Motion in Two or Three Dimensions

Topic 1b

A/P Patrick Chua

Learning Outcomes for Topic 1b

- Using vectors to represent the position and velocity of a particle in two or three dimensions.
- Finding the vector acceleration of a particle and interpreting the components of acceleration parallel to and perpendicular to a particle's path.
- Solving problems that involve the curved path followed by a projectile.
- Relating the velocities of a moving body as seen from two different frames of reference.

Overview of Topic 1b

- Introduction
- Position and velocity vectors
- Velocity
- Acceleration
- Components of acceleration
- Projectiles
- Relative velocity

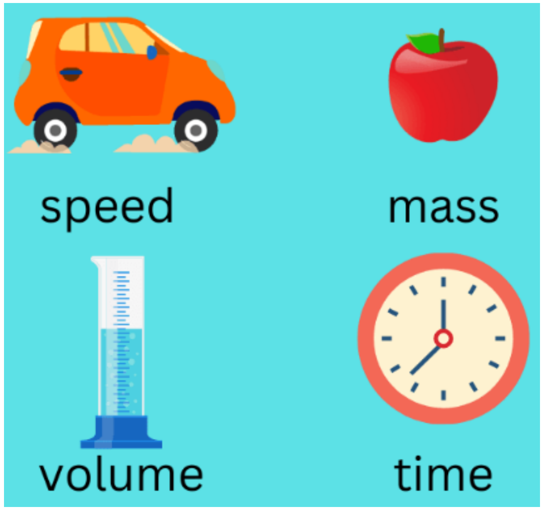
Introduction

- How do we describe the motion of a particle along a curved track?
- We need to extend our description of motion along a straight line to two and three dimensions.
- We will be merging motion along a straight line with vectors to describe motion along 2 or 3 dimensions, that is, in a plane or in space.

2D 3D

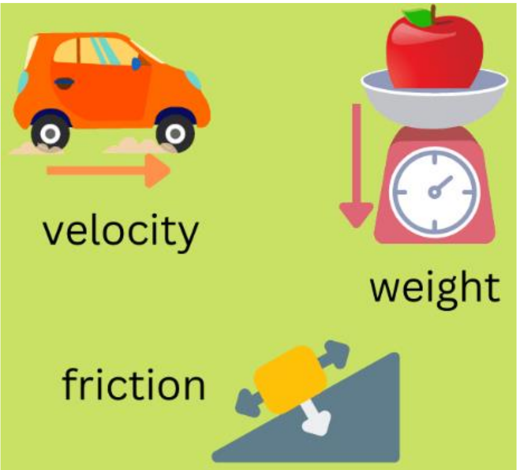
Scalars and vectors

Scalars and vectors are two kinds of quantities that are commonly used in physics and mathematics. Scalars are quantities that only have magnitude (or size), while vectors have both magnitude and direction. Some examples of scalars and vectors are as follows.



Scalars include : distance, length, speed, pressure, energy, temperature, time, mass, volume, density, heat, electrical resistance.

Vectors include : displacement (linear and angular), velocity, acceleration, force, torque, momentum, weight, gravity.

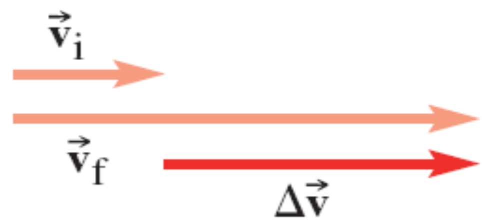


Velocity vectors

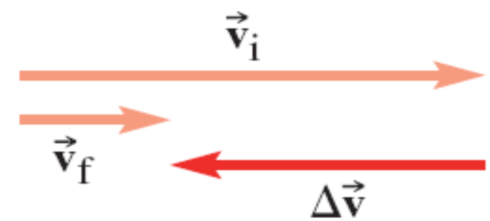
The direction of the change in velocity $\Delta\vec{v}$ is not necessarily the same as either the initial or the final velocity direction.

1. Change in speed

Increasing speed without changing direction

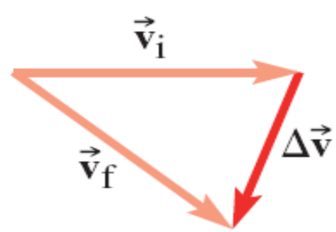


Decreasing speed without changing direction



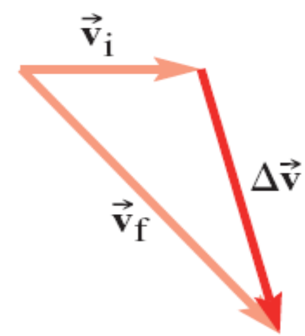
There are 3 types of changes in velocity i.e. a change in speed (magnitude), a change in direction, or a change in both.

