DUAL ARTIN GROUP PRESENTATIONS

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1. Introduction

We will begin by setting up our notation for Coxeter and Artin groups.

Let W and A respectively be the Coxeter and Artin groups associated to some Coxeter system Γ. In this document, the specific Coxeter system is arbitrary and constant.

Our Coxeter and Artin groups have a conventional finite generating set S. In this document, n will always denote $|S|$, the rank of our Coxeter system. We will consider S to be a free generating set, with F_S the free group on S. For any quotient G of F_S (particularly W or A), we denote the quotient surjection $\pi_{(S,G)} \colon F_S \to G$. Where S is obvious from context, we may drop it from this notation in favour of $\pi_G \colon F_S \to G$. For shorthand, we will denote $\pi_W(S)$ and $\pi_A(S)$ by S_W and S_A respectively.

To specify a dual Artin group, we must in principle choose a Coxeter element in W. We will always denote this choice w. We will denote the generating simple reflections W by S. The elements of S are indexed by the naturals from 1 to n such that w is a lexicographic product, i.e. $w = \pi_W(s_1 s_2 \cdots s_n)$.

Let $q \colon A \to W$ denote the natural surjection associated to these groups. For each $s \in S$, we denote the *positive lift* of s by $\overline{s} \in A$. This is a section of q restricted to the generating set of A such that $\overline{q(\alpha)} = \alpha$ for each α in the (positive) generating set of A. Accordingly, we may identify the generating set of A with $\overline{S} = \{\overline{s} \mid s \in S\}$. It is tempting to extend $s \mapsto \overline{s}$ to a map $W \to A$, but this would only be well-defined on words in S, not elements of W.

Let $R \coloneqq (S_W)^W \subseteq W$, $T \coloneqq (S_A)^A \subseteq A$ and $U \coloneqq S^{F_S}$ denote all conjugates of generators in W, A and F_S respectively. We call R the *reflections* of W.

Let B_n denote the braid group on n strands with standard generating set $\{\sigma_i\}_{1 \le i < n}$. Given any group G, there is an action of B_n on G^n where

$$\sigma_i \cdot (g_1, \dots g_n) = \left(g_1, \dots, g_{i-1}, g_{i+1}^{(g_i^{-1})}, g_i, g_{i+2}, \dots, g_n \right).$$

This is called the *Hurwitz action*. This action preserves the index-wise product $(g_1, \dots, g_n) \mapsto g_1 g_2 \cdots g_n$. Given some $X \subseteq G$, if $X^G = G$, then the Hurwitz action is also well-defined on X^n, and this makes any such X a B_n-set. In particular, we wish to consider this action on S^n, T^n and U^n.

Let $p_1 \colon G^n \to G$ denote generic projection to the first coordinate of a tuple. We are interested in the tuples that occur due to the Hurwitz action. The set $B_n \cdot \tau$ is the set of all such tuples for some tuple τ. The set $\mathrm{HurRef}(\tau)$ is the set of all $g \in G$ that occur in any position in $B_n \cdot \tau$. By choosing braids that move elements to the first position in the tuple, we see that $\mathrm{HurRef}(\tau) = p_1(B_n \cdot \tau)$.

Associated to a tuple, there is a monoid construction, which we will call the *Hurwitz monoid*.

Definition 1.1. *Let G be some group, and let τ be some tuple in G^k. Let X denote* $\mathrm{HurRef}(\tau)$. *Suppose there in there is a tuple in $B_k \cdot \tau$ that begins r_1, r_2. Let r_3 denote the element of X equal in G to $r_1^{r_2^{-1}}$. We call any such (r_1, r_2, r_3) a* Hurwitz triple. *In this context, we define the* Hurwitz monoid *I associated to τ as follows. Decorate elements $x \in X$ as $[x]$ to distinguish them as generators of I rather than elements of G.*

$$I \coloneqq \langle [X] \mid \{[r_1][r_2] = [r_3][r_1] \mid (r_1, r_2, r_3) \textit{ is a Hurwitz triple}\} \rangle.$$

We call the corresponding group, the *Hurwitz group*.

