# DUAL ARTIN GROUP PRESENTATIONS

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## 1. Notation and setup

We will begin by setting up our notation for Coxeter and Artin groups.

Let $W$ and $A$ respectively be the Coxeter and Artin groups associated to some Coxeter system $\Gamma$. In this document, the specific Coxeter system is arbitrary and constant.

Our Coxeter and Artin groups have a conventional finite generating set which we associate to the set $S$. In this document, $n$ will always denote $|S|$, the rank of our Coxeter system. Rather than $S$ being a subset of $W$ or $A$, we will consider $S$ to be a free generating set, with $F_S$ the free group on $S$. For any quotient $G$ of $F_S$ (particularly $W$ or $A$), we denote the quotient surjection $\pi_{(S,G)}\colon F_S \to G$. Where $S$ is obvious from context, we may drop it from this notation in favour of $\pi_G\colon F_S \to G$. For shorthand, we will denote $\pi_W(S)$ and $\pi_A(S)$ by $S_W$ and $S_A$ respectively.

To specify a dual Artin group, we must in principle choose a Coxeter element in $W$. We will always denote this choice $w$. We will denote the generating simple reflections $W$ by $S$. The elements of $S$ will always be denoted $s_i$ for $1 \leq i \leq n$ such that $w$ is a lexicographic product, i.e. $w = \pi_W(s_1 s_2 \cdots s_n)$. For any object (set or tuple) $X$ with such an indexing $x_i \in X$, and function $f$, we may use the notation $\{f(x_i)\}$ as shorthand for $\{f(x_i) \mid x_i \in X\}$.

Let $R \coloneqq (S_W)^W \subseteq W$ denote all conjugates of generators in $W$. We call $R$ the *reflections* of $W$.

## 2. Hurwitz groups and the dual Artin group

**Definition 2.1.** *Let $G$ be a group and $X$ be a set. A $G$-*set *is a set $X$ with an accompanying $G$ action, denoted $g \cdot x$ for each $g \in G$ and $x \in X$.*

**Definition 2.2.** *Given a group $G$, suppose we have two $G$-sets, $X$, and $X'$. Now suppose we have a map $f\colon X \to X'$. This map is a $G$-*set morphism *iff*

$$f(g \cdot x) = g \cdot f(x)$$

*for all $g \in G$ and $x \in X$.*

Let $B_n$ denote the braid group on $n$ strands with standard generating set $\{\sigma_i\}$.

**Definition 2.3** (The Hurwitz action)**.** *Given any group $G$, there is an action of $B_n$ on $G^n$, called the* Hurwitz action *where*

$$\sigma_i \cdot (g_1, \dots g_n) = \left( g_1, \dots, g_{i-1}, g_{i+1}^{(g_i^{-1})}, g_i, g_{i+2}, \dots, g_n \right)$$

*for all $1 \leq i < n$. This also determines the action of $\sigma_i^{-1}$.*

This action preserves the index-wise product $(g_1, \ldots, g_n) \mapsto g_1 g_2 \cdots g_n$. Given some $X \subseteq G$, if $X^G = G$, then the Hurwitz action is also well-defined on $X^n$, and this makes any such $X$ a $B_n$-set in the sense of Definition 2.1. Given a map $f\colon X \to Y$, let $f^{(n)}$ denote the map between $n$-tuples $(x_1, \ldots, x_n) \mapsto (f(x_1), \ldots, f(x_n))$. If we set $\tau = \pi_W^{(n)}(s_1, \ldots, s_n)$, then by [IS10], we know that $B_n \cdot \tau$ gives every minimal factorisation of $w$ in to elements of $R$.

*Remark* 2.4. Given two groups $G_1$ and $G_2$, and a homomorphism $\varphi\colon G_1 \to G_2$, the map $\varphi^{(n)}\colon G_1^n \to G_2^n$ is a $B_n$-set morphism.

Let $p_1\colon G^n \to G$ denote generic projection to the first coordinate of a tuple. We are interested in the tuples that occur due to the Hurwitz action. The set of tuples $B_n \cdot \tau$ is the orbit of $\tau$. The set $\mathrm{HurRef}(\tau)$ is the set of all $g \in G$ that occur in any position in $B_n \cdot \tau$. By choosing braids that move elements to the first position in the tuple, we see that $\mathrm{HurRef}(\tau) = p_1(B_n \cdot \tau)$.

Associated to a tuple, there is a monoid construction, which we will call the *Hurwitz monoid.*

**Definition 2.5.** *Let $G$ be some group, and let $\tau$ be some tuple in $G^k$. Suppose there in there is a tuple in $B_k \cdot \tau$ that begins $r_1, r_2$. Let $s$ denote the element of* $\mathrm{HurRef}(\tau)$ *equal in $G$ to $r_1 r_2 r_1^{-1}$. We call any such $(r_1, r_2, s)$ a* Hurwitz triple*. In this context, we define the* Hurwitz monoid *$I$ associated to $\tau$ as follows. Decorate elements $x \in \mathrm{HurRef}(\tau)$ as $[x]$ to distinguish them as generators of $I$ rather than elements of $G$.*

$$I \coloneqq \langle\, [\mathrm{HurRef}(\tau)] \mid \{[r_1][r_2] = [s][r_1] \mid (r_1, r_2, s) \text{ is a Hurwitz triple}\}\,\rangle.$$

We call the corresponding group, the *Hurwitz group.* We denote the Hurwitz group associated to a tuple $\tau$ by $H(\tau)$.

