YEAR 1 ANNUAL PROGRESS REVIEW REPORT

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Here, I will outline my PhD research progress since my program began in October 2024. I will begin by outlining a relevant literature review, where I will assume an understanding of the definitions of Coxeter and Artin groups, as well as related constructions. For an introduction to Coxeter groups, the Tits cone and other geometric constructions, see [Hum90]. For a brief overview of Artin groups dual Artin groups see [PS21, Sections 1,2]. And for a review of the $K(\pi, 1)$ and related problems, see [Par14].

1. Literature Review

Let (W, S) denote a Coxeter system where W is the Coxeter group and S is its generating reflections. Associated to this system is an edge labelled graph Γ. The vertices of Γ correspond to elements of S. There is an edge labelled m connecting the vertices corresponding to $s, t \in S$ if there is a relation between s and t in W of the form $(st)^m = 1$. We use W_Γ and A_Γ as a shorthand for the Coxeter and Artin groups associated to a Coxeter system with graph Γ. Note that Γ also defines a Coxeter system, so we may use this notation in place of (W, S), still denoting the generating set by S.

Let $|\Gamma|$ denote the number of vertices in Γ. Associated to a Coxeter system Γ, there is the so-called *Tits cone*, denoted T, which is a subset of $\mathbb{R}^{|\Gamma|}$. The Tits cone sees a canonical way of realising W_Γ as a linear group. The action of $S \subseteq W_\Gamma$ on T is by reflections (corresponding to a possibly non-standard inner product) through hyperplanes intersecting T. All conjugates of S in W_Γ similarly act by reflections, and all such elements define a hyperplane intersecting T. Let H denote this set of hyperplanes. It can be shown that H separates T into regions which are simplicial cones and are fundamental domains for the action of W_Γ.

With this picture in mind, we define the complexified hyperplane arrangement $\overline{Y}$.

$$\overline{Y} \coloneqq (T \times T) \setminus \bigcup_{h \in H} h \times h.$$

The action of W_Γ on T preserves pointwise the union of H, so we have an action of W_Γ on $\overline{Y}$ and can consider the quotient space $Y \coloneqq W_\Gamma \backslash \overline{Y}$. It is known by [vdL83] that the fundamental group of Y is the corresponding Artin group A_Γ. The $K(\pi, 1)$ conjecture, attributed to Arnold, Brieskorn, Pham, and Thom, states that Y always has contractible universal cover, that is, Y is a $K(A_\Gamma, 1)$ space. This was known in the case where A_Γ was the braid group by work of Fox and Neuwirth in [FN62] and proven in greater generality for all finite W_Γ by Deligne in [Del72]. The conjecture was proven for certain other classes of Γ, including Large type and RAAGs in [Hen85] and [CD16] respectively.

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In 2021 Paolini and Salvetti proved the conjecture in the case where W_Γ is affine [PS21], which constituted significant progress on the problem. Central to their proof was the so-called *dual Artin group*, which we will denote $A_\Gamma^\vee$ and define shortly. The construction of $A_\Gamma^\vee$ is due to Bessis in [Bes03], which contextualises an alternative presentation of the braid group first developed by Birman, Ko and Lee in [BKL98].

Given a Coxeter system Γ with generators S, we define R to be all conjugates of S in W_Γ. We can consider R as a generating set for W_Γ and denote the corresponding Cayley graph $\mathrm{Cay}(W_\Gamma, R)$. A *Coxeter element* of W_Γ is any product of all the elements of S (each occurring exactly once) in any order. Fixing some Coxeter element w, we define $C_{w,R}$ to be the complete subgraph of $\mathrm{Cay}(W_\Gamma, R)$ consisting of all geodesics from the identity to w.

Definition 1.1. *Given a Coxeter system Γ with reflection set $R \subseteq W_\Gamma$ and Coxeter element $w \in W_\Gamma$, a dual Artin group is defined as follows*

$$A_\Gamma^\vee := \langle \{r \in R \mid r \text{ is an edge in } C_{w,R}\} \mid \{\text{loops in } C_{w,R}\} \rangle.$$

Note that I previously said *the dual Artin group*, but in the definition said *a dual Artin group*. This is because it is not known in general if $A_\Gamma^\vee$ depends on our choice of Coxeter element, though it is conjectured to not depend on that choice. This is reflected in the absence of w in our notation $A_\Gamma^\vee$.