Lemma 1.2. *Given a group G and a tuple $\tau \in G^k$, the Hurwitz group H associated to τ is generated by the elements of τ.*

Proof. Let $\ell \colon B_n \to \mathbb{N}$ denote minimum word length with respect to the standard generating set for B_n. Let $x \in \mathrm{HurRef}(\tau) \setminus \{\tau_i\}$. Let $\beta \in B_n$ be the minimum length braid such that $\beta \cdot \tau = (\mu, \cdots, \mu_n)$ and $x = \mu_i$ for some i. We will show that such an $[x]$ can be written in terms of generators appearing in $\gamma \cdot \tau$ where γ is a braid with $\ell(\gamma) < \ell(\beta)$. Induction then completes the proof.

Since $x \notin \{\tau_j\}$, we know $\ell(\beta) > 0$. Recall our notation for the set of standard generators for B_n is $\{\sigma_j\}$. Since β is a braid of minimum length with respect to these generators, we know that there is a factorisation of β in the $\{\sigma_j\}$ such that the last factor is either σ_i or σ_{i-1}^{-1}, since these are the only generators whose action puts something new in the i^{th} position of the tuple.

Suppose the last factor of β is σ_i. Let $(\nu_1, \ldots, \nu_n) = \beta\sigma_i^{-1} \cdot \tau$, so $[x] = [\mu_i] = [\nu_{i+1}\nu_i\nu_{i+1}^{-1}]$. Since $\ell(\beta\sigma_i^{-1}) \leq \ell(\beta) - 1$, if we can express x in terms of the set $\{[\nu_j]\}$ then we would be done. Since we can bring ν_i and ν_{i+1} to the start of the tuple, we have the equation $[x] = [\nu_{i+1}][\nu_i][\nu_{i+1}]^{-1}$ in H.

We make a similar argument if the last factor of β is σ_{i-1}^{-1}. $\square$

The following is not a standard definition.

Definition 1.3. *Given a Coxeter element $w \in W$, the* dual Artin group *associated to w is defined to be the Hurwitz group associated to any $(r_1, \ldots, r_n)$ such that all $r_i \in R$ and $r_1 r_2 \cdots r_n = w$.*

The obvious tuple to consider is $(\pi_W(s_1), \ldots, \pi_W(s_n))$. Since our choice of w is constant, we will denote our dual Artin group $A^\vee$ (omitting any mention of w).

Let $a, b \in G$ for some group G. Let $\Pi(a, b\,;k)$ denote the alternating product of a and b, beginning with a, and of length k, where a and b are any group elements. For example $\Pi(a, b\,; 5) = ababa$.

Lemma 1.4. *Let G be a group such that there exists a homomorphism $\varphi \colon A \to G$ from our Artin group. Consider the tuple $\tau = (\varphi \circ \pi_A)^n(s_1, \ldots, s_n) \in G^n$. There is a natural surjection from A to H, the Hurwitz group associated to τ.*

Proof. For each i, let t_i denote $\pi_A(s_i)$ and u_i denote $\varphi(t_i)$. We will show that the map $t_i \mapsto u_i$ extends to a homomorphism. To do so, we need to show that any Artin-like equation of the form $\Pi(u_i, u_j\,; k) = \Pi(u_j, u_i\,; k)$ associated to a defining relation in A also holds in H.

For any $i < j$, there is a braid β such that $\beta \cdot (u_1, \dots, u_n)$ begins with u_i, u_j. Thus, without loss of generality, we can assume for the following argument that $i = 1$ and $j = 2$ Let us examine what happens when we keep acting by σ_1 on $(u_1, \dots, u_n)$.

Acting once we get the relation

$$[u_2 u_1 u_2^{-1}] = [u_2][u_1][u_2]^{-1}.$$

Once more, with some substitution using the above

$$[u_1 u_2 u_1 u_2^{-1} u_1^{-1}] = [u_1][u_2 u_1 u_2^{-1}][u_1]^{-1} = [u_1][u_2][u_1][u_2]^{-1}[u_1]^{-1}.$$

□

So the index-wise product of $\pi^n_W(\tau)$ is w. In this context, π^n_G is a B_n-set morphism from F^n_S to G^n, where G is either W or A. It is true that $\pi^n_W(B_n \cdot \tau) = B_n \cdot \pi^n_W(\tau)$ gives every minimal factorisation of w in to elements of R.

1.1. **Some facts about the objects involved.** Here we will state some facts and try to point to the relevant place to read up on them.

