PROOF THAT NON-CROSSING LIFTS OF REFLECTIONS ARE UNIQUE

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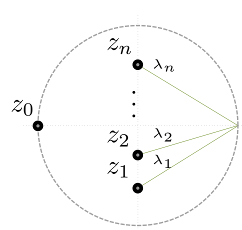
1. Introduction

Let F_n denote the free group of rank n, generated by $A \coloneqq \{a_1, \ldots, a_n\}$. Let U_n denote the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$. This should be thought of as the *universal Coxeter group of rank* n, since all rank n Coxeter groups are naturally a quotient of U_n. Let $S \coloneqq \{s_1, \ldots, s_n\}$ be the natural generating set for U, such that each s_i generates the i^{th} free factor. For the whole of this document, n will be constant and arbitrary, so we will drop n from the above two notations, preferring just F and U.

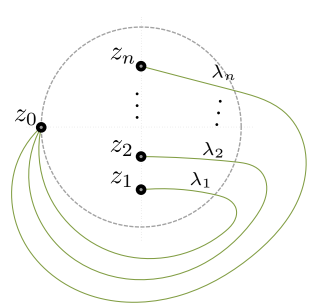
A reflection in F is a conjugate of any f_i in F. Denote the set of all reflections in F by R. Similarly, a reflection in U is a conjugate of any s_i in U. Denote the set of all reflections in U by T. So, a reflection in U is some word us_iu^{-1}, where $u = u_1 \ldots u_k$ is a word in S. We may assume us_iu^{-1} is a reduced word. Thus, since every generator in S is of order 2, we may assume that u is a positive word where each factor has exponent $+1$, i.e. each $u_i \in S$, and each u_i is not equal to u_{i+1}.

Since we will be using this notation a lot, let $s_i : w$ denote ws_iw^{-1}. Let $\varphi\colon F \to U$ be the surjective homomorphism defined by $a_i \mapsto s_i$. A lift of a reflection $s_i : u$ in U is a choice of a positive or negative power for each u_i in u, i.e. a choice of a length $2k+1$ element in $\varphi^{-1}(s_i : u) \cap R$.

Let $\mathbb{C}_n$ denote $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$. Our free group F is $\pi_1(\mathbb{C}_n, z_0)$ for any $z_0 \notin \{z_1, \ldots, z_n\}$. We make the choice of $\{z_1, \ldots, z_n\}$ to be n evenly spaced points on the imaginary axis which are within the unit circle, and $z_0 = -1$. We draw a line λ_i from each z_i to z_0. For reasons that will become clear, we draw these lines in a roundabout way, but then for diagramatic readability, we draw that as on the left diagram below.



equivalent to



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In the following, let each ϵ_i be in $\{\pm 1\}$. A loop l in $\pi_1(\mathbb{C}_n, z_0)$ represents the element $v = f_{\alpha_1}^{\epsilon_1} \cdots f_{\alpha_k}^{\epsilon_k} \in F$, if l passes through each λ_i in the order specified in u, passing from bottom to top if $\epsilon_i = 1$ and top to bottom if $\epsilon_i = -1$. We say that some

$v \in F$ is non-crossing if there exists a non-crossing loop which represents the free reduction of v. One can see pictorially that any non-crossing $u = f_{\alpha_1}^{\epsilon_1} \cdots f_{\alpha_k}^{\epsilon_k} \in F$ must be *square free*. So all $f_{\alpha_i} \neq f_{\alpha_{i+1}}$ and all $\epsilon_k \in \{\pm 1\}$ in this setting.

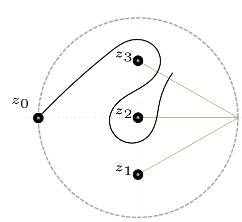
Let l_i represent a loop just around z_i that remains a close distance around z_i for its entirety.

Lemma 1.1. *Let $v \in F$, and let $f_i \in A$, which is the generating set for F. Any non-crossing loop representing $f_i : v$ is homotopy equivalent to $\gamma l_i \gamma^{-1}$, where gamma is a non-crossing representative of $v\Omega$ concatenated with a path close to z_i*

We will want to talk about specific paths in $\mathbb{C}_n$, not just loops up to homotopy. For this, we use the following definition,

Definition 1.2. *Let $\Omega \notin \{z_0, \ldots, z_n\}$ denote an* end symbol*. Let v be some element in F. A path representing $v\Omega$ is a path (not a loop) that starts at z_0, crosses each λ_i in the correct order and orientation as above, but then does not continue back to z_0. Once the path has crossed the final λ_k (in the correct orientation), it stops a short distance away from that crossing.*

We say that $v\epsilon$ is non-crossing if there exists a non-crossing path that represents $u\Omega$. For example, the following is a non-crossing path that represents $f_3^{-1} f_2 f_3 \Omega$.

