Rolling without Slipping

- Let's assume that we have a round object with radius R rolling without slipping.
- This object has special relationships between its linear and angular quantities:
  - Displacement and angular displacement:

    $$r = R\theta$$

  - Velocity and angular velocity:

    $$v = R\omega$$

  - Acceleration and angular acceleration:

    $$a = R\alpha$$

Moment of Inertia

- Now let's take our expression for the moment of inertia for discrete particles and extend it to continuous extended objects.
- We approximate our extended object as a collection of small, identically sized cubes of volume *V* of density *ρ:*

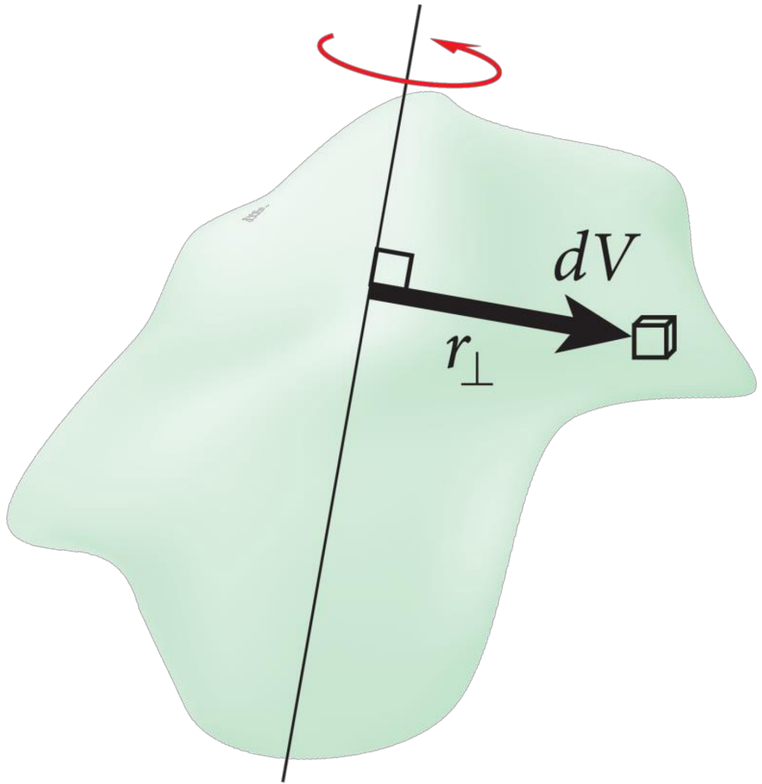
$$m_i = V\rho(\vec{r}_i) \Rightarrow I = \sum_{i=1}^{n} \rho(\vec{r}_i) r_i^2 V$$

- Letting the volume of the cubes go to 0:

$$\boxed{I = \int_V r_\perp^2 \rho(\vec{r})\, dV}$$

- The total mass of the object is:

$$M = \int_V \rho(\vec{r})\, dV$$



Moment of Inertia

- If we take our results for the moment of inertia and assume that the density is constant, we get:
  - The moment of inertia for constant density ρ

