Topic 4
Linear Transformation & Eigendecomposition

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Outline

- Definition of a Linear Transformation
- Determinant of a Matrix
- Inverse of a Matrix
- Solving A**v** = **b** using Matrix Inversion
- Eigenvalues & Eigenvectors
- Eigendecomposition of a Matrix

Recap: Matrix Notation of a System of Linear Eqns

A system of 2 linear equations can be written using matrices in the form:

$$\begin{aligned} 2x + 3y &= 1 \\ x - 7y &= -14 \end{aligned} \longrightarrow \begin{bmatrix} 2 & 3 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$$A\mathbf{v} = \mathbf{b}$$

where A is a matrix of coefficients, **v** is a vector of variables and **b** is a vector of constants.

Recap: Matrix Notation of a System of Linear Eqns

Similarly, a system of n linear equations can be written using matrices in the form:

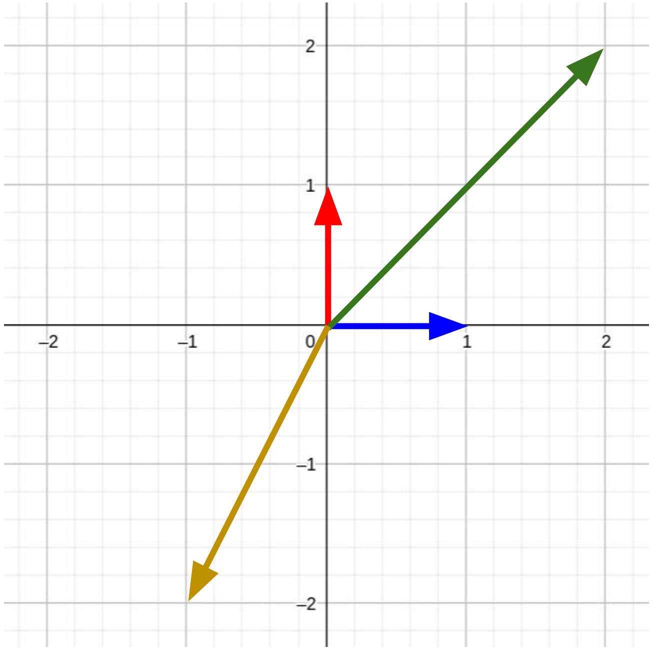
$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A\mathbf{v} = \mathbf{b}$$

From another perspective, one can say matrix A 'transforms' a vector **v** into another vector **b**. This is an example of a **linear transformation** in linear algebra.

Linear Transformation of a Vector

Graphically, lets see what happens in such a linear transformation by an example:



$$A\mathbf{v} = \mathbf{u}$$

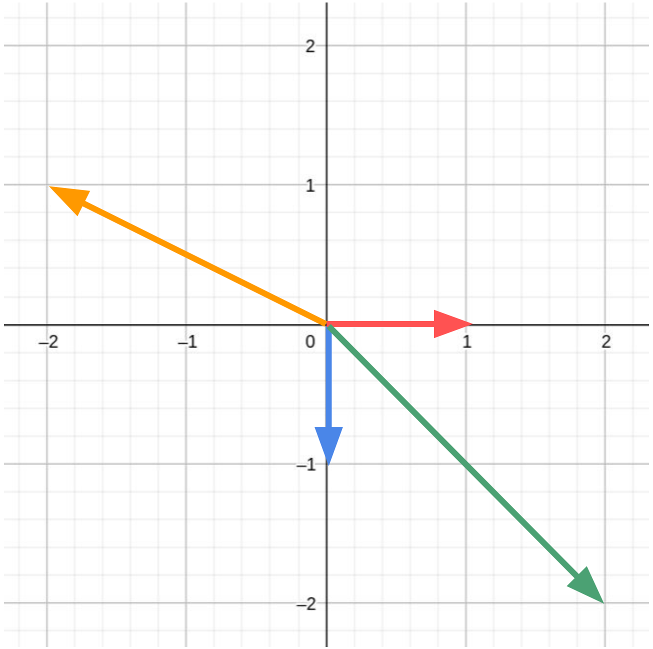
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$



Linear Transformation of a Vector

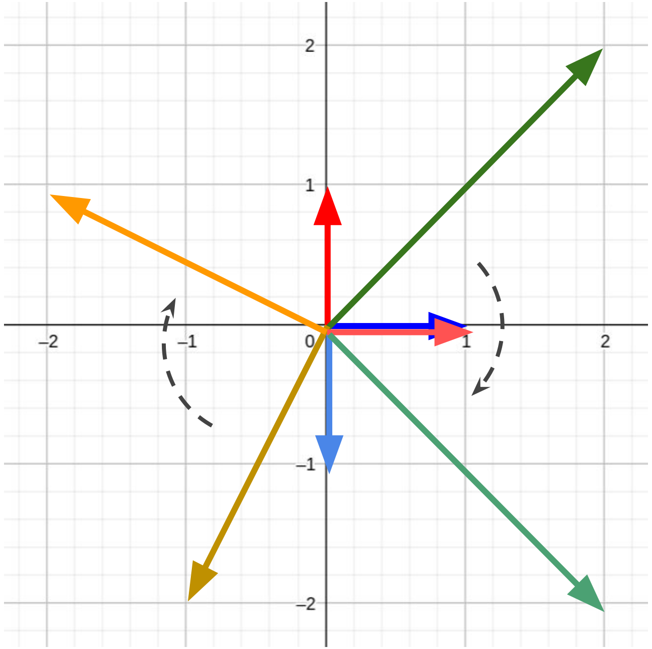
The linear transformation given by matrix A represents 90° rotation clockwise.

$$A\mathbf{v} = \mathbf{u}$$

$$\begin{bmatrix} \boxed{\begin{matrix} 0 \\ -1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[\mathbf{A}_1 \quad \mathbf{A}_2] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\mathbf{A}_1} \qquad [\mathbf{A}_1 \quad \mathbf{A}_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\mathbf{A}_2}$$



In fact, just by looking at how the basis vectors, **i** and **j**, changes, we can visualize how the entire 'vector space' would be changed by the transformation. The columns of A represent the transformed basis vectors.