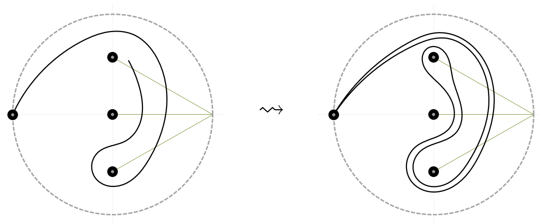


A path (possibly a path or a loop) p in $\mathbb{C}_n$ intersects the λ_i at certain points. We can order these points by closeness to z_n. We say that a crossing of p across λ_i is *close*, if that crossing is the closest to z_i along all of p.

Lemma 1.3. *Let $v \in F$, and let $f_i \in A$, which is the generating set for F. The following are equivalent.*

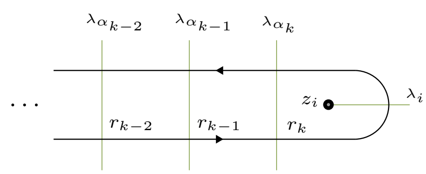
1. *$f_i : v$ is non-crossing.*
2. *There exists a non-crossing path representing $v f_i \Omega$ such that the final crossing of λ_i is close.*
3. *There exists a non-crossing path for $v f_i \Omega$ such that we can alter the path just before and after the central crossing of λ_i, so that this altered path is a non-crossing representative for $v f_i^{-1} \Omega$.*

Proof. We first show (2) $\implies$ (1). To construct a non-crossing path for $f_i : v$, we follow the path given by (2), then turn in a positive orientation around z_i, then follow backwards along our original path, keeping it close on our left as we go back to z_0. We follow this procedure in the following example.



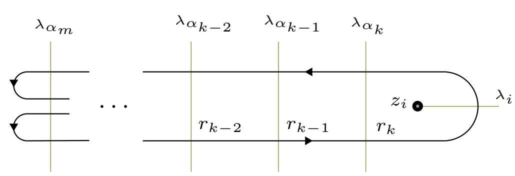
We now show (1) $\implies$ (2). Let $v = f_{\alpha_1}^{\epsilon_1} f_{\alpha_2}^{\epsilon_2} \cdots f_{\alpha_k}^{\epsilon_n}$, which we may assume is freely reduced. Let l be a non-crossing loop representing $f_i : v$. We work from the middle outwards to construct a picture of what l could look like.

Before turning around z_i, l passed through λ_{α_k} and right after turning around z_i, it passed through λ_{α_k} in the opposite direction. Similarly, it passed through $\lambda_{\alpha_{k-1}}$ just before passing through λ_{α_k}, and just after in the opposite direction, and so on. This gives us the following picture, where l is the line in black.



This picture is misleading in many ways. For instance, there is no reason to assume that $\lambda_x \neq \lambda_y$ for $x \neq y$. We have also neglected to show whether each z_{α_j} is above or below each λ_{α_j}. This is because we do not know any of the values of ϵ_j. However, the important features of this picture are the regions, denoted $r_0, \dots, r_k$ which are bounded by l and the λ lines. By working from the centre of $f_i : v$, we can see that these regions exist exactly as they are depicted in the above picture.

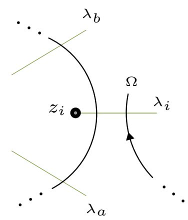
Now, assume that somewhere in l, there is a turn closer to z_i, which occurs inside region r_k. To reach region r_k, we must go via the regions r_{k-1}, r_{k-2} and so on. Suppose we turn back inside at index m, as depicted in the following picture.



This would involve both ends of l going back in to r_m, having just passed out of region r_m. The only way this could happen is if there was a subword $f_{\alpha_m} f_{\alpha_m}^{-1}$ or $f_{\alpha_m}^{-1} f_{\alpha_m}$ somewhere in l. We know this is not the case because v is a reduced word.

We now show (2) $\implies$ (3). Suppose a non-crossing path p representing $v f_i \Omega$ such that the final crossing of λ_i is close. Since the crossing is close, we can alter the last part of the path path by moving the crossing point arbitrarily close to z_i. We then move the path so that it goes to the left of z_i and passes back through λ_i from top to bottom.

We finally show (3) $\implies$ (2). Suppose that all non-crossing paths representing $v f_i \Omega$ have a non-close final crossing of λ_i. Then at their ends, all non-crossing paths representing $v f_i \Omega$ look like the following picture, where Ω signifies the end of the path.



