Topic 6
Derivatives

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Outline

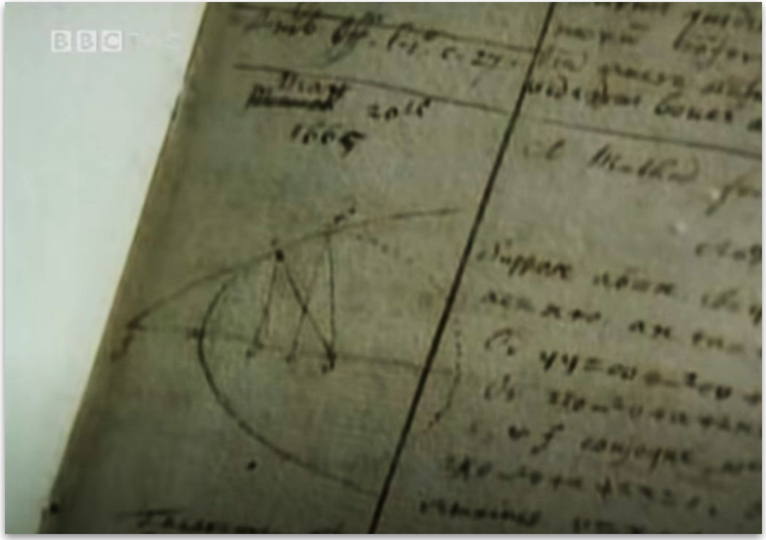
- Birth of Calculus: The Tangent Problem
- Derivatives of Elementary Functions
- Differentiation Rules
- Implicit & Logarithmic Differentiation
- Derivative of the Inverse Function
- Higher-order Derivatives

Birth of Calculus: The Tangent Problem

In 20th May 1665, **Isaac Newton** wrote in his notebook (named ‘Waste Book’) the techniques of finding the equation of a tangent line (& normal line) at a point on a curve.

In this topic, we are going use limits and derive the slope of the tangent. This seemingly simple idea of finding slopes connects to the entirety of calculus and its vast sea of applications.

One of the most powerful applications today, machine learning, works by finding the slope vector of a multivariable function and traversing *opposite* to that direction (gradient descent). You will learn it in Math 2 & 3.



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Birth of Calculus: The Tangent Problem

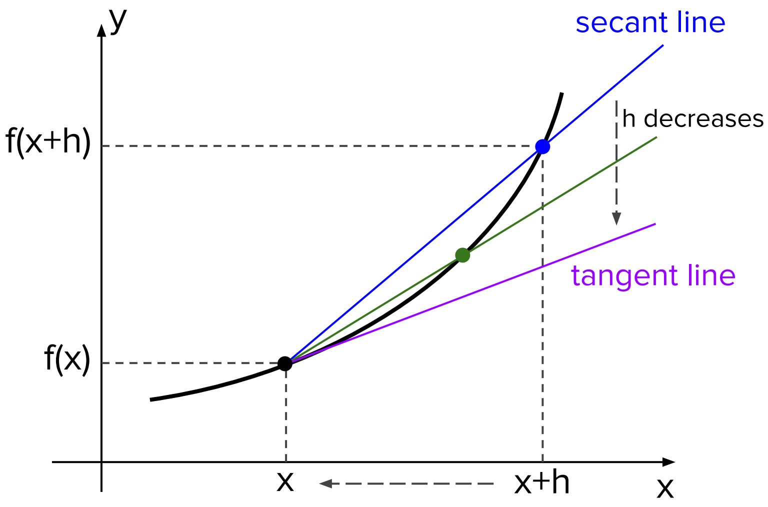
Consider a function f(x) as shown. The slope of the secant line connecting 2 points on the curve is

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}$$

When the point at x + h approaches the point at x, or as h approaches 0, then the secant line approaches the tangent line. So the slope of the tangent is

$$m = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

which gives the indeterminate form 0/0 but this slope clearly exists from the graph.



Hence, we shall use the limit evaluation techniques covered in the last topic to evaluate the slope of the tangent which is dependent on the actual function f(x). But, what has the slope of a tangent got to do with anything?

Notice that the slope of the secant represents the change of a function output wrt a change in its input, i.e.

$$m_{\text{sec}} = \frac{f(x+h)-f(x)}{h} = \frac{\Delta f}{\Delta x}$$

And when the change in the input approaches 0 (called **infinitesimal change, dx**), one can expect a corresponding **infinitesimal change in the output, df**. Hence,

$$m = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \frac{\mathrm{d}f}{\mathrm{d}x}$$

which is called the derivative of a function. So, differentiation is all about finding the slope of the tangent. That’s calculus.

Displacement & Velocity

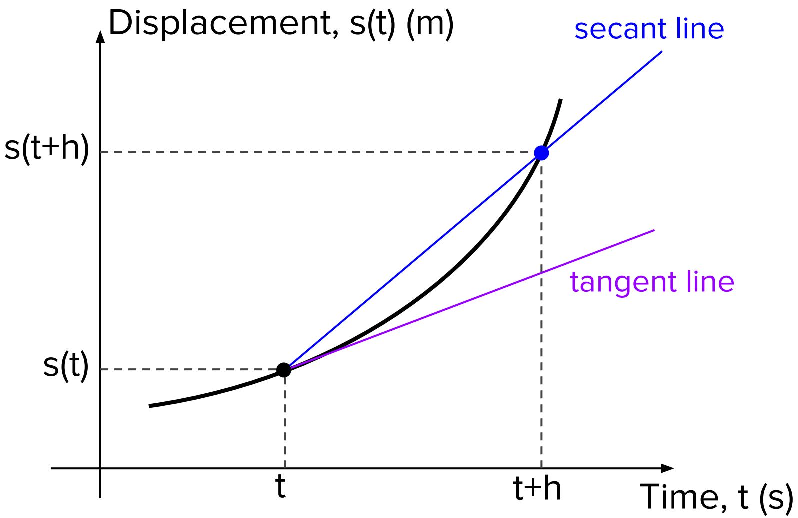
To make the last explanation more intuitive, consider an object moving with the displacement function of time, s(t). The slope of the secant line is

