FYS4560;
Elementary Particle Physics

Final Project

Higher Dimensions;
Theoretical and Experimental Aspects

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Abstract

The Randall-Sundrum (RS) model of compacted dimensions will be studied, together with the postulate of the graviton particle. Quark-anti-quark production of the simplest possible massive graviton (1st order tower of Kaluza-Klein excitations) will be calculated.

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1 Introduction

Higher dimensions, also called extra dimensions, are physical models for the dimensionality of our universe, mostly suggested with the goal of explaining the hierarchy problem of the standard model. There are *many* theories for higher dimensions. The most recognised are:

- *Large extra dimensions*: The often-heard theory that gravity acts through several dimensions, therefore becoming weaker. It originates from the ADD model as an attempt to solve the hierarchy problem[1].
- *Warped extra dimensions*: Describing our universe as a five-dimensional anti-de Sitter space, and claiming the SM particles are localized on a (3 + 1)-dimensional brane(s).
- *Universal extra dimensions*: All particles move universally through the extra dimensions, unlike to two other models where only gravity propagates through them.

Obviously, a thorough description of any of these models is near impossible for such a small paper, let alone all the models together. Therefore, a brief outline of the theory behind the two currently most promising[2] models will be given.
The first is the large extra dimension model by Arkani-Hamed, Dimopoulos, and Dvali (ADD). Originally, it was proposed as a model to explain the hierarchy problem (why the weak force is 10^{32} times stronger than gravity, among other problems). The extra dimensions[3] are then suggested as planes into which gravity, assumed just as strong as the other forces, spreads. Therefore gravity becomes "diluted", while the known SM particles stay in (1,3)-spacetime.
The second model is the warped extra dimension model by Randall and Sundrum (RS), made due to disliking the current universal extra dimensions models. They assumed that, rather than having universal extra dimensions in which all particles propagate, there is a small extra dimension. This means they model our world as a 5-dimensional anti-de Sitter space[4]. By small, it means the extra dimension has a large curvature, or is *warped*. From general relativity, gravity and curvature are very much the same thing, and therefore the extra dimension, called the Planckbrane, can easily host gravitons.
A question that then springs to mind is why exactly gravitons and extra dimensions are connected (other than gravitons "carrying" gravity). If the standard model is expanded, but without inclusion of extra dimensions, to include a graviton field, then measuring it would be, at best, very optimistic[5]. Should any extra dimensional model be true, it would certainly be desirable to prove it by measurement. Finding a particle with the properties of the graviton would mean that at least *some* extra dimensions model is true.

2 Kaluza-Klein theory

The reason Kaluza-Klein theory is discussed is because one requires knowledge of the so-called "Kaluza-Klein towers"; massive excitations of an expansion model[6] of the spacetime metric. While RS1 only considers a single extra dimension, ADD works for different numbers of dimensions. However, for the sake of simplicity, and comparison, only one extra dimension will be considered.
Before adding extra dimensions, the question of how get so-called "massive gravity"; a massive field carries the gravitational force. First, one starts with linearised gravity. Problems in general relativity can be approximated by perturbing Minkowski spacetime;

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

[1]The "hierarchy problem" is the problem in explaining why gravity and the weak force are so weak compared to QED and QCD.

[2]"Promising" in the sense that they provide measurable outcomes, and fit very well with what we already know from the standard model. The problem is that so far they have predicted nothing. Finding a graviton would possibly confirm either theory.

[3]Today, 6 dimensions is the most common expansion.

[4]This will be explained later on.

[5]Refer to "Can Gravitons be Detected?" by Rothman and Boughn (https://arxiv.org/pdf/gr-qc/0601043v3.pdf) for an impression of the problem.

