PROOF THAT NON-CROSSING LIFTS ARE UNIQUE

SEAN O'BRIEN

1. Introduction

Let F_n denote the free group of rank n, generated by $A \coloneqq \{a_1, \dots, a_n\}$. Let U_n denote the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$. This should be thought of as the *universal Coxeter group of rank n*, since all rank n Coxeter groups are naturally a quotient of U_n. Let $S \coloneqq \{s_1, \dots, s_n\}$ be the natural generating set for U, such that each s_i generates the i^{th} free factor. For the whole of this document, n will be constant and arbitrary, so we will drop n from the above two notations, preferring just F and U.

A reflection in F is a conjugate of any f_i in F. Denote the set of all reflections in F by R. Similarly, a reflection in U is a conjugate of any s_i in U. Denote the set of all reflections in U by T. So, a reflection in U is some word wu_iw^{-1}, where $w = u_1 \dots u_k$ is a word in S. We may assume ws_iw^{-1} is a reduced word. Thus, since every generator in S is of order 2, we may assume that w is a positive word where each factor has exponent $+1$, i.e. each $u_i \in S$, and each u_i is not equal to u_{i+1}.

Since we will be using this notation a lot, let $s_i : w$ denote ws_iw^{-1}. Let $\varphi\colon F \to U$ be the surjective homomorphism defined by $a_i \mapsto u_i$. A lift of a reflection $s_i : w$ in U is a choice of a positive or negative power for each u_i in w, i.e. a choice of a length $2k+1$ element in $\varphi^{-1}(s_i : w) \cap R$.

Let $\mathbb{C}_n$ denote $\mathbb{C} \setminus \{z_1, \dots, z_n\}$. Our free group F is $\pi_1(\mathbb{C}_n, z_0)$ for any $z_0 \notin \{z_1, \dots, z_n\}$. We make the choice of $\{z_1, \dots, z_n\}$ to be n evenly spaced points on the imaginary axis and within the unit circle, and $z_0 = -1$. We draw a line λ_i from each z_i to z_0. For reasons that will become clear, we draw these lines in a roundabout way.

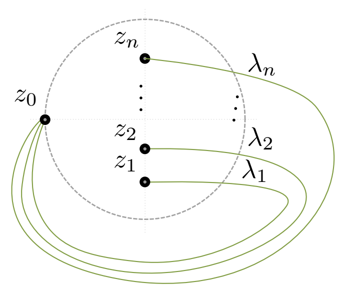

Figure 1. Arrangement of $n+1$ points on the sphere, illustrating $\tilde{\mathbb{C}}_n$.

We should think of these lines meeting z_0 somehow, as in the We can put these points on the sphere.

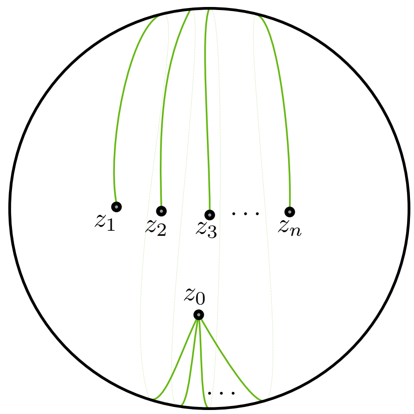

FIGURE 2. Arrangement of $n+1$ points on the sphere, illustrating $\tilde{\mathbb{C}}_n$.

1.1. **The conjecture and the objects involved.** Coxeter groups emerge as generalisations of reflection groups. A Coxeter group is defined by a particular group presentation. The data of this presentation is typically encoded by a labelled graph. The group W, coupled with the data of its presentation is called a Coxeter system, denoted (W, S) where S is the generating set of W. Given a Coxeter system (W, S), we can construct a different group G_W, called the Artin group associated to W.

For affine Coxeter groups W, the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of $\mathrm{Isom}(\mathbb{E})$, the group of isometries on a Euclidean space $\mathbb{E}$. We will consider $\mathbb{E}$ as $\mathbb{R}^n$ without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S. Within W, we consider the set of all reflections R (not necessarily finite). To each reflection $r \in R$ there is a corresponding codimension–1 space $H_r \subset \mathbb{E}$ that is the plane of reflection of r. We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of $\mathbb{R}^n$.

