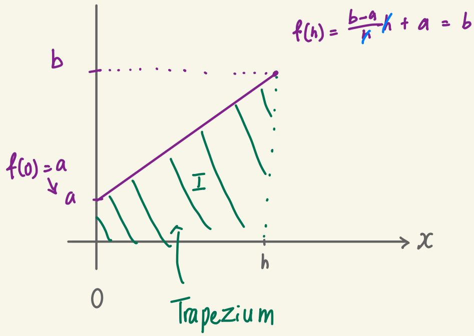
1. Sketch the area represented by the integral below and identify the shape. Then, set up the Riemann sum using the right endpoint and evaluate the Riemann integral to derive the area formula. Repeat using the left endpoint.

$$I = \int_0^h \overbrace{\frac{b-a}{h}x + a}^{f(x)}\,dx$$

ANS: Trapezium. $\displaystyle\sum_{i=1}^{n}\left[\frac{(b-a)i}{n}+a\right]\frac{h}{n}$. **Area** $= \dfrac{(a+b)h}{2}$



$$\Delta x = \frac{h-0}{n} = \frac{h}{n}$$

$$x_0 = 0, \quad x_i = 0 + i\Delta x = 0 + \frac{ih}{n} = \frac{ih}{n}$$

$$\Rightarrow \sum_1^n f(x_i)\,\Delta x = \sum_1^n \left[\frac{b-a}{\cancel{h}}\left(\frac{i\cancel{h}}{n}\right) + a\right]\left(\frac{h}{n}\right)$$

$$= h\left[\frac{b-a}{n^2}\underline{\sum_1^n i} + \frac{a}{n}\underline{\sum_1^n 1}\right]$$

$$= h\left[\frac{b-a}{n^2}\cdot\underline{\frac{n(n+1)}{2}} + \underline{\frac{a}{n}n}\right]$$

$$= h\left[\frac{b-a}{2} + \frac{b-a}{2n} + a\right]$$

$$I = \lim_{n\to\infty}\sum_1^n f(x_i)\,\Delta x = \lim_{n\to\infty}\left\{h\left[\frac{b-a}{2} + \frac{b-a}{2n} + a\right]\right\}$$

$$= h\left[\frac{b-a}{2} + a\right]$$

$$= \underbrace{h\left[\frac{a+b}{2}\right]}_{\text{area formula of trapezium}}$$

2. Evaluate the definite integrals below by setting up the Riemann sum using the right endpoint and evaluating the Riemann integral.

a) $\int_0^1 x^3\,dx$

b) $\int_{-2}^2 -3x\,dx$

c) $\int_{-3}^0 2x^2\,dx$

d) $\int_{-1}^3 x^3\,dx$

ANS: **a)** ¼. **b)** 0. **c)** 18. **d)** 20

a) $\int_0^1 x^3\,dx$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = \underbrace{x_0}_{0} + i\,\Delta x = 0 + \frac{i}{n} = \frac{i}{n}$$

$$\sum_1^n \underbrace{f(x_i)}_{x_i^3}\Delta x = \sum_1^n \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n} = \sum_1^n \frac{i^3}{n^4}$$

$$= \frac{1}{n^4}\sum_1^n i^3$$

$$= \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

$$(n^2+n)(n^2+n)$$

$$n^4+2n^3+n^2$$

$$= \frac{1}{n^4}\left[\frac{n^4+2n^3+n^2}{4}\right]$$

$$= \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}$$

$$\lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right) = \int_0^1 x^3\,dx$$

$$= \frac{1}{4}$$

b) $\int_{-2}^{2} -3x\,dx$

$$\Delta x = \frac{2-(-2)}{n} = \frac{4}{n} \qquad x_i = -2 + \frac{4i}{n}$$

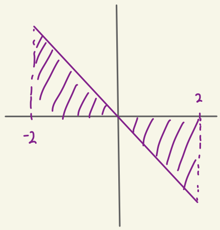
$$-3\sum_{1}^{n} f(x_i)\,\Delta x = -3\sum_{1}^{n}\left(-2+\frac{4i}{n}\right)\left(\frac{4}{n}\right)$$

$$= -3\sum_{1}^{n}\left(\frac{-8}{n}+\frac{16i}{n^2}\right)$$

$$= \frac{24}{n}\sum_{1}^{n} 1 - \frac{48}{n^2}\sum_{1}^{n} i$$

$$= \frac{24}{n}(n) - \frac{48}{n^2}\left[\frac{n(n+1)}{2}\right] = 24 - 24 - \frac{24}{n}$$