It is known that any Coxeter element has word length $|\Gamma|$ in R. Thus, the dual Artin group construction depends on the set of length $|\Gamma|$ R–factorisations of a Coxeter element, and all of S occurs in such a minimal factorisation. In fact, it turns out to only depend on which $r \in R$ appear in such factorisations, as we will show.

By [Bes03, BW02] it is known that $A_\Gamma^\vee \cong A_\Gamma$ where W_Γ is finite. The corresponding result for affine W_Γ was first proved in [MS17] and also proved in [PS21]. Recently, in [Res24], Resteghini proved that if Γ_1 and Γ_2 satisfy $A_{\Gamma_1}^\vee \cong A_{\Gamma_1}$ and $A_{\Gamma_2}^\vee \cong A_{\Gamma_2}$, then the free product also satisfies this isomorphism problem, i.e. $A_{\Gamma_1 * \Gamma_2}^\vee \cong A_{\Gamma_1 * \Gamma_2}$, where $\Gamma_1 * \Gamma_2$ is constructed by taking the disjoint union of Γ_1 and Γ_2 and joining the two graphs by edges labelled by ∞.

It is shown in [DPS24], that all infinite triangle groups (which happen to all be hyperbolic, in the Coxeter group sense) also satisfy this isomorphism problem. Apart from these results mentioned, there are no other classes of Γ where it is known that $A_\Gamma^\vee \cong A_\Gamma$.

A vital tool in the understanding of the set of reflections in R which occur in minimal R-factorisations of Coxeter element is the so-called *Hurwitz action.* Given a factorisation $r_1 r_2, \cdots, r_n = w$ of a Coxeter element w, we can form a new factorisation by acting on the tuple $(r_1, \ldots, r_n)$ by an element of the n-strand braid group B_n. Let $\sigma_1, \ldots, \sigma n - 1$ denote the standard generators for B_n. The Hurwitz action of σ_1 on $(r_1, \ldots, r_n)$ is defined as

$$\sigma_1 \cdot (r_1, \ldots, r_n) := (r_1 r_2 r_1^{-1}, r_1, r_3, \ldots, r_n),$$

the inverse is

$$\sigma_1^{-1} \cdot (r_1, \ldots, r_n) := (r_2, r_2^{-1} r_1 r_2, r_3, \ldots, r_n),$$

and we extend this action similarly to all other $\sigma_k^{\pm 1}$ for $2 \leq k \leq n$. The Hurwitz action generates new R-factorisations because R is closed under conjugation.

In [Bes03], Bessis shows that the Hurwitz action is transitive on all minimal R-factorisations for a Coxeter element $w \in W_\Gamma$, where W_Γ is finite. This was the first published proof, but this was previously proved by Deligne in a letter to Looijenga [Del74]. Igusa and Schiffler generalised this result to all Γ in [IS10] and a relatively concise re-proof of that result can be found in [BDSW14]. The Hurwitz action gives us a completely combinatorial way to enumerate the minimal R-factorisations of a Coxeter element, although to my knowledge no progress has been made tackling this problem using purely combinatorial tools.

Since there is a natural inclusion of S in to the generating set of $A_\Gamma^\vee$, it is sensible to consider the extension of this inclusion, which we will denote $\varphi\colon A_\Gamma \to A_\Gamma^\vee$. It happens that φ is a homomorphism, which is straightforward to see using the Hurwitz action, and is spelled out in [MS17, Proposition 10.1]. So in fact, to understand whether $A_\Gamma \cong A_\Gamma^\vee$, we need to understand whether φ is an injection, in which case A_Γ and $A_\Gamma^\vee$ are in some sense naturally isomorphic.

The importance of the dual Artin group construction in recent proofs solidifies its relevance as an object of study. It gives a different way to study Artin groups, and the programme of proving statements about $A_\Gamma^\vee$, then proving $A_\Gamma^\vee \cong A_\Gamma$ has seen some success as tactic to solve Artin group problems. The general lack of results for very obvious questions around dual Artin groups also adds to the intrigue.

