DUAL ARTIN GROUP PRESENTATIONS

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1. Notation and setup

We will begin by setting up our notation for Coxeter and Artin groups.

Let W and A respectively be the Coxeter and Artin groups associated to some Coxeter system Γ. In this document, the specific Coxeter system is arbitrary and constant.

Our Coxeter and Artin groups have a conventional finite generating set which we associate to the set S. In this document, n will always denote $|S|$, the rank of our Coxeter system. Rather than S being a subset of W or A, we will consider S to be a free generating set, with F_S the free group on S. For any quotient G of F_S (particularly W or A), we denote the quotient surjection $\pi_{(S,G)} \colon F_S \to G$. Where S is obvious from context, we may drop it from this notation in favour of $\pi_G \colon F_S \to G$. For shorthand, we will denote $\pi_W(S)$ and $\pi_A(S)$ by S_W and S_A respectively.

To specify a dual Artin group, we must in principle choose a Coxeter element in W. We will always denote this choice w. We will denote the generating simple reflections W by S. The elements of S will always be denoted s_i for $1 \le i \le n$ such that w is a lexicographic product, i.e. $w = \pi_W(s_1 s_2 \cdots s_n)$. For any object (set or tuple) X with such an indexing $x_i \in X$, and function f, we may use the notation $\{f(x_i)\}$ as shorthand for $\{f(x_i) \mid x_i \in X\}$.

Let $R \coloneqq (S_W)^W \subseteq W$, $T \coloneqq (S_A)^A \subseteq A$ and $U \coloneqq S^{F_S}$ denote all conjugates of generators in W, A and F_S respectively. We call R the *reflections* of W.

2. Hurwitz groups and the dual Artin group

Let B_n denote the braid group on n strands with standard generating set $\{\sigma_i\}_{1\le i<n}$.

Definition 2.1 (The Hurwitz action)**.** *Given any group G, there is an action of B_n on G^n, called the* Hurwitz action *where*

$$\sigma_i \cdot (g_1, \dots g_n) = \left(g_1, \dots, g_{i-1}, g_{i+1}^{(g_i^{-1})}, g_i, g_{i+2}, \dots, g_n\right)$$

for all $1 \le i < n$. This also determines the action of σ_i^{-1}.

This action preserves the index-wise product $(g_1, \dots, g_n) \mapsto g_1 g_2 \cdots g_n$. Given some $X \subseteq G$, if $X^G = G$, then the Hurwitz action is also well-defined on X^n, and this makes any such X a B_n-set. If we set $\tau = \pi_W^n(s_1, \dots, s_n)$, then by [IS10], we know that $B_n \cdot \tau$ gives every minimal factorisation of w in to elements of R.

Remark 2.2. Given two groups G_1 and G_2, and a homomorphism $\varphi \colon G_1 \to G_2$, the map $\varphi^n \colon G_1^n \to G_2^n$ is a B_n-set morphism.

Let $p_1\colon G^n \to G$ denote generic projection to the first coordinate of a tuple. We are interested in the tuples that occur due to the Hurwitz action. The set $B_n \cdot \tau$ is the set of all such tuples for some tuple τ. The set $\mathrm{HurRef}(\tau)$ is the set of all $g \in G$ that occur in any position in $B_n \cdot \tau$. By choosing braids that move elements to the first position in the tuple, we see that $\mathrm{HurRef}(\tau) = p_1(B_n \cdot \tau)$.

Associated to a tuple, there is a monoid construction, which we will call the *Hurwitz monoid.*

Definition 2.3. *Let G be some group, and let τ be some tuple in G^k. Let X denote* $\mathrm{HurRef}(\tau)$*. Suppose there in there is a tuple in $B_k \cdot \tau$ that begins r_1, r_2. Let r_3 denote the element of X equal in G to $r_1^{r_2^{-1}}$. We call any such (r_1, r_2, r_3) a* Hurwitz triple. *In this context, we define the* Hurwitz monoid *I associated to τ as follows. Decorate elements $x \in X$ as $[x]$ to distinguish them as generators of I rather than elements of G.*

$$I \coloneqq \langle\, [X] \mid \{[r_1][r_2] = [r_3][r_1] \mid (r_1, r_2, r_3) \text{ is a Hurwitz triple}\}\,\rangle.$$

We call the corresponding group, the *Hurwitz group.* We denote the Hurwitz group associated to a tuple τ by $H(\tau)$.