The configuration space is realised as the complement of the complexification of all such hyperplanes H_r. It is known by work of Brieskorn [Bri71] that the fundamental group of Y_W is G_W. Thus, proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [Sal87, Sal94], there is a CW–complex X_W called the Salvetti complex that is homotopy equivalent to Y_W. Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W. Because of this, the Salvetti complex is the starting point in a chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W. This is a space realised using a certain poset structure on subsets of W. To this poset structure there is an associated group called the dual Artin group, denoted W_w. It was already known (by a now standard construction due to Garside [Gar69], extended by other authors, see [CMW02]) that K_W was a classifying space for the dual Artin group for finite W. In [PS21], the authors extend this result to affine W. Thus, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that $W_w \cong G_W$.

In the following section, we will identify the intermediate spaces used in proving $X_W \simeq K_W$.

1.2. **Proof overview.** Here we will compile several main results from [PS21] in to two theorems. The concern of this work is Theorem 1.1 which proves that the *Salvetti complex* X_W is homotopy equivalent to the *interval complex* K_W. A *Coxeter element* is a non–repeating product of all the elements of S. A choice of order on S corresponds to a choice of Coxeter element. Constructing an interval complex associated to (W, S) involves making such a choice of Coxeter element $w \in W$.

For a subset $T \subseteq S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T and only with relations explicitly involving elements of T. A *parabolic Coxeter element* w_T is a product of all elements of T that respects the order of multiplication in a Coxeter element $w \in W$. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements w_T with $T \subseteq S$ such that W_T is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W. Given a CW–complex X, we can encode some information of how cells of X attach to each other in a poset called the *face poset* of X, denoted $\mathcal{F}(X)$. Connected components of preimages $\eta^{-1}(d)$ of a certain poset map $\eta\colon K_W \to \mathbb{N}$ have a linear structure as subposets of $\mathcal{F}(K_W)$. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

Theorem 1.1 ([PS21])**.** *Given an affine Coxeter system (W, S), the configuration space Y_W is homotopy equivalent to the order complex K_W.*

Proof. By **??** the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$X_W \overset{(a)}{\simeq} X'_W \overset{(b)}{\simeq} K'_W \overset{(c)}{\simeq} K_W \tag{1}$$

Where the results are gathered from the following sources:
(a) **??** [PS21, Theorem 5.5]
(b) **??** [PS21, Theorem 8.14]
(c) **??** [PS21, Theorem 7.9] □

In [PS21], another main result is shown.

Theorem 1.2 ([PS21, Theorem 6.6])**.** *Given an affine Coxeter system (W, S), corresponding affine type Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual Artin group W_w.*

We have that $\pi_1(Y_W) \cong G_W$ by [Bri71]. Thus, considering $\pi_1(Y_W)$ and combining Theorems 1.1 and 1.2 gives

$$Y_W \simeq K(G_W, 1)$$
$$G_W \cong W_w$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type Artin groups [Bes03] and affine Artin groups [MS17].

References

[Bes03] David Bessis. The dual braid monoid. *Annales Scientifiques de l'École Normale Supérieure*, 36(5):647–683, September 2003.

[Bri71] E. Brieskorn. Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. *Inventiones mathematicae*, 12(1):57–61, March 1971.

[CMW02] Ruth Charney, John Meier, and Kim Whittlesey. Bestvina's normal form complex and the homology of Garside groups, March 2002.

[Gar69] Frank Arnold Garside. The braid group and other groups. *The Quarterly Journal of Mathematics*, 20(1):235–254, January 1969.

[MS17] Jon McCammond and Robert Sulway. Artin groups of Euclidean type. *Inventiones mathematicae*, 210(1):231–282, October 2017.

[PS21] Giovanni Paolini and Mario Salvetti. Proof of the K(pi,1) conjecture for affine Artin groups. *Inventiones mathematicae*, 224(2):487–572, May 2021.

[Sal87] M. Salvetti. Topology of the complement of real hyperplanes in ℂN. *Inventiones mathematicae*, 88(3):603–618, October 1987.

[Sal94] Mario Salvetti. The Homotopy Type of Artin Groups. *Mathematical Research Letters*, 1(5):565–577, September 1994.

Email address: `2812920O@student.gla.ac.uk`