$$I = \rho \int_V r_\perp^2 \, dV$$

  - The mass for constant density ρ

$$M = \rho \int_V dV = \rho V$$

  - Final result

$$\boxed{I = \frac{M}{V} \int_V r_\perp^2 \, dV}$$

Moment of Inertia

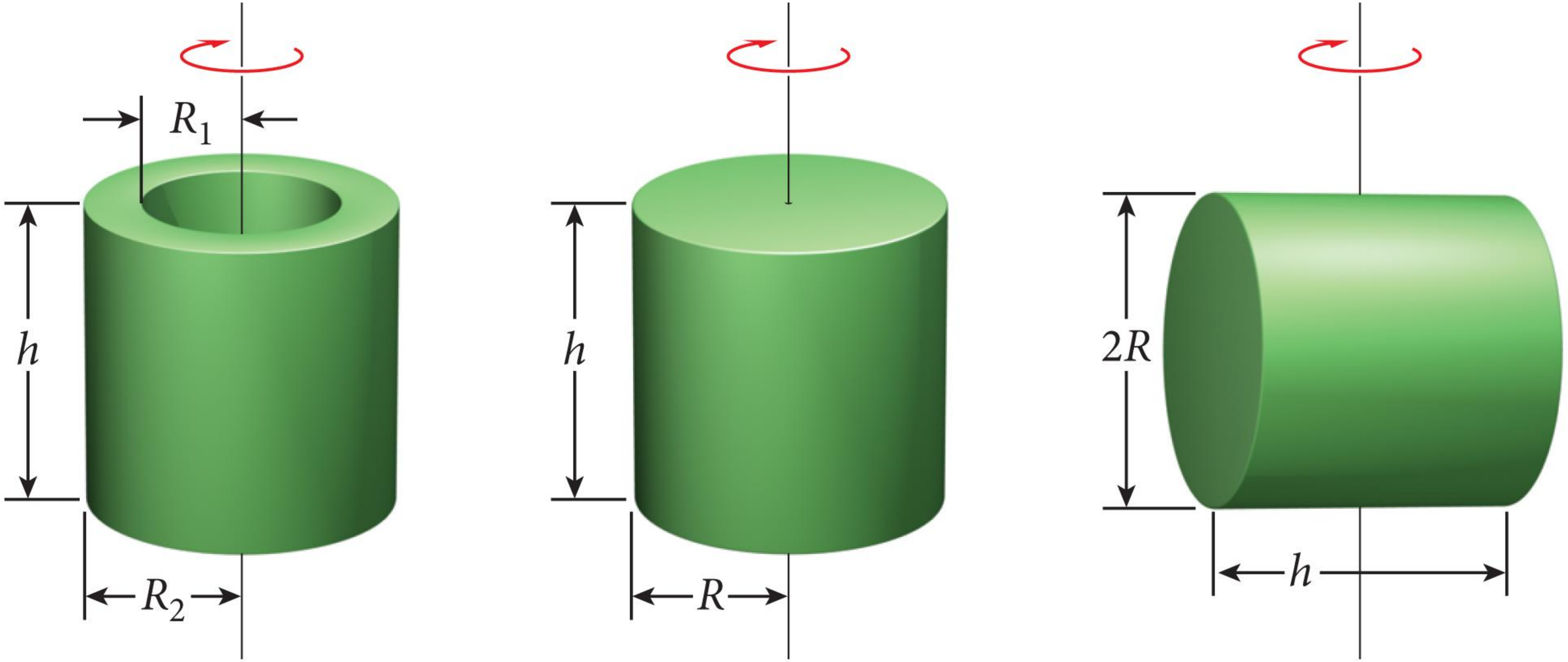
- We can use the formula we just derived to calculate the moment of inertia of an object with respect to rotation about the center of mass:

$$I = \frac{M}{V} \int_V r_\perp^2 \, dV$$

- For convenience, the location of the center of mass is chosen as the origin of the coordinate system.

- We chose our coordinate system wisely to minimize the computational work.

