EDE1012 MATHEMATICS 2

Tutorial 3

Multivariable Functions & Partial Derivatives

1. Determine the domain and range of the following functions. Describe their level sets.

a) $z(x,y) = (x-1)^2 + (y-2)^2 - 3$

b) $y(x,t) = 4x^2 + 9y^2$

c) $T(x,y) = \sqrt{16 - x^2 - y^2}$

d) $f(x,y) = y/x^2$

e) $g(x,y) = e^{-(x^2+y^2)} + 1$

f) $\sigma(x,y,z) = \ln\left(x^2 + 2y^2 + 2z^2\right)$

ANS: **a)** Domain is $\mathbb{R}^2$. Range is [-3, ∞). Level curve is a circle centred at (1, 2) with radius $\sqrt{c+3}$ for c > -3, and a point at (1, 2) for c = -3. **b)** Domain is $\mathbb{R}^2$. Range is [0, ∞). Level curve is an ellipse $4x^2 + 9y^2 = c$ centred at (0, 0) for c > 0, and a point at (0, 0) for c = 0. **c)** Domain is $x^2 + y^2 \leq 16$. Range is [0, 4]. Level curve is a circle centred at (0, 0) with radius $\sqrt{16 - c^2}$ for c in [0, 4), and a point at (0, 0) for c = 4. **d)** Domain is $\mathbb{R}^2$ | x ≠ 0. Range is $\mathbb{R}$. Level curve is a parabola $y = cx^2$ for c ≠ 0, and the x-axis for c = 0. **e)** Domain is $\mathbb{R}^2$. Range is (1, 2]. Level curve is a circle with radius $\sqrt{-\ln(c-1)}$ centred at (0, 0) for c in (1, 2), and a point at (0, 0) for c = 2. **f)** Domain is $\mathbb{R}^3$ | (x, y, z) ≠ (0, 0, 0). Range is $\mathbb{R}$. Level surface is an ellipsoid $x^2 + 2y^2 + 2z^2 = e^c$ centred at (0, 0, 0).

2. Determine the limits below. (https://openstax.org/books/calculus-volume-3@7650bff/pages/4-2-limits-and-continuity)

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2 + 10y^2 + 4}{4x^2 - 10y^2 + 6}$

b) $\displaystyle\lim_{(x,y)\to(\pi/4,\,1)} \frac{y\tan x}{y+1}$

c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^4}{x^2 + 2y^2}$

e) $\displaystyle\lim_{(x,y)\to(0^+,0^+)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

f) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - y^2 - z^2}{x^2 + y^2 - z^2}$

g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

ANS: **a)** ⅔. **b)** ½. **c)** 2. **d)** 0. **e)** 0. **f)** DNE. **g)** DNE.

3. Evaluate the limits below.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2}$

b) $\displaystyle\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right)$

c) $\displaystyle\lim_{(x,y)\to(1/2,\,2/3)} \frac{(x-1/2)^3}{(2x-1)^2+(3y-2)^2}$

ANS: **a)** DNE. **b)** 1. **c)** 0.

4. Determine the region where each function below is continuous. Sketch the region for (e).

a) $f(x,y)=\begin{cases} e^x\cos y, & (x,y)\neq(0,\pi) \\ 1, & (x,y)=(0,\pi) \end{cases}$

b) $f(x,y)=\begin{cases} \dfrac{1}{y(x-1)}, & (x,y)\neq(1,0) \\ 0, & (x,y)=(1,0) \end{cases}$

c) $f(x,y)=\begin{cases} \dfrac{\sin(xy)}{3xy}, & (x,y)\neq(0,0) \\ 1/3, & (x,y)=(0,0) \end{cases}$

d) $f(x,y,z)=\dfrac{y}{x^2+z^2-1}$

e) $f(x,y,z)=x\sqrt{4-y^2-z^2}$

ANS: **a)** $\mathbb{R}^2$ | (x, y) ≠ (0, π). **b)** $\mathbb{R}^2$ | x ≠ 1, y ≠ 0. **c)** $\mathbb{R}^2$ | x ≠ 0, y ≠ 0. **d)** $\mathbb{R}^3$ | $x^2 + z^2 \neq 1$.
e) In the infinite cylindrical rod $y^2 + z^2 \leq 4$.

