DUAL ARTIN GROUP PRESENTATIONS

SEAN O'BRIEN

1. Notation and setup

We will begin by setting up our notation for Coxeter and Artin groups.

Let W and A respectively be the Coxeter and Artin groups associated to some Coxeter system Γ. In this document, the specific Coxeter system is arbitrary and constant.

Our Coxeter and Artin groups have a conventional finite generating set which we associate to the set S. In this document, n will always denote $|S|$, the rank of our Coxeter system. Rather than S being a subset of W or A, we will consider S to be a free generating set, with F_S the free group on S. For any quotient G of F_S (particularly W or A), we denote the quotient surjection $\pi_{(S,G)}\colon F_S \to G$. Where S is obvious from context, we may drop it from this notation in favour of $\pi_G\colon F_S \to G$. For shorthand, we will denote $\pi_W(S)$ and $\pi_A(S)$ by S_W and S_A respectively.

To specify a dual Artin group, we must in principle choose a Coxeter element in W. We will always denote this choice w. We will denote the generating simple reflections W by S. The elements of S will always be denoted s_i for $1 \le i \le n$ such that w is a lexicographic product, i.e. $w = \pi_W(s_1 s_2 \cdots s_n)$. For any object (set or tuple) X with such an indexing $x_i \in X$, and function f, we may use the notation $\{f(x_i)\}$ as shorthand for $\{f(x_i) \mid x_i \in X\}$.

Let $R \coloneqq (S_W)^W \subseteq W$ denote all conjugates of generators in W. We call R the *reflections* of W.

2. Hurwitz groups and the dual Artin group

Definition 2.1. *Let G be a group and X be a set. A G-*set *is a set X with an accompanying G action, denoted $g \cdot x$ for each $g \in G$ and $x \in X$.*

Definition 2.2. *Given a group G, suppose we have two G-sets, X, and X'. Now suppose we have a map $f\colon X \to X'$. This map is a G-*set morphism *iff*

$$f(g \cdot x) = g \cdot f(x)$$

for all $g \in G$ and $x \in X$.

Let B_n denote the braid group on n strands with standard generating set $\{\sigma_i\}$.

Definition 2.3 (The Hurwitz action)**.** *Given any group G, there is an action of B_n on G^n, called the* Hurwitz action *where*

$$\sigma_i \cdot (g_1, \dots g_n) = \left(g_1, \dots, g_{i-1}, g_{i+1}^{(g_i^{-1})}, g_i, g_{i+2}, \dots, g_n \right)$$

for all $1 \le i < n$. This also determines the action of σ_i^{-1}.

This action preserves the index-wise product $(g_1, \ldots, g_n) \mapsto g_1 g_2 \cdots g_n$. Given some $X \subseteq G$, if $X^G = G$, then the Hurwitz action is also well-defined on X^n, and this makes any such X a B_n-set in the sense of Definition 2.1. Given a map $f\colon X \to Y$, let $f^{(n)}$ denote the map between n-tuples $(x_1, \ldots, x_n) \mapsto (f(x_1), \ldots, f(x_n))$. If we set $\tau = \pi_W^{(n)}(s_1, \ldots, s_n)$, then by [IS10], we know that $B_n \cdot \tau$ gives every minimal factorisation of w in to elements of R.

Remark 2.4. Given two groups G_1 and G_2, and a homomorphism $\varphi\colon G_1 \to G_2$, the map $\varphi^{(n)}\colon G_1^n \to G_2^n$ is a B_n-set morphism.

Let $p_1\colon G^n \to G$ denote generic projection to the first coordinate of a tuple. We are interested in the tuples that occur due to the Hurwitz action. The set of tuples $B_n \cdot \tau$ is the orbit of τ. The set $\mathrm{HurRef}(\tau)$ is the set of all $g \in G$ that occur in any position in $B_n \cdot \tau$. By choosing braids that move elements to the first position in the tuple, we see that $\mathrm{HurRef}(\tau) = p_1(B_n \cdot \tau)$.

Associated to a tuple, there is a monoid construction, which we will call the *Hurwitz monoid*.