Throughout, let $W = (W, S)$ be some Coxeter system. Let $\{\{\}s_i\}_{1 \le i \le n}$ be some ordering on S and let $h = s_1 s_2 \cdots s_n$ be a choice of Coxeter element resulting from the ordering on S. From here forward, fix $W = (W, S)$ and h. Let $T = S^W$ be the set of reflections in W. This is all conjugations of S in W. We define the dual Artin group $A^\vee = A^\vee(W, S, h)$, as in [Res24] or [PS21].

Let B_n denote the braid group on n strands generated by $\sigma_1, \sigma_2, \dots \sigma_{n-1}$ in the usual way. Let $x^y = yxy^{-1}$ denote conjugation in groups.

Definition 1.5 (The Hurwitz action)**.** *Let X be some set on which we can define conjugates which lie within X. Define a group action of B_n on X^n in the following way.*

$$\sigma_i \cdot (t_1, \dots t_n) = \left(t_1, \dots t_{i-1}, t_{i+1}^{t_i}, t_i, t_{i+2} \dots t_n\right).$$

Any group G is a fine candidate for X. A subset that is closed under conjugation will also work, for example the set of reflections $T \subseteq W$ as defined above.

Theorem 1.6. *We can map T^n to W by multiplying all the elements in the tuple in order. Considering this map, the action of B_n is transitive on minimal T–factorisations of h.*

We will use the symbol h as to refer to both a Coxeter element $h \in W$, and a specific factorisation of h, realised as a tuple, $h \in T^n \subseteq W^n$. Let $T_0 \coloneqq T \cap (B_n \cdot h)$ denote all reflections that occur in any position in a minimal T–factorisation of h.

Theorem 1.7. *For each $r \in T_0$, and for each $i \in \{1, \dots n\}$ there exists a minimal T–factorisation $(t_1, \dots t_n) \in (B_n \cdot h)$ such that $t_i = r$. So every reflection that appears, appears in every possible position.*

Theorem 1.8. *The above theorem also applies to subwords. For each $(r_1, \dots r_n) \in (B_n \cdot h)$, each $i \le n$ and each $j \le n - i$, there exists a $(t_1, \dots t_n) \in (B_n \cdot h)$ such that $(t_j, t_{j+1}, \dots, t_{j+i}) = (r_1, \dots, r_i)$.*

Theorem 1.9 ([MS17, Proposition 10.1])**.** *The generating W generating set S is contained in T_0 and this is also a generating set for $A^\vee$. The natural inclusion of $\overline{S} \subseteq A$ in to $A^\vee$ induces a surjective homomorphism.*

Theorem 1.10 ([Bes04, Lemma 7.11])**.** *Let $[t]$ denote an abstract generator coming from some $t \in T \cap (B_n \cdot h)$. The dual Artin group has the following presentation.*

$$A^\vee(W,S,h) \cong \langle T_0 \mid \{[tt't][t][t']^{-1}[t]^{-1} \mid (t,t',x_3,\dots x_n) \in B_n \cdot h\} \rangle.$$

We call such $[tt't][t][t']^{-1}[t]^{-1}$ *dual braid relations.*

The following is a basic group theoretic fact, but it will be useful to explicitly state and prove it. For some set S, let F_S be the free group generated by S.

Lemma 1.11. *Suppose we have the groups G_1 and G_2 such that G_1 has a presentation $G_1 \cong \langle S \mid R \rangle$ where R is a set of words in $S \cup S^{-1}$ that are the identity in G_1. Further, suppose we are given some map $f\colon S \to G_2$. For all $s \in S$, define $f(s^{-1})$ to be $f(s)^{-1}$. If for each word $s_1 s_2 \cdots s_n \in R$ we have that $f(s_1)f(s_1)\cdots f(s_n) = 1$ in G_2 then we can extend f to a homomorphism $h\colon G_1 \to G_2$ where $h|_S = f$ considering S as a subset of G_1.*

