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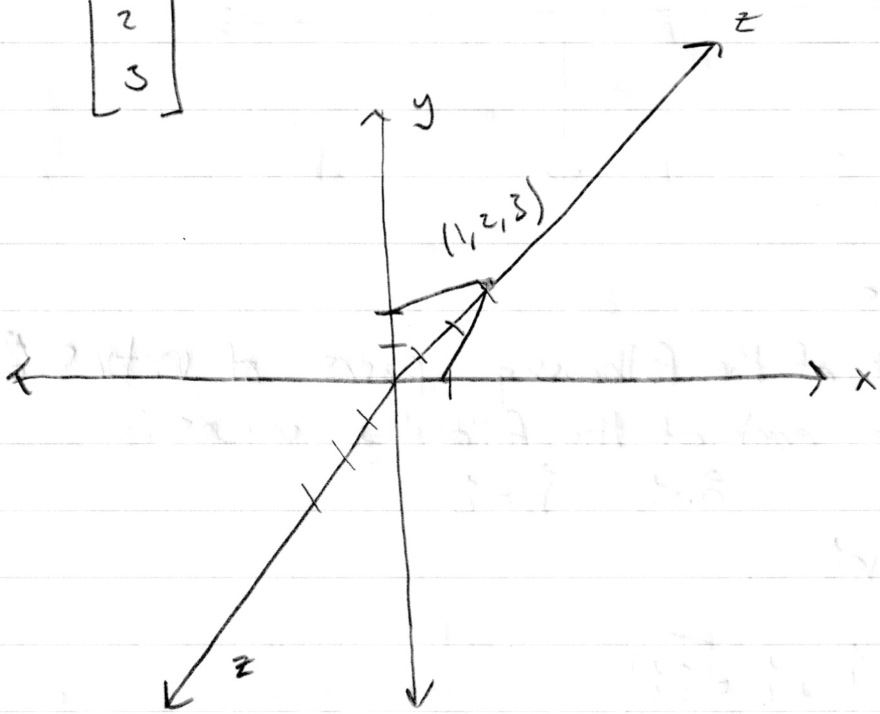
9/16/20
COMP 464