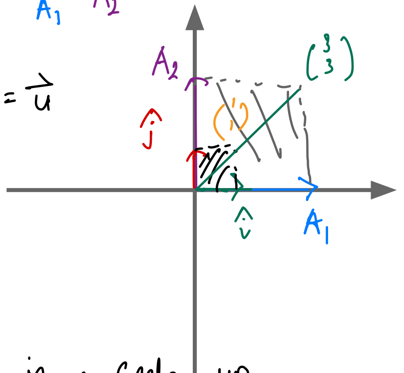
Linear Transformation

Exercise: Describe the following linear transformations (Hint: Draw a parallelogram formed by the basis vectors before and after the transformation.)

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

A_1 A_2

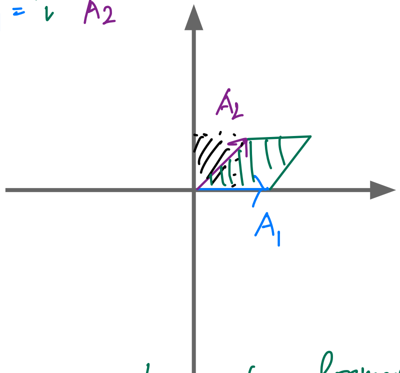
$3I\vec{v} = \vec{u}$



A is a scale-up transformation by 3x.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

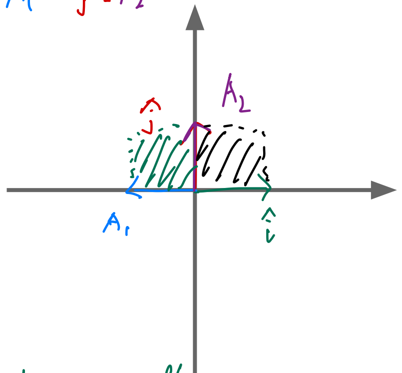
$A_1 = \hat{i}$ A_2



A is a shearing transformation to the right

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

A_1 $\hat{j} = A_2$



A is a reflection transformation about y-axis

Formal Definition of a Linear Transformation

A linear transformation, T: $\boldsymbol{V} \mapsto \boldsymbol{U}$ is one that takes an input vector space $\boldsymbol{V}$ and produces an output vector space $\boldsymbol{U}$, such that it satisfies:

For any $\mathbf{v}_1, \mathbf{v}_2 \in \boldsymbol{V}$ and any scalar $k \in \mathbb{R}$,

1. $T(\mathbf{v}_1+\mathbf{v}_2) = T(\mathbf{v}_1)+\mathrm{T}(\mathbf{v}_2)$
2. $T(k\mathbf{v}) = kT(\mathbf{v})$

A linear transformation is also known as a linear mapping or linear function.

Linear Transformation

Example: Show that the function defined by T(**v**) = A**v**, where A is a 2 x 2 matrix, is a linear transformation.

Check the 2 conditions :

① $T(\vec{v}_1 + \vec{v}_2) = A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = T(\vec{v}_1) + T(\vec{v}_2)$ ✓

② $T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v})$ ✓

so $T(\vec{v}) = A\vec{v}$ is a linear transformation

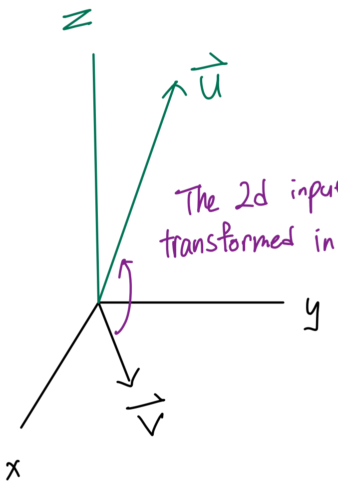
Linear Transformation

Exercise: Explain the mapping defined by T: $\mathbb{R}^n \mapsto \mathbb{R}^m$, T(**v**) = A**v**, where A is an m x n matrix. Draw a graphical representation for m = 3 and n = 2. Is it a linear transformation?

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(\vec{v}) = A\vec{v} = \vec{u}$

$$\underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{(3\times 2)} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{(2\times 1)} = \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{(3\times 1)}$$



The 2d input vector v is transformed into a 3d input vector u

$T(\vec{v}) = A\vec{v}$ is also a linear transformation here since the 2 conditions stated earlier also satisfy

Matrix Multiplication as Composition of Linear Transformations

One can also interpret that matrix multiplication corresponds to a composition of linear transformations. For example, given the linear transformations defined by

$$T_1(\mathbf{v}) = A\mathbf{v}, \quad T_2(\mathbf{v}) = B\mathbf{v}$$

The composition of linear transformations gives

$$T_1(T_2(\mathbf{v})) = A(B\mathbf{v}) = AB\mathbf{v}, \quad T_2(T_1(\mathbf{v})) = B(A\mathbf{v}) = BA\mathbf{v} \neq T_1(T_2(\vec{v}))$$

which corresponds to the multiplication of matrices A and B. Geometrically, this means when matrices are multiplied, the transformations to a vector space are "cascaded". Eg.

Special case of $AB\vec{v} = BA\vec{v}$

since $AB\vec{v} = 3IB\vec{v}$

$= 3BI\vec{v}$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} \mathbf{v}$$

(3x Scaling) (Shear) = (3x Scaling & Shear)

Determinant of a Matrix

The determinant of a 2 x 2 square matrix is a scalar that represents the (signed) area scale factor of a linear transformation in $\mathbb{R}^2$. It can be computed from the matrix elements according to:

$$\det(A) \text{ or } |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example, for the scaling matrix below, its determinant is 9 which means the area of the vector space is expanded by 9 times after the scaling transformation. Correspondingly, for the shearing matrix, its determinant is 1 since the vector space is neither expanded nor contracted by the shear transformation. If the vector space is "flipped" by the transformation, then the determinant is negative.

$$\det \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 9 \qquad \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \qquad \det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -1$$

The formal derivation of the determinant is presented later in the matrix inverse.