Definition 2.5. *Let G be some group, and let τ be some tuple in G^k. Suppose there in there is a tuple in $B_k \cdot \tau$ that begins r_1, r_2. Let s denote the element of* $\mathrm{HurRef}(\tau)$ *equal in G to $r_1 r_2 r_1^{-1}$. We call any such (r_1, r_2, s) a* Hurwitz triple. *In this context, we define the* Hurwitz monoid *I associated to τ as follows. Decorate elements $x \in X$ as $[x]$ to distinguish them as generators of I rather than elements of G.*

$$I \coloneqq \langle [X] \mid \{[r_1][r_2] = [s][r_1] \mid (r_1, r_2, s) \text{ is a Hurwitz triple}\} \rangle.$$

We call the corresponding group, the *Hurwitz group*. We denote the Hurwitz group associated to a tuple τ by $H(\tau)$.

Remark 2.6. The relations in the above definition are naturally associated to σ_1 acting on a tuple beginning with r_1, r_2. However, we can also get the relation associated to σ_1^{-1} acting on the same tuple. Suppose we have this tuple beginning with r_1, r_2 in $B_n \cdot \tau$. If we then act by σ_1^{-1}, our tuple begins with r_2, t, where t is the element in $\mathrm{HurRef}(\tau)$ that is equal to $r_2^{-1} r_1 r_2$ in G. The relation associated to this tuple beginning r_2, t is

$$[r_2][t] = [t^{-1} r_2 t][r_2]$$

where $t^{-1} r_2 t = r_1$.

Lemma 2.7. *Suppose we have two groups G_1 and G_2 and a homomorphism $\varphi\colon G_1 \to G_2$. Let $\tau_1 = (g_1, \ldots, g_n) \in G^n$ and $\tau_2 = \varphi^n(\tau)$ be two tuples. If φ is injective on* $\mathrm{HurRef}(\tau_1)$*, then $H(\tau_1) \cong H(\tau_2)$.*

Proof. By Remark 2.4, φ^n is a B_n-set morphism from $B_n \cdot \tau_1$ to $B_n \cdot \tau_2$. If φ is injective on $\mathrm{HurRef}(\tau_1)$, then φ^n is injective on $B_n \cdot \tau$, so $\varphi^n|_{B_n \cdot \tau_1}\colon B_n \cdot \tau_1 \to B_n \cdot \tau_2$ is a B_n-set isomorphism. Since $H(\tau_1)$ and $H(\tau_2)$ are determined by the B_n-sets $B_n \cdot \tau_1$ and $B_n \cdot \tau_2$ respectively, the map $[x] \to [\varphi(x)]$ for all $x \in \mathrm{HurRef}(\tau_1)$ defines an isomorphism $H(\tau_1) \to H(\tau_2)$. □

Lemma 2.8. *Given a group G and a tuple $\tau \in G^k$, we have that $H(\tau)$ is generated by $\{[\tau_i]\}$.*

Proof. Let $\ell\colon B_n \to \mathbb{N}$ denote minimum word length with respect to the standard generating set $\{\sigma_i\}$ for B_n. Let $x \in \mathrm{HurRef}(\tau) \setminus \{\tau_i\}$. Let $\beta \in B_n$ be the minimum length braid such that $\beta \cdot \tau = (\mu, \cdots, \mu_n)$ and $x = \mu_k$ for some k. We will show that such an $[x]$ can be written in terms of generators appearing in $\gamma \cdot \tau$ where γ is a braid with $\ell(\gamma) < \ell(\beta)$. Induction then completes the proof.

Since $x \notin \{\tau_i\}$, we know $\ell(\beta) > 0$. Since β is a braid of minimum length with respect to these generators, we know that there is a factorisation of β in the $\{\sigma_i\}$ such that the last factor is either σ_k or σ_{k-1}^{-1}, since these are the only generators whose action puts something new in the k^{th} position of the tuple.