Turning while keeping speed constant



2. Change in direction

Turning while increasing speed

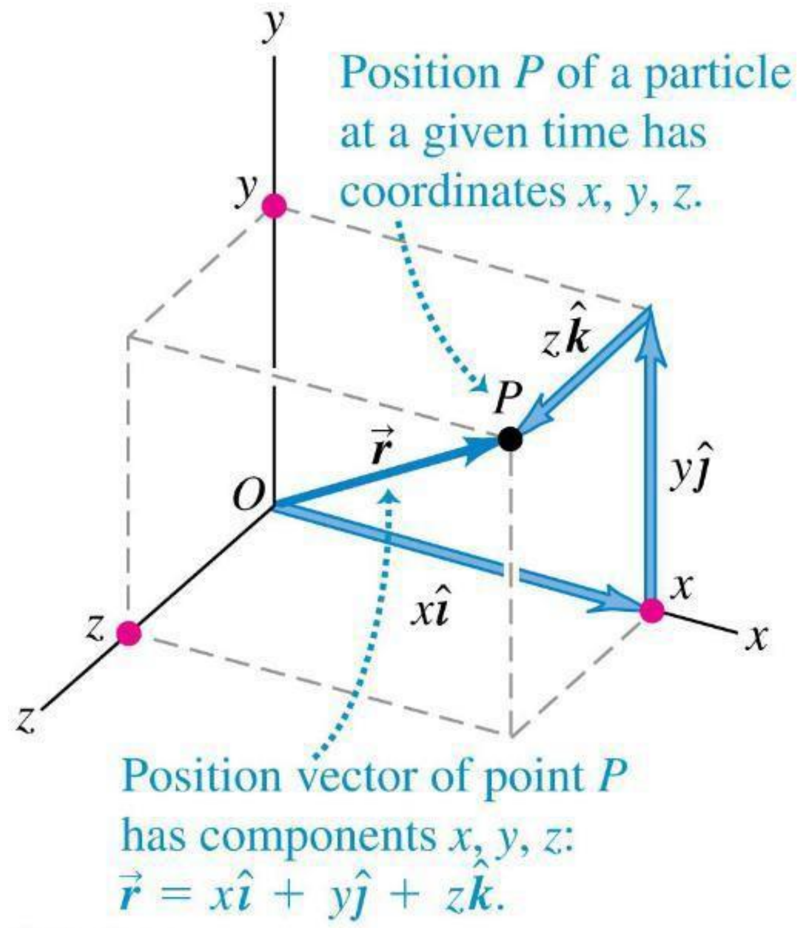


3. Change in speed and direction

Position and velocity vectors

- A particle at a point P at a certain instant can be represented by a **position vector ($\vec{r}$)** that is a vector from the origin to P.

  Position vector has its x, y and z components as shown



Velocity

- Particle moves from P_1 to P_2 in a time interval Δt.
- We define **average velocity** as displacement divided by time.

Change in the particle's position vector

Average velocity vector of a particle during time interval from t_1 to t_2

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

Final position minus initial position

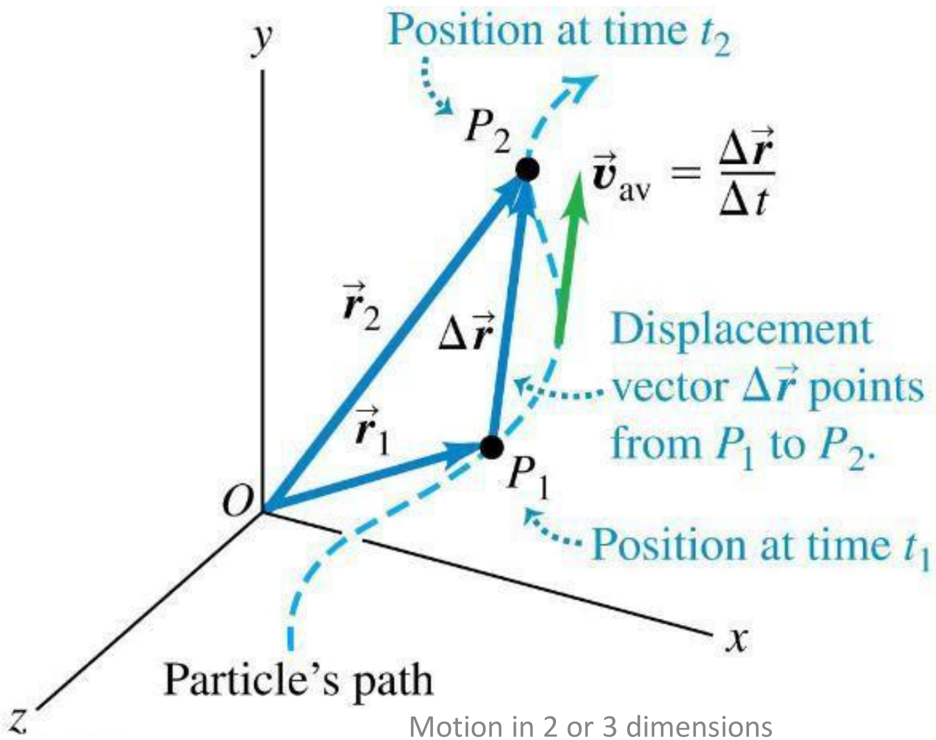
Time interval

Final time minus initial time

- If you consider only ***x*** component of the velocity, the velocity would be linear (discussed earlier)

Velocity

- The **average velocity** between two points is the displacement divided by the time interval between the two points
- It has the same direction as the displacement



Instantaneous velocity

- **Instantaneous velocity** is the *instantaneous* rate of change of position with time

The **instantaneous velocity vector** of a particle ...

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

... equals the limit of its average velocity vector as the time interval approaches zero ...

... and equals the instantaneous rate of change of its position vector.