Determinant of a Matrix

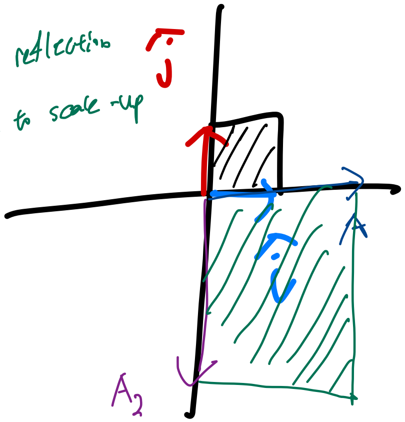
Example: Determine the determinant of matrix A below and justify its value graphically.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = -4$$

due to reflection

due to scale-up



Since A is a 4x scale up with reflection transformation, the det is -4 which agrees with the transformation being applied

ANS: det(A) = -4

Determinant of a Matrix

For a 3 x 3 matrix, the determinant represents the (signed) volume scale factor of a linear transformation in $\mathbb{R}^3$. It can be computed by:

$$
\begin{aligned}
|A| &= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b\begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c\begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} \\
&= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= a(ei - fh) - b(di - fg) + c(dh - eg)
\end{aligned}
$$

Row 1 has been used as a ‘pivot’ above. Note that other rows or columns can also be used as pivots as well (preferably one with more zero elements).

Determinant of a Matrix

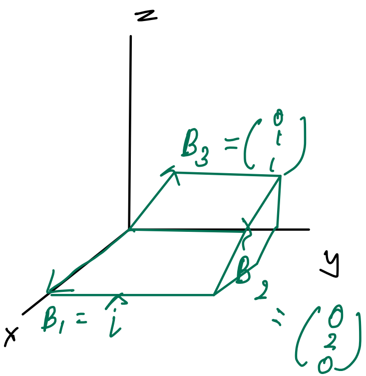
Example: Determine the determinant of matrix B and justify its value graphically.

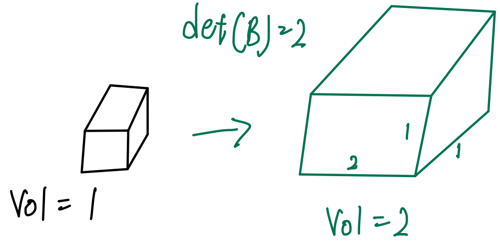
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(B) = 1\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$$

$$= 1\big(2(1) - 0(1)\big) - 0 + 0$$

$$= 2$$

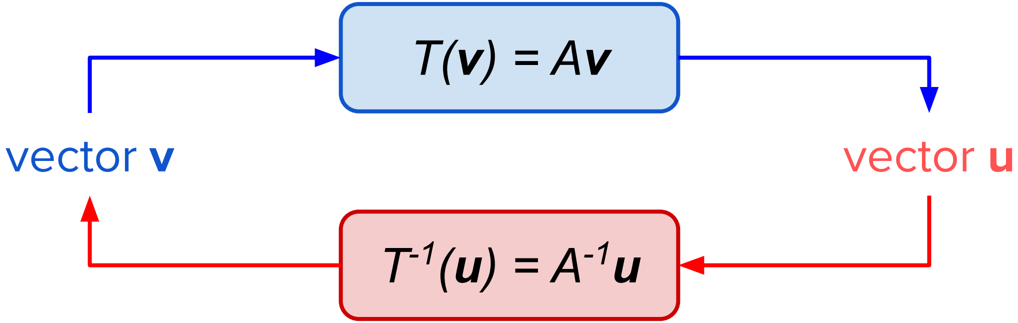




ANS: det(B) = 2

Inverse of a Linear Transformation

Similar to the inverse of a function, the inverse of a linear transformation **might exist** such that:



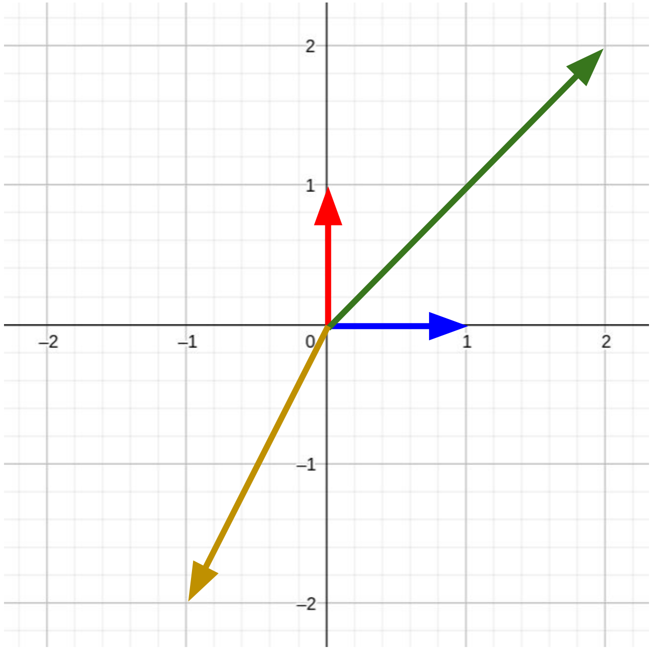
which follows the definition of an inverse given by:

$$T\left(T^{-1}(\mathbf{v})\right) = T^{-1}(T(\mathbf{v})) = \mathbf{v}$$

$$AA^{-1}\mathbf{v} = A^{-1}A\mathbf{v} = \mathbf{v}$$

Inverse of a Linear Transformation

Graphically, by a previous example, this means:



$$A\mathbf{v} = \mathbf{u}$$

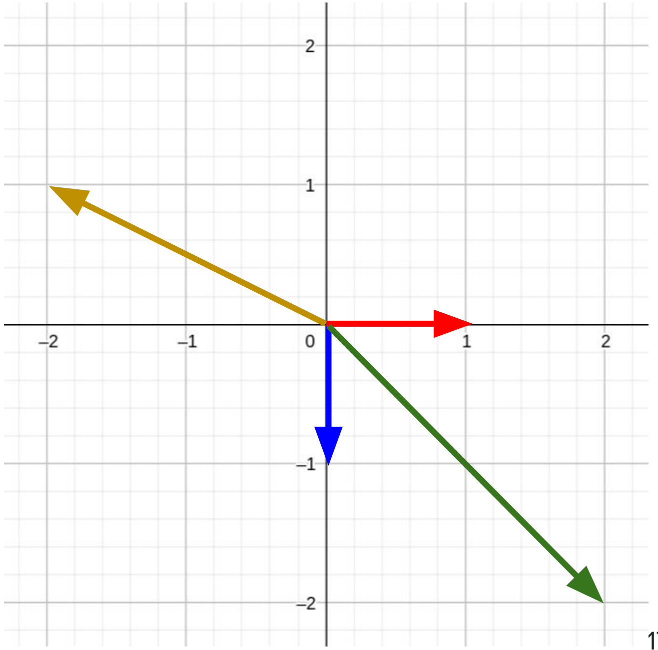
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

