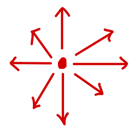
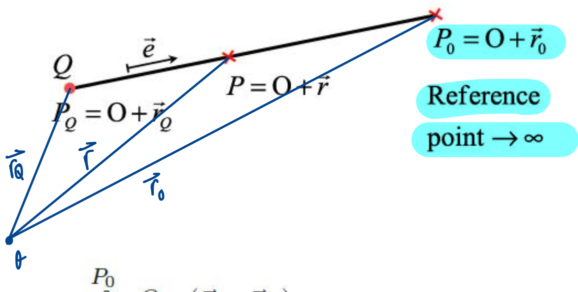
1.4.4 Coulomb-Potential of a point charge

(i) Let us calculate the electric potential of a point charge Q located at a position $P_Q = O + \vec{r}_Q$ in free space. The point charge generates the electric field (cf. equation (1.4))

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{(\vec{r}-\vec{r}_Q)}{|\vec{r}-\vec{r}_Q|^3}.$$



For an arbitrary point $P = O + \vec{r}$ we take the straight line passing through the points P_Q and P. We will use this line to calculate the path integral from P to the reference point P_0. P_0 is also placed on this line; eventually, it will be shifted to infinity.



So we have to calculate the path integral

$$\Phi(\vec{r}) = \Phi(\vec{r}_0) + \int_P^{P_0} \vec{E} \cdot d\vec{r} = \Phi(\vec{r}_0) + \int_P^{P_0} \frac{Q}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}_Q)}{|\vec{r}-\vec{r}_Q|^3} \cdot d\vec{r}.$$

$$-\int_{P_0}^{P} \vec{E}\,d\vec{r} = \int_P^{P_0} \vec{E}\,d\vec{r}$$

Curve is a straight line, which connect $P(\vec{r}_Q)$, $P(\vec{r})$ and $P(\vec{r}_0)$

Parameterize

$$\vec{r}(\lambda) = \vec{r}_Q + \lambda \cdot \vec{e}$$

$P(\vec{r})$ $P_0(\vec{r}_0)$

$\downarrow$

$\lambda \in [\lambda_1, \lambda_2]$

Integration is from $P(\vec{r}) \to P_0(\vec{r}_0)$

$\lambda_1 = |\vec{r} - \vec{r}_Q|$ $\qquad \lambda_2 = |\vec{r}_0 - \vec{r}_Q|$

Parameterize $d\vec{r}$ $\qquad d\vec{r} \to \vec{t}\, d\lambda$

$\uparrow$ tangent vector $= \vec{e}$

$$\left(d\vec{r} = \frac{d\vec{r}(s)}{ds} \cdot ds = \vec{t} \cdot ds \right)$$

$$d\vec{r} = \vec{e} \cdot d\lambda$$

· Parameterize vector field

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}_Q)}{|\vec{r}-\vec{r}_Q|^3} = \frac{Q}{4\pi\varepsilon_0} \frac{(\vec{r}_Q + \lambda\vec{e} - \vec{r}_Q)}{|\vec{r}_Q + \lambda\vec{e} - \vec{r}_Q|^3} = \frac{Q}{4\pi\varepsilon_0} \frac{\lambda \cdot \vec{e}}{\lambda^3} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\lambda^2} \cdot \vec{e}$$

Solve path integral:

$$\int_{P(\vec{r})}^{P_0} \vec{E}\,d\vec{r} = \int_{\lambda_1}^{\lambda_0} \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{\lambda^2} \cdot \underbrace{\vec{e} \cdot \overbrace{\vec{e} \cdot d\lambda}^{d\vec{r}}}_{1} = \int_{\lambda_1}^{\lambda_0} \frac{Q}{4\pi\varepsilon_0} \frac{1}{\lambda^2}\, d\lambda = \left[\frac{-Q}{4\pi\varepsilon_0} \frac{1}{\lambda} \right]_{\lambda_1}^{\lambda_0}$$

$$\phi = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{\lambda_0}+\frac{1}{\lambda_1}\right)$$

Resubstitute λ:

$$\phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{|\vec{r}_0-\vec{r}_Q|}+\frac{1}{|\vec{r}-\vec{r}_Q|}\right)$$

$\phi(\vec{r}_0)$ has to defined still : we shift $\vec{r}_0 \to \infty$ and we define ϕ_0 to be zero for $\vec{r}_0 \to \infty$

$$\Rightarrow \phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{|\vec{r}-\vec{r}_Q|}$$

if $\vec{r}_Q$ is located in the origin of coordinate system $\Rightarrow \vec{r}_Q = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow \phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0|\vec{r}|}$$

$$\sim \frac{1}{|\vec{r}|} = \frac{1}{r}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{r}|^3}\cdot\vec{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{r}|^2}\cdot\vec{e}_r$$

