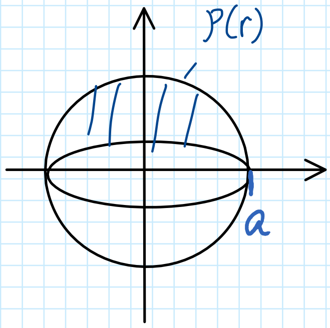
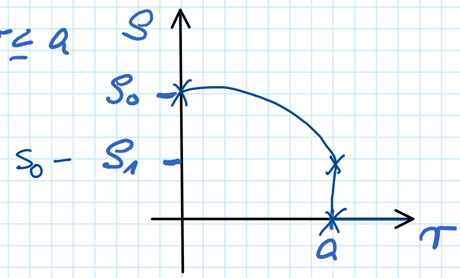
Example problem: Gauss's law - calculation of electric field everywhere in space for a given spherical charge distribution



Given: spherical, positions-dependent charge distribution:

$$\rho = \rho_0 - \rho_1 \left(\frac{r}{a}\right)^2 \quad \text{für } r \le a$$

$$\rho = 0 \quad \text{for } r > a$$

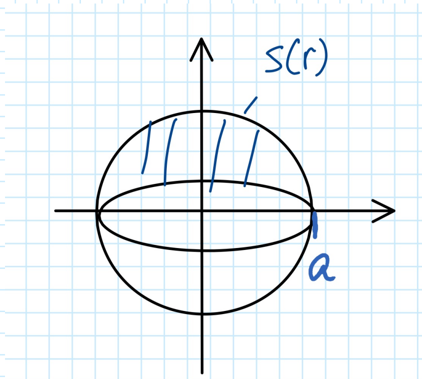


- Calculate electric displacement field $\vec{D}$ and electric field $\vec{E}$ for the above given space charge distribution everywhere in space $0 < r < \infty$
- distinguish between the different regions, when applying Gauss's law
- Use spherical coordinates for solving the problem

· spherical symmetry ⇒ $\vec{D}, \vec{E}$ show no dependence on θ, φ (symmetric)

⇒ $\vec{D}, \vec{E} \Rightarrow E_r, D_r$; they depend only on r

⇒ $\vec{E}(\vec{r}) = E_r \cdot \vec{e}_r$, $\vec{D}(\vec{r}) = D_r \cdot \vec{e}_r$



· Calculate $\vec{D}$: Apply Gauss's Law: $\underbrace{\int_{\partial V} \vec{D}\, d\vec{a}}_{\mathrm{I}} = \underbrace{\int_V \rho(\vec{r})\, d^3 r}_{\mathrm{II}} = \int_V \rho(\vec{r})\, dV$

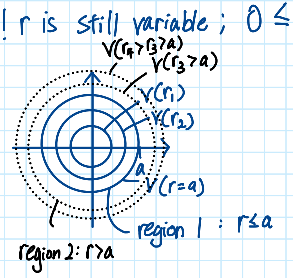
I $\int_{\partial V} \vec{D} \cdot d\vec{a} = \int_{\partial V} D_r \cdot \vec{e}_r \cdot d\vec{a} = \int_{\partial V} D_r \cdot \vec{e}_r \cdot da \cdot \vec{e}_r$

$$= \int_0^{2\pi}\int_0^{\pi} D_r \cdot r^2 \sin\theta \, d\theta \, d\varphi = D_r r^2 \int_0^{2\pi} [-\cos\theta]_0^{\pi} \, d\varphi$$

$$= D_r r^2 \int_0^{2\pi} \underbrace{[1+1]}_{2} \, d\varphi = 2 D_r \cdot r^2 \, [\varphi]_0^{2\pi}$$

$$= 4\pi D_r \cdot r^2$$

! r is still variable ; $0 \le r \le \infty$



II $\int_V \rho(\vec{r})\, dV$ → Two regions: $r \le a \Rightarrow \rho(\vec{r})$

$r > a \Rightarrow 0$

goal is to calculate $Q(V(r))$

$Q(V(r_1)) < Q(V(r_2)) < Q(V(r=a))$

$Q(V(r_3)) = Q(V(r=a)) = Q(V(r_4))$

Region 1: $\int_V \rho(\vec{r})\, dv = Q(V(\vec{r}))$

$r \le a$

$$Q(V(\vec{r})) = \underbrace{\int_0^{2\pi}\int_0^{\pi}}\int_{r'=0}^{r} \rho(r')\, r'^2 \sin\theta\, dr'\, \underbrace{d\theta\, d\varphi}_{=4\pi} \qquad \rho \neq f(\theta,\varphi)$$