90° clockwise rotation

90° anti-clockwise rotation

$$A^{-1}\mathbf{u} = \mathbf{v}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



Inverse of a Matrix

$A\vec{v} = \vec{b} \rightsquigarrow [A | \vec{b}] \xrightarrow{\text{row ops}} [I | \vec{v}]$

$AA^{-1} = I \quad [A | I] \xrightarrow{\text{row ops}} [I | A^{-1}]$

The inverse of a matrix can be derived from from Gauss-Jordan elimination (row operations).

For example, a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, row ops on the augmented matrix give:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{\frac{a}{c}R_2 - R_1} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{c} & -1 & \frac{a}{c} \end{array}\right] \xrightarrow{\frac{c}{ad-bc}R_2} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{\frac{1}{b}R_1 - R_2} \left[\begin{array}{cc|cc} \frac{a}{b} & 0 & \frac{ad}{b(ad-bc)} & \frac{-a}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \xrightarrow{\frac{b}{a}R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{matrix}}_{A^{-1}}$$

So the inverse of A is:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} A_{adj}$$

Inverse of a Matrix

A square n x n matrix A is called invertible if there exist a square n x n matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix and A^{-1} is known as the inverse of matrix A given by:

$$A^{-1} = \frac{1}{\det(A)} A_{adj}$$

A_{adj} is known as the adjugate (or adjoint) of matrix A given by the transpose of its cofactor matrix.

For 2x2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A_{adj} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ Simply swap the diagnosis & multiply -1 to the off diagonals

Inverse of a Matrix

Example: Determine the inverse of matrix A and verify using the definition of the inverse. In layman, what does the inverse of A do?

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} A_{adj} = \frac{1}{2-0}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} //$$

$$\text{Check : } AA^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\left(\frac{1}{2}\right)\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

since A is a horizontal elongation with shearing transformation, A^{-1} reverses the transformation performed by A.

ANS: $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

Inverse of a Matrix

For a 3 x 3 matrix A, obtaining its inverse is much more tedious. The formula is:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \qquad A^{-1} = \frac{1}{\det(A)} A_{adj}$$

$$A_{adj} = C^T = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^T$$

where C is known as the cofactor matrix.

Inverse of a Matrix

Exercise: Determine the inverse of the matrix below and check using the definition of the inverse.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$A_{adj} = C = \begin{bmatrix} (0-0) & -(0+3) & (0+6) \\ -(0-0) & (0-3) & -(0-0) \\ (0+2) & -(-1-2) & (-2-0) \end{bmatrix}^T = \begin{bmatrix} 0 & -3 & 6 \\ 0 & -3 & 0 \\ 2 & 3 & -2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 0 & 2 \\ -3 & -3 & 3 \\ 6 & 0 & -2 \end{bmatrix}$$

$$\det(A) = 3\begin{vmatrix} 0 & 1 \\ -2 & -1 \end{vmatrix} - 0 + 0$$

$$= 3(0(-1) - (-2)(1))$$

$$= 3(2)$$

$$= 6$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 0 & 2 \\ -3 & -3 & 3 \\ 6 & 0 & -2 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}\left(\frac{1}{6}\right)\begin{bmatrix} 0 & 0 & 2 \\ -3 & -3 & 3 \\ 6 & 0 & -2 \end{bmatrix}$$

ANS: $A^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 0 & 2 \\ -3 & -3 & 3 \\ 6 & 0 & -2 \end{bmatrix}$

Singular Matrix

A square n x n matrix A is known to be singular, or not invertible, if its determinant is zero. This happens when the linear transformation defined by T(**v**) = A**v** results in a vector space of zero area/volume.

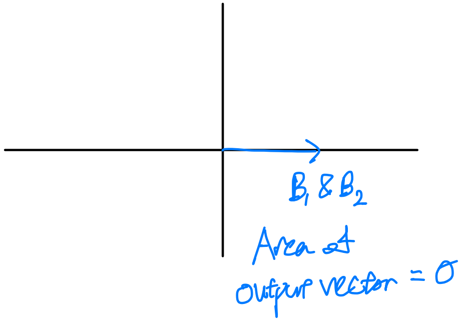
Example: Matrices A and B below are both singular. Justify graphically.

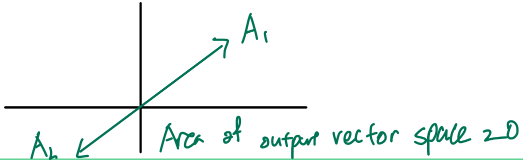
$$A = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\det(A) = -2 - (-2)$

$= 0$

$\det(B) = 0$





Inverse of a Matrix

Exercise: Determine the inverse of the following matrices, if it exists.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & 8 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\det(A) = 1(1-16) - 0 + 3(4+1)$$
$$= -15 + 15$$
$$= 0$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \cdot C^T, \text{ since } \det(A) = 0$$

A^{-1} is undefined

$$\det(B) = -1(4-2) - 0 + 3(4-1)$$
$$= -2 + 9$$
$$= 7$$

$$C^T = \begin{bmatrix} (4-2) & -(0-6) & (0-3) \\ -(8-1) & (-4-3) & -(-1-6) \\ (4-1) & -(-2-0) & (-1-0) \end{bmatrix}^T = \begin{bmatrix} 2 & 6 & -3 \\ -7 & -7 & 7 \\ 3 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & -7 & 3 \\ 6 & -7 & 2 \\ -3 & 7 & -1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -7 & 3 \\ 6 & -7 & 2 \\ -3 & 7 & -1 \end{bmatrix}$$

ANS: A^{-1} DNE, $B^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -7 & 3 \\ 6 & -7 & 2 \\ -3 & 7 & -1 \end{bmatrix}$

Using Matrix Inverse to Solve a SLE

Since a system of linear equations can be written in matrix notation as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A\mathbf{v} = \mathbf{b}$$

Using the inverse of matrix A, the vector of unknown variables, **v**, can be solved by:

$$A\mathbf{v} = \mathbf{b}$$
$$A^{-1}A\mathbf{v} = A^{-1}\mathbf{b}$$
$$\boxed{\mathbf{v} = A^{-1}\mathbf{b}}$$

Using Matrix Inverse to Solve a SLE

Notice that in $\mathbf{v} = A^{-1}\mathbf{b}$, because **b** is a constant vector, **v** is only defined if matrix A is invertible. And in this case, **v** must be a unique solution. Hence, matrix inversion can only be used to solve consistent SLEs that are linearly independent where the rank of matrix A equals the number of unknown variables (in **v**).