- The components of the instantaneous velocity are $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$ (note directions)
- *The instantaneous velocity is a vector*

$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

Instantaneous velocity

- Magnitude of the instantaneous velocity is

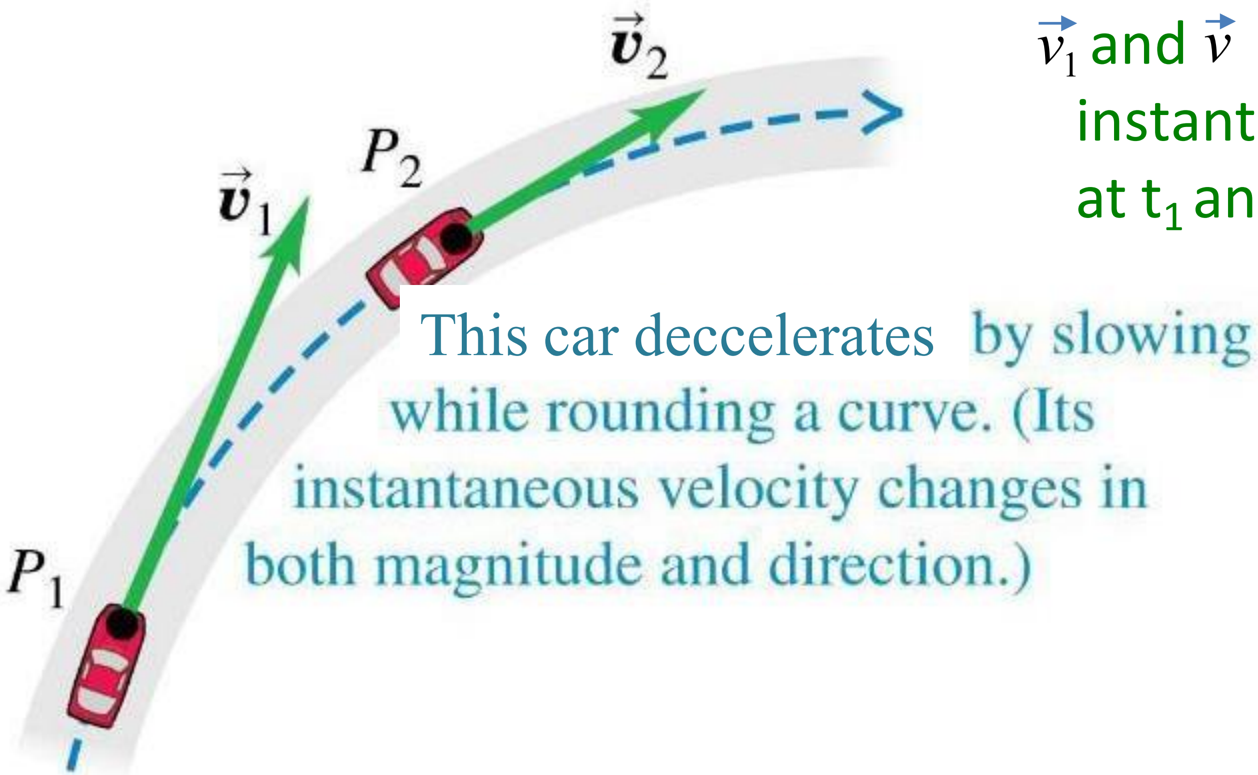
$$\left|\vec{v}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- The magnitude gives the instantaneous speed of the particle.

- Instantaneous velocity has direction which is along the tangent to the particle's path at the particle's position.