$$= 4\pi \int_0^r (\rho_0 - \rho_1)\left(\frac{r'}{a}\right)^2 r'^2\, dr'$$

$$= 4\pi \int_0^r \left(\rho_0 r'^2 - \rho_1\left(\frac{r'^4}{a^2}\right)\right) dr' = 4\pi\left[\frac{1}{3}\rho_0 r'^3 - \frac{1}{5}\rho_1 \frac{r'^5}{a^2}\right]$$

$$= 4\pi r'^3\left[\frac{1}{3}\rho_0 - \frac{1}{5}\rho_1 \frac{r'^2}{a^2}\right]_0^r$$

$$Q(V(r)) = 4\pi r^3\left(\frac{1}{3}\rho_0 - \frac{1}{5}\rho_1 \frac{r^2}{a^2}\right)$$

Gauss's Law : $\mathrm{I} = \mathrm{II}$ $\vec{D}$-field inside the sphere:

$$\cancel{4\pi r^2} D_r = \cancel{4\pi r^3}\left(\frac{1}{3}\rho_0 - \frac{1}{5}\rho_1 \frac{r^2}{a^2}\right)$$

$$D_r = \frac{1}{3}\rho_0 \cdot r - \frac{1}{5}\rho_1 \cdot \frac{r^3}{a^2} = \frac{1}{3}\rho_0 \cdot a\left(\frac{r}{a}\right) - \frac{1}{5}\rho_1 a\left(\frac{r}{a}\right)^3$$

$\vec{D} = D_r \cdot \vec{e_r}$ for $\vec{r} \le a$

$\vec{D}$-Field outside sphere: (region 2) $r > a$

$Q(V(r=a)) = \int_V \rho(\vec{r}) \cdot dV =$ total charge on the sphere

$$= \int_V \rho(\vec{r}) \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho(\vec{r})\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

take result for $Q(V(\vec{r}))$ $\downarrow$ Insert $r = a$

$$= 4\pi a^3\left(\frac{1}{3}\rho_0 - \frac{1}{5}\frac{\rho_1}{a^2} \cdot a^2\right)$$

$$= 4\pi a^3\left(\frac{1}{3}\rho_0 - \frac{1}{5}\rho_1\right)$$

$= Q_{total}$ on the sphere

gauss's Law : $\mathrm{I} = \mathrm{II}$

$$4\pi r^2 D_r = Q(V(a)) = \text{Constant}$$

for $r > a$ $\boxed{D_r = \frac{1}{4\pi r^2} Q(V(a))}$

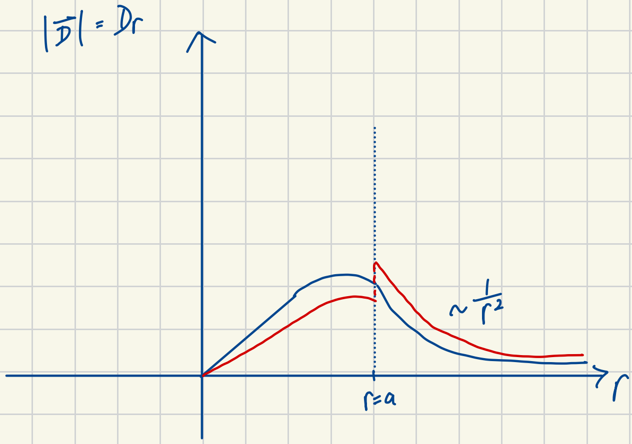
(Compare Coulomb field of point charge: $D_r = \frac{1}{4\pi}\frac{Q}{r^2}$

$\vec{D} = D_r \cdot \vec{e_r}$

$$\vec{D} = D_r \cdot \vec{e_r} \quad \begin{cases} D_r = \frac{1}{3}\rho_0 \cdot a \cdot \left(\frac{r}{a}\right) - \frac{1}{5}\rho_1 \cdot a \cdot \left(\frac{r}{a}\right)^3 & \text{for } r \le a \\ D_r = \frac{1}{r^2} a^3 \left(\frac{1}{3}\rho_0 - \frac{1}{5}\rho_1\right) & \text{for } r > a \end{cases}$$

$\vec{D}$ field is Continuous at $r = a$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0 \varepsilon_r}$$



$\varepsilon_{inside} > \varepsilon_{outside}$

$\vec{E_i} = \frac{D_r}{\varepsilon_{r_i}}$ $\quad D_r = \varepsilon_0 \varepsilon_r \vec{E}$

$|\vec{E_o}| = \frac{D_r}{\varepsilon_{r_o}}$

outside

for $r = a$

$|\vec{E_i}| < |\vec{E_o}|$