[6]This expansion depends on the model, and is where RS and ADD differ.

where $g_{\mu\nu}$ is the approximated spacetime metric, $\eta_{\mu\nu}$ is the Minkowski metric, and $h_{\mu\nu}$ is the perturbation, and is what eventually was suggested to be gravitons. The Lagrangian density will then have a interactive term on the form $h_{\mu\nu}T^{\mu\nu}$, and a kinetic term for $h_{\mu\nu}$. The field $h_{\mu\nu}$ can be given mass by saying the Lagrangian density also has a term $ah_{\mu\nu}h^{\mu\nu} + b(\eta_{\mu\nu}h^{\mu\nu})^2$. Markus Fierz and Wolfgang Pauli then, in 1939, wrote an article[7] in which they proved that $a = -b$ in order to avoid unphysical results. The result is the Fierz-Pauli Lagrangian for massive gravity[8] [SOURCE]:

$$\frac{1}{\kappa^2}\sqrt{|g|}R = \frac{1}{4}\left(\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} - \partial^\mu h^\nu_\nu \partial_\mu h^\rho_\rho - 2\partial^\nu h_{\nu\mu}\partial_\rho h^{\rho\mu} + 2\partial_\nu h^{\nu\mu}\partial_\mu h^\rho_\rho\right) + \mathcal{O}(\kappa) \tag{2}$$

Now, additional spatial dimensions may be added. These dimensions will not change the form of equation 2, so one can simply let indices change from greek to latin, i.e. $\mu \to a$, $\nu \to b$, etc..., where $a, b, \ldots \in \{4+d\}$. It is then assumed the perturbative field has a form

$$\hat{h}_{ab} = V_d^{-1/2}\begin{pmatrix} h_{\mu\nu} + \eta_{\mu\nu}\phi & A_{\mu i} \\ A_{\nu j} & 2\phi_{ij} \end{pmatrix} \tag{3}$$

where V_d is the volume of the compactified space, $\eta_{\mu\nu}\phi$ is a Weyl rescaling and $A_{\mu i}$ is some tensor field. Since the fields are compact, one can claim periodicity of the field, such that a Fourier expansions is possible:

$$h_{\mu\nu}(x, y) = \sum_{\substack{n=\{n_1,n_2,\ldots,n_5\}; \\ n_i \in \mathbb{Z}\,\forall\, i}} h_{n,\mu\nu}(x)Y_n(y) \tag{4}$$

where $Y_n(y)$ are orthogonal, normalized eigenfuntions of the Laplace operator on the internal space[9]:

$$\Delta_{K_d}Y_n(\theta) = \frac{\lambda_n}{R^2}Y_n(\theta) \tag{5}$$

where R is the "characteristic size" of the space K_d (tied to the volume V_d). This series expansion is called the Kalulza-Klein (KK) tower of modes, and one value of n is called mode n, and the corresponding term the n'th excitation. The field $h_{\mu\nu}(x, y)$ has to satisfy the d'Alembert equation, and in doing this one finds it necessary to redefine the fields $h_{\mu\nu}(x, y)$, $A_{\nu j}$, and ϕ_{ij}, to $\tilde{h}_{n,\mu\nu}$, $\tilde{A}_{n,\mu i}$, and $\tilde{\phi}_{n,ij}$. The detailed steps are omitted, as they are many and non-intuitive. These new fields, when put into the Lagrangian, will give mass eigenstates. This redefinition of fields can be thought of as an analogue to the rotations from the CKM matrix in QCD theory (needed for mass eigenstates in the Lagrangian), but it is not the same (not a rotation). From the equation of motion one can go one to show that the masses are:

$$m_n^2 = \frac{4\pi^2 n^2}{R^2} \tag{6}$$

A relation that will be of importance later is the reduction formula:

$$M_{Pl}^2 = V_d M^{d+2} \tag{7}$$

where $M_{Pl} = G_{N(4)}^{-1/2}$ is the 4-dimensional Planck mass and $M^{d+2} = G_{N(4+d)}^{-1/(4+d)}$ is the fundamental mass scale in the new model. The relation is derived by demanding the Einstein-Hilbert action to be the same with and without the new dimension(s)[10], and performing the integrals by using KK mode expansion on the integrand. I.e one demands:

$$S_{E(4)} = S_{E(d+4)} \tag{8}$$

where

[7] Fierz, Markus; Pauli, Wolfgang (1939). "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field". Proc. Roy. Soc. Lond. A173: 211–232.

[8] Note that this is *not* the only way to get a massive gravity Lagrangian, and it has some problems as well. It serves well as an introductory example, however.

[9] The extra dimension/space, denoted K_d.