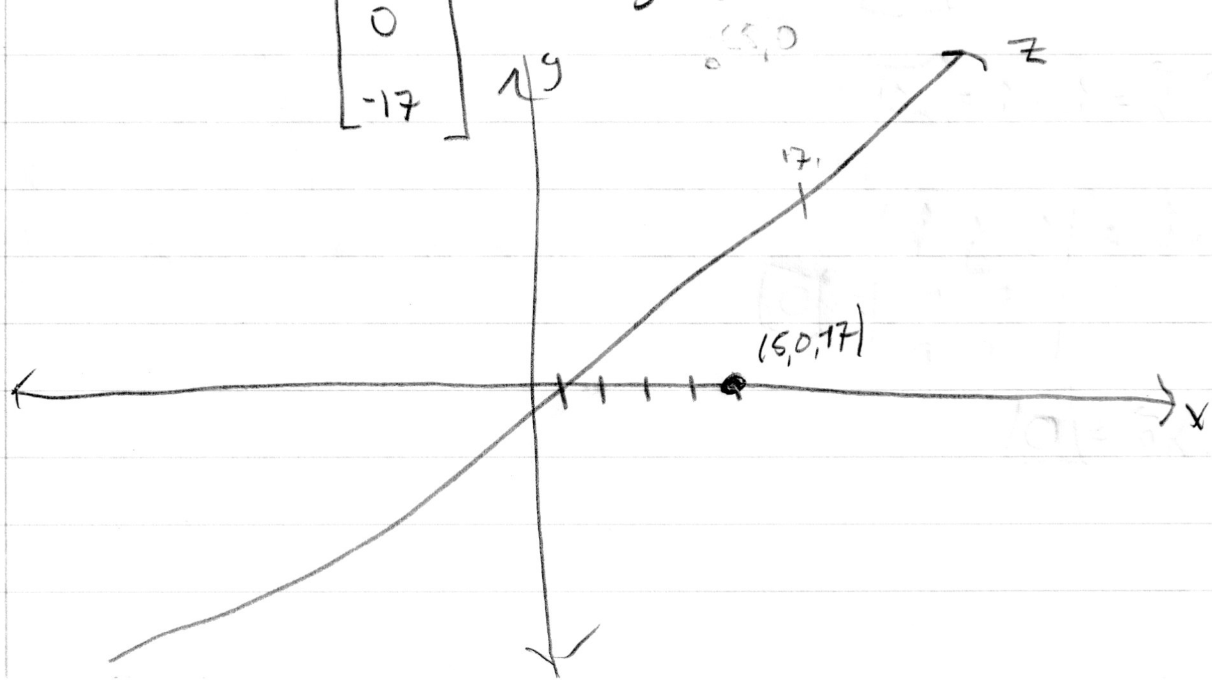
HOMEWORK 2

2-1 a) Rewrite each of the following vectors in as a column vector.

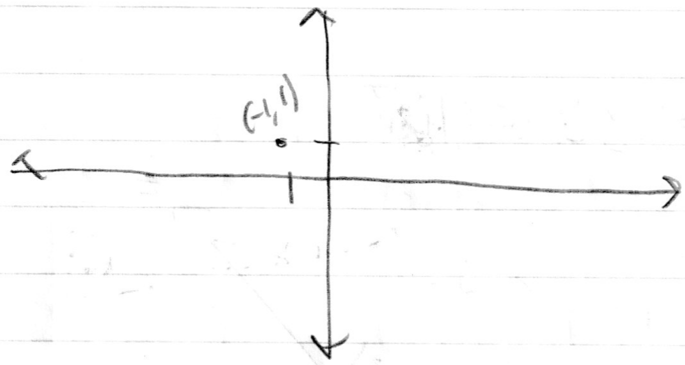
$(1,2,3) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\hat{i} + 2\hat{j} + 3\hat{k}$



$(5,0,-17) = \begin{bmatrix} 5 \\ 0 \\ -17 \end{bmatrix} = 5\hat{i} + 0\hat{j} + (-17)\hat{k}$



$$(-1,1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\hat{i} + 1\hat{j}$$



b) Operations

For each of the following pairs of vectors $\vec{a}$ and $\vec{b}$, Calculate each of the following values:

$$\vec{a} = \hat{i} \qquad \vec{b} = \hat{i}$$

$$k\vec{a} = k\hat{i}$$

$$\vec{a} + \vec{b} = \hat{i} + \hat{i} = \boxed{2\hat{i}}$$

$$\vec{a} - \vec{b} = \hat{i} - \hat{i} = \boxed{0\hat{i}}$$

$$\vec{a} \cdot \vec{b} = \hat{i} \cdot \hat{i} = \boxed{2\hat{i}}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \boxed{0}$$

$$\vec{b} \times \vec{a} = \boxed{0}$$

$\vec{a} = (1, 0, -1), \quad \vec{b} = (1, 0, 1)$

$k\vec{a} = k\hat{i} - k\hat{j}$

$\vec{a} + \vec{b} = 2\hat{i}$

$\vec{a} - \vec{b} = -2\hat{k}$

$\vec{a} \cdot \vec{b} = (\hat{i} - \hat{k}) \cdot (\hat{i} + \hat{k}) = k - 1 = 0$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \hat{i}(8) - \hat{j}(2) + \hat{k}(0) = \boxed{-2\hat{j}}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \hat{i}(0) - \hat{j}(-2) + \hat{k}(0) = \boxed{2\hat{j}}$$

c) Magnitude and Unit Vectors

For each of the following vectors $\vec{a}$, calculate the magnitude $\|\vec{a}\|$ and the unit vector $\hat{u}_{\vec{a}}$

$\vec{a} = 4\hat{k} = 0\hat{i} + 0\hat{j} + 4\hat{k}$

$\|\vec{a}\| = \sqrt{0^2 + 0^2 + 4^2} = 4$

$$\hat{u}_{\vec{a}} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{4\hat{k}}{4} = \boxed{\hat{k}}$$

