# A Potential for a Three-Field AdS/QCD Model

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### Abstract

The Anti-de Sitter Space/Conformal Field Theory (AdS/CFT) correspondence may offer new and useful insights into the non-perturbative regime of strongly coupled gauge theories such as Quantum Chromodynamics (QCD). We present an AdS/CFT-inspired model that describes the spectra of light mesons. The conformal symmetry is broken by a background dilaton field, and chiral symmetry breaking and linear confinement are described by a chiral condensate field. These background fields, along with a background glueball condensate field, are derived from a potential. We describe the construction of the potential, and the calculation of the meson spectra, which match experimental data well. We also argue that the presence of the third background field is necessary to properly describe the meson spectra. The outlook for application of this model to finite temperature systems is also discussed.

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# 1  Introduction

The Anti-de Sitter Space/Conformal Field Theory (AdS/CFT) correspondence is a useful mathematical tool for the analysis of strongly-coupled gauge theories. This correspondence correspondence establishes a connection between an $n$-dimensional Super-Yang Mills Theory and a weakly-coupled gravitational theory in $n + 1$ dimensions. Calculations that are analytically intractable in the field theory can be related to results from the gravity theory using an effective dictionary developed over the past decade. Quantum chromodynamics (QCD) is a strongly-coupled gauge theory at hadronic scales, making it a candidate for the application of the gauge/gravity correspondence. It is not known whether a gravitational dual to QCD exists, but there has been much work on models that capture its key features. The bottom-up approach assumes the existence of such a dual, modeling features of QCD by an effective five-dimensional gravity theory. Linear confinement in QCD sets a scale that is encoded in a cut-off of the fifth dimension in the AdS/QCD model. So-called soft-wall models use a dilaton field as an effective cut-off to limit the penetration of the meson fields into the bulk. The simplest soft-wall models use a quadratic dilaton to recover the linear Regge trajectories, while models that modify the UV behavior of the dilaton more accurately model the ground state masses.

The soft-wall models typically include at least two background fields: the aforementioned dilaton, and a chiral condensate field that corresponds to the chiral symmetry breaking in the gauge theory. These models use parametrizations for the background dilaton and chiral fields that are not derived as the solution to any equations of motion. A well-defined action would provide a set of background equations from which these fields can be derived, and may suggest how the model can be derived from a top-down approach. In addition, this action provides access to the thermal properties of the model through perturbation of the geometry.

In this paper, we expand upon previous work to find a suitable potential for the background fields of a soft-wall AdS/QCD model. After demonstrating the limitations of models including a dilaton and chiral field alone, we suggest the inclusion of a background glueball field. We then construct a potential that satisfies the necessary UV and IR limits, and use this potential to generate numerically the background fields and calculate the resulting meson spectra.

# 2  Review and Motivation

We assume that four-dimensional QCD can be modeled by the following five-dimensional action, written in the string frame:

$$\mathcal{S} = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ e^{-2\Phi} \mathcal{L}_{grav} + e^{-\Phi} \mathcal{L}_{meson} \right], \qquad (1)$$

where $\Phi$ is the dilaton and the metric is asymptotically AdS, $g_{MN} = z^{-2}\eta_{MN}$, with the AdS curvature defined to be unity. The gravitational Lagrangian includes the

dilaton and chiral condensate background fields

$$\mathcal{L}_{grav} = R_s + 4\partial_\mu \Phi \partial^\mu \Phi - \tfrac{1}{2}\partial_\mu \chi \partial^\mu \chi - V(\phi, \chi). \tag{2}$$

The scalar fields $\Phi, \chi$ are dimensionless.

It is easier to search for the background fields in the Einstein frame, where the gravitational action takes the canonical form

$$\mathcal{S}_E = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g_E} \left( R_E - \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2}\partial_\mu \chi \partial^\mu \chi - V(\phi, \chi) \right). \tag{3}$$

The string and Einstein frame metrics are related by the conformal transformation

$$g^{string}_{\mu\nu} = e^{2\phi/\sqrt{6}} g^E_{\mu\nu}. \tag{4}$$

We have also used a rescaled version of the dilaton field,

$$\phi = \sqrt{\frac{8}{3}}\Phi. \tag{5}$$

Previous work showed how to construct a potential for such a gravity-dilaton-chiral system, assuming that the fields have power-law behavior, which is accurate in both the UV and IR limits. One of the equations of motion is independent of the choice of potential,

$$\dot{\chi}^2 = \frac{\sqrt{6}}{z^2}\frac{d}{dz}(z^2\dot{\phi}). \tag{6}$$

To obtain linear confinement, the dilaton should have quadratic behavior in the IR limit, $\phi(z) = \lambda z^2$. The chiral field should have have linear behavior, $\chi(z) = Az$, where $A$ sets the large-$n$ mass splitting between the axial-vector and vector mesons in the model. Inserting this into (6), we find that the chiral field behaves as

$$\chi(z) = 6^{3/4}\sqrt{\lambda}z, \tag{7}$$

which removes one of the independent parameters of the original GKK model. Using the phenomenological value of $\lambda$, which determines the slope of the Regge trajectories, we find a mass splitting that is much too large.

Because this problem arises in the equation that is independent of the potential, this issue cannot be resolved by the choice of potential in the two-field model. Models that derive the field behavior using the superpotential method suffer from the same problem.