My research focuses on the dual Artin group isomorphism problem. Specifically, I am interested in the class of Γ which have hyperbolic signature, for which W_Γ act naturally on $\mathbb{H}^{|\Gamma|-1}$. This is because rank 4 hyperbolic Coxeter groups (W_Γ acting naturally on $\mathbb{H}^3$) represent a sensible next frontier for the dual Artin group isomorphism problem. I also work a lot with rank 3 hyperbolic Coxeter groups as they provide a more accessible The most frequent setting in which I work is the projectivisation of the Tits cone T, which in this case is a simplicial tiling of Hyperbolic space. In this setting, we explore the geometry of the reflections $R \subseteq W_\Gamma$ with the aim of better understanding $A_\Gamma^\vee$.

2. My research progress

Resteghini's paper [Res24] was released as a preprint in October 2024, so my research began by reading those results. In doing so, I developed a group theoretic construction that would significantly clarify some of his intermediate results, as well as provide a new tool for understanding $A_\Gamma^\vee$.

Let $\mathrm{MinFact}_R(x)$ denote all $r \in R$ that occur in a minimal R-factorisation of a chosen Coxeter element w. It is clear that we can restrict the relations in Definition 1.1 to all loops that start at the identity, go all the way up to w, then go back down to the identity, i.e. we have

$$A_\Gamma^\vee \cong \langle \mathrm{MinFact}_R(x) \mid \{ r_1 \cdots r_n = r_1' \cdots r_n' \mid r_1 \cdots r_n = r_1' \cdots r_n' = w \} \rangle.$$

Using the fact that the Hurwitz action is transitive on such factorisations, we can further restrict this to

$$A_\Gamma^\vee \cong \langle \mathrm{MinFact}_R(x) \mid \{ rsr^{-1} = t \mid rsr^{-1} = t \text{ holds in } W_\Gamma \} \rangle$$

This was noted [Bes04, Lemma 7.11].

The striking form of the these relations warrants investigation of such groups in general. We develop a new construction relevant to this in the following section.

2.1. **Using Languages to choose relations.** Let $F\colon \mathbf{Set} \to \mathbf{Grp}$ be the free functor and denote its action on objects as $F_S \coloneqq F(S)$ and its action on morphisms as $f_* \coloneqq F(f\colon S \to T)$. To each quotient G of F_S we associate the surjective homomorphism $\pi_{(S,G)}\colon F_S \to G$ which is projection. In this context, we can frame a basic group theoretic fact.

Basic fact 2.1. *Suppose $G_1 \cong \langle S_1 \,|\, R_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Given a map $f\colon S_1 \to S_2$, if there exists a map h such that the following diagram commutes, then h is a homomorphism.*

$$\begin{tikzcd} S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "f"] & F_{S_1} \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "f_*"] & G_1 \arrow[d, "h", color=red] \\ S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \end{tikzcd} \tag{1}$$

Now consider (1), but replace F_{S_1} with some subset $Q \subseteq F_{S_1}$. We will now construct a group, which we will denote G^Q, where such a diagram defines a homomorphism from G^Q to G_2.

Definition 2.2 (Group with relations visible in Q)**.** *Suppose we have a group $G \cong \langle S \rangle$. Fix some $Q \subseteq F_S$ such that $S \subseteq Q$. Let $\pi \coloneqq \pi_{(S,G)}$. We have the following maps.*

$$S \xhookrightarrow{\;i\;} Q \xrightarrow{\;\pi\;} \pi(Q)\ .$$

Define the group G with relations visible in Q*, to be*

$$G^Q \coloneqq \langle \pi(Q) \mid \{\pi(q) = (\pi \circ i)_*(q) \mid q \in Q\} \rangle.$$

Do not think of the generators $\pi(Q)$ as elements of G. They are abstract generators. Specifically, G^Q is a quotient of $F_{\pi(Q)}$. Thus, our relations should be equations in $F_{\pi(Q)}$, which they are. G^Q is generated by $\pi(S) \subseteq \pi(Q)$. G is a quotient of G^Q, which we will explore later.

