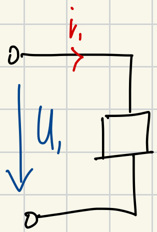
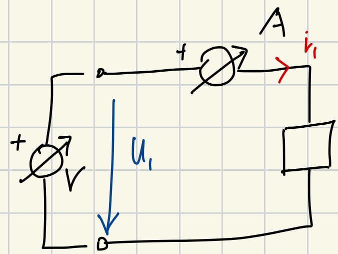
Chapter 1

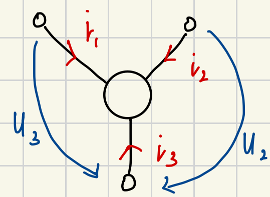
1.1 Reference Directions



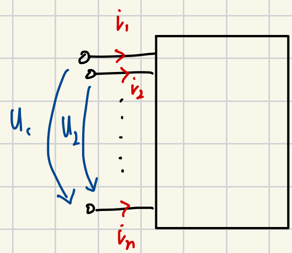
associated reference directions



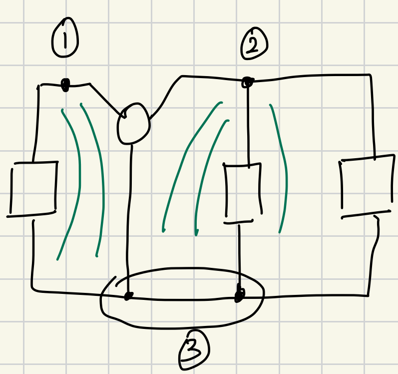
two-terminal



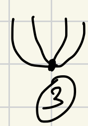
three-terminal



n-terminal



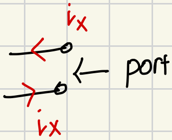
node is defined as a connection between 2 elements



3 nodes

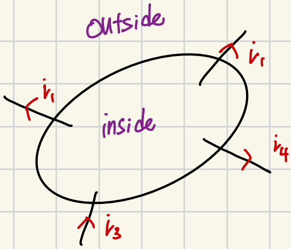
5 branches

1.2 Ports



i_x flowing into first terminal and flowing out of second terminal

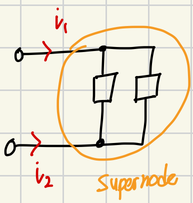
1.3 KCL



take into account the directions (out ⊕ , in ⊖)

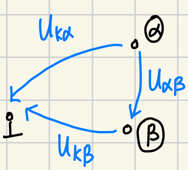
$i_1 + i_2 + i_4 - i_3 = 0$

$$KCL : \sum_{j \in \text{branches connected to node}} i_j = 0$$

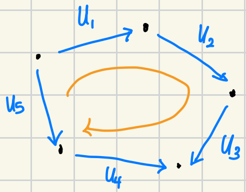


KCL : $i_1 + i_2 = 0$

1.4 KVL



KVL: $u_{\alpha\beta} = u_{k\alpha} - u_{k\beta}$

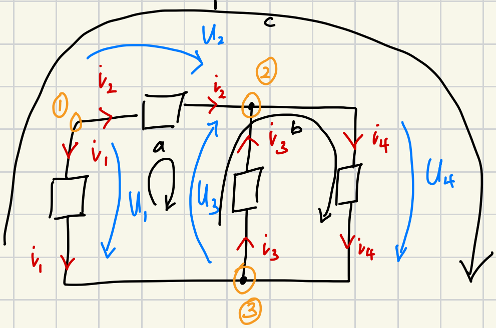


KVL :

$$U_1 + U_2 + U_3 - U_4 - U_5 = 0$$

KVL :

$$\sum_{\substack{j \in \text{branches} \\ \text{of loops}}} U_j = 0$$



$$\left.\begin{aligned} &\text{KCL①}: i_1 + i_2 = 0 \\ &\text{KCL②}: -i_3 + i_4 - i_2 = 0 \end{aligned}\right\} + = i_1 - i_3 + i_4 = 0$$

$$\text{KCL③}: i_1 - i_3 + i_4 = 0$$

$$\left.\begin{aligned} &\text{KVL ⓐ}: -U_1 + U_2 - U_3 = 0 \\ &\text{KVL ⓑ}: U_3 + U_4 = 0 \end{aligned}\right\} + = -U_1 + U_2 + U_4 = 0$$

$$\text{KVL ⓒ}: U_2 + U_4 - U_1 = 0$$

1.5 Number of Linearly independent Kirchhoff's Eqns

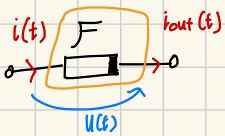
n nodes, e.g., $n=3$

$n-1$ linearly KCL eqn, e.g., $n-1 = 2$ eqns

b branches, e.g., $b=4$

$b-(n-1)$ linearly $\overset{\text{ind.}}{\wedge}$ KVL eqns, eg, $b-(n-1) = 4-(3-1) = 2$ eqns

2 Resistive One Ports



KCL : $i(t) - i_{out}(t) = 0$

$i_{out}(t) = i(t)$

two - terminal = one-ports

Characteristics

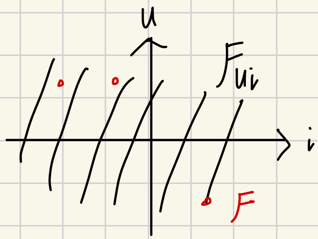
$\mathcal{F} \subseteq \mathcal{F}_{ui}$

$\mathcal{F}_{ui} = \{ (u, i) \mid \frac{u}{1V} \in \mathbb{R}, \frac{i}{1A} \in \mathbb{R} \}$

units

$\mathcal{F} = \{ (u,i) \mid (u,i)$ is operating point of $\mathcal{F} \}$

Every one port is
a collection of point in $\mathcal{F}_{ui}$ plane



2.1 Explicit Representation

$$\frac{i}{i_0} = \arctan \frac{u}{u_0} = \tan^{-1}\left(\frac{u}{u_0}\right) \text{ for } G$$

in general, one-port $\mathcal{F}$:

Voltage - controlled repr. :

$$i = g_{\mathcal{F}}(u)$$

$$G : i = g_G(u) = i_0 \arctan \frac{u}{u_0}$$

Current - Controlled repr. :

$$u = r_{\mathcal{F}}(i)$$

$$G : u = u_0 \tan\left(\frac{i}{i_0}\right)$$

$$i \in i_0\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

2.2 Properties of One-Ports

2.2.1 Bilateral Property

$\forall (u, i) \in \mathcal{F} : (-u, -i) \in \mathcal{F}$

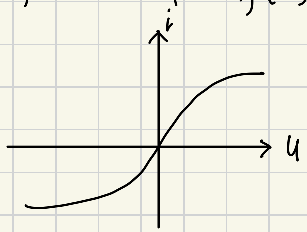
$\Longleftrightarrow$

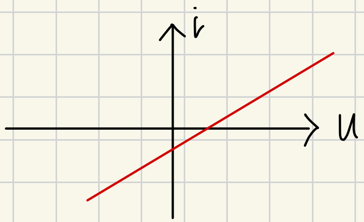
bilateral $\mathcal{F}$

$i = g_{\mathcal{F}}(u) : \quad -i = g_{\mathcal{F}}(-u)$

$\qquad\qquad\qquad\quad i = -g_{\mathcal{F}}(-u)$

$u = r_{\mathcal{F}}(i) : \quad u = -r_{\mathcal{F}}(-i)$





not - bilateral

2.2.2 Power

$P(t) = u(t) \cdot i(t)$

$\exists (u,i) \in F : ui < 0$

$\Leftrightarrow$ active

$\forall (u,i) \in F : ui \geq 0$

$\Leftrightarrow$ passive

$\forall (u,i) \in F : ui = 0$

$\Leftrightarrow$ lossless

$\exists (u,i) \in F : ui \neq 0$

$\Leftrightarrow$ lossy (opposite of lostless)

2.2.3 Source-Free

$(0,0) \in F$: source-free F

2.2.4 Duality

duality constant : R_d

$(u,i) \in F$:

$u^d = R_d i \qquad (u^d, i^d) \in F^d$

$i^d = \frac{1}{R_d} u$

e.g., $i = g_F(u)$

$\downarrow R_d$

$\frac{1}{R_d} u^d = g_F(R_d i^d)$

$u^d = R_d\, g_F(R_d i^d)$

$$u^d = r_F^d(i^d)$$

$$u = r_F(i) \curvearrowright R_d$$

$$R_d i^d = r_F\left(\frac{1}{R_d} u^d\right)$$

$$i^d = \frac{1}{R_d} r_F\left(\frac{1}{R_d} u^d\right)$$

$$i^d = g_F^d(u^d)$$

2.3 Strictly linear one ports

$(u, i) \in \mathcal{F}$:

$\forall k \in \mathbb{R} : (ku, ki) \in \mathcal{F}$

$\forall (u_1, i_1), (u_2, i_2) \in \mathcal{F}$:

$(u_1 + u_2, i_1 + i_2) \in \mathcal{F}$

$k = -1$:

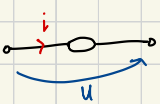
$\forall (u, i) \in \mathcal{F} : (-u, -i) \in \mathcal{F}$

$\Leftrightarrow$ bilateral

$k = 0$

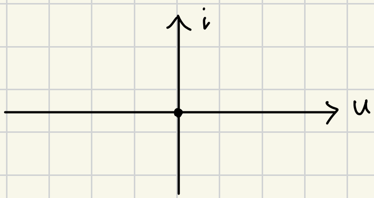
$(0, 0) \in \mathcal{F} \Leftrightarrow$ source free

2.3.1 Nullator



$u = 0, \ i = 0$

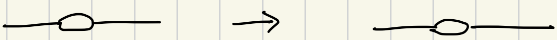
$\mathcal{F}_0 = \{(0,0)\}$



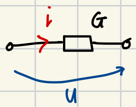
$P = u \cdot i = 0 \Leftrightarrow$ lossless

$$u=0,\ i=0 \xrightarrow{Rd} u^d=0,\ i^d=0$$

Rd



2.3.2 Strictly linear Resistors



$i = Gu$

↓

Conductance

$$G = \frac{i}{u}\Big|_{u\neq 0}$$

$$= \frac{1A}{1V} = 1S$$

$$i = Gu$$

$$\downarrow Rd$$

$$\frac{1}{Rd}u^d = GRd\, i^d$$

$$u^d = \underbrace{Rd^2}_{Rd} G i^d$$

$u = Ri$

↓

resistance

$$R = \frac{1}{G} \qquad \frac{1V}{1A} = 1\Omega$$

2.3.2.1 Open Circuits and short circuits

$i = Gu$

$G = 0$

$i = 0$

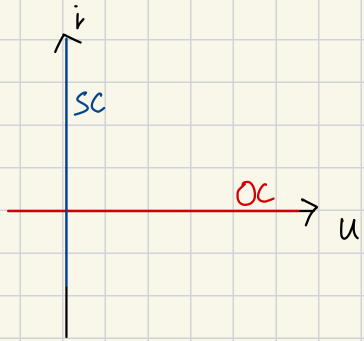
Open Circuit OC

$u = Ri$

$R = 0$

$u = 0$

Short Circuit SC



OC, SC : lossless

define:

$$\frac{1}{0\,\Omega} = \infty\,S \qquad \text{OC: } G=0,\ R=\infty$$

$$\frac{1}{0\,S} = \infty\,\Omega \qquad \text{SC: } R=0,\ G=\infty$$

2.3.2.2 Ohmic Resistors

$0 \leq R \leq \infty$ or $0 \leq G \leq \infty$

$$u = R \cdot i$$

$$P = R \cdot i \cdot i$$

$$= \underbrace{R}_{\geq 0} \cdot \underbrace{i^2}_{\geq 0} \geq 0$$

$\Rightarrow$ passive

2.3.2.3 Negative Resistors

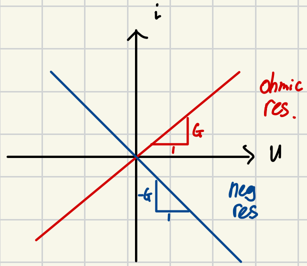
$-\infty < R < 0$

$-\infty < G < 0$

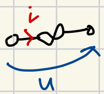
$$u = Ri$$

$$P = \underbrace{R}_{<0}\underbrace{i^2}_{\geq 0} \leq 0$$

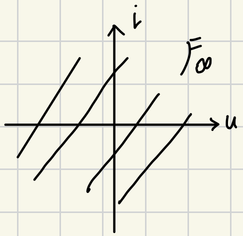
$\Rightarrow$ active



2.3.3 Norator



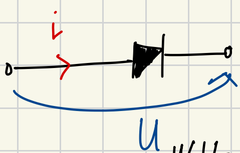
$$F_\infty = F_{ui}$$



no eqn.

u arbitrary, i arbitrary

2.4 pn-Junction Diode



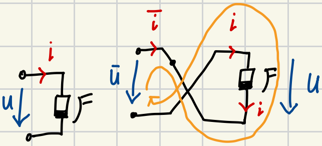
$$i = I_s\left(e^{u/U_T} - 1\right)$$

thermal voltage

$$u = U_T \ln\left(\frac{i}{I_s} + 1\right)$$

2.5 Basic One-Port Circuits

2.5.1 Polarity Reversal



$(u,i) \in F \; : (\bar{u},\bar{i}) = (-u,-i) \in \bar{F}$

$i = g_F(u) : \quad \bar{i} = -g_F(-\bar{u})$

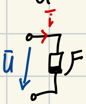
$= g_F(\bar{u}) \; :$ bilateral

KCL: $\bar{i} + i = 0$

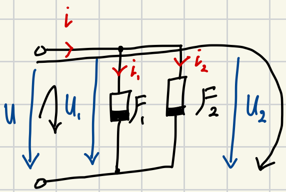
KVL: $-\bar{u} + u = 0$

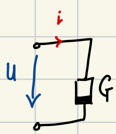
$\bar{u} = -u , \quad \bar{i} = -i$

$\bar{u} = -r_F(-\bar{i})$



2.5.2 Parallel-Connection





KCL: $i - i_1 - i_2 = 0$

KVL: $-u + u_1 = 0$

$-u + u_2 = 0$

$i = i_1 + i_2$

$u = u_1$

$u = u_2$

$\rightarrow u = u_1 = u_2$

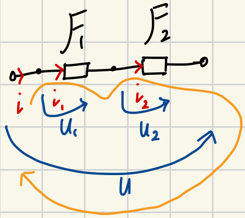
Voltage-Control

$i_1 = g_{F_1}(u_1)$

$i_2 = g_{F_2}(u_2)$

$i = g_{F_1}(u) + g_{F_2}(u) = g_G(u)$

2.5.3 series Connection



KCL: $-i + i_1 = 0$

$-i_1 + i_2 = 0$

KVL: $u_1 + u_2 - u = 0$

$u = u_1 + u_2$

$i = i_1 = i_2$

series Connection is dual to parallel

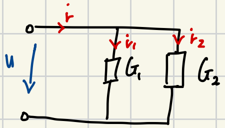
$u = r_{F_1}(i) + r_{F_2}(i)$

2.5.4 Connection of Strictly linear Resistors

Parallel sum: $\frac{1}{a \| b} = \frac{1}{a} + \frac{1}{b}$

$$a \| b = \frac{ab}{a+b}$$

2.5.4.1 Parallel Connection



$i = i_1 + i_2$

Ohm: $i = G_1 u + G_2 u$

$i = (G_1 + G_2) u$

$= G u$

$G = G_1 + G_2$

$$R = \frac{1}{G} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\Rightarrow R = R_1 \parallel R_2$$

$$i = Gu = (G_1 + G_2)u$$

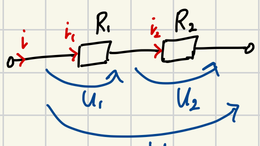
$$i_1 = G_1 u$$

$$i_2 = G_2 u$$

$$\frac{i_1}{i} = \frac{G_1 u}{(G_1+G_2)u} = \frac{G_1}{G_1+G_2} \qquad \frac{i_2}{i_1} = \frac{G_2}{G_1}$$