Lemma 2.4. *Suppose we have two groups G_1 and G_2 and a homomorphism $\varphi\colon G_1 \to G_2$. Let $\tau_1 = (g_1, \ldots, g_n) \in G^n$ and $\tau_2 = \varphi^n(\tau)$ be two tuples. If φ is injective on* $\mathrm{HurRef}(\tau_1)$*, then $H(\tau_1) \cong H(\tau_2)$.*

Proof. By Remark 2.2, φ^n is a B_n-set morphism from $B_n \cdot \tau_1$ to $B_n \cdot \tau_2$. If φ is injective on $\mathrm{HurRef}(\tau_1)$, then φ^n is injective on $B_n \cdot \tau$, so $\varphi^n|_{B_n\cdot\tau_1} \colon B_n \cdot \tau_1 \to B_n \cdot \tau_2$ is a B_n-set isomorphism. Since $H(\tau_1)$ and $H(\tau_2)$ are determined by the B_n-sets $B_n \cdot \tau_1$ and $B_n \cdot \tau_2$ respectively, the map $[x] \to [\varphi(x)]$ for all $x \in \mathrm{HurRef}(\tau_1)$ defines an isomorphism $H(\tau_1) \to H(\tau_2)$. $\square$

Lemma 2.5. *Given a group G and a tuple $\tau \in G^k$, we have that $H(\tau)$ is generated by $\{\tau_i\}$.*

Proof. Let $\ell\colon B_n \to \mathbb{N}$ denote minimum word length with respect to the standard generating set for B_n. Let $x \in \mathrm{HurRef}(\tau) \setminus \{\tau_i\}$. Let $\beta \in B_n$ be the minimum length braid such that $\beta \cdot \tau = (\mu, \cdots, \mu_n)$ and $x = \mu_i$ for some i. We will show that such an $[x]$ can be written in terms of generators appearing in $\gamma \cdot \tau$ where γ is a braid with $\ell(\gamma) < \ell(\beta)$. Induction then completes the proof.

Since $x \notin \{\tau_j\}$, we know $\ell(\beta) > 0$. Recall our notation for the set of standard generators for B_n is $\{\sigma_j\}$. Since β is a braid of minimum length with respect to these generators, we know that there is a factorisation of β in the $\{\sigma_j\}$ such that the last factor is either σ_i or σ_{i-1}^{-1}, since these are the only generators whose action puts something new in the i^{th} position of the tuple.

Suppose the last factor of β is σ_i. Let $(\nu_1, \ldots, \nu_n) = \beta\sigma_i^{-1} \cdot \tau$, so $[x] = [\mu_i] = [\nu_{i+1}\nu_i\nu_{i+1}^{-1}]$. Since $\ell(\beta\sigma_i^{-1}) \leq \ell(\beta) - 1$, if we can express x in terms of the set $\{[\nu_j]\}$ then we would be done. Since we can bring ν_i and ν_{i+1} to the start of the tuple, we have the equation $[x] = [\nu_{i+1}][\nu_i][\nu_{i+1}]^{-1}$ in $H(\tau)$.

We make a similar argument if the last factor of β is σ_{i-1}^{-1}. $\square$

Remark 2.6. The above lemma tells us that there is a presentation for any $H(\tau)$ in just the elements $\{[\tau_i]\}$. When we transform the defining relations in to relations in the set $\{[\tau_i]\}$, they have a very particular form. For example, considering the tuple $(a, b) \in G^2$, we have the relation

$$[aba^{-1}] = [a][b][a]^{-1}.$$

We see that this relation allows us to commute the square bracket decorations through the word that we get from the Hurwitz action, and every such defining relation can be re-written in this way. However, note that the data that defines the relation is the specific word aba^{-1} that emerges by performing the Hurwitz action as if a and b are free group elements. If it was true that $aba^{-1} \overset{G}{=} xyz$, for some arbitrary $x, y, z \in G$, we would *not* necessarily have the relation $[aba^{-1}] = [x][y][z]$, and there is no reason to assume that any of $[x]$, $[y]$ or $[z]$ are even generators in the defining presentation for $H(\tau)$.

