuals for the data y_i and x_i

ALGEBRA

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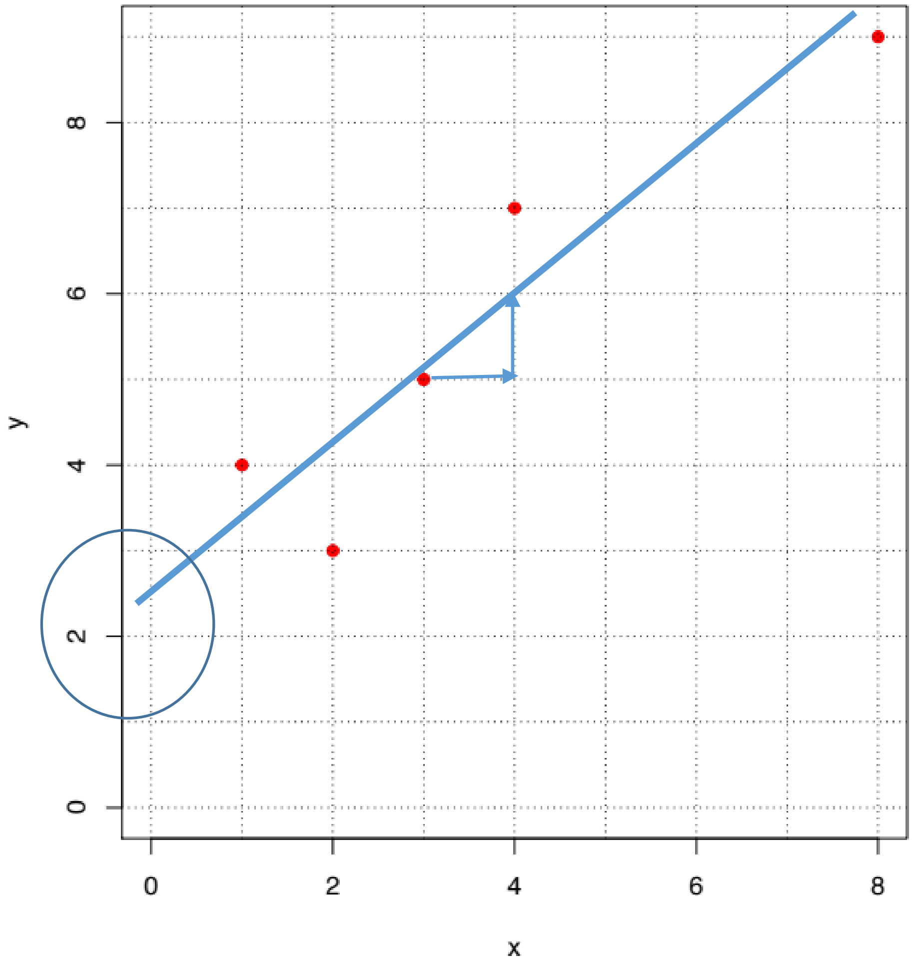
b_1

Variance of x

Let’s give this a try…

- Let’s plot this
- Now we “guesstimate” the line
- Now we “guesstimate” b_0 and b_1:
- Intercept: something 2.2
- Slope: close enough to 1
- But what’s with ?

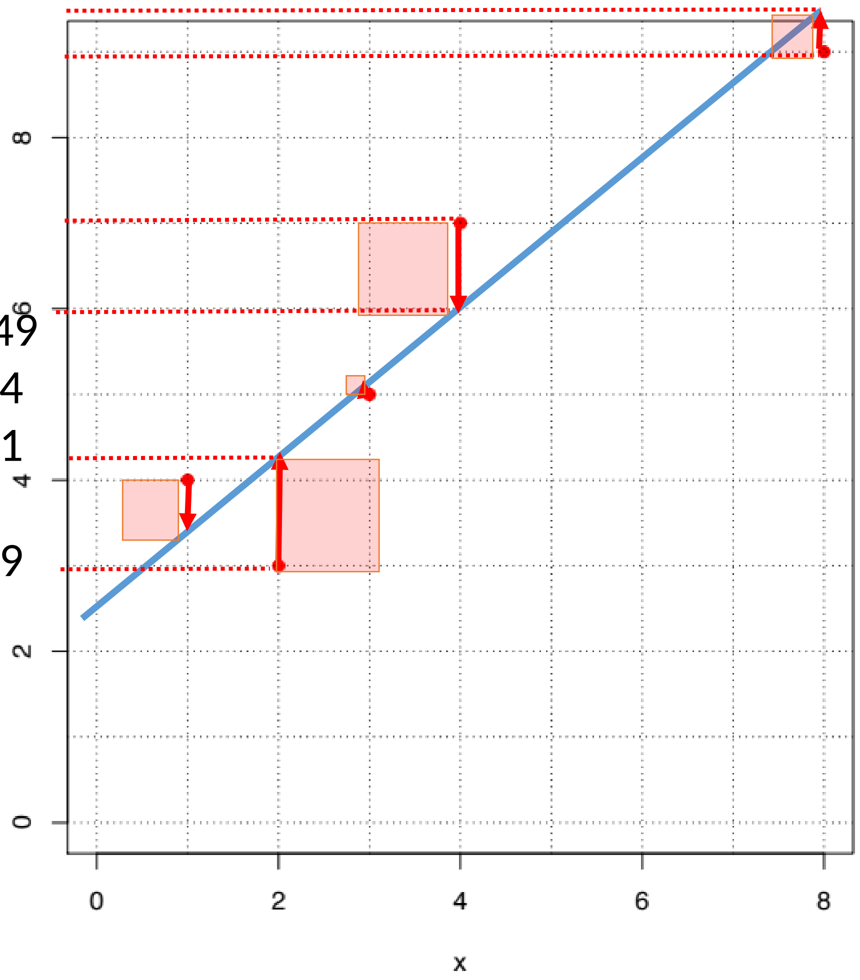


$$y_i = 2.2 + 1\,x_i + \varepsilon_i$$

Let’s give this a try:

-0.7 -> 0.49
1.2 -> 1.44
0.1 -> 0.01
1 -> 1
0.3 -> 0.09

SS: 3.03



$$b_0 = \overline{y} - b_1 \overline{x}$$

b_1

$$y_i = 2.2 + 1 x_i + \varepsilon_i$$

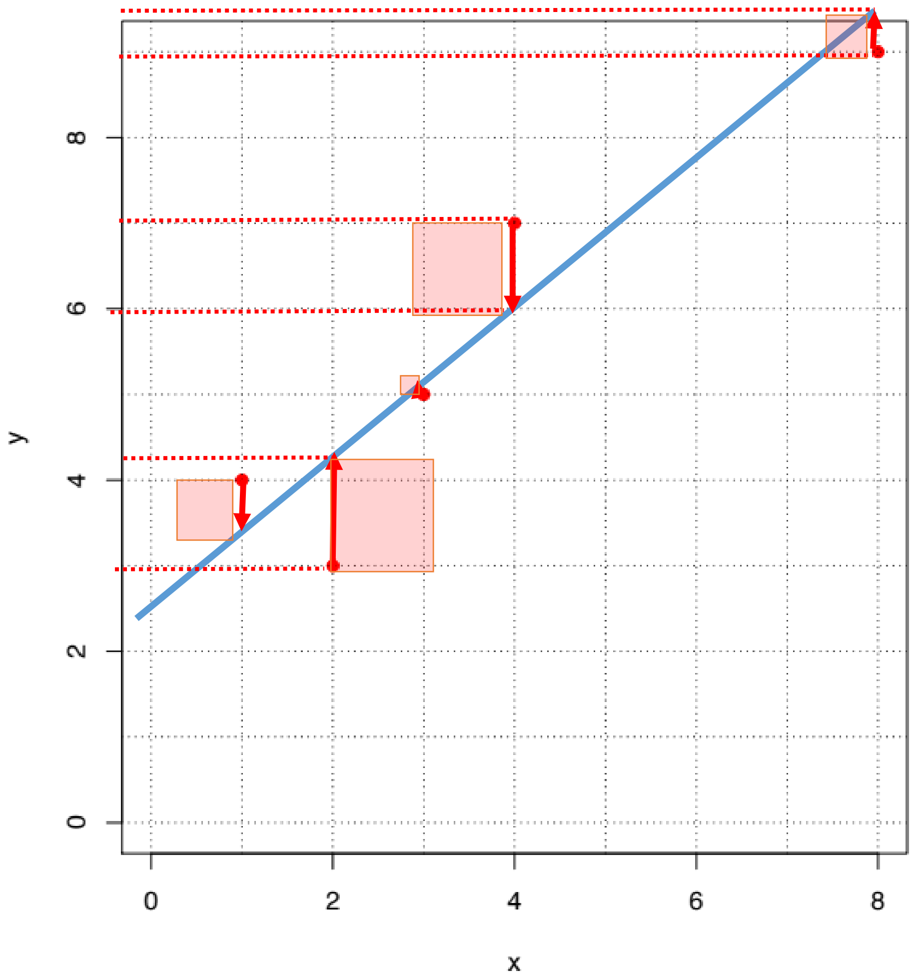
SS: 3.03

Let’s give this a try:

x, y
1,4
2,3
3.5
4,7
8,9

$$b_0 = \overline{y} - b_1 \overline{x}$$

b_1



$$y_i = 2.2 + 1\, x_i + \varepsilon_i$$

SS: 3.03

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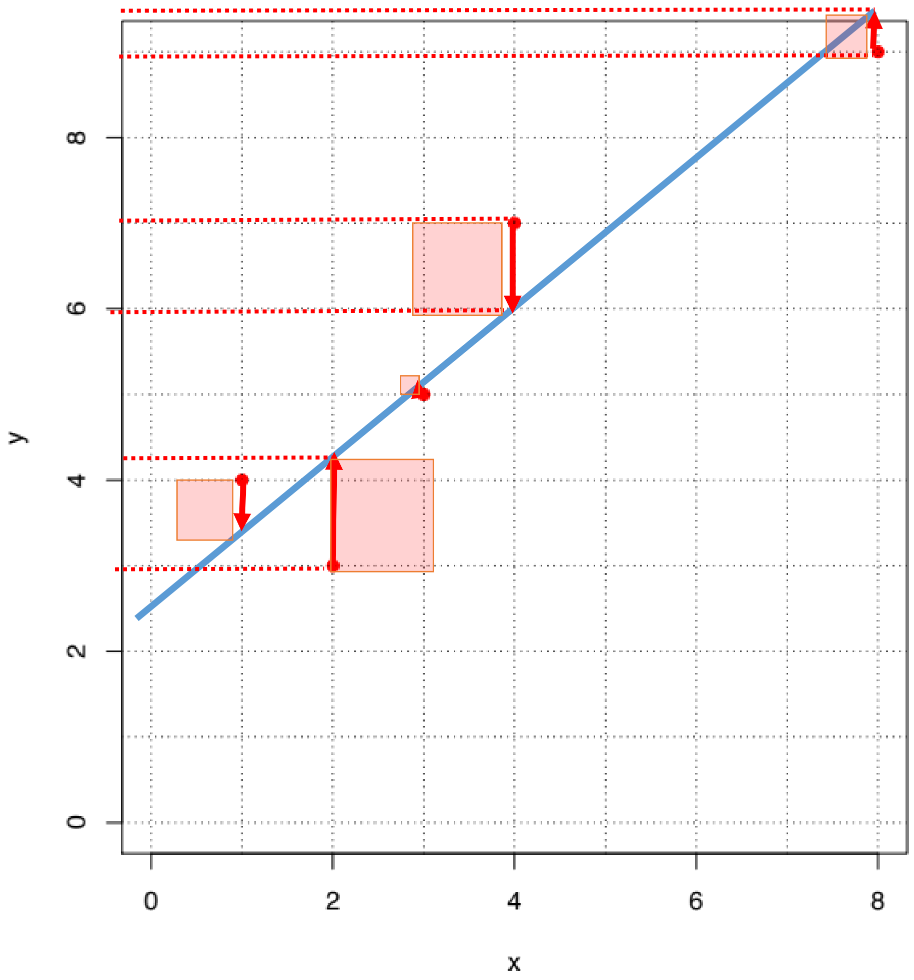
2,3
3.5
4,7
8,9



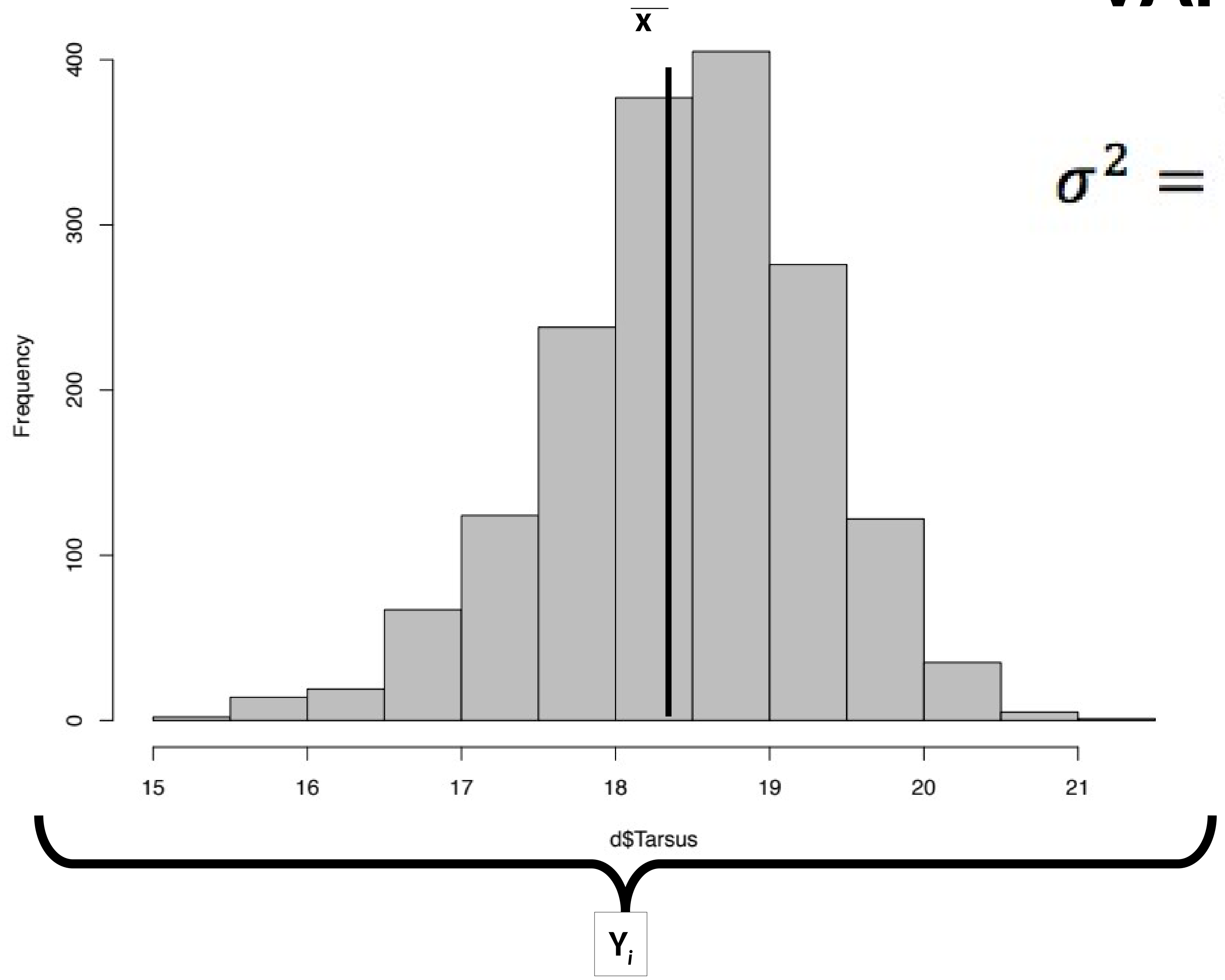
$$b_0 = \bar{y} - b_1 \bar{x}$$

b_1

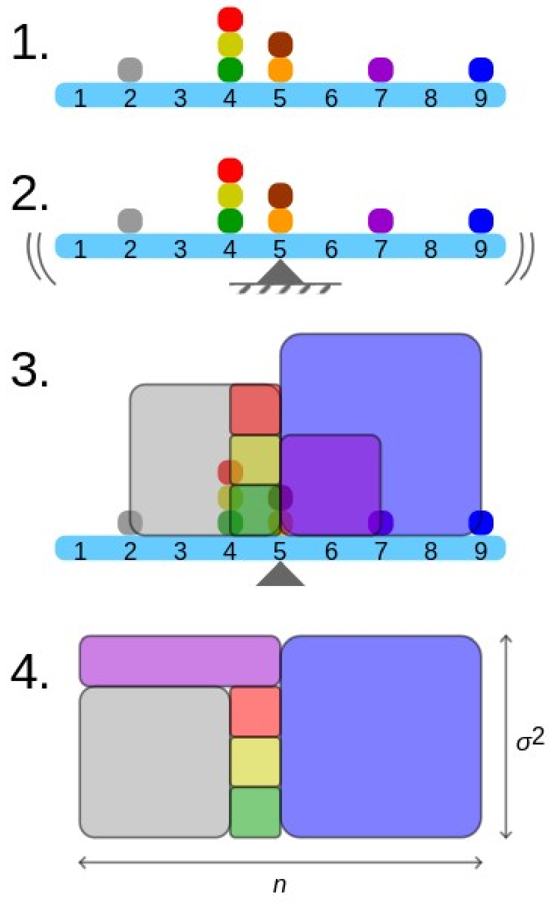
$$y_i = 2.2 + 1\, x_i + \varepsilon_i$$

SS: 3.03

VARIANCE



$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}$$



Let's give this a try:

x, y
1,4
2,3
3.5
4,7
8,9

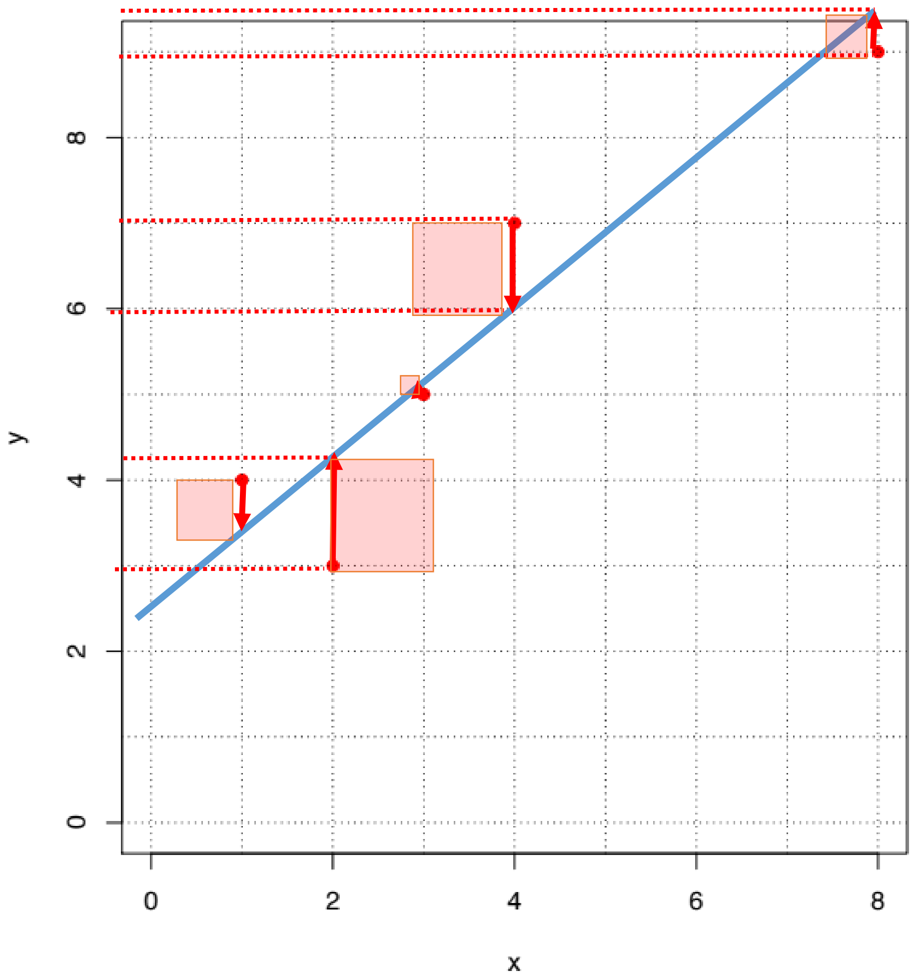
$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

=

=

$$b_0 = \bar{y} - b_1 \bar{x}$$

b_1



$$y_i = 2.2 + 1\,x_i + \varepsilon_i$$

SS: 3.03

Let’s give this a try:

x, y
1,4
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=

=

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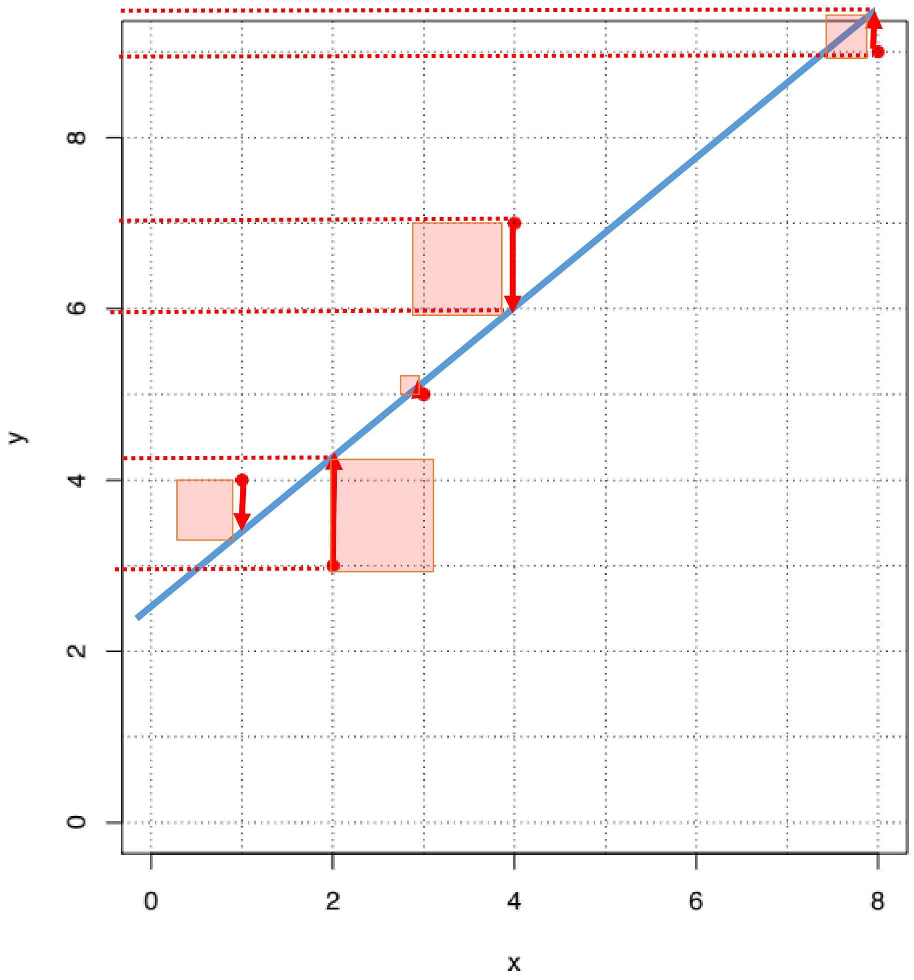
b_1



$$y_i = 2.2 + 1\, x_i + \varepsilon_i$$

SS: 3.03

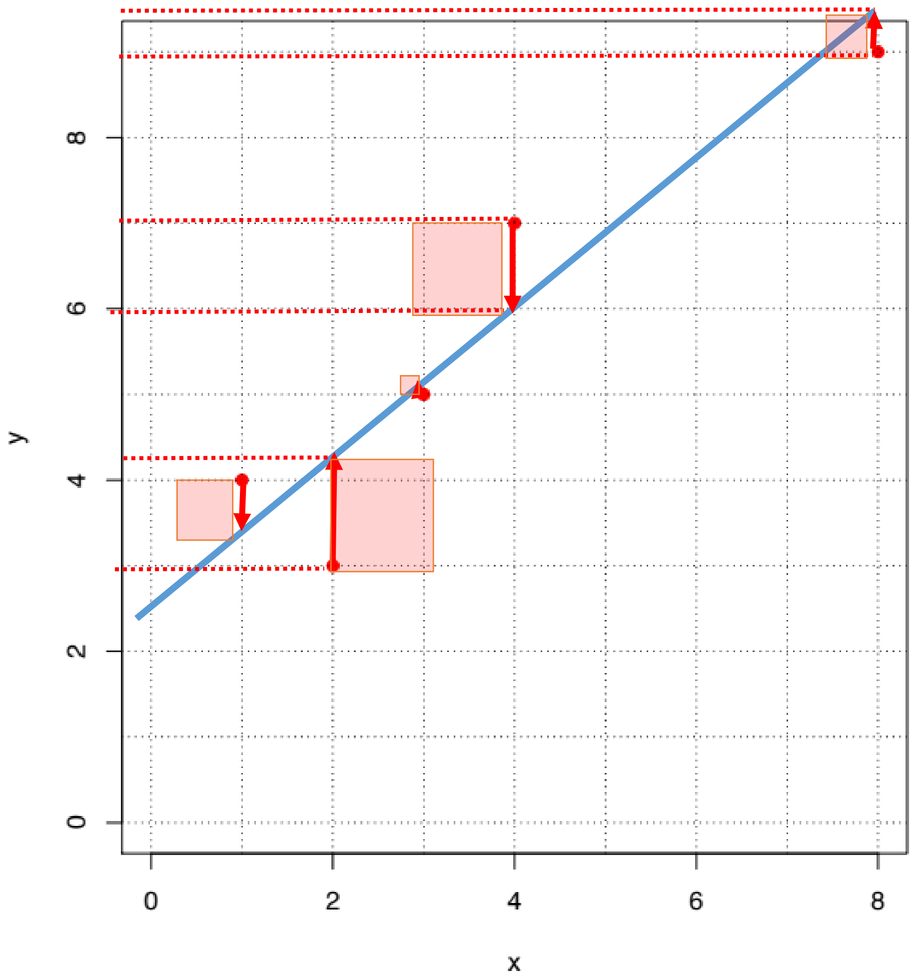
Let's give this a try:

x, y
1,4
2,3
3.5
4,7
8,9

$$Cov(x,y) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n}$$

$$b_0 = \bar{y} - b_1 \bar{x}$$

b_1



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2,3
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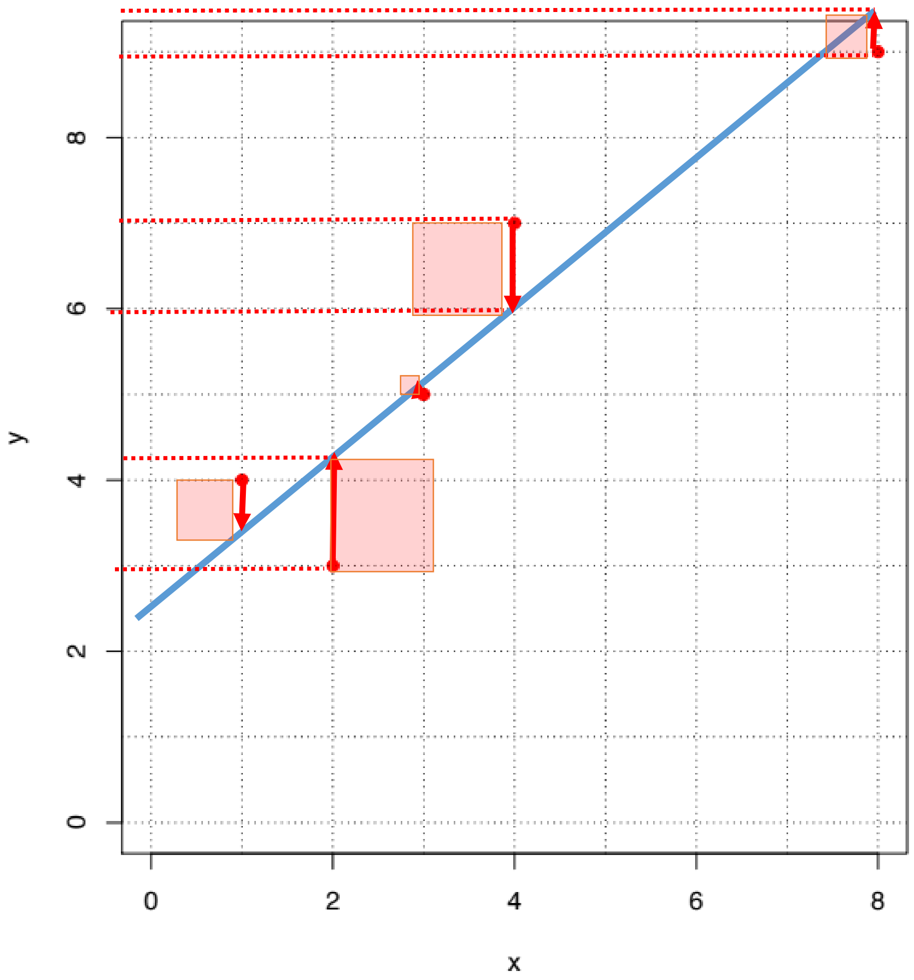
8,9

$$Cov(x,y)=\frac{\sum(x_i-\overline{x})(y_i-\overline{y})}{n}$$

=

$$b_0=\overline{y}-b_1\overline{x}$$

b_1



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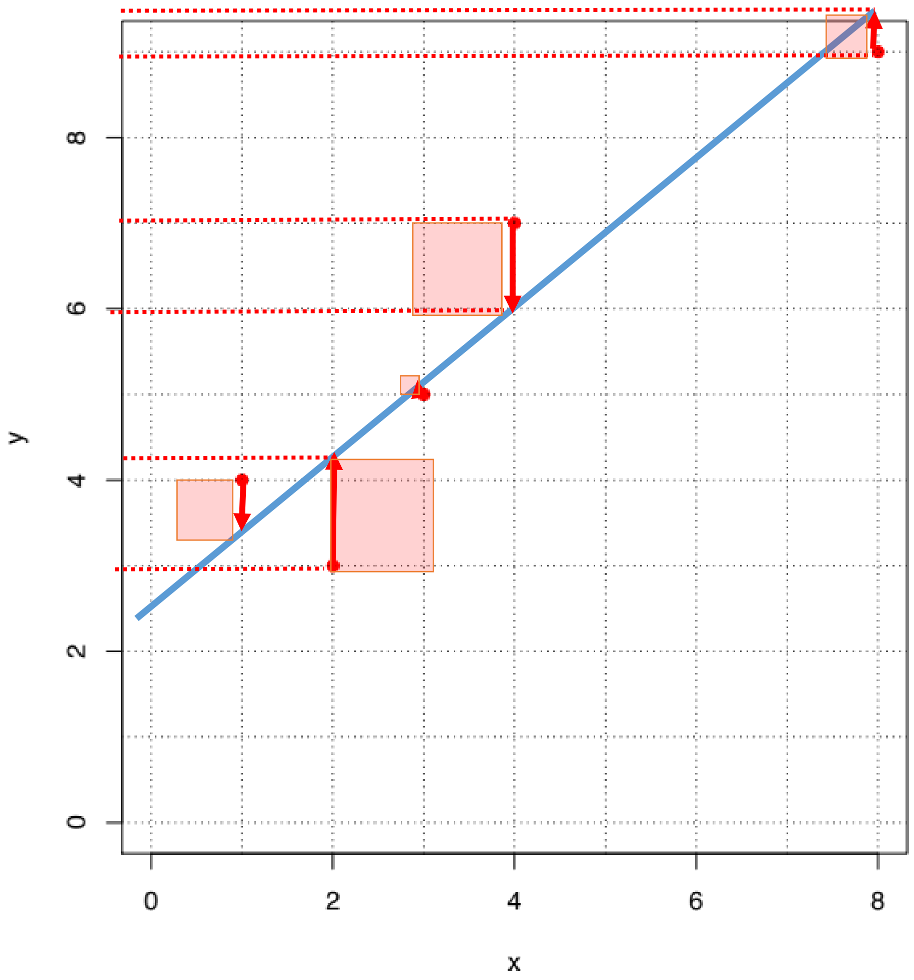