$$\lim_{n\to\infty}\left(24-24-\frac{24}{n}\right) = 0$$



Odd function

$\therefore \int_{-2}^{2} -3x\,dx = 0$

c) $\int_{-3}^{0} 2x^2\,dx$

$$\Delta x = \frac{0-(-3)}{n} = \frac{3}{n} \qquad x_i = 0 + i\Delta x = \frac{3i}{n}$$

$$2\sum_{1}^{n} f(x_i)\,\Delta x = 2\sum_{1}^{n}\left(\frac{3i}{n}\right)^2\frac{3}{n}$$

$$= 2\sum_{1}^{n}\left(\frac{27i^2}{n^3}\right)$$

$$= \frac{54}{n^3}\sum_{1}^{n} i^2$$

$$= \frac{54}{n^3}\left[\frac{\overbrace{n(n+1)(2n+1)}^{2n^3+3n^2+n}}{6}\right]$$

$$= 18 + \frac{27}{n} + \frac{9}{n^2}$$

$$\lim_{n\to\infty}\left(18+\frac{27}{n}+\frac{9}{n^2}\right) = \int_{-3}^{0} 2x^2\,dx$$

$$= 18$$

d) $\int_{-1}^{3} x^3 \, dx$

$$\Delta x = \frac{3-(-1)}{n} = \frac{4}{n} \qquad x_i = x_0 + i\Delta x = -1 + \frac{4i}{n}$$

$$\sum_{1}^{n} \left(-1+\frac{4i}{n}\right)^3 \frac{4}{n} = \sum_{1}^{n}\left(\frac{64i^3}{n^3} - \frac{48i^2}{n^2} + \frac{12i}{n} - 1\right)\frac{4}{n}$$

$$= \sum_{1}^{n}\left(\frac{256i^3}{n^4} - \frac{192i^2}{n^3} + \frac{48i}{n^2} - \frac{4}{n}\right)$$

$$\left(-1+\frac{4i}{n}\right)\left(-1+\frac{4i}{n}\right)\left(-1+\frac{4i}{n}\right)$$

$$\left(1-\frac{8i}{n}+\frac{16i^2}{n^2}\right)$$

$$\left(-1+\frac{8i}{n}-\frac{16i^2}{n^2}+\frac{4i}{n}-\frac{32i^2}{n^2}+\frac{64i^3}{n^3}\right.$$

$$= \frac{256}{n^4}\sum_{1}^{n} i^3 - \frac{192}{n^3}\sum_{1}^{n} i^2 + \frac{48}{n^2}\sum_{1}^{n} i - \frac{4}{n}\sum_{1}^{n} 1$$

$$= \frac{256}{n^4}\overbrace{\left[\frac{n(n+1)}{2 \;\, 4}\right]^2}^{n^4+O(n^3)} - \frac{192}{n^3}\left[\overbrace{\frac{n(n+1)(2n+1)}{6}}^{2n^3+O(n^2)}\right] + \frac{48}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{4}{n}(n)$$

$$= 64 + \frac{O(n^3)}{n^4} - 64 - \frac{O(n^2)}{n^3} + 24 + \frac{24}{n} - 4$$

$$\lim_{n\to\infty}\left(64 + \frac{O(n^3)}{n^4} - 64 - \frac{O(n^2)}{n^3} + 24 - \frac{24}{n} - 4\right) = 64 - 64 + 24 - 4$$

$$= \underline{\underline{20}}$$

3. (https://openstax.org/books/calculus-volume-1/pages/5-2-the-definite-integral)
Without evaluating the Riemann integral, determine each area represented by each definite integral below using area formulas.

a) $\displaystyle\int_0^3 3 - x\,dx$

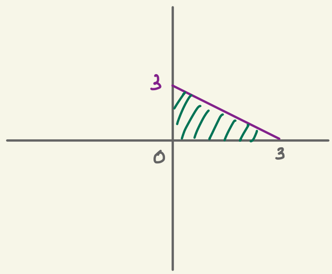
b) $\displaystyle\int_{-2}^6 3 - |x-3|\,dx$

c) $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\,dx$

d) $\displaystyle\int_6^{12} \sqrt{36-(x-6)^2}\,dx$

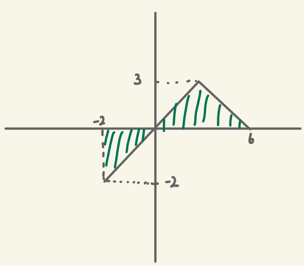
ANS: **a)** 9/2. **b)** 7. **c)** 2π. **d)** 9π.