Lemma 2.3. *Let $G \cong \langle S \rangle$ be a group and consider the following setup.*

$$S \xhookrightarrow{\;i_S\;} F_S \xrightarrow{\;\pi_{(S,G)}\;} G \xhookrightarrow{\;i_G\;} F_G \xrightarrow{\;\pi_{(G,G)}\;} G$$

The following maps are equal.

$$\pi_{(G,G)} \circ i_G \circ \pi_{(S,G)} = \pi_{(G,G)} \circ (\pi_{(S,G)} \circ i_S)_*.$$

Proof. Both maps are homomorphisms which agree with $\pi_{(S,G)}$ on the generating set S. $\square$

Proposition 2.4. *Suppose we have two groups $G_1 \cong \langle S_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Fix some $Q \subseteq F_{S_1}$ such that $S \subseteq Q$. Let $\overline{Q} \coloneqq \pi_{(S_1,G_1)}(Q)$. If in the following diagram (2), if there exists a map f that makes the diagram commute, then there is a* *homomorphism* $h\colon (G_1)^Q \to G_2$.

(2)
$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[d, red, "\exists f?"] \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2
\end{tikzcd}$$

I will include the proof only because I find it quite satisfying.

Proof. Suppose we have such a map f. We use the following commutative diagram to define some inclusion maps and as a reference for the setup.

(3)
$$\begin{tikzcd}
S_1 \arrow[r, hook, "i_{S_1}"] \arrow[d, "\theta"] & Q \arrow[r, "\pi_{(S_1,G_1)}"] \arrow[d, "\theta_*"] & \overline{Q} \arrow[r, hook, "i_{\overline{Q}}"] \arrow[d, "f"] & F_{\overline{Q}} \arrow[d, "f_*"] & \\
S_2 \arrow[r, hook, "i_{S_2}"] & F_{S_2} \arrow[r, "\pi_{(S_2,G_2)}"] & G_2 \arrow[r, hook, "i_{G_2}"] & F_{G_2} \arrow[r, "\pi_{(G_2,G_2)}"] & G_2
\end{tikzcd}$$

We can construct a homomorphism if $\pi_{(G_2,G_2)} \circ f_*(R) = \{1\}$, where R are the relations for G^Q, defined in Definition 2.2. For some $q \in Q$, the corresponding element in R equalling the identity is $i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1}$. Using that $\pi_{(G_2,G_2)} \circ f_*$ is a homomorphism, we have

$$\begin{aligned}
&\pi_{(G_2,G_2)} \circ f_* \left(i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q)(\pi_{(S_1,G_1)} \circ i_1)_*(q)^{-1} \right) = \\
&\left(\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) \right) \left(\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1} \right)_* (q)^{-1} \right).
\end{aligned}$$

We will concentrate on each factor separately.

First we consider the first factor. Since $\pi_{(S_1,G_1)}(q) \in \overline{Q}$, by the rightmost commuting square of (3), we have $f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q)$. This gives us

$$\pi_{(G_2,G_2)} \circ f_* \circ i_{\overline{Q}} \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q).$$

We then use the middle commuting square of (3) to give us

$$\pi_{(G_2,G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1,G_1)}(q) = \pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q).$$

We then use Lemma 2.3 and functoriality, giving us

$$\begin{aligned}
\pi_{(G_2,G_2)} \circ i_{G_2} \circ \pi_{(S_2,G_2)} \circ \theta_*(q) &= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \right)_* \circ \theta_*(q) \\
&= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta \right)_* (q)
\end{aligned}$$

We now concentrate on the second factor. Using functoriality, then the middle commuting square of (3), we get

$$\begin{aligned}
\pi_{(G_2,G_2)} \circ f_* \circ \left(\pi_{(S_1,G_1)} \circ i_{S_1} \right)_* (q)^{-1} &= \pi_{(G_2,G_2)} \circ \left(f \circ \pi_{(S_1,G_1)} \circ i_{S_1} \right)_* (q)^{-1} \\
&= \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1} \right)_* (q)^{-1}.
\end{aligned}$$

Then, we use the leftmost commuting square of (3) to get

$$\pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ \theta_* \circ i_{S_1} \right)_* (q)^{-1} = \pi_{(G_2,G_2)} \circ \left(\pi_{(S_2,G_2)} \circ i_{S_2} \circ \theta \right)_* (q)^{-1}.$$

This is the inverse of the left factor. ∎

Remark 2.5. If we set $Q = S_1 \subseteq F_{S_1}$ (the minimum subset allowed), then G^Q is F_{S_1}. In this case, f always exists (and is θ) and Proposition 2.4 tells us the standard theorem about homomorphisms from the free group.