Current divider rule

2.5.4.2 Series Connection



$$U = U_1 + U_2$$

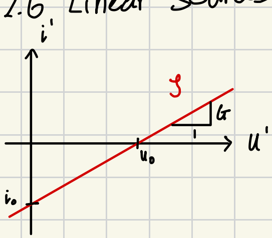
$$i = i_1 + i_2$$

$$R = R_1 + R_2$$

$$\frac{U_1}{U} = \frac{R_1}{R_1+R_2}$$

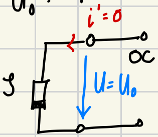
$$\frac{U_2}{U_1} = \frac{R_2}{R_1}$$

2.6 Linear sources

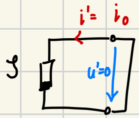


non-source free

U_0 : $i = 0 \rightarrow$ OC



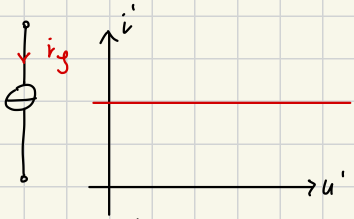
$-i_0$: $u' = 0 \rightarrow$ SC



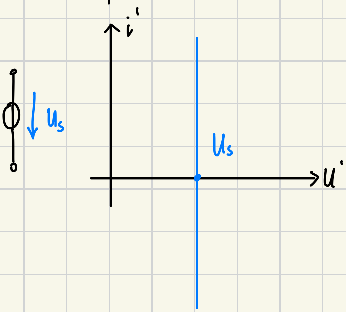
$$G = \frac{i_0}{u_0} \quad \ldots \text{ internal conductance}$$

$$R = \frac{U_0}{i_0} = \frac{1}{G} \quad \ldots \text{ internal resistance}$$

2.6.1 Independent Sources

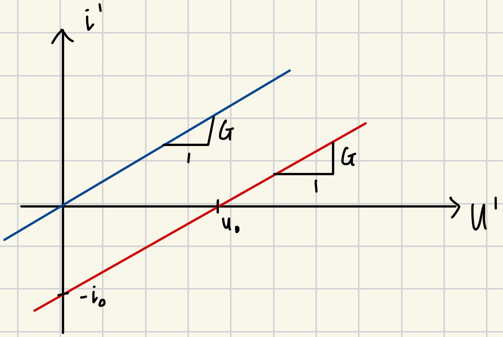


current source

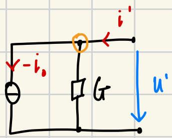


voltage source

2.6.2 internal structure



$i' = \underline{Gu' - i_0}$

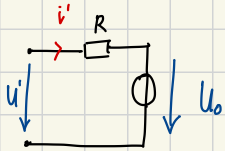


$i' + i_0 = Gu'$

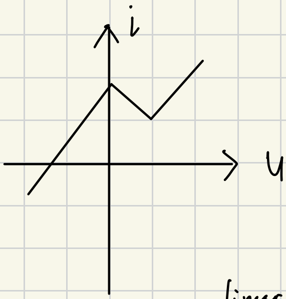
$u' + \frac{1}{G} i' + \frac{1}{G} i_0$

$\frac{1}{G} = R, \quad G = \frac{i_0}{u_0}$

$u' = Ri' + u_0$

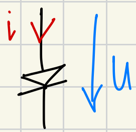


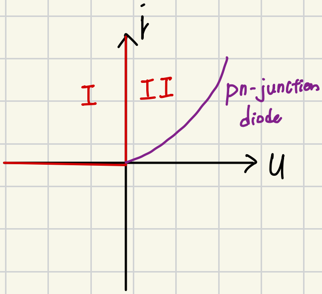
2.7 Piecewise linear One - Ports



2.7.1 Piecewise linear Diodes

2.7.1.1 Ideal Diodes

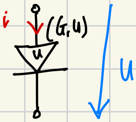


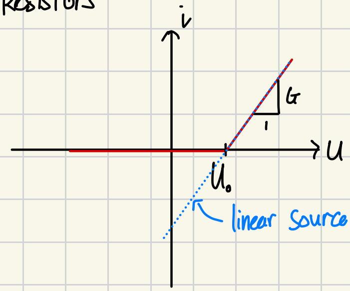


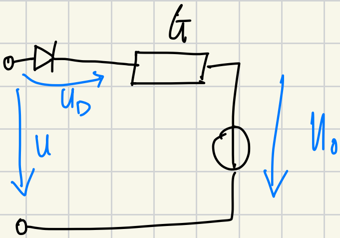
I: $i=0$ for $u \leq 0$

II: $u=0$ for $i \geq 0$

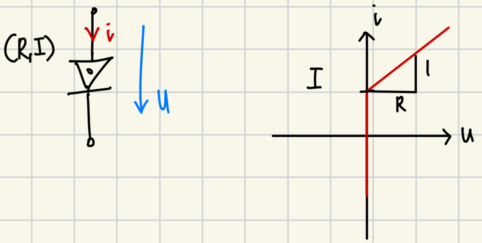
2.7.1.2 Concave Resistors

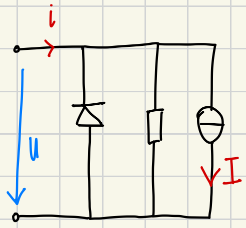




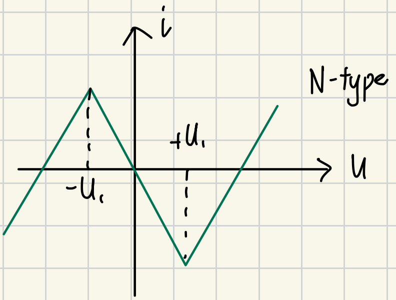


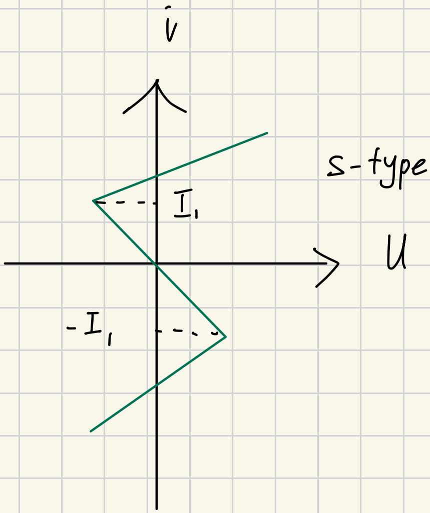
2.7.1.2 Convex resistor



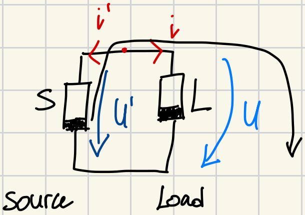


2.7.2 Real Negative Resistors





2.8 one - Port Circuits



2.8.1 Operating point

KCL : $i' + i = 0$

KVL : $-u' + u = 0$

Rearranging them

$$i' = -i$$
$$u' = u$$

$$(u', i') \in S$$

$$(u, i) \in L$$

$$(u, -i) \in S$$

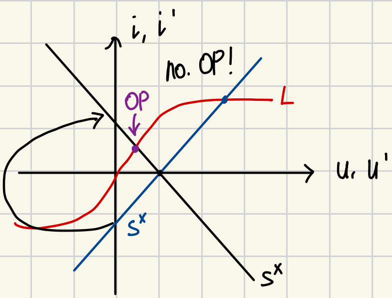
$$S = \{(u,i) \mid (u,-i) \in S\}$$

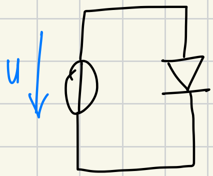
external source characteristic

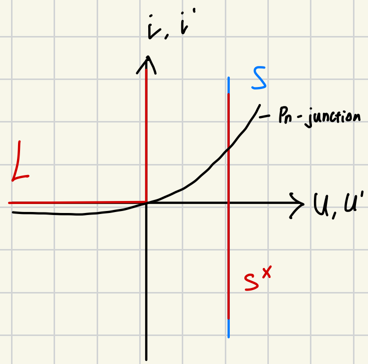
$$(u,i) \in S^{x} \ \& \ (u,i) \in L$$

operating point

$$\mathcal{OP} = S^{x} \cap L$$





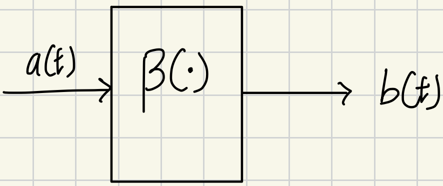


$$\mathcal{OP} = \{\} = \emptyset$$

2.8.2 Large Signal Analysis

system input : $a(t)$

" output : $b(t)$



$$b(t) = \beta(a(t))$$

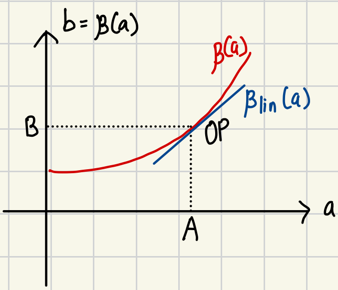
non-linear

2.8.3 Linearization

$$b(t) = \beta(a(t))$$

$$\approx k(a(t) - A) + B = \beta_{lin}(a(t))$$

$$OP : B = \beta(A)$$



$$k = \left. \frac{d\beta(a)}{da} \right|_{a=A}$$

$\beta(a)$

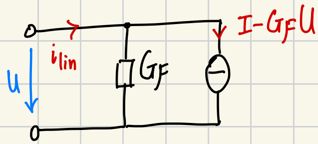
$$\rightarrow \beta_{lin}(a) = \left.\frac{d\beta(a)}{da}\right|_{a=A} \cdot (a-A) + B \quad \rightarrow \text{impt. formula}$$

$i = g_F(u)$, $I = g_F(U)$: OP

$$i_{lin} = \left.\frac{dg_F(u)}{du}\right|_{u=U} \cdot (u-U) + I$$

$$= G_F(u-U) + I$$

$$= G_F u + I - G_F U$$

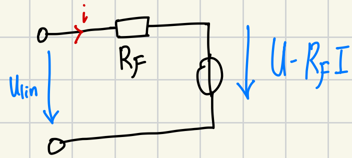


$U = r_F(i)$, OP: $U = r_F(I)$

$$u_{lin} = \left.\frac{dr_F(i)}{di}\right|_{i=I} \cdot (i-I) + U$$

$$R_F = \left.\frac{dr_F(i)}{di}\right|_{i=I}$$

$$u_{lin} = R_F i + U - R_F I$$



$g_F(\cdot)$ is the inverse function of $r_F(\cdot)$

$$G_F = \frac{1}{R_F}$$

$$i_{lin} = G_F u + I - G_F U$$

$$\frac{1}{G_F} \cdot i_{lin} = u + \frac{1}{G_F} I - U$$

$$u = R_F \cdot i_{lin} + U - R_F \cdot I$$

2.8.4 Small Signal Analysis

input $a(t) = A + \Delta a(t)$

output $b(t) = B + \Delta b(t)$

$$b(t) = B(a(t))$$

$$b_{lin}(t) = \boxed{\left.\frac{d\,B(a)}{a}\right|_{a=A}} \cdot (a-A) + B$$

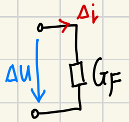
$$b_{lin}(t) = K(a(t) - A) + B$$

$$b_{lin}(t) - B = K(a(t) - A)$$

$$\Delta b(t) = K \Delta a(t)$$

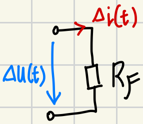
$$i(t) \approx g_{F,lin}(u) = I + G_F(u - U)$$

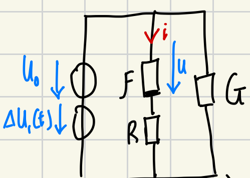
$$\Delta i(t) = G_F(\Delta u(t))$$



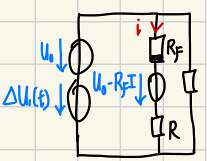
$$u_{lin}(t) = U + R_F(i - I)$$

$$\Delta u(t) = R_F \cdot \Delta i(t)$$

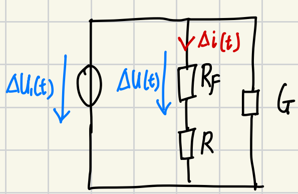




$U = V_F(i)$ lin



Small signals:



Strictly linear two ports:

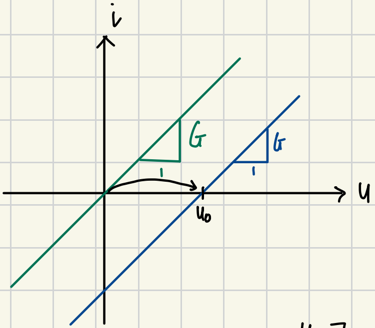
2 measurements:

$$\underset{\sim}{U} = [\underline{u}^{(1)}, \underline{u}^{(2)}]$$

$$\underset{\sim}{I} = [\underline{i}^{(1)}, \underline{i}^{(2)}]$$

$$\begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{U} \\ \underset{\sim}{I} \end{bmatrix} \underline{c}, \quad \underline{c} \in \mathbb{R}^2$$

3.3 Linear two ports



$$\begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{U} \\ \underset{\sim}{I} \end{bmatrix} \cdot \underline{c} + \begin{bmatrix} \underline{u}_0 \\ \underline{i}_0 \end{bmatrix} \quad \underline{c} \in \mathbb{R}^2$$

<u>three</u> measurements

$$\begin{bmatrix} \underline{u}_0 \\ \underline{i}_0 \end{bmatrix} = \begin{bmatrix} \underline{u}^{(1)} \\ \underline{i}^{(1)} \end{bmatrix}$$

$$\underset{\sim}{U} = [\underline{u}^{(2)} - \underline{u}^{(1)}, \underline{u}^{(3)} - \underline{u}^{(1)}]$$

$$\underset{\sim}{i} = [\underline{i}^{(2)} - \underline{i}^{(1)}, \underline{i}^{(3)} - \underline{i}^{(1)}]$$

$$\underline{u} = \underset{\sim}{U}\underline{c} + \underline{u}_0$$

$$\underline{i} = \underset{\sim}{I}\underline{c} + \underline{i}_0$$

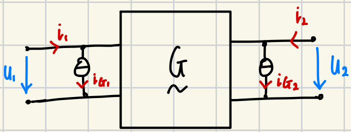
$$\underline{u} - \underline{u}_0 = \underset{\sim}{U}\underline{c}$$

$$\underline{c} = \underset{\sim}{U}^{-1}(\underline{u} - \underline{u}_0)$$

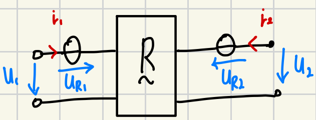
$$\underline{i} = \underset{\sim}{I}\,\underset{\sim}{U}^{-1}(\underline{u} - \underline{u}_0) + \underline{i}_0$$

$$= \underbrace{\underset{\sim}{I}\,\underset{\sim}{U}^{-1}}_{\underset{\sim}{G}}\,\underline{u} \underbrace{- \underbrace{\underset{\sim}{I}\,\underset{\sim}{U}^{-1}}_{\underset{\sim}{G}}\,\underline{u}_0 + \underline{i}_0}_{\underline{i}_G}$$

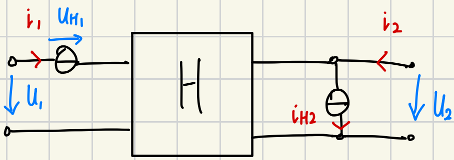
$$\underline{i} = \underset{\sim}{G}\,\underline{u} + \underline{i}_G$$



$$\underline{u} = \underset{\sim}{R}\,\underline{i} + \underline{u}_R$$



$$\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \underset{\sim}{H}\begin{bmatrix} i_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} u_{H_1} \\ i_{H_2} \end{bmatrix}$$



3.4 Properties of two ports

3.4.1 Power

$P_1 = u_1 i_1$

$P_2 = u_2 i_2$

$P = P_1 + P_2$

$= u_1 i_1 + u_2 i_2$

3.4.1.1 losslessness

$$\forall \begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} \in F : \ p = u_1 i_1 + u_2 i_2 = \underline{u}^T \underline{i} = 0$$

Strictly linear two port

$$\begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{U} \\ \underset{\sim}{I} \end{bmatrix} \underline{c} \ , \qquad \underline{c} \in \mathbb{R}^2$$

$$\underline{u} = \underset{\sim}{U}\underline{c} \ , \quad \underline{i} = \underset{\sim}{I}\underline{c}$$

$$p = \underline{u}^T \underline{i} = (\underset{\sim}{U}\underline{c})^T \underset{\sim}{I}\underline{c}$$

$$= \underline{c}^T \underset{\sim}{U}^T \underset{\sim}{I} \underline{c} = 0 \quad \forall \underline{c} \in \mathbb{R}^2$$

$$= \frac{1}{2}\underline{c}^T \left(\underset{\sim}{U}^T \underset{\sim}{I} + \underset{\sim}{I}^T \underset{\sim}{U} \right) \underline{c} = 0$$

$$\underset{\sim}{U}^T \underset{\sim}{I} + \underset{\sim}{I}^T \underset{\sim}{U} = \underset{\sim}{0} \iff \text{lossness}$$

losslessness:

$$\forall \begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} \in \mathcal{F} : \underline{u}^T \underline{i} = 0$$

strictly linear two-port:

$$\underset{\sim}{U}^T \underset{\sim}{I} + \underset{\sim}{I}^T \underset{\sim}{U} = \underset{\sim}{0}$$

3.4.1.2 Activity & Passive

passive:

$$\forall \begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} \in \mathcal{F} : \underline{u}^T \underline{i} \geq 0$$

active:

$$\exists \begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} \in \mathcal{F} : \underline{u}^T \underline{i} < 0$$

3.4.2 Duality

$$\underline{u} \xrightarrow{R_d} R_d \underline{i}^d$$

$$\underline{i} \xrightarrow{R_d} \frac{1}{R_d} \underline{u}^d$$

$$\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \underset{\sim}{H} \begin{bmatrix} i_1 \\ u_2 \end{bmatrix}$$

$$\downarrow R_d$$

$$\begin{bmatrix} R_d i_1^d \\ \frac{1}{R_d} u_2^d \end{bmatrix} = \underset{\sim}{H} \begin{bmatrix} \frac{1}{R_d} u_1^d \\ R_d i_2^d \end{bmatrix}$$

$$\begin{bmatrix} i_1^d \\ u_2^d \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{R_d} & 0 \\ 0 & R_d \end{bmatrix} \underset{\sim}{H} \begin{bmatrix} \frac{1}{R_d} & 0 \\ 0 & R_d \end{bmatrix}}_{\underset{\sim}{H}'^d} \begin{bmatrix} u_1^d \\ i_2^d \end{bmatrix}$$

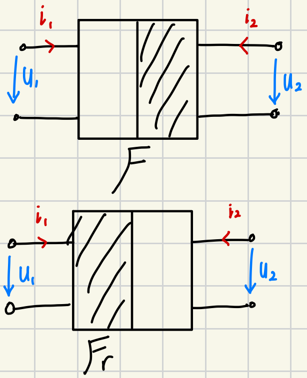
$$\underline{u} = \underset{\sim}{R}\, \underline{i}$$

$$\downarrow R_d$$

$$R_d\, \underline{i}^d = \underset{\sim}{R} \cdot \frac{1}{R_d} \underline{u}^d$$

$$\underline{i}^d = \underbrace{\frac{1}{R_d^2} \underset{\sim}{R}}_{\underset{\sim}{G}^d} \cdot \underline{u}^d$$

3.4.3 Reversibility (symmetry)



permutation matrix

$$\underset{\sim}{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\underset{\sim}{P} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

$$\underline{u} \xrightarrow{\text{rev.}} \underset{\sim}{P}\underline{u}$$

$$\underline{i} \xrightarrow{\text{rev.}} \underset{\sim}{P}\,\underline{i}$$

$$\underline{i} = \underset{\sim}{G}\,\underline{u}$$

$$\downarrow \text{rev}$$

$$\underset{\sim}{P}\underline{i} = \underset{\sim}{G}\,\underset{\sim}{P}\underline{u}$$

$$\underset{\sim}{P}^{-1} = \underset{\sim}{P}$$

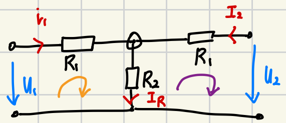
$$\underset{\sim}{i} = \underbrace{\underset{\sim}{P}\underset{\sim}{G}\underset{\sim}{P}}\underset{\sim}{u}$$

if $\underset{\sim}{G}$ is symmetric

if $\underset{\sim}{P}\underset{\sim}{R}\underset{\sim}{P} = R$, F is symmetric

$$\Rightarrow \underset{\sim}{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{11} \end{bmatrix}$$

$$\underset{\sim}{A} = \underset{\sim}{A}'$$



KCL: $i_1 + i_2 - i_R = 0$

KVL: $-u_1 + R_1 i_1 + R_2 i_R = 0$

KVL: $-R_2 i_R - R_1 i_2 + u_2 = 0$

$i_R = i_1 + i_2$

$u_1 = R_1 i_1 + R_2 i_R = (R_1 + R_2) i_1 + R_2 i_2$

$u_2 = R_2 i_1 + (R_1 + R_2) i_2$

$$\underset{\sim}{R} = \begin{bmatrix} R_1 + R_2 & R_2 \\ R_2 & R_1 + R_2 \end{bmatrix}$$