Proof. We can define a homomorphism $f'\colon F_S \to G_2$ recursively by setting $f'(1) = 1$, $f'(s^{\pm 1}) = f(s)^{\pm 1}$ for all $s \in S$ and then setting $f'(s_1 s_2 \cdots s_n) = f'(s_1)f'(s_2 \cdots s_n)$ for all words $s_1 s_2 \cdots s_n \in \{S \cup S^{-1}\}_*$. Let N be the minimal normal subgroup in F_S which contains the elements of R such that $G_1 \cong F_S/N$. By assumption, for each $r \in R$ we have $f'(r) = 1$. So $R \subseteq \ker(f')$. So $N \subseteq \ker(f')$. By the universal property of the quotient we have the following commutative diagram where the unique map h is the required homomorphism.

$$\begin{tikzcd} F_S \arrow[r, "f'"] \arrow[d, "q"'] & G_2 \\ F_S/N \cong G_1 \arrow[ur, "\exists! h"'] & \end{tikzcd}$$

□

Note that if $\langle \mathrm{Im}(f) \rangle = G_2$ then h is a surjection.

2. A proof in [Res24]

We will overview a proof in [Res24] while trying to avoid unnecessary detail. The generators $S = \{s_1, \dots, s_n\}$ have a natural bijection with generators of A. Let $\overline{s_i}$ denote the generator in A corresponding to $s_i \in S \subseteq W$. We have an action of B_n on A^n defined in the exact same way as in Definition 1.5. We will consider the orbit of $\overline{h} = (\overline{s_1}, \dots, \overline{s_n})$ under this action.

Definition 2.1. *Let G be a group. Let $\rho_n\colon G^n \to G$ be projection on to coordinate n. We define the map $- \odot -\colon B_n \times G^n \to G$ as follows*

$$\tau \odot (g_1, \dots, g_n) = \rho_n(\tau \cdot (g_1, \dots, g_n)).$$

Remark 2.2. $G^n \neq G$ so this is not a group action!

Definition 2.3. *Define the equivalence relation $\sim$ on B_n by*

$$\tau \sim \tau' \iff \tau \odot h = \tau' \odot h.$$

Definition 2.4. *Define the equivalence relation $\dot\sim$ on B_n by*

$$\tau \mathrel{\dot\sim} \tau' \iff \tau \odot \overline{h} = \tau' \odot \overline{h}.$$

Theorem 2.5 ([Res24, Propsition 3.9])**.** *The B_n relations $\sim$ and $\dot{\sim}$ are the same iff $A^\vee$ is isomorphic to A.*

Proof. We will only concentrate on the $\implies$ direction. Furthermore, it is easy enough to see that $\tau \dot{\sim} \tau' \implies \tau \sim \tau'$. As such, we will only prove that if $\tau \sim \tau' \implies \tau \dot{\sim} \tau'$ for all $\tau, \tau' \in B_n$, then $A \cong A^\vee$.

Define $\psi \colon T_0 \to A$ in the following way. For $t \in T_0$, there exists a $\tau \in B_n$ such that $\tau \circledcirc h = t$. Define $\psi(t) = \tau \circledcirc \overline{h}$. We will show this does not depend on our choice τ. Suppose that $\tau' \circledcirc h$ is also t. Then $\tau \sim \tau'$. Then, by assumption $\tau \dot{\sim} \tau'$, so $\tau' \circledcirc \overline{h} = \tau \circledcirc \overline{h}$.

So we have a map from the generating set of $A^\vee$ to A. Recall the presentation of $A^\vee$ from Theorem 1.10. Let R be the set of relations (words equal to the identity) from that presentation. As in Lemma 1.11, if we can show that the natural extension of ψ to R, maps all of R to the identity in A, then we can extend ψ to a homomorphism $\psi \colon A^\vee \to A$.

Let t, t' and $tt't$ be as in Theorem 1.10. Let τB_n be such that

$$\tau \cdot h = (x_1, x_2, \dots, x_{n-2}, t, t').$$

Let $\tau \cdot \overline{h} = (z_1, \dots z_n)$. Thus, we have the following.

$$\begin{aligned} \psi(t') &= \tau \cdot \overline{h} = z_n \\ \psi(t) &= (\sigma_{n-1}\tau) \cdot \overline{h} = z_{n-1} \\ \psi(tt't) &= (\sigma_{n-1}^2\tau) \cdot \overline{h} = z_{n-1} z_n z_{n-1}^{-1} \end{aligned}$$

Thus, $\psi([tt't][t][t']^{-1}[t]^{-1}) = 1$ and ψ extends to a homomorphism $\psi \colon A^\vee \to A$.