$$m_{\text{sec}} = \frac{s(t+h) - s(t)}{h} = \frac{\Delta s}{\Delta t} = v_{avg} \quad \left(\frac{m}{s}\right)$$

which represents an average velocity from t to t + h. Then, the slope of the tangent line

$$m = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \frac{\mathrm{d}s}{\mathrm{d}t} = v(t) \quad \left(\frac{m}{s}\right)$$

represents the 'instantaneous' velocity at time t. Velocity is the **rate of change** of displacement wrt time.



Essence of Differentiation

Evaluating the derivative of a function, called differentiation, is finding the $\underbrace{\text{instantaneous}}_{\text{infinitesimal}}$ rate of change of the function wrt its input variable.

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

Lagrange's notation

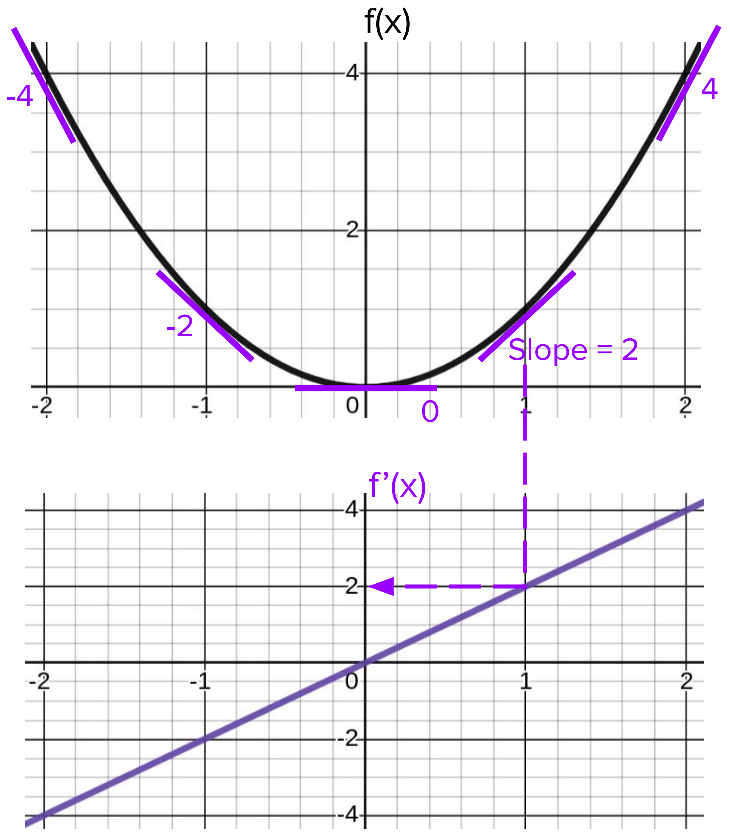
Leibniz's notation

Derivative of x^2

We shall be deriving the derivatives of all elementary functions. Let's begin with **$f(x) = x^2$**. The derivative is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$
$$= \lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h} = \lim_{h\to 0}\frac{2hx+h^2}{h}$$
$$= \lim_{h\to 0} 2x+h = 2x$$

This means the slope of the tangent on f(x) increases linearly with x, as observed on the graph. By visualizing the tangents on f(x), we can deduce how the graph of f'(x) looks like.



Derivative of √x

$$f(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{2}}$$

Exercise: Derive from first principles (using the limit operator) that the derivative of **f(x) = √x** is

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f(x) = \sqrt{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Derivative of x^n

Using binomial expansion, the derivative of **f(x) = xⁿ** can be derived to be

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left[\cancel{x^n} + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - \cancel{x^n}}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} \left\{ nx^{n-1} + \cancel{\binom{n}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}} \right\} = nx^{n-1}
\end{aligned}
$$

Note that the above formula applies to **power functions of any real number n** as well. This will be proven later in logarithmic differentiation.

Linearity Properties of the Derivative

Since the derivative is a limit operation, the following linearity properties follow naturally from those of the limit operator.

a) $$\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) \pm g(x)\} = f'(x) \pm g'(x)$$

b) $$\frac{\mathrm{d}}{\mathrm{d}x}\{kf(x)\} = kf'(x)$$

Exercise: Differentiate directly the function below.

$$f(x) = 3x^7 - \frac{\sqrt{x}}{2}$$

$$f'(x) = 21x^6 - \frac{1}{4\sqrt{x}}$$

Derivative of sin(x)

From first principles, the derivative of **f(x) = sin(x)** can be derived to be

Compound angle formula

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\ \cos h + \cos x\ \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x\ \sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x\,(0) + \cos x\,(1) = \cos x
\end{aligned}
$$

Using limits from the last topic on the squeeze theorem.

The same approach can be used to derive the derivative of f(x) = cos(x). You can try it in Tutorial 6.

Derivative of e^x

From first principles, the derivative of **f(x) = e^x** can be derived to give

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = \lim_{h \to 0} \frac{e^x \left(e^h - 1\right)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

To evaluate the limit at the last step, we make the substitution $k = e^h - 1$, so

$$e^h = 1 + k \rightarrow h = \underline{\ln(1+k)}$$

And notice that as h approaches 0, k approaches 0 as well.