For SLEs that are inconsistent or have infinite solutions, we can depend on Gauss-Jordan elimination.

Using Matrix Inverse to Solve a SLE

Example: Solve the following SLEs using the matrix inverse.

a) $3x - 4y = 1$
$2x + 3y = 12$

$$\underbrace{\begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

$$= \frac{1}{17} \begin{pmatrix} 3 & 4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 12 \end{pmatrix} = \frac{1}{17} \begin{bmatrix} 51 \\ 34 \end{bmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

b) $x + z = 5$
$2x - 2y - z = 2$
$x = 3$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$$

$$\det(A) = 2$$

$$C^T = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -1 & 0 \\ 2 & 3 & -2 \end{bmatrix}^T \qquad A^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 2 \\ -1 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$$

ANS: **a)** x = 3, y = 2. **b)** x = 3, y = 1, z = 2.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & 2 \\ -1 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

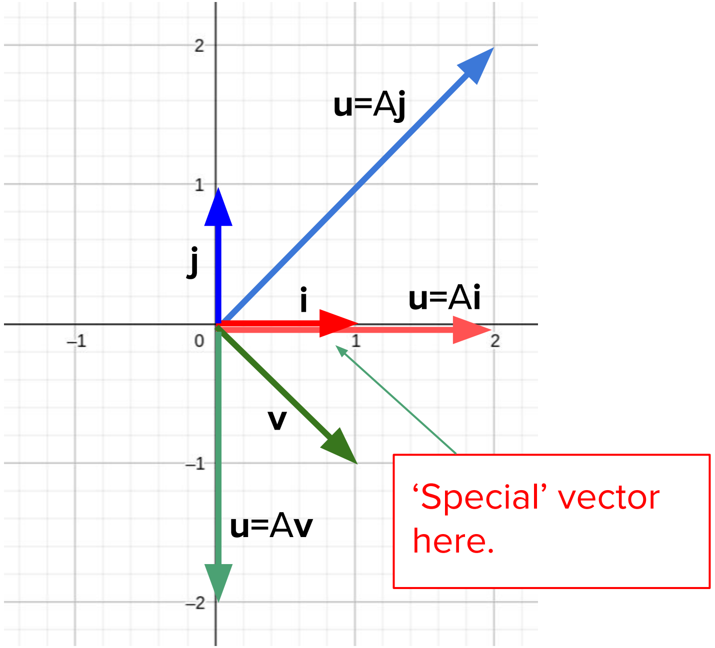
$\not=$

'Special' Vectors in a Linear Transformation

Usually, in a linear transformation, most input vectors **v** would undergo a rotation to become the output vector **u**. For example, the shear + scale transformation given by A gives

$$\mathbf{u} = A\mathbf{v} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \mathbf{v}$$

$$\rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \quad (\text{rotated CW by } 45^\circ)$$



From the graph, it looks like most input vectors are rotated by matrix A. However, notice that the vector $[1 \ \ 0]^T$ remains on its span.

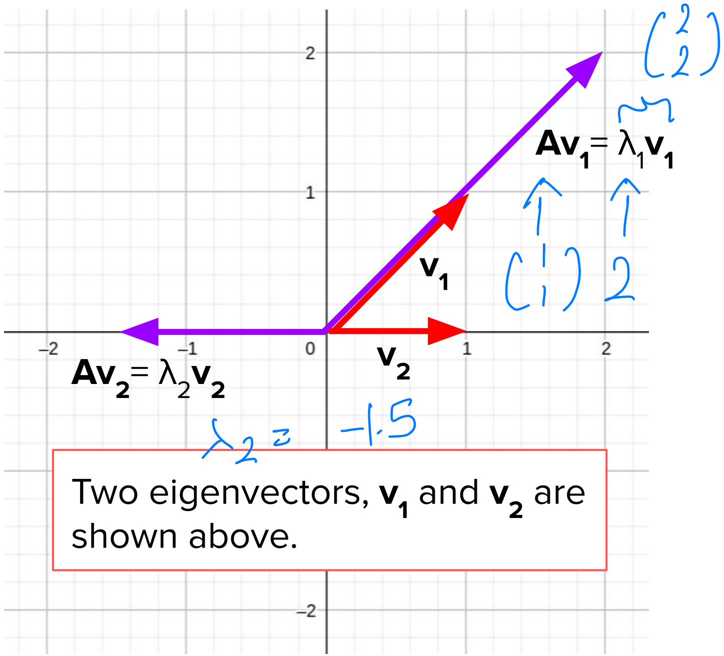
Eigenvalues & Eigenvectors

In a linear transformation T(**v**) = A**v**, there might be some 'special' vectors **v** that satisfy:

$$A\mathbf{v} = \lambda \mathbf{v}$$

Eigenvalue Eigenvector

The geometrical meaning of the above equation means that eigenvector v only scales by eigenvalue λ when transformed by matrix A. This implies eigenvectors do not change their orientation during the transformation, except possibly reversing their direction.