a) $\displaystyle\int_0^3 3 - x\,dx$



$I = \frac{1}{2} \times 3 \times 3 = \frac{9}{2}$

b) $\displaystyle\int_{-2}^6 3 - |x-3|\,dx$



$I = \frac{1}{2}(6)(3) - \frac{1}{2}(2)(2)$

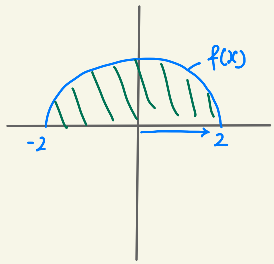
$= 7$

$f(x)$

c) $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\, dx$

from $x^2 + y^2 = 4$

$\Rightarrow y = \pm\sqrt{4 - x^2}$



$I = \frac{1}{2}\pi(2^2)$

$= 2\pi$

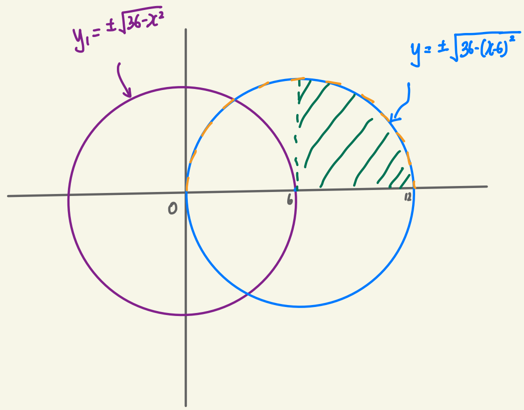
$f(x)$

d) $\displaystyle\int_{6}^{12} \sqrt{36 - (x-6)^2}\, dx$

$y_1 = \pm\sqrt{36 - x^2}$

$y = \pm\sqrt{36 - (x-6)^2}$

shift right by 6



$I = \frac{1}{4}\pi(6^2) = 9\pi$

4. Express each limit below as a (right) Riemann integral.

a) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^5}{n^6}$

b) $\lim_{n\to\infty}\frac{5}{n}\sum_{i=1}^{n}\frac{1}{1-\left(2+\frac{5i}{n}\right)}$

c) $\lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\ln\left(1+\frac{2i}{n}\right)$

d) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3n}{n^2+i^2}$

ANS: **a)** $\int_0^1 3x^5\,dx$. **b)** $\int_2^7 \frac{1}{1-x}\,dx$. **c)** $\int_1^3 2\ln x\,dx$. **d)** $\int_0^1 \frac{3}{1+x^2}\,dx$.

a) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^5}{n^6}$

$$= \lim_{n\to\infty}\sum_{1}^{n}\left(\frac{3i^5}{n^5}\right)\frac{1}{n}$$

Let $\Delta x = \frac{1}{n}$ & $x_0 = a = 0$

$b = 1$

$$\Rightarrow x_i = x_0 + i\Delta x = 0 + \frac{i}{n}$$
$$= \frac{i}{n}$$

so $f(x_i) = 3\left(\frac{i}{n}\right)^5 = 3x_i^5$

$$\therefore \lim_{n\to\infty}\sum_{1}^{n}\left(\frac{3i^5}{n^5}\right)\frac{1}{n} = \int_0^1 3x^5\,dx$$

b) $\lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \frac{1}{1-(2+\frac{5i}{n})}$

$2+\frac{5i}{n}$

Let $\Delta x = \frac{5}{n}$, $x_0 = a = 2$, $b = 7$

$x_i = x_0 + i\Delta x$

$= 2 + i(\frac{5}{n})$

$= 2 + \frac{5i}{n}$

$f(x_i) = \frac{1}{1-(2+\frac{5i}{n})} = \frac{1}{1-x_i}$

$\Rightarrow \int_2^7 \frac{1}{1-x}\,dx$

c) $\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \ln\left(1+\frac{2i}{n}\right) = 2\,\frac{2}{n}\sum_{1}^{n} \ln\left(1+\frac{2i}{n}\right)$