Derivative of e^x

Then, the limit can be rewritten as

$$\lim_{h\to 0}\frac{e^h-1}{h}=\lim_{k\to 0}\frac{\underline{k}}{\underline{\ln(1+k)}}=\lim_{k\to 0}\frac{k}{\ln(1+k)}\left(\frac{1/k}{1/k}\right)$$

$$=\lim_{k\to 0}\frac{1}{\frac{1}{k}\ln(1+k)}=\lim_{k\to 0}\frac{1}{\ln(1+k)^{\frac{1}{k}}}$$

$$=\frac{1}{\ln\left[\lim_{k\to 0}(1+k)^{\frac{1}{k}}\right]}$$

Now, if we let k = 1/n, which means as k approaches 0, n approaches ∞, the above limit becomes the familiar

$$\lim_{k\to 0}(1+k)^{\frac{1}{k}}=\underline{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e}$$

From Topic 5.

Derivative of e^x

Hence,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \frac{1}{\ln e} = 1$$

Finally, the derivative of **f(x) = e^x** is

$$f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x$$

Interestingly, f(x) = ce^x is the only class of non-zero functions where the derivative is equal to the function itself. Perhaps this demonstrates another reason why **e = 2.71828...** is such a special number, because if the base is changed, i.e. f(x) = a^x ($a \neq e$), then the prior statement is no longer true. We shall derive the derivative of f(x) = a^x later in logarithmic differentiation.

Product Rule

For a function that is a product of two other functions, like f(x)g(x), we can derive its derivative from first principles as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)+\overbrace{f(x+h)g(x)-f(x+h)g(x)}^{0}}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]}{h} \\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= f(x)g'(x)+g(x)f'(x)
\end{aligned}
$$

which is called the product rule.

Product Rule

Exercise: By applying the product rule directly, evaluate the derivative of each function below.

a) $f(x) = \underbrace{x^3}_{g(x)} \underbrace{\sin x}_{h(x)}$

$f'(x) = \underbrace{3x^2 \sin x}_{hg'} + \underbrace{x^3 \cos x}_{gh'}$

$= x^2(3\sin x + x\cos x)$

b) $p(x) = f(x)\overbrace{g(x)h(x)}^{u(x)}$

$p'(x) = f(x)u'(x) + f'(x)u(x)$

$= f(x)[g(x)h'(x) + g'(x)h(x)] + f'(x)g(x)h(x)$

ANS: a) $f'(x) = x^2(3\sin x + x\cos x)$ b) $p'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

Quotient Rule

For a function that is a quotient of two other functions, like f(x) / g(x), we can derive its derivative called the quotient rule as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\right]\right\} \\
&= \lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{f(x+h)g(x)-f(x)g(x+h)+\underbrace{f(x)g(x)-f(x)g(x)}_{=0}}{g(x+h)g(x)}\right]\right\} \\
&= \lim_{h\to 0}\left\{\frac{1}{g(x+h)g(x)}\left[\frac{g(x)[f(x+h)-f(x)]-f(x)[g(x+h)-g(x)]}{h}\right]\right\} \\
&= \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left[g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right] \\
&= \frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}
\end{aligned}$$

Quotient Rule

Exercise: By applying the quotient rule directly, evaluate the derivative of each function below.

a) $f(x) = \tan x = \frac{\sin x}{\cos x}$

$$f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

b) $g(x) = \frac{\tan x}{x^2}$

$$g'(x) = \frac{x^2 \sec^2 x - 2x \tan x}{x^4}$$

$$= \frac{x \sin^2 x - 2\tan x}{x^3}$$

ANS: a) $f'(x) = \sec^2 x$ b) $g'(x) = \frac{x \sec^2 x - 2\tan x}{x^3}$

Chain Rule

For a composite function, f(g(x)), its derivative can be derived from first principles to be

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h \to 0} \left\{ \frac{f(g(x+h)) - f(g(x))}{h} \left[\frac{g(x+h) - g(x)}{g(x+h) - g(x)} \right] \right\} \\
&= \lim_{h \to 0} \left\{ \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \left[\frac{g(x+h) - g(x)}{h} \right] \right\} \\
&= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x} = f'(g) g'(x)
\end{aligned}
$$

$\lim_{\Delta g \to 0}$

which is called the chain rule. This is one of the most important differentiation rules.

Chain Rule

To use the chain rule, there are two approaches demonstrated as follows. For example, to differentiate

$$h(x) = \sin\left(x^2\right)$$

the first approach (more beginner) is to recognize the composite function as

$$h(x) = f(g(x)) \quad \text{where} \quad f(g) = \sin g, \quad g(x) = x^2$$

Differentiating, we get

$$f'(g) = \cos g, \quad g'(x) = 2x$$
$$\rightarrow h'(x) = f'(g)g'(x) = \cos g \cdot 2x = 2x\cos\left(x^2\right)$$

Chain Rule

The second approach (faster) is to differentiate from the ‘outer’ function towards the ‘inner’ function, i.e.

$$h'(x) = \left[\sin\left(x^2\right)\right]'\left(x^2\right)' = \cos\left(x^2\right) \cdot 2x = 2x\cos\left(x^2\right)$$

Ignore the ‘inner’ function when differentiating the ‘outer’ function

Another example of using this approach is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan^2\left(x^3\right) &= \left[\left(\tan\left(x^3\right)\right)^2\right]'\left[\tan\left(x^3\right)\right]'\left(x^3\right)' \\ &= 2\tan\left(x^3\right)\cdot\sec^2\left(x^3\right)\cdot 3x^2 \\ &= 6x^2\tan\left(x^3\right)\sec^2\left(x^3\right)\end{aligned}$$

where we used $\frac{\mathrm{d}}{\mathrm{d}x}f(g(h(x))) = f'(g)g'(h)h'(x)$.