To resolve this problem, we suggest to add an additional scalar field to the model, $G$, representing the glueball condensate. This field must be linear in the IR for linear confinement, and go as $G \sim z^4$ in the UV to match the operator dimension in the AdS/CFT dictionary.

# 3   Setup

Consider the action in the string frame for three fields: $\phi$, $\chi$ and $G$ representing the dilaton, a chiral field, and a glueball field with zero mass

$$\mathcal{S}_{string} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} e^{-2\Phi} \left( R_s \partial_\mu \Phi \partial^\mu \Phi - \tfrac{1}{2}\partial_\mu \chi \partial^\mu \chi - \tfrac{1}{2}\partial_\mu G \partial^\mu G - e^{-4\Phi/3} V(\phi, \chi, G) \right) \tag{8}$$

The potential in the Einstein frame, where the action has its canonical form, is

$$V(\phi, \chi, G) = e^{2\phi/\sqrt{6}} \, \tilde{V}(\phi, \chi, G) \tag{9}$$

with

$$\tilde{V} = -12 + 4\sqrt{6}\phi + a_0\phi^2 - \tfrac{3}{2}\chi^2 + \tilde{U} \tag{10}$$

Here $\tilde{U}$ is more than quadratic in the fields. The dilaton mass is undetermined and is not connected to the dimension of the corresponding operator, as discussed by Kapusta and Springer. It is related to the parameter $a_0$ by $a_0 = \frac{1}{2}\left[(m_\phi L)^2 - 8\right]$. The potential should be an even function of $\chi$.

The equations of motion can be written as

$$\dot{\chi}^2 + \dot{G}^2 = \frac{\sqrt{6}}{z^2} \frac{d}{dz}(z^2 \dot{\phi}) \tag{11}$$

$$\tilde{U} = \tfrac{1}{2}\sqrt{6}z^2\ddot{\phi} - \tfrac{3}{2}(z\dot{\phi})^2 - 3\sqrt{6}z\dot{\phi} - 4\sqrt{6}\phi - a_0\phi^2 + \tfrac{3}{2}\chi^2 \tag{12}$$

$$\frac{\partial \tilde{U}}{\partial \phi} = 3z\dot{\phi} - 2a_0\phi \tag{13}$$

$$\frac{\partial \tilde{U}}{\partial \chi} = z^2\ddot{\chi} - 3z\dot{\chi}\left(1 + \frac{z\dot{\phi}}{\sqrt{6}}\right) + 3\chi \tag{14}$$

$$\frac{\partial \tilde{U}}{\partial G} = z^2\ddot{G} - 3z\dot{G}\left(1 + \frac{z\dot{\phi}}{\sqrt{6}}\right) \tag{15}$$

We assume that the potential has no explicit dependence on the coordinate $z$, so the equations 13-15 are not independent, and we can eliminate one.

## 3.1   IR Limit

The requirement of linear confinement requires a solution in the large $z$ limit of the form

$$\phi = \lambda z^2 \tag{16}$$
$$\chi = Az \tag{17}$$
$$G = Bz. \tag{18}$$

Substitution into (11) gives

$$A^2 + B^2 = 6\sqrt{6}\lambda \tag{19}$$

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The $\lambda$ is fixed by the slope of the linear trajectory and $A$ is fixed by the axial-vector – vector mass difference. It is useful to write these as

$$A = 6^{3/4}\sqrt{\lambda}\cos\theta$$

$$B = 6^{3/4}\sqrt{\lambda}\sin\theta, \tag{20}$$

where $\theta$ now becomes the parameter controlling the axial-vector – vector mass splitting. Inserting (18) into (12-15) suggests the following terms in our ansatz for the potential

$$\tilde{U} = a_1\phi\chi^2 + a_2\phi G^2 + a_3\chi^4 + a_4 G^4 + a_5\chi^2 G^2 + a_6 G^2\tanh(g\phi). \tag{21}$$

We see that there must be a $G^2$ term in the IR limit, but this is forbidden in the weak-field limit because the glueball condensate field is massless. To circumvent this, we propose the term $G^2\tanh(g\phi)$ with $g > 0$. In the weak field limit this goes to $g\phi G^2$, which is acceptable. The $\tanh$ is suggested by (9), and it suggests a rapid exponential transition from the weak field to the strong field limits that is supported by phenomenology. By substitution one finds the following constraints on the parameters:

$$\tilde{U} \to 6 + a_0 + 6\sqrt{6}\left(\cos^2\theta\, a_1 + \sin^2\theta\, a_2\right)$$
$$+6^3\left(\cos^4\theta\, a_3 + \sin^4\theta\, a_4 + \cos^2\theta\sin^2\theta\, a_5\right) = 0 \tag{22}$$

$$\frac{\partial\tilde{U}}{\partial\chi} \to 2a_1 + 24\sqrt{6}\cos^2\theta\, a_3 + 12\sqrt{6}\sin^2\theta\, a_5 + \sqrt{6} = 0 \tag{23}$$

$$\frac{\partial\tilde{U}}{\partial G} \to 2a_2 + 24\sqrt{6}\sin^2\theta\, a_4 + 12\sqrt{6}\cos^2\theta\, a_5 + \sqrt{6} = 0 \tag{24}$$

$$\frac{\partial\tilde{U}}{\partial G} \to a_6 = -\tfrac{3}{2} \tag{25}$$

We have chosen to exclude (13) because it is not independent. The parameter $a_6$ is determined, and the others will be determined by an examination of the UV limit.