$\vec{a} = 3\hat{i} + 5\hat{j} + -4\hat{k}$

$\|\vec{a}\| = \sqrt{3^2+5^2+(-4)^2} = \sqrt{50}$

$$\hat{u}_{\vec{a}} = \frac{\vec{a}}{\|\vec{a}\|} = \boxed{\frac{1}{\sqrt{50}}(3\hat{i}+5\hat{j}-4\hat{k})}$$

$\vec{a} = \langle 1, -1 \rangle = \hat{i} + (-1)\hat{j}$

$\|\vec{a}\| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$$\hat{u}_{\vec{a}} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{2}}\langle 1, -1 \rangle = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \boxed{\frac{1}{\sqrt{2}}\hat{i} + \left(-\frac{1}{\sqrt{2}}\right)\hat{j}}$$

$\vec{a} = \langle 2, -2, 0 \rangle = 2\hat{i} + (-2)\hat{j} + 0\hat{k}$

$\|\vec{a}\| = \sqrt{2^2+(-2)^2+0^2} = 2\sqrt{2}$

$$\hat{u}_{\vec{a}} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{2\sqrt{2}}\langle 2, -2, 0 \rangle = \left\langle \frac{2}{2\sqrt{2}}, \frac{-2}{2\sqrt{2}}, \frac{0}{2\sqrt{2}} \right\rangle$$

$$= \boxed{\frac{1}{\sqrt{2}}\hat{i} + \left(-\frac{1}{\sqrt{2}}\right)\hat{j}}$$

d) Dot Product Properties

1) $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

We know that $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

$= \left(\sqrt{a_1^2+a_2^2+b_3^2}\right)\left(\sqrt{b_1^2+b_2^2+b_3^2}\right)\cos(\theta)$

swap

$= \left(\sqrt{b_1^2+b_2^2+b_3^2}\right)\left(\sqrt{a_1^2+a_2^2+a_3^2}\right)\ \overset{0}{\cancel{\cos(-\theta)}}$

$= \|\vec{b}\|\,\|\vec{a}\|$

$= \vec{b}\cdot\vec{a}$ ✓

2) associativity of dot product:

$\vec{a}\cdot\vec{b}$ is a scalar quantity, this shows that This is not associative because the dot product between a scalar $(a\cdot b)$ and a vector c is $\boxed{\text{not defined.}}$

3) Formula for magnitude

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(a_1\hat{j}+a_2\hat{j}+a_3\hat{k})}$$

$$= \boxed{\sqrt{a_1^2+a_2^2+a_3^2}}$$

4) Find formula for $\cos\theta$

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

$$\cos\theta = \boxed{\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}}$$

e) 1) Is the cross product commutative?

$A \times B = -(B \times A)$ ← so cross product is not commutative

2) Is the cross product associative?

Consider two non-zero vectors a, b.

$$(a\times a)\times b = 0 \times b = 0$$

$$a\times(a\times b) \neq 0$$

So... $\boxed{(a\times a)\times b \neq a\times(a\times b)}$

This proves that the cross product is $\boxed{\text{NOT}}$ associative.

3) $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin(\theta)\vec{n}$

$\vec{a} \times \vec{b} = ||\vec{a}|| ||\vec{b}|| \hat{n}$

$n = \dfrac{\vec{a} \times \vec{b}}{||\vec{a}|| \times ||\vec{b}||}$ → take inverse of sine

$$\boxed{\sin\theta = \frac{||\vec{a} \times \vec{b}||}{||\vec{a}|| \cdot ||\vec{b}||}.}$$

4) consider $\vec{a} = (2,2,3)$, $\vec{b} = (1,0,1)$, $\vec{c} = (-1,3,4)$ example