*Remark* 2.6. The relations in the above definition are naturally associated to $\sigma_1$ acting on a tuple beginning with $r_1, r_2$. However, we can also get the relation associated to $\sigma_1^{-1}$ acting on the same tuple. Suppose we have this tuple beginning with $r_1, r_2$ in $B_n \cdot \tau$. If we then act by $\sigma_1^{-1}$, our tuple begins with $r_2, t$, where $t$ is the element in $\mathrm{HurRef}(\tau)$ that is equal to $r_2^{-1} r_1 r_2$ in $G$. The relation associated to this tuple beginning $r_2, t$ is

$$[r_2][t] = [t^{-1} r_2 t][r_2]$$

where $t^{-1} r_2 t = r_1$.

**Lemma 2.7.** *Suppose we have two groups $G_1$ and $G_2$ and a homomorphism $\varphi\colon G_1 \to G_2$. Let $\tau_1 = (g_1, \ldots, g_n) \in G^n$ and $\tau_2 = \varphi^n(\tau)$ be two tuples. If $\varphi$ is injective on* $\mathrm{HurRef}(\tau_1)$*, then $H(\tau_1) \cong H(\tau_2)$.*

*Proof.* By Remark 2.4, $\varphi^n$ is a $B_n$-set morphism from $B_n \cdot \tau_1$ to $B_n \cdot \tau_2$. If $\varphi$ is injective on $\mathrm{HurRef}(\tau_1)$, then $\varphi^n$ is injective on $B_n \cdot \tau$, so $\varphi^n|_{B_n \cdot \tau_1}\colon B_n \cdot \tau_1 \to B_n \cdot \tau_2$ is a $B_n$-set isomorphism. Since $H(\tau_1)$ and $H(\tau_2)$ are determined by the $B_n$-sets $B_n \cdot \tau_1$ and $B_n \cdot \tau_2$ respectively, the map $[x] \to [\varphi(x)]$ for all $x \in \mathrm{HurRef}(\tau_1)$ defines an isomorphism $H(\tau_1) \to H(\tau_2)$. $\square$

Let $\ell\colon B_n \to \mathbb{N}$ denote minimum braid length with respect to the standard generating set $\{\sigma_i\}$. We can associate a related length for elements of $\mathrm{HurRef}(\tau)$.

**Definition 2.8.** *Given some tuple $\tau$, each $r \in \mathrm{HurRef}(\tau)$ has an associated* reflection braid length *defined as*

$$\ell_\tau(r) := \min\{\ell(\beta) \mid r \in \beta \cdot \tau\}.$$

**Lemma 2.9.** *Given a group $G$ and a tuple $\tau \in G^k$, we have that $H(\tau)$ is generated by $\{[\tau_i]\}$.*

*Proof.* Let $x \in \mathrm{HurRef}(\tau) \setminus \{\tau_i\}$. Let $\beta \in B_n$ be a minimum length braid such that $\beta \cdot \tau = (\mu, \cdots, \mu_n)$ and $x = \mu_k$ for some $k$, i.e. a $\beta$ such that $\ell(\beta) = \ell_\tau(r)$. We will show that such an $[x]$ can be written in terms of generators appearing in $\gamma \cdot \tau$ where $\gamma$ is a braid with $\ell(\gamma) < \ell(\beta)$. Induction then completes the proof.

Since $x \notin \{\tau_i\}$, we know $\ell(\beta) > 0$. Since $\beta$ is a braid of minimum length with respect to these generators, we know that there is a factorisation of $\beta$ in the $\{\sigma_i\}$ such that the last factor is either $\sigma_k$ or $\sigma_{k-1}^{-1}$, since these are the only generators whose action puts something new in the $k^{\text{th}}$ position of the tuple.

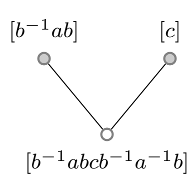
Suppose the last factor of $\beta$ is $\sigma_k$. Let $(\nu_1, \ldots, \nu_n) = \beta\sigma_k^{-1} \cdot \tau$, so $[x] = [\mu_k] = [\nu_{k+1}\nu_k\nu_{k+1}^{-1}]$. Since $\ell(\beta\sigma_k^{-1}) < \ell(\beta)$, if we can express $x$ in terms of the set $\{[\nu_i]\}$ then we would be done. Since we can bring $\nu_k$ and $\nu_{k+1}$ to the start of the tuple, we have the equation $[x] = [\nu_{k+1}][\nu_k][\nu_{k+1}]^{-1}$ in $H(\tau)$.

We make a similar argument if the last factor of $\beta$ is $\sigma_{i-1}^{-1}$. $\square$

The above lemma tells us that there is a presentation for any $H(\tau)$ in just the elements $\{[\tau_i]\}$. We now wish to develop this presentation for $H(\tau)$.