## 3.2 UV Limit

Next we look for a solution in the small $z$ limit. The AdS/CFT dictionary dictates that the leading-order UV behavior of the chiral and glueball condensate fields is determined by their dimension. Note also that we are working in the chiral limit where the quark mass is zero. We start by examining only the leading-order terms

$$\chi = \Sigma_0 z^3 \tag{26}$$
$$G = G_0 z^4. \tag{27}$$

Substitution into (11) and imposing the boundary condition $\phi(0) = 0$ gives

$$\phi = \frac{\sqrt{6}}{28}\Sigma_0^2 z^6 + \frac{\sqrt{6}}{27}G_0^2 z^8 \tag{28}$$

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Using only this leading-order behavior in (12-15), the system of equations is inconsistent, as there are more equations from matching powers of $z$ than unknown parameters.

To solve this problem, try adding a term $\Sigma_n z^n$ to $\chi$. Substituting into (11) and keeping only the lowest-order cross-term we find the additional term in $\phi$

$$\Delta\phi = \frac{\sqrt{6}n\Sigma_0\Sigma_n}{(n+4)(n+3)}z^{n+3} \tag{29}$$

From (12) we find that

$$\tilde{U} = -\tfrac{3}{2}(z\dot{\phi})^2 - a_0\phi^2 + 3\frac{n^3 - 13n + 12}{(n+4)(n+3)}\Sigma_0\Sigma_n z^{n+3} \tag{30}$$

Since the $\phi^2$ terms start out as $z^{12}$, $z^{14}$, $z^{16}$, and so do the terms in the potential, the $n$ can only take the values 9, 11, etc. This term contributes only to the equation for $\partial\tilde{U}/\partial\chi$.

$$\frac{\partial\tilde{U}}{\partial\chi} = -9\Sigma_0\left(\frac{3}{14}\Sigma_0^2 + \frac{8}{27}G_0^2 z^2\right)z^9 + (n-3)(n-1)\Sigma_n z^n \tag{31}$$

By power counting both $n = 9$ and $n = 11$ can contribute.

There could also be higher order terms in $G$ such as $G_m z^m$. This leads to the additional term in $\phi$

$$\Delta\phi = \frac{8mG_0G_m}{\sqrt{6}(m+5)(m+4)}z^{m+4} \tag{32}$$

It contributes to the equation for $\partial\tilde{U}/\partial G$ as

$$\frac{\partial\tilde{U}}{\partial G} = -12G_0\left(\frac{3}{14}\Sigma_0^2 + \frac{8}{27}G_0^2 z^2\right)z^{10} + m(m-4)G_n z^m \tag{33}$$

The choice $m = 8$ is not possible as there is no term of the same order to balance it. Terms with $m = 10$ and $m = 12$ are possible. These new terms cannot affect the equation for $\partial\tilde{U}/\partial\phi$ nor can they contribute to the equation for $\partial\tilde{U}/\partial\chi$. Considering higher order terms in both $\chi$ and $G$ leads to

$$\tilde{U} = -\tfrac{3}{2}(z\dot{\phi})^2 - a_0\phi^2 + 3\frac{n^3 - 13n + 12}{(n+4)(n+3)}\Sigma_0\Sigma_n z^{n+3} + \frac{4m(m-4)}{m+4}G_0G_m z^{m+4} \tag{34}$$

The appearance of these terms can be understood by writing the following schematic expansions.

$$\chi \sim \Sigma_0 z^3 + \Sigma_0^3 z^9 + G_0^2\Sigma_0 z^{11} + \cdots$$
$$G \sim G_0 z^4 + \Sigma_0^2 G_0 z^{10} + G_0^3 z^{12} + \cdots$$

That is, $\chi$ is an odd function of $\Sigma_0$ and $G$ is an odd function of $G_0$. These are the symmetries in the equations of motion. They also follow the spirit of the AdS/CFT correspondence in terms of the dimensionality of the operators and the powers of $z$.

Including now $m = 10$ and 12, and $n = 9$ and 11, we have the following set of equations in the small $z$ limit:

$$
\begin{aligned}
\tilde{U}_{\text{LHS}} &= 3\Sigma_0^4 z^{12}\left[4\frac{\Sigma_9}{\Sigma_0^3} - \frac{(54 + a_0)}{2^3 \cdot 7^2}\right] \\
&+ \frac{1}{7}\Sigma_0^2 G_0^2 z^{14}\left[120\frac{G_{10}}{\Sigma_0^2 G_0} + 120\frac{\Sigma_{11}}{\Sigma_0 G_0^2} - \frac{(72 + a_0)}{9}\right] \\
&+ 2G_0^4 z^{16}\left[12\frac{G_{12}}{G_0^3} - \frac{(96 + a_0)}{3^5}\right]
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\tilde{U}_{\text{RHS}} &= \Sigma_0^4 z^{12}\left[\frac{\sqrt{6}}{28}a_1 + a_3\right] \\
&+ \Sigma_0^2 G_0^2 z^{14}\left[\frac{\sqrt{6}}{27}a_1 + \frac{\sqrt{6}}{28}(a_2 + ga_6) + a_5\right] \\
&+ G_0^4 z^{16}\left[\frac{\sqrt{6}}{27}(a_2 + ga_6) + a_4\right]
\end{aligned} \tag{36}
$$