Moment of Inertia



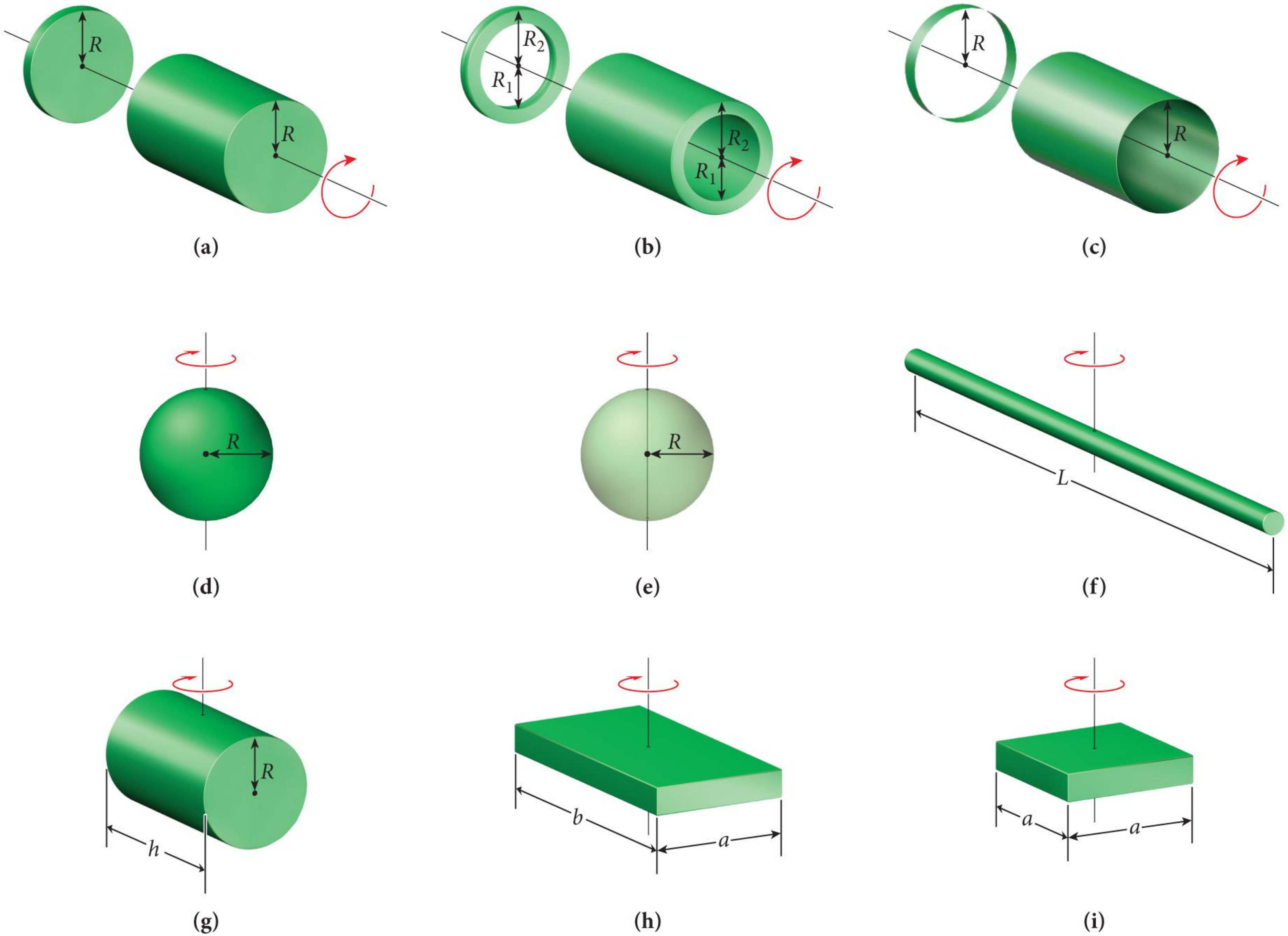
(a)	(b)	(c)
$M = \pi(R_2^2 - R_1^2)h\rho$	$M = \pi R^2 h\rho$	$M = \pi R^2 h\rho$
$I = \frac{1}{2}M(R_1^2 + R_2^2)$	$I = \frac{1}{2}MR^2$	$I = \frac{1}{4}MR^2 + \frac{1}{12}Mh^2$