Let φ denote the homomorphism in Theorem 1.9. Note that $\psi \circ \varphi|_{\overline{S}} = \mathrm{Id}_{\overline{S}}$, thus $\psi \circ \varphi$ is the identity ψ is an isomorphism. □

This proof gives the distinct impression of "what just happened". Most striking, is how in showing ψ was a homomorphism, at no point did anything to do with A play a part. All we needed was that A was a group, and it all popped out. In fact, the key step was the existence of a map ψ, we got homomorphism for free. This is due to the structure of the relations in Theorem 1.10. We wish to now abstract the construction of the dual braid group emerging in Theorem 1.10 and develop that in to a reconstruction of the above proof.

3. Masking group relations using languages

Let $F \colon \mathbf{Set} \to \mathbf{Grp}$ be the free functor and denote its action on objects as $F_S \coloneqq F(S)$ and its action on morphisms as $f_* \coloneqq F(f \colon S \to T)$. To each quotient G of F_S we associate the surjective homomorphism $\pi_{(S,G)} \colon F_S \to G$ which is projection. In this context, we can frame a basic group theoretic fact. We write $G \cong \langle S \mid R \rangle$ if $G \cong F_S/N(R)$ where $N(R)$ is the minimal normal subgroup in F_S that contains R. We write $G \cong \langle S \rangle$ if G is isomorphic to some quotient of F_S.

Theorem 3.1. *Suppose we have two groups G_1 and G_2. Suppose that $G_1 \cong \langle S \mid R \rangle$ where R is a subset of F_S for which $\pi_{(S,G_1)}(R) = \{1\}$. Given a map $f \colon S \to G_2$, if $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\} \subseteq G_2$, then f defines a homomorphism $h \colon G_2 \to G_2$.*

Proof. We have $G_1 \cong F_S/N(R)$. By assumption, $R \subseteq \ker(\pi_{(G_2,G_2)} \circ f_*)$, which is normal, thus $N(R) \subseteq \ker(\pi_{(G_2,G_2)} \circ f_*)$. By the universal property of the quotient we have the following commutative diagram where the unique map h is the required homomorphism.

$$\begin{tikzcd}
F_S \arrow[r, "\pi_{(G_2,G_2)} \circ f_*"] \arrow[d, "\pi_{(S,G_1)}"'] & G_2 \\
F_S/N(R) \cong G_1 \arrow[ur, "\exists! h"'] &
\end{tikzcd}$$

□

The above theorem defines a homomorphism $h\colon \langle S \mid R \rangle \cong G_1 \to G_2$ when a map $f\colon S \to G_2$ is compatible with the relations in G_1 and G_2. If we also had knowledge of a generating set for G_2, we could construct homomorphisms in a different way.

Theorem 3.2. *Suppose $G_1 \cong \langle S_1 \mid R_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Given a map $f\colon S_1 \to S_2$, if there exists a map h such that the following diagram commutes, then h is a homomorphism.*

$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "f"] & F_{S_1} \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "f_*"] & G_1 \arrow[d, red, "h"] \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2
\end{tikzcd} \tag{1}$$

Proof. Consider some $a, b \in G_1$. Since $\pi_{(S_1,G_1)}$ is surjective, we can choose some $\overline{a} \in \pi^{-1}_{(S_1,G_1)}(a)$ and $b \in \pi^{-1}_{(S_1,G_1)}(b)$. Since $\pi_{(S_1,G_1)}$ is a homomorphism, we have $\overline{a}\overline{b} \in \pi^{-1}_{(S_1,G_1)}(ab)$. So $h(ab) = h \circ \pi_{(S_1,G_1)}(\overline{a}\overline{b})$. By the commutative diagram, this is $\pi_{(S_2,G_2)} \circ f_*(\overline{a}\overline{b})$. Since $\pi_{(S_2,G_2)} \circ f_*$ is a homomorphism, this is

$$\begin{aligned}
\pi_{(S_2,G_2)} \circ f_* \left(\overline{a}\right) \pi_{(S_2,G_2)} \circ f_* \left(\overline{b}\right) = \\
h \circ \pi_{(S_1,G_1)} \left(\overline{a}\right) h \circ \pi_{(S_1,G_1)} \left(\overline{b}\right) = \\
h(a)h(b)
\end{aligned}$$

□

Corrolary 3.3. *A map $h\colon G_1 \to G_2$ is a homomorphism if and only if we can construct the relevant commuting diagram* (1)*.*

Proof. Theorem 3.2 gives us the $\Longleftarrow$ direction. If we consider $G \cong \langle G_2 \rangle$ we can construct the $\Longrightarrow$ direction. □

Remark 3.4. If we construct a homomorphism using Theorem 3.1, then we get the middle commuting square of (1) automatically.