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=

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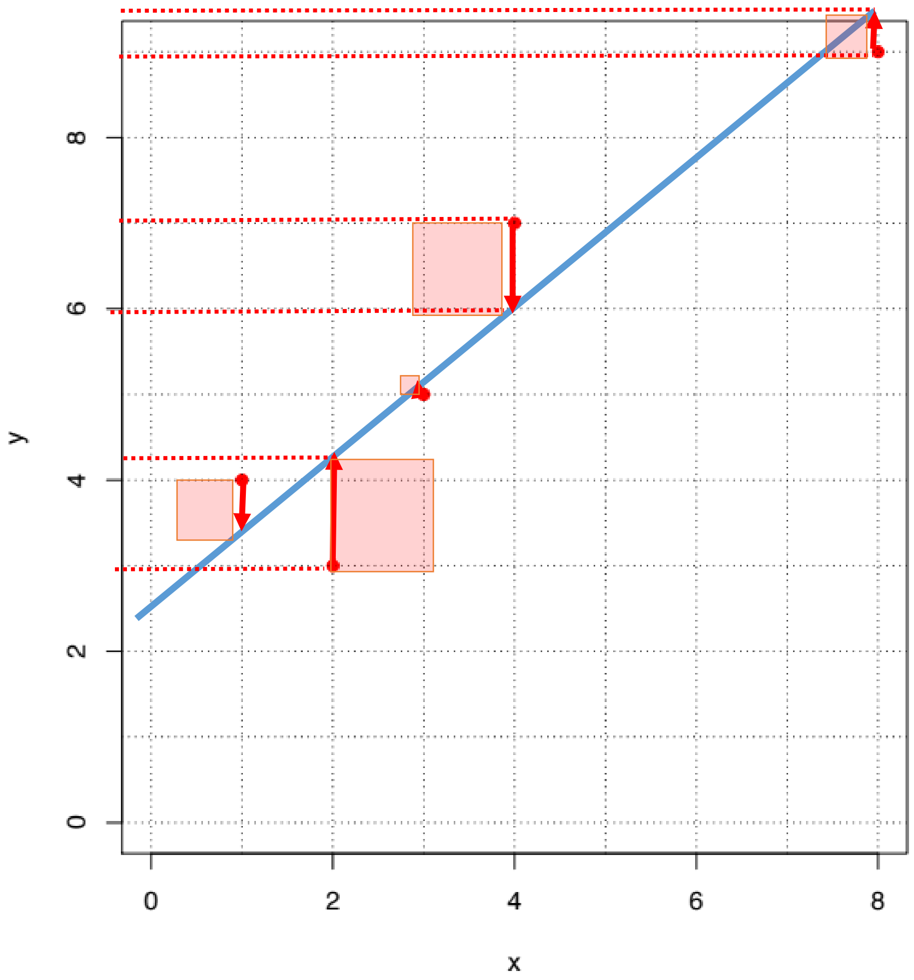
b_1



$$y_i = 2.2 + 1\,x_i + \varepsilon_i$$

SS: 3.03

Let’s give this a try:

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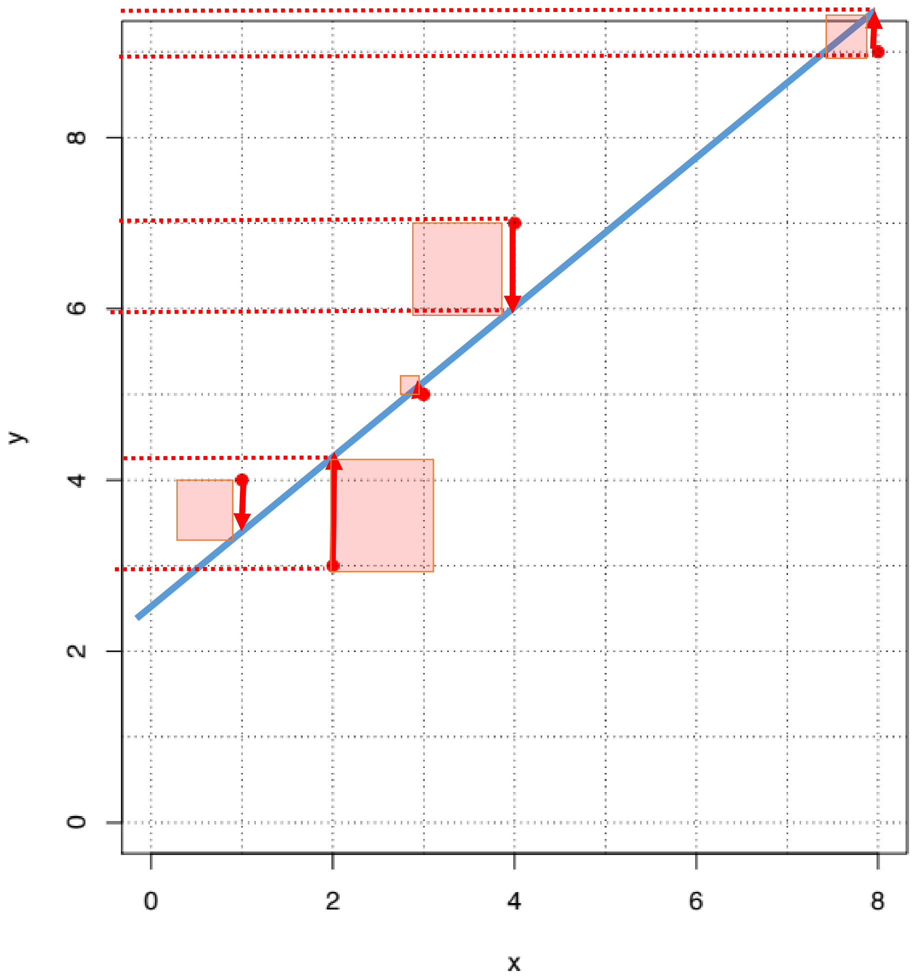
2,3
3.5
4,7
8,9

$$Cov(x,y)=\frac{\sum(x_i-\overline{x})(y_i-\overline{y})}{n}$$

=



$$y_i = 2.2 + 0.83\,x_i + \varepsilon_i$$

b_1

SS: 3.03

Let’s give this a try:

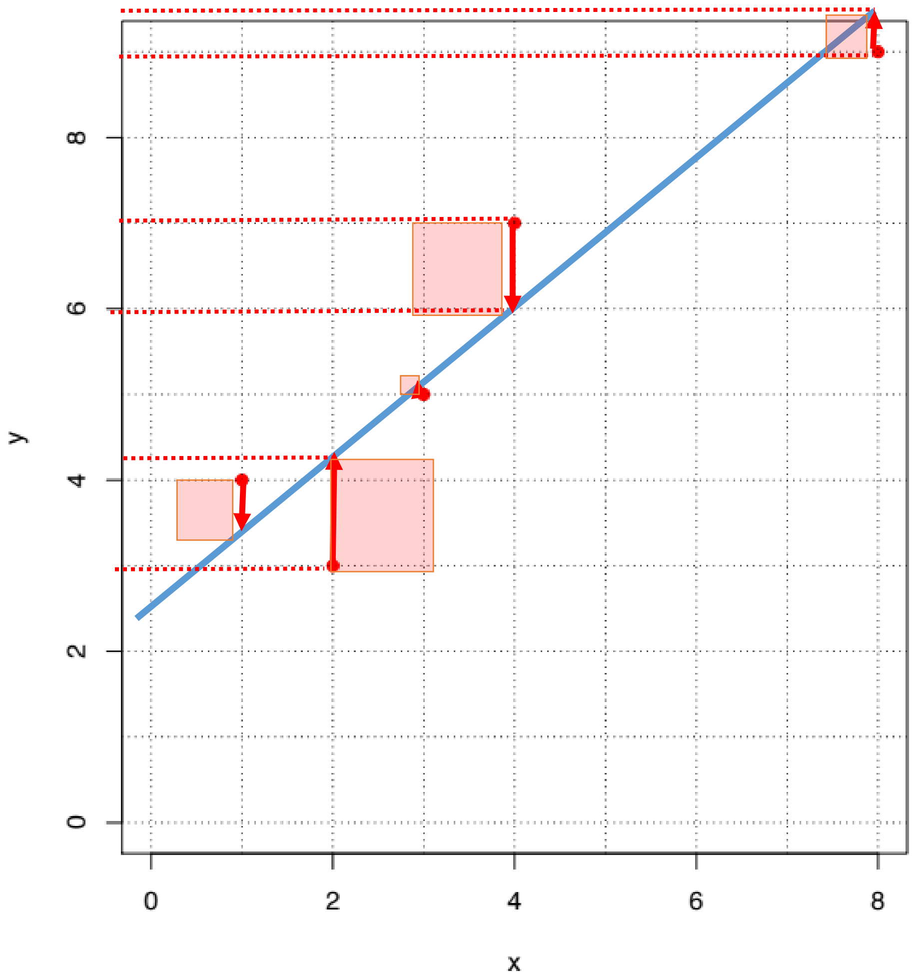
x, y
1,4
2,3
3.5
4,7
8,9

$$Cov(x,y)=\frac{\sum (x_i-\overline{x})(y_i-\overline{y})}{n}$$

=

= 5.6 – 0.83 * 3.6

b_1



$$y_i = 2.2 + 0.83\, x_i + \varepsilon_i$$

SS: 3.03

Let's give this a try:

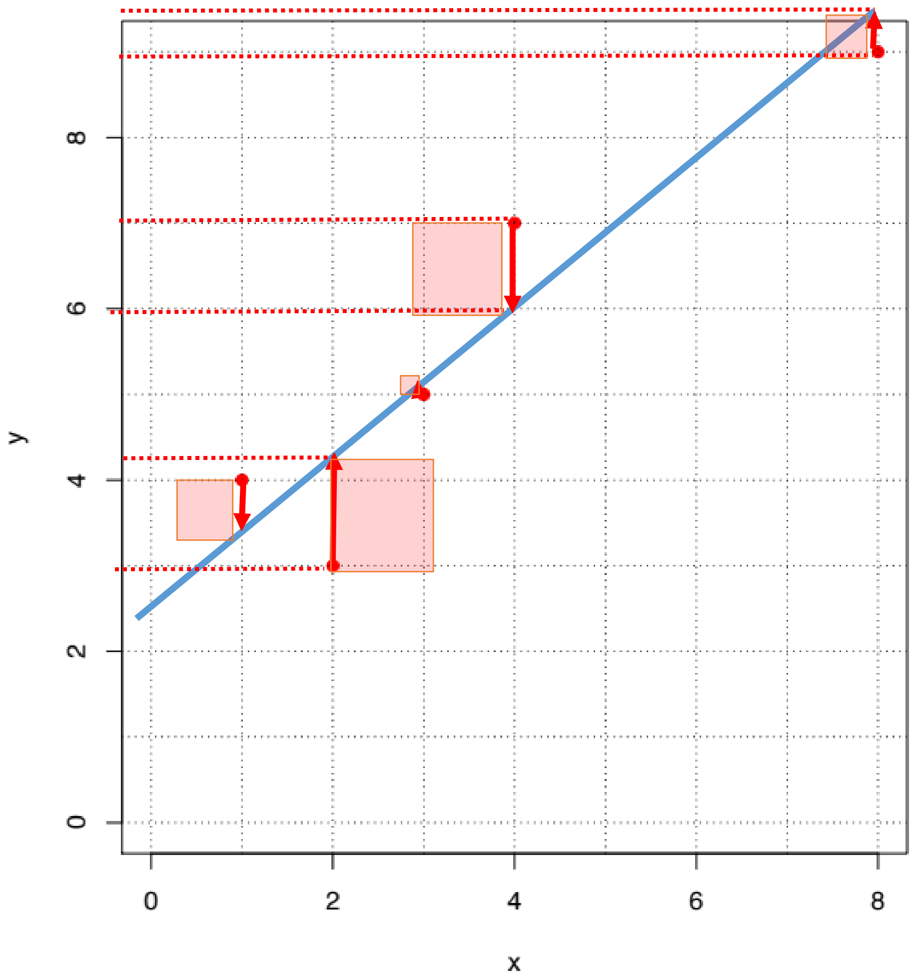
x, y
1,4
2,3
3.5
4,7
8,9

$$Cov(x,y) = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{n}$$

=

= 5.6 – 0.83 * 3.6
= 5.6 – 2.99
= 2.5

$$b_1$$



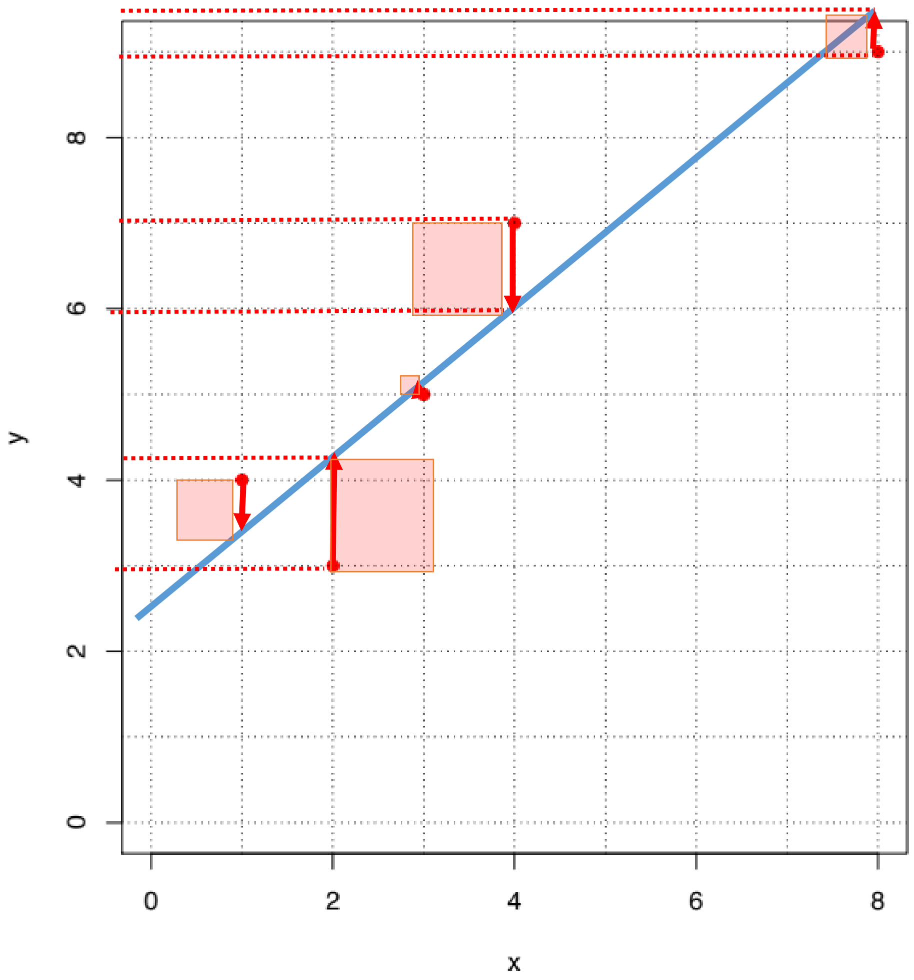
$$y_i = 2.2 + 0.83\, x_i + \varepsilon_i$$

SS: 3.03

R^2

- Coefficient of determination
- Proportion of how much variance in y is explained by x

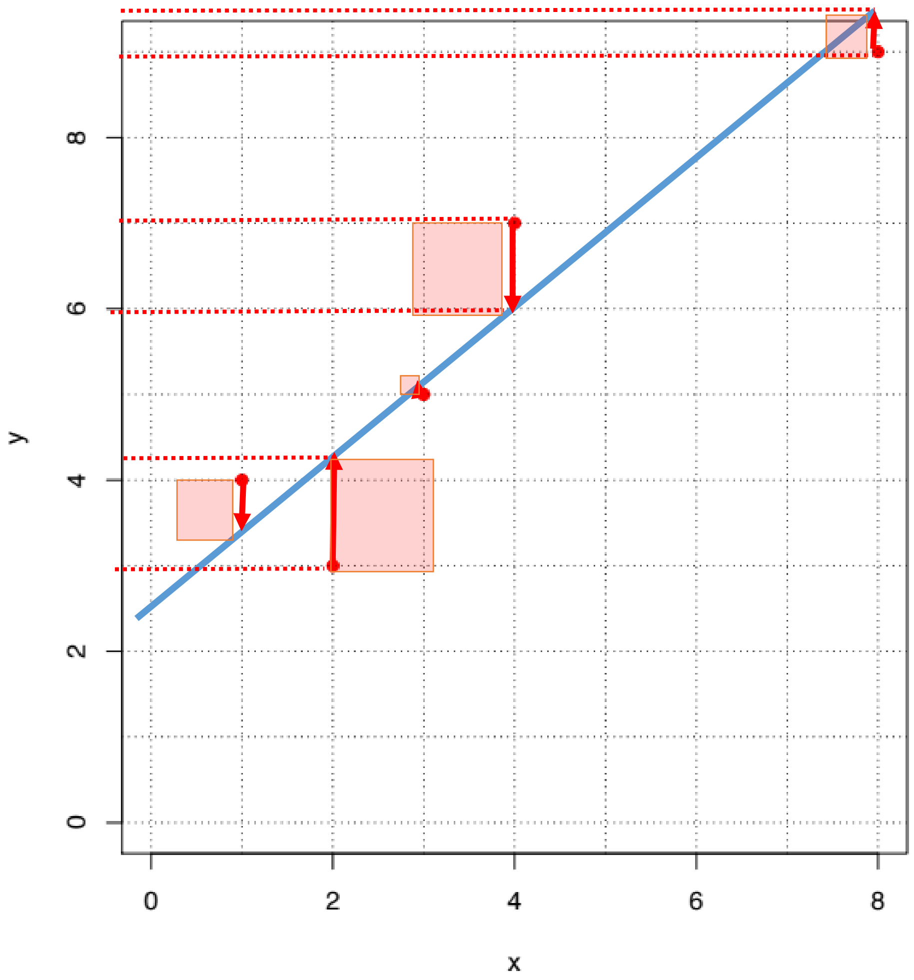
$$R^2 = 1 - \frac{SS_{residuals}}{SS_{total}}$$



R^2

- Coefficient of determination
- Proportion of how much variance in y is explained by x