5. Evaluate all 1st-order and 2nd-order partial derivatives of each function below.

a) $z(x,y) = (x-1)^2 + (y-2)^2 - 3$

c) $g(x,y) = x^y$

b) $f(x,y) = y/x^2$

d) $w(x,t) = e^{xt}\ln(xt)$

e) $h(x,y) = \displaystyle\int_x^y g(t)\,dt$

ANS: **a)** $z_x = 2(x-1),\ z_y = 2(y-2),\ z_{xx} = 2,\ z_{yy} = 2,\ z_{xy} = z_{yx} = 0.$

b) $f_x = -2y/x^3,\ f_y = 1/x^2,\ f_{xx} = 6y/x^4,\ f_{yy} = 0,\ f_{xy} = f_{yx} = -2/x^3.$

c) $g_x = yx^{y-1},\ g_y = x^y\ln x,\ g_{xx} = y(y-1)x^{y-2},\ g_{yy} = x^y(\ln x)^2,$ $g_{xy} = g_{yx} = x^{y-1} + yx^{y-1}\ln x.$

d) $w_x = e^{xt}[t\ln(xt) + 1/x],\ w_t = e^{xt}[x\ln(xt) + 1/t],$ $w_{xx} = e^{xt}\left[t^2\ln(xt) + 2t/x - 1/x^2\right],\ w_{tt} = e^{xt}\left[x^2\ln(xt) + 2x/t - 1/t^2\right],$ $w_{xt} = w_{tx} = e^{xt}[(xt+1)\ln(xt) + 2].$

e) $h_x = -g(x),\ h_y = g(y),\ h_{xx} = -g'(x),\ h_{yy} = g'(y),\ h_{xy} = h_{yx} = 0.$

6. Use Geogebra to illustrate the first-order partial derivatives of the function below and illustrate their values at (-1, 2).

$$f(x,y) = 9 - x^2 - y^2$$

$f_y = -2y = 4$

$f_x = -2x = 2$

ANS: $f_x(-1, 2) = 2$, $f_y(-1, 2) = -4$.

7. **Data Analysis Problem**: The temperature values in an air-conditioned room are measured at various positions (x, y, z) in meters from a corner on the floor and given in the table below. Estimate the rate of change of temperature at the location (2, 2, 2) in each direction.

x (m)	y (m)	z (m)	T (°C)	x (m)	y (m)	z (m)	T (°C)
2	2	2	25	2	3	2	25
1	2	2	24	2	2	3	23
2	1	2	25	1	1	1	24
2	2	1	27	3	3	3	25
3	2	2	26	2	3	3	26

ANS: $T_x(2, 2, 2) \approx 1$ °C/m, $T_y(2, 2, 2) \approx 0$ °C/m, $T_z(2, 2, 2) \approx -2$ °C/m

$T_x = \dfrac{\Delta T}{\Delta x} = \dfrac{26-25}{1} \approx 1\,°C/m$

$T_y = \dfrac{25-25}{1} \approx 0\,°C/m$

$T_z = \dfrac{23-25}{1} \approx -2\,°C/m$

8. Evaluate ∂z/∂x at the point (1, 1, 1) from the equation below where z is an implicit function of x & y.

$$xy + xz^3 - 2yz = 0$$

ANS: -⅕.

9. Evaluate the 1st-order derivative / partial derivatives using the chain rule for each set of functions below.

a) $f(x,y) = x^2 + y^2,\ x(t) = t,\ \ y(t) = t^2$

b) $g(x,y,z) = \sin(xyz),\ x(t) = 1 - 3t,\ y(t) = e^{-t},\ z(t) = 2t$

c) $z(x,y) = \tan^{-1}\dfrac{x}{y},\ x(r,\theta) = r\cos\theta,\ \ y(r,\theta) = r\sin\theta$

d) $w(x,y,z) = xy + xz + yz,\ x(u,v) = u + v,\ y(u,v) = u - v,\ z(u,v) = uv$

ANS: **a)** $f'(t) = 2t + 4t^3$. **b)** $g'(t) = 2e^{-t}\left(3t^2 - 7t + 1\right)\cos\left[2te^{-t}(1-3t)\right]$.
c) $z_r = 0,\ z_\theta = -1$. d) $w_u(u,v) = 2u(1+2v),\ w_v(u,v) = 2\left(u^2 - v\right)$.

10. Given the following information, determine $w_s(0, 0)$ and $w_t(0, 0)$.

$$w(s,t) = F(x(s,t),\ y(s),\ z(2\sin t))$$

$$x(0,0) = 2,\ y(0) = 4,\ z(0) = 1,\ x_s(0,0) = -1,\ x_t(0,0) = 3,\ y'(0) = 1,\ z'(0) = 8,$$
$$F_x(0,0,2) = 2,\ F_y(0,0,2) = -9,\ F_z(0,0,2) = 2,\ F_x(2,4,1) = 6,\ F_y(2,4,1) = 0,$$
$$F_z(2,4,1) = 5, F_x(2,4,2) = -1,\ F_y(2,4,2) = 2,\ F_z(2,4,2) = 3$$

ANS: $w_s(0, 0) = -6$, $w_t(0, 0) = 98$.

11. Determine the region where each function below is differentiable.

a) $f(x,y) = x^2\sqrt{y-2}$

b) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

ANS: a) $\mathbb{R}^2$ | y > 2. b) $\mathbb{R}^2$ | (x, y) ≠ (0,0).

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-3/pages/4-1-functions-of-several-variables
2) https://openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity
3) https://openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives
4) https://openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations
5) https://openstax.org/books/calculus-volume-3/pages/4-5-the-chain-rule

End of Tutorial 3

(Email to youliangzheng@gmail.com for assistance.)