Moment of Inertia

- If *R* is the largest perpendicular distance of any part of the rotating object from the axis of rotation, then the moment of inertia is always related to the mass of an object by:

$$I = cMR^2, \text{ with } 0 < c \leq 1$$

Moment of Inertia



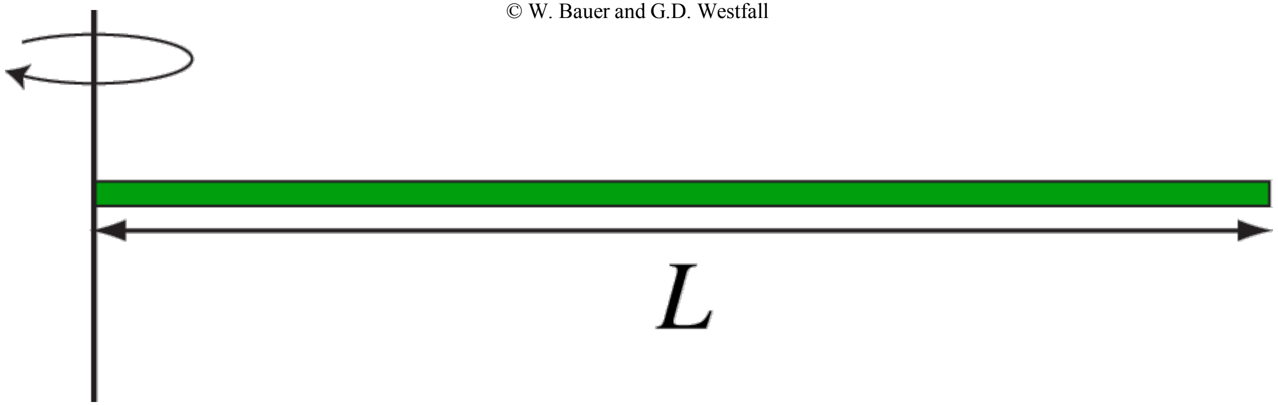
Moment of Inertia

Table 10.1 The Moment of Inertia and Value of Constant *c* for the Objects Shown in Figure 10.10. All Objects Have Mass *M*

Object	I	c
a) Solid cylinder or disk	$\frac{1}{2}MR^2$	$\frac{1}{2}$
b) Thick hollow cylinder or wheel	$\frac{1}{2}M(R_1^2 + R_2^2)$	
c) Hollow cylinder or hoop	MR^2	1
d) Solid sphere	$\frac{2}{5}MR^2$	$\frac{2}{5}$
e) Hollow sphere	$\frac{2}{3}MR^2$	$\frac{2}{3}$
f) Thin rod	$\frac{1}{12}ML^2$	
g) Solid cylinder perpendicular to symmetry axis	$\frac{1}{4}MR^2 + \frac{1}{12}Mh^2$	
h) Flat rectangular plate	$\frac{1}{12}M(a^2 + b^2)$	
i) Flat square plate	$\frac{1}{6}Ma^2$	

Moment of Inertia

- Consider a uniform rod with length *L* and mass *M* as shown below. What is the moment of inertia of this rod if rotated about one end?



$$I = \frac{M}{V}\int_V r_\perp^2 \, dV$$

$$V = AL$$

$$dV = A dx$$

$$r_\perp = x$$

$$I = \frac{M}{AL}\int_L x^2 A\,dx = \frac{M}{L}\int_0^L x^2\,dx = \frac{M}{L}\frac{L^3}{3}$$