3.4.4 Reciprocity & linear two ports

Symmetry of transfer Characteristics

$$\underset{\sim}{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$u_1 = r_{11} i_1 + r_{12} i_2$

$u_2 = r_{21} i_1 + r_{22} i_2$

reciprocity :

$r_{12} = r_{21}$

$g_{12} = g_{21}$

page 42:

$h_{12} = -h_{21}$

$h'_{12} = -h'_{21}$

$\det \underset{\sim}{A} = 1$

$\det \underset{\sim}{A}' = 1$

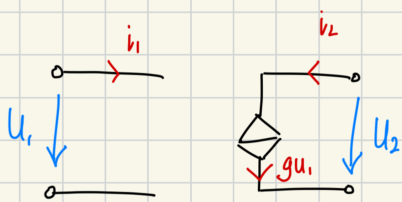
reciprocal two-port

$$\underset{\sim}{U}^T \underset{\sim}{I} - \underset{\sim}{I}'^T \underset{\sim}{U} = \underset{\sim}{0}$$

different from lossnessness

3.5 Special two-ports

3.5.1 Controlled sources



Voltage-controlled current source

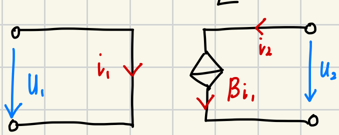
$i_1 = 0$

$i_2 = g u_1$

$$\underset{\sim}{G} = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}$$

Alternative

$$\underset{\sim}{A} = \begin{bmatrix} 0 & -\frac{1}{g} \\ 0 & 0 \end{bmatrix}$$



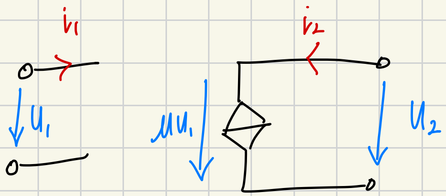
CCCS

$U_1 = 0$

$i_2 = \beta i_1$

$$\underset{\sim}{H} = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}$$

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{\beta} \end{bmatrix}$$



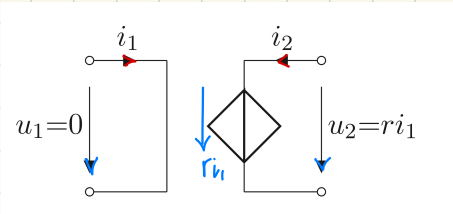
VCVS

$i_1 = 0$

$U_2 = \mu U_1$

$$\underset{\sim}{H}' = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix}$$

$$\underset{\sim}{A} = \begin{bmatrix} \frac{1}{\mu} & 0 \\ 0 & 0 \end{bmatrix}$$



$U_1 = 0$

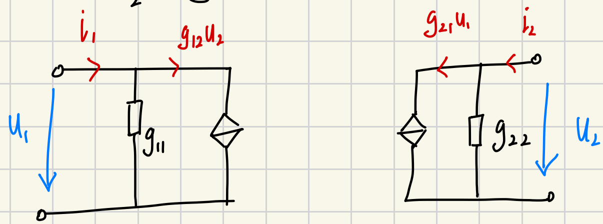
$U_2 = ri_1$

$$\underset{\sim}{R} = \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}$$

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 0 \\ \frac{1}{r} & 0 \end{bmatrix}$$

$$i_1 = g_{11} u_1 + g_{12} u_2$$

$$i_2 = g_{21} u_1 + g_{22} u_2$$

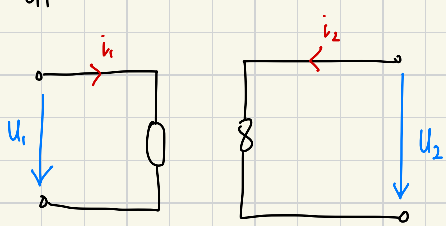


3.5.2 Nullor

VCCS: $\underset{\sim}{A}_{VCCS} = \begin{bmatrix} 0 & -\frac{1}{g} \\ 0 & 0 \end{bmatrix}$

$$\lim_{g \to \infty} \underset{\sim}{A}_{VCCS} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underset{\sim}{0}$$

$u_1 = 0, \quad i_1 = 0$



3.5.3 Transformer

lossless & reciprocal

$$\underset{\sim}{U}^T \underset{\sim}{I} + \underset{\sim}{I}^T \underset{\sim}{U} = \underset{\sim}{0}$$

$$\underset{\sim}{U}^T \underset{\sim}{I} - \underset{\sim}{I}^T \underset{\sim}{U} = \underset{\sim}{0}$$

$$\underset{\sim}{U} = \underset{\sim}{R}\underset{\sim}{I}$$

$$\underset{\sim}{I}^T \underset{\sim}{R}^T I + \underset{\sim}{I}^T \underset{\sim}{R} \underset{\sim}{I}$$

$$\underset{\sim}{R}^T + \underset{\sim}{R} = 0$$

$$\left.\begin{aligned} \underset{\sim}{R} &= -\underset{\sim}{R}^T \\ \underset{\sim}{R} &= \underset{\sim}{R}^T \end{aligned}\right\} \underset{\sim}{R} = \underset{\sim}{0}$$

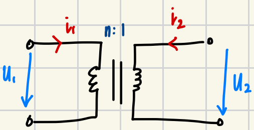
$$\underset{\sim}{G} = \underset{\sim}{0}$$

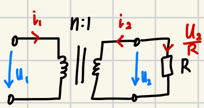
$$\underset{\sim}{H}_{trans} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix}$$

$$\underset{\sim}{H}'_{trans} = \begin{bmatrix} 0 & -\frac{1}{n} \\ \frac{1}{n} & 0 \end{bmatrix}$$

$$u_1 = n u_2$$

$$i_2 = -n i$$





$$i_2 = -\frac{u_2}{R}$$

$$= -n i_1$$

$$i_1 = \frac{1}{n} \cdot \left(\frac{u_2}{R}\right)$$

$$u_2 = \frac{1}{n} \cdot u_1$$

$$i_1 = \frac{1}{n^2} \frac{u_1}{R}$$

$= n^2 R$ at port 1

3.5.4 gyrator

lossless & <u>not</u> reciprocal

$$\underset{\sim}{u}^T \underset{\sim}{I} + \underset{\sim}{I}^T \underset{\sim}{u} = \underset{\sim}{0}$$

$$\underset{\sim}{u}^T \underset{\sim}{I} - \underset{\sim}{I}^T \underset{\sim}{u} \neq \underset{\sim}{0}$$

$$\underset{\sim}{R} = -\underset{\sim}{R}^T$$

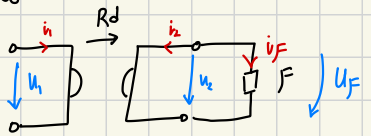
$$\Rightarrow \underset{\sim}{R}_{gyra} = \begin{bmatrix} 0 & -Rd \\ Rd & 0 \end{bmatrix}$$

$$u_1 = -Rd\, i_2$$

$$u_2 = Rd\, i_1$$

$$i_1 = \frac{1}{Rd} u_2$$

$$\underset{\sim}{G}_{gyra} = \begin{bmatrix} 0 & +Gd \\ -Gd & 0 \end{bmatrix}$$



$i_2 = -i_F$, $u_2 = u_F$

$(u_{F_1}, i_F) \in F$

$\downarrow Rd$

$(Rd\, i_1 , \frac{1}{Rd} u_1) \in F$

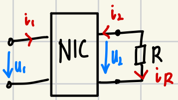
$(u_1 , i_1) \in F^d$

3.5.5 negative immitance Converter (NIC)

$U_1 = -kU_2$

$i_1 = -\frac{1}{k} i_2 \Rightarrow -i_2 = k i_1$

$\underset{\sim}{A}_{NIC} = \begin{bmatrix} -k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$



ohm: $U_2 = R i_R \leftarrow i_R = -i_2$

$U_2 = R(-i_2)$

$U_1 = -k U_2$

$= -kR(-i_2)$

$U_1 = -kRki_1$

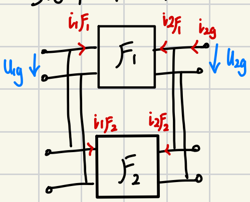
$= \underbrace{-k^2R}_{<0} i_1$

$\Rightarrow$ negative resistor at port 1

3.6 Connection of two ports

F_1 & F_2

3.6.1 Parallel Connection



parallel connection of ports 1:

$U_{1g} = U_{1F_1} = U_{1F_2}$

$i_{1g} = i_{1F_1} + i_{1F_2}$

port 2: $U_{2g} = U_{2F_1} = U_{2F_2}$

$i_{2g} = i_{2F_1} + i_{2F_2}$

$\underline{i}_g = \begin{bmatrix} i_{1g} \\ i_{2g} \end{bmatrix}$ $\underline{i}_{F_1} = \begin{bmatrix} i_{1F_1} \\ i_{2F_1} \end{bmatrix}$ $\underline{i}_{F_2} = \begin{bmatrix} i_{1F_2} \\ i_{2F_2} \end{bmatrix}$

$$\underline{i}_{F_1} = \underline{g}_{F_1}(\underline{u}_{F_1}) = \underline{g}_{F_1}(\underline{u}_g)$$

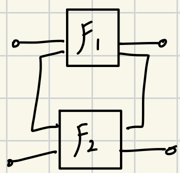
$$\underline{i}_{F_2} = \underline{g}_{F_2}(\underline{u}_{F_2}) = \underline{g}_{F_2}(\underline{u}_g)$$

$$\begin{aligned} \underline{i}_g &= \underline{i}_{F_1} + \underline{i}_{F_2} \\ &= \underline{g}_{F_1}(\underline{u}_g) + \underline{g}_{F_2}(\underline{u}_g) \\ &= \underline{g}_g(\underline{u}_g) \end{aligned}$$

$$\underline{g}_g = \underline{g}_{F_1} + \underline{g}_{F_2}$$

$$\underset{\sim}{G}_g = \underset{\sim}{G}_{F_1} + \underset{\sim}{G}_{F_2}$$

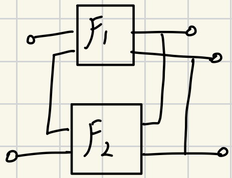
3.6.2 series Connection



$$\underline{r}_g = \underline{r}_{F_1} + \underline{r}_{F_2}$$

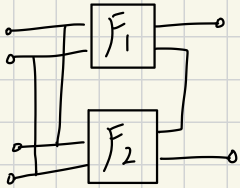
$$\underset{\sim}{R}_g = \underset{\sim}{R}_{F_1} + \underset{\sim}{R}_{F_2}$$

3.6.3 Hybrid Connections



$$\underline{h}_g = \underline{h}_{F_1} + \underline{h}_{F_2}$$

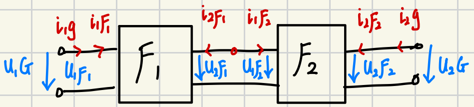
$$\underset{\sim}{H}_g = \underset{\sim}{H}_{F_1} + \underset{\sim}{H}_{F_2}$$



$$\underline{h}'_g = \underline{h}'_{F_1} + \underline{h}'_{F_2}$$

$$\underset{\sim}{H}'_g = \underset{\sim}{H}'_{F_1} + \underset{\sim}{H}'_{F_2}$$

3.6.4 Cascade Connection



KCL: $i_{2F_1} + i_{1F_2} = 0$

$$\underline{i_{2F_1} = -i_{1F_2}}$$

$$\underline{u_{2F_1} = u_{1F_2}}$$

$$\begin{bmatrix} \underline{u_{1F_2}} \\ \underline{i_{1F_2}} \end{bmatrix} = g_{F_2}\left(\begin{bmatrix} u_{2F_2} \\ -i_{2F_2} \end{bmatrix}\right)$$

$$= \underbrace{\begin{bmatrix} u_{2F_1} \\ -i_{2F_1} \end{bmatrix}}$$

$$\begin{bmatrix} u_{1F_1} \\ i_{1F_1} \end{bmatrix} = \underline{a}_{F_1}\left(\underbrace{\begin{bmatrix} u_{2F_1} \\ -i_{2F_1} \end{bmatrix}}\right)$$

$$= \underline{a}_{F_1}\left(\underline{a}_{F_2}\begin{bmatrix} u_{2F_2} \\ -i_{2F_2} \end{bmatrix}\right)$$

$$\begin{bmatrix} u_{1g} \\ i_{1g} \end{bmatrix} = \underline{a}_{F_1}\left(\underline{a}_{F_2}\begin{bmatrix} u_{2g} \\ -i_{2g} \end{bmatrix}\right)$$

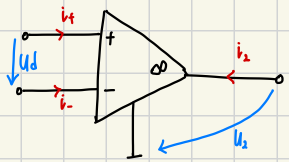
$$\begin{bmatrix} u_{1g} \\ i_{1g} \end{bmatrix} = \underset{\sim}{A}_{F_1} \cdot \underset{\sim}{A}_{F_2} \cdot \begin{bmatrix} u_{2g} \\ -i_{2g} \end{bmatrix}$$

4 Operational Amplifier

4.1 modelling

4.1.1 Idealized

Non-lineas Model



$i_+ = i_- = 0$

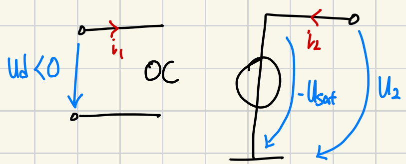
$$U_2 = \begin{cases} -U_{sat} & U_d < 0 \quad (\mathrm{I}) \\ +U_{sat} & U_d > 0 \quad (\mathrm{III}) \end{cases}$$

sat ⇒ saturation

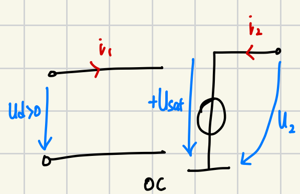
$|U_2| \leq U_{sat}$ for $U_d = 0$ (II)

4.1.2 Saturation Regions

(I) : $U_2 = -U_{sat}$ for $U_d < 0$



(III) : $U_2 = +U_{sat}$ $U_d > 0$



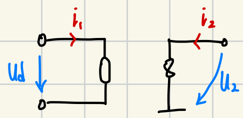
4.1.3.1 Linear Region

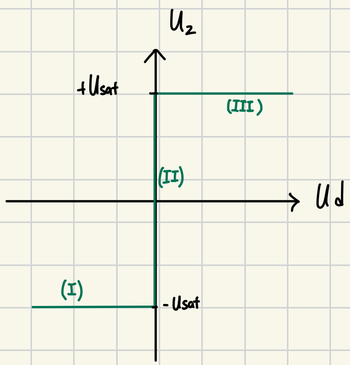
(II) : $u_d = 0, \; i_1 = 0$

$|u_2| \leq U_{sat}$

nullator

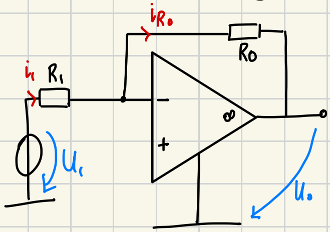
norator at port 2



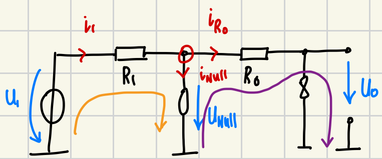


4.2 Op-Amp Circuits

4.2.1 Inverting Amplifier



(II) Linear region:



KVL: $-U_1 + R_1 i_1 + \overset{0}{\cancel{U_{Null}}} = 0$

$U_1 = R_1 i_1 \Rightarrow i_1 = \frac{1}{R_1} U_1$

KCL: $i_1 - i_{R_0} - \overset{0}{\cancel{i_{Null}}} = 0$

$i_1 = i_{R_0} = \frac{1}{R_1} U_1$

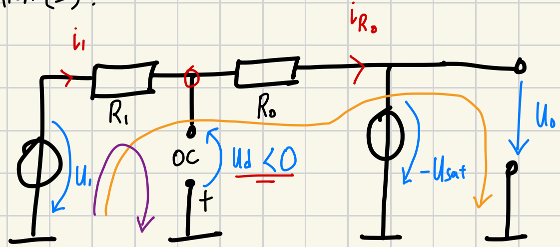
KVL: $-\overset{0}{\cancel{U_{Null}}} + R_0 i_{R_0} + U_0 = 0$

$U_0 = -R_0 i_{R_0} = -\frac{R_0}{R_1} U_1$

$\Rightarrow$ gain $= -\frac{R_0}{R_1} < 0$

inverting amplifier

saturation (I) :



$i_1 = i_{R_0}$

KVL : $-U_1 + R_1 i_1 + R_0 i_1 - U_{sat} = 0$

$$i_1 = \frac{1}{R_0 + R_1}(U_1 + U_{sat})$$

KVL : $-U_1 + R_1 i_1 - U_d = 0$

$$U_d = -U_1 + R_1 i_1$$

$$= -U_1 + R_1 \frac{1}{R_0 + R_1}(U_1 + U_{sat})$$

$$= -U_1 + \frac{R_1}{R_0 + R_1} \cdot U_1 + \frac{R_1}{R_0 + R_1} \cdot U_{sat}$$

$$= -\frac{R_0}{R_0 + R_1} \cdot U_1 + \frac{R_1}{R_0 + R_1} U_{sat} < 0$$

$$\frac{R_0}{R_0 + R_1} U_1 > \frac{R_1}{R_0 + R_1} U_{sat}$$

$$U_1 > \frac{R_1}{R_0} U_{sat}$$

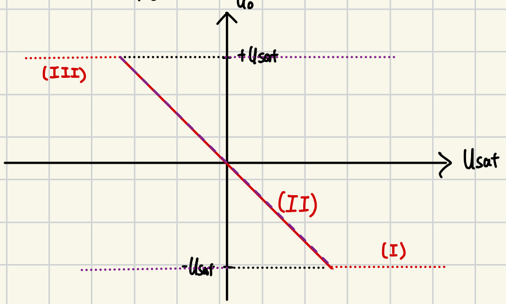
Saturation (III)

$-U_{sat} \rightarrow +U_{sat}$

$U_d < 0 \rightarrow U_d > 0$

$U_0 = +U_{sat}$

for $U_1 < -\frac{R_1}{R_0} U_{sat}$

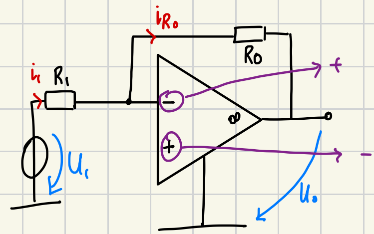


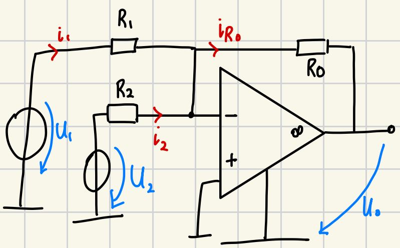
$>$

$U_1 < -\frac{R_1}{R_0} U_{sat}$

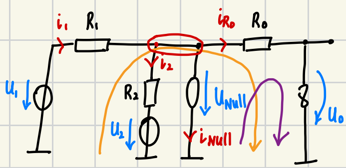
$<$

$U_1 > \frac{R_1}{R_0} U_{sat}$





linear region (II) :



