Newton's Second Law for Rotation

- We noted that the moment of inertia *I* is the rotational equivalent of mass.
- Let's consider a point particle of mass *M* moving around an axis at a distance *R.*
- If we multiply the moment of inertia times the angular acceleration we get:

$$I a = (R^2 M) a = RM(R a) = RMa = RF_{\text{net}}$$

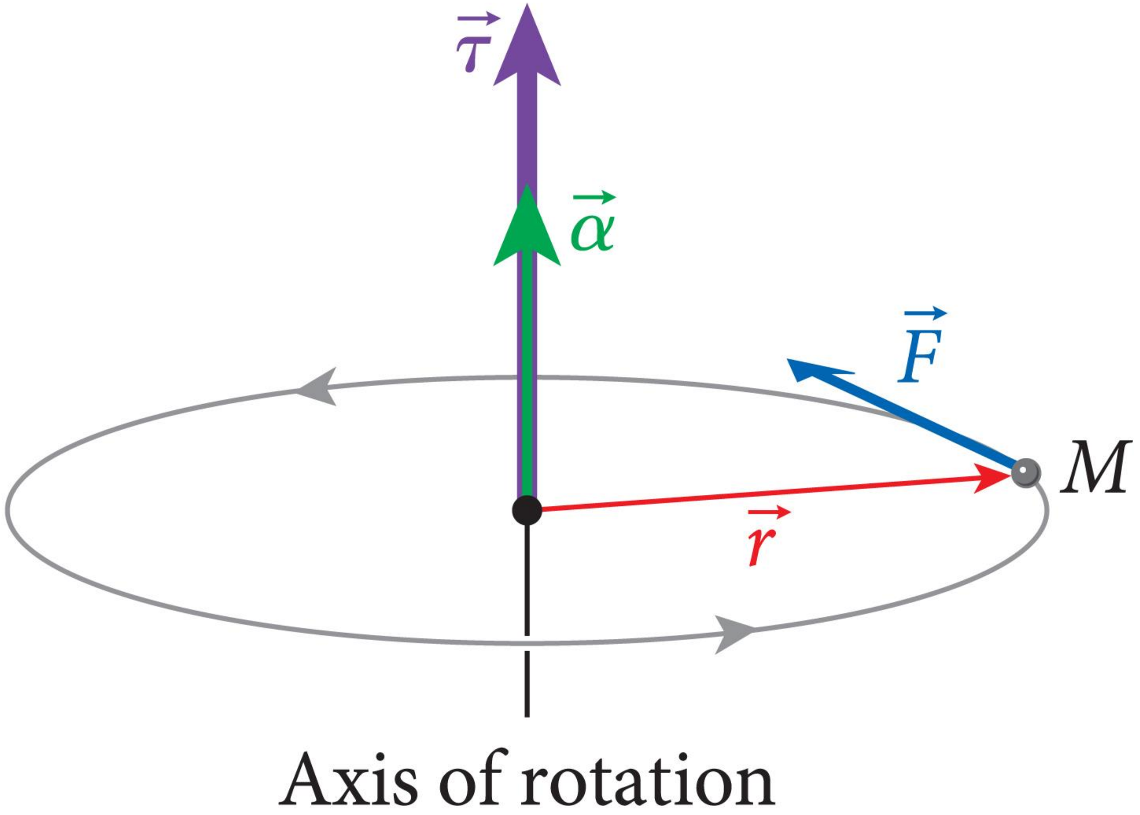
- We can see that: $t = I a$

- Which is in analogy with Newton's Second Law: $F = ma$

Newton’s Second Law for Rotation

- We can combine these results to get:

$$\vec{\tau} = \vec{r} \times \vec{F}_{\text{net}} = I\vec{\alpha}$$



- This result is for a point particle, but also holds for extended objects.

Newton's Second Law for Rotation

- You are trying to put a new roll of toilet paper into its holder in the bathroom.
- However, you drop the roll, managing to hold onto just the first sheet.
- On its way to the floor, the toilet paper roll unwinds, as shown to the right.

PROBLEM:

- How long does it take the roll of toilet paper to hit the ground, if it was released from a height of 0.73 m?
- The roll has an inner radius R_1 = 2.7 cm, an outer radius R_2 = 6.1 cm, and a mass of 274 g.

Newton's Second Law for Rotation

SOLUTION:

- If the roll of toilet paper simply falls with the acceleration of gravity, the time it takes to hit the ground is:

$$t_{\text{free}} = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2\cdot(0.73\text{ m})}{9.81\text{ m/s}^2}} = 0.39\text{ s}$$

- However, because we are holding onto the first sheet, the toilet paper unwinds on its way down.
- The toilet paper "rolls" without slipping.
- The acceleration will be different from free-fall.