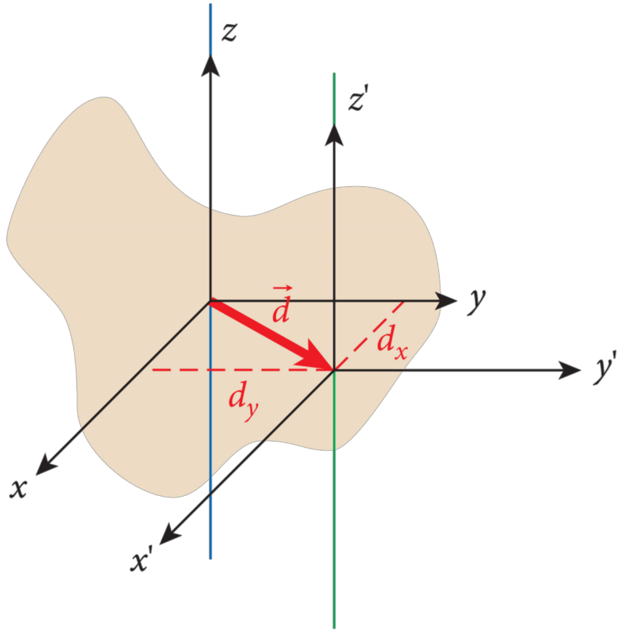
$$I = \frac{1}{3}ML^2$$

- A) (1/3)ML^2
- B) (1/5)ML^2
- C) (1/6)ML^2
- D) (1/10)ML^2
- E) (1/12)ML^2

Moment of Inertia

Parallel Axis Theorem

- Now that we have solved the problem of the moment of inertia of a body rotating on an axis through the center of mass, let's get the moment of inertia for bodies rotating on an axis that does not go through the center of mass.



- Set up axes of our coordinate system in such a way that it has axes **parallel** to the axes of the system for which we already calculated the moment of inertia of rotation about center of mass!

Moment of Inertia

- Relationship between old and new coordinates:

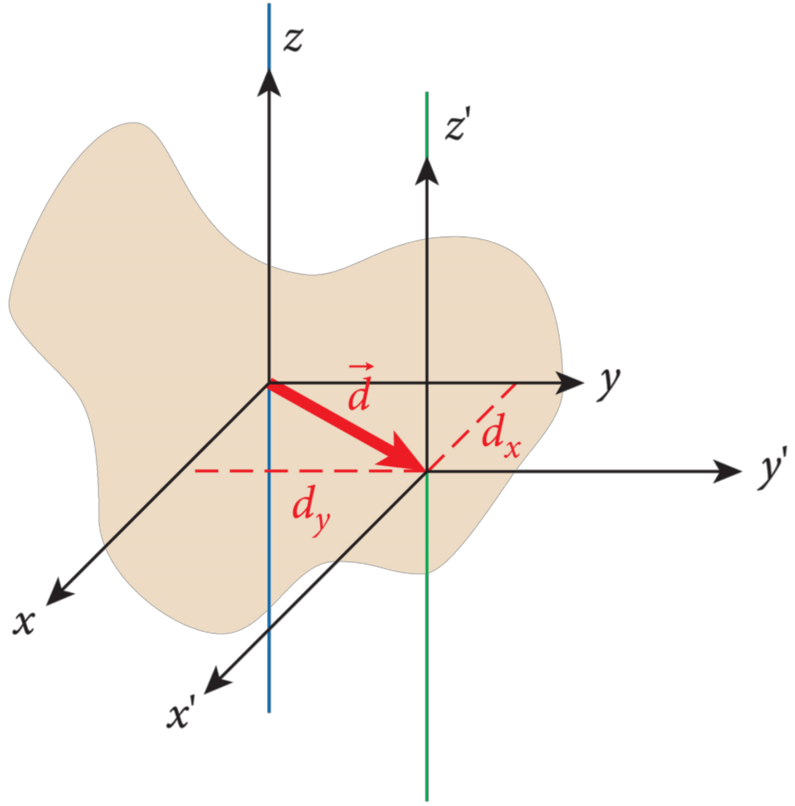
$$x' = x - d_x;\;\; y' = y - d_y;\;\; z' = z$$

- Perpendicular distance in the new coordinates:

$$\begin{aligned} r'^2_{\perp} &= x'^2 + y'^2 = (x - d_x)^2 + (y - d_y)^2 = \\ &= x^2 - 2xd_x + d_x^2 + y^2 - 2yd_y + d_y^2 \\ &= (x^2 + y^2) + (d_x^2 + d_y^2) - 2xd_x - 2yd_y \\ &= r_{\perp}^2 + d^2 - 2xd_x - 2yd_y \end{aligned}$$

- Moment of inertia about new axis:

$$\begin{aligned} I_{\parallel} &= \int_V (r'_{\perp})^2 \rho\, dV \\ &= \int_V r_{\perp}^2 \rho\, dV + d^2 \int_V \rho\, dV - 2d_x \int_V x\rho\, dV - 2d_y \int_V y\rho\, dV \end{aligned}$$