KVL: $-U_2 + R_2 i_2 + \overset{0}{\cancel{U_{Null}}} = 0$

$i_2 = \frac{1}{R_2} U_2$

$i_1 = \frac{1}{R_1} U_1$

KCL: $i_1 + i_2 - \overset{0}{\cancel{i_{Null}}} - i_{R0} = 0$

$i_{R0} = i_1 + i_2$

KVL: $-U_{Null} + R_0 i_{R_0} + U_0 = 0$

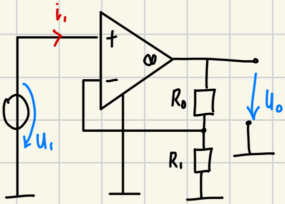
$$U_0 = -R_0\, i_{R_0} = -R_0(i_1 + i_2)$$

$$U_0 = \underbrace{-\frac{R_0}{R_1}}_{\alpha_1} U_1 = \underbrace{-\frac{R_0}{R_1}}_{\alpha_2} U_2$$

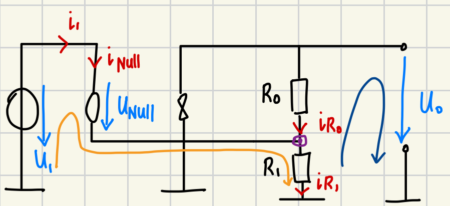
$$U_0 = \alpha_1 U_1 + \alpha_2 U_2$$

$$= [\alpha_1 , \alpha_2] \cdot \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

4.2.2 Non-Inverting Amplifier



Linear region (II):



KVL: $-U_1 + \overset{0}{\cancel{U_{Null}}} + R_1 i_{R_1} = 0$

$$i_{R_1} = \frac{U_1}{R_1}$$

KCL: $\overset{0}{\cancel{i_{Null}}} + i_{R_0} - i_{R_1} = 0$

$$i_{R_0} = i_{R_1} = \frac{U_1}{R_1}$$

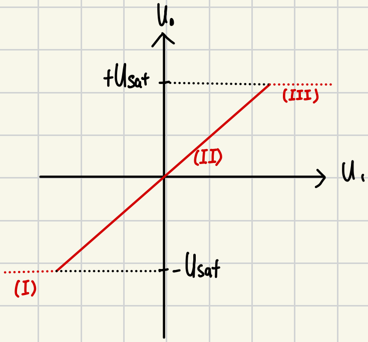
KVL: $-R_1 i_{R_1} - R_0 i_{R_0} + U_0 = 0$

$$U_0 = R_1 i_{R_1} + R_0 i_{R_0}$$

$$= \frac{R_0 + R_1}{R_1} U_1$$

$$= \underbrace{\left(1 + \frac{R_0}{R_1}\right)}_{\geq 1} U_1$$

$$gain = 1 + \frac{R_0}{R_1}$$

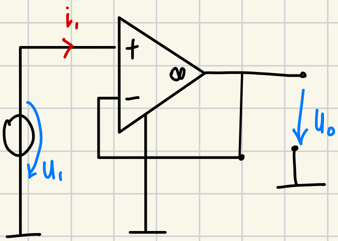


4.2.2.1 Voltage Follower

R_0 : SC

R_1 : OC

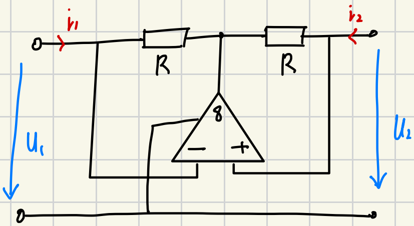
$\Rightarrow gain = 1$



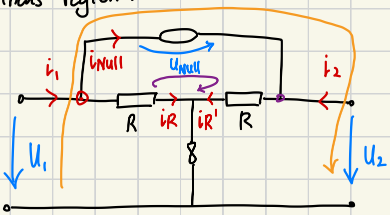
$U_o = U_1$

$i_1 = 0$

4.2.3 Negatirie immittance Converter (NIC)



Lineas region:



KVL: $-U_1 + \overset{0}{\cancel{U_{Null}}} + U_2 = 0$

$$U_1 = U_2$$

KCL: $i_1 - \overset{0}{\cancel{i_{Null}}} - i_R = 0$

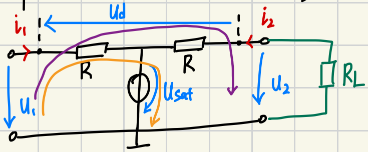
$$i_1 = i_R$$

KCL: $i_2 = i_R'$

KVL: $-Ri_R + U_{Null} + Ri_R' = 0$

$$\left.\begin{array}{l} i_R = i_R' \\ i_1 = i_2 \\ U_1 = U_2 \end{array}\right\} \text{NIC with } K = -1$$

$\Rightarrow$ ability to build negative resistance positive saturation (III) :



KVL : $-U_1 + R i_1 + U_{sat} = 0$

$U_1 = R i_1 + U_{sat}$

$U_2 = R i_2 + U_{sat}$

KVL: $-U_1 - U_d + U_2 = 0$

$U_d = U_2 - U_1 > 0$

$R i_2 + U_{sat} - R i_1 - U_{sat} > 0$

$i_2 - i_1 > 0$

$i_2 > i_1$ for (III)

due to R_L:

$U_2 = R_2(-i_2)$

Lineas Region

$i_1 = i_2$

$U_2 = U_1 = R_2(-i_2)$

$U_1 = \underbrace{-R_L}_{\text{neg. resistor}} i_1$

Saturated region (III):

$U_1 = R i_1 + U_{sat}$

$U_2 = -R_L i_2$

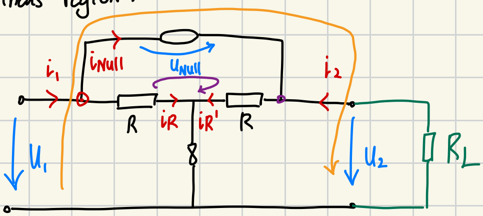
$(= R i_2 + U_{sat})$

$-R_L i_2 = R i_2 + U_{sat}$

$(R + R_L) i_2 = U_{sat}$

$i_2 = -\dfrac{U_{sat}}{R + R_L}$

Lineas region :



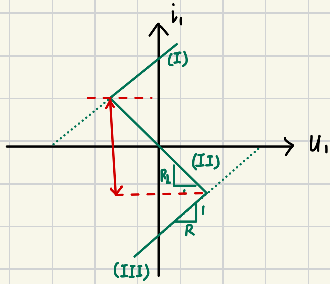
$$U_2 = \frac{R_L}{R+R_L} U_{sat}$$

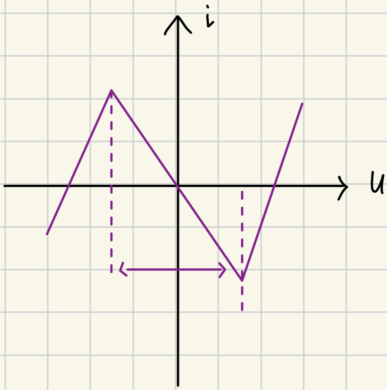
$$U_d = U_2 - U_1$$

$$= \frac{R_L}{R+R_L} U_{sat} - R i_1 - U_{sat}$$

$$= \frac{-R}{R+R_L} U_{sat} - R i_1 > 0$$

$$i_1 < \frac{-1}{R+R_L} U_{sat}$$

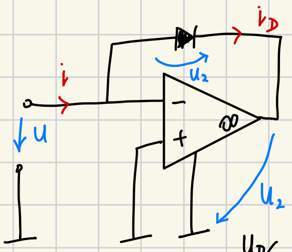




+ & - terminals exchanged

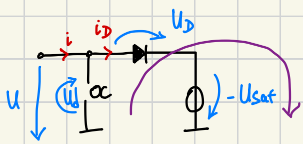
4.2.4 Piecewise Linear Resistors

4.2.4.1 Ideal Diode



$$i_D = I_s\left(e^{u_D/u_T} - 1\right)$$

negative saturation (I):



$i_D = i$ KVL : $U_d + U_D - U_{sat} = 0$

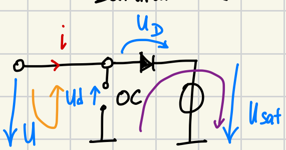
$$U_d = -U_D + U_{sat} < 0$$

$$U_D > U_{sat}$$

$U_{sat} = 12.5V$ $I_s = 10^{-17}A$, $U_T = 25mV$

$$i_D \approx 10^{200} A \cdot e^{u/u_T} \lightning \longrightarrow \text{impossible}$$

positive saturation (III)



KVL: $U_d + U_D + U_{sat} = 0$

$$U_d = -U_D - U_{sat} > 0$$

$$U_D < -U_{sat}$$

$$\Rightarrow i_D = 0$$

$i = i_D = 0$

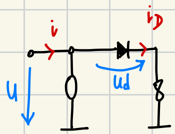
KVL: $U + U_d = 0$

$U = -U_d$, $U_d > 0$

$\Rightarrow U < 0$

if voltage is negative, current = 0 for ideal diode

linear region(II)



nullator:

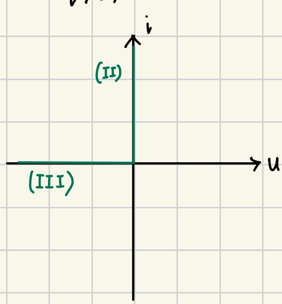
$U = 0$

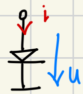
$i_D = 0$

pn-junction diode:

$i_D \geq 0$

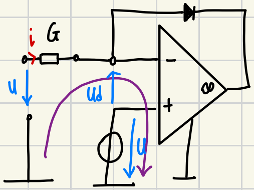
$i > 0$, $U = 0$





ideal diode is connected to reference node

4.2.4.2 Concave Resistors



KVL : $-u + \frac{i}{G} - u_d + U = 0$

$u = U + \frac{i}{G} - u_d$

linear region : $u_d = 0$

$u = U + \frac{i}{G}$

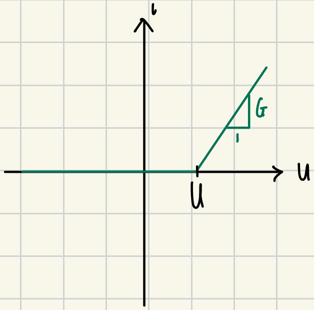
Sat. region (III) : $u_d > 0$

$u_d = U - u + \frac{i}{G} > 0$

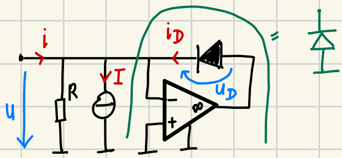
$i = 0$:

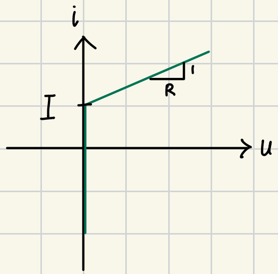
$U - u > 0$

$u < U$



4.2.4.3 Convex Resistors

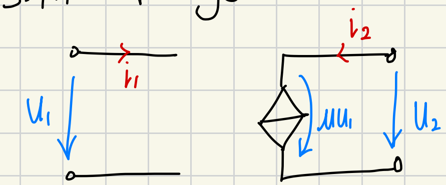




4.3 linear op-Amp Circuits

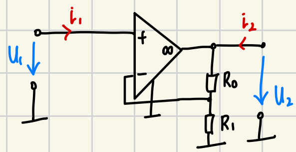
4.3.1 Controlled sources

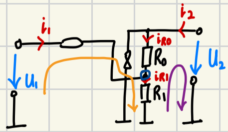
4.3.1.1 Voltage - Controlled Voltage Source



$i_1 = 0$

$u_2 = \mu u_1$





KVL: $-u_1 + \overset{0}{\cancel{u_{Null}}} + R_1 i_{R1} = 0$

$u_1 = R_1 i_{R1}$

KCL: $i_{Null} + i_{R_0} - i_{R_1} = 0$

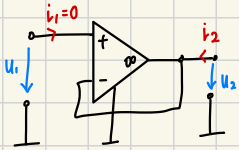
$i_{R_0} = i_{R_1}$

KVL: $-u_2 + R_0 i_{R_0} + R_1 i_{R1} = 0$

$u_2 = R_1 i_{R1} - R_0 i_{R_0}$

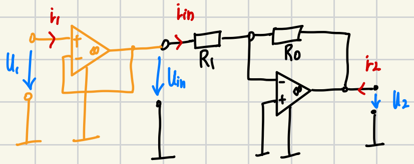
$= \frac{R_0 + R_1}{R_1} u_1$

$\mu = \frac{R_0 + R_1}{R_1} \geq 1$



$$U_2 = -\frac{R_0}{R_1} U_{in}$$

$$i_{in} = \frac{U_{in}}{R_1} \neq 0$$



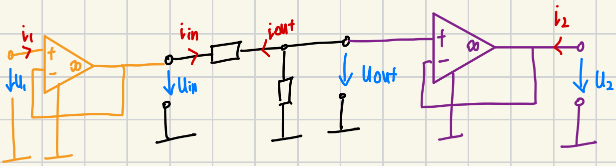
Voltage followes:

$$i_1 = 0$$

$$U_{in} = U_1$$

$$U_2 = -\frac{R_0}{R_1} U_1$$

$$\mu \leq 0$$



for $i_{out} = 0$: $i_{in} = \frac{U_{in}}{R_0 + R_1} \neq 0$

Volt. foll: $i_1 = 0$, $U_{in} = U_1$

Volt. foll2: $i_{out} = 0$, $U_2 = U_{out}$

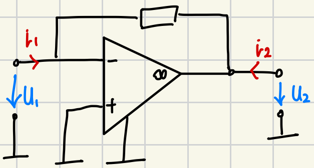
Volt. div. rule:

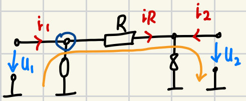
$$\frac{U_{out}}{U_{in}} = \frac{R_0}{R_0 + R_1}$$

$$U_2 = \frac{R_0}{R_0 + R_1} U_1$$

$$0 \leq \mu \leq 1$$

4.3.1.2 Current controlled voltage source





$u_1 = 0$

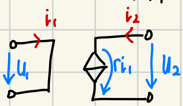
KCL: $i_1 + i_R - \underbrace{\cancel{i_{null}}}_{0} = 0$

$i_R = -i_1$

KVL: $-u_1 - R i_R + u_2 = 0$

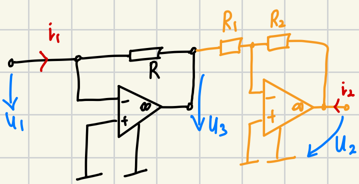
$u_2 = R i_R$

$= -R i_1$



$u_1 = 0 \ , \quad u_2 = r i_1$

$r = -R \leq 0$



$u_3 = -R i_1$

inv. ampl: $u_2 = -\frac{R_2}{R_1} u_3$

$u_2 = \frac{R_2}{R_1} R i_1$

$r = \frac{R_2 \cdot R}{R_1} \geq 0$

5 General circuit Analysis

5.1 Incidence Matrices

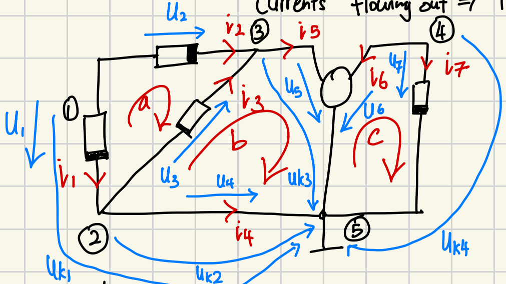
b branches

n nodes

Set up KCL eqns.

for all nodes except reference node & Currents going in $\Rightarrow -$

Currents flowing out $\Rightarrow +$



branch currents :

$\underline{i} = [i_1, i_2, i_3, i_4, i_5, i_6, i_7]^T$

1: $i_1 + i_2 = 0$

2: $-i_1 + i_3 + i_4 = 0$

3: $i_5 - i_2 - i_3 = 0$

4: $i_6 + i_7 = 0$

$$\underbrace{\begin{array}{ccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ \end{array}}_{}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & & & & & \\ -1 & 0 & 1 & 1 & & & \\ 0 & -1 & -1 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{\underset{\sim}{A}} \cdot \underline{i} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

$\underset{\sim}{A}$: node incidence matrix

KCL: $\underset{\sim}{A}\,\underline{i} = \underline{0}$

KVL

a: $-u_1 + u_2 - u_3 = 0$

b: $u_3 + u_5 - u_4 = 0$

c: $-u_6 + u_7 = 0$

$$\underline{u} = [u_1, u_2, u_3, u_4, u_5, u_6, u_7]$$

$$\underset{\sim}{B} = \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{array} \\ \begin{bmatrix} -1 & 1 & -1 & & & & \\ & & 1 & -1 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$\underset{\sim}{B}\,\underline{u} = \underline{0}$$

branches(b) = 7

nodes (n) = 5

$n-1 = 4$ KCL eqns

$\underline{b-(n-1)} = 3$ KVL eqn

b. eqns.

$$\underline{u_k} = [u_{k1}, u_{k2}, u_{k3}, u_{k4}]^T$$

$u_1 = u_{k1} - u_{k2}$

$u_2 = u_{k1} - u_{k3}$

$u_3 = u_{k2} - u_{k3}$

$u_4 = u_{k2} - u_{k5} = u_{k2}$

ref. node $= 0$

$u_5 = u_{k3}$

$u_6 = u_{k4}$

$u_7 = u_{k4}$

$$\underline{u} = \underbrace{\begin{array}{c} \begin{array}{cccc} u_{k1} & u_{k2} & u_{k3} & u_{k4} \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}}_{\to\ \underset{\sim}{A}^T} \cdot \underline{u}_k$$

$$\underline{u} = \underset{\sim}{A}^T \underline{u}_k$$

$$\underset{\sim}{A}\,\underline{i} = \underline{0}$$

5.5.1 Tellegen's Theorem

$$\underline{u}^T \underline{i} \overset{KVL}{=} (\underset{\sim}{A}^T \underline{u}_k)^T \underline{i}$$

$$= \underline{u}_k^T \underline{A}\,\underline{i} \overset{KCL}{=} \underline{u}_k^T \underline{0} = 0$$

$\underline{u}^T \underline{i} = 0$ Tellegen's Theorem

$$\underset{\sim}{A}\,\underset{\sim}{B}^T = \underset{\sim}{0}$$

5.2 Tableau Equations

linear elements :