[10] This is because the new dimension(s) must reproduce what we observe, and 4D spacetime fits very well with observations.

$$S_{E(D)} = \int d^D x \sqrt{-G_D} \frac{1}{16\pi G_{N(D)}} \mathcal{R}^D(G_{ab}) \tag{9}$$

With the principles of the KK tower and massive gravity, one can start to consider theories that are based on extra dimensions.

3 The Arkani-Hamed-Dimopoulos-Dvali model

The ADD model is mainly based on 3 features:

- There exists d new spatial compact dimensions, with compactification volume V_d.
- The Planck scale is very low, at the order of one TeV,
- The SM degrees of freedom are localized on a 3D-brane, stretching along 3 non-compact spatial dimensions (i.e. the SM particles move in normal spacetime, not in the new dimension(s)).

The idea is that the electroweak scale is the only fundamental scale in the universe, and that the true Planck scale is actually of the same order. from the reduction formula, one could introduce d new spatial dimensions and find their size, i.e.

$$\frac{M_{Pl}^2}{M^{d+2}} = V_d \sim R^d \quad , \quad \mathcal{O}(M^{d+2}) = \mathcal{O}(m_{EW}) \tag{10}$$

where R is the compactification radius as before. With new dimensions, Newton's law for gravitational force would instead be proportional to $r^{-(d+2)}$, which is of course not true. However, since the extra dimensions are compact, only objects with distances less $r << R$ feel this new force, while it still goes as r^{-2} for $r >> R$. Therefore, for $d = 1$, the radius is of the same order as the earth-moon distance, and therefore d can't be equal one. $d = 2$ is not possible either. For $d > 2$, the radius is so small that van der Waal forces prevent us from conducting "table-top" experiments, and is where particle physics experiments at CERN become relevant. If ADD is true, then it should be possible to determine the radius by measuring the graviton, which is a direct consequence of extra dimensions. Since the mass m_n is inversely proportional to the size, it would be possible to determine the new Planck scale.

In the ADD model, the standard metric is expanded upon by introducing a field $\hat{h}_{ab}(x,\theta)$ such that:

$$\tilde{g}_{ab}(x,\theta) = \eta_{ab} + \frac{2}{M^{1+d/2}} \hat{h}_{ab}(x,\theta) \tag{11}$$

4 The Randall-Sundrum model

4.1 The hierarchy problem in RS1

The RS model assumes that there are two points on the $S^1/\mathbb{Z}^2$ ordibfold in which 3-branes are compactified. What this means is that there are two 3-dimensional, non-compact branes that are connected at every point by a "circle", see figure SOME FIG.

The two branes, 1 and 2, are located on points $\theta = 0$ and $\theta = \pi$, which means the branes are seperated by a distance $2L = 2R$, where R is the circle radius.
As mentioned, the model set out to explain the hierarchy problem, and will be briefly explained how below. Firstly, the action of the model is given by the sum of the Hilbert-Einstein action and the matter "part":

$$S_5 = S_E + S_M = \int d^4 x \int_{-L}^{L} dy \sqrt{-\tilde{G}} (M^3 \mathcal{R}_5 - \Lambda_5) \tag{12}$$

where Λ_5 is the five dimensional cosmological constant. In order to match real world observations, the new metric must uphold Poinecaré invariance, which can be shown to lead to:

$$ds^2 = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \tag{13}$$

where $A(y)$ is called the *warp factor*. Solving the Einstein equations lets one find that $A(y) = \pm ky$, where $k \equiv \sqrt{\frac{-\Lambda}{12M^3}}$ is a constant. Since the orbifold abides $\mathbb{Z}^2$ symmetry, $A(y) = A(-y) \rightarrow A(y) = k|y|$. There is a problem with action above, which is that it does not include the energy densities of the two branes, that are:

$$S_1 = \int_{B_1}\int_{S^1/\mathbb{Z}^2} d^4x dy \sqrt{-\tilde{G}(x,y)}\lambda_1\delta(y) \tag{14}$$

$$S_2 = \int_{B_2}\int_{S^1/\mathbb{Z}^2} d^4x dy \sqrt{-\tilde{G}(x,y)}\lambda_2\delta(y-L) \tag{15}$$

where $\tilde{G}(x,0) = G_1$ and $\tilde{G}(x,L) = G_2$ have been used. So, the total action for the space is $S = S_E + S_M + S_1 + S_2$. To fulfil the Einstein equations, one needs to impose $\lambda_1 = -\lambda_2 = 12kM^3$. THINK ON THIS