$$
\left(\frac{\partial \tilde{U}}{\partial \chi}\right)_{\text{LHS}} = 3\Sigma_0^3 z^9\left[-\frac{9}{14} + 16\frac{\Sigma_9}{\Sigma_0^3}\right] + 8\Sigma_0 G_0^2 z^{11}\left[-\frac{1}{3} + 10\frac{\Sigma_{11}}{\Sigma_0 G_0^2}\right] \tag{37}
$$

$$
\left(\frac{\partial \tilde{U}}{\partial \chi}\right)_{\text{RHS}} = \Sigma_0^3 z^9\left[\frac{\sqrt{6}}{14}a_1 + 4a_3\right] + \Sigma_0 G_0^2 z^{11}\left[\frac{2\sqrt{6}}{27}a_1 + 2a_5\right] \tag{38}
$$

$$
\tag{39}
$$

$$
\left(\frac{\partial \tilde{U}}{\partial G}\right)_{\text{LHS}} = 6\Sigma_0^2 G_0 z^{10}\left[-\frac{3}{7} + 10\frac{G_{10}}{\Sigma_0^2 G_0}\right] + 32G_0^3 z^{12}\left[-\frac{1}{9} + 3\frac{G_{12}}{G_0^3}\right] \tag{40}
$$

$$
\left(\frac{\partial \tilde{U}}{\partial G}\right)_{\text{RHS}} = \Sigma_0^2 G_0 z^{10}\left[\frac{\sqrt{6}}{14}(a_2 + ga_6) + 2a_5\right] + G_0^3 z^{12}\left[\frac{2\sqrt{6}}{27}(a_2 + ga_6) + 4a_4\right] \tag{41}
$$

Altogether, from both the UV and IR limits, there are ten independent equations for the twelve parameters $a_0 - a_6$, $\Sigma_9$, $\Sigma_{11}$, $G_{10}$, $G_{12}$, and $g$. We take $g$ as the free parameter to use as the rate of transition from small $z$ to large $z$. The parameters in the potential are found to be

$$
a_0 = \frac{3}{2}\frac{1}{6 + \sin^2\theta}\left[120 + 62\sin^2\theta + 63\sqrt{6}g\sin^2\theta\right] \tag{42}
$$

$$
a_1 = -\frac{3\sqrt{6}}{4}\frac{1}{6 + \sin^2\theta}\left[12 + 8\sin^2\theta + 9\sqrt{6}g\sin^2\theta\right] \tag{43}
$$

$$
a_2 = -\frac{\sqrt{6}}{4}\frac{1}{6 + \sin^2\theta}\left[32 + 24\sin^2\theta + 3\sqrt{6}g(9\sin^2\theta - 2)\right] \tag{44}
$$

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$$2a_3 \cos^2 \theta + a_5 \sin^2 \theta \;=\; \frac{1}{24} \frac{1}{6 + \sin^2 \theta} \left[ 24 + 22 \sin^2 \theta + 27\sqrt{6}g \sin^2 \theta \right] \tag{45}$$

$$2a_4 \sin^2 \theta + a_5 \cos^2 \theta \;=\; \frac{1}{24} \frac{1}{6 + \sin^2 \theta} \left[ 20 + 22 \sin^2 \theta + 3\sqrt{6}g(9 \sin^2 \theta - 2) \right] \tag{46}$$

$$a_6 \;=\; -\frac{3}{2} \tag{47}$$

The coefficients $a_0$, $a_1$, $a_2$ and $a_6$ are determined, while there are two equations for the three coefficients $a_3$, $a_4$ and $a_5$. That leaves $a_5$ as a free parameter, to be fit numerically, along with $g$, $\theta$, $G_0$, $\Sigma$, and $\lambda$.

# 4 Numerical Solution

Using the above potential, we seek a numerical solution that simultaneously satisfies the UV and IR limits. We use equations (11, 14, 15), which allows for an additional term in the potential, $\Delta\tilde{U}$, such that

$$\frac{\partial \Delta\tilde{U}}{\partial \chi} = \frac{\partial \Delta\tilde{U}}{\partial G} = 0, \tag{48}$$

which will be determined from the numerical solution.

The differential equations represent a stiff system, and treatment of the problem as an initial value problem leads to numerical instabilities. We treat it instead as a boundary value problem, using Dirichlet boundary conditions at both boundaries. A relaxation method is used in combination with input approximations for the background fields, which are then iterated to find a stable solution to the system with the given boundary conditions. Because the system is nonlinear, the solution found is not guaranteed to be unique.

The IR boundary is chosen to be sufficiently large to capture the infrared behavior and to give accurate Regge behavior large-$n$ radial excitations of the mesons. The UV boundary should approach zero, but it cannot reach zero because of the singularity in the equations of motion. This becomes a problem because equation (11) allows constant and divergent terms

$$\Delta\phi(z) = c_1 + c_2 z^{-1}. \tag{49}$$

Symbolically, these terms can be set to zero by enforcing the Dirichlet boundary condition $\phi(0) = 0$, but this is impossible to enforce numerically. Creative choice of UV boundary conditions can eliminate one, but not both of these unwanted terms without affecting the chiral and glueball fields. The behavior of the numerical solutions suggests that the desired UV behavior is an unstable solution to the equations, and therefore difficult or impossible to find with this iterative method.