Chain Rule

Exercise: Apply the chain rule and differentiate the following functions.

a) $f(x) = \sqrt{3x^2 + x}$

$f'(x) = \frac{1}{2}(3x^2+x)^{-\frac{1}{2}} \cdot (6x+1)$

$= \frac{6x+1}{2\sqrt{3x^2+x}}$

b) $g(x) = \csc x = \frac{1}{\sin x}$

$= (\sin x)^{-1}$

$g'(x) = -(\sin x)^{-2} \cdot \cos x$

$= -\frac{\cos x}{\sin^2 x}$

c) $h(x) = e^{\tan(\pi x)}$

$h'(x) = e^{\tan(\pi x)} \cdot \sec^2(\pi x) \cdot \pi$

$= \pi \sec^2(\pi x)\, e^{\tan(\pi x)}$

ANS: a) $f'(x) = \dfrac{6x+1}{2\sqrt{3x^2+x}}$ b) $g'(x) = -\cot x \csc x$ (can also be obtained by quotient rule). c) $h'(x) = \pi \sec^2(\pi x) e^{\tan(\pi x)}$

Combining Rules of Differentiation

The various rules of differentiation can be applied together in a logical sequence to evaluate a derivative. Examples are:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\sin(3x)\} = \overbrace{2x\sin(3x) + x^2[\sin(3x)]'}^{\text{Product rule}}$$
$$= 2x\sin(3x) + x^2\underbrace{[3\cos(3x)]}_{\text{Chain rule}}$$
$$= x[2\sin(3x) + 3x\cos(3x)]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{\frac{\cos x}{x}} = \underbrace{\left(\frac{\cos x}{x}\right)' e^{\frac{\cos x}{x}}}_{\text{Chain rule}} = \underbrace{\left(\frac{-x\sin x - \cos x}{x^2}\right)}_{\text{Quotient rule}} e^{\frac{\cos x}{x}}$$

Combining Rules of Differentiation

Exercise: Differentiate the following functions.

a) $f(x) = x^2 e^{3x}$

chain rule

product rule

$f'(x) = 2xe^{3x} + x^2\left[e^{3x} \cdot 3\right]$

$= 2xe^{3x} + 3x^2e^{3x}$

$= xe^{3x}(2 + 3x)$

b) $g(x) = 5\tan\left(\overbrace{x^2e^{3x}}^{f(x)}\right)$

$g'(x) = 5\sec^2(x^2e^{3x}) \cdot f'(x)$

$= 5xe^{3x}(2 + 3x)\sec^2(x^2e^{3x})$

ANS: a) $f'(x) = xe^{3x}(2 + 3x)$ b) $g'(x) = 5f'(x)\sec^2\left(x^2e^{3x}\right)$

Implicit Differentiation

So far, we only looked at the derivatives of functions that can be defined explicitly, i.e.

$$y = f(x) \to \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

However, there are functions that are defined implicitly in the general form

$$F(x, y(x)) = c$$

called an implicit equation. For example, the equation of a circle

$$F(x, y(x)) = x^2 + [y(x)]^2 = r^2$$

contains the implicit function(s) y(x). Now, the question is, can we obtain the derivative dy/dx in an implicit equation?

Implicit Differentiation

Since differentiation is a linear operator (because the limit operator is), we can differentiate term by term in an implicit equation and use the chain rule when necessary. For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2 + y^2 = 1\} \to 2x + \overbrace{2y}^{g'(y)} \cdot y' = 0$$

$g(x) = [y(x)]^2$

$$\to y' = \frac{-x}{y}$$

If we isolate y(x) and differentiate explicitly, we get

$$y_1(x) = \sqrt{1 - x^2} \to y_1'(x) = \frac{-x}{\sqrt{1 - x^2}} = \frac{-x}{y_1}$$

$$y_2(x) = -\sqrt{1 - x^2} \to y_2'(x) = \frac{x}{\sqrt{1 - x^2}} = \frac{-x}{y_2}$$

which give the same derivative from implicit differentiation as logically expected.

Implicit Differentiation

So, implicit differentiation is simply differentiating with using the chain rule on the 'function terms', followed by isolating the derivative.

Exercise: Determine dy/dx for the functions below.

a) $\overbrace{y\cos x}^{y(x)\cos x} + x\sin y = 1$

$\frac{d}{dx}$ product rule

$[y'\cos x + y(-\sin x)] + [\sin y + x(\cos y \cdot y')] = 0$

$y'(\cos x + x\cos y) = y\sin x - \sin y$

$y' = \dfrac{y\sin x - \sin y}{\cos x + x\cos y}$

b) $\sqrt{xy} + 3x^2 - y^3 = 3$

$(xy)^{\frac{1}{2}}$

product rule

$\frac{d}{dx} \to \frac{1}{2}(xy)^{-1/2}[xy' + y(1)] + 6x - 3y^2 \cdot y' = 0$

$y'\left[\frac{x}{2\sqrt{xy}} - 3y^2\right] = -\frac{y}{2\sqrt{xy}} - 6x$

$\times 2\sqrt{xy}$

$y'[x - 6y^2\sqrt{xy}] = -y - 12x\sqrt{xy}$

$y' = \dfrac{-y - 12x\sqrt{xy}}{x - 6y^2\sqrt{xy}}$

ANS: a) $\dfrac{dy}{dx} = \dfrac{y\sin x - \sin y}{x\cos y + \cos x}$ b) $\dfrac{dy}{dx} = \dfrac{12x\sqrt{xy} + y}{6y^2\sqrt{xy} - x}$