4.2 KK mode expansion/Finding gravitons

Much in the same manner as for QED and other gauge theories, one can create gravitational bosons. This means doing a metric transform, given by:

$$ds^2 = e^{-2k|y|}(\eta_{\mu\nu} + \tilde{h}_{\mu\nu}(x,y))dx^\mu dx^\nu + (1+\phi(x))dy^2 \tag{16}$$

However, this means there will be cross-terms ($\phi\tilde{h}_{\mu\nu}$-terms) in the Lagrangian that prevent mass eigenstates. It can be shown[11] that it can be diagonalized. One will end up with two fields, $h_{\mu\nu}$ and φ, which will be used hereafter.
Then, a KK mode expansion is done on $h_{\mu\nu}(x,y)$, giving

$$h_{\mu\nu}(x,y) = \sum_{n=0}^{\infty} h_{\mu\nu}^{(n)}(x)\frac{\chi_n(y)}{R}, \tag{17}$$

where $\chi_0(y) = 2\sqrt{kR}e^{-2k|y|}$ and

$$\chi_n(y) = N_n\left[C_1 Y_2\left(\frac{m_n}{k}e^{k|y|}\right) + C_2 J_2\left(\frac{m_n}{k}e^{k|y|}\right)\right] \quad , \quad n \neq 0 \tag{18}$$

While this seems out of the blue, this is the result of diagonalizing the Lagrangian, which is a rather intricate procedure. The functions χ_n are eigenfunction of an equation met during the diagonalising of the Lagrangian. J_2 and Y_2 are the Bessel functions of the first and second kind, respectively. The constants C_1 and C_2 can be determined by the boundary conditions on the vacuum energy terms of the energy-momentum tensor (the delta functions),

$$T_{ab} = \lambda_1\sqrt{G_1}g_{\mu\nu}^{(1)}\delta_a^\mu\delta_b^\nu\delta(y) + \lambda_2\sqrt{G_2}g_{\mu\nu}^{(2)}\delta_a^\mu\delta_b^\nu\delta(y-\pi R), \tag{19}$$

which give $C_1 = Y_1\left(\frac{m_n}{k}\right)$ and $C_2 = -J_1\left(\frac{m_n}{k}\right)$ SOMETHING MORE HERE

The $h_{\mu\nu}^{(0)}(x)$ field describes the massless graviton, while the $n \geq 1$ states are the massive KK modes, and finally φ describes the massless *radion*.

[11] See http://arxiv.org/abs/hep-th/0105304v3, section 2, for a rigorous derivation.

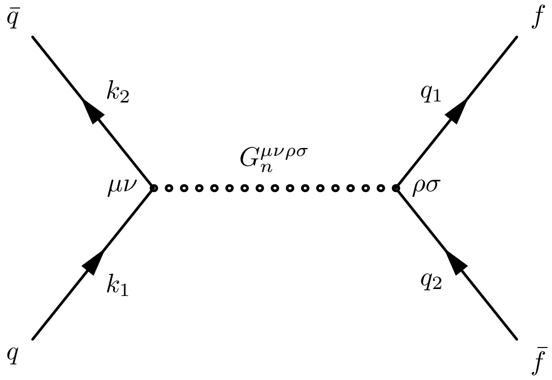

Figure 1: $q\bar{q} \rightarrow f\bar{f}$

5 Graviton production at the LHC

6 Conclusions

A Calculations

A detailed calculation for the channel $q\bar{q} \rightarrow l^+l^-$ is as follows.
The n'th resonance spin-2 KK graviton propagator is given by:

$$G_n^{\mu\nu\rho\sigma} = i\frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{k_G^2 - m_n^2 + i\epsilon} \tag{20}$$

