

PROOF THAT NON-CROSSING LIFTS OF REFLECTIONS ARE UNIQUE

SEAN O'BRIEN

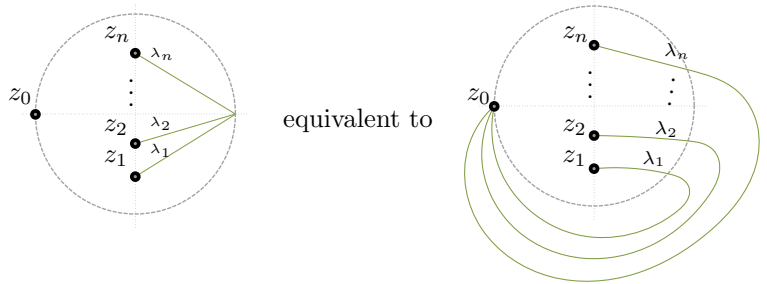
1. INTRODUCTION

Let F_n denote the free group of rank n , generated by $A := \{a_1, \dots, a_n\}$. Let U_n denote the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$. This should be thought of as the *universal Coxeter group of rank n* , since all rank n Coxeter groups are naturally a quotient of U_n . Let $S := \{s_1, \dots, s_n\}$ be the natural generating set for U , such that each s_i generates the i^{th} free factor. For the whole of this document, n will be constant and arbitrary, so we will drop n from the above two notations, preferring just F and U .

A reflection in F is a conjugate of any f_i in F . Denote the set of all reflections in F by R . Similarly, a reflection in U is a conjugate of any s_i in U . Denote the set of all reflections in U by T . So, a reflection in U is some word us_iu^{-1} , where $u = u_1 \dots u_k$ is a word in S . We may assume us_iu^{-1} is a reduced word. Thus, since every generator in S is of order 2, we may assume that u is a positive word where each factor has exponent $+1$, i.e. each $u_i \in S$, and each u_i is not equal to u_{i+1} .

Since we will be using this notation a lot, let $s_i : w$ denote ws_iw^{-1} . Let $\varphi : F \rightarrow U$ be the surjective homomorphism defined by $a_i \mapsto s_i$. A lift of a reflection $s_i : u$ in U is a choice of a positive or negative power for each u_i in u , i.e. a choice of a length $2k + 1$ element in $\varphi^{-1}(s_i : u) \cap R$.

Let \mathbb{C}_n denote $\mathbb{C} \setminus \{z_1, \dots, z_n\}$. Our free group F is $\pi_1(\mathbb{C}_n, z_0)$ for any $z_0 \notin \{z_1, \dots, z_n\}$. We make the choice of $\{z_1, \dots, z_n\}$ to be n evenly spaced points on the imaginary axis which are within the unit circle, and $z_0 = -1$. We draw a line λ_i from each z_i to z_0 . For reasons that will become clear, we draw these lines in a roundabout way, but then for diagrammatic readability, we draw that as on the left diagram below.



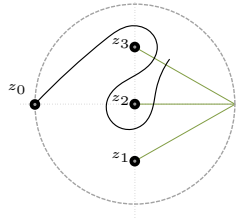
In the following, let each ϵ_i be in $\{\pm 1\}$. A loop l in $\pi_1(\mathbb{C}_n, z_0)$ represents the element $v = f_{\alpha_1}^{\epsilon_1} \dots f_{\alpha_k}^{\epsilon_k} \in F$, if l passes through each λ_i in the order specified in u , passing from bottom to top if $\epsilon_i = 1$ and top to bottom if $\epsilon_i = -1$. We say that

some $v \in F$ is non-crossing if there exists a non-crossing loop which represents v . One can see pictorially that any non-crossing $u = f_{\alpha_1}^{\epsilon_1} \dots f_{\alpha_k}^{\epsilon_k} \in F$ must be *square free*. So all $f_{\alpha_i} \neq f_{\alpha_{i+1}}$ and all $\epsilon_k \in \{\pm 1\}$ in this setting.