Remark 2.6. If we set $Q = F_{S_1}$, then G^Q is G, f is g and Proposition 2.4 tells us nothing more than Basic fact 2.1.

We see that if $\theta\colon S_1 \to S_2$ is surjective, then the homomorphism $h\colon (G_1)^Q \to G_2$ resulting from Proposition 2.4 is surjective.

Observing the form of Definition 2.2, it is clear that $\pi_{(S,G)}(S)$ is a generating set for G^Q, that is to say that $\pi_{(\overline{Q},G^Q)} \circ \left(\pi_{(S,G)} \circ i_S\right)_*$ is surjective. If we set $Q = F_S$, then $G^Q \cong G$, so G can be realised as a quotient of G^Q via a surjection $p_Q\colon G^Q \to G$ in a way such that the following diagram commutes.

$$\begin{tikzcd} & F_S \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\ G^Q \arrow[rr, "p_Q"] & & G \end{tikzcd}$$

Furthermore, in the following diagram, it can be shown that the bottom map is injective.

(4)
$$\begin{tikzcd} & Q \arrow[dl, "\pi_{(S,G^Q)}"'] \arrow[dr, "\pi_{(S,G)}"] & \\ \pi_{(S,G^Q)}(Q) \arrow[rr, hook, "p_Q"] & & \pi_{(S,G)}(Q) \end{tikzcd}$$

Lemma 2.7. *Given some group $G \cong \langle S \rangle$ and Q such that $S \subseteq Q \subseteq F_S$, we have that G^Q is isomorphic to the following group by extending the natural identification of generators.*

(5) $$X \coloneqq \langle S \mid \{q_1 = q_2 \mid q_1, q_2 \in Q,\ \pi_{(S,G)}(q_1) = \pi_{(S,G)}(q_2)\} \rangle$$

We could explore this construction a bit further, but I will return to the context of Artin and dual Artin groups. In summary, we have the following.

Given a Coxeter system Γ, with generating set S, let $\overline{w}$ be some product of all the elements of S in F_S, i.e. $\pi_{(S,W_\Gamma)}(\overline{w})$ is a Coxeter element in W_Γ. We call such a $\overline{w}$ a *pre-Coxeter element*. We can consider reflections in F_S as all conjugates of S, we denote this by $\overline{R}$. We can also act on $\overline{R}$-factorisations of $\overline{w}$ by the Hurwitz action all within F_S. Doing so, we get a subset $\overline{R}$ that appear in minimal $\overline{R}$-factorisations of $\overline{w}$, we denote this set by $\mathrm{MinFact}_{\overline{R}}(\overline{w})$.

Using Lemma 2.7, we have the following.

Lemma 2.8. *Given a Coxeter system Γ generated by S and some pre-Coxeter element $\overline{w} \in F_S$. We have*
$$A_\Gamma^\vee \cong W_\Gamma^Q,$$
where $Q = \mathrm{MinFact}_{\overline{R}}(\overline{w})$, obtained by application of the Hurwitz action in F_S.

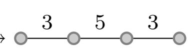
Then, by using Proposition 2.4, we have the following.

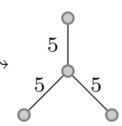
Proposition 2.9. *Suppose we have a Coxeter system Γ with generating set S and some pre-Coxeter element $\overline{w} \in F_S$. Let $Q = \mathrm{MinFact}_{\overline{R}}(\overline{w})$. If in the following* *diagram, the bottom map q (a restriction of the standard projection from A_Γ to W_Γ) is injective, then $A_\Gamma^\vee \cong A_\Gamma$.*

$$\begin{tikzcd} & Q \arrow[dl, "\pi_{(S,A_\Gamma)}"'] \arrow[dr, "\pi_{(S,W_\Gamma)}"] & \\ \pi_{(S,A_\Gamma)}(Q) \arrow[rr, "q"] & & W_\Gamma \end{tikzcd}$$

This presents a new way of tackling the dual Artin group isomorphism problem, and suggests further study of $\mathrm{MinFact}_{\overline{R}}(\overline{w}) \subseteq F_S$.