Lemma 2.7. *Given a group G and a tuple $\tau = (g_1, \dots, g_n)$, the map $[g_i] \mapsto g_i$ defines a surjection $H(\tau) \to \langle \{g_i\} \rangle \subseteq G$.*

Proof. This is true because every relation of the form discussed in Remark 2.6 is trivially true in G. $\square$

Let $a, b \in G$ for some group G. Let $\Pi_k(a, b)$ denote the alternating product of a and b, beginning with a, and of length k. For example $\Pi_5(a, b) = ababa$.

Lemma 2.8. *Let G be a group such that there exists a homomorphism $\varphi\colon A \to G$ from our Artin group A. Consider the tuple $\tau = (\varphi \circ \pi_A)^n(s_1, \dots, s_n) \in G^n$. The map $\pi_A(s_i) \mapsto [\varphi \circ \pi_A(s_i)]$ defines a surjection from A to $H(\tau)$.*

Proof. For each i, let t_i denote $\pi_A(s_i)$ and u_i denote $\varphi(t_i)$. We will show that the map $t_i \mapsto [u_i]$ extends to a homomorphism. To do so, we need to show that any Artin-like equation of the form $\Pi_k(t_i, t_j) = \Pi_k(t_j, t_j)$ associated to a defining relation in A also holds in $H(\tau)$.

For any $i < j$, there is a braid β such that $\beta \cdot (u_1, \dots, u_n)$ begins with u_i, u_j. Thus, without loss of generality, we can assume for the following argument that $i = 1$ and $j = 2$. We only care about what happens at the first two places of this tuple, so we will only consider the Hurwitz action on the duple (u_1, u_2). Consider the function $f_n(a, b) := \Pi_{n+1}(a, b)\Pi_n(a, b)^{-1}$. One can compute that $f_n(a, b)^{f_{n-1}(a,b)^{-1}} = f_{n+1}(a, b)$. Acting once by σ_1 on our duple we have

$$\sigma_1 \cdot (u_1, u_2) = (u_1 u_2 u_1^{-1}, u_1) = (f_1(u_1, u_2), f_0(u_1, u_2)).$$

Thus, for $k > 0$, we have

$$\sigma_1^k \cdot (u_1, u_2) = (f_k(u_1, u_2), f_{k-1}(u_1, u_2)).$$

So for each $k \geq 0$, we have a generator $[f_k(u_1, u_2)]$ in H, and as in Remark 2.6 the corresponding relation

$$[f_k(u_1, u_2)] = f_k([u_1], [u_2]).$$

Suppose that A has the relation $\Pi_k(t_1, t_2) = \Pi_k(t_2, t_1)$, then in H we have

$$\begin{gathered} f_k([u_1], [u_2]) = [f_k(u_1, u_2)] = [\varphi(f_k(t_1, t_2))] = \\ [\varphi(\Pi_{k+1}(t_1, t_2)\Pi_k(t_1, t_2)^{-1}] = [\varphi\left(\Pi_{k+1}(t_1, t_2)\Pi_k(t_2, t_1)^{-1}\right)] = \\ [\varphi(t_1)] = [u_1]. \end{gathered}$$

So we have the relation $\Pi_k([u_1], [u_2]) = \Pi_k([u_2], [u_1])$ in $H(\tau)$ as required. This map is surjective by Lemma 2.5. $\square$

Corrolary 2.9. *Let* $\tau = \pi_A^n(s_1, \ldots, s_n)$*. We have that* $H(\tau) \cong A$*.*

Proof. Lemma 2.8 gives a homomorphism $a\colon A \to H(\tau)$, and Lemma 2.7 gives a homomorphism $b\colon H(\tau) \to A$. The composition $b \circ a$ is the identity on a generating set for A. $\square$

The following is not a standard definition, but it is equivalent to the standard definition by [Bes04, Lemma 7.11].