Strictly linear resistance, linear sources

$$\underset{\sim}{M}\,\underline{u} + \underset{\sim}{N}\,\underline{i} = \underline{e}$$

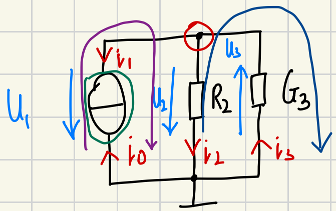
$$\underset{\sim}{B}\,\underline{u} = \underline{0}$$

$$\underset{\sim}{A}\,\underline{i} = \underline{0}$$

$$\begin{array}{c} b \\ \\ b \end{array}\underbrace{\overbrace{\begin{bmatrix} \underset{\sim}{B} & \underline{0} \\ \underline{0} & \underset{\sim}{A} \\ \underset{\sim}{M} & \underset{\sim}{N} \end{bmatrix}}^{\underset{\sim}{T}}}_{2b \times 2b} \cdot \underbrace{\begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix}}_{2b} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{e} \end{bmatrix}$$

$$\underset{\sim}{T} \cdot \begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{e} \end{bmatrix}$$

$$\begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} = \underset{\sim}{T}^{-1} \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{e} \end{bmatrix}$$



KCL : $i_1 + i_2 - i_3 = 0$

KVL : $-U_1 + U_2 = 0$

KVL : $-U_2 - U_3 = 0$

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

Tellegen : $\underset{\sim}{A} \cdot \underset{\sim}{B}^T = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$i_1 + i_0 = 0 \Rightarrow i_1 = -i_0$

Ohm : $U_2 = R_2 i_2$

$U_2 - R_2 i_2 = 0$

Ohm : $i_3 = G_3 U_3$

$-G_3 U_3 + i_3 = 0$

$$
\begin{array}{c}
\\ \text{KVL} \\ \\ \text{KCL} \\ \\ \\ \\
\end{array}
\underbrace{
\begin{array}{c}
\begin{array}{cccccc} u_1 & u_2 & u_3 & i_1 & i_2 & i_3 \end{array} \\
\left[\begin{array}{ccc:ccc}
-1 & 1 & 0 & & & \\
0 & -1 & -1 & & \underset{\sim}{0} & \\ \hdashline
 & \underline{0}^T & & 1 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & -R_2 & 0 \\
0 & 0 & -G_3 & 0 & 0 & 1
\end{array}\right]
\end{array}}_{\underset{\sim}{T}}
\cdot
\begin{bmatrix} \underline{u} \\ \\ \underline{i} \end{bmatrix}
=
\begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \hdashline -i_0 \\ 0 \\ 0 \end{bmatrix}
$$

$\underset{\sim}{M}$ $\qquad$ $\underset{\sim}{N}$

5.2.1 Node Analysis

$$\underset{\sim}{A}\,\underline{i} = \underline{0} \qquad \underline{u} = \underset{\sim}{A}^T \underline{u}_k$$

$$\underset{\sim}{M}\,\underline{u} + \underset{\sim}{N}\,\underline{i} = \underline{e} \qquad \downarrow\ \underline{u} - \underset{\sim}{A}^T \underline{u}_k = 0$$

$$
\begin{array}{r}
b \\ n-1 \\ b
\end{array}
\underbrace{
\left[\begin{array}{c:c:c}
-\underset{\sim}{A}^T & \underset{\sim}{1} & \underset{\sim}{0} \\ \hdashline
\underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{A} \\ \hdashline
\underset{\sim}{0} & \underset{\sim}{M} & \underset{\sim}{N}
\end{array}\right]}_{2b+n-1 \ \times \ 2b+n-1}
\cdot
\underbrace{\begin{bmatrix} \underline{u}_k \\ \underline{u} \\ \underline{i} \end{bmatrix}}_{2b+n-1}
=
\begin{bmatrix} \underline{0} \\ \hdashline \underline{0} \\ \hdashline \underline{e} \end{bmatrix}
$$

5.3 Nodal Analysis

$$\underline{u} = \underset{\sim}{A}^T \underline{u}_k$$

$$\underset{\sim}{A}\,\underline{i} = 0$$

$$\underset{\sim}{M}\,\underline{u} + \underset{\sim}{N}\,\underline{i} = \underline{e}$$

$$\underset{\sim}{N}\,i = -\underset{\sim}{M}\,\underline{u} + \underline{e}$$

$\underset{\sim}{N}^{-1}$ exists

Voltage-control elements

$$\underline{i} = -\underset{\sim}{N}^{-1}\underset{\sim}{M}\,\underline{u} + \underset{\sim}{N}^{-1}\underline{e}$$

$$\underset{\sim}{A}\,\underline{i} = \underline{0}$$

$$\underset{\sim}{A}\,\underline{i} = -A\underset{\sim}{N}^{-1}\underset{\sim}{M}\,\underline{u} + \underset{\sim}{A}\underset{\sim}{N}^{-1}\underline{e} = \underline{0}$$

$$-A\underset{\sim}{N}^{-1}\underset{\sim}{M}\,\underset{\sim}{A}^T\underline{u}_k + \underset{\sim}{A}\underset{\sim}{N}^{-1}\underline{e} = 0$$

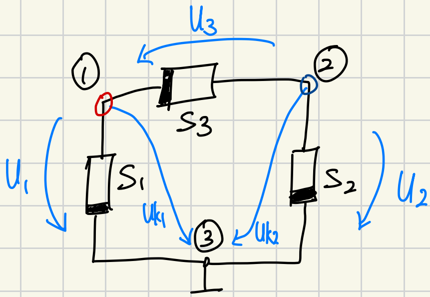
$$\underbrace{-A\underset{\sim}{N}^{-1}\underset{\sim}{M}\,\underset{\sim}{A}^T\underline{u}_k}_{\underset{\sim}{G}_k \underline{u}_k} = -\underset{\sim}{A}\underset{\sim}{N}^{-1}\underline{e}$$

$$\underset{\sim}{G}_k\,\underline{u}_k = \underline{i}_q$$

node conductance matrix

node current source vector

Arrow goes direction of bar

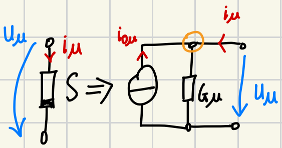


$KCL\ 1:\quad i_1 - i_2 = 0$

$KCL\ 2:\quad i_2 + i_3 = 0$

$$\underset{\sim}{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

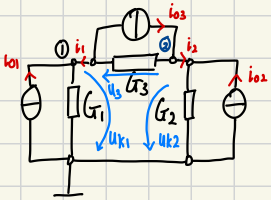
$$\underline{u}_k = \begin{bmatrix} U_{k1} \\ U_{k2} \end{bmatrix} ; \quad \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}}_{\underset{\sim}{A}^T} \cdot \begin{bmatrix} U_{k1} \\ U_{k2} \end{bmatrix}$$



$KCL:\quad G_\mu \cdot U_\mu - i_\mu - i_{0\mu} = 0$

$$G_\mu \cdot U_\mu - i_\mu = i_{0\mu}$$

$$G_1 \cdot U_1 - i_1 = i_{01}$$
$$G_2 \cdot U_2 - i_2 = i_{02}$$
$$G_3 \cdot U_3 - i_3 = i_{03}$$



$$\begin{bmatrix} -\tilde{A}^T & \tilde{1} & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{A} \\ \tilde{0} & \tilde{M} & \tilde{N}\end{bmatrix} \cdot \begin{bmatrix} \underline{u_k} \\ \underline{u} \\ \underline{i}\end{bmatrix} = \begin{bmatrix}\underline{0}\\ \underline{0}\\ \underline{e}\end{bmatrix}$$

$\tilde{1} \rightarrow$ identity matrix

$$\left[\begin{array}{cc:ccc:ccc} \multicolumn{2}{c:}{\overbrace{\qquad}^{-A^T}} & \multicolumn{3}{c:}{\overbrace{\qquad\qquad}^{\tilde{1}}} & & & \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline & & & & & 1 & 0 & -1 \\ \multicolumn{2}{c:}{\tilde{0}} & \multicolumn{3}{c:}{\tilde{0}} & 0 & 1 & 1 \\ \hdashline & & G_1 & & & -1 & & \\ & & & G_2 & & & -1 & \\ & & & & G_3 & & & -1 \\ & & \multicolumn{3}{c}{\tilde{M}} & \multicolumn{3}{c}{\tilde{N}} \end{array}\right] \cdot \begin{bmatrix} u_{k1} \\ u_{k2} \\ u_1 \\ u_2 \\ u_3 \\ i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i_{01} \\ i_{02} \\ i_{03} \end{bmatrix}$$

$\tilde{A}$

$$\tilde{G}_k = -\tilde{A}\tilde{N}^{-1}\tilde{M}\tilde{A}^T$$

$$= - \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} G_1 & & \\ & G_2 & \\ & & G_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \\ -G_3 & G_3 \end{bmatrix} = \begin{bmatrix} G_1 + G_3 & -G_3 \\ -G_3 & G_2 + G_3 \end{bmatrix}$$

if element lies betwen 4 node, it appears $\underset{\sim}{2}$ times

1 will be negtive

$$\underline{i_q} = -\underset{\sim}{A}\underset{\sim}{N}^{-1}\underline{e} = -\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \cdot \begin{bmatrix} i_{01} \\ i_{02} \\ i_{03} \end{bmatrix}$$

$$= \begin{bmatrix} i_{01} - i_{03} \\ i_{02} + i_{03} \end{bmatrix} \Rightarrow \begin{bmatrix} G_1 + G_3 & -G_3 \\ -G_3 & G_2 + G_3 \end{bmatrix} \cdot \begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix} = \begin{bmatrix} i_{01} - i_{03} \\ i_{02} + i_{03} \end{bmatrix}$$

5.4 Direct Formulation of Node Conductance Matrix

$$\underset{\sim}{G_k} \cdot \underline{u_k} = \underline{i_q}$$

VCCS:

$$i_1 = 0, \quad i_2 = g u_1$$

$$\underset{\sim}{G}_{vccs} = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}$$

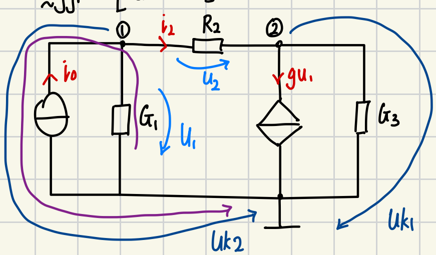
$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underset{\sim}{G}_{vccs} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

gyrator:

$$i_1 = G_d u_2$$

$$i_2 = -G_d u_1$$

$$\underset{\sim}{G}_{gyr} = \begin{bmatrix} 0 & G_d \\ -G_d & 0 \end{bmatrix}$$



ohm: $u_2 = R_2 i_2$

$$\Rightarrow i_2 = \frac{1}{R_2} u_2$$

KVL : $-u_1 + u_{k1} = 0$

$$u_1 = u_{k1}$$

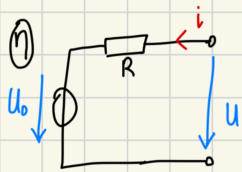
$$g u_1 = g u_{k1}$$

$$\underbrace{\begin{bmatrix} G_1 + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} + g & G_3 + \frac{1}{R_2} \end{bmatrix}}_{\underset{\sim}{G}_u} \cdot \underbrace{\begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix}}_{\underline{u}_k} = \underbrace{\begin{bmatrix} i_0 \\ 0 \end{bmatrix}}_{\underline{i}_q}$$

5.5 Non- Voltage controlled Element

5.5.1 Source Transform

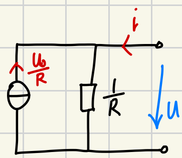
node between the resistor and source



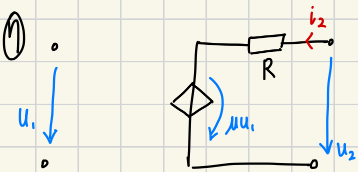
$i_0 = \frac{u_0}{R}, G = \frac{1}{R}$

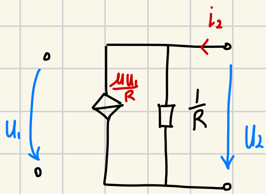
$\longleftrightarrow$

$u_0 = \frac{i_0}{G}, R = \frac{1}{G}$



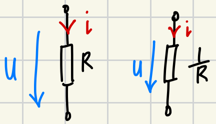
no n !





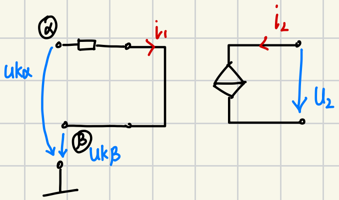
no n !

5.5.2 Ohm's Law



$i = \frac{1}{R} u$

CCCS :

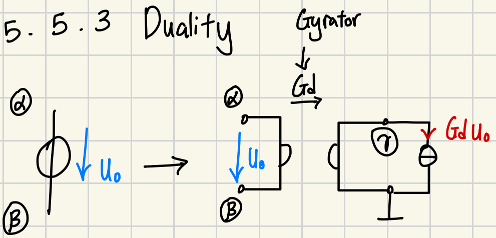


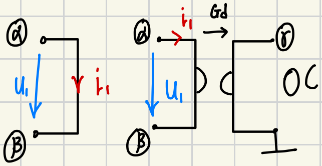
$i_1 = G(U_{k\alpha} - U_{k\beta}$

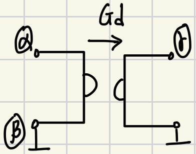
$i_2 = \beta i_1$

$= \beta G (u_{k\alpha} - u_{k\beta})$

5.5.3 Duality

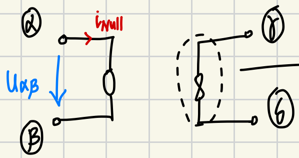






$$
\begin{array}{cc}
\underset{\sim}{G}_k \begin{bmatrix} & Gd \\ -Gd & \end{bmatrix} & \begin{array}{c} \alpha \\ \gamma \end{array} \\
\begin{array}{cc} U_{k\alpha} & U_{k\beta} \end{array} &
\end{array}
$$

5.5.4 Nullors

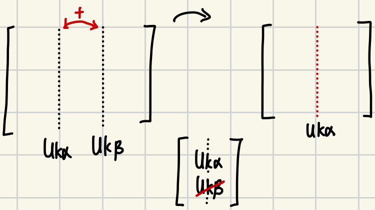


Supernode as voltage and current unknown

KCL γ + KCL δ

$u_{\alpha\beta} = 0, \; i_{Null} = 0$

$u_{k\alpha} = u_{k\beta}$



Nullator reduce columns

Norators reduce rows

6. Reactive elements

Charge:

$$q(t) = \int_{-\infty}^{t} i(t')dt'$$

$$= q(t_0) + \int_{t_0}^{t} i(t')\,dt'$$

flux:

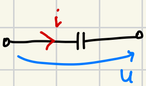
$$\Phi(t) = \Phi(t_0) + \int_{t_0}^{t} u(t')dt'$$

derivative wrt time

$$i(t) = \frac{dq(t)}{dt} = \mathring{q}(t)$$

$$u(t) = \frac{d\Phi(t)}{dt} = \mathring{\Phi}(t)$$

Capacitors:



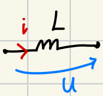
$$q(t) = \int_{-\infty}^{t} i(t')dt'$$

$$q(t) = C \cdot u(t)$$

C: Capacitance

$$i(t) = \mathring{q}(t) = C\mathring{u}(t)$$

inductor:



$$\Phi(t) = \int_{-\infty}^{t} u(t')dt'$$

$\Phi(t) = L\, i(t)$

L: inductance

$u(t) = \dot{\Phi}(t)$

$= L\dot{i}(t)$

6.1 Duality

$u \xrightarrow{Rd} Rd\, i^d$; $i \xrightarrow{Rd} \frac{1}{Rd} \cdot u^d$

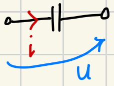
$$q(t) = \int_{-\infty}^{t} i(t')\,dt'$$

$$\downarrow Rd$$

$$\int_{-\infty}^{t} \frac{1}{Rd} u^d(t')\,dt' = \frac{1}{Rd}\int_{-\infty}^{t} u^d(t')\,dt'$$

$$= \frac{1}{Rd}\Phi^d(t)$$

$\Phi(t) \xrightarrow{Rd} Rd\, q^d(t)$

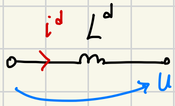


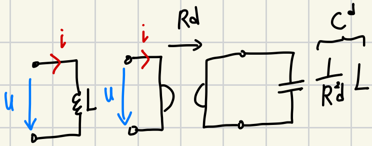
$$q(t) = C u(t)$$

$$\downarrow Rd$$

$$\frac{1}{Rd}\Phi^d(t) = C\, Rd\, i^d(t)$$

$$\Phi^d(t) = \underbrace{Rd^2 C}_{L^d}\, i^d(t)$$





6.2 Properties of reactances

6.2.1 Continuity

State variables :

u, q Capacitors

i, Φ inductors

do not jump

6.2.2 Energy

power : $p(t) = u(t) \cdot i(t)$

$$W_c(t_1, t_2) = \int_{t_1}^{t_2} p(t)\, dt$$

$$= \int_{t_1}^{t_2} u(t)\, i(t)\, dt$$

$i(t) = C\dot{u}(t)$

$$W_c(t_1, t_2) = \int_{t_1}^{t_2} u(t)\, C\dot{u}(t)\, dt$$

$$= \int_{u(t_1)}^{u(t_2)} C \cdot u\, du = \frac{C}{2}\left(u^2(t_2) - u^2(t_2)\right)$$

$$W_C(u_1,u_2) = \frac{C}{2}(u_2^2 - u_1^2)$$

$$W_C(q_1,q_2) = \frac{1}{2C}(q_2^2 - q_1^2)$$

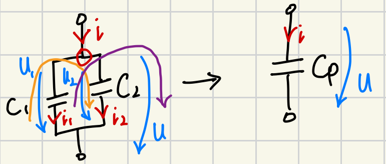
for $u_1 = 0$, $u_2 = u$

$$E(u) = W_C(0,u)$$
$$= \frac{C}{2}u^2$$

$$W_L(i_1,i_2) = \frac{L}{2}(i_2^2 - i_1^2)$$

$$E_L(i) = \frac{L}{2}i^2$$

6.3 Connection of Reactances



KCL: $-i + i_1 + i_2 = 0$

$i = i_1 + i_2$

$\int dt$

$q(t) = q_1(t) + q_2(t)$

$C_p u(t) = C_1 u_1(t) + C_2 u_2(t)$

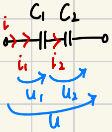
KVL: $-U_1 + U = 0$

KVL: $-U_2 + U = 0$

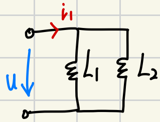
$$U = U_1 = U_2$$

$$C_p u = C_1 u + C_2 u$$

$$C_p = C_1 + C_2$$



$\xrightarrow{Rd}$



$U = U_1 + U_2$

$i = i_1 + i_2$

$q = q_1 = q_2$

$q = C_s u$

$\downarrow$ series

$U = \frac{1}{C_s} q$

$\frac{1}{C_s} i = \frac{1}{C_1} i_1 + \frac{1}{C_2} i_2$

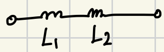
$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2}$

$C_s = C_1 \| C_2$

$C_s = C_1 \| C_2$

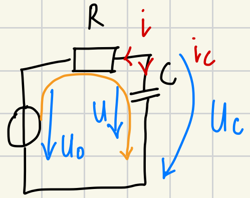
$\downarrow$

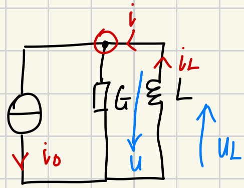
$L_p = L_1 \| L_2$



$L_s = L_1 + L_2$

7. linear first order circuits





KCL: $i_0 - Gu_L - i_L = 0$

KVL $= -u_0 + Ri_C + u_C = 0$

$u_L = L\dot{i}_L$

$i_C = C\dot{u}_C$

$i_0 - G(L\dot{i}_L) - i_L = 0$

$-u_0 + R(C\dot{u}_C) + u_C = 0$

$\dot{i}_L = -\frac{1}{GL} i_L + \frac{1}{GL} i_0$

$\dot{u}_C = -\frac{1}{RC} u_C + \frac{1}{RC} u_0$

$\tau = GL$

$\tau = RC$

$\boxed{\dot{x}(t) = -\frac{1}{\tau} x(t) + \frac{1}{\tau} v(t)}$

$x = i_L, \quad v = i_0$

$x = u_C, \ v = u_0$

7.1 General Excitation

$\dot{x} = -\frac{1}{\tau} x + \frac{1}{\tau} v$

homogeneous Case

$\dot{x} = -\frac{1}{\tau} x$

$\frac{dx}{dt} = -\frac{1}{\tau} x$

$\frac{dx}{x} = -\frac{dt}{\tau}$

$\int \frac{dx}{x} = -\int \frac{dt}{\tau}$

integr. const.