2.2. **Coxeter arrangements in $\mathbb{H}^3$.** Since the second half of the 1st semester, I have been investigating Coxeter arrangements of rank 3 and 4 Coxeter groups of hyperbolic signature. The nature of this investigation has been informed by the fact that a lot of progress in this field has been made by looking at relevant pictures. Accordingly, I have developed a set of tools to visualise Coxeter arrangements in $\mathbb{H}^2$ and $\mathbb{H}^3$ using Mathematica. Developing such tools also has the advantage of providing tangible examples on which to try ideas, and was a good way to learn hyperbolic geometry. In the following, let G_{353} and G_{555} be Coxeter groups corresponding to the following Γ.

$G_{353} \rightsquigarrow$ 

, $G_{555} \rightsquigarrow$ 

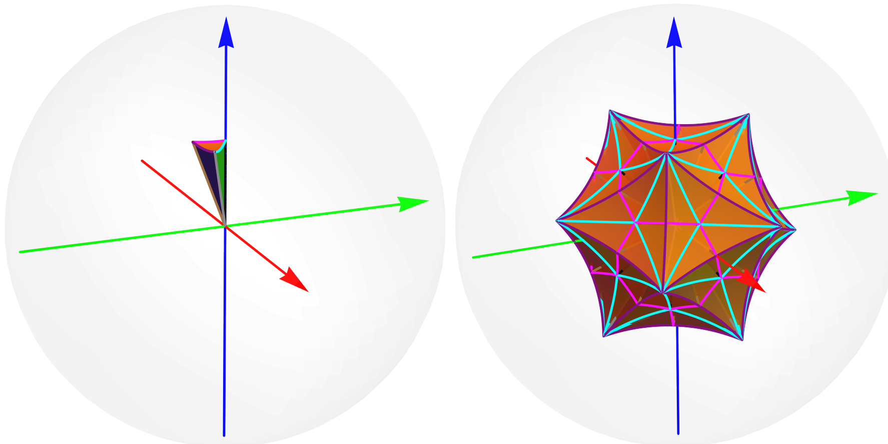
It turns out that G_{353} is the only rank 4 Coxeter system for which the $K(\pi, 1)$ conjecture is open. It also happens to be geometrically well-behaved, in that its fundamental domain in $\mathbb{H}^3$ is compact and of finite volume.

The geometry of a rank 4 hyperbolic Coxeter group is determined by 6 interplanar angles. In Euclidean space, if these interplanar angles do correspond to some tetrahedron, then it is relatively simple to find a specific tetrahedron that realises these angles. In $\mathbb{H}^3$, this is more complicated, since angles are specific scale. We developed a way to symbolically compute tetrahedrons in $\mathbb{H}^3$ corresponding to any appropriate Coxeter group.

We used this to realise a specific model for G_{353}. In Fig. 1b, we show 120 cells (many of which are at the back of the plot and not visible) in the Coxeter arrangement for G_{353} that have a vertex at the origin. The chosen fundamental chamber has an edge along the z-axis in blue. The faces of the fundamental chamber were coloured red, green, blue and orange, and we considered the Coxeter element that corresponded to the product of those faces in that order. The edges of the fundamental chamber were also coloured in a unique way. In Fig. 1b, we only see orange faces since the cells shown are the orbit under the subgroup generated by the red, green and blue faces of the fundamental chamber, and we see 120 cells because that is the order of that parabolic subgroup.

In Fig. 2, we see a cut of the Coxeter arrangement of G_{353} along a plane in $\mathbb{H}^3$. For this, we initially plotted significantly more cells of the Coxeter arrangement than in Fig. 1b. This is an irregular tiling of $\mathbb{H}^2$. In the case of G_{353}, taking different cuts reveals the same tiling up to translation.

In Fig. 3, we see the Coxeter axis corresponding to the product of the red, green, blue and orange planes of the fundamental chamber in that order. Alongside this, we see some cells that lie along the Coxeter axis. Such cells are important as any



(A) Our chosen fundamental chamber for G_{353}.