$1+\frac{2i}{n}$

Let $\Delta x = \frac{2}{n}$, $x_0 = a = 1$, $b = 3$

$x_i = x_0 + i\Delta x$

$= 1 + \frac{2i}{n}$

$f(x_i) = 2\ln\left(1+\frac{2i}{n}\right) = 2\ln(x)$

$\Rightarrow \int_1^3 2\ln x\,dx$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$

d) $\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{3n}{n^2+i^2}}$

$= \dfrac{3n}{n^2\left(1+\frac{i^2}{n^2}\right)} = \dfrac{1}{n} \cdot \underbrace{\dfrac{3}{1+\left(\frac{i}{n}\right)^2}}_{f\left(\frac{i}{n}\right)}$

Let $\Delta x = \dfrac{1}{n}$, $a = x_0 = 0$, $b = 1$

$\Rightarrow x_i = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$, $f(x_i) = \dfrac{3}{1+x_i^2}$

$= \int_0^1 f(x)\,dx = \int_0^1 \dfrac{3}{1+x^2}\,dx$

(not unique)

OR:

$\dfrac{3}{n} \cdot \dfrac{1}{1+\left(\frac{i}{n}\right)^2}$

Let $\Delta x = \dfrac{3}{n}$, $a = x_0 = 0 \to b = 3$

$\Rightarrow x_i = 0 + \dfrac{3i}{n} = \dfrac{3i}{n} \to f(x_i) = \dfrac{1}{1+\frac{1}{9}\left(\frac{3i}{n}\right)^2} = \dfrac{1}{1+\frac{x_i^2}{9}}$

$\Rightarrow I = \int_0^3 \dfrac{1}{1+x^2/9}\,dx$

5. Prove the following integrals for even and odd functions over a symmetric interval. Hint: Use a change of variables, y = -x, where appropriate.

$$\int_{-a}^{a} f_e(x)dx = 2\int_{0}^{a} f_e(x)dx,$$
$$\int_{-a}^{a} f_o(x)dx = 0$$

$$LHS = \int_{-a}^{0} f_e(x)\,dx + \int_{0}^{a} f_e(x)\,dx$$

since $f_e(x)$ is even, $f(-x) = f(x)$

$$\Rightarrow \underbrace{\int_{-a}^{0} f_e(-x)\,dx}_{I_1} + \int_{0}^{a} f_e(x)\,dx$$

Let $y = -x \Rightarrow \frac{dy}{dx} = -1$

At $x = -a$, $y = a$ | $dy = -dx$

$x = 0$, $y = 0$ | $dx = -dy$

$$I_1 = \int_{a}^{0} f_e(y)(-dy) = \int_{0}^{a} f_e(y)(dy)$$

$$\Rightarrow LHS = \int_{0}^{a} f_e(y)dy + \int_{0}^{a} f_e(x)dx = 2\int_{0}^{a} f_e(x)dx$$

(since the 2 integrals are the same)

$$\int_{-a}^{a} f_o(x)dx = 0$$

$$LHS = \int_{-a}^{0} f_0(x)\,dx + \int_{0}^{a} f_0(x)\,dx$$

since $f_0(x)$ is odd, $f_0(-x) = -f_0(x)$

$\Rightarrow f_0(x) = -f_0(-x)$

$$\Rightarrow \overbrace{\int_{-a}^{0} -f_0(-x)\,dx}^{I_1} + \int_{0}^{a} f_0(x)\,dx$$

Let $\underbrace{y = -x}$ $\rightarrow \frac{dy}{dx} = -1$

At $x = -a$, $y = a$ | $dy = -dx \rightarrow dx = -dy$

$x = 0$, $y = 0$

$$I_1 = -\int_{a}^{0} f_0(y)\,(-dy)$$

$$= \int_{a}^{0} f_0(y)\,(dy)$$

$$= -\int_{0}^{a} f_0(y)\,(dy)$$

$$\Rightarrow LHS = -\int_{0}^{a} f_0(y)\,(dy) + \int_{0}^{a} f_0(x)\,dx$$

$= 0$ (since the 2 integrals are the same)