Definition 2.10. *Given a Coxeter element* $w \in W$*, the* dual Artin group associated *to* w*, denoted* $A_w^\vee$*, is defined to be the Hurwitz group associated to any* $(r_1, \ldots, r_n)$ *such that all* $r_i \in R$ *and* $r_1 r_2 \cdots r_n = w$*.*

The obvious tuple to consider is $(\pi_W(s_1), \ldots, \pi_W(s_n))$. Since our choice of w is constant, we will denote our dual Artin group $A^\vee$ (omitting any mention of w).

Theorem 2.11 ([Bes06])**.** *Let* $\tau = \pi_A^n(s_1, \ldots, s_n)$*. If the standard surjection* $q\colon A \to W$ *is injective on* $\mathrm{HurRef}(\tau)$*, then* $A^\vee \cong A$*.*

Proof. By the corollary above, $A \cong H(\tau)$, and by the hypothesis and Lemma 2.4 we have $A^\vee \cong H(\tau)$. $\square$

3. The construction G^Q

Let $F\colon \mathbf{Set} \to \mathbf{Grp}$ be the free functor and denote its action on objects as $F_S \coloneqq F(S)$ and its action on morphisms as $f_* \coloneqq F(f\colon S \to T)$. To each quotient G of F_S we associate the surjective homomorphism $\pi_{(S,G)}\colon F_S \to G$ which is projection. In this context, we can frame a basic group theoretic fact. We write $G \cong \langle\, S \mid R\, \rangle$ if $G \cong F_S/N(R)$ where $N(R)$ is the minimal normal subgroup in F_S that contains R. We write $G \cong \langle\, S\, \rangle$ if G is isomorphic to some quotient of F_S.

Basic fact 3.1. *Suppose we have two groups* G_1 *and* G_2*. Suppose that* $G_1 \cong \langle\, S \mid R\, \rangle$ *where* R *is a subset of* F_S *for which* $\pi_{(S,G_1)}(R) = \{1\}$*. Given a map* $f\colon S \to G_2$*, if* $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\} \subseteq G_2$*, then* f *defines a homomorphism* $h\colon G_2 \to G_2$*.*

The above theorem defines a homomorphism $h\colon \langle\, S \mid R\, \rangle \cong G_1 \to G_2$ when a map $f\colon S \to G_2$ is compatible with the relations in G_1 and G_2. If we also had knowledge of a generating set for G_2, we could construct homomorphisms in a different way.

Basic fact 3.2. *Suppose* $G_1 \cong \langle\, S_1 \mid R_1\, \rangle$ *and* $G_2 \cong \langle\, S_2\, \rangle$*. Given a map* $f\colon S_1 \to S_2$*, if there exists a map* h *such that the following diagram commutes, then* h *is a homomorphism.*

$$
\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "f"] & F_{S_1} \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "f_*"] & G_1 \arrow[d, "h"] \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2
\end{tikzcd}
\tag{1}
$$

Remark 3.3. If we construct a homomorphism using Basic fact 3.1, then we get the middle commuting square of (1) automatically.

Now consider (1), but replace F_{S_1} with some subset $Q \subseteq F_{S_1}$. We will now construct a group, which we will denote G^Q, where such a diagram defines a homomorphism from G^Q to G_2.

Definition 3.4 (Group with relations visible in Q)**.** *Suppose we have a group $G \cong \langle S \rangle$. Fix some $Q \subseteq F_S$ such that $S \subseteq Q$. Let $\pi \coloneqq \pi_{(S,G)}$. We have the following maps.*

$$S \xhookrightarrow{i} Q \xrightarrow{\pi} \pi(Q) \ .$$

Define the group G with relations visible in Q*, to be*

$$G^Q \coloneqq \langle \pi(Q) \mid \{\pi(q) = (\pi \circ i)_*(q) \mid q \in Q\} \rangle.$$

Do not think of the generators $\pi(Q)$ as elements of G. They are abstract generators. Specifically, G^Q is a quotient of $F_{\pi(Q)}$. Thus, our relations should be equations in $F_{\pi(Q)}$, which they are. G^Q is generated by $\pi(S) \subseteq \pi(Q)$. G is a quotient of G^Q, which we will explore later.