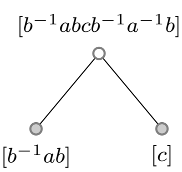
Suppose the last factor of β is σ_k. Let $(\nu_1, \ldots, \nu_n) = \beta\sigma_k^{-1} \cdot \tau$, so $[x] = [\mu_k] = [\nu_{k+1}\nu_k\nu_{k+1}^{-1}]$. Since $\ell(\beta\sigma_k^{-1}) < \ell(\beta)$, if we can express x in terms of the set $\{[\nu_i]\}$ then we would be done. Since we can bring ν_k and ν_{k+1} to the start of the tuple, we have the equation $[x] = [\nu_{k+1}][\nu_k][\nu_{k+1}]^{-1}$ in $H(\tau)$.

We make a similar argument if the last factor of β is σ_{i-1}^{-1}. $\square$

The above lemma tells us that there is a presentation for any $H(\tau)$ in just the elements $\{[\tau_i]\}$. We now wish to develop this presentation for $H(\tau)$.

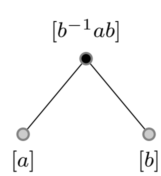
To every relation $[r_1][r_2] = [s][r_1]$ set out in Definition 2.5 we can associate a braid β such that $\beta \cdot \tau$ begins with r_1, r_2. The relation shows how to express $[s] = [r_1r_2r_1^{-1}]$ in terms of $[r_1]$ and $[r_2]$. The proof above shows us that we can express $[r_1]$ and $[r_2]$ in terms of elements in $\mathrm{HurRef}(\tau)$ which result from braids of lower length, and so on. We can express this structure using a decorated binary tree in the following way

Suppose for now our tuple τ is $(a, b, c) \in G^3$. We can act on this by $\sigma_2\sigma_1^{-1}$ to get $(b, b^{-1}abcb^{-1}a^{-1}b, b^{-1}ab)$. Consider now the generator $[b^{-1}abcb^{-1}a^{-1}b]$, and the relation $[b^{-1}ab][c] = [b^{-1}abcb^{-1}a^{-1}b][b^{-1}ab]$ in $H(\tau)$. Let us express this relation by the following tree, which should be understood as the encoding relation $[b^{-1}abcb^{-1}a^{-1}b] = [b^{-1}ab][c][b^{-1}ab]^{-1}$.



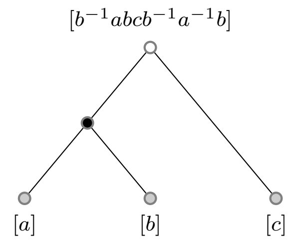
The top node is white, which specifies that the last factor in the right-hand side of the relation is an inverse. This corresponds to the positive braid σ_2 that *created* the new generator $[b^{-1}abcb^{-1}a^{-1}b]$. Note that the planar order in this picture ($[b^{-1}ab]$ on the left and $[c]$ on the right) is important.

Now consider the following tree, which should understood as encoding the relation $[b^{-1}ab] = [b]^{-1}[a][b]$.



The top node is black, which specifies that the first factor in the right-hand side of the relation is an inverse. This corresponds to the negative braid σ_1^{-1} that *created* the new generator $[b^{-1}ab]$.

We can put these two trees together.



This tree encodes the relation $[b^{-1}abcb^{-1}a^{-1}b] = \left([b]^{-1}[a][b]\right)[c]\left([b]^{-1}[a][b]\right)^{-1}$, by parsing relations as we move down the tree.

The proof of Lemma 2.8 tells us that any $r \in \mathrm{HurRef}(\tau)$, and thus any relation, can be expressed using such a tree, where the bottom leaf nodes are labelled by elements of $\{[\tau_i]\}$.