Now consider (1), but replace F_{S_1} with some subset $Q \subseteq F_{S_1}$. We will now construct a group, which we will denote G^Q, where such a diagram defines a homomorphism from G^Q to G_2.

Definition 3.5 (Group with relations visible in Q)**.** *Suppose we have a group $G \cong \langle S \rangle$. Fix some $Q \subseteq F_S$ such that $S \subseteq Q$. Let $\pi \coloneqq \pi_{(S,G)}$. We have the following maps.*

$$S \xhookrightarrow{i} Q \xrightarrow{\pi} \pi(Q) \ .$$

Define the group G with relations visible in Q, *to be*

$$G^Q \coloneqq \langle \pi(Q) \mid \{\pi(q) = (\pi \circ i)_*(q) \mid q \in Q\} \rangle.$$

Do not think of the generators $\pi(Q)$ as elements of G. They are abstract generators. Specifically, G^Q is a quotient of $F_{\pi(Q)}$. Thus, our relations should be equations in $F_{\pi(Q)}$, which they are. G^Q is generated by $\pi(S) \subseteq \pi(Q)$. G is a quotient of G^Q, which we will explore later.

Lemma 3.6. *Let $G \cong \langle S \rangle$ be a group and consider the following setup.*

$$S \xhookrightarrow{i_S} F_S \xrightarrow{\pi_{(S,G)}} G \xhookrightarrow{i_G} F_G \xrightarrow{\pi_{(G,G)}} G$$

The following maps are equal.

$$\pi_{(G,G)} \circ i_G \circ \pi_{(S,G)} = \pi_{(G,G)} \circ (\pi_{(S,G)} \circ i_S)_*.$$

Proof. Both maps are homomorphism which agree with $\pi_{(S,G)}$ on the generating set S. $\square$

Theorem 3.7. *Suppose we have two groups $G_1 \cong \langle S_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Fix some $Q \subseteq F_{S_1}$ such that $S \subseteq Q$. Let $\overline{Q} \coloneqq \pi_{(S_1,G_1)}(Q)$. If in the following diagram* (2)*, if there exists a map f that makes the diagram commute, then there is a homomorphism $h\colon (G_1)^Q \to G_2$.*

(2)
$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[d, "\exists f?"] \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2
\end{tikzcd}$$

Proof. Suppose we have such a map f. Note that $\overline{Q}$ is a generating set for G^Q, thus we have the setup of Theorem 3.1. We use the following commutative diagram to define some inclusion maps and as a reference for the setup.

(3)
$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] \arrow[d, "f"] & F_{\overline{Q}} \arrow[d, "f_*"] & \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \arrow[r, hook, "i_{G_2}"] & F_{G_2} \arrow[r, "\pi_{(G_2,G_2)}"] & G_2
\end{tikzcd}$$

We can construct a homomorphism if $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\}$, where R are the relations for G^Q, defined in Definition 3.5. For some $q \in Q$, the corresponding element in R equalling the identity is $i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1}$. Using that $\pi_{(G_2,G_2)} \circ f_*$ is a homomorphism, we have

$$\pi_{(G_2,G_2)} \circ f_* \left(i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1} \right) =$$
$$\left(\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)\right) \left(\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1}\right)_* (q)^{-1}\right).$$

We will concentrate on each factor separately.