Call the path p. The lines λ_a and λ_b are the lines that occur adjacent to the closer crossing of λ_i. Let p_{close} denote the segment of p that is between λ_a and λ_b. It is possible that $\lambda_a = \lambda_b$, but this case does not cause issues for our argument. Irrespective of the orientation of the crossing of λ_a and λ_b by p, the end of p is inside a region bounded by p_{close}, $_a$, and λ_b. This region does not contain z_i. We see that there is no way to alter the end of the path so that we cross λ_i from top to bottom at the end of the path, unless we cross λ_a or λ_b first. □

Lemma 1.4. *Let $v \in F$ and let $f_i \in A$. The following are equivalent.*

(1)

Theorem 1.5. *Let $u \in U$ and $s_i \in S$. If there is a non-crossing lift $f_i : v$ for $s_i : u$, then it is unique.*

Proof. We do this by inducting on the length of u, which we denote $\ell(u)$. The statement is vacuous when $\ell(u) = 0$. Assume this is true for all s_i and $\ell(u_0) < k$. Now suppose we have some reflection $s_i : u$ with $u = s_{\alpha_1} \cdots s_{\alpha_k}$. And suppose this reflection has a non-crossing lift. There exists a maximal strict prefix $u' = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_j}$ of u such that $s_{\alpha_j} : s_{\alpha_1} \cdots s_{\alpha_{j-1}}$ has a non-crossing lift. By induction and by Lemma 1.3, there is a unique non-crossing lift for $u_1 \cdots u_j \Omega$. Thus, to complete our proof, we need to show that the choices of travelling positively or negatively around $u_j, \ldots, u_k$ are unique. We have a choice of how to travel around □

1.1. **The conjecture and the objects involved.** Coxeter groups emerge as generalisations of reflection groups. A Coxeter group is defined by a particular group presentation. The data of this presentation is typically encoded by a labelled graph. The group W, coupled with the data of its presentation is called a Coxeter system, denoted (W, S) where S is the generating set of W. Given a Coxeter system (W, S), we can construct a different group G_W, called the Artin group associated to W.

For affine Coxeter groups W, the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of $\mathrm{Isom}(\mathbb{E})$, the group of isometries on a Euclidean space $\mathbb{E}$. We will consider $\mathbb{E}$ as $\mathbb{R}^n$ without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S. Within W, we consider the set of all reflections R (not necessarily finite). To each reflection $r \in R$ there is a corresponding codimension–1 space $H_r \subset \mathbb{E}$ that is the plane of reflection of r. We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of $\mathbb{R}^n$.

The configuration space is realised as the complement of the complexification of all such hyperplanes H_r. It is known by work of Brieskorn [Bri71] that the fundamental group of Y_W is G_W. Thus, proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [Sal87, Sal94], there is a CW–complex X_W called the Salvetti complex that is homotopy equivalent to Y_W. Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W. Because of this, the Salvetti complex is the starting point in a chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W. This is a space realised using a certain poset structure on subsets of W. To this poset structure there is an associated group called the dual Artin group, denoted W_w. It was already known (by a now standard construction due to Garside [Gar69], extended by other authors, see [CMW02]) that K_W was a classifying space for the dual Artin group for finite W. In [PS21], the authors extend this result to affine W. Thus, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that $W_w \cong G_W$.

In the following section, we will identify the intermediate spaces used in proving $X_W \simeq K_W$.

1.2. **Proof overview.** Here we will compile several main results from [PS21] in to two theorems. The concern of this work is Theorem 1.6 which proves that the *Salvetti complex* X_W is homotopy equivalent to the *interval complex* K_W. A *Coxeter element* is a non–repeating product of all the elements of S. A choice of order on S corresponds to a choice of Coxeter element. Constructing an interval complex associated to (W, S) involves making such a choice of Coxeter element $w \in W$.

For a subset $T \subseteq S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T and only with relations explicitly involving elements of T. A *parabolic Coxeter element* w_T is a product of all elements of T that respects the order of multiplication in a Coxeter element $w \in W$. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements w_T with $T \subseteq S$ such that W_T is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W. Given a CW–complex X, we can encode some information of how cells of X attach to each other in a poset called the *face poset* of X, denoted $\mathcal{F}(X)$. Connected components of preimages $\eta^{-1}(d)$ of a certain poset map $\eta\colon K_W \to \mathbb{N}$ have a linear structure as subposets of $\mathcal{F}(K_W)$. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

Theorem 1.6 ([PS21]). *Given an affine Coxeter system (W, S), the configuration space Y_W is homotopy equivalent to the order complex K_W.*

Proof. By **??** the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$X_W \overset{(a)}{\simeq} X'_W \overset{(b)}{\simeq} K'_W \overset{(c)}{\simeq} K_W \tag{1}$$

Where the results are gathered from the following sources:

(a) **??** [PS21, Theorem 5.5]
(b) **??** [PS21, Theorem 8.14] □
(c) **??** [PS21, Theorem 7.9]

In [PS21], another main result is shown.

Theorem 1.7 ([PS21, Theorem 6.6]). *Given an affine Coxeter system* (W, S)*, corresponding affine type Artin group* G_W *and Coxeter element* $w \in W$*, the complex* K_W *is a classifying space for the dual Artin group* W_w*.*

We have that $\pi_1(Y_W) \cong G_W$ by [Bri71]. Thus, considering $\pi_1(Y_W)$ and combining Theorems 1.6 and 1.7 gives

$$Y_W \simeq K(G_W, 1)$$
$$G_W \cong W_w$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type Artin groups [Bes03] and affine Artin groups [MS17].

References

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