Moment of Inertia

- Need to evaluate integrals individually:

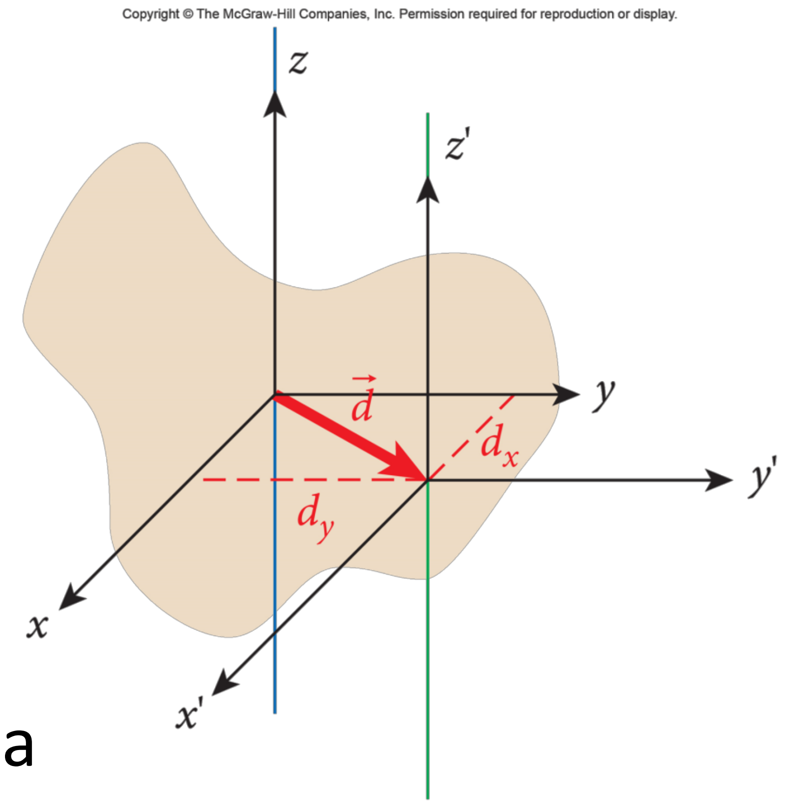
$$I_{\parallel} = \int_V r_{\perp}^2 \rho\, dV + d^2 \int_V \rho\, dV - 2d_x \int_V x\rho\, dV - 2d_y \int_V y\rho\, dV$$

- 1st: *I* about c.m.
- 2nd: d^2M.
- 3rd & 4th: location of *x*-and *y*-coordinate of c.m. by construction

  these integrals are 0.

- Final result:

$$\boxed{I_{\parallel} = I_{\text{cm}} + Md^2}$$



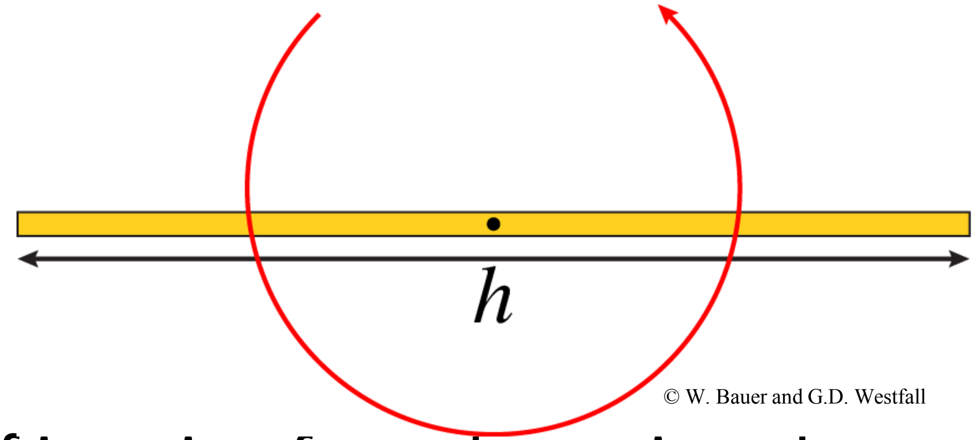
- General way of writing any moment of inertia

