EDE1012 MATHEMATICS 2

Tutorial 7

Introduction to Laplace Transform

1. Determine the Laplace transform of the following functions:

a) [redacted]

b) [redacted]

c) [redacted]

d) [redacted]

e) [redacted]

~~f)~~ [redacted]

~~g)~~ [redacted]

ANS: **a)** $F(s) = \frac{2}{s+3} - \frac{24}{s^5}$. **b)** $F(s) = \frac{3s-10}{s^2+25}$. **c)** $F(s) = \frac{2}{(s+1)^3} + \frac{6}{(s+1)^2} + \frac{9}{s+1}$.
d) $F(s) = \frac{1}{(s-1)^2+1}$. **e)** $F(s) = 9e^{-3s}$. **f)** $F(s) = \frac{e^{-\pi s}}{s^2} - e^{-2\pi s}(\frac{1}{s^2} + \frac{\pi}{s})$.
g) $F(s) = \frac{2\omega^3}{(s^2+\omega^2)^2}$.

DIY

~~2.~~ Using integration, determine the Laplace transform of f(t). Show that it is equivalent to that obtained by shifting in s-domain as well as that using the derivative of the Laplace transform.

[redacted]

ANS: $F(s) = \frac{2}{(s-1)^3}$

~~3.~~ Determine the Laplace transform of the function below, where k and ω are constants.

[redacted]

$g(t)$

Approach 2

$t^n g(t)$	$(-1)^n G^{(n)}(s)$

correction.

ANS: $\cancel{F}H(s) = \frac{(s-k)^2 - \omega^2}{\left[(s-k)^2+\omega^2\right]^2}$.

$G(s)$

Approach 1

$H(s) = (-1)^1 \frac{d}{ds}\left(\underbrace{\frac{s-k}{(s-k)^2+\omega^2}}_{G(s)}\right)$

OR: $h(t) = e^{kt}\underbrace{t\cos(\omega t)}_{f(t)}$

$f(t) \downarrow F(s)$

$= F(s-k)$

Q1 f) $f(t) = \begin{cases} t-\pi, & \pi < t < 2\pi \\ 0, & \text{otherwise} \end{cases}$

on at π off at 2π

$$= \underbrace{(t-\pi)u(t-\pi)}_{\text{on at } \pi} - \underbrace{(t-\pi)u(t-2\pi)}_{\text{off at } 2\pi},$$

$g(t-a)u(t-a)$	$e^{-as}G(s)$

$$= \underbrace{(t-\pi)}_{g(t-\pi)}u(t-\pi) - (t-2\pi+\pi)u(t-2\pi)$$

$g(t-\pi)$

$\downarrow$

$g(t) = t$

$\downarrow$

$G(s) = \frac{1}{s^2}$

$\downarrow$

$$F(s) = e^{-\pi s} \cdot \frac{1}{s^2} - e^{-2\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right) //$$

g) $f(t) = \sin(\omega t) - \omega t \underbrace{\cos(\omega t)}_{g(t)}$

$n=1$

$t^n g(t)$	$(-1)^n G^{(n)}(s)$

$$F(s) = \frac{\omega}{s^2+\omega^2} - \omega\left[(-1)^1 \frac{d}{ds}\overbrace{\left(\frac{s}{s^2+\omega^2}\right)}^{G(s)}\right]$$

$$= \frac{\omega}{s^2+\omega^2} + \omega\underbrace{\left(\frac{(s^2+\omega^2)(1) - s(2s)}{(s^2+\omega^2)^2}\right)}$$

$$= \frac{\omega(s^2+\omega^2)}{(s^2+\omega^2)^2} + \omega\left(\frac{\omega^2-s^2}{(s^2+\omega^2)^2}\right) = \frac{2\omega^3}{(s^2+\omega^2)^2} //$$

4. Using both integration and the t-domain shifting property., determine the Laplace transform of the following function. Are they equivalent?

[redacted]

ANS: $F(s) = e^{-2s}\left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}\right)$. Yes.