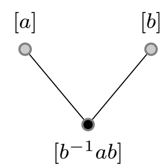
To every relation $[r_1][r_2] = [s][r_1]$ set out in Definition 2.5 we can associate a braid $\beta$ such that $\beta \cdot \tau$ begins with $r_1, r_2$. The relation shows how to express $[s] = [r_1 r_2 r_1^{-1}]$ in terms of $[r_1]$ and $[r_2]$. The proof above shows us that we can express $[r_1]$ and $[r_2]$ in terms of elements in $\mathrm{HurRef}(\tau)$ which result from braids of lower length, and so on. We can express this structure using a decorated binary tree in the following way

Suppose for now our tuple $\tau$ is $(a, b, c) \in G^3$. We can act on this by $\sigma_2\sigma_1^{-1}$ to get $(b, b^{-1}abcb^{-1}a^{-1}b, b^{-1}ab)$. Consider now the generator $[b^{-1}abcb^{-1}a^{-1}b]$, and the relation $[b^{-1}ab][c] = [b^{-1}abcb^{-1}a^{-1}b][b^{-1}ab]$ in $H(\tau)$. Let us express this relation by the following tree, which should be understood as the encoding relation $[b^{-1}abcb^{-1}a^{-1}b] = [b^{-1}ab][c][b^{-1}ab]^{-1}$.




The bottom node is white, which specifies that the last factor in the right-hand side of the relation is an inverse. This corresponds to the positive braid $\sigma_2$ that *created* the new generator $[b^{-1}abcb^{-1}a^{-1}b]$. Note that the planar order in this picture ($[b^{-1}ab]$ on the left and $[c]$ on the right) is important.

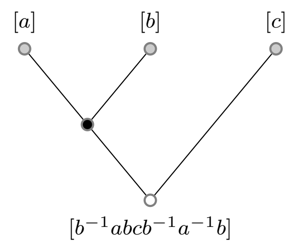
Now consider the following tree, which should understood as encoding the relation $[b^{-1}ab] = [b]^{-1}[a][b]$.




The top node is black, which specifies that the first factor in the right-hand side of the relation is an inverse. This corresponds to the negative braid $\sigma_1^{-1}$ that *created* the new generator $[b^{-1}ab]$.

We can put these two trees together.

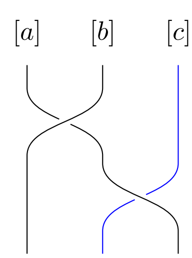
(1)




This tree encodes the relation $[b^{-1}abcb^{-1}a^{-1}b] = \left([b]^{-1}[a][b]\right)[c]\left([b]^{-1}[a][b]\right)^{-1}$, by parsing relations as we move up the tree. Let us explicitly state how these trees relate to the objects we have already discussed.

Given some tuple $\tau$, each $r \in \mathrm{HurRef}(\tau)$ occurs in some position in $\beta \cdot \tau$ for some braid $\beta$. This $r$ was some conjugate of other elements of $\mathrm{HurRef}(\tau)$, this gives us the bottom node and the two outgoing edges of the tree, and we continue in this fashion. The proof of Lemma 2.9 tells us that any $r \in \mathrm{HurRef}(\tau)$, can be expressed using such a tree, where the top leaf nodes are labelled by elements of $\{[\tau_i]\}$. The information that specifies the tree is the braid $\beta$ and which element (which index) of $\beta \cdot \tau$ is the relevant $r$. Note that we do not assert any kind of uniqueness associated to this tree. This is because we do not assert uniqueness of the braid $\beta$.

There is a pictorial (from the braid picture) way to draw these trees. Suppose we are dealing with the braid $\sigma_2\sigma_1^{-1}$ as in the above example. Since this is acting from the left, let us draw the reversed braid $\sigma_1^{-1}\sigma_2$, and highlight the relevant strand which has index 2 in $\sigma_2\sigma_1^{-1} \cdot \tau$ in blue.




For now, this blue strand is our *strand of interest.* Follow this strand of interest upwards until it passes under another strand or reaches the top of the braid. We choose the colour of the node according to how the strand of interest passes under the other strand.

Once we cross under and make this node in the tree, we follow the two strands that were involved in that crossing. Separately, we consider each of these strands as our strands of interest and move up the tree generating the daughter trees from our initial node in the same fashion.

In our example, the initial blue strand crosses under strand 1, creating a white node. One strand coming from that node goes to the top of the braid to the label $[c]$ without any (under) crossings. The other strand does have an under crossing, which corresponds to the black node in (1).