Lemma 2.9. *Let G be a group. Let $\tau = (s_1, \dots, s_n) \in F_S^n$ and let $\pi\colon F_S \to G$ denote a homomorphism. Let $i\colon S \hookrightarrow F_S$ denote the natural inclusion. The Hurwitz group associated to the tuple $\pi^{(n)}(\tau)$ is isomorphic to the following group presentation.*

$$\langle\, \pi(\mathrm{HurRef}(\tau)) \mid \{\pi(q) = (\pi \circ i)_*(q) \mid q \in \mathrm{HurRef}(\tau)\}\,\rangle.$$

Remark 2.10. The above lemma tells us that there is a presentation for any $H(\tau)$ in just the elements $\{[\tau_i]\}$. When we transform the defining relations in to relations in the set $\{[\tau_i]\}$, they have a very particular form. For example, considering the tuple $(a, b) \in G^2$, we have the relation

$$[aba^{-1}] = [a][b][a]^{-1}.$$

We see that this relation allows us to commute the square bracket decorations through the word that we get from the Hurwitz action, and every such defining relation can be re-written in this way. However, note that the data that defines the relation is the specific word aba^{-1} that emerges by performing the Hurwitz action as if a and b are free group elements. If it was true that $aba^{-1} \overset{G}{=} xyz$, for some arbitrary $x, y, z \in G$, we would *not* necessarily have the relation $[aba^{-1}] = [x][y][z]$, and there is no reason to assume that any of $[x]$, $[y]$ or $[z]$ are even generators in the defining presentation for $H(\tau)$. Section 3 and Lemma 3.5 expand on these observations.

Lemma 2.11. *Given a group G and a tuple $\tau = (g_1, \dots, g_n)$, the map $[g_i] \mapsto g_i$ defines a surjection $H(\tau) \to \langle\, \{g_i\}\,\rangle \subseteq G$.*

Proof. This is true because every relation of the form discussed in Remark 2.10 is trivially true in G. ☐

Let $a, b \in G$ for some group G. Let $\Pi_k(a, b)$ denote the alternating product of a and b, beginning with a, and of length k. For example $\Pi_5(a, b) = ababa$.

Lemma 2.12. *Let G be a group such that there exists a homomorphism $\varphi\colon A \to G$ from our Artin group A. Consider the tuple $\tau = (\varphi \circ \pi_A)^n(s_1, \ldots, s_n) \in G^n$. The map $\pi_A(s_i) \mapsto [\varphi \circ \pi_A(s_i)]$ defines a surjection from A to $H(\tau)$.*

Proof. For each i, let t_i denote $\pi_A(s_i)$ and u_i denote $\varphi(t_i)$. We will show that the map $t_i \mapsto [u_i]$ extends to a homomorphism. To do so, we need to show that any Artin-like equation of the form $\Pi_k(t_i, t_j) = \Pi_k(t_j, t_j)$ associated to a defining relation in A also holds in $H(\tau)$.

For any $i < j$, there is a braid β such that $\beta \cdot (u_1, \ldots, u_n)$ begins with u_i, u_j. Thus, without loss of generality, we can assume for the following argument that $i = 1$ and $j = 2$. We only care about what happens at the first two places of this tuple, so we will only consider the Hurwitz action on the duple (u_1, u_2). Consider the function $f_n(a, b) := \Pi_{n+1}(a, b)\Pi_n(a, b)^{-1}$. One can compute that $f_n(a, b)^{f_{n-1}(a,b)^{-1}} = f_{n+1}(a, b)$. Acting once by σ_1 on our duple we have

$$\sigma_1 \cdot (u_1, u_2) = (u_1u_2u_1^{-1}, u_1) = (f_1(u_1, u_2), f_0(u_1, u_2)).$$

Thus, for $k > 0$, we have

$$\sigma_1^k \cdot (u_1, u_2) = (f_k(u_1, u_2), f_{k-1}(u_1, u_2)).$$

So for each $k \geq 0$, we have a generator $[f_k(u_1, u_2)]$ in H, and as in Remark 2.10 the corresponding relation

$$[f_k(u_1, u_2)] = f_k([u_1], [u_2]).$$

Suppose that A has the relation $\Pi_k(t_1, t_2) = \Pi_k(t_2, t_1)$, then in H we have

$$\begin{gathered} f_k([u_1], [u_2]) = [f_k(u_1, u_2)] = [\varphi(f_k(t_1, t_2))] = \\ [\varphi(\Pi_{k+1}(t_1, t_2)\Pi_k(t_1, t_2)^{-1}] = [\varphi\left(\Pi_{k+1}(t_1, t_2)\Pi_k(t_2, t_1)^{-1}\right)] = \\ [\varphi(t_1)] = [u_1]. \end{gathered}$$