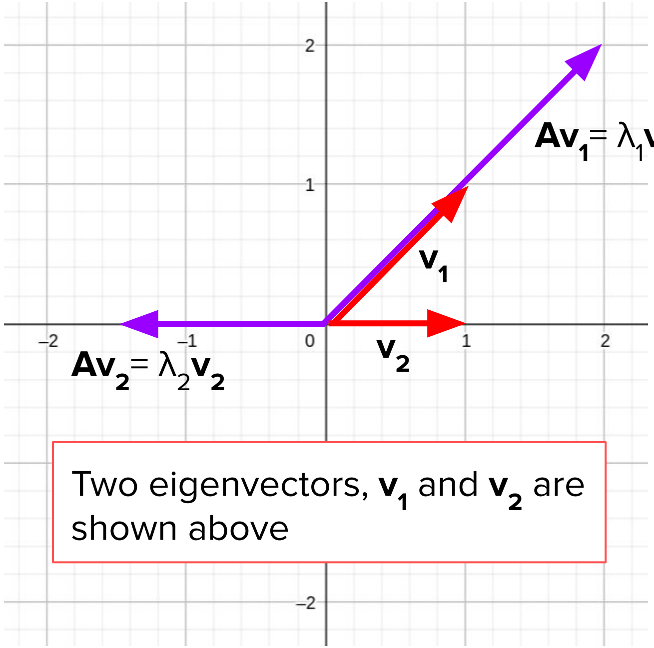
Two eigenvectors, $\mathbf{v_1}$ and $\mathbf{v_2}$ are shown above.

Eigenvalues & Eigenvectors

Example: Write down the eigenvalues and eigenvectors from the graph shown.

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_1 = 2$

$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = -1.5$



Two eigenvectors, $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown above

Solving for Eigenvalues & Eigenvectors

Given the transformation matrix A, from the eigenvector equation, we have:

$$A\mathbf{v} = \lambda\mathbf{v} = \lambda I\mathbf{v}$$

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \leftarrow \text{Zero vector}$$

$$\mathbf{v} = (A - \lambda I)^{-1}\mathbf{0}$$

Notice that if the inverse of (A-λI) exists, then the eigenvector would just be the zero vector (trivial solution). Since we do not want that, we need:

$$\det(A - \lambda I) = 0$$

such that $(A-\lambda I)^{-1}$ would not exist. In this case, we could possibly find the non-zero eigenvector/s **v**.

Solving for Eigenvalues & Eigenvectors

Hence, we first solve the 'characteristic polynomial equation' from:

$$\det(A - \lambda I) = 0$$

for the eigenvalues, λ_i, then substitute each λ_i into:

$$(A - \lambda_i I)\mathbf{v}_i = \mathbf{0}$$

and solve for each eigenvector, $\mathbf{v}_i$.

Note that eigenvectors are not unique. All vectors along the same span can be an eigenvector.

The following example will clarify the above.

Solving for Eigenvalues & Eigenvectors

Example: Given the matrix A below, determine its eigenvalues and eigenvectors. Sketch the eigenvectors and see if they change their orientation.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

To get λs:

$$\det(A - \lambda I) = 0$$

$$(3-\lambda)(2-\lambda) - 0 = 0$$

ANS: $\lambda_{1,2} = 2, 3$. $\mathbf{v}_{1,2} = [-1\ \ 1]^T, [1\ \ 0]^T$.

$$\lambda_1 = 2 \quad , \quad \lambda_2 = 3$$

To get $\vec{v}_1$, for $\lambda_1 = 2$,

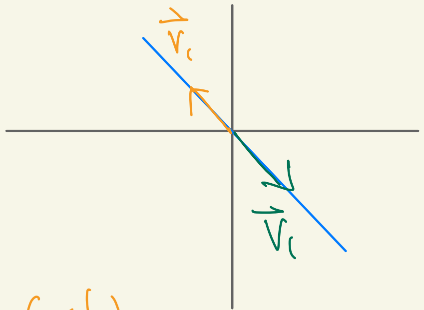
$$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$$

$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x + y = 0$$

$$y = -x$$



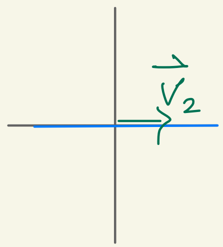
Choose $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

To get $\vec{v}_2$, for $\lambda_2 = 3$,

$$(A - \lambda_2 I)\vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$0x + \underbrace{y = 0}_{x\text{-axis}} \rightarrow \text{Choose } \vec{V}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Check: $A\vec{v} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$$= 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Check: $A\vec{v}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solving for Eigenvalues & Eigenvectors

Exercise: Determine the eigenvalues and eigenvectors of matrix B.

$$B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ -2 & 1 & -1 \end{bmatrix} \rightarrow B - \lambda I = \begin{bmatrix} 1-\lambda & 3 & 0 & + \\ 0 & 2-\lambda & 0 & - \\ -2 & 1 & -1-\lambda & + \end{bmatrix}$$

$$Tr(B) = 1+2-1 = 2 = \lambda_1 + \lambda_2 + \lambda_3$$

$$\det(B - \lambda I) = (-1-\lambda)\left[(1-\lambda)(2-\lambda) - 0\right] = 0$$

$$\lambda_{1,2,3} = -1, 1, 2$$

ANS: $\lambda_{1,2,3}$ = -1, 1, 2. $\mathbf{v}_{1,2,3}$ = $[0\ \ 0\ \ 1]^T$, $[1\ \ 0\ \ -1]^T$, $[9\ \ 3\ \ -5]^T$.

To find $\vec{v}_1$ for $\lambda_1 = -1$

$$(B-\lambda I)\vec{v}_1 = \begin{bmatrix}2 & 3 & 0\\ 0 & 3 & 0\\ -2 & 1 & 0\end{bmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \vec{0}$$

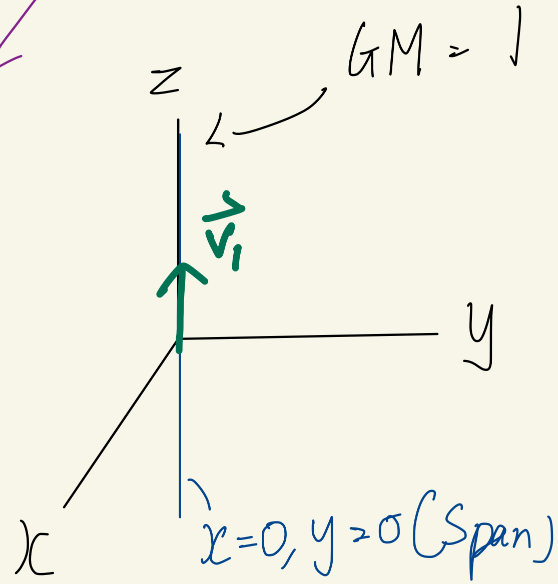
$2x+3y=0$

$x=0$

$3y=0 \to y=0$

$-2x+\cancel{y}=0 \to x=0$

$\therefore$ choose $\vec{v}_1 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$