We can also associate trees to relations coming from Definition 2.5. Here we must take note of Remark 2.11. If a relation is naturally associated to a $\sigma_i^{-1}$ crossing, then the bottom node of our tree should be black, even though every relation in We have a way to generate Given some relation $[r_1][r_2] = [s][r_1]$ as in Definition 2.5, there is a braid $\beta$ such that the set

$$X = \{p_1(\beta \cdot \tau), p_2(\beta \cdot \tau), p_1(\sigma_1\beta \cdot \tau), p_2(\sigma_1\beta \cdot \tau)\}$$

contains all the generators present in that relation. Choose an element $x \in X$ such that $\ell_\tau(x)$ is maximal. Then the tree associated to that $x$ and whichever of $\beta \cdot \tau$ or $\sigma_1\beta \cdot \tau$ it occurs in encodes the relevant relation, as well as the relations of all the other reflections and relations used to get there. This is the tree we associate to the relation $[r_1][r_2] = [s][r_1]$. As with elements of $\mathrm{HurRef}(\tau)$, we do not assert any kind of uniqueness associated to this tree. This is because we do not assert uniqueness of the braid chosen from $\beta$ or $\sigma_1\beta$.

However, note that not all trees correspond to a relation seen in Definition 2.5.

**Lemma 2.10.** *Let $G$ be a group. Let $\tau = (s_1, \ldots, s_n) \in F_S^n$ and let $\pi\colon F_S \to G$ denote a homomorphism. Let $i\colon S \hookrightarrow F_S$ denote the natural inclusion. The Hurwitz group associated to the tuple $\pi^{(n)}(\tau)$ is isomorphic to the following group presentation.*

$$J \coloneqq \langle\, \pi(\mathrm{HurRef}(\tau)) \mid \{\pi(q) = (\pi \circ i)_*(q) \mid q \in \mathrm{HurRef}(\tau)\}\,\rangle.$$

*Proof.* Let $I$ denote the group presentation as defined in Definition 2.5. The generating set of $I$ is exactly $[\mathrm{HurRef}(\pi(\tau))]$. By Remark 2.4, we have $\pi(\mathrm{HurRef}(\tau)) = \mathrm{HurRef}(\pi^{(n)}(\tau))$. Thus, the bracketing map $b\colon x \mapsto [x]$ is a bijection between the generating sets of $J$ and $I$. Using this bijection $b$, we need to show that the relations of $I$ are consequences of the relations of $J$ and visa-versa.

Let us begin by showing that the relations of $J$ are consequences of the relations of $I$. By the argument preceding this lemma, each $q \in \mathrm{HurRef}(\tau)$, and each relation in $I$ corresponds to a tree. For each relation $R$ in $I$, we have a relation-tree $T(R)$. For each node in any such tree, we can take the subtree beneath that node, which also corresponds to a relation in $I$. For a node $n$ in $T(R)$, let $R(n)$ denote the relation in $I$ corresponding to the subtree beneath $n$ in $T(R)$.

Let $q$ be some word in $\mathrm{HurRef}(\tau)$. If $q \in S$, then the corresponding relation in $I$ is trivial, so we may assume $q \notin S$. Let $T$ be a tree corresponding to $q$. Let $n$ be a node in $T$, and let $n_L$ and $n_R$ be the two daughter nodes of $n$. These two daughter nodes correspond to words $q_L$ and $q_R$ in $\mathrm{HurRef}(\tau)$. We will show that if the relations corresponding to $q_L$ and $q_R$ in $J$ are true in $I$, then the relation corresponding to

$q$ is also true in $I$. This part of the proof will then follow using induction down the tree. The trivial relations $[s_i] = [s_i]$ corresponding to $q \in S \subseteq \mathrm{HurRef}(\tau)$ form the base case for our induction.

Recall that $b$ is our bracketing map which is a bijection between generating sets. So, we have by assumption that $[\pi(q_L)] = (b \circ \pi \circ i)(q_L)$ and $[\pi(q_R)] = (b \circ \pi \circ i)_*(q_R)$ in $I$. Assume the node at the top of the tree corresponding to $q$ is white. Thus, $q = q_L q_R q_L^{-1}$, and we have the relation $[\pi(q)] = [\pi(q_L)][\pi(q_R)][\pi(q_L)]^{-1}$ in $I$. So in $I$ we have the following equation.

$$\begin{aligned}
&[\pi(q)] &&= \\
&[\pi(q_L)][\pi(q_R)][\pi(q_L)]^{-1} &&= \\
&(b \circ \pi \circ i)_*(q_L)(b \circ \pi \circ i)_*(q_R)(b \circ \pi \circ i)_*(q_L)^{-1} &&= \\
&(b \circ \pi \circ i)_*(q_L q_R q_L^{-1}) &&= \\
&(b \circ \pi \circ i)_*(q).
\end{aligned}$$

We make a similar argument if the node corresponding the word $q$ is black. This completes the proof that the relations in $J$ are consequences of the relations in $I$.

Now we will show that the relations in $I$ are consequences of the relations in $J$. Let $T$ now denote the tree corresponding to some relation in $I$. □

*Remark* 2.11. The above lemma tells us that there is a presentation for any $H(\tau)$ in just the elements $\{[\tau_i]\}$. When we transform the defining relations in to relations in the set $\{[\tau_i]\}$, they have a very particular form. For example, considering the tuple $(a, b) \in G^2$, we have the relation

$$[aba^{-1}] = [a][b][a]^{-1}.$$

We see that this relation allows us to commute the square bracket decorations through the word that we get from the Hurwitz action, and every such defining relation can be re-written in this way. However, note that the data that defines the relation is the specific word $aba^{-1}$ that emerges by performing the Hurwitz action as if $a$ and $b$ are free group elements. If it was true that $aba^{-1} \overset{G}{=} xyz$, for some arbitrary $x, y, z \in G$, we would *not* necessarily have the relation $[aba^{-1}] = [x][y][z]$, and there is no reason to assume that any of $[x]$, $[y]$ or $[z]$ are even generators in the defining presentation for $H(\tau)$. Section 3 and Lemma 3.5 expand on these observations.