So we have the relation $\Pi_k([u_1], [u_2]) = \Pi_k([u_2], [u_1])$ in $H(\tau)$ as required. This map is surjective by Lemma 2.8. $\square$

Corrolary 2.13. *Let $\tau = \pi_A^n(s_1, \ldots, s_n)$. We have that $H(\tau) \cong A$.*

Proof. Lemma 2.12 gives a homomorphism $a\colon A \to H(\tau)$, and Lemma 2.11 gives a homomorphism $b\colon H(\tau) \to A$. The composition $b \circ a$ is the identity on a generating set for A. $\square$

The following is not a standard definition, but it is equivalent to the standard definition by [Bes04, Lemma 7.11].

Definition 2.14. *Given a Coxeter element $w \in W$, the* dual Artin group *associated to w, denoted $A_w^\vee$, is defined to be the Hurwitz group associated to any $(r_1, \ldots, r_n)$ such that all $r_i \in R$ and $r_1r_2 \cdots r_n = w$.*

The obvious tuple to consider is $(\pi_W(s_1), \ldots, \pi_W(s_n))$. Since our choice of w is constant, we will denote our dual Artin group $A^\vee$ (omitting any mention of w).

Corrolary 2.15. *The natural map $\psi\colon [S_W] \to W$ which acts as $[\pi_W(s_i)] \mapsto \pi_W(s_i)$ defines a surjective homomorphism $A^\vee \to W$.*

Proof. This directly follows from Lemma 2.11. $\square$

Corrolary 2.16. *The map natural map $\varphi\colon S_A \to A^\vee$ which acts as $\pi_A(s_i) \mapsto [q \circ \pi_A(s_i)] = [\pi_W(s_i)]$ extends to a surjective homomorphism $\varphi\colon A \to A^\vee$.*

Proof. This directly follows from Lemma 2.12. ☐

Theorem 2.17 ([Bes06]). *Let $\tau = \pi_A^n(s_1, \ldots, s_n)$. If the natural surjection $q\colon A \to W$ is injective on* $\mathrm{HurRef}(\tau)$*, then $A^\vee \cong A$.*

Proof. By Corollary 2.13, we have $A \cong H(\tau)$, and by the hypothesis and Lemma 2.7 we have $A^\vee \cong H(\tau)$. ☐

3. The construction G^Q

Let $F\colon \mathbf{Set} \to \mathbf{Grp}$ be the free functor and denote its action on objects as $F_S \coloneqq F(S)$ and its action on morphisms as $f_* \coloneqq F(f\colon S \to T)$. To each quotient G of F_S we associate the surjective homomorphism $\pi_{(S,G)}\colon F_S \to G$ which is projection. In this context, we can frame a basic group theoretic fact. We write $G \cong \langle S \mid R \rangle$ if $G \cong F_S/N(R)$ where $N(R)$ is the minimal normal subgroup in F_S that contains R. We write $G \cong \langle S \rangle$ if G is isomorphic to some quotient of F_S.

Basic fact 3.1. *Suppose we have two groups G_1 and G_2. Suppose that $G_1 \cong \langle S \,|\, R \rangle$ where R is a subset of F_S for which $\pi_{(S,G_1)}(R) = \{1\}$. Given a map $f\colon S \to G_2$, if $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\} \subseteq G_2$, then f defines a homomorphism $h\colon G_2 \to G_2$.*

The above theorem defines a homomorphism $h\colon \langle S \,|\, R \rangle \cong G_1 \to G_2$ when a map $f\colon S \to G_2$ is compatible with the relations in G_1 and G_2. If we also had knowledge of a generating set for G_2, we could construct homomorphisms in a different way.

Basic fact 3.2. *Suppose $G_1 \cong \langle S_1 \,|\, R_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Given a map $f\colon S_1 \to S_2$, if there exists a map h such that the following diagram commutes, then h is a homomorphism.*

$$\begin{tikzcd} S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "f"] & F_{S_1} \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "f_*"] & G_1 \arrow[d, "h"] \\ S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \end{tikzcd} \tag{1}$$

Remark 3.3. If we construct a homomorphism using Basic fact 3.1, then we get the middle commuting square of (1) automatically.