Newton's Second Law for Rotation

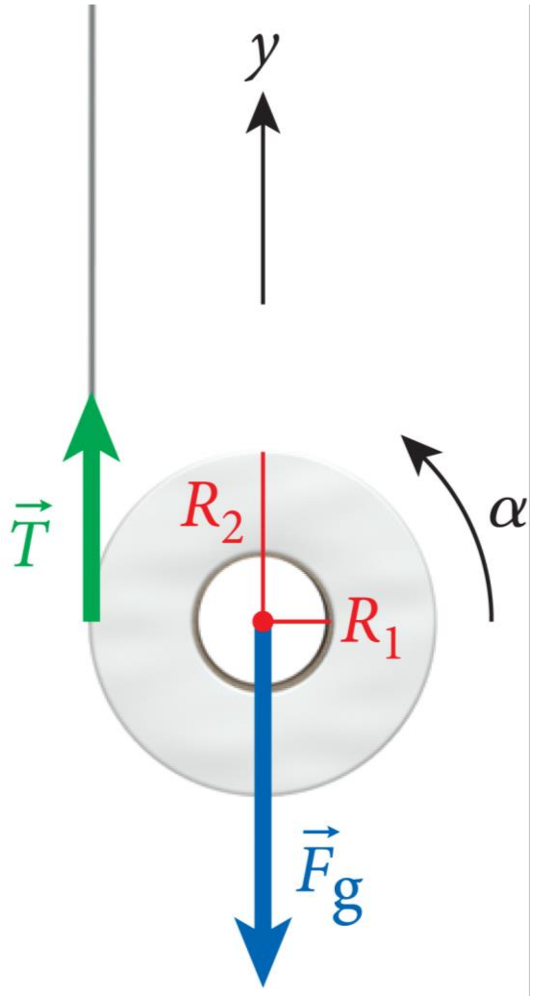
- Newton's Second Law then allows us to connect the net force acting on the toilet paper to the acceleration of the roll:

$$T - mg = ma_y$$

- The tension and the acceleration are both unknown so we need a second equation.
- We get this second equation from the rotational motion of the roll:

$$\tau = I\alpha \text{ where}$$

$$I = \tfrac{1}{2}m(R_1^2 + R_2^2)$$



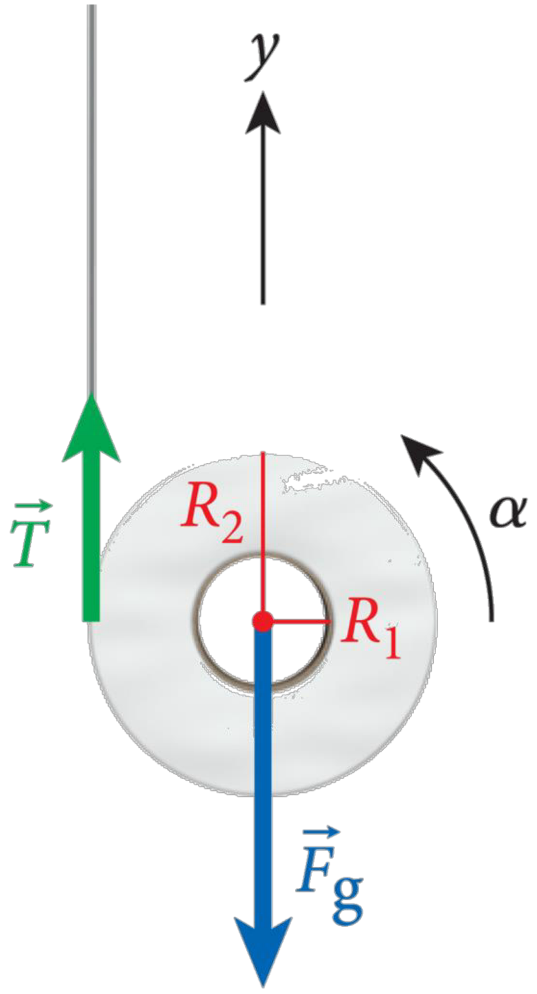
Newton’s Second Law for Rotation

- We know:

$$a_y = R_2\alpha$$

- We need to be careful with the direction of the angular acceleration.
- A positive acceleration in the *y*-direction corresponds to a counterclockwise angular acceleration.
- We define a counterclockwise angular acceleration as positive.
- The torque is then:

$$\tau = -R_2 T$$



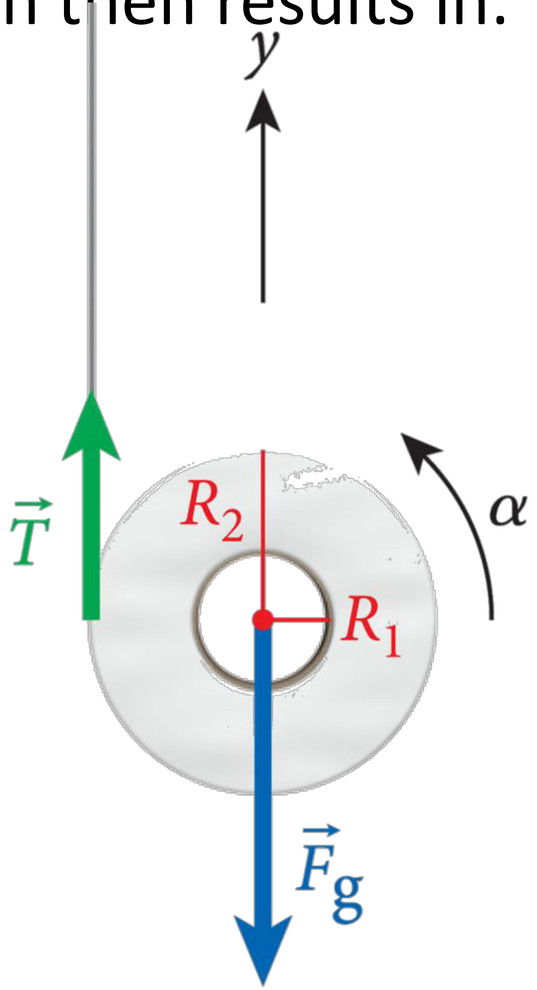
Newton’s Second Law for Rotation

- Newton’s Second Law for rotational motion then results in:

$$\tau = I\alpha$$

$$-R_2 T = (\tfrac{1}{2} m(R_1^2 + R_2^2)) \frac{a_y}{R_2}$$

$$-T = \tfrac{1}{2} m \left(1 + \frac{R_1^2}{R_2^2} \right) a_y$$



Newton's Second Law for Rotation

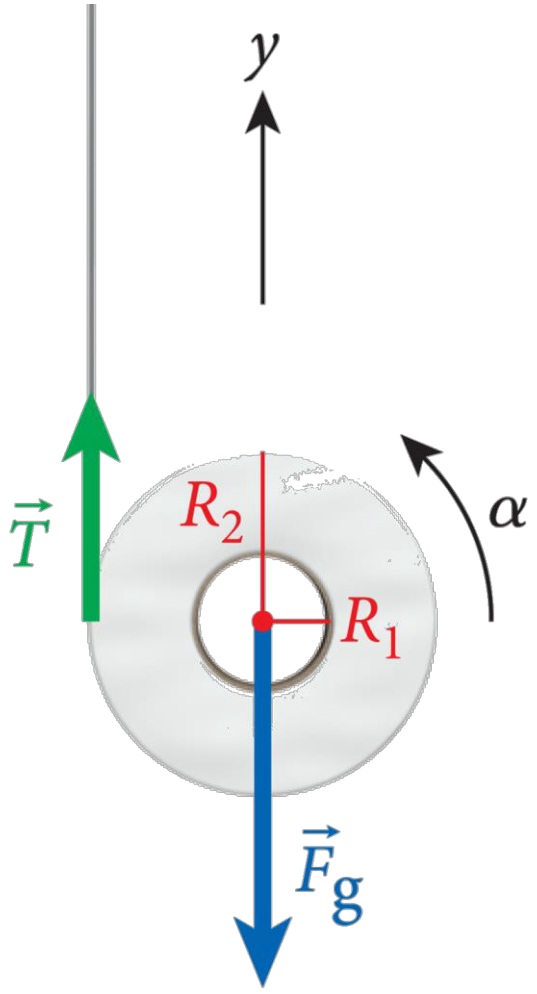
- We can add the two equations:

$$T - mg = ma_y \quad \text{and} \quad -T = \tfrac{1}{2}m\left(1 + \frac{R_1^2}{R_2^2}\right)a_y$$

- Which gives us:

$$-mg = \tfrac{1}{2}m\left(1 + \frac{R_1^2}{R_2^2}\right)a_y + ma_y \quad \Rightarrow$$

$$a_y = -\frac{g}{\dfrac{3}{2} + \dfrac{R_1^2}{2R_2^2}}$$



Newton's Second Law for Rotation

- Putting in numbers we get the acceleration:

$$a = -\frac{9.81\ \text{m/s}^2}{\dfrac{3}{2} + \dfrac{(2.7\ \text{cm})^2}{2(6.1\ \text{cm})^2}} = -6.14\ \text{m/s}^2$$

- Our fall time is then:

$$t = \sqrt{\frac{2y_0}{(-a_y)}} = \sqrt{\frac{2\cdot(0.73\ \text{m})}{6.14\ \text{m/s}^2}} = 0.49\ \text{s}$$

compared with 0.39 s for free-fall