**Lemma 2.12.** *Given a group $G$ and a tuple $\tau = (g_1, \ldots, g_n)$, the map $[g_i] \mapsto g_i$ defines a surjection $H(\tau) \to \langle \{g_i\} \rangle \subseteq G$.*

*Proof.* This is true because every relation of the form discussed in Remark 2.11 is trivially true in $G$. □

Let $a, b \in G$ for some group $G$. Let $\Pi_k(a, b)$ denote the alternating product of $a$ and $b$, beginning with $a$, and of length $k$. For example $\Pi_5(a, b) = ababa$.

**Lemma 2.13.** *Let $G$ be a group such that there exists a homomorphism $\varphi\colon A \to G$ from our Artin group $A$. Consider the tuple $\tau = (\varphi \circ \pi_A)^n(s_1, \ldots, s_n) \in G^n$. The map $\pi_A(s_i) \mapsto [\varphi \circ \pi_A(s_i)]$ defines a surjection from $A$ to $H(\tau)$.*

*Proof.* For each $i$, let $t_i$ denote $\pi_A(s_i)$ and $u_i$ denote $\varphi(t_i)$. We will show that the map $t_i \mapsto [u_i]$ extends to a homomorphism. To do so, we need to show that any Artin-like equation of the form $\Pi_k(t_i,t_j) = \Pi_k(t_j,t_j)$ associated to a defining relation in $A$ also holds in $H(\tau)$.

For any $i < j$, there is a braid $\beta$ such that $\beta \cdot (u_1, \ldots, u_n)$ begins with $u_i, u_j$. Thus, without loss of generality, we can assume for the following argument that $i = 1$ and $j = 2$. We only care about what happens at the first two places of this tuple, so we will only consider the Hurwitz action on the duple $(u_1, u_2)$. Consider the function $f_n(a,b) := \Pi_{n+1}(a,b)\Pi_n(a,b)^{-1}$. One can compute that $f_n(a,b)^{f_{n-1}(a,b)^{-1}} = f_{n+1}(a,b)$. Acting once by $\sigma_1$ on our duple we have

$$\sigma_1 \cdot (u_1,u_2) = (u_1u_2u_1^{-1}, u_1) = (f_1(u_1,u_2), f_0(u_1,u_2)).$$

Thus, for $k > 0$, we have

$$\sigma_1^k \cdot (u_1,u_2) = (f_k(u_1,u_2), f_{k-1}(u_1,u_2)).$$

So for each $k \geq 0$, we have a generator $[f_k(u_1,u_2)]$ in $H$, and as in Remark 2.11 the corresponding relation

$$[f_k(u_1,u_2)] = f_k([u_1],[u_2]).$$

Suppose that $A$ has the relation $\Pi_k(t_1,t_2) = \Pi_k(t_2,t_1)$, then in $H$ we have

$$f_k([u_1],[u_2]) = [f_k(u_1,u_2)] = [\varphi(f_k(t_1,t_2))] =$$
$$[\varphi(\Pi_{k+1}(t_1,t_2)\Pi_k(t_1,t_2)^{-1}] = [\varphi\left(\Pi_{k+1}(t_1,t_2)\Pi_k(t_2,t_1)^{-1}\right)] =$$
$$[\varphi(t_1)] = [u_1].$$

So we have the relation $\Pi_k([u_1],[u_2]) = \Pi_k([u_2],[u_1])$ in $H(\tau)$ as required. This map is surjective by Lemma 2.9. □

**Corrolary 2.14.** *Let $\tau = \pi_A^n(s_1,\ldots,s_n)$. We have that $H(\tau) \cong A$.*

*Proof.* Lemma 2.13 gives a homomorphism $a\colon A \to H(\tau)$, and Lemma 2.12 gives a homomorphism $b\colon H(\tau) \to A$. The composition $b \circ a$ is the identity on a generating set for $A$. □

The following is not a standard definition, but it is equivalent to the standard definition by [Bes04, Lemma 7.11].

**Definition 2.15.** *Given a Coxeter element $w \in W$, the* dual Artin group *associated to $w$, denoted $A_w^\vee$, is defined to be the Hurwitz group associated to any $(r_1,\ldots,r_n)$ such that all $r_i \in R$ and $r_1r_2\cdots r_n = w$.*

The obvious tuple to consider is $(\pi_W(s_1),\ldots,\pi_W(s_n))$. Since our choice of $w$ is constant, we will denote our dual Artin group $A^\vee$ (omitting any mention of $w$).