Lemma 3.5. *Let $Q = \mathrm{HurRef}(s_1, \ldots, s_n)$. The dual Artin group $A^\vee$ is isomorphic to W^Q.*

Proof. Let us use π as shorthand for $\pi_{(S,W)}$. We have that $\pi(Q)$ is the defining generating set for both $A^\vee$ and W^Q. Thus, it makes sense to compare relations in $A^\vee$ and W^Q as objects of the same type. The (non-trivial) defining relations of $A^\vee$ and W^Q are both in bijection with $Q \setminus S$.

TODO (do this)

Let $\tau = (s_1, \ldots, s_n)$. We will show that the defining relations for each of these groups are in one to one correspondence. We will show this by induction on braid length $\ell \colon B_n \to \mathbb{N}$ (with respect to the standard generating set $\{\sigma_i\}$).

We can build up the relators for $A^\vee$ or W^Q by successively considering braids of longer length. To each relator for $A^\vee$ or W^Q there is an associated $k \in \mathbb{N}$ which corresponds to the minimum braid length such that $\ell(\beta) = k$ and $\beta \cdot \pi^n(\tau)$ sees the generator of

There is a bijection and identification of relations of $A^\vee$ and W^Q for all relations (or $q \in Q$) occurring due to braids of length 1. We see this in Remark 2.6. This constitutes our base case.

Now suppose that every relation in $A^\vee$ due to braids of length k or less can be re-written as $\pi(q) = (i \circ \pi)_*(q)$ for some $q \in Q$ occurring due to a braid of length k or less. $\square$

Lemma 3.6. *Let $G \cong \langle S \rangle$ be a group and consider the following setup.*

$$S \xhookrightarrow{i_S} F_S \xrightarrow{\pi_{(S,G)}} G \xhookrightarrow{i_G} F_G \xrightarrow{\pi_{(G,G)}} G$$

The following maps are equal.

$$\pi_{(G,G)} \circ i_G \circ \pi_{(S,G)} = \pi_{(G,G)} \circ (\pi_{(S,G)} \circ i_S)_*.$$

Proof. Both maps are homomorphism which agree with $\pi_{(S,G)}$ on the generating set S. $\square$

Theorem 3.7. *Suppose we have two groups $G_1 \cong \langle S_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Fix some $Q \subseteq F_{S_1}$ such that $S \subseteq Q$. Let $\overline{Q} \coloneqq \pi_{(S_1,G_1)}(Q)$. If in the following diagram* (2)*, if there exists a map f that makes the diagram commute, then there is a* *homomorphism* $h\colon (G_1)^Q \to G_2$.

$$\begin{tikzcd} S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[d, red, "\exists f?"] \\ S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \end{tikzcd} \tag{2}$$

Proof. Suppose we have such a map f. Note that $\overline{Q}$ is a generating set for G^Q, thus we have the setup of Basic fact 3.1. We use the following commutative diagram to define some inclusion maps and as a reference for the setup.

$$\begin{tikzcd} S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] \arrow[d, "f"] & F_{\overline{Q}} \arrow[d, "f_*"] & \\ S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \arrow[r, hook, "i_{G_2}"] & F_{G_2} \arrow[r, "\pi_{(G_2,G_2)}"] & G_2 \end{tikzcd} \tag{3}$$

We can construct a homomorphism if $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\}$, where R are the relations for G^Q, defined in Definition 3.4. For some $q \in Q$, the corresponding element in R equalling the identity is $i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1}$. Using that $\pi_{(G_2,G_2)} \circ f_*$ is a homomorphism, we have

$$\pi_{(G_2,G_2)} \circ f_* \left(i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1} \right) =$$
$$\left(\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) \right) \left(\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1} \right)_* (q)^{-1} \right).$$

We will concentrate on each factor separately.