We will want to talk about specific paths in \mathbb{C}_n , not just loops up to homotopy. For this, we use the following definition,

Definition 1.1. Let $\Omega \notin \{z_0, \dots, z_n\}$ denote an end symbol. Let v be some element in F . A path representing $v\Omega$ is a path (not a loop) that starts at z_0 , crosses each λ_i in the correct order and orientation as above, but then does not continue back to z_0 . Once the path has crossed the final λ_k (in the correct orientation), it stops a short distance away from that crossing.

We say that $v\epsilon$ is non-crossing if there exists a non-crossing path that represents $u\Omega$. For example, the following is a non-crossing path that represents $f_3^{-1}f_2f_3\Omega$.

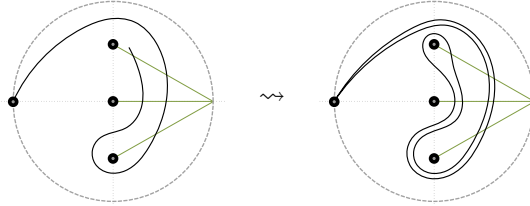


A path (possibly a path or a loop) p in \mathbb{C}_n intersects the λ_i at certain points. We can order these points by closeness to z_n . We say that a crossing of p across λ_i is *close*, if that crossing is the closest to z_i along all of p .

Lemma 1.2. Let $v \in F$, and let $f_i \in A$, which is the generating set for F . The following are equivalent.

- (1) $f_i : v$ is non-crossing.
- (2) There is a non-crossing path representing $vf_i\Omega$ such that the final crossing of λ_i is close.

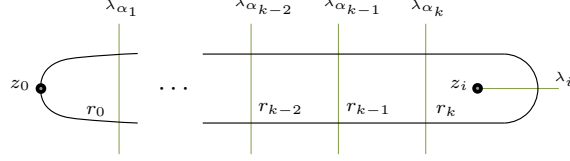
Proof. (2) \implies (1) is simple. To construct a non-crossing path for $f_i : v$, we follow the path given by (2), then turn in a positive orientation around z_i , then follow backwards along our original path, keeping it close on our left as we go back to z_0 . We follow this procedure in the following example.



We now show (1) \implies (2). Let $v = f_{\alpha_1}^{\epsilon_1} f_{\alpha_2}^{\epsilon_2} \dots f_{\alpha_k}^{\epsilon_k}$. Let l be a non-crossing loop representing $f_i : v$. We work from the middle outwards to construct a picture of what l could look like.

Before turning around z_i , l passed through λ_{α_k} and right after turning around z_i , it passed through λ_{α_k} in the opposite direction. Similarly, it passed through $\lambda_{\alpha_{k-1}}$ just before passing through λ_{α_k} , and just after in the opposite direction, and

so on. This gives us the following picture, where l is the submarine shaped line in black.



This picture is misleading in many ways. For instance, there is no reason to assume that $\lambda_x \neq \lambda_y$ for $x \neq y$. However, the important features of this picture are the regions, denoted r_0, \dots, r_k which are bounded by l and the λ lines. By the above argument, these regions exist exactly as they are depicted in the above picture, in that there are no features inside these regions, apart from z_i inside r_k .

Now assume there is a part of l that crosses λ_i closer to z_i . Being non crossing, this part of l is square free, so must exit (on both ends) the region r_k through λ_{α_k} . Then both ends of this part of l are in r_{k-1} , and by the same argument both ends must exit r_{k-1} through $\lambda_{\alpha_{k-1}}$, and so on, leading to a contradiction. \square

Theorem 1.3. *Let $u \in U$ and $s_i \in S$. If there is a non-crossing lift $f_i : v$ for $s_i : u$, then it is unique.*

Proof. We do this by inducting on the length of u , which we denote $\ell(u)$. The statement is vacuous when $\ell(u) = 0$. Assume this is true for all s_i and $\ell(u_0) < k$. Now suppose we have some reflection $s_i : u$ with $u = u_1 \cdots u_{k+1}$ where each $u_i \in S$. And suppose this reflection has a non-crossing lift. There exists a maximal strict prefix $u' = u_1 u_2 \cdots u_j$ of u such that $u_j : u_1 \cdots u_{j-1}$ has a non-crossing lift. By induction and by Lemma 1.2, there is a unique non-crossing lift for $u_1 \cdots u_j \Omega$. \square

1.1. The conjecture and the objects involved. Coxeter groups emerge as generalisations of reflection groups. A Coxeter group is defined by a particular group presentation. The data of this presentation is typically encoded by a labelled graph. The group W , coupled with the data of its presentation is called a Coxeter system, denoted (W, S) where S is the generating set of W . Given a Coxeter system (W, S) , we can construct a different group G_W , called the Artin group associated to W .