**Corrolary 2.16.** *The natural map $\psi\colon [S_W] \to W$ which acts as $[\pi_W(s_i)] \mapsto \pi_W(s_i)$ defines a surjective homomorphism $A^\vee \to W$.*

*Proof.* This directly follows from Lemma 2.12. □

**Corrolary 2.17.** *The map natural map $\varphi\colon S_A \to A^\vee$ which acts as $\pi_A(s_i) \mapsto [q \circ \pi_A(s_i)] = [\pi_W(s_i)]$ extends to a surjective homomorphism $\varphi\colon A \to A^\vee$.*

*Proof.* This directly follows from Lemma 2.13. □

**Theorem 2.18** ([Bes06])**.** *Let $\tau = \pi_A^n(s_1,\ldots,s_n)$. If the natural surjection $q\colon A \to W$ is injective on* HurRef$(\tau)$*, then $A^\vee \cong A$.*

*Proof.* By Corollary 2.14, we have $A \cong H(\tau)$, and by the hypothesis and Lemma 2.7 we have $A^\vee \cong H(\tau)$. □

## 3. The construction $G^Q$

Let $F\colon \mathbf{Set} \to \mathbf{Grp}$ be the free functor and denote its action on objects as $F_S \coloneqq F(S)$ and its action on morphisms as $f_* \coloneqq F(f\colon S \to T)$. To each quotient $G$ of $F_S$ we associate the surjective homomorphism $\pi_{(S,G)}\colon F_S \to G$ which is projection. In this context, we can frame a basic group theoretic fact. We write $G \cong \langle S \mid R \rangle$ if $G \cong F_S/N(R)$ where $N(R)$ is the minimal normal subgroup in $F_S$ that contains $R$. We write $G \cong \langle S \rangle$ if $G$ is isomorphic to some quotient of $F_S$.

**Basic fact 3.1.** *Suppose we have two groups $G_1$ and $G_2$. Suppose that $G_1 \cong \langle S | R \rangle$ where $R$ is a subset of $F_S$ for which $\pi_{(S,G_1)}(R) = \{1\}$. Given a map $f\colon S \to G_2$, if $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\} \subseteq G_2$, then $f$ defines a homomorphism $h\colon G_2 \to G_2$.*

The above theorem defines a homomorphism $h\colon \langle S \,|\, R \rangle \cong G_1 \to G_2$ when a map $f\colon S \to G_2$ is compatible with the relations in $G_1$ and $G_2$. If we also had knowledge of a generating set for $G_2$, we could construct homomorphisms in a different way.

**Basic fact 3.2.** *Suppose $G_1 \cong \langle S_1 | R_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Given a map $f\colon S_1 \to S_2$, if there exists a map $h$ such that the following diagram commutes, then $h$ is a homomorphism.*

$$\begin{tikzcd} S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "f"] & F_{S_1} \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "f_*"] & G_1 \arrow[d, red, "h"] \\ S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \end{tikzcd} \tag{2}$$

*Remark* 3.3. If we construct a homomorphism using Basic fact 3.1, then we get the middle commuting square of (2) automatically.

Now consider (2), but replace $F_{S_1}$ with some subset $Q \subseteq F_{S_1}$. We will now construct a group, which we will denote $G^Q$, where such a diagram defines a homomorphism from $G^Q$ to $G_2$.

**Definition 3.4** (Group with relations visible in $Q$)**.** *Suppose we have a group $G \cong \langle S \rangle$. Fix some $Q \subseteq F_S$ such that $S \subseteq Q$. Let $\pi \coloneqq \pi_{(S,G)}$. We have the following maps.*

$$S \xhookrightarrow{\;i\;} Q \xrightarrow{\;\pi\;} \pi(Q) \ .$$

*Define the* group $G$ with relations visible in $Q$*, to be*

$$G^Q \coloneqq \langle \pi(Q) \mid \{\pi(q) = (\pi \circ i)_*(q) \;\mid\; q \in Q\} \rangle.$$

Do not think of the generators $\pi(Q)$ as elements of $G$. They are abstract generators. Specifically, $G^Q$ is a quotient of $F_{\pi(Q)}$. Thus, our relations should be equations in $F_{\pi(Q)}$, which they are. $G^Q$ is generated by $\pi(S) \subseteq \pi(Q)$. $G$ is a quotient of $G^Q$, which we will explore later.

**Lemma 3.5.** *Let $Q = \mathrm{HurRef}(s_1, \dots, s_n)$. The dual Artin group $A^\vee$ is isomorphic to $W^Q$.*

*Proof.* TODO ∎

**Lemma 3.6.** *Let $G \cong \langle S \rangle$ be a group and consider the following setup.*

$$S \xhookrightarrow{i_S} F_S \xrightarrow{\pi_{(S,G)}} G \xhookrightarrow{i_G} F_G \xrightarrow{\pi_{(G,G)}} G$$

*The following maps are equal.*

$$\pi_{(G,G)} \circ i_G \circ \pi_{(S,G)} = \pi_{(G,G)} \circ (\pi_{(S,G)} \circ i_S)_*.$$

*Proof.* Both maps are homomorphism which agree with $\pi_{(S,G)}$ on the generating set $S$. □

**Theorem 3.7.** *Suppose we have two groups $G_1 \cong \langle S_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Fix some $Q \subseteq F_{S_1}$ such that $S \subseteq Q$. Let $\overline{Q} \coloneqq \pi_{(S_1,G_1)}(Q)$. If in the following diagram (3), if there exists a map $f$ that makes the diagram commute, then there is a homomorphism $h\colon (G_1)^Q \to G_2$.*

(3)
$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[d, red, "\exists f?"] \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2
\end{tikzcd}$$

*Proof.* Suppose we have such a map $f$. Note that $\overline{Q}$ is a generating set for $G^Q$, thus we have the setup of Basic fact 3.1. We use the following commutative diagram to define some inclusion maps and as a reference for the setup.