Acceleration

- Acceleration of a particle moving in space describes how the velocity changes



$\vec{v}_1$ and $\vec{v}$ are the instantaneous velocities at t_1 and t_2

Acceleration

- **The average acceleration** during the time interval from t_1 to t_2 is

Change in the particle's velocity

Average acceleration vector of a particle during time interval from t_1 to t_2

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

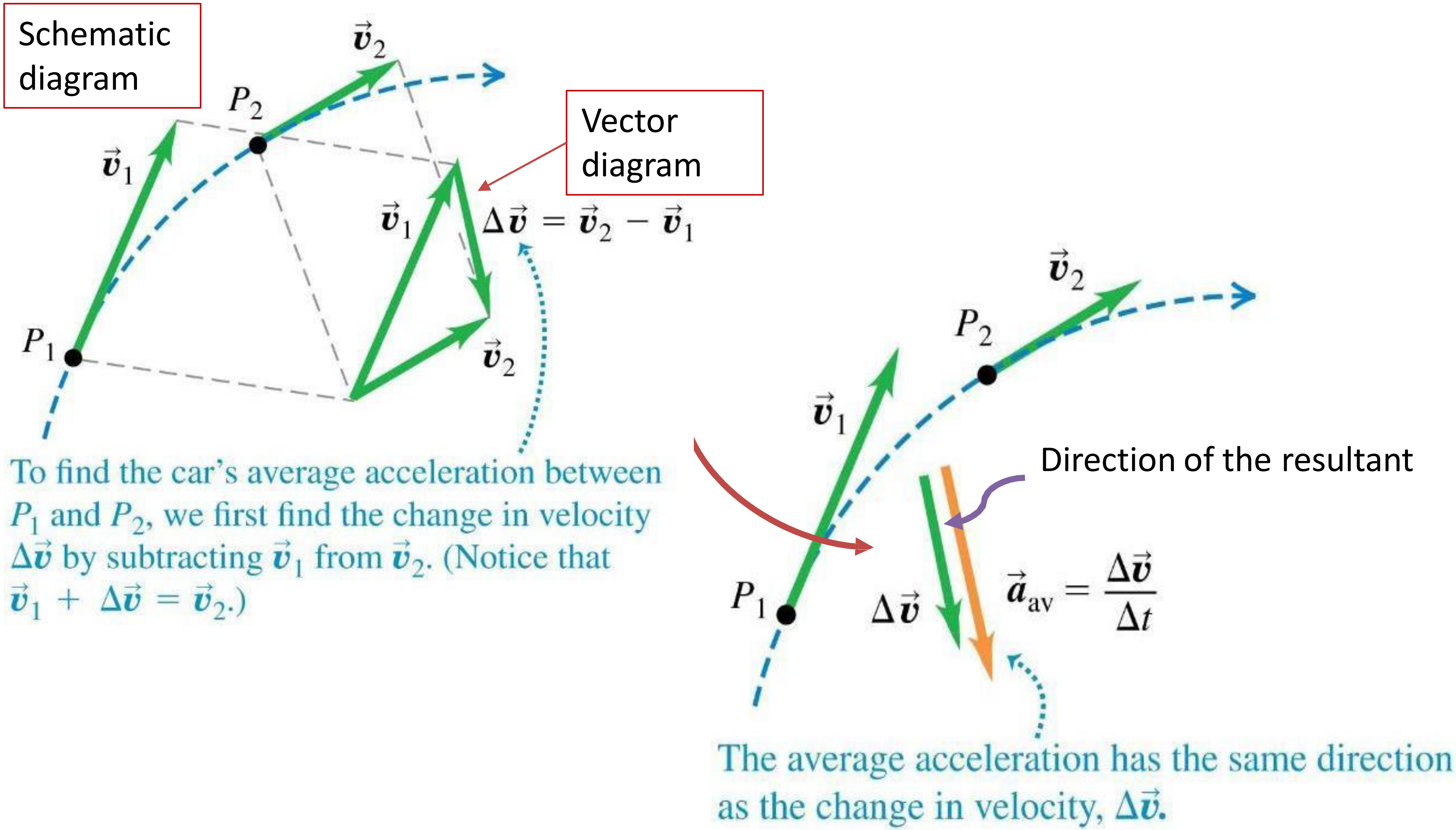
Final velocity minus initial velocity

Time interval

Final time minus initial time

- The change in the velocity (refer to previous slide) is obtained by subtracting velocity vectors.

Acceleration



Instantaneous acceleration

- **Instantaneous acceleration** is the instantaneous rate of change of velocity with time:

The **instantaneous acceleration vector** of a particle ...

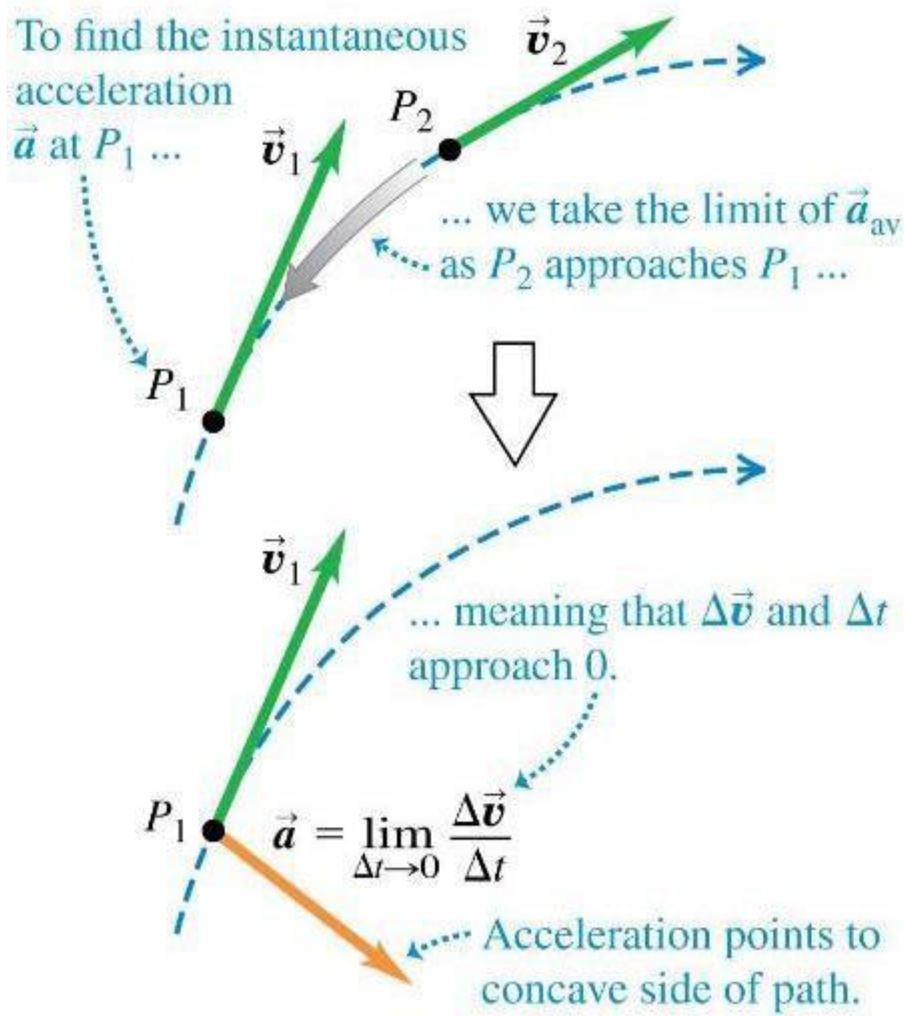
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

... equals the limit of its average acceleration vector as the time interval approaches zero ...