First we consider the first factor. Since $\pi_{(S_1,G_1)}(q) \in \overline{Q}$, by the rightmost commuting square of (3), we have $f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q)$. This gives us

$$\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q).$$

We then use the middle commuting square of (3) to give us

$$\pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q).$$

We then use Lemma 3.6 and functoriality, giving us

$$\begin{aligned}\pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q) &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2}\right)_* \circ \theta_*(q) \\ &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta\right)_* (q)\end{aligned}$$

We now concentrate on the second factor. Using functoriality, then the middle commuting square of (3), we get

$$\begin{aligned}\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1}\right)_* (q)^{-1} &= \pi_{(G_2,G_2)} \circ \left(f \circ \pi_{(S_1,G_1)} \circ i_{S_1}\right)_* (q)^{-1} \\ &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1}\right)_* (q)^{-1}.\end{aligned}$$

Then, we use the leftmost commuting square of (3) to get

$$\pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1}\right)_* (q)^{-1} = \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta\right)_* (q)^{-1}.$$

This is the inverse of the left factor. □

Remark 3.8. If we set $Q = S_1 \subseteq F_{S_1}$ (the minimum subset allowed), then G^Q is F_{S_1}. In this case, f always exists (and is θ) and Theorem 3.7 tells us the standard theorem about homomorphisms from the free group.

Remark 3.9. If we set $Q = F_{S_1}$, then G^Q is G, f is g and Theorem 3.7 tells us nothing more than Theorem 3.2.

Corrolary 3.10. *The homomorphism h resulting from Theorem 3.7 makes the following diagram (4) commute. Thus, considering S_1 as a generating set for $(G_1)^Q$, h is an extension of θ, as in Theorem 3.2.*

$$(4) \qquad \begin{array}{ccccccccc} S_1 & \xhookrightarrow{i_{S_1}} & Q & \xrightarrow{\pi_{(S_1,G_1)}} & \overline{Q} & \xhookrightarrow{i_{\overline{Q}}} & F_{\overline{Q}} & \xrightarrow{\pi_{(\overline{Q},(G_1)^Q)}} & (G_1)^Q \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta_*} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle h} \\ S_2 & \xhookrightarrow{i_{S_2}} & F_{S_2} & \xrightarrow{\pi_{(S_2,G_2)}} & G_2 & \xhookrightarrow{i_{G_2}} & F_{G_2} & \xrightarrow{\pi_{(G_2,G_2)}} & G_2 \end{array}$$

Proof. By Remark 3.4, we get the rightmost square in (4). □

We see that if $\theta\colon S_1 \to S_2$ is surjective, then the homomorphism $h\colon (G_1)^Q \to G_2$ resulting from Theorem 3.7 is surjective.

Observing the form of Definition 3.5, it is clear that $\pi_{(S,G)}(S)$ is a generating set for G^Q, that is to say that $\pi_{(\overline{Q},G^Q)} \circ \left(\pi_{(S,G)} \circ i_S\right)_*$ is surjective. If we set $Q = F_S$, then $G^Q \cong G$, so G can be realised as a quotient of G^Q via a surjection $p_Q\colon G^Q \to G$ in a way such that the following diagram commutes.

$$\begin{array}{ccc} & F_S & \\ \swarrow{\scriptstyle \pi_{(S,G^Q)}} & & \searrow{\scriptstyle \pi_{(S,G)}} \\ G^Q & \xrightarrow{p_Q} & G \end{array}$$

Furthermore, the following diagram commutes.

$$\begin{tikzcd} Q \arrow[r, "\pi_{(S,G)}"] \arrow[rrrr, bend right, "\pi_{(S,G)}"'] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] & F_{\overline{Q}} \arrow[r, "\pi_{(\overline{Q},G^Q)}"] & G^Q \arrow[r, "p_Q"] & G \end{tikzcd} \tag{5}$$

TODO fix this

This is because $i_{\overline{Q}}$ is a restriction of $i_G \colon G \to F_G$, which is a homomorphism. So both the bottom and top maps in (5) are restrictions of homomorphisms. And the top map agrees with the bottom map on the generating set $S \subseteq Q$.

Thus, if $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_1)$, then necessarily $\pi_{(S,G^Q)}(q_1) = \pi_{(S,G^Q)}(q_2)$, thus p_Q is injective on $\pi_{(\overline{Q},G^Q)} \circ i_{\overline{Q}} \circ \pi_{(S,G)}(Q) = \pi_{(S,G^Q)}(Q)$. This can be expressed in the following diagram, where the bottom map is injective.

$$\begin{tikzcd} & Q \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\ \pi_{(S,G^Q)}(Q) \arrow[rr, hook, "p_Q"] & & \pi_{(S,G)}(Q) \end{tikzcd} \tag{6}$$