(4)
$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] \arrow[d, "f"] & F_{\overline{Q}} \arrow[d, "f_*"] & \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \arrow[r, hook, "i_{G_2}"] & F_{G_2} \arrow[r, "\pi_{(G_2,G_2)}"] & G_2
\end{tikzcd}$$

We can construct a homomorphism if $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\}$, where $R$ are the relations for $G^Q$, defined in Definition 3.4. For some $q \in Q$, the corresponding element in $R$ equalling the identity is $i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1}$. Using that $\pi_{(G_2,G_2)} \circ f_*$ is a homomorphism, we have

$$\pi_{(G_2,G_2)} \circ f_* \left(i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1}\right) =$$
$$\left(\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)\right) \left(\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1}\right)_* (q)^{-1}\right).$$

We will concentrate on each factor separately.

First we consider the first factor. Since $\pi_{(S_1,G_1)}(q) \in \overline{Q}$, by the rightmost commuting square of (4), we have $f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q)$. This gives us

$$\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q).$$

We then use the middle commuting square of (4) to give us

$$\pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q).$$

We then use Lemma 3.6 and functoriality, giving us

$$\begin{aligned}\pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q) &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2}\right)_* \circ \theta_*(q)\\ &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta\right)_* (q)\end{aligned}$$

We now concentrate on the second factor. Using functoriality, then the middle commuting square of (4), we get

$$\begin{aligned}\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1}\right)_* (q)^{-1} &= \pi_{(G_2,G_2)} \circ \left(f \circ \pi_{(S_1,G_1)} \circ i_{S_1}\right)_* (q)^{-1}\\ &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1}\right)_* (q)^{-1}.\end{aligned}$$

Then, we use the leftmost commuting square of (4) to get

$$\pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1}\right)_* (q)^{-1} = \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta\right)_* (q)^{-1}.$$

This is the inverse of the left factor. □

*Remark* 3.8. If we set $Q = S_1 \subseteq F_{S_1}$ (the minimum subset allowed), then $G^Q$ is $F_{S_1}$. In this case, $f$ always exists (and is $\theta$) and Theorem 3.7 tells us the standard theorem about homomorphisms from the free group.

*Remark* 3.9. If we set $Q = F_{S_1}$, then $G^Q$ is $G$, $f$ is $g$ and Theorem 3.7 tells us nothing more than Basic fact 3.2.

**Corrolary 3.10.** *The homomorphism $h$ resulting from Theorem 3.7 makes the following diagram* (5) *commute. Thus, considering $S_1$ as a generating set for $(G_1)^Q$, $h$ is an extension of $\theta$, as in Basic fact 3.2.*

(5)
$$\begin{tikzcd} S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] \arrow[d, "f"] & F_{\overline{Q}} \arrow[r, "\pi_{(\overline{Q},(G_1)^Q)}"] \arrow[d, "f_*"] & (G_1)^Q \arrow[d, "h"] \\ S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \arrow[r, hook, "i_{G_2}"] & F_{G_2} \arrow[r, "\pi_{(G_2,G_2)}"] & G_2 \end{tikzcd}$$

*Proof.* By Remark 3.3, we get the rightmost square in (5). □

We see that if $\theta\colon S_1 \to S_2$ is surjective, then the homomorphism $h\colon (G_1)^Q \to G_2$ resulting from Theorem 3.7 is surjective.

Observing the form of Definition 3.4, it is clear that $\pi_{(S,G)}(S)$ is a generating set for $G^Q$, that is to say that $\pi_{(\overline{Q},G^Q)} \circ \left(\pi_{(S,G)} \circ i_S\right)_*$ is surjective. If we set $Q = F_S$, then $G^Q \cong G$, so $G$ can be realised as a quotient of $G^Q$ via a surjection $p_Q\colon G^Q \to G$ in a way such that the following diagram commutes.

$$\begin{tikzcd} & F_S \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\ G^Q \arrow[rr, "p_Q"] & & G \end{tikzcd}$$

**Lemma 3.11.** *The projection $p_Q$ restricts to an injection on the image of $\pi_{(S,G)}(Q)$, i.e. in the following diagram, the bottom map is an injection.*

$$\begin{tikzcd} & Q \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\ \pi_{(S,G^Q)}(Q) \arrow[rr, hook, "p_Q"] & & \pi_{(S,G)}(Q) \end{tikzcd}$$

*Proof.* The following diagram commutes.