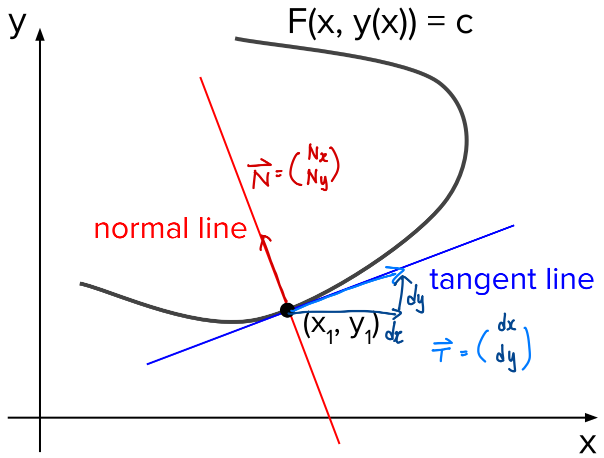
Tangent & Normal Lines to a Curve

Since the slope of the tangent line to a curve is dy/dx, the equation of the tangent line at a point (x_1, y_1) on the curve can be given by the point-slope form to be

$$y - y_1 = m(x - x_1),\ \ m = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1,y_1}$$

For the normal line, the slope is $\mathbf{m_n = -1/m}$, so the equation of the normal line at the point point (x_1, y_1) is

$$y - y_1 = m_n(x - x_1),\ \ m_n = -1/\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1,y_1}$$



Exercise: Using the dot product, show that $m_n = -1/m$.

$\vec{T} \cdot \vec{N} = 0$

$\begin{pmatrix} dx \\ dy \end{pmatrix} \cdot \begin{pmatrix} N_x \\ N_y \end{pmatrix} = 0$

$dx N_x + dy N_y = 0$

$\underbrace{\frac{N_y}{N_x}}_{m_n} = -\frac{dx}{dy} = -1/\frac{dy}{dx} = -1/m$

Tangent & Normal Lines to a Curve

Exercise: Determine the equations of the tangent and normal lines to the curve defined below at the point (-1, 2). Verify your answers in Desmos.

$$x^2 - xy + y^2 = 7$$

$\frac{d}{dx} \rightarrow 2x - [xy' + y] + 2y \cdot y' = 0$

$y'(-x+2y) = y - 2x$

$y' = \frac{y-2x}{-x+2y}$

at $(-1, 2)$ $y' = \frac{2+2}{1+4}$

$= \frac{4}{5} = m$

Eqn of tangent is:

$y - 2 = \frac{4}{5}(x+1)$

$y = \frac{4}{5}x + \frac{14}{5}$

Eqn of normal is:

$m_n = -\frac{1}{\frac{4}{5}}$

$= -\frac{5}{4}$

$y - 2 = -\frac{5}{4}(x+1)$

$y = -\frac{5}{4}x + \frac{3}{4}$

ANS: $y = \frac{4}{5}x + \frac{14}{5}$ (Tangent), $y = -\frac{5}{4}x + \frac{3}{4}$ (Normal)

Derivative of ln(x) & $\log_a x$

The derivative of **y(x) = ln(x)** can be derived using implicit differentiation.

$$y = \ln x \rightarrow x = e^y$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \rightarrow 1 = y' e^y$$
$$\rightarrow y' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

And for **f(x) = $\log_a$x**, its derivative can be worked out to be

$$f(x) = \log_a x = \frac{\ln x}{\ln a}$$
$$\rightarrow f'(x) = \frac{1}{\ln a}\frac{d}{dx}\ln x = \frac{1}{x \ln a}$$

Derivative of ln(x) & $\log_a x$

$(x^2+1)^{-\frac{1}{2}}$

Exercise: Differentiate the following functions.

a) $f(x) = \ln\left(2x^4 - 1\right)^3$

$= 3\ln(2x^4-1)$

$f'(x) = 3\left[\frac{1}{2x^4-1} \cdot 8x^3\right]$

$= \frac{24x^3}{2x^4-1}$

b) $g(x) = \log_{10}\frac{1}{\sqrt{x^2+1}}$ $= \log_{10}(x^2+1)^{-1/2}$

$= -\frac{1}{2}\log_{10}(x^2+1)$

Chain rule

$g'(x) = -\frac{1}{\cancel{2}}\left[\frac{1}{(x^2+1)\ln 10} \cdot \cancel{2}x\right]$

$= -\frac{x}{(x^2+1)\ln 10}$

ANS: a) $f'(x) = \frac{24x^3}{2x^4-1}$ b) $g'(x) = \frac{-x}{(x^2+1)\ln 10}$

Derivative of Inverse Trigonometric Functions

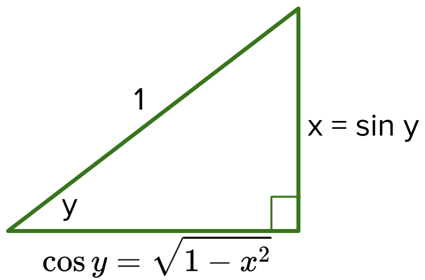
The derivatives of **sin^{-1} x**, **cos^{-1} x** and **tan^{-1} x** can be derived using implicit differentiation.