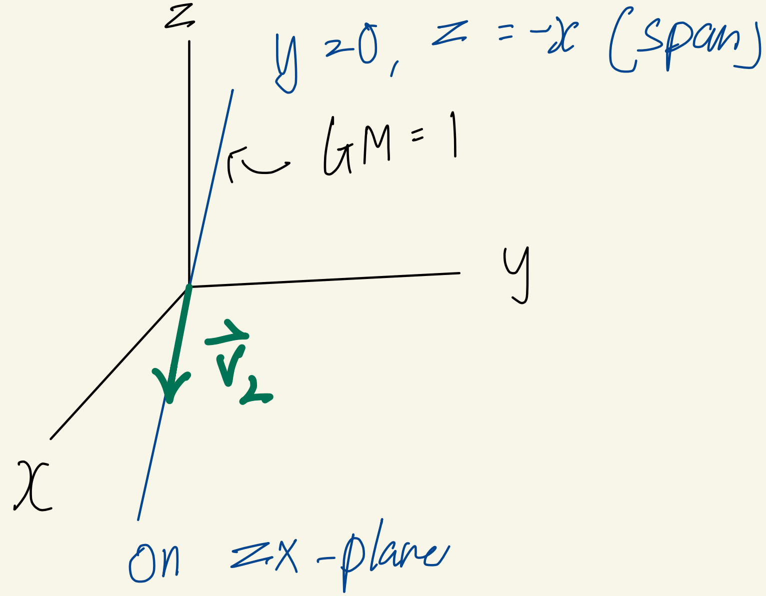
To find $\vec{v}_2$ for $\lambda_2 = 1$

$$(B-\lambda_2 I)\vec{v}_2 = \begin{bmatrix}0 & 3 & 0\\ 0 & 1 & 0\\ -2 & 1 & -2\end{bmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \vec{0}$$

$y=0$

$-2x+\cancel{y}-2z=0 \to z=-x$

Choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$



To find $\vec{v}_3$ for $\lambda_3 = 2$

$$(B - \lambda_3 I) = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ -2 & 1 & -3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-x + 3y = 0 \to x = 3y$

$-2x + y - 3z = 0$

$-2(3y) + y - 3z = 0$

$-5y - 3z = 0 \to z = -\frac{5}{3}y$

Choose $\vec{v}_3 = \begin{pmatrix} 3 \\ 1 \\ -5/3 \end{pmatrix}$

or

Choose $\vec{v}_3 = \begin{pmatrix} 9 \\ 3 \\ -5 \end{pmatrix}$

no fractions

Trace of a Matrix

The trace of a square matrix is defined as the sum of its diagonal elements, which is always equal to the sum of its eigenvalues.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \longrightarrow Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Verify the above for the previous examples.

Solving for Eigenvalues of a 2x2 Matrix

Exercise: For any 2x2 matrix below, show that its eigenvalues are the roots of the quadratic equation: $\lambda^2 - Tr(A)\ \lambda + \det(A) = 0.$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= ad - (a+d)\lambda + \lambda^2 - bc$$

$$= \lambda^2 - \underbrace{(a+d)}_{Tr(A)}\lambda + \underbrace{(ad - bc)}_{\det(A)} = 0$$

Algebraic & Geometric Multiplicity of Eigenvalues

The algebraic multiplicity of an eigenvalue λ_i is the largest integer k for which

GM

$$(\lambda - \lambda_i)^k$$

is a factor of the characteristic polynomial from:

$$\det(A - \lambda I) = 0$$

AM

The geometric multiplicity of the eigenvalue λ_i is the dimension of the eigenspace solved from the eigenvector equation:

$$(A - \lambda_i I)\,\mathbf{v_i} = \mathbf{0}$$

The following exercise will clarify the above definitions.

Solving for Eigenvalues & Eigenvectors

Exercise: Determine the eigenvalues and eigenvectors of A. Describe the eigenspaces geometrically.

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ -3 & 3 & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} -1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ -3 & 3 & 2-\lambda \end{bmatrix} = (2-\lambda)\left[(-1-\lambda)(2-\lambda) - 0\right] = 0$$

$$(2-\lambda)^2 \, \underbrace{(-1-\lambda)} = 0$$

$\to$ AM $= 2$

$$\lambda_{2,3} = 2 \qquad \lambda_1 = -1$$

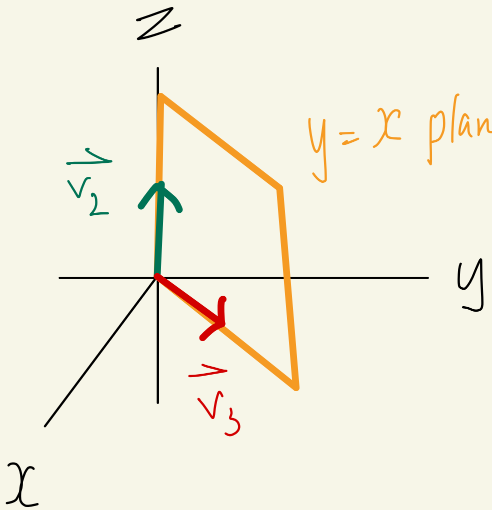
$\lambda_{1,2,3}$ = -1, 2, 2. $\mathbf{v}_{1,2,3}$ = $[1\ \ 0\ \ 1]^T$, $[1\ \ 1\ \ 0]^T$, [$[0\ \ 0\ \ 1]^T$.