$AB = (1,0,1) - (2,2,3) = (1-2, 0-2, 1-3) = (-1,-2,-2)$
$BC = (-1,3,4) - (1,0,1) = (-2,3,3)$

$AB \times BC = \boxed{(0,7,-7)}$ ← normal plane

MULTIPLICATION

2-2 a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1\cdot a + 0\cdot c & 1\cdot b + 0\cdot d \\ 0\cdot a + 1\cdot c & 0\cdot b + 1\cdot d \end{bmatrix} = \boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$

b) $\begin{bmatrix} 3 & 4 \\ -6 & -8 \end{bmatrix} \times \begin{bmatrix} 12 & 16 \\ -9 & -12 \end{bmatrix} = \begin{bmatrix} 3\cdot 12 + 4\cdot(-9) & 3\cdot 16 + 4\cdot(-12) \\ -6\cdot 12 + (-8)\cdot(-9) & -6\cdot 16 + (-8)\cdot(-12) \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} a1 + b1 + c1 & 0 & 0 \\ d1 + e1 + f1 & 0 & 0 \\ g1 + h1 + i\cdot 1 & 0 & 0 \end{bmatrix}}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \boxed{\begin{bmatrix} 1a + 1d + 1g & 1b + 1e + 1h & 1c + 1j + hi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$

b) TRANSFORMING COLUMN VECTORS

$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \\ 1 \end{bmatrix}$

any extra value (component) in vector can effect the answer to the vector.

$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$

2-3a) Simplify

$$\hat{j}^2 = \hat{j}\cdot\hat{j} = |\hat{j}||\hat{j}|\cos(0)^\circ = \boxed{1}$$

$$\hat{j}\cdot\hat{i} = |\hat{j}||\hat{i}|\cos(90^\circ) = \boxed{0}$$

$$\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos(90) = \boxed{0}$$

$$\hat{j}\hat{k}\hat{i} = (\hat{j}\cdot\hat{k})\cdot\hat{i} = (|\hat{j}||\hat{k}|\cos 90^\circ)\cdot\hat{i} = 0\cdot\hat{i} = \boxed{0}$$

$$\hat{k}\hat{j}\hat{i} = (\hat{k}\cdot\hat{j})\cdot\hat{i} = (|\hat{k}|\cdot|\hat{j}|\cos(90^\circ))\cdot\hat{i} = 0\cdot\hat{i} = \boxed{0}$$

$$(\hat{i}+\hat{j})(\hat{i}-\hat{j}) = \hat{i}\cdot\hat{i} - \hat{i}\cdot\hat{j} + \hat{j}\cdot\hat{i} - \hat{j}\cdot\hat{j} = \hat{i}\cdot\hat{i} - \hat{j}\cdot\hat{j} = \cancel{0} = 1 - 1 = \boxed{0}$$

$$\left(\frac{\sqrt{2}}{2} + \frac{\hat{i}\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} - \frac{\hat{i}\sqrt{2}}{2}\right) = \frac{2}{4} - \left(\frac{\sqrt{2}}{2}\right)^2\cdot\hat{i}\cdot\hat{i}$$

$$= \frac{1}{2} - \frac{2}{4}\cdot 1 = \frac{1}{2} - \frac{1}{2} = \boxed{0}$$

b) Rotations

1) Rotation by π radians about $\hat{x}$ axis

$\hat{i}\hat{j}\hat{k} = \langle 1, 0, 0 \rangle = \hat{i}$

$\sin(\frac{\pi}{2}) = 0$, $\cos(\frac{\pi}{2}) = -0.5$

$$\boxed{0 + -0.5\hat{i} = q}$$

2) Rotation of $90°$ about z- axis

$$\underset{-1}{\cos(180)} + \underset{0}{\sin(180)}(1\hat{k}) = \boxed{-1}$$

$w^2 + a^2 + b^2 + c^2 = -1$

$$\underbrace{\cos^2(\theta) + \sin(\theta)}(\underbrace{a\hat{i} + b\hat{j} + c\hat{k}})$$

$$\frac{\hat{q}}{\|\hat{q}\|} = \boxed{\|\hat{q}\| = \sqrt{w^2 + a^2 + b^2 + c^2}}$$

3) Rotation by $\pi/3$ radians about axis $(4, -3, 12)$

$$\underset{1/2}{\cos(\pi/3)} + \underset{.866}{\sin(\pi/3)}(1\hat{k}) =$$

NOT complete

c) INVERSE

$$q = q_0 + \hat{i}q_1 + \hat{j}q_2 + \hat{k}q_3$$

$$q^{-1} = \frac{q_0 - \hat{i}q_1 + \hat{j}q_2 + \hat{k}q_3}{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