(B) A part of the Coxeter arrangement for G_{353} consisting of 120 copies of the fundamental chamber that are positioned at the origin.

FIGURE 1. Realising the Coxeter arrangement for G_{353}.

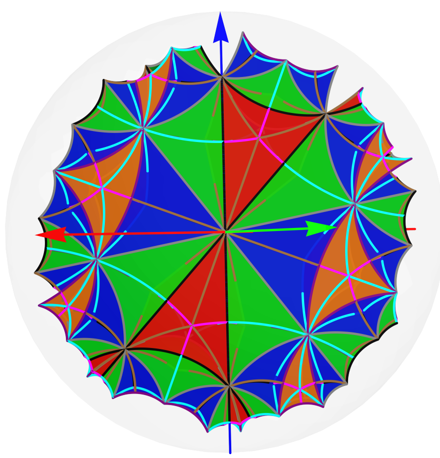


FIGURE 2. A cut of the Coxeter arrangement of G_{353} along a plane in $\mathbb{H}^3$.

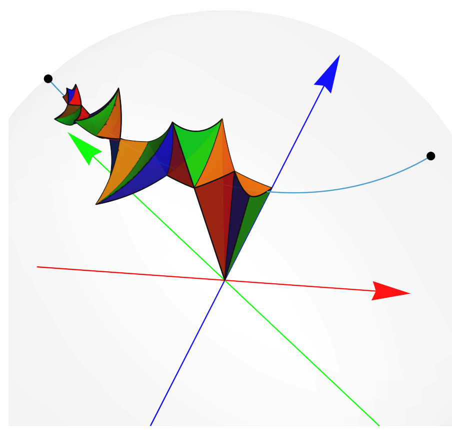


FIGURE 3. The Coxeter axis Corresponding to a Coxeter element in G_{353}, as well as some cells that lie along this Coxeter axis.

reflection factorisation of a Coxeter element that makes a tetrahedron must be one of these cells.

We were not able to use these tools to prove anything about G_{353}, but in developing these plots and the geometric tools needed to make them, I gained a lot of intuition and knowledge for these hyperbolic arrangements. We were also able to compute some closed forms for certain geometric constructions in $\mathbb{H}^2$ and $\mathbb{H}^3$ useful for future calculations. For example, we may denote planes in $\mathbb{H}^3$ as $p = (\mathbf{n}, r)$, where $\mathbf{n}$ and r are a normal vector and non-negative real respectively, such that the circle of intersection of p with the unit sphere is the same as the circle of intersection of the plane $\mathbf{n} \cdot \mathbf{x} = r$ with the unit sphere. In this notation, the reflection of the plane $p_1 = (\mathbf{n_1}, r)$ through $p_2 = (\mathbf{n_2}, s)$ with $\mathbf{n_1} = (x, y, z)$ and $\mathbf{n_2} = (a, b, c)$ is

$$R_{p_2}(p_1) = \Bigg(\mathrm{Sign}\Big(r(1+s^2) - 2s(\mathbf{n_1}\cdot\mathbf{n_2})\Big) \text{ Normalise}\Big[\\
2ars + (1-s^2)x - 2(a^2x + aby + acz), \\
2brs + (1-s^2)y - 2(b^2y + abx + bcz), \\
2crs + (1-s^2)z - 2(c^2z + acx + bcy) \Big], \\
\sqrt{\frac{r^2(1+s^2)^2 - 4rs(1+s^2)(\mathbf{n_1}\cdot\mathbf{n_2}) + 4s^2((-1+c^2)(-1+y^2) + 2acxz + (-1+2c^2)z^2 + 2by(ax+cz) + b^2(-1+2y^2+z^2))}{1 + s^4 - 4rs(\mathbf{n_1}\cdot\mathbf{n_2}) - 4rs^3(\mathbf{n_1}\cdot\mathbf{n_2}) + s^2(2 + 4r^2 - 4y^2 + 8acxz - 4z^2 + 8by(ax+cz) + 4b^2(-1+2y^2+z^2) + 4c^2(-1+y^2+2z^2))}} \Bigg).$$

We also investigated

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