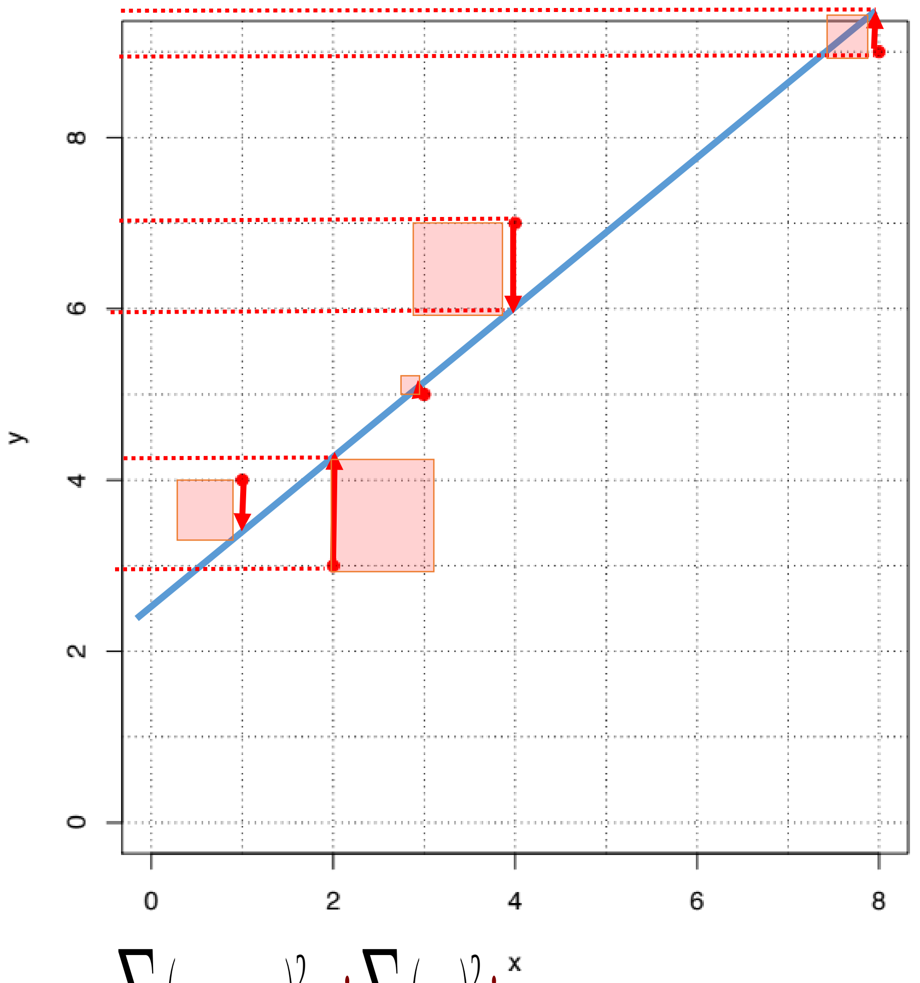
$$R^2 = 1 - \frac{SS_{residuals}}{SS_{total}}$$



Wait, what?

R^2

- Coefficient of determination
- Proportion of how much variance in y is explained by x



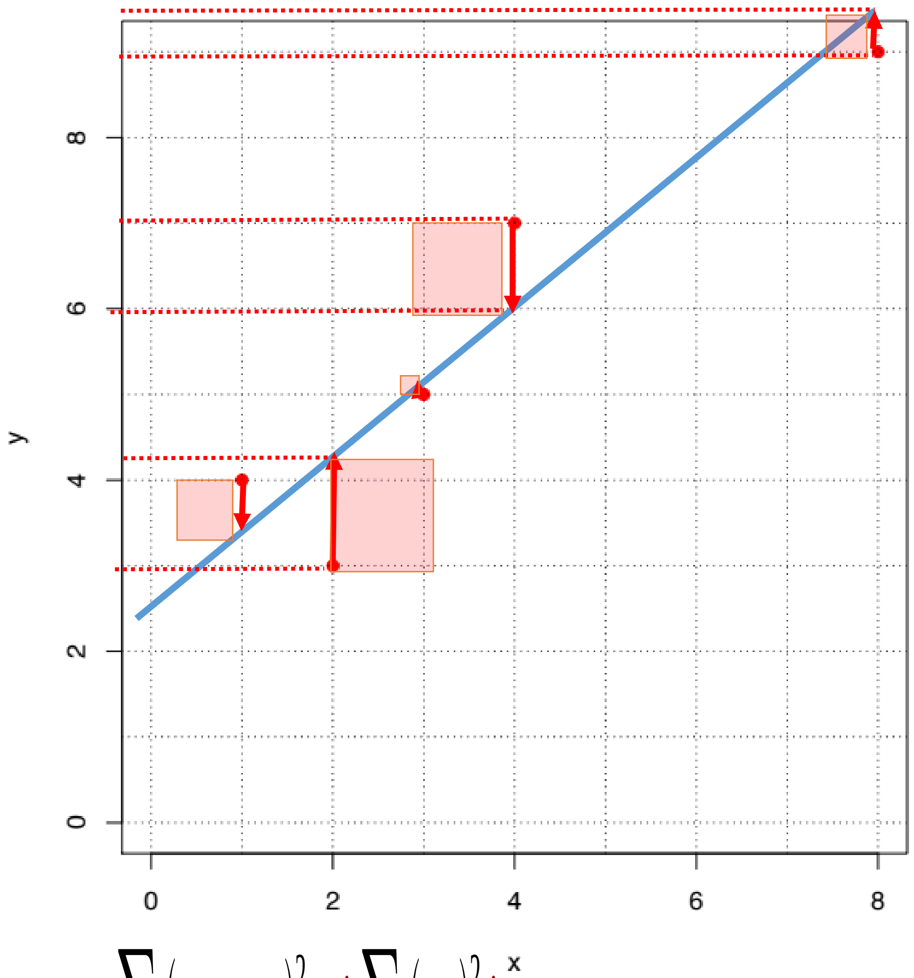
$$R^2 = 1 - \frac{SS_{residuals}}{SS_{total}}$$

We know what SS_{res} is – the residual sum of squares. $\sum (y_i - x_i)^2 = ¿ \sum (\varepsilon_i)^2 ¿$

R^2

- Coefficient of determination
- Proportion of how much variance in y is explained by x

$$R^2 = 1 - \frac{SS_{residuals}}{SS_{total}}$$



We know what SS_{res} is – the residual sum of squares

$$\sum (y_i - x_i)^2 = ¿ \sum (\varepsilon_i)^2 ¿$$

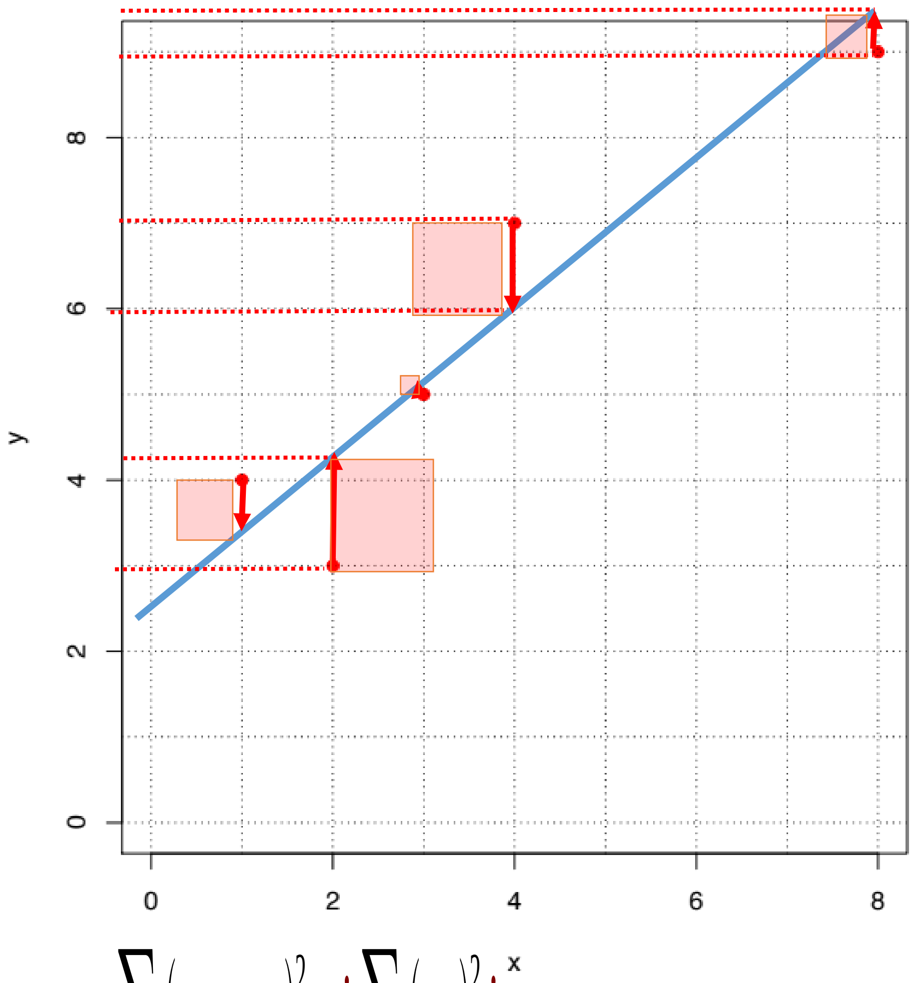
The total sum of squares is this:

$$\sum (y_i - \overline{y})^2$$

R^2

- Coefficient of determination
- Proportion of how much variance in y is explained by x

$$R^2 = 1 - \frac{SS_{residuals}}{SS_{total}}$$



We know what SS_{res} is – the residual sum of squares

$$\sum (y_i - x_i)^2 = ¿ \sum (\varepsilon_i)^2 ¿$$

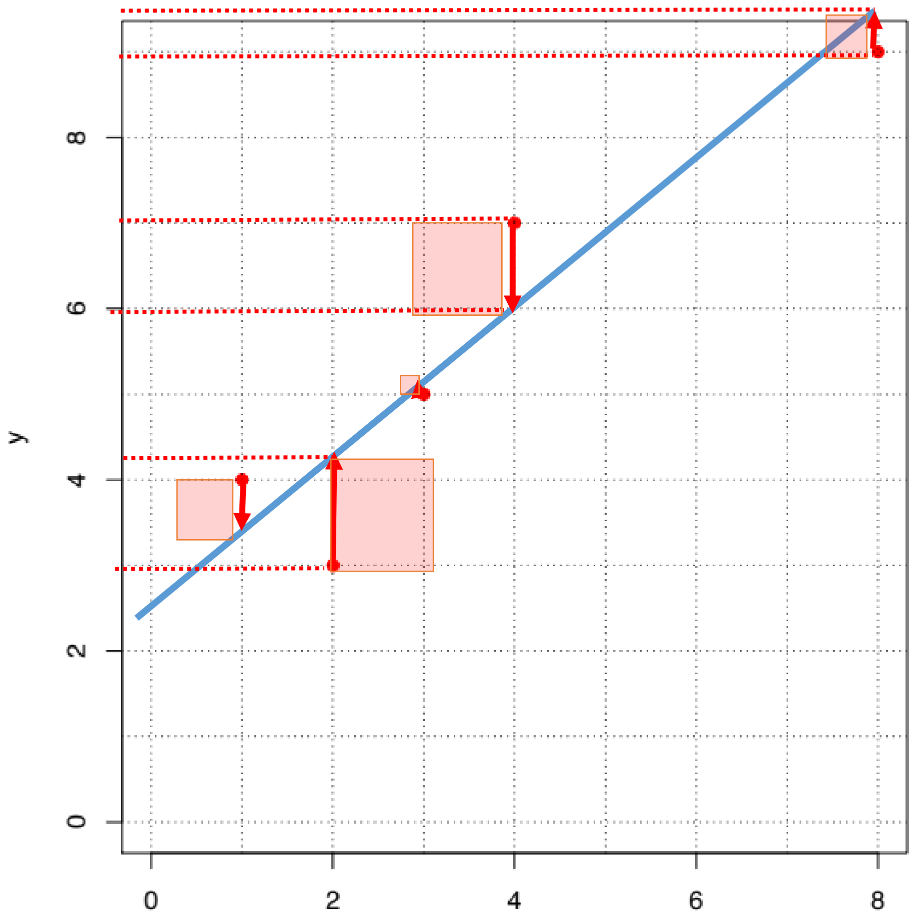
The total sum of squares is this:

$$\sum (y_i - \overline{y})^2 = \sigma^2 * (n-1)$$

R^2

- Coefficient of determination
- Proportion of how much variance in y is explained by x

$$R^2 = 1 - \frac{SS_{residuals}}{SS_{total}}$$



x, y
1,4
2,3
3.5
4,7
8,9

We know what SS_{res} is – the residual sum of squares $\sum (y_i - x_i)^2 = ¿ \sum (\varepsilon_i)^2 ¿^x$

3.03

The total sum of squares is this: $\sum (y_i - \bar{y})^2 = \sigma^2 * (n-1)$

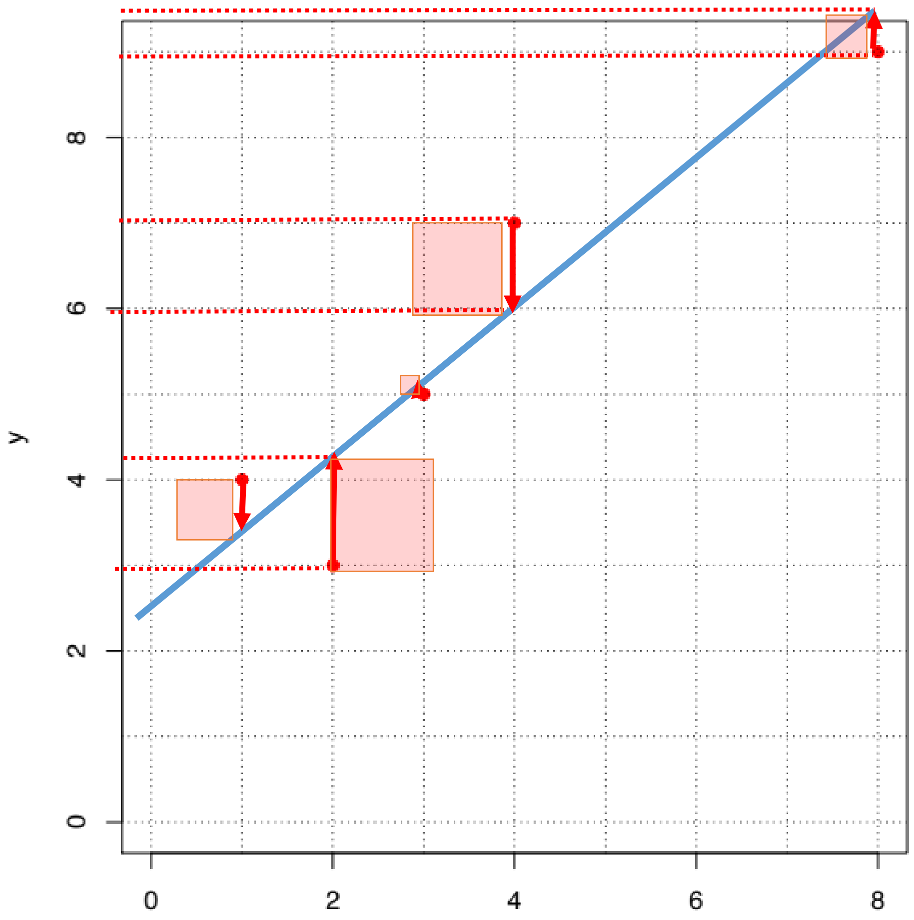
5.8 * 4
23.2

R^2

- Coefficient of determination
- Proportion of how much variance in y is explained by x

$$R^2 = 1 - \frac{SS_{residuals}}{SS_{total}}$$

= 1 – 3.03/23.2
= 0.87



x, y
1,4
2,3
3,5
4,7
8,9

We know what SS_{res} is – the residual sum of squares $\sum (y_i - x_i)^2 = \text{¿} \sum (\varepsilon_i)^2 \text{¿}^{x}$ 3.03

The total sum of squares is the nominator of : $\sum (y_i - \bar{y})^2 = \sigma^2 * (n-1)$ 5.8 * 4 23.2

Linear regression:

- Minimizing sum of squared residuals of line
- Then get b_1 and b_0
- Calculate R^2 to assess how much variance in the response variable is explained by the explanatory variable

Exercise – no hand-out

R^2 = 0.87

-0.7
1.2
0.1
1
0.
SS: 3.03

- Run a linear regression in R with x and y as we've used them here.

```
x<-c(1,2,3,4,8)
y<c(4,3,5,7,9)
model1 <- (lm(y~x))
model1
summary(model1)
anova(model1)
resid(model1)
cov(x,y)
var(x)
plot(y~x)
```

x, y
1,4
2,3
3.5
4,7
8,9