Theorem 3.11. *Given some group $G \cong \langle S \rangle$ and Q such that $S \subseteq Q \subseteq F_S$, we have that G^Q is isomorphic to the following group by extending the natural identification of generators.*

$$X \coloneqq \langle S \mid \{q_1 = q_2 \mid q_1, q_2 \in Q,\ \pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)\} \rangle \tag{7}$$

Proof. We begin by showing that Id_S extends to a homomorphism $a \colon X \to G^Q$. Using Theorem 3.1, it suffices to show $\pi_{(S,G^Q)}(q_1 q_2^{-1}) = 1$ for all $q_1, q_2 \in Q$ such that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$. Using (6), if $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$ then $\pi_{(S,G^Q)}(q_1) = \pi_{(S,G^Q)}(q_2)$. Then, since $\pi_{(S,G^Q)}$ is a homomorphism, we have

$$\pi_{(S,G^Q)}\left(q_1 q_2^{-1}\right) = \pi_{(S,G^Q)}(q_1)\pi_{(S,G^Q)}(q_2)^{-1} = 1.$$

Now we work to show that Id_S also extends to a homomorphism $b \colon G^Q \to X$. Using Theorem 3.7, it is sufficient to show that there exists a map f to make the following diagram commute.

$$\begin{tikzcd} & Q \arrow[dl, "\pi_{(S,G)}"'] \arrow[dr, "\pi_{(S,X)}"] & \\ G \arrow[rr, "f"] & & X \end{tikzcd}$$

Such an f exists if for all $q_1, q_2 \in Q$ such that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$, we also have $\pi_{(S,X)}(q_1) = \pi_{(S,X)}(q_2)$. This is immediately true on inspection of the defining presentation for X.

Thus, we have two homomorphisms, $a \colon X \to G^Q$, and $b \colon G^Q \to X$. Furthermore, by construction $(a \circ b)|_S = \mathrm{Id}_S$, thus $a \circ b$ is the identity and a is an isomorphism. □

We now consider altering Q. If we increase Q to some Q' where $Q \subseteq Q'$, we see that both G^Q and $G^{Q'}$ are generated by the image of S, but the relations for $G^{Q'}$ are a superset of the relations for G^Q. Thus, $G^{Q'}$ can be realised as a quotient of G^Q via a surjective homomorphism p, for which the following diagram commutes.

$$
\begin{tikzcd}
 & F_S \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{\left(S,G^{Q'}\right)}"] & \\
G^Q \arrow[rr, "p"] & & G^{Q'}
\end{tikzcd} \tag{8}
$$

Conversely, if Id_S can extend to a homomorphism in the opposite direction $G^{Q'} \to G^Q$, then G^Q and $G^{Q'}$ are isomorphic. We can use Theorem 3.7 to find when this is possible.

Theorem 3.12. *Given some group $G \cong \langle S \rangle$, and $Q, Q' \subseteq F_S$ such that $S \subseteq Q \subseteq Q'$, if there exists a map f that makes the following diagram commute, then $G^Q \cong G^{Q'}$.*

$$
\begin{tikzcd}
 & Q' \arrow[dl, "\pi_{(S,G)}"'] \arrow[dr, "\pi_{(S,G^Q)}"] & \\
\pi_{(S,G)}(Q') \arrow[rr, "f"] & & \pi_{(S,G^Q)}(Q')
\end{tikzcd}
$$

Proof. By Theorem 3.7, this we know that Id_S extends to a homomorphism $a\colon G^{Q'} \to G^Q$. Using $p\colon G^Q \to G^{Q'}$ in (8) gives us a homomorphism in the opposite direction such that $(p \circ a)\,|_S = \mathrm{Id}_S$, thus a is an isomorphism. $\square$

A pleasant feature of Theorem 3.12 is that we only need to understand the behaviour of $\pi_{(S,G)}$ and $\pi_{(S,G^Q)}$ on Q' in order to know if we can increase Q to Q'. We do not need to understand how $\pi_{\left(S,G^{Q'}\right)}$ behaves. This suggests the possibility of extending Q inductively.

We have a well–defined notion of Q being maximal. If $Q \subseteq F_S$ is such that any $Q' \supset Q$ results in a $G^{Q'} \ncong G^Q$, then we say Q is maximal. We also have an equivalent notion of Q being minimal.

It is unclear whether a minimal or maximal Q is unique.

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