PROOF THAT NON-CROSSING LIFTS ARE UNIQUE

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1. Introduction

We can put these points on the sphere.

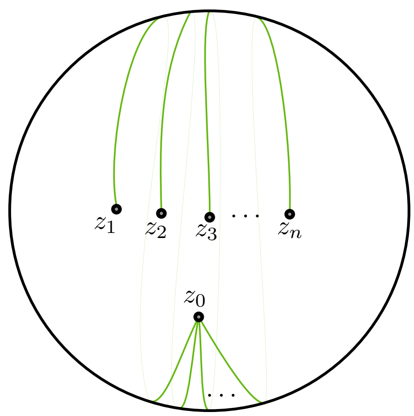

Figure 1. Arrangement of $n+1$ points on the sphere, illustrating $\tilde{\mathbb{C}}_n$.

1.1. **The conjecture and the objects involved.** Coxeter groups emerge as generalisations of reflection groups. A Coxeter group is defined by a particular group presentation. The data of this presentation is typically encoded by a labelled graph. The group W, coupled with the data of its presentation is called a Coxeter system, denoted (W, S) where S is the generating set of W. Given a Coxeter system (W, S), we can construct a different group G_W, called the Artin group associated to W.

For affine Coxeter groups W, the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of Isom($\mathbb{E}$), the group of isometries on a Euclidean space $\mathbb{E}$. We will consider $\mathbb{E}$ as $\mathbb{R}^n$ without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S. Within W, we consider the set of all reflections R (not necessarily finite). To each reflection $r \in R$ there is a corresponding codimension–1 space $H_r \subset \mathbb{E}$ that is the plane of reflection of r. We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of $\mathbb{R}^n$.

The configuration space is realised as the complement of the complexification of all such hyperplanes H_r. It is known by work of Brieskorn [Bri71] that the fundamental group of Y_W is G_W. Thus, proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [Sal87, Sal94], there is a CW–complex X_W called the Salvetti complex that is homotopy equivalent to Y_W. Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W. Because of this, the Salvetti complex is the starting point in a chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W. This is a space realised using a certain poset structure on subsets of W. To this poset structure there is an associated group called the dual Artin group, denoted W_w. It was already known (by a now standard construction due to Garside [Gar69], extended by other authors, see [CMW02]) that K_W was a classifying space for the dual Artin group for finite W. In [PS21], the authors extend this result to affine W. Thus, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that $W_w \cong G_W$.

In the following section, we will identify the intermediate spaces used in proving $X_W \simeq K_W$.

1.2. **Proof overview.** Here we will compile several main results from [PS21] in to two theorems. The concern of this work is Theorem 1.1 which proves that the *Salvetti complex* X_W is homotopy equivalent to the *interval complex* K_W. A *Coxeter element* is a non–repeating product of all the elements of S. A choice of order on S corresponds to a choice of Coxeter element. Constructing an interval complex associated to (W, S) involves making such a choice of Coxeter element $w \in W$.

For a subset $T \subseteq S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T and only with relations explicitly involving elements of T. A *parabolic Coxeter element* w_T is a product of all elements of T that respects the order of multiplication in a Coxeter element $w \in W$. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements w_T with $T \subseteq S$ such that W_T is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W. Given a CW–complex X, we can encode some information of how cells of X attach to each other in a poset called the *face poset* of X, denoted $\mathcal{F}(X)$. Connected components of preimages $\eta^{-1}(d)$ of a certain poset map $\eta\colon K_W \to \mathbb{N}$ have a linear structure as subposets of $\mathcal{F}(K_W)$. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

Theorem 1.1 ([PS21])**.** *Given an affine Coxeter system (W, S), the configuration space Y_W is homotopy equivalent to the order complex K_W.*

Proof. By **??** the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$X_W \overset{\text{(a)}}{\simeq} X'_W \overset{\text{(b)}}{\simeq} K'_W \overset{\text{(c)}}{\simeq} K_W \tag{1}$$

Where the results are gathered from the following sources:

(a) **??** [PS21, Theorem 5.5]
(b) **??** [PS21, Theorem 8.14] □
(c) **??** [PS21, Theorem 7.9]

In [PS21], another main result is shown.

Theorem 1.2 ([PS21, Theorem 6.6])**.** *Given an affine Coxeter system* (W, S)*, corresponding affine type Artin group* G_W *and Coxeter element* $w \in W$*, the complex* K_W *is a classifying space for the dual Artin group* W_w*.*

We have that $\pi_1(Y_W) \cong G_W$ by [Bri71]. Thus, considering $\pi_1(Y_W)$ and combining Theorems 1.1 and 1.2 gives

$$Y_W \simeq K(G_W, 1)$$
$$G_W \cong W_w$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type Artin groups [Bes03] and affine Artin groups [MS17].

References

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