DIY

5. Rewrite the following piecewise function using the unit-step function and evaluate its Laplace transform.

[redacted]

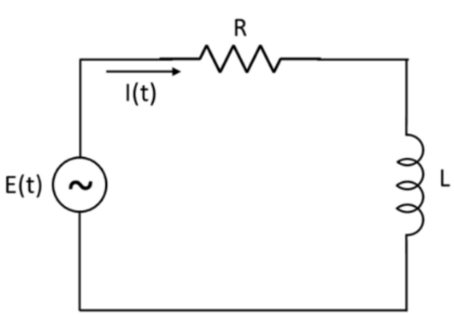
ANS: $G(s) = \frac{1}{e(s+1)}e^{-s} + \left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} - \frac{1}{e^2(s+1)}\right]e^{-2s}$

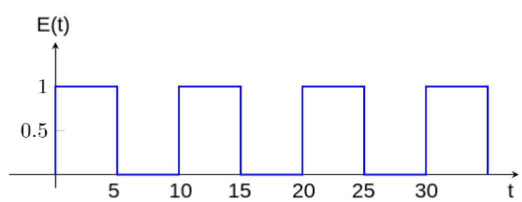
6. Determine the Laplace transform of the function below. (Hint: You might need the compound angle formula.)

[redacted]

ANS: $F(s) = \frac{e^2 e^{-s}}{s-2} - \frac{e^{-\pi s}}{s^2+1}$

7. Determine the Laplace transform of the periodic voltage supply of the resistor-inductor circuit below.





ANS $L\{E(t)\} = \sum_{n=0}^{\infty} \frac{e^{-10n} - e^{-(10n+5)}}{s}$.

DIY

4. Using both $\overbrace{\text{integration}}^{\text{DIY}}$ and the t-domain shifting property., determine the Laplace transform of the following function. Are they equivalent?

$$f(t) = \begin{cases} 0, & t < 2 \\ (t-1)^2, & t \geq 2 \end{cases} \rightarrow F(s) = \int_2^{\infty} (t-1)^2 e^{-st}\, dt$$

since $\int_0^2 0 e^{-st}\, dt = 0$.

ANS: $F(s) = e^{-2s}\left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}\right)$. Yes.

$$f(t) = (t-1)^2 u(t-2) = (t-2+1)^2 u(t-2)$$

$$= [\underbrace{(t-2)^2}_{g_1(t-2)} + 2(t-2) + 1]\, u(t-2)$$

$g_1(t-2) \downarrow g_1(t) = t^2 \downarrow G_1(s)$

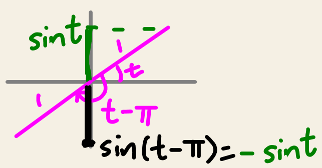
$g(t-a)u(t-a)$	$e^{-as}G(s)$

$$F(s) = e^{-2s}\left[\underbrace{\frac{2}{s^3}}_{G_1(s)} + \frac{2}{s^2} + \frac{1}{s}\right] //$$

6. Determine the Laplace transform of the function below. (Hint: You might need the compound angle formula.)

$$f(t) = \begin{cases} e^{2t}, & 1 \leq t < \pi \\ \sin t + e^{2t}, & t \geq \pi \end{cases}$$

ANS: $F(s) = \frac{e^2 e^{-s}}{s-2} - \frac{e^{-\pi s}}{s^2+1}$



$$f(t) = e^{2t}u(t-1) + (\sin t + \cancel{e^{2t}} - \cancel{e^{2t}})\, u(t-\pi)$$

$$= e^{2(t-1+1)} u(t-1) + \sin(\underbrace{t-\pi}_{x} + \underbrace{\pi}_{y})\, u(t-\pi)$$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$$= e^2 \underbrace{e^{2(t-1)}}_{g(t-1)} u(t-1) + [\underbrace{\sin(t-\pi)}_{g(t-\pi)} \underbrace{\cos\pi}_{-1} + \cancel{\cos(t-\pi) \underbrace{\sin\pi}_{0}}]\, u(t-\pi)$$

$g(t-1) \downarrow g(t) = e^{2t} \downarrow G(s)$

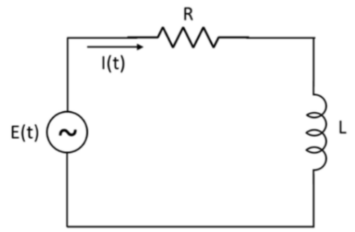
$g(t-\pi) \downarrow g(t) = \sin t \downarrow G(s)$