a) $$F(x) = \int_0^x \underbrace{t^3 - 2}_{f(t)}\,dt$$

$$\underbrace{F(t)\Big|_0^x}= F(x) - F(0)$$

$$F'(x) = \frac{d}{dx}\left[F(x) - F(0)\right]$$

$$= f(x)$$

$$= x^3 - 2$$

b) $$f(x) = \int_x^1 \underbrace{e^{2\sin t}}_{\bar{f}(x)}\,dx$$

$$f(x)\Big|_x^1 = F(1) - F(x)$$

$$f'(x) = \frac{d}{dx}\left[F(1) - F(x)\right]$$

$$= -\bar{f}(x)$$

$$= -e^{2\sin x}$$

c) $$g(t) = \int_{-2}^{\ln t} \underbrace{e^{3x}}_{\bar{g}(t)}\,dx$$

$$g(t)\Big|_{-2}^{\ln t} = g(\ln t) - g(-2)$$

$$g'(t) = \frac{d}{dt}\left[g(\ln t) - g(-2)\right]$$

$$= \bar{g}(\ln t) \cdot \frac{1}{t}$$

$$= e^{3\ln t} \cdot \frac{1}{t}$$

$$= \frac{e^{\ln t^3}}{t}$$

$$= \frac{t^3}{t} = t^2$$

d) $$h(y) = \int_y^{2y} \underbrace{\sqrt{1 - x^2}}_{\bar{h}(x)}\,dx$$

$$h(y)\Big|_y^{2y} = h(2y) - h(y)$$

$$h'(y) = \frac{d}{dy}\left[h(2y) - h(y)\right]$$

$$= \bar{h}(2y) \cdot 2 - \bar{h}(y)$$

$$= \sqrt{1 - (2y)^2} \cdot 2 - \sqrt{1 - y^2}$$

$$= 2\sqrt{1 - 4y^2} - \sqrt{1 - y^2}$$

e) $$p(t) = \int_{e^t}^{-2} \underbrace{\ln u^2}_{\bar{p}(t)} \, du$$

$$p(t) \Big|_{e^t}^{-2} = p(-2) - p(e^t)$$

$$p'(t) = \frac{d}{dt}\left[p(-2) - p(e^t) \right]$$

$$= -\bar{p}(e^t) \cdot e^t$$

$$= -\ln (e^t)^2 \cdot e^t$$

$$= -\ln e^{2t} \cdot e^t$$

$$= -2te^t$$

f) $$G(z) = \int_{\sqrt{z}}^{\tan z} \underbrace{\frac{1-v}{1+v^2}}_{g(z)} \, dv$$

$$G(z) \Big|_{\sqrt{z}}^{\tan z} = G(\tan z) - G(\sqrt{z})$$

$$G'(z) = \frac{d}{dz}\left[G(\tan z) - G(\sqrt{z}) \right]$$

$$= g(\tan z) \cdot \sec^2 z - g(\sqrt{z}) \cdot \frac{1}{2\sqrt{z}}$$

$$= \frac{1-\tan z}{\underbrace{1+\tan^2 z}_{\cancel{\sec^2 z}}} \cdot \cancel{\sec^2 z} - \left(\frac{1-\sqrt{z}}{1+(\sqrt{z})^2} \right) \cdot \frac{1}{2\sqrt{z}}$$

$$= 1 - \tan z - \frac{1-\sqrt{z}}{2\sqrt{z}(1+z)}$$

$\sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta - \tan^2\theta = 1$

$\csc^2\theta - \cot^2\theta = 1$

6 g) $g(x) = \int_0^x x f(t)\,dt$

$$g'(x) = \frac{d}{dx}\left\{x \underbrace{\int_0^x f(t)\,dt}_{F(x)}\right\}$$

Product rule

$$= F(x) + x F'(x)$$

$$= \int_0^x f(t)\,dt + x f(x)$$

6 h) $h(x) = \int_3^{g(x)} x f(t)\,dt$

$$h'(x) = \frac{d}{dx}\left\{x \underbrace{\int_3^{g(x)} f(t)\,dt}_{F(x)}\right\}$$

$$= F(x) + x \frac{d}{dx}\left[F(g(x)) - F(3)\right]$$

$$= \int_3^{g(x)} f(t)\,dt + x\left[f(g(x)) \cdot g'(x) - 0\right]$$

$$= \int_3^{g(x)} f(t)\,dt + x\left[f(g(x)) \cdot g'(x)\right]$$

7. By evaluating the physical meaning of an integral, answer the following questions.

a) Given that I'(t) represents the new daily COVID19 cases, state the meaning of

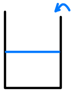
new cases per day

$$\text{Total Change in total cases from day 0 to day 30} \rightarrow \int_0^{30} \underbrace{I'(t)\,dt}_{dI \leftarrow \text{small change in total cases}}$$

b) If Q(t) represents the volumetric flow rate of water flowing into an empty tank in cubic meters per second, state the meaning of