For y(x) = sin^{-1} x,

$$y = \sin^{-1} x \to x = \sin y$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \to 1 = y' \cos y$$

$$\to y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$



Exercise: Derive the derivative of tan^{-1} x.

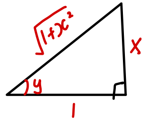
$y = \tan^{-1} x \to x = \tan y$) $\frac{d}{dx}$

$\sec^2 y \cdot y' = 1$

$y' = \frac{1}{\sec^2 y}$

$= \cos^2 y$

$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$



$\tan y = \frac{x}{1} = x$

ANS: a) $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$

Logarithmic Differentiation

Logarithmic differentiation builds on implicit differentiation and is needed for evaluating derivatives of non-elementary functions in the form

$$h(x) = f(x)^{g(x)}$$

Taking the natural log (most convenient) and applying implicit differentiation gives

$$\begin{aligned}
\ln h(x) &= \ln \left\{ f(x)^{g(x)} \right\} = g(x) \ln f(x) \\
\rightarrow \frac{h'(x)}{h(x)} &= g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \\
\rightarrow h'(x) &= f(x)^{g(x)} \left[g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right]
\end{aligned}$$

The above process is known as logarithmic differentiation.

Logarithmic Differentiation

Exercise: Using logarithmic differentiation, find the derivative of each function below.

a) $f(x) = x^{\sqrt{x}}$

$\ln f = \sqrt{x}\ln x$

$\frac{d}{dx} \to \frac{f'}{f} = \sqrt{x}\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}}\ln x$

$f'(x) = f\left[\sqrt{x}\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}}\ln x\right]$

$= x^{\sqrt{x}}\left[\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}}\ln x\right]$

$= x^{\sqrt{x}-1}\left[1 + \ln\sqrt{x}\right]$

b) $g(x) = x^{\ln x}\cos x$

$\ln g = \ln\left(x^{\ln x}\cdot\cos x\right) = \ln x^{\ln x} + \ln(\cos x)$

$= (\ln x)^2 + \ln(\cos x)$

$\frac{d}{dx} \to \frac{g'}{g} = 2\ln x\left(\frac{1}{x}\right) + \frac{1}{\cos x}(-\sin x)$

$g'(x) = x^{\ln x}\cos x\left[\frac{2\ln x}{x} - \tan x\right]$

ANS: a) $f'(x) = \frac{x^{\sqrt{x}}}{\sqrt{x}}\left(1+\ln\sqrt{x}\right)$ b) $g'(x) = x^{\ln x}\cos x\left(\frac{2\ln x}{x} - \tan x\right)$

Derivative of a^x & x^r

We can also use logarithmic differentiation to derive the derivative of the exponential function with any base a > 1.

$$f(x) = a^x \to \ln f(x) = x \ln a$$
$$\frac{d}{dx} \to \frac{f'(x)}{f(x)} = \ln a$$
$$\to f'(x) = f(x) \ln a = a^x \ln a$$

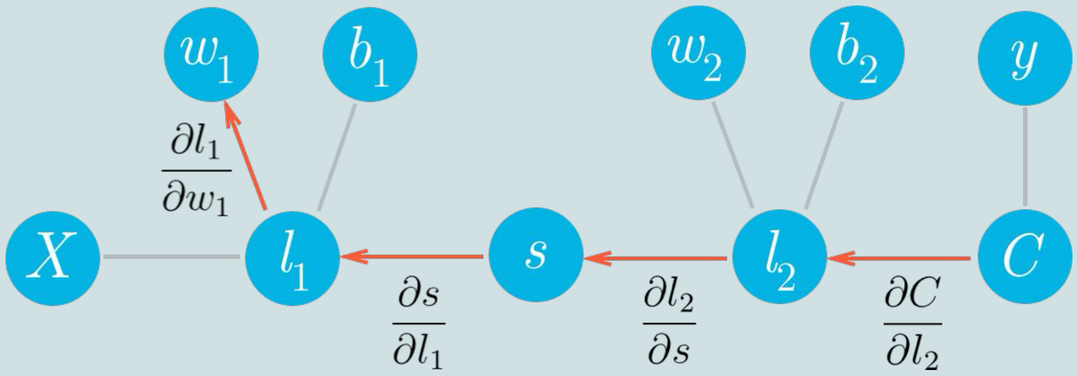
Exercise: Using logarithmic differentiation, show that the derivative of **f(x) = x** where **r is a real number** is

$$f'(x) = rx^{r-1}$$

$$\ln f = r \ln x$$
$$\frac{d}{dx} \to \frac{f'}{f} = \frac{r}{x}$$
$$f'(x) = f(x) \cdot \frac{r}{x} = x^r\left(\frac{r}{x}\right) = rx^{r-1}$$

Since r is a constant

Take a moment to appreciate the importance of the chain rule. Implicit and logarithmic differentiation would not be possible without it. The chain rule is also critical in many multivariable applications (Math 2) such as artificial neural networks in ML.