The $G\bar{\psi}\psi$-coupling is given by (again, spin-2):

$$-\frac{i\kappa}{8}\left[\gamma_\mu(k_1+k_2)_\nu + \gamma_\nu(k_1+k_2)_\mu - 2\eta_{\mu\nu}(\not{k}_1 + \not{k}_2 - 2m_f)\right] \tag{21}$$

Therefore, the diagram shown in figure 1 has effective amplitudes:

$$\begin{aligned}
i\mathcal{M}(q\bar{q} \rightarrow f\bar{f}) &= -\sum_n \frac{i\kappa^2}{64}\bar{u}_q\Big[\gamma_\rho(q_1+q_2)_\sigma + \gamma_\sigma(q_1+q_2)_\rho - 2\eta_{\rho\sigma}(\not{q}_1 + \not{q}_2 - 2m_f)\Big]v_q \\
&\qquad \times\left[\frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{k_G^2 - m_n^2 + i\epsilon}\right] \\
&\qquad \times\bar{v}_k\left[\gamma_\mu(k_1+k_2)_\nu + \gamma_\nu(k_1+k_2)_\mu - 2\eta_{\mu\nu}(\not{k}_1 + \not{k}_2 - 2m_f)\right]u_k \\
&= -i\frac{\pi C_4}{4}\bar{u}_q\Big[\gamma_\rho(q_1+q_2)_\sigma + \gamma_\sigma(q_1+q_2)_\rho - 2\eta_{\rho\sigma}(\not{q}_1 + \not{q}_2 - 2m_f)\Big]v_q \\
&\qquad \times\bar{v}_k\left[\gamma^\rho(k_1+k_2)^\sigma + \gamma^\sigma(k_1+k_2)^\rho + \frac{4}{3}\eta^{\sigma\rho}\right]u_k \\
&= -i\frac{\pi C_4}{2}\bar{u}_q\Big[(q_1+q_2)_\mu(k_1+k_2)^\mu\gamma_\nu v_q\bar{v}_k\gamma^\nu + (\not{k}_1 + \not{k}_2)v_q\bar{v}_k(\not{q}_1 + \not{q}_2) \\
&\qquad - 2(\not{q}_1 + \not{q}_2)v_q\bar{v}_k(\not{k}_1 + \not{k}_2) + 4\Big(m_q(\not{q}_1 + \not{q}_2)v_q\bar{v}_k + m_f v_q\bar{v}_k(\not{k}_1 + \not{k}_2)\Big) \\
&\qquad - \frac{32}{3}m_f m_q v_q\bar{v}_k\Big]u_k
\end{aligned} \tag{22}$$

where $C_4 \equiv \frac{\kappa^2}{8\pi}D(s)$, $D(s) \equiv \sum_n \frac{1}{k_G^2 - m_n^2 + i\epsilon}$. The above expression can be rewritten as

$$i\mathcal{M}(q\bar{q} \to f\bar{f}) = -i\frac{\pi C_4}{2}\bar{u}_q\Big[(q_1+q_2)_\mu(k_1+k_2)^\mu\gamma_\nu v_q\bar{v}_k\gamma^\nu + (\not{k}_1+\not{k}_2)v_q\bar{v}_k(\not{q}_1+\not{q}_2) - \frac{8}{3}m_f m_q v_q\bar{v}_k \\ - 2(\not{q}_1+\not{q}_2-2m_f)v_q\bar{v}_k(\not{k}_1+\not{k}_2-2m_q)\Big]u_k \tag{23}$$

The last term in the above equation is actually just the Dirac equation in momentum space,

$$(\not{k} - m_f)u_f(k) = 0, \tag{24}$$

and therefore equals zero. The amplitude is then

$$i\mathcal{M}(q\bar{q} \to f\bar{f}) = -i\frac{\pi C_4}{2}\bar{u}_q\left[(q_1+q_2)_\mu(k_1+k_2)^\mu\gamma_\nu v_q\bar{v}_k\gamma^\nu + (\not{k}_1+\not{k}_2)v_q\bar{v}_k(\not{q}_1+\not{q}_2) - \frac{8}{3}m_f m_q v_q\bar{v}_k\right]u_k \tag{25}$$

for the spin-2 massive KK graviton propagators.