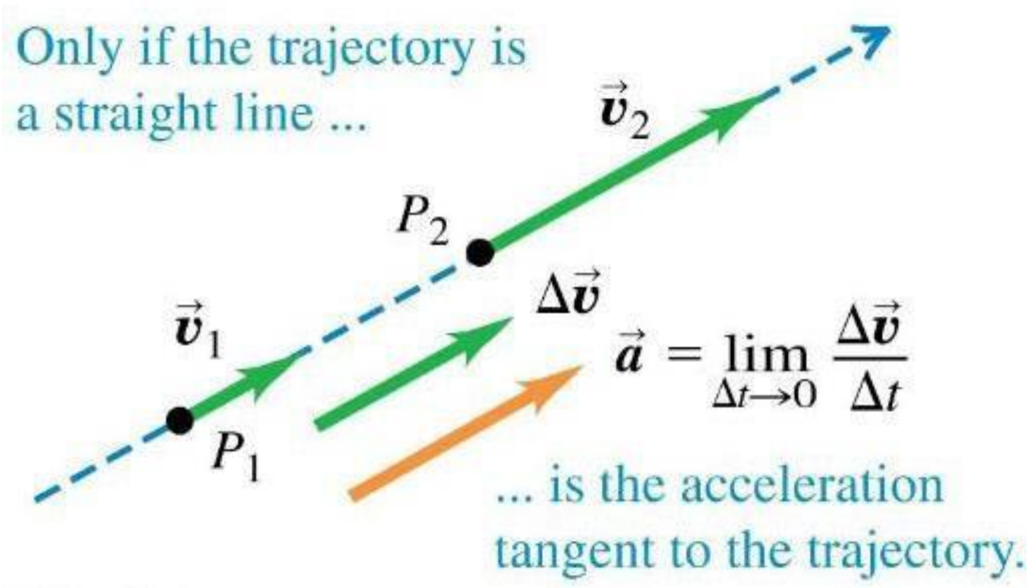
... and equals the instantaneous rate of change of its velocity vector.

- The velocity vector is always tangent to the particle's path.
- The instantaneous acceleration is directed towards inside of the path of the particle and not tangential, why?
- (acceleration is tangent to the path only if it object moves is straight line)

Instantaneous acceleration



Instantaneous acceleration



Components of acceleration

- Each component of the following acceleration vector

$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ is: (3D representation)

Each **component of** a particle's **instantaneous acceleration vector** ...

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt}$$

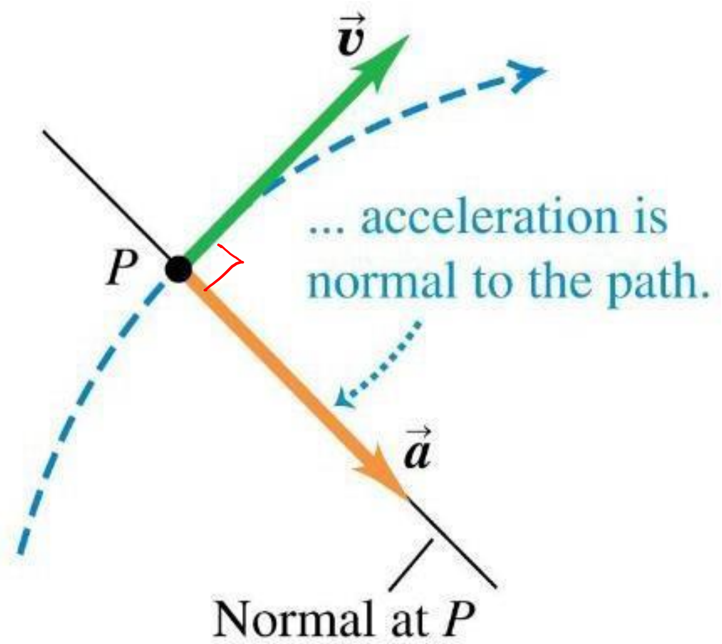
... equals the instantaneous rate of change of its corresponding velocity component.

- In terms of unit vectors

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

Parallel and perpendicular components

- Another useful way is to think of ***a*** in terms of one component parallel to the path and another component perpendicular to the path