$$\begin{tikzcd}
& \overline{Q} \arrow[dr, "i_{\overline{Q}}"] & & \\
Q \arrow[ur, "\pi_{(S,G)}"] \arrow[rr, "(\pi_{(S,G)}\circ i_S)_*"] \arrow[rrr, bend right, "\pi_{(S,G^Q)}"'] & & F_{\overline{Q}} \arrow[r, "\pi_{(\overline{Q},G^Q)}"] & G^Q
\end{tikzcd}$$

The top triangle commutes due to the defining relations for $G^Q$, and the bottom triangle commutes because both maps are restrictions of homomorphisms that agree on the generating set $S$. Now suppose for some $q_1, q_2 \in Q$ that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$. The above diagram tells us that $\pi_{(S,G^Q)}(q_1) = \pi_{(S,G^Q)}(q_2)$. $\square$

**Lemma 3.12.** *Given some group $G \cong \langle S \rangle$ and $Q$ such that $S \subseteq Q \subseteq F_S$, we have that $G^Q$ is isomorphic to the following group by extending the natural identification of generators.*

$$X \coloneqq \langle S \mid \{q_1 = q_2 \mid q_1, q_2 \in Q,\ \pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)\} \rangle \tag{6}$$

*Proof.* We begin by showing that $\mathrm{Id}_S$ extends to a homomorphism $a\colon X \to G^Q$. Using Basic fact 3.1, it suffices to show $\pi_{(S,G^Q)}(q_1 q_2^{-1}) = 1$ for all $q_1, q_2 \in Q$ such that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$. Using Lemma 3.11, if $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$ then $\pi_{(S,G^Q)}(q_1) = \pi_{(S,G^Q)}(q_2)$. Then, since $\pi_{(S,G^Q)}$ is a homomorphism, we have

$$\pi_{(S,G^Q)}\left(q_1 q_2^{-1}\right) = \pi_{(S,G^Q)}(q_1)\pi_{(S,G^Q)}(q_2)^{-1} = 1.$$

Now we work to show that $\mathrm{Id}_S$ also extends to a homomorphism $b\colon G^Q \to X$. Using Theorem 3.7, it is sufficient to show that there exists a map $f$ to make the following diagram commute.

$$\begin{tikzcd}
& Q \arrow[dl, "\pi_{(S,G)}"'] \arrow[dr, "\pi_{(S,X)}"] & \\
G \arrow[rr, "f"] & & X
\end{tikzcd}$$

Such an $f$ exists if for all $q_1, q_2 \in Q$ such that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$, we also have $\pi_{(S,X)}(q_1) = \pi_{(S,X)}(q_2)$. This is immediately true on inspection of the defining presentation for $X$.

Thus, we have two homomorphisms, $a\colon X \to G^Q$, and $b\colon G^Q \to X$. Furthermore, by construction $(a \circ b)|_S = \mathrm{Id}_S$. $\square$

We now relate this back to the dual Artin group $A^\vee$. Firstly, our construction gives an alternative presentation for $A^\vee$.

**Corrolary 3.13.** *Let $S$ denote the standard generating set for our Artin group $A$, and let $Q = \mathrm{HurRef}(s_1, \dots, s_n)$. The dual Artin group $A^\vee$ has the following presentation.*

$$A^\vee \cong \langle S \mid \{q_1 = q_2 \mid q_1, q_2 \in Q, \pi_{(S,W)}(q_1) = \pi_{(S,W)}(q_2)\} \rangle.$$

*Proof.* This follows from Lemma 3.5 and Lemma 3.12. $\square$

Using Lemma 2.13, Lemma 3.5 and Theorem 3.7, we can re-prove Theorem 2.18, that injectivity of $q\colon A \to W$ on $\pi_A(Q)$ implies $A^\vee \cong A$. Now, using our construction, we can prove the inverse direction.

**Theorem 3.14.** *Let $\tau = \pi_A^n(s_1, \ldots, s_n)$ and let $Q = \mathrm{HurRef}(\tau)$. The Artin group $A$ is naturally isomorphic to the dual Artin group $A^\vee$ iff the natural projection $q\colon A \to W$ is injective on $\pi_A(Q)$.*

*Proof.* Theorem 2.18 already shows that if $q$ is injective on $\pi_A(Q)$, then $A^\vee \cong A$.

For the other direction, consider the following commutative diagram.

$$\begin{tikzcd}
 & F_S \arrow[dl, "\pi_A"'] \arrow[d, "\pi_{(S,{}^W Q)}"] \arrow[dr, "\pi_W"] & \\
A \arrow[r, "\varphi"] \arrow[rr, bend right, "q"'] & A^\vee \arrow[r, "\psi"] & W
\end{tikzcd}$$

We get $\varphi$ and $\psi$ from Corollary 2.17 and Corollary 2.16 respectively. Note that $\psi$ is the same as $p_Q$ as discussed above. Commutativity is easy to check by comparing these maps when restricted to generating sets.

If we replace $F_S$ with $Q$, then Lemma 3.11 tells us that the corresponding restriction of $\psi$ is injective. So, if $\varphi$ is an isomorphism, in particular an injection, then $q = \psi \circ \varphi$ is an injection on $\pi_A(Q)$. □

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