Circular Motion

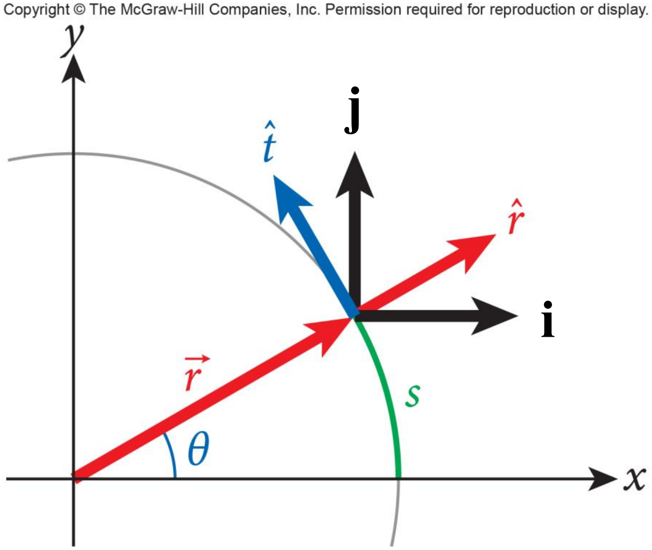
Revision

What We Will Learn

- The motion of objects traveling in a circle rather than in a straight line can be described using coordinates based on radius and angle rather than Cartesian coordinates.
- There is a relationship between linear motion and circular motion.
- Circular motion can be described in terms of the angular coordinate, angular frequency, and period.
- An object undergoing circular motion can have angular velocity and angular acceleration.

Polar Coordinates

- We can specify the position vector by giving its x- and y-components.
- We can also specify the same vector by giving two other variables: r and θ.
- The relationship between Cartesian coordinates and polar coordinates is:



$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Polar Coordinates

WORKED EXAMPLE

Plot the point indicated by the polar coordinates $(3, -\pi/2)$

[Ans: (0, -3)]

Angular Coordinates and Angular Displacement

- Like x and y, θ can be positive or negative, but θ is periodic:
  - A complete turn around the circle (360° or 2π rad) returns the position vector back to the same point.

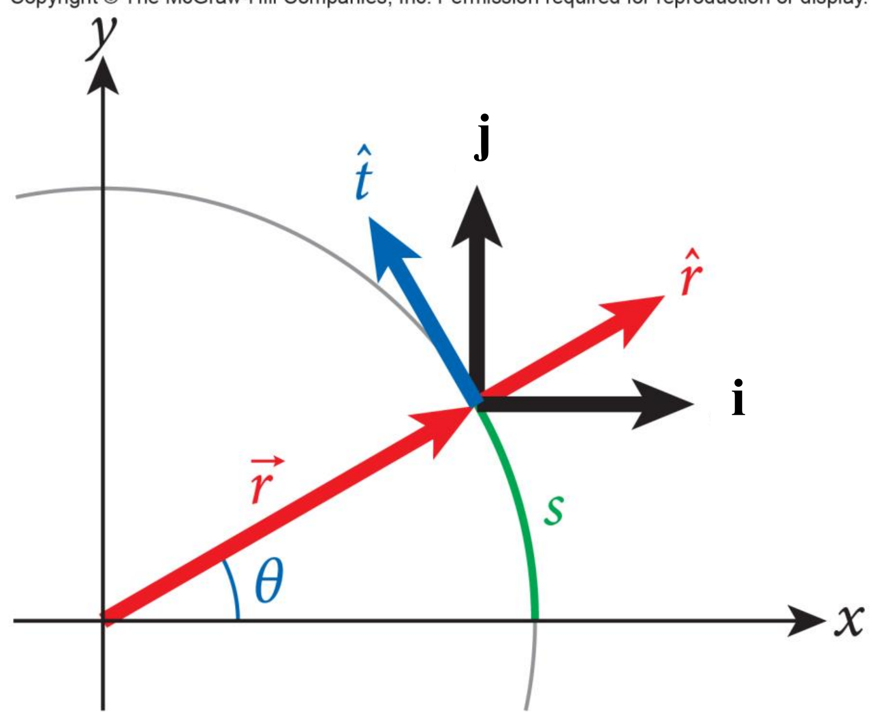
- We define the angular displacement as:

$$\Delta\theta = \theta_2 - \theta_1$$

- We define the arc length to be the distance traveled along the circular path:

$$s = r\theta$$

where θ is measured in radians.



Angular Coordinates and Angular Displacement

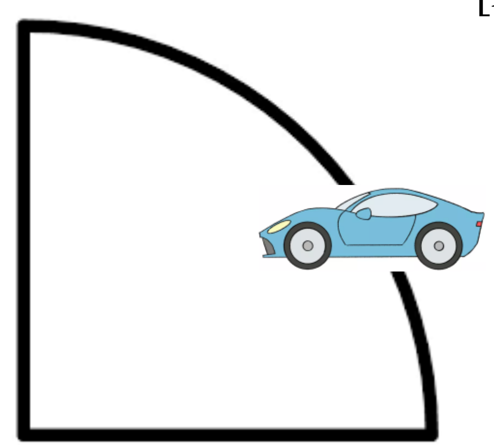
WORKED EXAMPLE

Plot the point indicated by the Cartesian coordinates (-2, 0)

[Ans: (2, π)]

Determine the distance covered by a car travelling along a circular path of 100 km radius as it travels East to North

[Ans: 157 km]



Angular Velocity

- Rate of change of displacement is velocity.
- Rate of change of angular displacement is angular velocity.