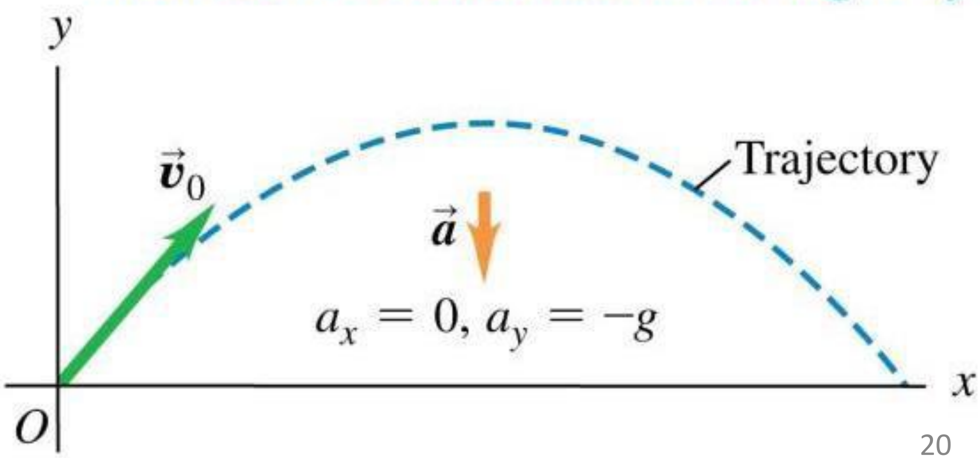


Projectiles

- Projectile:
  - any body with an initial velocity
  - that follows a path is determined entirely by gravitational acceleration and air resistance
  - motion is confined to vertical plane

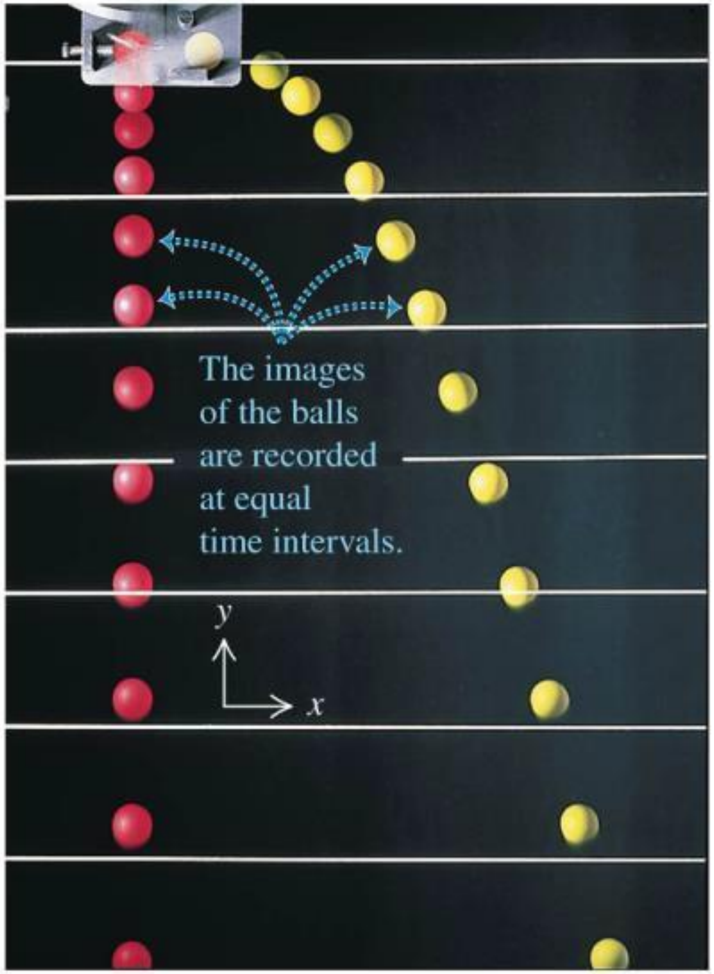
We neglect air resistance and curvature of Earth

- A projectile moves in a vertical plane that contains the initial velocity vector $\vec{v}_0$.
- Its trajectory depends only on $\vec{v}_0$ and on the downward acceleration due to gravity.



Projectiles – x and y motions are separable

- The red ball is dropped at the same time that the yellow ball is fired horizontally



We can analyze projectile motion as horizontal motion with constant velocity and vertical motion with constant acceleration:

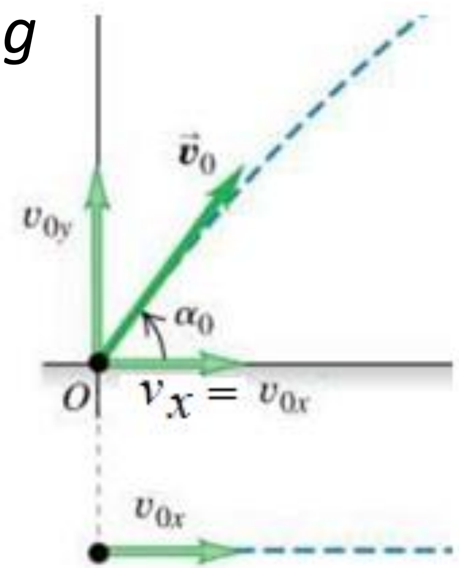
$$a_x = 0, \quad a_y = -g$$

We can treat x and y coordinates separately

Projectiles – x and y motions are separable

- Suppose that at $t = 0$, and our particle is at x_0 and y_0.
- Its initial velocity v_0 and components v_{0x} and v_{0y}
- Components of acceleration are $a_x = 0$ and $a_y = -g$
- Applying principles of motion along a straight line (x axis) $v_x = v_{0x}$

$$x = x_0 + v_{0x}t, \qquad a_x = 0$$



- In the y direction, again it is motion along a straight line

$$v_y = v_{0y} - gt$$

Apply 1st of law of motion.