As an alternative to direct solution, we parameterize the fields as follows:

$$\Psi(z) = \psi(z)_{UV} f(z) + \psi(z)_{IR} \left( 1 - f(z) \right), \tag{50}$$

where $f(z)$ is some function that transitions smoothly from 1 at small values of $z$ to 0 at large $z$, and $\psi(z)_{xx}$ represents the known UV and IR limits of the fields $\phi, \chi$, and $G$. The switching functions $f$ need not be the same for each field. We choose

$$f_\phi(z) = e^{-(\beta_1 z)^{10}} \tag{51}$$

$$f_\chi(z) = e^{-(\beta_2 z)^4} \tag{52}$$

$$f_G(z) = e^{-(\beta_3 z)^5}. \tag{53}$$

The powers of the exponential are chosen to be greater than the known power-law behavior of the fields in the UV limit, so as to not interfere with this behavior. The $\beta_i$ will be set by numerical fitting.

In all, we have nine parameters to be set numerically. The first constraint is to obtain the best global visual fit to the meson spectrum. We do not simply do a chi-squared fitting to the experimental data because the measurement error for the ground state $\rho$ meson is so much smaller than for the others that this would effectively act as the only constraint. Secondly, we seek to minimize the error in the finite-difference approximations to equations $11, 14$, and $15$.

Three of the parameters are most phenomenologically relevant: $\lambda$, which controls the slope of the meson spectra in the large-$n$ limit; $\theta$, which controls the mass splitting between the $a_1$ and $\rho$ mesons, and $\beta_2$, which controls the location of the "bend" in the $a_1$ spectrum. For each set of these parameters, the other parameters are set by a routine that minimizes the error in the equations of motion. The parameters found are shown in Table 1.
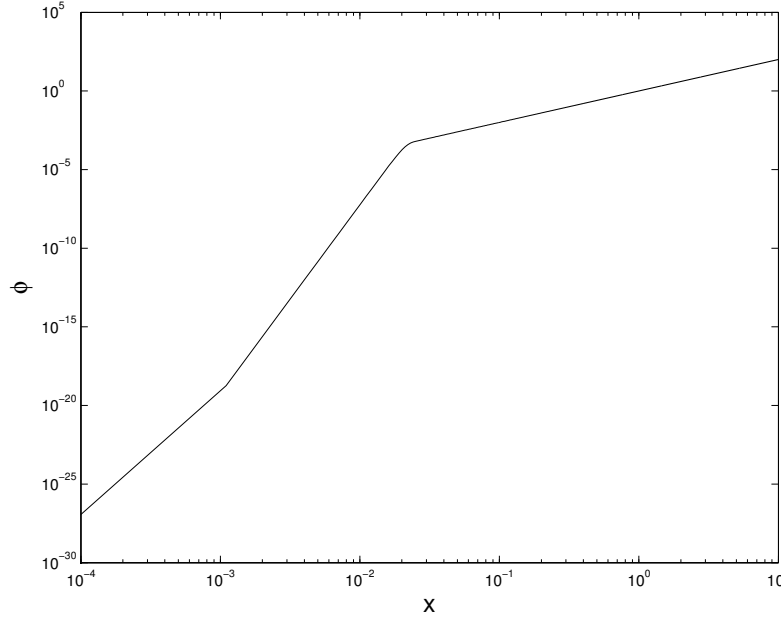


Figure 1: *Log-log plot of the dilaton field $\phi$. The coordinate $x$ is a dimensionless re-scaling of the conformal coordinate, $x = \sqrt{\lambda} z$.*
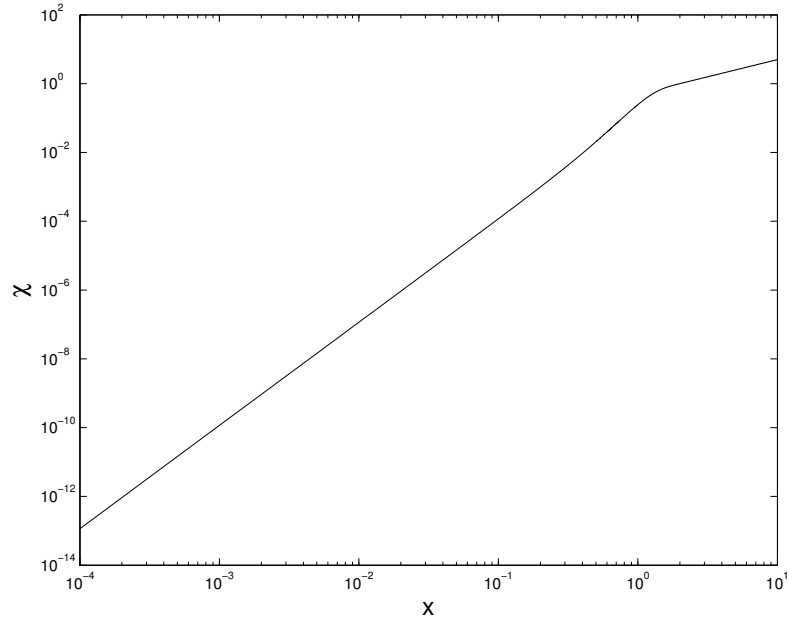
Figure 2: *Log-log plot of the chiral field $\chi$.*