Solve $\vec{v}_2$ for λ_2

$$(A - \lambda_2 I)\vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-3x + 3y = 0 \rightarrow y = x$



$y = x$ plane GM = 2

Choose $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Eigendecomposition / Diagonalization

One important application of eigenvalues and eigenvectors is the eigendecomposition (or diagonalization) of a transformation matrix. From the eigenvector equation,

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$[A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n]$$

$$A\underbrace{[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]}_{\text{matrix of eigenvectors, } P} = \overbrace{[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]}^{P} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{D}$$

p^{-1}

$$AP = PD \; P^{-1}$$

$$A = PDP^{-1}$$

So matrix A is ‘decomposed’ into 3 matrices containing eigenvalues and eigenvectors, hence called **eigendecomposition**.

Eigendecomposition / Diagonalization

So, to obtain the eigendecomposition (or diagonalization) of a matrix, we just put the matrix in the form:

$$A = PDP^{-1}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

(matrix of column eigenvectors)

$$P^{-1} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}^{-1}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

(**diagonal** matrix of eigenvalues)

Eigendecomposition is also known as spectral decomposition. Some matrices are non-diagonalizable and will be addressed in Math 3.

Eigendecomposition / Diagonalization

Example: From the previous example, diagonalize matrix A. Verify that $PDP^{-1} = A$.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$\lambda_1 = 2$, $\lambda_2 = 3$

$\vec{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

* order of λs must be same as $\vec{v}$s

$$P = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A = PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

ANS: $A = PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Eigendecomposition / Diagonalization

A matrix in its diagonalized form is easy to manipulate in math operations. Eg, to compute A^2, we have:

$$\begin{aligned} A^2 &= AA = \left(PDP^{-1}\right)\left(PDP^{-1}\right) \\ &= PD\left(P^{-1}P\right)DP^{-1} \\ &= PD(I)DP^{-1} \\ &= PDDP^{-1} = PD^2P^{-1} \end{aligned}$$

Eg) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$

Since D is a diagonal matrix, D^2 just means squaring each diagonal element.

By repeating the above, we can get $A^n = PD^nP^{-1}$. So to obtain A raised to power n, simply raise each element of D to the same power.

Elementary Functions of a Diagonalizable Matrix

Combined with Maclaurin series, we can get the various functions of A, such as e^A, ln (A), sin(A) etc. For example, we can derive:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

From Maclaurin series for e^x.
(Topic 7. Don't worry about it for now.)

$$= \sum_{n=0}^{\infty} \frac{PD^nP^{-1}}{n!} = P\left(\sum_{n=0}^{\infty} \frac{D^n}{n!}\right)P^{-1}$$

$$= Pe^DP^{-1} = P\begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix} P^{-1}$$

The same can be done for other elementary functions of a diagonalizable matrix A. Simply apply the function to each eigenvalue in the diagonal matrix D.

Elementary Functions of a Diagonalizable Matrix

Example: From the previous example, compute $7e^A$. Compute A^{-1} & verify

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad A = PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$7e^A = 7Pe^DP^{-1}$$

$$= 7\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1} = PD^{-1}P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{\det A} A_{Adj}$$

ANS: $7e^A = 7\begin{bmatrix} e^3 & -e^2 + e^3 \\ 0 & e^2 \end{bmatrix}$

Elementary Functions of a Diagonalizable Matrix

Exercise: From an earlier exercise, diagonalize B. Hence, compute cos(B).

$$B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ -2 & 1 & -1 \end{bmatrix}$$

ANS: $\lambda_{1,2,3}$ = -1, 1, 2. $\mathbf{v}_{1,2,3}$ = $[0\ \ 0\ \ 1]^T$, $[1\ \ 0\ \ -1]^T$, $[9\ \ 3\ \ -5]^T$. $\cos B = \begin{bmatrix} \cos(1) & 3\,[\cos(2) - \cos(1)] & 0 \\ 0 & \cos(2) & 0 \\ \cos(-1) - \cos(1) & \frac{-4\cos(-1)+9\cos(1)-5\cos(2)}{3} & \cos(-1) \end{bmatrix}$

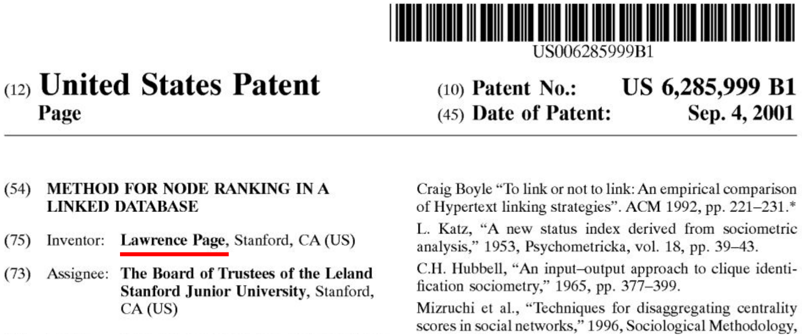
Avengers Endgame (2019)

https://youtu.be/5h3G4vKCiFE

*By evaluating the **eigenvalue** of a particle factoring in spectral decomp. (**eigendecomposition**) on an inverted Mobius strip, one can possibly time travel.*

Tony Stark

Seriously?



US006285999B1

(12) **United States Patent**
Page

(10) **Patent No.: US 6,285,999 B1**
(45) **Date of Patent: Sep. 4, 2001**

(54) **METHOD FOR NODE RANKING IN A LINKED DATABASE**

(75) Inventor: **Lawrence Page**, Stanford, CA (US)

(73) Assignee: **The Board of Trustees of the Leland Stanford Junior University**, Stanford, CA (US)

Craig Boyle "To link or not to link: An empirical comparison of Hypertext linking strategies". ACM 1992, pp. 221–231.*
L. Katz, "A new status index derived from sociometric analysis," 1953, Psychometricka, vol. 18, pp. 39–43.
C.H. Hubbell, "An input–output approach to clique identification sociometry," 1965, pp. 377–399.
Mizruchi et al., "Techniques for disaggregating centrality scores in social networks," 1996, Sociological Methodology,

End of Topic 4

*Sometimes, solving an **eigenvector** problem with an **eigenvalue** of 1 makes one a multibillionaire.*

$$A\mathbf{v} = \mathbf{v}$$

Matrix of links to webpages

Ranking eigenvector of webpages

Cofounder of Google, Lawrence Page

PageRank: A Trillion Dollar Algorithm
https://youtu.be/JGQe4kiPnrU