$\frac{m^3}{s}$

$1\,m^3/s$ means $1\,m^3$ of water flowing into the tank in 1s



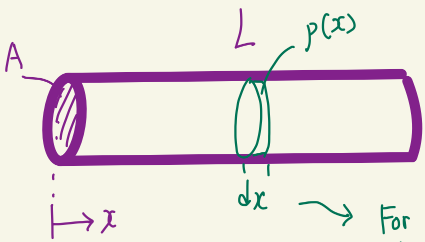
Total water vol in tank at time t

$\left(\frac{m^3}{s}\right)(s) = m^3$

$$\underbrace{\rho \text{Vol}}_{\text{Total water mass at } t} = \rho \int_0^t \underbrace{Q(t)\,dt}_{dv \leftarrow \text{small change of vol of water in tank}}$$

where ρ is the density of water in kg/m³.

c) A rod has length L and cross-sectional area A. If its material density function is ρ(x), where x is the position along the rod from one end, state an expression for determining the mass of the rod. What is the rate of change of mass wrt position? $\frac{dm}{dx}$



If $\rho = \text{const}$, then mass $m = \rho \underbrace{AL}_{\text{Vol}}$

→ For the disc of small length dx, the density can be ρ(x) which is "constant" over the disc

$$\text{disc mass, } dm = \rho(x)\underbrace{A\,dx}_{\text{disc vol}}$$

$$\text{Total rod mass} = \int_0^L dm = \int_0^L \rho(x)\,A\,dx$$

$$= A\int_0^L \rho(x)\,dx$$

$$\text{Rod mass from 0 to } x,\ m(x) = A\int_0^x \rho(x)\,dx$$

$$\Rightarrow \frac{dm}{dx} = \frac{d}{dx}\left\{A\int_0^x \rho(x)\,dx\right\} \overset{\text{FTC}}{=} A\,\rho(x)$$

8. Show that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\frac{1}{i+n}}=\ln 2$$

$$\frac{1}{n\left(\frac{i}{n}+1\right)} = \frac{1}{n}\cdot\frac{1}{\left(\frac{i}{n}+1\right)}$$

Let $\Delta x = \frac{1}{n}$, $a = x_0 = 1$, $b = 2$

$$\Rightarrow x_i = 1+\frac{i}{n} \rightarrow f(x_i) = \frac{1}{1+\frac{i}{n}} = \frac{1}{x_i}$$

$$\Rightarrow \int_1^2 \frac{1}{x}\,dx = \ln 2 - \overbrace{\ln 1}^{0}$$

$$= \ln 2$$

9. For a linear spring with stiffness k, show that the work done in compressing or extending the spring over distance x is given by

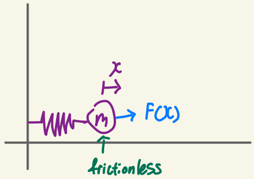
$$W = \frac{1}{2}kx^2$$

using integration. So what is the elastic potential energy stored in the spring?

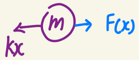
$PE = W = \frac{kx^2}{2}$

By conservation of energy

ANS: Elastic potential energy = $kx^2/2$.



FBD



$\sum F = 0$

$F(x) = kx$

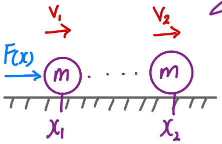
So work done, $W = \int_0^x F(x)\,dx$

$$= \int_0^x kx\,dx$$

$$= \frac{kx^2}{2}\Bigg|_0^x$$

$$= \frac{kx^2}{2}$$

10. A force is applied on a mass m causing it to undergo an acceleration a(x) from position x_1 to x_2. Using integration, show that the change in kinetic energy of the mass is



$\sum F = ma$ not necessarily const

$F(x) = ma$

$$\Delta KE = \frac{1}{2}m\left(v_2^2 - v_1^2\right).$$

By conservation of energy,

$$W = \Delta KE = \int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_2} m\,a(x)\,dx$$

Hint: Use a change of variables.