First we consider the first factor. Since $\pi_{(S_1,G_1)}(q) \in \overline{Q}$, by the rightmost commuting square of (3), we have $f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q)$. This gives us

$$\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q).$$

We then use the middle commuting square of (3) to give us

$$\pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q).$$

We then use Lemma 3.6 and functoriality, giving us

$$\begin{aligned} \pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q) &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \right)_* \circ \theta_*(q) \\ &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta \right)_* (q) \end{aligned}$$

We now concentrate on the second factor. Using functoriality, then the middle commuting square of (3), we get

$$\begin{aligned} \pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1} \right)_* (q)^{-1} &= \pi_{(G_2,G_2)} \circ \left(f \circ \pi_{(S_1,G_1)} \circ i_{S_1} \right)_* (q)^{-1} \\ &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1} \right)_* (q)^{-1}. \end{aligned}$$

Then, we use the leftmost commuting square of (3) to get

$$\pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1} \right)_* (q)^{-1} = \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta \right)_* (q)^{-1}.$$

This is the inverse of the left factor. □

Remark 3.8. If we set $Q = S_1 \subseteq F_{S_1}$ (the minimum subset allowed), then G^Q is F_{S_1}. In this case, f always exists (and is θ) and Theorem 3.7 tells us the standard theorem about homomorphisms from the free group.

Remark 3.9. If we set $Q = F_{S_1}$, then G^Q is G, f is g and Theorem 3.7 tells us nothing more than Basic fact 3.2.

Corrolary 3.10. *The homomorphism h resulting from Theorem 3.7 makes the following diagram (4) commute. Thus, considering S_1 as a generating set for $(G_1)^Q$, h is an extension of θ, as in Basic fact 3.2.*

$$
\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] \arrow[d, "f"] & F_{\overline{Q}} \arrow[r, "\pi_{(\overline{Q},(G_1)^Q)}"] \arrow[d, "f_*"] & (G_1)^Q \arrow[d, "h"] \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \arrow[r, hook, "i_{G_2}"] & F_{G_2} \arrow[r, "\pi_{(G_2,G_2)}"] & G_2
\end{tikzcd}
\tag{4}
$$

Proof. By Remark 3.3, we get the rightmost square in (4). $\square$

We see that if $\theta\colon S_1 \to S_2$ is surjective, then the homomorphism $h\colon (G_1)^Q \to G_2$ resulting from Theorem 3.7 is surjective.

Observing the form of Definition 3.4, it is clear that $\pi_{(S,G)}(S)$ is a generating set for G^Q, that is to say that $\pi_{(\overline{Q},G^Q)} \circ \left(\pi_{(S,G)} \circ i_S\right)_*$ is surjective. If we set $Q = F_S$, then $G^Q \cong G$, so G can be realised as a quotient of G^Q via a surjection $p_Q\colon G^Q \to G$ in a way such that the following diagram commutes.

$$
\begin{tikzcd}
 & F_S \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\
G^Q \arrow[rr, "p_Q"] & & G
\end{tikzcd}
$$

Furthermore, the following diagram commutes.

$$
\begin{tikzcd}
Q \arrow[r, "\pi_{(S,G)}"] \arrow[rrrr, bend right, "\pi_{(S,G)}"'] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] & F_{\overline{Q}} \arrow[r, "\pi_{(\overline{Q},G^Q)}"] & G^Q \arrow[r, "p_Q"] & G
\end{tikzcd}
\tag{5}
$$

TODO fix this

This is because $i_{\overline{Q}}$ is a restriction of $i_G\colon G \to F_G$, which is a homomorphism. So both the bottom and top maps in (5) are restrictions of homomorphisms. And the top map agrees with the bottom map on the generating set $S \subseteq Q$.

Thus, if $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_1)$, then necessarily $\pi_{(S,G^Q)}(q_1) = \pi_{(S,G^Q)}(q_2)$, thus p_Q is injective on $\pi_{(\overline{Q},G^Q)} \circ i_{\overline{Q}} \circ \pi_{(S,G)}(Q) = \pi_{(S,G^Q)}(Q)$. This can be expressed in the following diagram, where the bottom map is injective.

$$
\begin{tikzcd}
 & Q \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\
\pi_{(S,G^Q)}(Q) \arrow[rr, hook, "p_Q"] & & \pi_{(S,G)}(Q)
\end{tikzcd}
\tag{6}
$$

Theorem 3.11. *Given some group $G \cong \langle\, S \,\rangle$ and Q such that $S \subseteq Q \subseteq F_S$, we have that G^Q is isomorphic to the following group by extending the natural identification of generators.*

$$
X \coloneqq \langle\, S \mid \{q_1 = q_2 \mid q_1, q_2 \in Q,\ \pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)\}\,\rangle
\tag{7}
$$