$$I = (cR^2 + d^2)M \quad (0 < c \leq 1, \text{ depending on } I_{\text{cm}})$$

Moment of Inertia

- The moment of inertia of a rod with mass *M* and length *h* rotating about its center of mass is:

$$I_{\text{cm}} = \frac{1}{12} M h^2$$

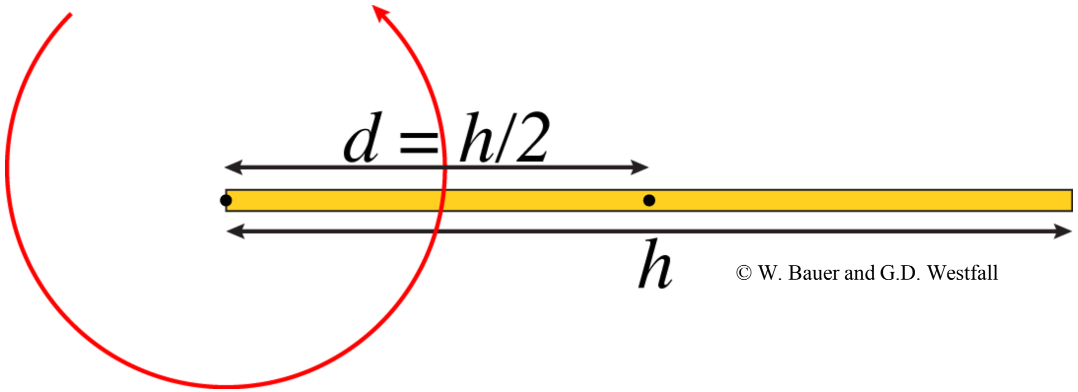


- What is the moment of inertia of a rod rotating about one of its ends.
- We can get the result using the parallel-axis theorem:

$$I_{\parallel} = I_{\text{cm}} + M d^2$$

Moment of Inertia

- The moment of inertia of a rod with mass *M* and length *h* rotating about its end is:



$$I_{\parallel} = I_{\mathrm{cm}} + Md^2 = \frac{1}{12}Mh^2 + M\left(\frac{h}{2}\right)^2$$

$$I_{\parallel} = \frac{1}{12}Mh^2 + \frac{1}{4}Mh^2 = \frac{1}{12}Mh^2 + \frac{3}{12}Mh^2$$

$$I_{\parallel} = \frac{1}{3}Mh^2$$

Rolling without Slipping

- A rolling object has both translational kinetic energy and rotational kinetic energy:

$$K = K_{\text{trans}} + K_{\text{rot}} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

- We use our knowledge of the relationship between the linear and angular quantities to get:

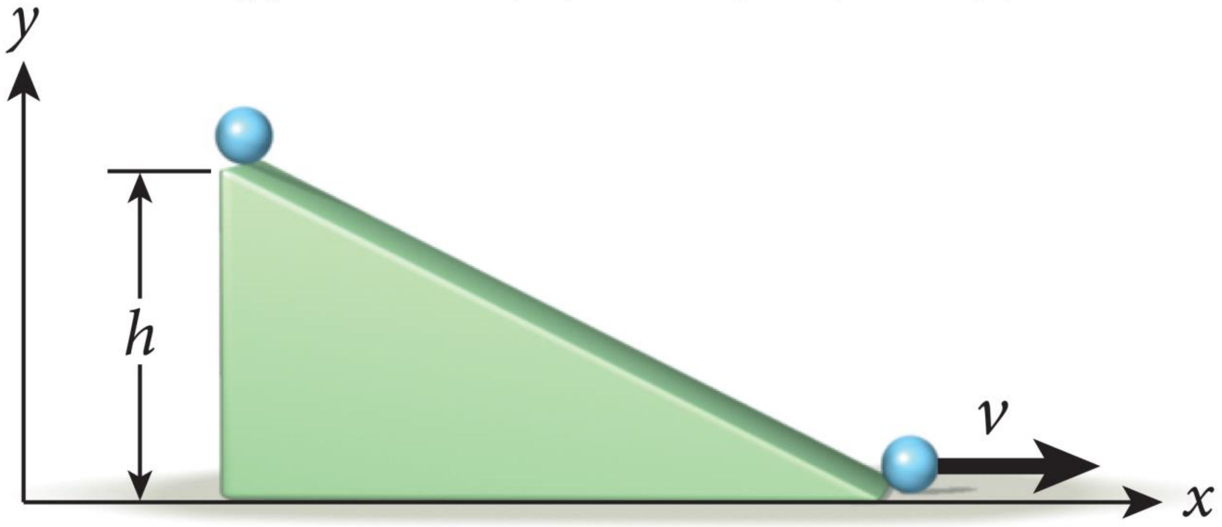
$$\begin{aligned} K &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}(cR^2m)(v/R)^2 \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2c \\ &= (1+c)\tfrac{1}{2}mv^2 \end{aligned}$$

$$(0 < c \leq 1)$$

Sphere Rolling Down an Inclined Plane

PROBLEM:

- A solid sphere with mass 5.15 kg and radius 0.340 m starts from rest at a height 2.10 m above the base of an inclined plane and rolls down under the influence of gravity.
- What is the linear speed of the center of mass of the sphere just as it leaves the incline and rolls onto a horizontal surface?



Rolling without Slipping

- At the top of the incline, the sphere is at rest and has zero kinetic energy.
- At the top, its total energy is equal to the potential energy *mgh:*

$$E_{\text{top}} = K_{\text{top}} + U_{\text{top}} = 0 + mgh = mgh$$

- At the bottom of the incline, just as the sphere starts rolling on the horizontal surface, the potential energy is zero.
- At the bottom, the sphere has a total energy equal to its kinetic energy (sum of translational and rotational kinetic energy):

$$E_{\text{bottom}} = K_{\text{bottom}} + U_{\text{bottom}} = (1+c)\tfrac{1}{2}mv^2 + 0 = (1+c)\tfrac{1}{2}mv^2$$

Rolling without Slipping

- Conservation of energy gives us:

$$mgh = (1 + c)\tfrac{1}{2}mv^2$$

- Solving for the linear velocity gives us:

$$v = \sqrt{\frac{2gh}{1 + c}}$$

- Remembering that $c = 2/5$ for a sphere, we get an expression for the speed of the rolling sphere:

$$v = \sqrt{\frac{2gh}{1 + \frac{2}{5}}} = \sqrt{\frac{10}{7}gh}$$

- Putting in our numerical values gives us:

$$v = \sqrt{\frac{10}{7} \times 9.81\ \text{m/s}^2 \times 2.10\ \text{m}} = 5.42494\ \text{m/s}$$

Rolling without Slipping

- The acceleration of an object by gravity is independent of the mass of the object (free fall).
- What about rolling?
  - Does the mass matter?
  - Does the radius matter?

- Let’s look at the case of three objects with the same mass, same radius, but different distribution of mass, rolling down an inclined plane:
  - A solid sphere
  - A solid cylinder
  - A hollow cylinder
  - Which one will win?

Rolling without Slipping

- We can write the energy for each object:

$$E = K + U = K_0 + U_0$$

$$K_{\text{bottom}} = U_{\text{top}} \Rightarrow (1+c)\tfrac{1}{2}mv^2 = mg(h_0 - h) \Rightarrow$$

$$v = \sqrt{\frac{2gh}{1+c}}$$

$$c_{\text{sphere}} = \tfrac{2}{5} \qquad c_{\text{cylinder}} = \tfrac{1}{2} \qquad c_{\text{tube}} \approx 1$$

- $c_{\text{sphere}} = 0.4$, smallest denominator, highest v