Chain rule in backpropagation to update parameters in machine learning.
(https://medium.com/deeper-learning/glossary-of-deep-learning-backpropagation-e6d748d36a0e)

Table of Derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
x^r	rx^{r-1}	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
e^x	e^x	$\tan^{-1} x$	$\frac{1}{1+x^2}$
a^x	$a^x \ln a$	$kg(x)$	$kg'(x)$
$\ln x$	$\frac{1}{x}$	$g(x) \pm h(x)$	$g'(x) \pm h'(x)$
$\log_a x$	$\frac{1}{x \ln a}$	$g(x)h(x)$	$g'h + h'g$
$\sin x$	$\cos x$	$\frac{g(x)}{h(x)}$	$\frac{hg' - gh'}{h^2}$
$\cos x$	$-\sin x$	$g(h(x))$	$g'(h)h'(x)$
$\tan x$	$\sec^2 x$		

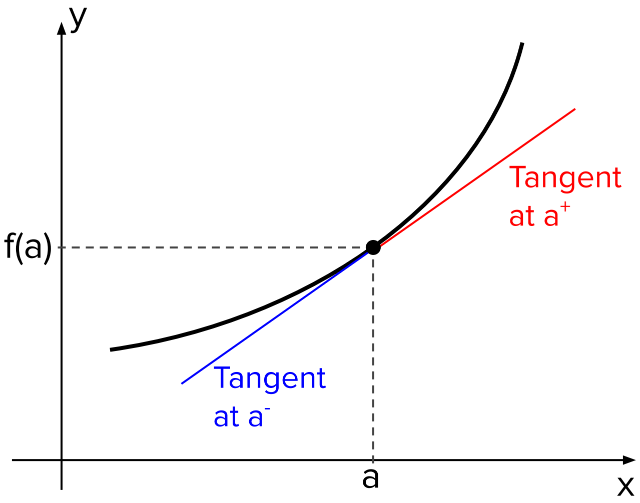
Differentiability of a Function

Having covered the various techniques of differentiation, now let's define the conditions of differentiability at a point. Since the derivative is the slope of the tangent, this implies that as long as a (non-vertical) tangent can be drawn at a point on a curve, the derivative must exist. So there are two conditions:

$$\text{(Continuity)} \qquad \lim_{x \to a} f(x) = f(a)$$

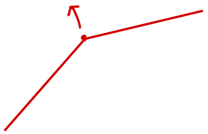
$$\text{(Collinearity of tangents)} \quad \lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x)$$

As long as the above two conditions are satisfied, we say the function is differentiable (smooth) at x = a. If they are satisfied for every x = a in (b, c), then the function is differentiable (smooth) in (b,c).



Differentiability of a Function

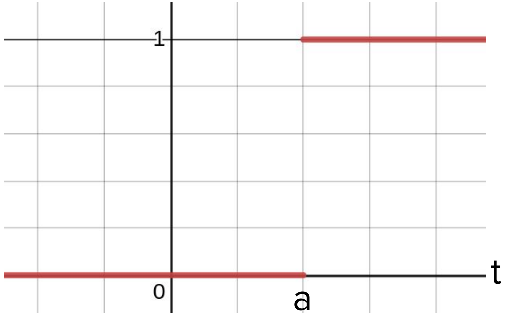
not differentiable (no tangent)



Example: Which function below is differentiable at x = a or t = a? Justify using the conditions of differentiability.

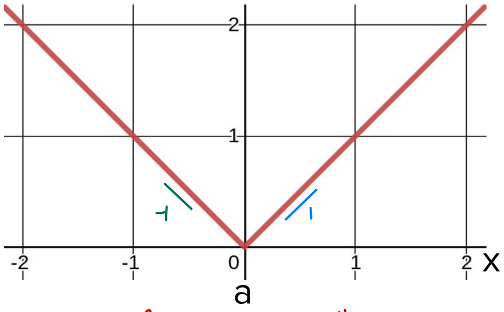
$\lim_{x\to a^-} f'(x) = -1 \neq \lim_{x\to a^+} f'(x) = 1$

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$



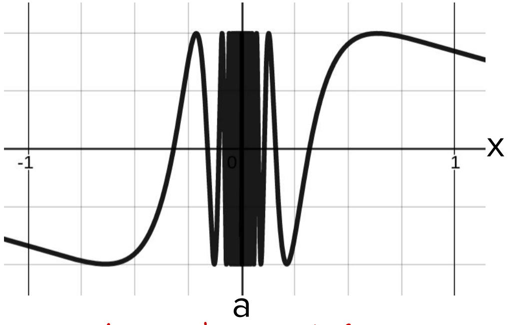
no, since function is not cont. at $x = a$ ($\lim_{t\to a} u(t-a)$ DNE)

$$f(x) = |x| = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases}$$



no, since $f(x)$ is not smooth at $x = a$

$$f(x) = \sin\left(\frac{1}{x}\right)$$



no, since $f(x)$ is discont. at $x = a$ ($\lim_{x\to a} f(x)$ DNE.)

Differentiability of a Function

Exercise: Determine the parameters a, b and c in order for the function to be differentiable in ℝ. Verify in Desmos.

$$f(x) = \begin{cases} ax^2 + b\sin(\pi x) + c, & x \le 0 \quad (f_1) \\ \ln(x+1), & x > 0 \quad (f_2) \end{cases} \xrightarrow{\frac{d}{dx}} f'(x) = \begin{cases} 2ax + \pi b\cos(\pi x), & x < 0 \\ \frac{1}{x+1}, & x > 0 \end{cases}$$

note that $f_1(x)$ & $f_2(x)$ are both differentiable in their own interval

so we need to check interval transition at $x=0$

① For continuity: $\underline{f(0) = \lim_{x\to0} f(x)}$ where $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} f_1(x) = a(0)+b(0)+c = c = f(0)$

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} f_2(x) = \ln(0+1) = 0$

$\therefore c = 0$ (LHS = RHS)

② For smoothness: $\lim_{x\to0^-} f'(x) = \lim_{x\to0^-} f_1'(x) = 2(0) + \pi b(1) = \pi b$

$\lim_{x\to0^+} f'(x) = \lim_{x\to0^+} f_2'(x) = \frac{1}{0+1} = 1$

$\lim_{x\to0^-} f'(x) = \lim_{x\to0^+} f'(x)$

$\pi b = 1$

$b = \frac{1}{\pi}$

ANS: a ∈ ℝ, b = 1/π, c = 0.