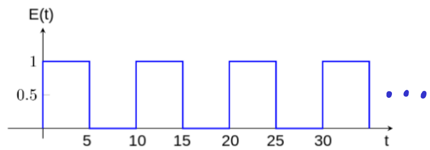
$g(t-a)u(t-a)$	$e^{-as}G(s)$

$\hookrightarrow g(t) \longrightarrow G(s)$

$$F(s) = e^2 \cdot \underbrace{\frac{1}{s-2}}_{G(s)} e^{-s} - \underbrace{\frac{1}{s^2+1}}_{G(s)} e^{-\pi s}$$

7. Determine the Laplace transform of the periodic voltage supply of the resistor-inductor circuit below.





correction

ANS $$L\{E(t)\} = \sum_{n=0}^{\infty} \frac{e^{-10ns} - e^{-(10n+5)s}}{s}.$$

$$E(t) = 1\,u(t) - u(t-5) + u(t-10) - u(t-15) + u(t-20) - u(t-25) + \cdots$$

$$= \sum_{n=0}^{\infty} \left[u(t-10n) - u(t-10n-5)\right]$$

$$= \sum_{n=0}^{\infty} \left[u(t-10n) - u(t-(10n+5))\right]$$

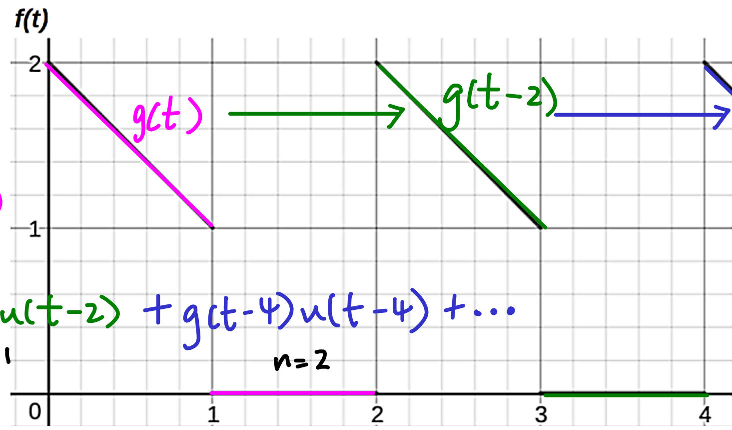
$$\mathcal{L}\{E(t)\} = \sum_{n=0}^{\infty} \mathcal{L}\left[u(t-10n) - u(t-(10n+5))\right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{-10ns}}{s} - \frac{e^{-(10n+5)s}}{s}\right] //$$

~~8.~~ DIY A periodic function f(t) is defined by the following waveform. Given that g(t) represents one cycle of f(t) in [0, 2], define g(t) using the unit-step function. Hence, determine the Laplace transform of f(t).

$$g(t) = \begin{cases} 2-t, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$$

$$= (2-t)u(t) - (2-t)u(t-1)$$



$$f(t) = \underbrace{g(t)u(t)}_{n=0} + \underbrace{g(t-2)u(t-2)}_{n=1} + \underbrace{g(t-4)u(t-4)}_{n=2} + \dots$$

$$= \sum_{n=0}^{\infty} g(t-2n)u(t-2n)$$

ANS: $g(t) = (2-t)u(t) - (2-t)u(t-1)$. $F(s) = [\frac{2}{s} - \frac{1}{s^2} + e^{-s}(\frac{1}{s^2} - \frac{1}{s})] \sum_{n=\cancel{1}\,0}^{\infty} e^{-2ns}$.

correction

For more practice problems (& explanations), check out:

1) https://math.libretexts.org/Bookshelves/Differential_Equations/Differential_Equations_for_Engineers_(Lebl)/6%3A_The_Laplace_Transform/6.E%3A_The_Laplace_Transform_(Exercises)

End of Tutorial 7

(Email to youliangzheng@gmail.com for assistance.)

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\left\{\sum_{n=0}^{\infty} g(t-2n)u(t-2n)\right\} = \sum_{n=0}^{\infty} \mathcal{L}\{g(t-2n)u(t-2n)\} = \sum_{n=0}^{\infty} \underline{\underline{G(s)}} \cdot e^{-2ns} \;//$$

where $G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{(2-t)u(t) - [2-(t-1+1)]u(t-1)\}$

$$= \mathcal{L}\{(2-t)u(t) - [1-(t-1)]u(t-1)\}$$

$$= \frac{2}{s} - \frac{1}{s^2} - e^{-s}\left[\frac{1}{s} - \frac{1}{s^2}\right] \;//$$