Now consider (1), but replace F_{S_1} with some subset $Q \subseteq F_{S_1}$. We will now construct a group, which we will denote G^Q, where such a diagram defines a homomorphism from G^Q to G_2.

Definition 3.4 (Group with relations visible in Q). *Suppose we have a group $G \cong \langle S \rangle$. Fix some $Q \subseteq F_S$ such that $S \subseteq Q$. Let $\pi \coloneqq \pi_{(S,G)}$. We have the following maps.*

$$S \xhookrightarrow{i} Q \xrightarrow{\pi} \pi(Q)\ .$$

Define the group G with relations visible in Q, *to be*

$$G^Q \coloneqq \langle \pi(Q) \mid \{\pi(q) = (\pi \circ i)_*(q) \mid q \in Q\} \rangle.$$

Do not think of the generators $\pi(Q)$ as elements of G. They are abstract generators. Specifically, G^Q is a quotient of $F_{\pi(Q)}$. Thus, our relations should be equations in $F_{\pi(Q)}$, which they are. G^Q is generated by $\pi(S) \subseteq \pi(Q)$. G is a quotient of G^Q, which we will explore later.

Lemma 3.5. *Let $Q = \mathrm{HurRef}(s_1, \ldots, s_n)$. The dual Artin group $A^\vee$ is isomorphic to W^Q.*

Proof. TODO $\square$

Lemma 3.6. *Let $G \cong \langle S \rangle$ be a group and consider the following setup.*

$$S \xhookrightarrow{i_S} F_S \xrightarrow{\pi_{(S,G)}} G \xhookrightarrow{i_G} F_G \xrightarrow{\pi_{(G,G)}} G$$

The following maps are equal.

$$\pi_{(G,G)} \circ i_G \circ \pi_{(S,G)} = \pi_{(G,G)} \circ (\pi_{(S,G)} \circ i_S)_*.$$

Proof. Both maps are homomorphism which agree with $\pi_{(S,G)}$ on the generating set S. $\square$

Theorem 3.7. *Suppose we have two groups $G_1 \cong \langle S_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Fix some $Q \subseteq F_{S_1}$ such that $S \subseteq Q$. Let $\overline{Q} := \pi_{(S_1,G_1)}(Q)$. If in the following diagram (2), if there exists a map f that makes the diagram commute, then there is a homomorphism $h\colon (G_1)^Q \to G_2$.*

$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[d, red, "\exists f?"] \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2
\end{tikzcd} \tag{2}$$

Proof. Suppose we have such a map f. Note that $\overline{Q}$ is a generating set for G^Q, thus we have the setup of Basic fact 3.1. We use the following commutative diagram to define some inclusion maps and as a reference for the setup.

$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] \arrow[d, "f"] & F_{\overline{Q}} \arrow[d, "f_*"] & \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \arrow[r, hook, "i_{G_2}"] & F_{G_2} \arrow[r, "\pi_{(G_2,G_2)}"] & G_2
\end{tikzcd} \tag{3}$$

We can construct a homomorphism if $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\}$, where R are the relations for G^Q, defined in Definition 3.4. For some $q \in Q$, the corresponding element in R equalling the identity is $i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1}$. Using that $\pi_{(G_2,G_2)} \circ f_*$ is a homomorphism, we have

$$\pi_{(G_2,G_2)} \circ f_* \left(i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1} \right) =$$
$$\left(\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) \right) \left(\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1} \right)_* (q)^{-1} \right).$$

We will concentrate on each factor separately.