Hence

1) $\boxed{0.5\ \hat{i} = q}$

2) $\boxed{\sqrt{c_0^2 + a^2 + b^2 + c^2}}$

3) NOT COMPLETE

d) Interpreting Encodings

b) Rotations

1) Rotation by π radians about $\hat{x}$ axis

$\hat{i}\hat{j}\hat{k} = \langle 1, 0, 0 \rangle = \hat{i}$

$\sin(\frac{\pi}{2}) = 0$, $\cos(\frac{\pi}{2}) = -0.5$

$$\boxed{0 + -0.5\hat{i} = q}$$

2) Rotation of 90° about z- axis

$\cos(180) + \sin(180)(1\hat{k}) = \boxed{-1}$

$-1 \qquad 0$

$w^2 + a^2 + b^2 + c^2 = -1$

$\underbrace{\cos^2(\theta) + \sin(\theta)}\underbrace{(a\hat{i} + b\hat{j} + c\hat{k})}$

$\frac{\hat{q}}{\|\hat{q}\|} = \boxed{\|\hat{q}\| = \sqrt{w^2 + a^2 + b^2 + c^2}}$

3) Rotation by $\pi/3$ radians about axis $(4, -3, 12)$

$\cos(\pi/3) + \sin(\pi/3)(1\hat{k}) =$

$1/2 \qquad .866$

NOT complete

e) Applying Quaternions

* Use quaternion $\hat{q}$ to rotate vector $\vec{v}$

$\hat{q}\,\hat{\vec{v}}\,\hat{q}^{-1}$

1) Rotate $5\hat{j}$ by $90°$ z axis. $90° = \pi/2$

$\hat{q}(5\hat{j})\hat{q}^{-1}$

$\hat{q} = \cancel{\cos}\overset{1/\sqrt{2}}{\frac{\pi}{4}} + \cancel{\sin}\ \pi/4 \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\hat{k} = \frac{1}{\sqrt{2}}(1+\hat{k})$

INVERSE $= 1/\sqrt{2}(1-\hat{k})$

$\frac{1}{\sqrt{2}}(1-\hat{k})(3\hat{j})\frac{1}{\sqrt{2}}(1+\hat{k}) = \frac{3}{2}(1+\hat{k})(\hat{j})(1-\hat{k})$

$= 3/2(1+\hat{k})(\hat{j}-\hat{i})$

JC

(2) Rotate by $\underline{\pi}$ radians about $\vec{y}$ axis. $(1,0,2)$

$q = \cos \pi/2 + \sin \pi/2 \, \underset{1}{(-\hat{j})}$

$\hat{q} = -\hat{j}$

$\hat{q}^{-1} = \hat{j}$

$(1,0,2)$ Rotation

$\hat{i} + 2\hat{k}$

$(\hat{q})(\hat{i}+2\hat{k})(q^{-1})$

$= -\hat{j}(\hat{i}+2\hat{k})(\hat{j})$

f) Quaternion & Rotation Properties

1) Create 4 quaternions

$a = (a_1, a_2, a_3, a_4)$ $\quad b = $ " " $\quad c = $ " " $\quad d = $ " "

so $c = a \cdot b = (a_1, a_2, a_3, a_4) \cdot (b_1, b_2, b_3, b_4)$

$c_1 = (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4)$

$c_2 = (a_1 b_2 \quad a_2 b_1 \quad a_3 b_4 \quad a_4 b_3)$

$c_3 = (a_1 b_3 \quad a_2 b_4 \quad a_3 b_1 \quad a_4 b_2)$

$c_4 = (a_1 b_4 \quad a_2 b_3 \quad a_3 b_2 \quad a_4 b_1)$

so $d = b \cdot a$

$d_1 = (b_1 a_1 - b_2 a_2 - b_3 a_3 - b_4 a_4)$

$d_2 = (b_1 a_2 + b_2 a_1 - b_3 a_4 + b_4 a_3)$ NOT EQUAL!

$c \neq d$

$ab \neq ba$ $\boxed{\text{NOT COMMUTATIVE}}$

2) Multiplication of quaternions is associative and distributes over vector addition.