Since $\frac{dx}{dt} = v \rightarrow dx = v\,dt$ & $a = \frac{dv}{dt}$

$$\Delta KE = m\int \frac{dv}{\cancel{dt}}\, v\,\cancel{dt} = m\int_{v_1}^{v_2} v\,dv$$

$$= m\frac{v^2}{2}\Big|_{v_1}^{v_2}$$

$$= \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

11. Evaluate each indefinite integral below.

a) $\int 4e^x - \frac{3}{x}\, dx$

b) $\int \frac{x^5 + 2x^3 + 3x + 4}{x^2}\, dx$

c) $\int \frac{1 - \cos^3 x}{\cos^2 x}\, dx$

d) $\int 7^x + \frac{\sin 2x}{\cos x}\, dx$

ANS: **a)** $4e^x - 3\ln x + c$. **b)** $\frac{x^4}{4} + x^2 + 3\ln x - \frac{4}{x} + c$. **c)** $\tan x - \sin x + c$. **d)** $\frac{7^x}{\ln 7} - 2\cos x + c$

a) $\int 4e^x - \frac{3}{x}\, dx = 4e^x - 3\ln x + C$

b) $\int \frac{x^5 + 2x^3 + 3x + 4}{x^2}\, dx = \int x^3 + 2x + \frac{3}{x} + \frac{4}{x^2}\, dx$

$= \frac{1}{4}x^4 + x^2 + 3\ln|x| - \frac{4}{x} + C$

c) $\int \frac{1 - \cos^3 x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx - \int \frac{\cos^{\cancel{3}} x}{\cancel{\cos^2 x}}\, dx$

$= \int \sec^2 x\, dx - \int \cos x\, dx$

$= \tan x - \sin x + C$

d) $\int 7^x + \frac{\sin 2x}{\cos x}\, dx$

$$= \int 7^x dx + \int \frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}} dx$$

$$= \int 7^x dx + \overbrace{\int 2\sin x \, dx}^{2\int \sin x dx}$$

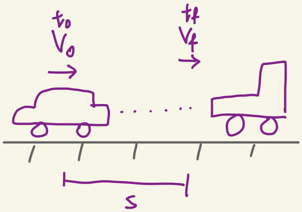
$$= \frac{7^x}{\ln 7} + 2(-\cos x) + C$$

$$= \frac{7^x}{\ln 7} - 2\cos x + C$$

12. **Mathematical Modelling:** Adaptive Cruise Control (ACC) (https://en.wikipedia.org/wiki/Adaptive_cruise_control)

A car activated with ACC is travelling along a straight road at velocity v_c. Suddenly, a truck travelling at a slower velocity v_T moved into the car's lane at distance d ahead. The ACC system senses the truck and activates the braking of the car. As a control engineer designing the ACC, answer the following questions.

a) Set up an integral that gives the braking distance, s, of the car from an initial velocity v_0 to the final velocity v_f. Hint: Use an appropriate change of variables.



$$S = \int_{t_0}^{t_f} V(t)\,dt$$

Since $\frac{dv}{dt} = a$, $\underline{dt = \frac{dv}{a}}$,

So $$S = \int_{t_0}^{t_f} V(t)\,\underline{dt} = \int_{v_0}^{v_f} \underline{v\left(\frac{dv}{a}\right)}$$

$$= \int_{v_0}^{v_f} \frac{v}{a}\,dv$$

b) If the car is to avoid a collision with the truck, determine the min constant deceleration that the ACC must activate on the car.

To avoid collision, $s \leq d$. At min deceleration, $s = d$ And velocity must slow from V_c to V_T

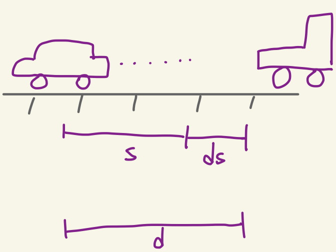
$$\Rightarrow d = s = \int_{V_c}^{V_T} \frac{v}{a}\, dv = \frac{1}{a}\int_{V_c}^{V_T} v\, dv = \frac{1}{a}\left(\frac{v^2}{2}\right)\Big|_{V_c}^{V_T}$$

$$= \frac{1}{a}\left(\frac{V_T^2 - V_c^2}{2}\right)$$

$$\Rightarrow a = \frac{V_T^2 - V_c^2}{2d} < 0 \quad \text{since } V_T < V_c$$

$$\therefore \text{ min deceleration} = \left|\frac{V_T^2 - V_c^2}{2d}\right|$$

c) If the car is to maintain a safe distance $d_s < d$ while cruising behind the truck, determine the min constant deceleration that the ACC must activate on the car. Is this value logically bigger or smaller than that in (b)?