Proof. We begin by showing that Id_S extends to a homomorphism $a\colon X \to G^Q$. Using Basic fact 3.1, it suffices to show $\pi_{(S,G^Q)}(q_1 q_2^{-1}) = 1$ for all $q_1, q_2 \in Q$ such that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$. Using (6), if $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$ then $\pi_{(S,G^Q)}(q_1) = \pi_{(S,G^Q)}(q_2)$. Then, since $\pi_{(S,G^Q)}$ is a homomorphism, we have

$$\pi_{(S,G^Q)}\left(q_1 q_2^{-1}\right) = \pi_{(S,G^Q)}(q_1)\pi_{(S,G^Q)}(q_2)^{-1} = 1.$$

Now we work to show that Id_S also extends to a homomorphism $b\colon G^Q \to X$. Using Theorem 3.7, it is sufficient to show that there exists a map f to make the following diagram commute.

$$\begin{tikzcd} & Q \arrow[dl, "\pi_{(S,G)}"'] \arrow[dr, "\pi_{(S,X)}"] & \\ G \arrow[rr, "f"] & & X \end{tikzcd}$$

Such an f exists if for all $q_1, q_2 \in Q$ such that $\pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)$, we also have $\pi_{(S,X)}(q_1) = \pi_{(S,X)}(q_2)$. This is immediately true on inspection of the defining presentation for X.

Thus, we have two homomorphisms, $a\colon X \to G^Q$, and $b\colon G^Q \to X$. Furthermore, by construction $(a \circ b)\,|_S = \mathrm{Id}_S$, thus $a \circ b$ is the identity and a is an isomorphism. $\square$

We now consider altering Q. If we increase Q to some Q' where $Q \subseteq Q'$, we see that both G^Q and $G^{Q'}$ are generated by the image of S, but the relations for $G^{Q'}$ are a superset of the relations for G^Q. Thus, $G^{Q'}$ can be realised as a quotient of G^Q via a surjective homomorphism p, for which the following diagram commutes.

(8)
$$\begin{tikzcd} & F_S \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{\left(S,G^{Q'}\right)}"] & \\ G^Q \arrow[rr, "p"] & & G^{Q'} \end{tikzcd}$$

Conversely, if Id_S can extend to a homomorphism in the opposite direction $G^{Q'} \to G^Q$, then G^Q and $G^{Q'}$ are isomorphic. We can use Theorem 3.7 to find when this is possible.

Theorem 3.12. *Given some group $G \cong \langle S \rangle$, and $Q, Q' \subseteq F_S$ such that $S \subseteq Q \subseteq Q'$, if there exists a map f that makes the following diagram commute, then $G^Q \cong G^{Q'}$.*

$$\begin{tikzcd} & Q' \arrow[dl, "\pi_{(S,G)}"'] \arrow[dr, "\pi_{(S,G^Q)}"] & \\ \pi_{(S,G)}(Q') \arrow[rr, "f"] & & \pi_{(S,G^Q)}(Q') \end{tikzcd}$$

Proof. By Theorem 3.7, this we know that Id_S extends to a homomorphism $a\colon G^{Q'} \to G^Q$. Using $p\colon G^Q \to G^{Q'}$ in (8) gives us a homomorphism in the opposite direction such that $(p \circ a)\,|_S = \mathrm{Id}_S$, thus a is an isomorphism. $\square$

A pleasant feature of Theorem 3.12 is that we only need to understand the behaviour of $\pi_{(S,G)}$ and $\pi_{(S,G^Q)}$ on Q' in order to know if we can increase Q to Q'.

We do not need to understand how $\pi_{\left(S, G^{Q'}\right)}$ behaves. This suggests the possibility of extending Q inductively.

We have a well–defined notion of Q being maximal. If $Q \subseteq F_S$ is such that any $Q' \supset Q$ results in a $G^{Q'} \ncong G^Q$, then we say Q is maximal. We also have an equivalent notion of Q being minimal.

It is unclear whether a minimal or maximal Q is unique.

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