The background fields that are obtained from this analysis are shown in Figures 1-3. The asymptotic power-law behavior of the fields is evident in the linear portions of the log-log scale plots shown. The "transition" behavior is most evident in the dilaton because of the large value of $\beta_1$, which controls the value of $z$ at which the field transitions from the UV limit to the IR limit.

| $\lambda^{1/2}$ | 430 MeV | $a_5$ | 25.342 |
|---|---|---|---|
| $\Sigma^{1/3}$ | 210 MeV | $\beta_1$ | 47.42 |
| $G_0^{1/4}$ | 397 MeV | $\beta_2$ | 0.80 |
| $\theta$ | 1.44 | $\beta_3$ | 0.782 |
| $g$ | 1.0 | | |

Table 1: Best fit parameters

We now analyze the "extra" term in the potential, $\Delta\tilde{U}$. We obtain this term numerically by subtracting the right-hand side of 12 from its left-hand side. This term can be approximated numerically as a function of the dilaton field,

$$\Delta\tilde{U}\left(\phi\right) = \alpha\phi e^{-(\phi-b)^2/c}. \tag{54}$$

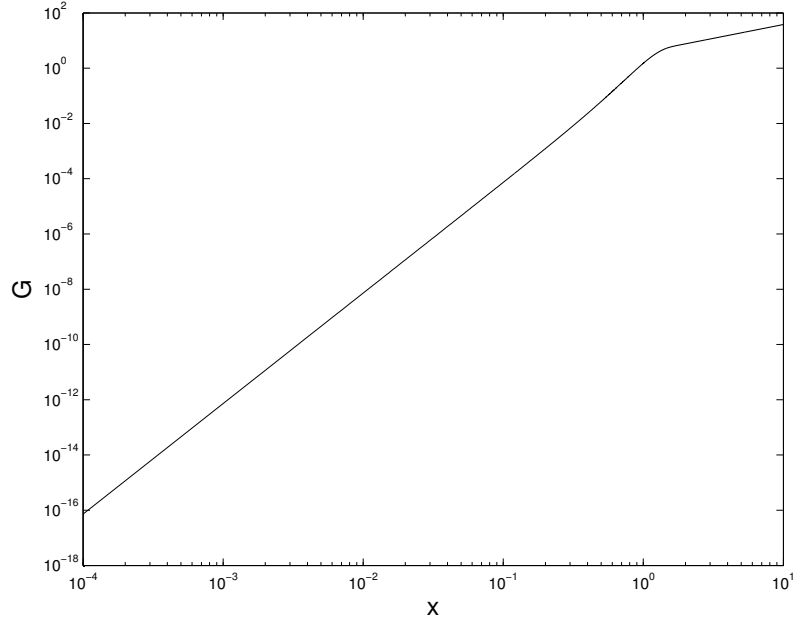The best-fit values for these parameters are shown in Table 4.

Figure 3: *Log-log plot of the glueball field $G$.*

| | |
|---|---|
| $\alpha$ | -77.07 |
| $b$ | 0.6819 |
| $c$ | 0.9219 |

Table 2: The parameters for the fitting to $\Delta \tilde{U}$

# 5  Meson Spectra

We use the GKK meson Lagrangian in the action (1),

$$\mathcal{L}_{meson} = |DX|^2 + m_X^2 \left|X\right|^2 - \kappa \left|X\right|^4 + \frac{1}{2g_5^2} \left(F_A^2 + F_V^2\right). \qquad (55)$$

The covariant derivative is defined as $D_M = \partial_M + i[V_M, X] - i\{A_M, X\}$. The $2 \times 2$ scalar field $X$ contains the scalar and pseudoscalar fields $(S, \pi)$, as well as a non-trivial vacuum expectation value, dual to the $q\bar{q}$ operator, which we have identified with the field $\chi(z)$. The AdS/CFT dictionary sets the mass for the fields according to the dimension of the dual operator,

$$m^2 L^2 = \Delta(\Delta - 4). \qquad (56)$$

The dimension of the $q\bar{q}$ operator is 3, so $m_X^2 = -3/L^2$.