$$\Rightarrow d - d_s = s = \int_{V_c}^{V_T} \frac{v}{a}\, dv = \frac{1}{a}\left(\frac{V_T^2 - V_c^2}{2}\right)$$

$$\Rightarrow a = \frac{V_T^2 - V_c^2}{2(d - d_s)}$$

$$\text{min deceleration} = \left|\frac{V_T^2 - V_c^2}{2(d - d_s)}\right| //$$

In this case, the min deceleration is bigger since logically the car must brake harder due to less distance to brake

$$\Delta x = \frac{2}{n} \qquad x_i = 0 + \frac{2i}{n}$$

$$b = 2, \; a = 0 \qquad x_i = \frac{2i}{n}$$

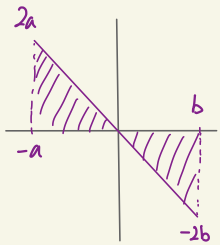
$$\int_0^2 1 - e^{-\frac{1}{2}\left(\frac{2i}{n}\right)}$$

$$\int_0^2 1 - e^{-x/2} \, dx$$

$$\Delta x = \frac{4}{n} \qquad x_i = -1 + \frac{4i}{n}$$

$$\sum_{1}^{n} \left(\frac{4}{n}\right) e^{2\left(\frac{4i}{n} - 1\right)}$$

$$\int_{-a}^{b} -2x \, dx$$



$$-2(-a) = 2a$$

$$-2(b) = -2b$$

$$= \frac{1}{2}(a)(2a)$$

$$- \frac{1}{2}(b)(2b)$$

$$= a^2 - b^2$$

$$\frac{d}{dx}\int_{x^2}^{e^{x/2}} \underbrace{\sin t}_{f(x)} \, dt$$

$$\frac{d}{dx}\left[f(e^{x/2}) - f(x^2) \right]$$

$$f(e^{x/2}) \cdot \frac{1}{2}e^{x/2} - f(x^2) \cdot 2x$$

$$\frac{1}{2}e^{x/2} \sin e^{x/2} - 2x \sin x^2$$

$$\frac{d}{dx}\left\{ \frac{1}{x}\int_{p(x)}^{k} f(t) \, dt \right\}$$

$$-\frac{1}{x^2}\int_{p(x)}^{k} f(t) \, dt + \frac{1}{x}\frac{d}{dx}\left[F(k) - F(p(x)) \right]$$

$$-\frac{x}{x^2}\underbrace{\left[f(p(x)) p'(x) \right]}$$

$$-\frac{1}{x^2}\int_{p(x)}^{k} f(t) dt + \frac{1}{x} \cdot \left[-F(p(x)) \cdot p'(x) \right]$$

$$-\frac{1}{x^2}\Big[$$

$$\theta'''(t)$$

$\frac{ds}{dt}$ = velocity

$\frac{dv}{dt}$ = acceleration

momentum = $m \cdot v$

$$\int a(t) = v$$
$$= \frac{t^3}{36}$$

$$\int_0^2 v(t) = s$$
$$= \frac{t^4}{144}\Big|_0^2$$

$$a(t) = \frac{t^2}{12}$$

$$\int Q(t)\, dt = \int 8t$$

$$= 4t^2$$

$$4t^2 = 36$$

$$t = 3$$

$$\int_1^2 \frac{8t^4 + 2}{t}\, dt$$

$$\int_1^2 8t^3 + \frac{2}{t}\, dt$$

$$= \left[2t^4 + 2\ln t \right] \Big|_1^2$$

$$= \left[2(2)^4 + 2\ln 2 \right] - \left[2(1)^4 + 2\ln(1) \right]$$

$$= 32 + 2\ln 2 - 2 + 0$$

$$= 30 + 2\ln 2$$

$$f(x) = \begin{cases} -\sin x & 0 \le x < \pi \\ \sec^2 x - 1 & \pi \le x \le 5\pi/4 \end{cases}$$

$$\int_0^{\pi} -\sin x = \cos x \Big|_0^{\pi}$$

$$= -2$$

$$\int_{\pi}^{5\pi/4} \sec^2 x - 1 \, dx$$

$$\int_{\pi}^{5\pi/4} \sec^2 x \, dx - \int_{\pi}^{5\pi/4} 1 \, dx$$

$$= \Big[\tan x\Big]\Big|_{\pi}^{5\pi/4} - \Big[x\Big]\Big|_{\pi}^{5\pi/4}$$

$$= \underset{1}{\tan\frac{5\pi}{4}} - \underset{0}{\tan\pi} - \underbrace{\left[5\pi/4 - \pi\right]}_{\frac{\pi}{4}}$$