Higher-order Derivatives

Since a derivative is itself a function f'(x), it can be further differentiated to give a higher-order derivative of the original function f(x). Hence we denote

$$
\begin{aligned}
f''(x) &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x} f'(x) \\
f'''(x) &= \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = \frac{\mathrm{d}}{\mathrm{d}x} f''(x) \\
&\vdots \\
f^{(n)}(x) &= \frac{\mathrm{d}^n f}{\mathrm{d}x^n} = \frac{\mathrm{d}}{\mathrm{d}x} f^{(n-1)}(x)
\end{aligned}
$$

where (n) represents the number of times f(x) is differentiated.

Higher-order Derivatives

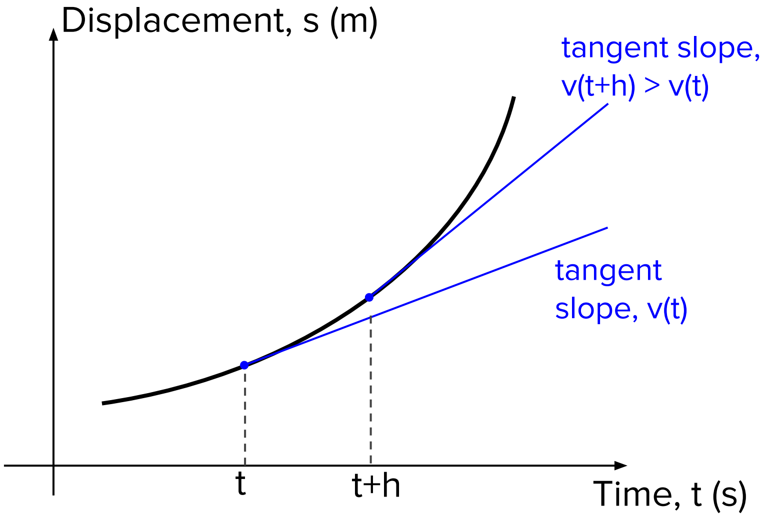
For example, similar to the the derivative of the displacement function of time being the velocity function, i.e.

$$s'(t) = v(t) \quad \left(\frac{m}{s}\right)$$

the derivative of the velocity function (2nd-order derivative of displacement) is the acceleration function.

$$s''(t) = v'(t) = a(t) \quad \left(\frac{m}{s^2}\right)$$

From the graph, we can deduce that the object is accelerating (a(t) > 0) because v(t) is increasing.



Higher-order Derivatives

Exercise: An object moves with the displacement function below. Determine its velocity and acceleration functions. The derivative of the acceleration function is called ‘jerk’. Explain if this name is appropriate.

$$s(t) = t\cos(\pi t),\ \ t \geq 0$$

$$v(t) = s'(t) = \cos(\pi t) + t\left[-\pi\sin(\pi t)\right]$$

$$a(t) = v'(t) = -\pi\sin(\pi t) - \pi\left[\sin(\pi t) + \pi t\cos(\pi t)\right]$$

$$= -2\pi\sin(\pi t) - \pi^2 t\cos(\pi t)$$

$$j(t) = a'(t)$$

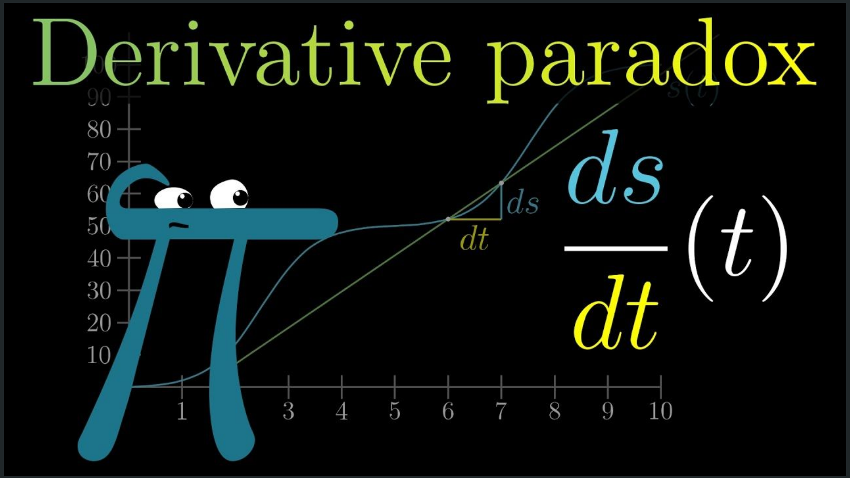
When there is a sudden change of acceleration, there is a sudden change in force applied to the body. This causes a jerking motion on the body, so the name is totally intuitive and appropriate.

ANS: $v(t) = \cos(\pi t) - \pi t\sin(\pi t),\ a(t) = -\pi[2\sin(\pi t) + \pi t\cos(\pi t)]$

End of Topic 6

*If they asked me, I would have called f'(t) an **infinitesimal** rate of change rather than instantaneous rate of change, since nothing changes in an instant (hence a paradox).*

But nobody asked me…



Essence of Calculus (3Blue1Brown)
https://youtube.com/playlist?list=PLZHQObOWTQDMsr9K-rj53DwVRMYO3t5Yr