First we consider the first factor. Since $\pi_{(S_1,G_1)}(q) \in \overline{Q}$, by the rightmost commuting square of (3), we have $f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q)$. This gives us

$$\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q).$$

We then use the middle commuting square of (3) to give us

$$\pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q).$$

We then use Lemma 3.6 and functoriality, giving us

$$\begin{aligned}\pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q) &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2}\right)_* \circ \theta_*(q) \\ &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta\right)_* (q)\end{aligned}$$

We now concentrate on the second factor. Using functoriality, then the middle commuting square of (3), we get

$$\begin{aligned}\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1}\right)_* (q)^{-1} &= \pi_{(G_2,G_2)} \circ \left(f \circ \pi_{(S_1,G_1)} \circ i_{S_1}\right)_* (q)^{-1} \\ &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1}\right)_* (q)^{-1}.\end{aligned}$$

Then, we use the leftmost commuting square of (3) to get

$$\pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1}\right)_* (q)^{-1} = \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta\right)_* (q)^{-1}.$$

This is the inverse of the left factor. □

Remark 3.8. If we set $Q = S_1 \subseteq F_{S_1}$ (the minimum subset allowed), then G^Q is F_{S_1}. In this case, f always exists (and is θ) and Theorem 3.7 tells us the standard theorem about homomorphisms from the free group.

Remark 3.9. If we set $Q = F_{S_1}$, then G^Q is G, f is g and Theorem 3.7 tells us nothing more than Basic fact 3.2.

Corrolary 3.10. *The homomorphism h resulting from Theorem 3.7 makes the following diagram* (4) *commute. Thus, considering S_1 as a generating set for $(G_1)^Q$, h is an extension of θ, as in Basic fact 3.2.*

$$\begin{tikzcd} S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] \arrow[d, "f"] & F_{\overline{Q}} \arrow[r, "\pi_{(\overline{Q},(G_1)^Q)}"] \arrow[d, "f_*"] & (G_1)^Q \arrow[d, "h"] \\ S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \arrow[r, hook, "i_{G_2}"] & F_{G_2} \arrow[r, "\pi_{(G_2,G_2)}"] & G_2 \end{tikzcd} \tag{4}$$

Proof. By Remark 3.3, we get the rightmost square in (4). □

We see that if $\theta\colon S_1 \to S_2$ is surjective, then the homomorphism $h\colon (G_1)^Q \to G_2$ resulting from Theorem 3.7 is surjective.

Observing the form of Definition 3.4, it is clear that $\pi_{(S,G)}(S)$ is a generating set for G^Q, that is to say that $\pi_{(\overline{Q},G^Q)} \circ \left(\pi_{(S,G)} \circ i_S\right)_*$ is surjective. If we set $Q = F_S$, then $G^Q \cong G$, so G can be realised as a quotient of G^Q via a surjection $p_Q\colon G^Q \to G$ in a way such that the following diagram commutes.

$$\begin{tikzcd} & F_S \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\ G^Q \arrow[rr, "p_Q"] & & G \end{tikzcd}$$

Lemma 3.11. *The projection p_Q restricts to an injection on the image of $\pi_{(S,G)}(Q)$, i.e. in the following diagram, the bottom map is an injection.*

$$\begin{tikzcd} & Q \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\ \pi_{(S,G^Q)}(Q) \arrow[rr, hook, "p_Q"] & & \pi_{(S,G)}(Q) \end{tikzcd}$$

Proof. The following diagram commutes.

$$\begin{tikzcd} & \overline{Q} \arrow[dr, "i_{\overline{Q}}"] & & \\ Q \arrow[ur, "\pi_{(S,G)}"] \arrow[rr, "(\pi_{(S,G)} \circ i_S)_*"] \arrow[rrr, bend right, "\pi_{(S,G^Q)}"'] & & F_{\overline{Q}} \arrow[r, "\pi_{(\overline{Q},G^Q)}"] & G^Q \end{tikzcd}$$

The top triangle commutes due to the defining relations for G^Q, and the bottom triangle commutes because both maps are restrictions of homomorphisms that agree on the generating set S. Now suppose for some $q_1, q_2 \in Q$ that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$. The above diagram tells us that $\pi_{(S,G^Q)}(q_1) = \pi_{(S,G^Q)}(q_2)$. $\square$