For affine Coxeter groups W , the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of $\text{Isom}(\mathbb{E})$, the group of isometries on a Euclidean space \mathbb{E} . We will consider \mathbb{E} as \mathbb{R}^n without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S . Within W , we consider the set of all reflections R (not necessarily finite). To each reflection $r \in R$ there is a corresponding codimension-1 space $H_r \subset \mathbb{E}$ that is the plane of reflection of r . We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of \mathbb{R}^n .

The configuration space is realised as the complement of the complexification of all such hyperplanes H_r . It is known by work of Brieskorn [Bri71] that the fundamental group of Y_W is G_W . Thus, proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [Sal87, Sal94], there is a CW-complex X_W called the Salvetti complex that is homotopy equivalent to Y_W . Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W . Because of this, the Salvetti complex is the starting point in a chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W . This is a space realised using a certain poset structure on subsets of W . To this poset structure there is an associated group called the dual Artin group, denoted W_w . It was already known (by a now standard construction due to Garside [Gar69], extended by other authors, see [CMW02]) that K_W was a classifying space for the dual Artin group for finite W . In [PS21], the authors extend this result to affine W . Thus, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that $W_w \cong G_W$.

In the following section, we will identify the intermediate spaces used in proving $X_W \simeq K_W$.

1.2. Proof overview. Here we will compile several main results from [PS21] in to two theorems. The concern of this work is Theorem 1.4 which proves that the *Salveti complex* X_W is homotopy equivalent to the *interval complex* K_W . A *Coxeter element* is a non-repeating product of all the elements of S . A choice of order on S corresponds to a choice of Coxeter element. Constructing an interval complex associated to (W, S) involves making such a choice of Coxeter element $w \in W$.

For a subset $T \subseteq S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T and only with relations explicitly involving elements of T . A *parabolic Coxeter element* w_T is a product of all elements of T that respects the order of multiplication in a Coxeter element $w \in W$. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements w_T with $T \subseteq S$ such that W_T is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W . Given a CW-complex X , we can encode some information of how cells of X attach to each other in a poset called the *face poset* of X , denoted $\mathcal{F}(X)$. Connected components of preimages $\eta^{-1}(d)$ of a certain poset map $\eta: K_W \rightarrow \mathbb{N}$ have a linear structure as subposets of $\mathcal{F}(K_W)$. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

Theorem 1.4 ([PS21]). *Given an affine Coxeter system (W, S) , the configuration space Y_W is homotopy equivalent to the order complex K_W .*

Proof. By ?? the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W . Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$(1) \quad X_W \stackrel{(a)}{\simeq} X'_W \stackrel{(b)}{\simeq} K'_W \stackrel{(c)}{\simeq} K_W$$

Where the results are gathered from the following sources:

- (a) ?? [PS21, Theorem 5.5]
- (b) ?? [PS21, Theorem 8.14]
- (c) ?? [PS21, Theorem 7.9]

□

In [PS21], another main result is shown.

Theorem 1.5 ([PS21, Theorem 6.6]). *Given an affine Coxeter system (W, S) , corresponding affine type Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual Artin group W_w .*

We have that $\pi_1(Y_W) \cong G_W$ by [Bri71]. Thus, considering $\pi_1(Y_W)$ and combining Theorems 1.4 and 1.5 gives

$$\begin{aligned} Y_W &\simeq K(G_W, 1) \\ G_W &\cong W_w \end{aligned}$$

for affine G_W .

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type Artin groups [Bes03] and affine Artin groups [MS17].

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Email address: 28129200@student.gla.ac.uk