$$v_y^2 = v_{0y}^2 - 2g\left(y - y_0\right)$$

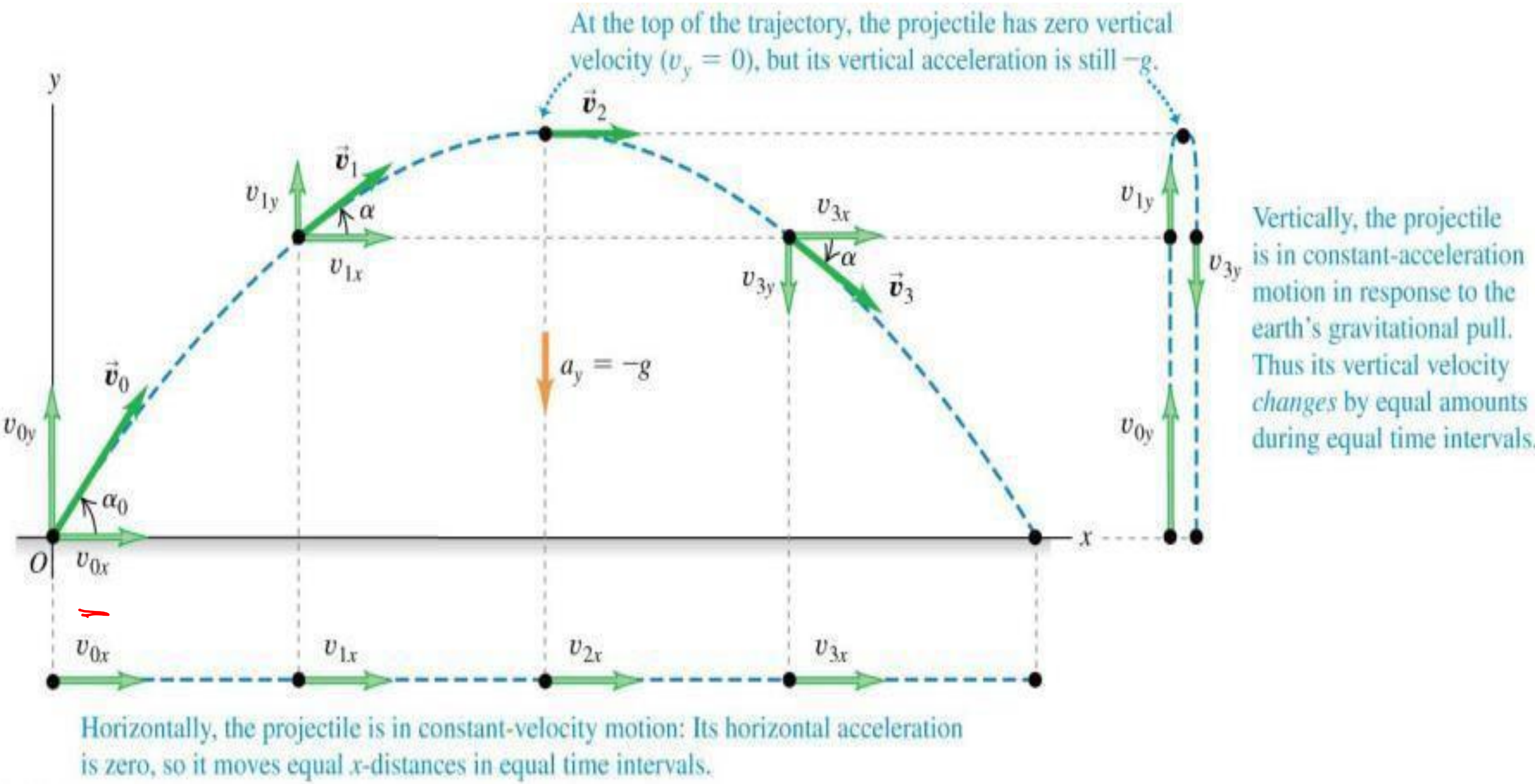
$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2;$$

Apply 2nd law of motion.

Apply 3rd law of motion.

Projectile motion

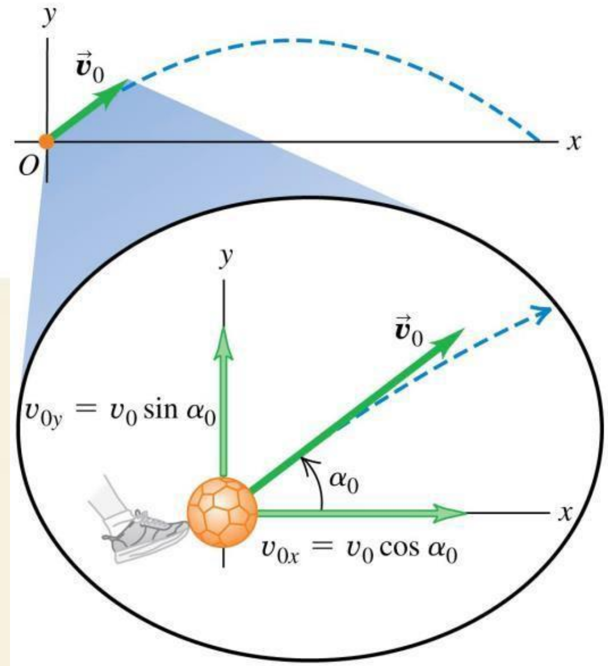
- Neglecting the air resistance, the trajectory of the projectile is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.