$\ln(x_{hom}) = -\frac{t}{\tau} + C_{int}$

$x_{hom}(t) = e^{-\frac{t}{\tau}} \cdot C$

$$X(t) = X_0 e^{-\frac{t-t_0}{\tau}} + \int_{t_0}^{t} \frac{1}{\tau} e^{-\frac{t-t'}{\tau}} \nu(t')\, dt'$$ Pg.103 (7.6)

$$U_C(t) = U_{C0}\, e^{-\frac{t-t_0}{RC}} + \int_{t_0}^{t} \frac{1}{RC} e^{-\frac{t-t'}{RC}} U_0(t')\, dt'$$

or

$$i_L(t) = i_L(t_0)\, e^{-\frac{t-t_0}{GL}} + \int_{t_0}^{t} \frac{1}{GL} e^{-\frac{t-t'}{GL}} i_0(t')\, dt'$$

7.2 Constant Excitation

$$\nu(t) \to X_\infty$$

$$\dot{X}(t) = -\frac{1}{\tau} X(t) + \frac{1}{\tau} X_\infty$$

$$X(t) = X_0 e^{-\frac{t-t_0}{\tau}} + \int_{t_0}^{t} \frac{1}{\tau} e^{-\frac{t-t'}{\tau}} X_\infty\, dt'$$

$$= X_0 e^{-\frac{t-t_0}{\tau}} + \left[\frac{1}{\tau} e^{-\frac{t-t'}{\tau}} (+\tau) X_\infty \right]_{t_0}^{t}$$

$$= X_0 e^{-\frac{t-t_0}{\tau}} + X_\infty - e^{-\frac{t-t_0}{\tau}} X_\infty$$

$$= X_\infty + (X_0 - X_\infty) e^{-\frac{t-t_0}{\tau}}$$

Stable Case : $\tau > 0$

e^{-1} approx $= 0.37 \approx \frac{1}{3}$

- $t \to \infty$: $X(t) \to X_\infty$
- $X(t_0+\tau) = X_\infty + (X_0 - X_\infty)\, e^{-\frac{t_0+\tau-t_0}{\tau}}$

$$\approx X_\infty + (X_0 - X_\infty) \frac{1}{3} \quad = 1 - \frac{2}{3}$$

$$= X_\infty + X_0\left(1 - \frac{2}{3}\right) - X_\infty\left(1 - \frac{2}{3}\right)$$

$$= X_0 + (X_\infty - X_0)\frac{2}{3}$$

$\downarrow$

moved $\frac{2}{3}$ of distance from X_∞ to X_0 towards X_∞

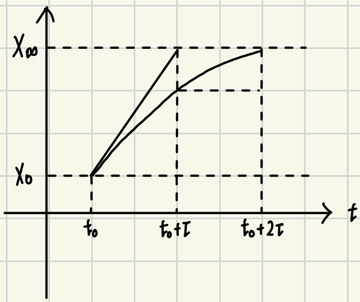
· linearization

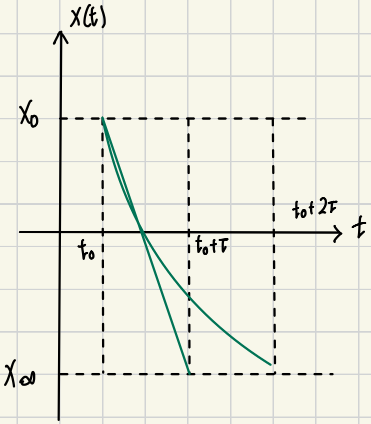
$$x_{lin}(t) = x(t_0) + \left.\frac{dx(t)}{dt}\right|_{t=t_0} (t-t_0)$$

$$= x_0 + \left(-\frac{1}{\tau}\right)(x_0 - x_\infty)(t-t_0)$$

$$= x_0 + \frac{x_\infty - x_0}{\tau}(t-t_0)$$

$$x_{lin}(t_0+\tau) = x_\infty$$



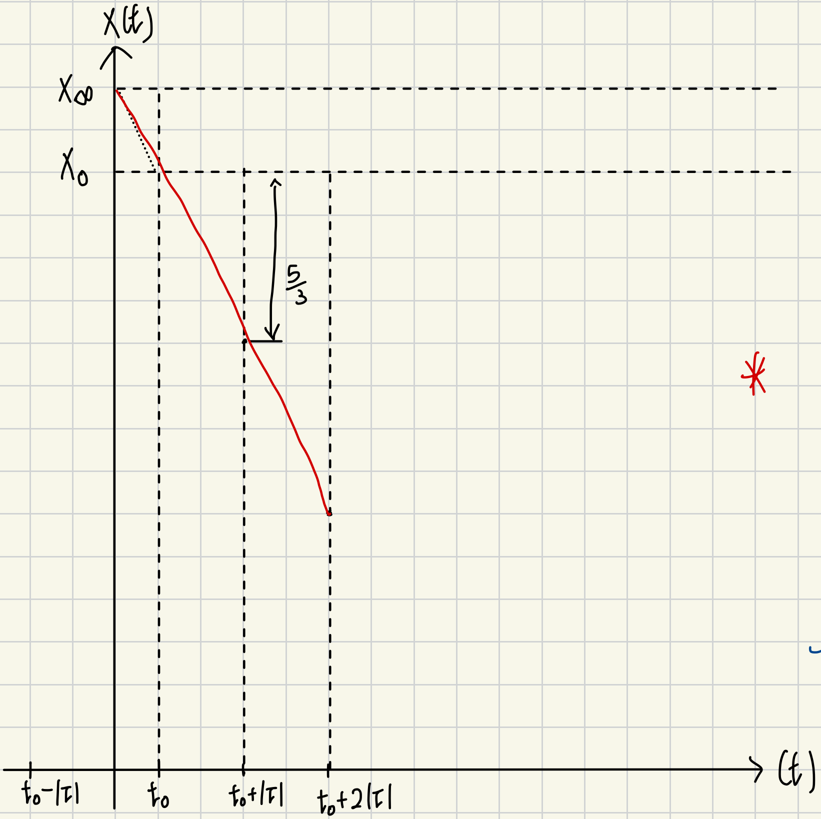


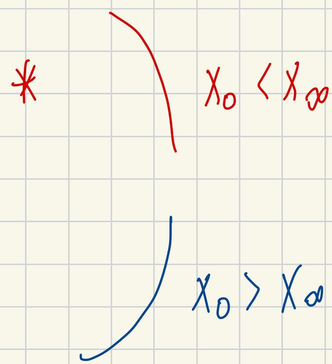
Unstable case: $\tau < 0$

$$X(t) = X_\infty + (X_0 - X_\infty)e^{\frac{t-t_0}{|\tau|}}$$

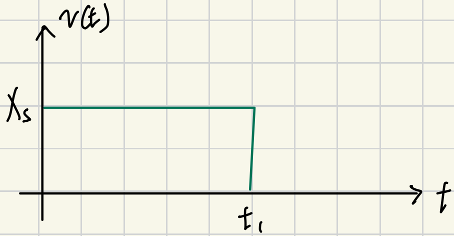
- $X(t_0 + |\tau|) \cong X_0 + (X_0 - X_\infty)\frac{5}{3}$
- $X_{lin}(t) \Rightarrow$
  tangent: $(t_0, X_0) - (t_0 - |\tau|, X_\infty)$
- $t \to \infty$: $X(t) \to +\infty$ for $X_0 > X_\infty$

  $X(t) \to -\infty$ for $X_0 < X_\infty$





7.3 Piecewise Constant Exci.

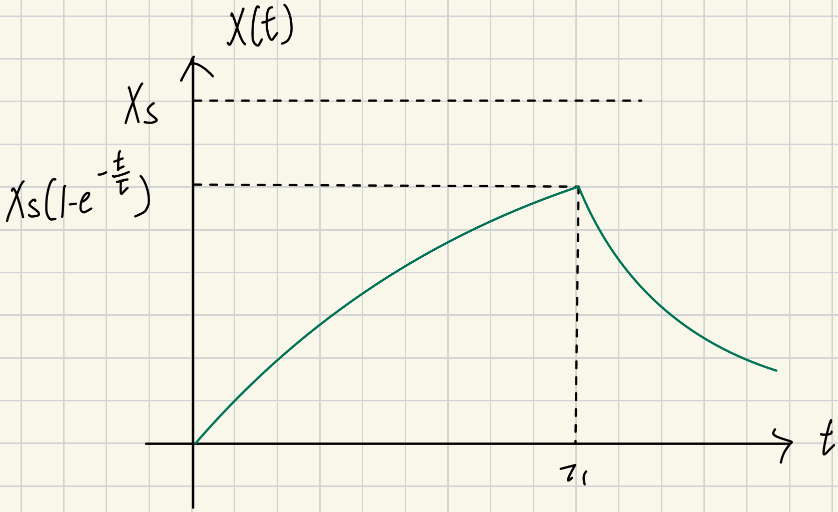


$$v(t) = \begin{cases} X_s & 0 \le t \le t_1 \\ 0 & \text{else} \end{cases}$$

$$t \geq 0 : \quad X(t) = X_\infty + (X_0 - X_\infty) e^{-\frac{t-t_0}{\tau}}$$

$$X_0 = X(0) = 0$$

$$= X_s - X_s e^{-\frac{t}{\tau}}$$



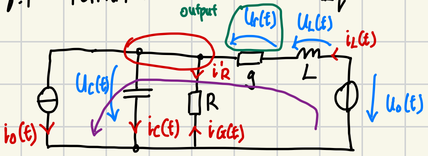
$$t \geq t_1 : \quad X_0 = X(t_1)$$

$$= X_s - X_s e^{-\frac{t_1}{\tau}}$$

$$X(t) = (X_s - X_s e^{-\frac{t_1}{\tau}}) \cdot e^{\frac{-t-t_1}{\tau}}$$

8 second order circuits

8.1 Formulation of State Eqn



KCL : $i_0 + i_C - i_L - i_G = 0$

KVL: $-u_0 + u_L + u_r + u_C = 0$

Ohm : $u_C = R i_R' = -R i_G$

$$i_G = -\frac{1}{R} u_C$$

ohm : $i_L = g u_r$

$$u_r = \frac{1}{g} i_L$$

$$i_0 + i_C - i_L + \frac{1}{R} u_C = 0$$

$$-u_0 + u_L + \frac{1}{g} i_L + u_C = 0$$

reactances: $i_C = C \dot{u}_C$

$$u_L = L \dot{i}_L$$

$$i_0 + C\dot{u}_C - i_L + \frac{1}{R} u_C = 0$$

$$-u_0 + L\dot{i}_L + \frac{1}{g} i_L + u_C = 0$$

$$\dot{u}_C = -\frac{1}{RC} u_C + \frac{1}{C} i_L - \frac{1}{C} i_0$$

$$\dot{i}_L = -\frac{1}{L} u_C - \frac{1}{gL} i_L + \frac{1}{L} u_0$$

$$\underline{x}(t) = \begin{bmatrix} u_C(t) \\ i_L(t) \end{bmatrix}$$

$$\underline{v}(t) = \begin{bmatrix} u_0(t) \\ i_0(t) \end{bmatrix}$$

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A} x + \underset{\sim}{B} \underline{v}$$

Xinchen

$$\underset{\sim}{A} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{1}{gL} \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$$

$$y(t) = u_r(t)$$
$$= \frac{1}{g} i_L(t)$$

$$y(t) = \underline{c}^T \underline{x}(t) + \underline{d}^T v(t)$$

$$\underline{c}^T = \left[0, \frac{1}{g}\right]$$

$$\underline{d}^T = [0, 0]$$

8.2 Solution of State Eqn

8.2.1 General Solution

$$\dot{x} = -\frac{1}{\tau}x + \frac{1}{\tau}v$$

$$x(t) = e^{-\frac{t-t_0}{\tau}}x(t_0) + \int_{t_0}^{t} e^{-\frac{t-t'}{\tau}}\frac{1}{\tau}v(t')dt' \quad \text{.... 1st order}$$

2nd order : $\underline{x}(t) = \exp(\underset{\sim}{A}(t-t_0))\,\underline{x}(t_0) + \int_{t_0}^{t}\exp(\underset{\sim}{A}(t-t'))\,\underset{\sim}{B}\,v(t')dt'$

Taylor series :

$$e^{\alpha t} = 1 + \alpha t + \frac{1}{2}(\alpha t)^2 + \frac{1}{6}(\alpha t)^3 + \ldots$$

$$= \sum_{j=0}^{\infty}\frac{1}{j!}(\alpha t)^j$$

$$\frac{de^{\alpha t}}{dt} = \alpha e^{\alpha t}$$

$$\exp(\underset{\sim}{A}t) = \sum_{j=0}^{\infty}\frac{1}{j!}\underbrace{(\underset{\sim}{A}t)^j}_{\underset{\sim}{A}^j\cdot t^j}$$

$$\frac{d\exp(\underset{\sim}{A}t)}{dt} = \underset{\sim}{A}\exp(\underset{\sim}{A}t)$$

$$= \exp(\underset{\sim}{A}t)\underset{\sim}{A}$$

$$\int\exp(\underset{\sim}{A}t)\,dt = \underset{\sim}{A}^{-1}\exp(\underset{\sim}{A}t)$$

#

8.2.2 Autonomous Case

$\underline{v}(t) \rightarrow \underline{v}_0$

$$\underline{X}(t) = \exp(\underset{\sim}{A}(t-t_0))\,\underline{X}(t_0) + \int_{t_0}^{t} \exp(\underset{\sim}{A}(t-t'))\,\underset{\sim}{B}\,\underline{v}_0\,dt'$$

$$\underline{X}(t) = \exp(\underset{\sim}{A}(t-t_0))\,\underline{X}(t_0) + \left[-\exp(\underset{\sim}{A}(t-t'))\cdot \underset{\sim}{A}^{-1}\underset{\sim}{B}\,\underline{v}_0\right]_{t_0}^{t}$$

$$= \exp(\underset{\sim}{A}(t-t_0))\,\underline{x}(t_0) + \left[-\underset{\sim}{A}^{-1}\underset{\sim}{B}\,\underline{v}_0 + \exp(\underset{\sim}{A}(t-t_0))\,\underset{\sim}{A}^{-1}\underset{\sim}{B}\,\underline{v}_0\right]$$

$$= -\underset{\sim}{A}^{-1}\underset{\sim}{B}\,\underline{v}_0 + \exp(\underset{\sim}{A}(t-t_0))\left(\underline{X}(t_0) + \underset{\sim}{A}^{-1}\underset{\sim}{B}\,\underline{v}_0\right)$$

autonomous :

$$\dot{\underline{X}} = \underset{\sim}{A}\,\underline{X} + \underset{\sim}{B}\,\underline{v}_0 \longrightarrow$$

equilibrium :

$$\dot{\underline{X}} = \underline{0}$$

$$\underline{0} = \underset{\sim}{A}\,\underline{X} + \underset{\sim}{B}\,\underline{v}_0$$

$$\underline{X}_\infty = -\underset{\sim}{A}^{-1}\underset{\sim}{B}\,\underline{v}_0$$

$$\underline{X}(t) = \underline{X}_\infty + \exp(\underset{\sim}{A}(t-t_0))(\underline{X}_0 - \underline{X}_\infty)$$

$$\underline{X}_0 = \underline{X}(t_0)$$

substitute $\underline{X} = \underline{X}' + \underline{X}_\infty$

$$\dot{\underline{X}} = \dot{\underline{X}}'$$

$$\dot{\underline{X}}' = \underset{\sim}{A}(\underline{x}' + \underline{X}_\infty) + \underset{\sim}{B}\,\underline{v}_0$$

$$= \underset{\sim}{A}\,\underline{x}' + \cancel{\underset{\sim}{A}(-\underset{\sim}{A}^{-1}\underset{\sim}{B}\,\underline{v}_0)} + \cancel{\underset{\sim}{B}\,\underline{v}_0}$$

$$\dot{\underline{X}}' = \underset{\sim}{A}\,\underline{x}' \;;\; \underline{X}'(t_0) = \underline{X}(t_0) - \underline{X}_\infty$$

$$\underline{x}'(t) = \exp(\underset{\sim}{A}(t-t_0))\,\underline{x}'(t_0)$$

$$\underline{X}(t) - \underline{X}_\infty = \exp(\underset{\sim}{A}(t-t_0))(\underline{X}(t_0) - \underline{X}_\infty)$$

$$\underline{X}(t) = \underline{X}_\infty + \exp(\underset{\sim}{A}(t-t_0))(\underline{X}_0 - \underline{X}_\infty)$$