We find the equations of motion for the various meson fields by varying the meson action. For the vector and axial-vector fields, we assume that the Kaluza-Klein modes
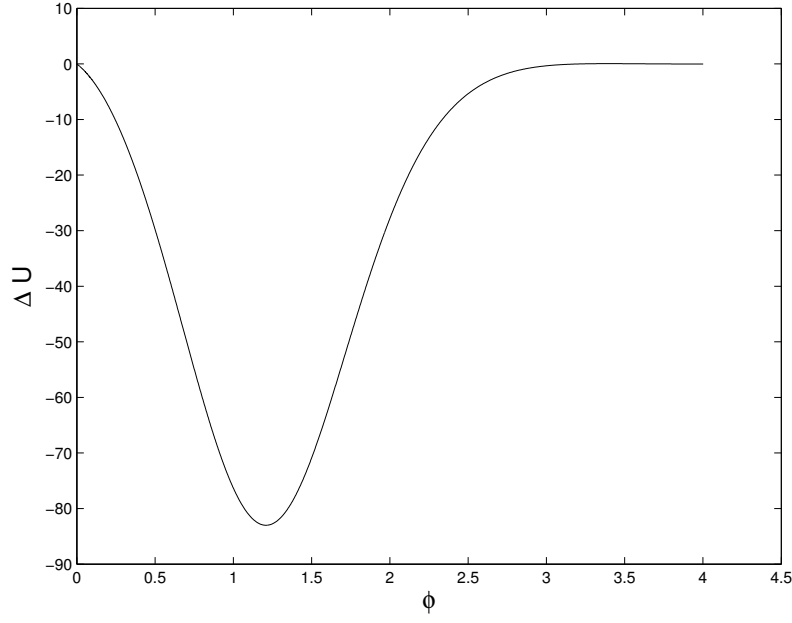
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Figure 4: *Plot of the "extra" term in the potential, $\Delta\tilde{U}(\phi)$.*

are separable from the 4D parts of the fields. The equation of motion in the axial gauge $\Psi_5 = 0$ is given by

$$-\partial_z^2\Psi_n + \omega'\partial_z\Psi_n + M_\Psi^2(z)\Psi_n = m_{\Psi_n}^2\Psi_n, \tag{57}$$

where $\omega = \phi(z) + log(z)$. The $z$-dependent mass term coefficient $M_\Psi = 0$ for the vector field, and $M_\Psi = g_5\chi/z$ for the axial field. The equation can be put in the Schrödinger form with the substitution $\Psi_n = e^{\omega/2}\psi_n$, resulting in

$$-\partial_z^2\psi_n + \left(\tfrac{1}{4}\omega'^2 - \tfrac{1}{2}\omega'' + M_\psi^2\right)\psi_n = m_{\Psi_n}^2\psi_n. \tag{58}$$

These equations are analytically solvable in the IR limit, but full analysis requires the use of a numerical shooting method to find the mass eigenvalues. This model finds a better phenomenological fit than the results presented in the GKK paper, particularly for the ground state $\rho$ meson, as shown in Figure 6. The scalar mesons are expected to mix with the scalar glueball field of this model, and this analysis is not performed here.

Finally, the pion spectrum is analyzed in the same way as in the KBK paper. We use the linear representation of the scalar field,

$$X = (\chi(z)/2 + S(x, z))\,\text{I} + i\pi(x, z)^a t^a, \tag{59}$$

where I is the $2 \times 2$ identity matrix. The pseudoscalar field $\pi$ mixes with the longitudinal component of the axial field, resulting in a set of coupled equations of motion.
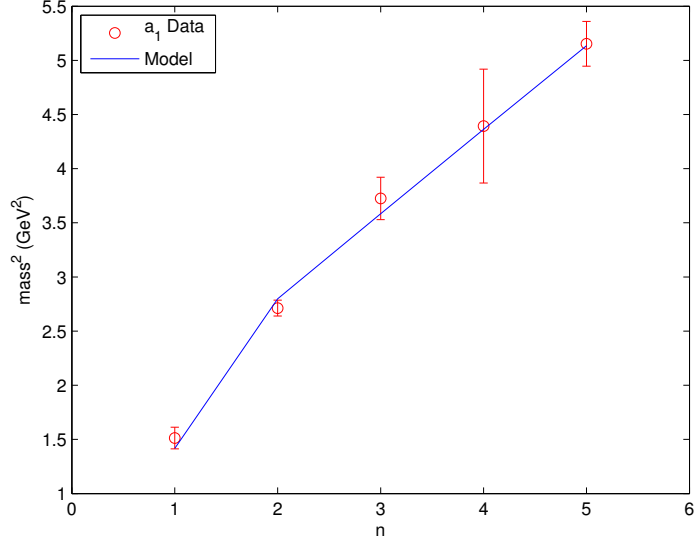
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Figure 5: $a_1$ meson spectrum.

Defining $A_\mu = A_{\mu\perp} + \partial_\mu\varphi$, where $\partial_\mu A^\mu = 0$, the equations of motion that result are

$$-\phi_n'' + \left(\frac{\phi'^2}{4} - \frac{\phi''}{2} + \frac{3}{4z^2} + \frac{\phi'}{2z} + \frac{g_5^2 L^2 \chi^2}{z^2}\right)\varphi_n = \frac{g_5^2 L \chi}{z}\pi_n \tag{60}$$

$$-\pi_n'' + \left(\frac{\phi'^2}{4} - \frac{\phi''}{2} + \frac{3}{4z^2} + \frac{3\phi'}{2z} - \frac{\kappa L^2 \chi^2}{2z^2} - m_n^2\right)\pi_n = -m_n^2 \frac{\chi L}{z}\varphi_n. \tag{61}$$

Using the same numerical technique, we find similar results, with the exception that the ground state pion is massless in this model, due to the assumption of massless quarks. The transition to the large-$n$ behavior occurs at $n \approx 4$. It is important to note that the experimental pion data was not used as an input to fit the parameters in Table 1. The quartic scalar term in (55) was added to get the proper chiral symmetry breaking behavior in a model without an action for the background fields. This term does not affect the spectra for the $\rho$ and $a_1$ mesons, but it does affect the scalar and pseudoscalar spectra. We have neglected this field in our current analysis by setting $\kappa = 0$, but it could be treated as an additional free parameter to fit the pion and $f_0$ spectra.