Projectiles – equations of motion

SINGAPORE INSTITUTE OF TECHNOLOGY

- Considering initial conditions x_0, $y_0 = 0$, equations of motion are :



Coordinates at time t of a **projectile** (positive y-direction is upward, and $x = y = 0$ at $t = 0$)

$$x = (v_0 \cos \alpha_0)t$$

$$y = (v_0 \sin \alpha_0)t - \tfrac{1}{2}gt^2$$

Speed at $t = 0$

Direction at $t = 0$

Time

Velocity components at time t of a **projectile** (positive y-direction is upward)

$$v_x = v_0 \cos \alpha_0$$

$$v_y = v_0 \sin \alpha_0 - gt$$

Speed at $t = 0$

Direction at $t = 0$

Acceleration due to gravity: Note $g > 0$.

Time

Projectiles – equations of motion

- We can get a lot of information from the above
  - Distance ***r*** from the origin

$$r = \sqrt{x^2 + y^2}$$

  - Magnitude of the projectile velocity (speed)

$$v = \sqrt{v_x^2 + v_y^2}$$

  - Direction of the velocity in term of α_0

$$\tan\alpha_0 = \frac{v_y}{v_x}$$

Projectiles – equations of motion

- Equation of the trajectory (parabolic in nature)

$$y = (\tan\alpha_0)x - \frac{g}{2v_0^2\cos^2\alpha_0}x^2$$

- The above equation is obtained by eliminating t from equations for x and y.

ie $x = v_0 \cos\alpha_0 t \rightarrow t = \frac{x}{v_0\cos\alpha_0}$ — ①

$y = (v_0\sin\alpha_0)t - \frac{1}{2}gt^2$ — ②

Sub ① into ② : $y = (v_0\sin\alpha_0)\left(\frac{x}{v_0\cos\alpha_0}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\alpha_0}\right)^2$

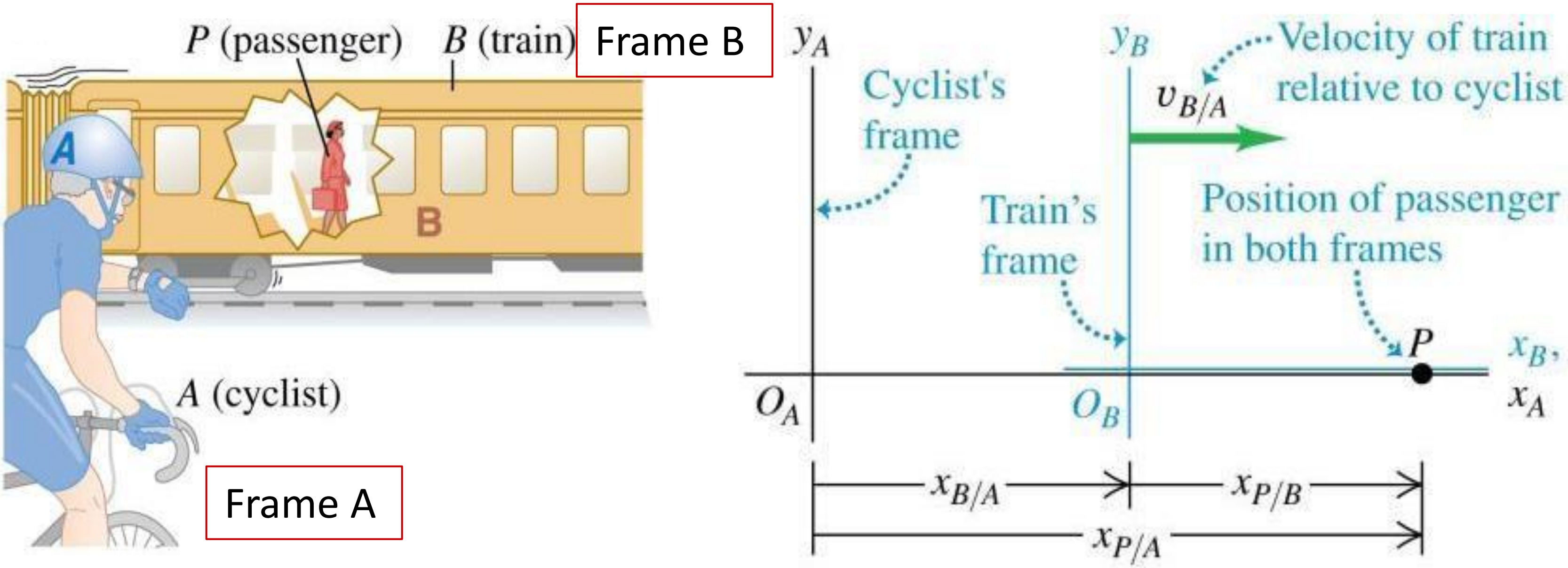
$= (\tan\alpha_0)x - \frac{g}{2v_0^2\cos^2\alpha_0}x^2$

Relative Velocity

- The velocity of a moving body seen by a particular observer is called the velocity ***relative*** to that observer, or simply the **relative velocity**.
- A **frame of reference** is a coordinate system plus a time scale.

Relative Velocity in one dimension

- If point *P* is moving relative to reference frame *A*, we denote the velocity of ***P* relative to frame *A* as $v_{P/A}$**.
- If *P* is moving relative to frame *B* and frame *B* is moving relative to frame *A*, then the *x*-velocity of *P* relative to frame *A* is $v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x}$.



End