Equipotential lines/Surface :

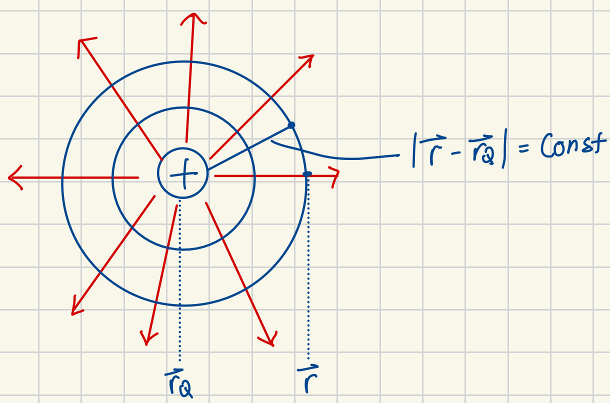
$$\phi(\vec{r}) = \text{Const.} = \phi_j = \frac{Q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_Q|} \qquad j = 1, 2, \dots$$

$$|\vec{r}-\vec{r}_Q| = \underbrace{\frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{\phi_j}}_{\text{Constant}}$$

$|\vec{r}-\vec{r}_Q| = \text{Const.}$

$\Downarrow$

defines a circle



Parametric representation of the straight path C from P to P_0 :

$$C : \vec{r}(\lambda) = \vec{r}_Q + \lambda\vec{e};\ \ \lambda_1 \leq \lambda \leq \lambda_0$$

$$\text{with } \vec{e} = \frac{\vec{r} - \vec{r}_Q}{|\vec{r} - \vec{r}_Q|};\ \ \lambda_1 = |\vec{r} - \vec{r}_Q|;\ \ \lambda_0 = |\vec{r}_0 - \vec{r}_Q|\,.$$

Tangential vector:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda} = \vec{e}\,.$$

Electric field in parametric representation:

$$\vec{E}(\vec{r}(\lambda)) = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{r}(\lambda) - \vec{r}_Q}{|\vec{r}(\lambda) - \vec{r}_Q|^3} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\lambda\vec{e}}{\lambda^3} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{e}}{\lambda^2}\,.$$

Path integral:

$$\int_P^{P_0} \vec{E} \cdot \mathrm{d}\vec{r} = \int_{\lambda_1}^{\lambda_0} \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{e}}{\lambda^2} \cdot \vec{e}\mathrm{d}\lambda = \int_{\lambda_1}^{\lambda_0} \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{\lambda^2}\mathrm{d}\lambda = \frac{Q}{4\pi\varepsilon_0} \cdot \left(-\frac{1}{\lambda_0} + \frac{1}{\lambda_1}\right)\,.$$

Hence, we obtain:

$$\Phi(\vec{r}) = \Phi(\vec{r}_0) + \frac{Q}{4\pi\varepsilon_0} \cdot \left(\frac{1}{|\vec{r} - \vec{r}_Q|} - \frac{1}{|\vec{r}_0 - \vec{r}_Q|}\right) \tag{1.16}$$

It is convenient to shift the reference point P_0 to infinity, $|\vec{r}_0| \to \infty$, and to set $\Phi(\vec{r}_0) = 0$; the result is:

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{|\vec{r} - \vec{r}_Q|}\,. \tag{1.17}$$

The equipotential surfaces are surfaces of concentric spheres with common center $\vec{r}_Q$:

$$\Phi(\vec{r}) = \text{const.} = \Phi_0 \ \Leftrightarrow\ |\vec{r} - \vec{r}_Q| = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{\Phi_0}$$

(ii) Coulomb-Potential of a discrete charge distribution

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Using the principle of linear superposition of fields (1.3) and equation (1.17), we obtain the electrostatic potential of a discrete distribution of point charges $(q_i, \vec{r}_i)_{i=1,\ldots,N}$:

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i|} \tag{1.18}$$

q_1 ·   · q_2 .... q_n

q_3 ·   · q_4

$$\phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_Q|}$$

$$\downarrow$$

$$\vec{r}_i \leftrightarrow q_i$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\Bigg(\underbrace{\frac{q_1}{|\vec{r}-\vec{r}_1|} + \frac{q_2}{|\vec{r}-\vec{r}_2|} + \ldots \frac{q_n}{|\vec{r}-\vec{r}_n|}}_{\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r}_i|}}\Bigg)$$