# 6   Conclusion

We discussed the construction of a potential for the background fields of a soft-wall AdS/QCD model. We showed the limitation of a model that contains only the dilaton and chiral condensate fields, and suggested a solution by adding a glueball condensate to the model. We analytically constructed a general potential $\tilde{U}(\phi, \chi, G)$ that recovers
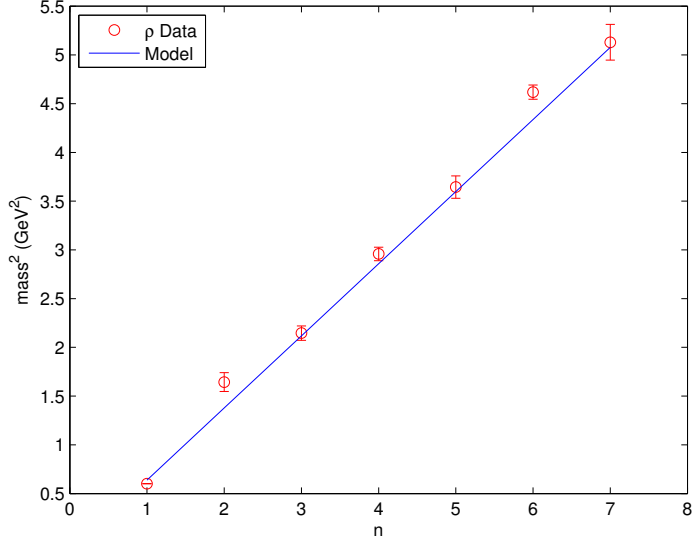
Figure 6: $\rho$ meson spectrum.

the necessary asymptotic behavior of the background fields. Using this as a basis, we numerically constructed a potential that solves the selected background equations to within an accuracy of $10^{-4}$. There is an additional allowed term in the potential, $\Delta\tilde{U}(\phi)$ that does not affect the equations that were used in the numerical procedure. This term was found numerically, and fit as a function of the dilaton alone. These background fields can then be used to find the spectra of the light vector, axial-vector, and pseudoscalar mesons. The model shows good phenomenological agreement with the experimental data for the meson spectra.

The potential as constructed is not guaranteed to be unique. If a different set of the background equations is chosen, the extra term would be expressed as a function of fields other than the dilaton. The parameterization in (52-53) could also be chosen differently, resulting in a different potential but making little difference to the resulting meson spectra. Finally, terms can be added that do not affect the equations of motion at all, i.e. terms such that

$$\Delta\tilde{U} = \Delta\frac{\partial\tilde{U}}{\partial\phi} = \Delta\frac{\partial\tilde{U}}{\partial\chi} = \Delta\frac{\partial\tilde{U}}{\partial G} = 0. \tag{62}$$

This work demonstrates the construction of a potential for the background fields of a soft-wall AdS/QCD model that captures several key features of QCD observed through meson spectra. The radially excited states of the light mesons have linear Regge trajectories. Chiral symmetry is not restored for highly-excited mesons, as seen in the constant mass-splitting of the vector and axial-vector mesons. The ground state pion is massless because the chiral symmetry is not explicitly broken with a non-zero quark mass.
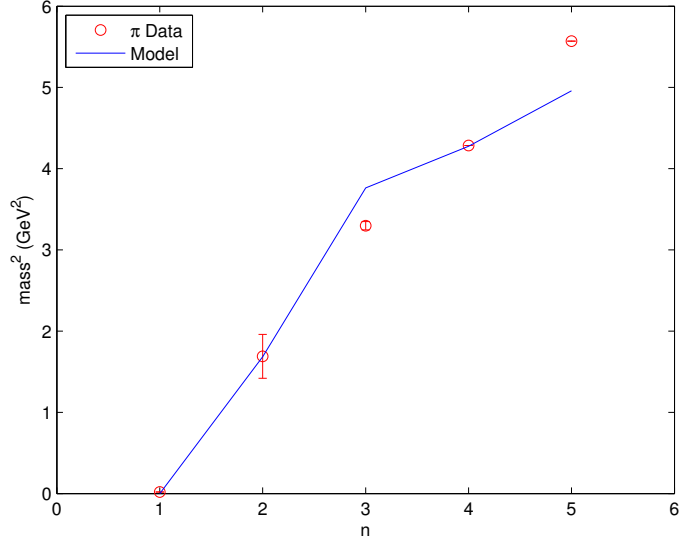
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Figure 7: *Pion spectrum. The ground state is massless in the chiral limit. The model was calculated with $\kappa = 0$ in the meson action. The $n = 4, 5$ states are unconfirmed and are found only in the extended version of the PDG.*

Future improvements to this model will include incorporating the quark mass by adding a linear term to the UV limit of the chiral condensate field. The scalar $f_0$ meson spectrum should be explored, which will require analyzing glueball mixing. This potential also opens the possibility of exploring the thermal properties of a model that has the correct chiral symmetry breaking behavior.

# Acknowledgments