Lemma 3.12. *Given some group $G \cong \langle S \rangle$ and Q such that $S \subseteq Q \subseteq F_S$, we have that G^Q is isomorphic to the following group by extending the natural identification of generators.*

$$X \coloneqq \langle S \mid \{q_1 = q_2 \mid q_1, q_2 \in Q,\ \pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)\} \rangle \tag{5}$$

Proof. We begin by showing that Id_S extends to a homomorphism $a\colon X \to G^Q$. Using Basic fact 3.1, it suffices to show $\pi_{(S,G^Q)}(q_1 q_2^{-1}) = 1$ for all $q_1, q_2 \in Q$ such that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$. Using Lemma 3.11, if $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$ then $\pi_{(S,G^Q)}(q_1) = \pi_{(S,G^Q)}(q_2)$. Then, since $\pi_{(S,G^Q)}$ is a homomorphism, we have

$$\pi_{(S,G^Q)}\left(q_1 q_2^{-1}\right) = \pi_{(S,G^Q)}(q_1)\pi_{(S,G^Q)}(q_2)^{-1} = 1.$$

Now we work to show that Id_S also extends to a homomorphism $b\colon G^Q \to X$. Using Theorem 3.7, it is sufficient to show that there exists a map f to make the following diagram commute.

$$\begin{tikzcd} & Q \arrow[dl, "\pi_{(S,G)}"'] \arrow[dr, "\pi_{(S,X)}"] & \\ G \arrow[rr, "f"] & & X \end{tikzcd}$$

Such an f exists if for all $q_1, q_2 \in Q$ such that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$, we also have $\pi_{(S,X)}(q_1) = \pi_{(S,X)}(q_2)$. This is immediately true on inspection of the defining presentation for X.

Thus, we have two homomorphisms, $a\colon X \to G^Q$, and $b\colon G^Q \to X$. Furthermore, by construction $(a \circ b)|_S = \mathrm{Id}_S$. $\square$

We now relate this back to the dual Artin group $A^\vee$. Firstly, our construction gives an alternative presentation for $A^\vee$.

Corrolary 3.13. *Let S denote the standard generating set for our Artin group A, and let $Q = \mathrm{HurRef}(s_1, \ldots, s_n)$. The dual Artin group $A^\vee$ has the following presentation.*

$$A^\vee \cong \langle S \mid \{q_1 = q_2 \mid q_1, q_2 \in Q, \pi_{(S,W)}(q_1) = \pi_{(S,W)}(q_2)\} \rangle.$$

Proof. This follows from Lemma 3.5 and Lemma 3.12. □

Using Lemma 2.12, Lemma 3.5 and Theorem 3.7, we can re-prove Theorem 2.17, that injectivity of $q\colon A \to W$ on $\pi_A(Q)$ implies $A^\vee \cong A$. Now, using our construction, we can prove the inverse direction.

Theorem 3.14. *Let $\tau = \pi_A^n(s_1, \ldots, s_n)$ and let $Q = \mathrm{HurRef}(\tau)$. The Artin group A is naturally isomorphic to the dual Artin group $A^\vee$ iff the natural projection $q\colon A \to W$ is injective on $\pi_A(Q)$.*

Proof. Theorem 2.17 already shows that if q is injective on $\pi_A(Q)$, then $A^\vee \cong A$.

For the other direction, consider the following commutative diagram.

$$\begin{tikzcd}
 & F_S \arrow[dl, "\pi_A"'] \arrow[d, "\pi_{(S,W^Q)}"] \arrow[dr, "\pi_W"] & \\
A \arrow[r, "\varphi"] \arrow[rr, bend right, "q"'] & A^\vee \arrow[r, "\psi"] & W
\end{tikzcd}$$

We get φ and ψ from Corollary 2.16 and Corollary 2.15 respectively. Note that ψ is the same as p_Q as discussed above. Commutativity is easy to check by comparing these maps when restricted to generating sets.

If we replace F_S with Q, then Lemma 3.11 tells us that the corresponding restriction of ψ is injective. So, if φ is an isomorphism, in particular an injection, then $q = \psi \circ \varphi$ is an injection on $\pi_A(Q)$. □

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Email address: 2812920O@student.gla.ac.uk