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

$$\omega = \lim_{\Delta t \to 0} \bar{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

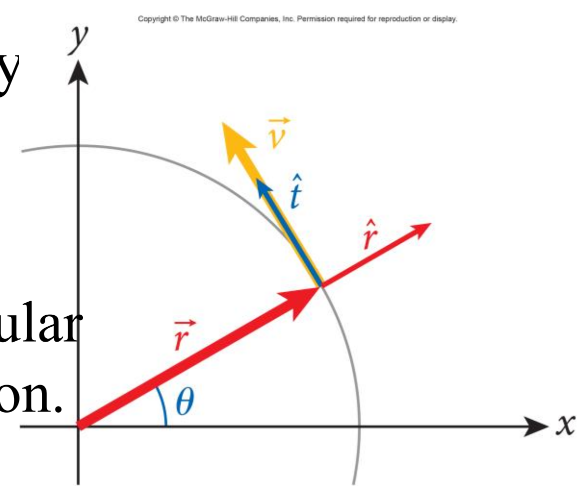
- Frequency, f, measures numbers of turns around the circle.
- Period T measures the time take to complete one revolution

$$\omega = 2\pi f = \frac{2\pi}{T}$$

- Relationship between linear and angular velocity

$$\vec{v} = r\omega\hat{t} \qquad v = r\omega$$

- Coordinate vector and velocity vector are perpendicular to each other at every point in time for circular motion.



Angular Velocity

WORKED EXAMPLE

$$V = Wr = \frac{2\pi}{T} r = \frac{2\pi}{87.58 \times \underset{\text{hrs}}{24} \times \underset{\text{mins}}{60} \times \underset{\text{seconds}}{60}}$$

Compute the linear velocity of Mercury as it orbits the Sun in 87.98 days at an average distance of 57 909 050 km.

[Ans: 47.8 km/s]

Hamilton's Mercedes has a top speed of 323 km/h during the Singapore Formula 1 Grand Prix. Determine the angular velocity of the tyre knowing its diameter is 720 mm.

[Ans: 249 /s]

Angular Acceleration

- The rate of change of angular velocity is the angular acceleration. $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$ $\alpha = \lim_{\Delta t \to 0} \bar{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
- The acceleration in circular motion has two components:
  - *Tangential acceleration* is related to the change in magnitude of the velocity.
  - *Radial acceleration* is related to the change in direction of the velocity.

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}(v\hat{t}) = \left(\frac{dv}{dt}\right)\hat{t} + v\left(\frac{d\hat{t}}{dt}\right)$$

$a_{\mathrm{t}} = r\alpha$ $\quad a_{\mathrm{c}} = v\omega = \frac{v^2}{r} = \omega^2 r$ Very important and useful equations

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^2 + \omega^4}$$

Angular Acceleration

Applet that allows visualization of velocity and acceleration vectors in linear, circular and elliptical motion.

https://phet.colorado.edu/sims/cheerpj/ladybug-motion-2d/latest/ladybug-motion-2d.html?simulation=ladybug-motion-2d

Angular Acceleration

WORKED EXAMPLE

If a rotating disc changes its angular speed at the rate of 60 rad/s for 10 seconds. Calculate its angular acceleration during this time.

$\alpha = \frac{\Delta W}{\Delta \theta} = \frac{60}{10} = 6 \text{ rad/s}^2$

[Ans: 6 rad/s^2]

A 900-kg car moving at 10 m/s takes a turn around a circle with a radius of 25.0 m. Determine the acceleration and the net force acting upon the car.

$a_c = \frac{v^2}{r} = \frac{10^2}{25} = 4 m/s^2$

[Ans: 4m/s^2, 3600 N]

$F_c = ma_c = 900 \times 4$

$= 3600N$

Centripetal Force

- The centripetal force is not another fundamental force of nature. It should not be drawn on a free body diagram.
- It is the inward force necessary to provide the centripetal acceleration necessary for circular motion.
- It has to point inward toward the circle's center.
- Its magnitude is the product of the mass of the object and the centripetal acceleration required to force the object onto a circular path:

$$F_\mathrm{c} = ma_\mathrm{c} = mv\omega = m\frac{v^2}{r} = m\omega^2 r$$

Centripetal Force

WORKED EXAMPLE

A van of 1,250 kg travelling at 50 m/s covers a curve of radius of 200 m. Calculate the centripetal force.

[Ans: 15625 N]

Circular and Linear Motion

- Relationship between linear and angular quantities. The radius r, of the circular path is constant and provides the connection between the two sets of quantities.

Quantity	Linear	Angular	Relationship
Displacement	s	θ	$s = r\theta$
Velocity	v	ω	$v = r\omega$
Acceleration	a	α	$a_t = r\alpha$ $a_c = r\omega^2$ $\mathbf{a} = r\alpha\mathbf{t} - r\omega^2\mathbf{r}$

Constant Angular Acceleration

- Kinematical equations for constant angular acceleration are obtained in complete analogy to those for linear motion with constant acceleration:

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \bar{\omega} t$$

$$\omega = \omega_0 + \alpha t$$

$$\bar{\omega} = \tfrac{1}{2}(\omega + \omega_0)$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

WORKED EXAMPLE

An athlete is rotating a discus in a circle of radius 80.0 cm at 10 rad/s and the angular speed is increasing at 50 rad/s^2. For this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

[Ans: 40 m/s^2, 80 m/s^2, 89.4 m/s^2]

You design an airplane propeller that is to turn at 2400 rpm. The forward airspeed of the plane is 75 m/s, and the speed of the propeller tips through the air must not exceed 270 m/s. Determine the maximum possible propeller radius and the corresponding acceleration of the propeller tip?

[Ans: 1.03 m, 6.5 $\times 10^4$ m/s^2]