8.2.3 Homogeneous State eqn

$$\dot{\underline{x}} = \underset{\sim}{A}\,\underline{x}$$

$$\underline{x}(t) = \exp(\underset{\sim}{A}(t-t_0))\,\underline{x}(t_0)$$

eigenvalues :

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$$

$$\det\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$

$$= \lambda^2 - \underbrace{(a_{11}+a_{22})}_{tr(\underset{\sim}{A})}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{=\det \underset{\sim}{A}}$$

$$T = trace(\underset{\sim}{A})$$

$$\Delta = \det(\underset{\sim}{A})$$

$$\lambda^2 - T\lambda + \Delta = 0$$

$$\lambda_{1/2} = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - \Delta}$$

I : $\left(\frac{T}{2}\right)^2 > \Delta$: $\lambda_1, \lambda_2 \in \mathbb{R}$

II : $\left(\frac{T}{2}\right)^2 = \Delta$: $\lambda_1 = \lambda_2 \in \mathbb{R}$

III : $\left(\frac{T}{2}\right)^2 < \Delta$: $\lambda_1 = \lambda_2^* \in \mathbb{C}$

Complex Conjugate

λ_i : eigenvalues

8.2.4 Normal Form

$\lambda_1 \neq \lambda_2 \in \mathbb{R}$

$\underset{\sim}{A}\, q_k = \lambda_k q_k$ $\quad q_k$: eigenvectors

$$\underset{\sim}{A}\, q_1 = \lambda_1 q_1$$

$$\underset{\sim}{A}\, q_2 = \lambda_2 q_2$$

$$\left[\underset{\sim}{A}\, q_1 , \underset{\sim}{A}\, q_2\right] = \left[\lambda_1 q_1 , \lambda_2 q_2\right]$$

$$\underset{\sim}{A} \underbrace{\left[q_1 , q_2 \right]}_{\underset{\sim}{Q}} = \underbrace{\left[q_1 , q_2 \right]}_{\underset{\sim}{Q}} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\underset{\sim}{\Lambda}}$$

$$\underset{\sim}{A}\,\underset{\sim}{Q} = \underset{\sim}{Q}\,\underset{\sim}{\Lambda}$$

$$\underset{\sim}{A} = \underset{\sim}{Q}\,\underset{\sim}{\Lambda}\,\underset{\sim}{Q}^{-1}$$

eigenvalue decomposition (EVD)

$$\mathring{\underline{x}} = \underset{\sim}{A}\,\underline{x}$$

EVD : $\mathring{\underline{x}} = \underset{\sim}{Q}\,\underset{\sim}{\Lambda}\,\underset{\sim}{Q}^{-1}\,\underline{x}$

$$\underbrace{\underset{\sim}{Q}^{-1}\mathring{\underline{x}}}_{\mathring{\xi}} = \underset{\sim}{\Lambda}\,\underbrace{\underset{\sim}{Q}^{-1}\underline{x}}_{\xi}$$

$\mathring{\xi} = \underset{\sim}{\Lambda}\,\xi \leftarrow$ xi

diagonalized : $\mathring{\xi}_1 = \lambda_1 \xi_1$

$\mathring{\xi}_2 = \lambda_2 \xi_2$

$$\xi_1(t) = e^{\lambda_1(t-t_0)} \cdot \xi_1(t_0)$$

$$\xi_2(t) = e^{\lambda_2(t-t_0)} \cdot \xi_2(t_0)$$

$$\xi(t) = \underbrace{\begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 \\ 0 & e^{\lambda_2(t-t_0)} \end{bmatrix}}_{\exp(\underset{\sim}{\Lambda}(t-t_0))} \cdot \xi(t_0)$$

$$\underset{\sim}{Q}^{-1} \underline{x}(t) = \exp(\underset{\sim}{\Lambda}(t-t_0)) \underset{\sim}{Q}^{-1} \underline{x}(t_0)$$

$$\underline{x}(t) = \underbrace{\underset{\sim}{Q} \exp(\underset{\sim}{\Lambda}(t-t_0)) \underset{\sim}{Q}^{-1}}_{\exp(\underset{\sim}{A}(t-t_0))} \overbrace{\underset{\sim}{Q}^{-1}\underline{x}(t_0)}^{\xi(t_0)}$$

$$= [q_1, q_2] \cdot \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 \\ 0 & e^{\lambda_2(t-t_0)} \end{bmatrix} \begin{bmatrix} \xi_1(t_0) \\ \xi_2(t_0) \end{bmatrix}$$

$$= \xi_1(t_0) e^{\lambda_1(t-t_0)} \cdot q_1 + \xi_2(t_0) e^{\lambda_2(t-t_0)} \cdot q_2$$

$$= \exp(\underset{\sim}{A}(t-t_0)) \underline{x}(t_0)$$

$$\underset{\sim}{A} q_k = \lambda_k q_k$$

$$(\underset{\sim}{A} - \underset{\sim}{I}\lambda_k) q_k = 0$$

$$\Rightarrow \det(\underset{\sim}{A} - \lambda_k \underset{\sim}{I}) = 0$$

$$\begin{bmatrix} a_{11} - \lambda_k & a_{12} \\ a_{21} & a_{22} - \lambda_k \end{bmatrix} \cdot \begin{bmatrix} q_{k1} \\ q_{k2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

non-zero

$$(a_{11} - \lambda_k) q_{k1} + a_{12} q_{k2} = 0$$

$$q_k = \begin{bmatrix} -a_{12} \\ a_{11} - \lambda_k \end{bmatrix}$$

8.2.5 Jordan Normal form

$\lambda_1 = \lambda_2 \in \mathbb{R}$

Special case :

$\underset{\sim}{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\underset{\sim}{A}$: non-diagonal : $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$

$\underset{\sim}{A}\,\underset{\sim}{q} = \lambda \underset{\sim}{q} \rightarrow$ eigenvector $\underset{\sim}{q}$

only <u>one</u> $\underset{\sim}{q}$

$\underset{\sim}{J} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

Jordan matrix

$\underset{\sim}{A}\,\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}' \cdot \underset{\sim}{J}$ (prime)

$\underset{\sim}{Q}' = [\underset{\sim}{q}, \underset{\sim}{q}'_2]$

$\underset{\sim}{A}\,\underset{\sim}{q} = \lambda \underset{\sim}{q}$

$\underset{\sim}{A}\,\underset{\sim}{q}'_2 = \underset{\sim}{q} + \lambda \underset{\sim}{q}'_2$

$(\underset{\sim}{A} - \underset{\sim}{I}\lambda)\,\underset{\sim}{q}'_2 = \underset{\sim}{q}$

$\underset{\sim}{A} = \underset{\sim}{Q}'\underset{\sim}{J}\,\underset{\sim}{Q}'^{-1}$

$\dot{\underline{x}} = \underset{\sim}{A}\,\underline{x}$

$\dot{\underline{x}} = \underset{\sim}{Q}'\underset{\sim}{J}\,\underset{\sim}{Q}'^{-1}\,\underline{x}$

$$\underbrace{\underset{\sim}{Q}^{-1}\mathring{\underline{x}}}_{} = \underset{\sim}{J}\underbrace{\underset{\sim}{Q}^{-1}\underline{x}}_{}$$

$$\mathring{\xi}' = \xi'$$

$$\mathring{\xi}' = \underset{\sim}{J}\xi' \qquad \xi' = \begin{bmatrix} \xi_1' \\ \xi_2' \end{bmatrix}$$

$$\mathring{\xi}_1' = \lambda\xi_1' + \xi_2'$$

$$\mathring{\xi}_2' = \lambda\xi_2'$$

$$\rightarrow \xi_2'(t) = e^{\lambda(t-t_0)}\xi_2'(t_0)$$

$$\mathring{\xi}_1' = \lambda\xi_1' + e^{\lambda(t-t_0)}\xi_2'(t_0)$$

$$\xi_1'(t) = e^{\lambda(t-t_0)}\xi_1'(t_0) + (t-t_0)e^{\lambda(t-t_0)}\xi_2'(t_0)$$

$$\mathring{\xi}_1' = \underline{\lambda e^{\lambda(t-t_0)}\xi_1'(t_0)} + e^{\lambda(t-t_0)}\xi_2'(t_0) + \underline{\lambda(t-t_0)e^{\lambda(t-t_0)}\xi_2'(t_0)}$$

$$= \lambda\xi_1' + \underbrace{e^{\lambda(t-t_0)}\xi_2'(t_0)}_{\xi_2'(t)}$$

$$\underline{x}(t) = \left(\xi_1'(t_0) + (t-t_0)\xi_2'(t_0)\right)e^{\lambda(t-t_0)}\underline{q} + \xi_2'(t_0)e^{\lambda(t-t_0)}\underline{q}_2'$$

recap:

$$\mathring{\underline{x}} = \underset{\sim}{A}\underline{x} + \underline{b}\underline{v}$$

$$\underline{x}(t) = \exp(\underset{\sim}{A}(t-t_0))\underline{x}(t_0) + \int_{t_0}^{t}\exp(\underset{\sim}{A}(t-t'))\underline{b}\,v(t')\,dt'$$

$$v(t) = \underline{v}_0$$

$$\mathring{\underline{x}} = \underset{\sim}{A}\underline{x} + \underset{\sim}{B}\underline{v}$$

$$= \underset{\sim}{A}\underline{x} + \underset{\sim}{B}\underline{v}_0$$

equilibrium: $\mathring{\underline{x}} = 0$; $0 = \underset{\sim}{A}\underline{x}_\infty + \underset{\sim}{B}\underline{v}_0$

$$\underline{x}_\infty = -\underset{\sim}{A}^{-1}\underset{\sim}{B}\underline{v}_0$$

$$\underline{x} + \underline{x}' + \underline{x}_\infty$$

$$\mathring{\underline{x}}' = \underset{\sim}{A}\,\underline{x}'$$

$$\underline{x}'(t) = \exp(\underset{\sim}{A}(t-t_0))\,\underline{x}'(t_0)$$

$$\underline{x}(t) = \underline{x}'(t) + \underline{x}_\infty$$

$$\mathring{\underline{x}} = \underset{\sim}{A}\,\underline{x}$$

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0 \Rightarrow \lambda_{1/2}$$

I) $\lambda_1 \neq \lambda_2 \in \mathbb{R}$

$$\underset{\sim}{A} = \underset{\sim}{Q}\,\underset{\sim}{\Lambda}\,\underset{\sim}{Q}^{-1}$$

$$\underset{\sim}{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2)$$

$$\underline{x}(t) = \underset{\sim}{Q}\exp(\underset{\sim}{\Lambda}(t-t_0))\,\underset{\sim}{Q}^{-1}\,\underline{x}(t_0)$$

$$= \underset{\sim}{Q}\begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 \\ 0 & e^{\lambda_2(t-t_0)} \end{bmatrix}\underset{\sim}{Q}^{-1}\,\underline{x}(t_0)$$

$$= \xi_1(t_0)\,e^{\lambda_1(t-t_0)} \cdot q_1 + \xi_2(t_0)\,e^{\lambda_2(t-t_0)}\,q_2$$

II) $\lambda_1 = \lambda_2 \in \mathbb{R}$:

$$\underset{\sim}{A} = \underset{\sim}{Q}'\,\underset{\sim}{J}\,\underset{\sim}{Q}'^{-1} \qquad \underset{\sim}{J} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

8.2.6 Real-Valued Normal Form

$$\lambda_1 = \alpha + j\beta \in \mathbb{C}$$

$$\lambda_2 = \alpha - j\beta = \lambda_1^* \in \mathbb{C} \qquad j = \sqrt{-1}$$

$$\mathring{\underline{x}} = \underset{\sim}{A}\,\underline{x}$$

$$\downarrow$$

$$\mathring{\underline{x}} = \underset{\sim}{Q}_{real}\,\underset{\sim}{\Lambda}_{real}\,\underset{\sim}{Q}^{-1}_{real}\,\underline{x}$$

$$\mathring{\xi}_{real} = \underset{\sim}{\Lambda}_{real}\,\xi_{real}$$

$$\xi_{real} = \underset{\sim}{Q}^{-1}_{real}\,\underline{x}$$

$$(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\, q_1 = \underline{0}$$

$$q_1 = q_r + j q_i$$

$$\underset{\sim}{Q}_{real} = [q_r, -q_i]$$

$$\underset{\sim}{\Lambda}_{real} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\text{EVD}: \underset{\sim}{A} = \underset{\sim}{Q}\underset{\sim}{\Lambda}\underset{\sim}{Q}^{-1}$$

$$\xi(t) = \underset{\sim}{Q}^{-1} \underline{x}(t)$$

$$\xi_{real} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \xi$$

$$= \begin{bmatrix} 2\,\mathrm{Re}\{\xi_1\} \\ 2\,\mathrm{Im}\{\xi_2\} \end{bmatrix}$$

$$\mathring{\xi} = \underset{\sim}{\Lambda}\, \xi$$

$$\xi_1(t) = e^{\lambda_1 (t-t_0)} \xi_1(t_0)$$

$$= e^{\alpha(t-t_0)} e^{j\beta(t-t_0)} \xi_1(t_0)$$

$$\xi_1(t_0) = k e^{j\theta} \leftarrow \text{angle of complex value}$$

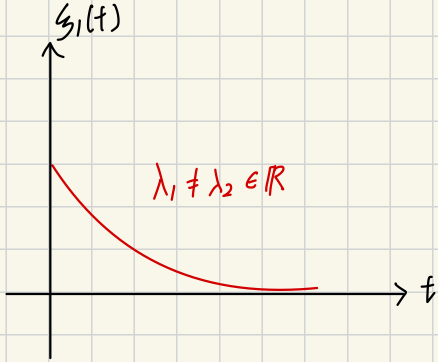
$$\xi_1(t) = e^{\alpha(t-t_0)} e^{j(\beta(t-t_0)+\theta)} \cdot k$$

$$\text{Euler's rule}: \quad e^{j\varphi} = \cos\varphi + j\sin\varphi$$

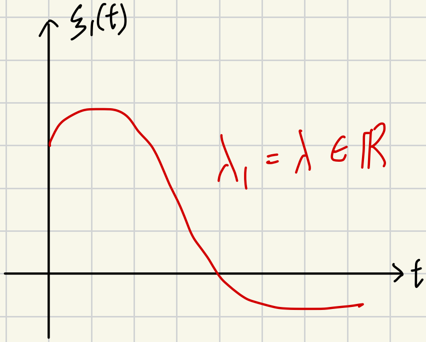
$$\xi_1(t) = e^{\alpha(t-t_0)} k \left[\cos(\beta(t-t_0)+\theta\right] + j\sin(\beta(t-t_0)+\theta) \,]$$

$$\xi_{real}(t) = \begin{bmatrix} 2\,\mathrm{Re}\{\xi_1\} \\ 2\,\mathrm{Im}\{\xi_1\} \end{bmatrix}$$

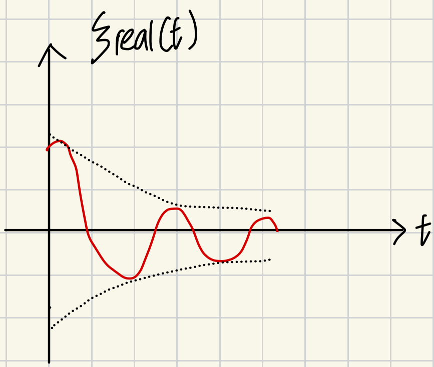
$$\xi_{real}(t) = 2k e^{\alpha(t-t_0)} \cdot \begin{bmatrix} \cos(\beta(t-t_0)+\theta) \\ \sin(\beta(t-t_0)+\theta) \end{bmatrix}$$



$\xi_1(t)$
$\lambda_1 \neq \lambda_2 \in \mathbb{R}$
t



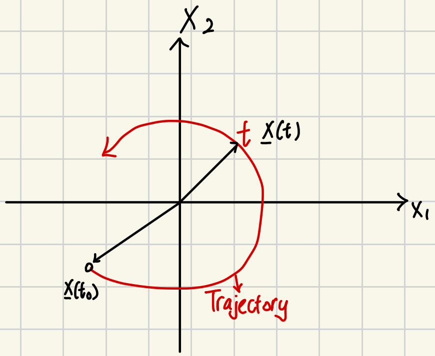
$\xi_1(t)$
$\lambda_1 = \lambda \in \mathbb{R}$
t



$\xi_{real}(t)$
t

8.3 Phase Portrait (qualitative properties)

8.3.1 Trajectory



8.3.2 Focii Case I & III

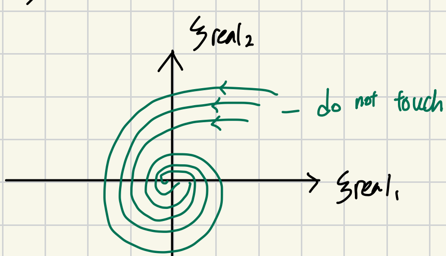
$$\lambda_1 = \alpha + j\beta$$

$$\alpha \neq 0,\ \beta > 0$$

for $t_0 = 0$;

$$\xi_{real_1}(t) = ke^{\alpha t}\cos(\beta t + \theta)$$

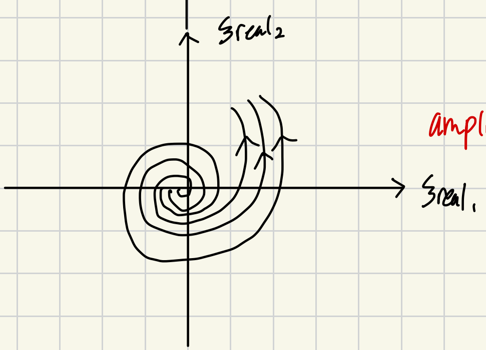
$$\xi_{real_2}(t) = ke^{\alpha t}\sin(\beta t + \theta)$$



stable focus

$\alpha < 0$

amplitude shrinks



amplitude expands

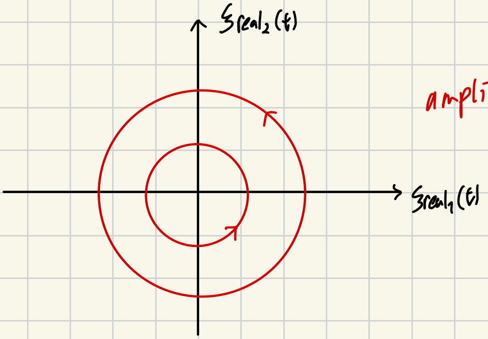
8.3.3 Center

$\lambda_1 = j\beta$

$\alpha = 0 \;,\; \beta > 0$

$\xi_{real_1}(t) = k\cos(\beta t + \theta)$

$\xi_{real_2}(t) = k\sin(\beta t + \theta)$



amplitude constant , centers are unstable

8.3.4 Nodes

$\lambda_1 \neq \lambda_2 \in \mathbb{R}$

$\lambda_1 \cdot \lambda_2 > 0$

normal forms :

$\xi_1(t) = e^{\lambda_1 t}\xi_1(0)$

$\xi_2(t) = e^{\lambda_2 t}\xi_2(0)$

$\frac{\xi_1(t)}{\xi_0(t)} = e^{\lambda_1 t}$

$\ln\frac{\xi_1(t)}{\xi_0(t)} = \lambda_1 t$

$\Rightarrow t = \frac{1}{\lambda_1}\ln\frac{\xi_1}{\xi_{01}}$

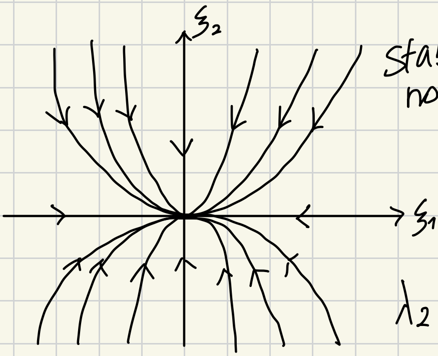
$$\xi_2 = e^{\lambda_2 \frac{1}{\lambda_1} \ln \frac{\xi_1}{\xi_{01}}} \cdot \xi_{02}$$

$$\alpha \ln b = \ln b^{\alpha}$$

$$\xi_2 = e^{\ln\left(\frac{\xi_1}{\xi_{01}}\right)^{\frac{\lambda_2}{\lambda_1}}} \xi_{02}$$

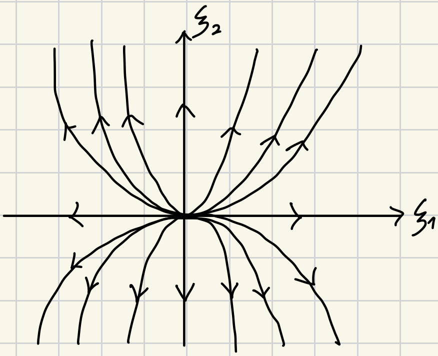
$$= \left(\frac{\xi_1}{\xi_{01}}\right)^{\frac{\lambda_2}{\lambda_1}} \xi_{02}$$

parabola



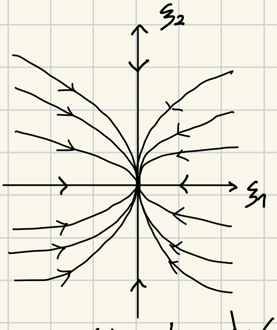
stable node

$\lambda_2 < \lambda_1 < 0$



unstable node

$\lambda_2 > \lambda_1 > 0$



stable node $\lambda_1 < \lambda_2 < 0$

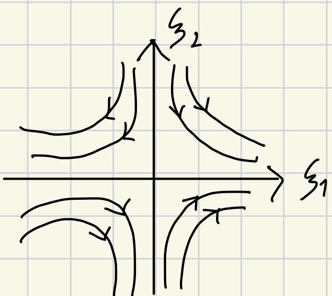
8.3.5 saddle point

$\lambda_1 \neq \lambda_2 \in \mathbb{R}$

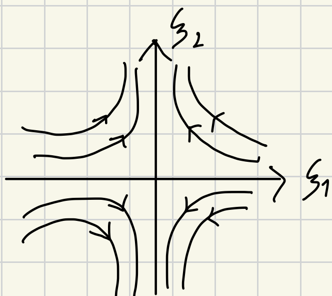
$\lambda_1 \cdot \lambda_2 < 0$

$$\xi_2 = \left(\frac{\xi_1}{\xi_{01}}\right)^{\frac{\lambda_2}{\lambda_1}} \xi_{02} \qquad \rightarrow < 0$$

$\rightarrow$ hyperbola



$\lambda_2 < 0 < \lambda_1$



$\lambda_1 < 0 < \lambda_2$

saddle points are unstable

Stability : $Re\{\lambda_1\} < 0$

$Re\{\lambda_2\} < 0$

$\downarrow$

Real

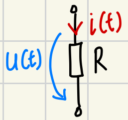
9 Complex Phasor Analysis

Linear circuits
+
Sinusoidal excitations

9.1 Sinusoidal input and steady state

$$a(t) = \underbrace{A_m}_{\substack{\text{amplitude} \\ A_m > 0}} \cos(\underbrace{\omega}_{\substack{\text{angular} \\ \text{frequency}}} t + \underbrace{\alpha}_{\text{angle}})$$

$\omega = 2\pi f$



$$u(t) = R\,i(t)$$

$$i(t) = I_m \cos(\omega t + \beta)$$

$$u(t) = \underbrace{R I_m}_{U_m} \cos(\omega t + \beta)$$

$$\dot{a}(t) = \frac{d}{dt} A_m \cos(\omega t + \alpha)$$

$$= A_m(-\sin(\omega t + \alpha) \cdot \omega)$$

$$= -\omega A_m \sin(\omega t + \alpha)$$

$$= -\omega A_m \cos(\omega t + \alpha - \frac{\pi}{2})$$

$$= \omega A_m \cos(\omega t + \alpha + \frac{\pi}{2})$$

Series Connection :

$$U_{tot}(t) = U_0(t) + U_2(t)$$

$$U_1(t) = U_{1m}\cos(\omega t)$$

$$U_2(t) = U_{2m}\cos(\omega t + \delta)$$

$$U_{tot}(t) = U_{1m}\cos(\omega t) + U_{2m}\cos(\omega t + \delta)$$

$$= U_{totm}\cos(\omega t + \varphi)$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$U_{1m}\cos(\omega t) + U_{2m}\cos(\omega t)\cos(\delta) - U_{2m}\sin(\omega t)\sin(\delta)$$

$$= U_{totm}\cos(\omega t)\cos\varphi - U_{totm}\sin(\omega t)\sin(\varphi)$$

$$U_{1m}\cancel{\cos(\omega t)} + U_{2m}\cancel{\cos(\omega t)}\cos\delta = U_{totm}\cancel{\cos(\omega t)}\cos\varphi$$

$$U_{1m} + U_{2m}\cos\delta = U_{totm}\cos\varphi$$

$$\cancel{-}U_{2m}\cancel{\sin(\omega t)}\sin\delta = \cancel{-}U_{totm}\cancel{\sin(\omega t)}\sin\varphi$$

$$U_{2m}\sin\delta = U_{totm}\sin\varphi$$

$$\frac{U_{totm}\sin\varphi}{U_{totm}\cos\varphi} = \tan\varphi = \frac{U_{2m}\sin\delta}{U_{1m} + U_{2m}\cos\delta}$$

$$\sin^2\varphi + \cos^2\varphi = 1$$

$$U_{totm}^2(\cos^2\varphi + \sin^2\varphi) = U_{totm}^2$$

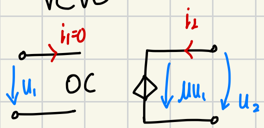
$$= (U_{1m} + U_{2m}\cos\delta)^2 + (U_{2m}\sin\delta)^2$$

$$= U_{1m}^2 + U_{2m}^2 + 2U_{1m}U_{2m}\cos\delta$$

for one frequency

if ≥ angular frequency must find one by one and add together

VCVS



$i_1 = 0, \quad u_2 = \mu u_1$

$$\underset{\sim}{H}' = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix}$$

9.2 Complex Phasors

$$a(t) = A_m \cos(\omega t + \alpha)$$

define phasor :

$$A = A_m e^{j\alpha}$$

then,

$$a(t) = Re\{A e^{j\omega t}\}$$

$$Re\{Ae^{j\omega t}\} = Re\{A_m e^{j\alpha} \cdot e^{j\omega t}\}$$

$$= Re\{A_m e^{j(\omega t + \alpha)}\}$$

$$= A_m \cos(\omega t + \alpha)$$

$$\boxed{\text{Euler's rule: } e^{j\varphi} = \cos\varphi + j\sin\varphi}$$

3 Lemmas

Lemma 1 : Uniqueness

$$\forall t : a(t) = b(t) \iff A = B$$

Lemma 2: Linearity

$$y(t) = \sum_{l=1}^{L} a_l \cdot x_l(t) \iff Y = \sum_{l=1}^{L} a_l X_l$$

Lemma 3: Differentiation

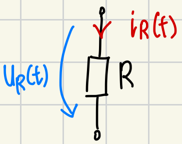
$$b(t) = \mathring{a}(t) = \frac{da(t)}{dt} \iff b(t) = \mathring{a}(t) = \frac{d}{dt} Re\{Ae^{j\omega t}\}$$

$$= Re\{A \cdot \frac{d}{dt} e^{j\omega t}\}$$

$$= Re\{j\omega A \cdot e^{j\omega t}\}$$

$$B = j\omega A$$

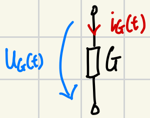
$$a_3 \overset{\circ\circ\circ}{x}(t) + a_2 \overset{\circ\circ}{x}(t) + a_1 \mathring{x}(t) + a_0 x(t) = w(t) \iff a_3 (j\omega)^3 X + a_2 (j\omega)^2 X + a_1 (j\omega) X + a_0 X = W$$



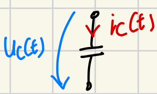
$$U_R(t) = R i_R(t) \overset{L_1, L_2}{\iff} U_R = R I_R$$

$$U_R(t) = Re\{U_R e^{j\omega t}\}$$

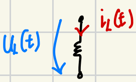
$$i_R(t) = Re\{I_R e^{j\omega t}\}$$



$$i_G(t) = G U_G(t) \overset{L_1, L_2}{\iff} I_G = G U_G$$

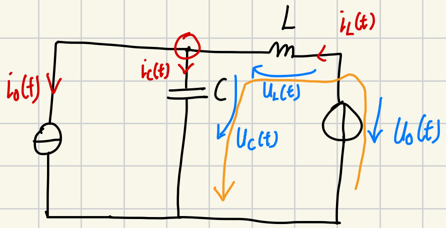


$$i_C = C \cdot \mathring{u}_C(t) \overset{L_1, L_2, L_3}{\iff} I_C = j\omega C U_C$$



$$U_L(t) = L \cdot \mathring{i}_L(t) \overset{L_1, L_2, L_3}{\iff} U_L = j\omega L I_L$$

9.2.1 Reactive Simple Circuit



KCL: $i_0(t) + i_C(t) - i_L(t) = 0$

KVL: $-U_0(t) + U_L(t) + U_C(t) = 0$

$$i_C(t) = i_L(t) - i_0(t)$$

$$U_L(t) = U_0(t) - U_C(t)$$

$$C\dot{U}(t) = i_L(t) - i_0(t) \quad \overset{L_1, L_2, L_3}{\Longleftrightarrow} \quad j\omega C U_C = I_L - I_0$$

$$L\dot{i}(t) = U_0(t) - U_C(t) \qquad\qquad j\omega L I_L = U_0 - U_C$$

$\times j\omega C$

$$j\omega C U_0 - j\omega C U_C = j\omega^2 LC I_L$$

+

$$j\omega C U_0 = I_L - I_0 + (j\omega)^2 LC I_L$$

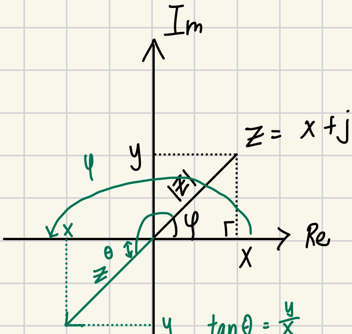
$$j\omega C U_0 + I_0 = (1 + (j\omega)^2 LC) I_L$$

$$I_L = \frac{I_0 + j\omega C U_0}{1 + (j\omega)^2 LC}$$

$$j\omega L \cdot \frac{I_0 + j\omega C U_0}{1 + (j\omega)^2 LC} = U_0 - U_C$$

$$U_C = U_0 - \frac{j\omega L I_0 + (j\omega)^2 C U_0}{1 + (j\omega)^2 LC}$$

$$= \frac{U_0 - j\omega L I_0}{1 + (j\omega)^2 LC}$$



$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\frac{y}{x}$

$\varphi = \tan^{-1}\frac{y}{x} + \pi$

Pythagoras:

$$|z|^2 = x^2 + y^2$$

$$= (x + jy)(x - jy)$$

$$= z \cdot z^*$$

$$\tan\varphi = \frac{y}{x}$$

$$= \frac{Im\{z\}}{Re\{z\}}$$

$$U_{c,m} = |U_c| = \sqrt{U_c \cdot U_c^*}$$

$$= \frac{|U_0 - j\omega L I_0|}{|1 - \omega^2 LC|}$$

$$I_{L,m} = \frac{|I_0 - j\omega C U_0|}{|1 - \omega^2 LC|}$$

$$\beta_u = \tan^{-1}\left(\frac{Im\{U_c\}}{Re\{U_c\}}\right)$$

$$U_0 , I_0 \in \mathbb{R}$$

$$\beta_u = \tan^{-1}\left(\frac{-j\omega L I_0}{U_0}\right)$$

$$\beta_i = \tan^{-1}\left(\frac{\omega C U_0}{I_0}\right)$$

$$U_c = U_{c,m} e^{j\beta_u}$$

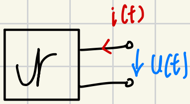
$$I_L = I_{L,m} e^{j\beta_i}$$

$$u_C(t) = Re\{U_C e^{j\omega t}\}$$

$$= U_{C,m} \cos(\omega t + \beta_u)$$

$$i_L(t) = I_{L,m} \cos(\omega t + \beta_i)$$

9.4 Network Functions



9.4.1 Two terminal function

$$u(t) = Re\{ U e^{j\omega t} \}$$

$$i(t) = Re\{ I e^{j\omega t} \}$$

impedance :

$$Z(j\omega) = \frac{U(j\omega)}{I(j\omega)}$$

admittance :

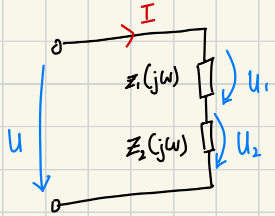
$$Y(j\omega) = \frac{1}{Z(j\omega)} = \frac{I(j\omega)}{U(j\omega)}$$

$$Z_R = R, \quad Y_R = \frac{1}{R}$$

$$Z_G = \frac{1}{G}, \quad Y_G = G$$

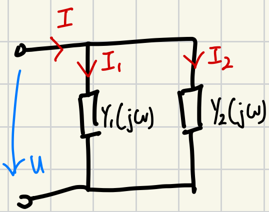
$$Z_C = \frac{1}{j\omega C}, \quad Y_C = j\omega C$$

$$Z_L = j\omega L, \quad Y_L = \frac{1}{j\omega L}$$



$$\frac{U_1(j\omega)}{U(j\omega)} = \frac{Z_1(j\omega)}{Z_1(j\omega)+Z_2(j\omega)}$$

$$\frac{U_2(j\omega)}{U_1(j\omega)} = \frac{Z_2(j\omega)}{Z_1(j\omega)}$$



$$\frac{I_1(j\omega)}{I(j\omega)} = \frac{Y_1(j\omega)}{Y_1(j\omega)+Y_2(j\omega)}$$

$$\frac{I_2(j\omega)}{I_1(j\omega)} = \frac{Y_2(j\omega)}{Y_1(j\omega)}$$

9.4.2 Transfer functions

relate quantities of different ports

e.g: $H(j\omega) = \dfrac{U_{out}(j\omega)}{U_{in}(j\omega)}$

$$U_{out} = H(j\omega) \cdot U_{in}(j\omega)$$

$$U_{out}(t) = \mathrm{Re}\{H(j\omega)\,U_{in}(j\omega)\,e^{j\omega t}\}$$

9.5 Energy & Power

$$P(t) = U(t) \cdot i(t)$$

$$E = \int_0^T p(t)\,dt$$

$T = \frac{2\pi}{\omega}$

average power

$$P_W = \frac{E}{T} = \frac{1}{T}\int_0^T p(t)\,dt$$

Complex power

$P = \frac{1}{2} \cdot U \cdot I^* \rightarrow$ complex Conjugate

$= P_W + j P_B$

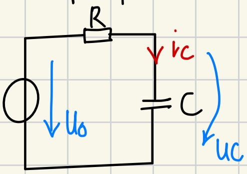
$$P_W = \mathrm{Re}\{P\}$$

$$P_B = \mathrm{Im}\{P\} \rightarrow \text{blind power}$$

apparent power:

$$S = |P|$$

Wrap up first order circuits



$$KVL: -U_0 + R i_C + U_C = 0$$

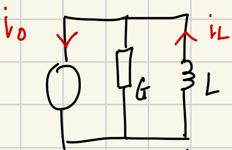
$$R i_C = -U_C + U_0$$

$$i_C = C \dot{u}_C$$

$$RC\dot{u}_C = -U_C + U_0$$

$$\dot{u}_C = -\frac{1}{RC} U_C + \frac{1}{RC} U_0$$

$$U_C(t) = U_C(t_0) e^{-\frac{t-t_0}{RC}} + \int_{t_0}^{t} \frac{1}{RC} e^{-\frac{t-t'}{RC}} U_0(t')\,dt'$$



$$\dot{i}_L = -\frac{1}{GL} i_L + \frac{1}{GL} i_0$$

$$i_L(t) = i_L(t_0) e^{-\frac{t-t_0}{GL}} + \int_{t_0}^{t} \frac{1}{GL} e^{-\frac{t-t'}{GL}} i_0(t')\,dt'$$

for $U_0(t) = U_\infty$ or $i_0(t) = i_\infty$

$$U_C(t) = U_\infty + (U_C(t_0) - U_\infty) e^{-\frac{t-t_0}{RC}}$$

$$i_L(t) = i_\infty + (i_L(t_0) - i_\infty) e^{-\frac{t-t_0}{GL}}$$

if RC or GL is ⊕, circuit is stable

Second order circuits

$$\underline{x} = \begin{bmatrix} u_{C1} \\ u_{C2} \end{bmatrix}, \begin{bmatrix} i_{L1} \\ i_{L2} \end{bmatrix}, \begin{bmatrix} u_C \\ i_L \end{bmatrix}$$

$$\mathring{\underline{x}} = \underset{\sim}{A}\,\underline{x} + \underline{b}\,v$$

find this

input vector

$$i_C = C\dot{u}_C$$
$$u_L = L\dot{i}_L$$

depending on which you need, find these

eigenvalues : $\det(\underset{\sim}{A} - \lambda \underset{\sim}{1}) = 0$

I) $\lambda_1 \neq \lambda_2 \in \mathbb{R}$

II) $\lambda_1 = \lambda_2 \in \mathbb{R} \rightarrow$ Jordan Normal form (won't be exam)

III) $\lambda_1 = \lambda_2^* \in \mathbb{C}$

$$\mathring{\xi}_1 = \lambda_1 \xi_1$$
$$\mathring{\xi}_2 = \lambda_2 \xi_2$$