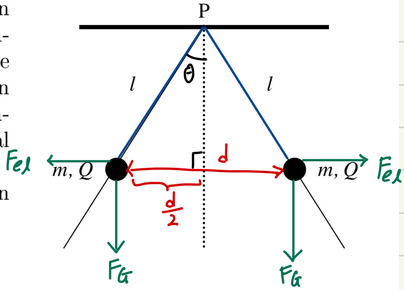
Problem 1 Charge Pendulum

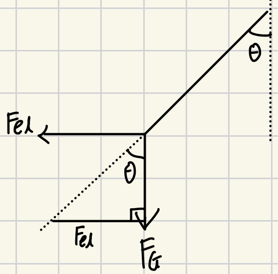
Two small metallic balls, each with mass m, are hanging on isolating strings with length l. The strings are fixed on a common suspension point P. Each ball carries an electric charge Q. Hence, the gravitational force (gravitational acceleration $g = 9.81\,\mathrm{m/s^2}$), the electrostatic repulsion force, and the suspension force act on the balls and balance each other in mechanical equilibrium.
Calculate the angle between the strings and the distance between the balls in the case that $m = 0.5\,\mathrm{g}$, $l = 1\,\mathrm{m}$, $Q = 10^{-8}\,\mathrm{C}$.
Hint: Assume that the angle of displacement is small.



Equilibrium of forces

$$\vec{F}_{el} = \frac{q}{4\pi\varepsilon_0} \cdot \sum_{i=1}^{n} \frac{q_i}{|\vec{r}-\vec{r}_i|^3} \cdot (\vec{r}-\vec{r}_i)$$

$$|F_{el}| = \frac{Q \cdot Q}{4\pi\varepsilon_0 d^2}$$



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{F_{el}}{F_G}$$

Gravitational force

$$\vec{F}_G = m \cdot \vec{g}$$

$$\tan\theta = \frac{|F_{el}|}{|F_G|}$$

$$\sin\theta = \frac{\frac{d}{2}}{l} \Rightarrow d = 2 \cdot l \cdot \sin\theta$$

$$|F_{el}| = \frac{Q^2}{4\pi\varepsilon_0 (2l\sin\theta)^2}$$

$$\tan\theta = \frac{\frac{Q^2}{4\pi\varepsilon_0 4 l^2 \sin^2\theta}}{m \cdot g} \quad \Big| \quad \tan\theta = \sin\theta = \theta$$

$$\theta = \frac{Q^2}{4\pi\varepsilon_0 4 l^2 m g\, \theta^2} \quad \Big| \cdot \theta^2$$

$$\theta^3 = \frac{Q^2}{16\pi\varepsilon_0 l^2 m g}$$

$$\theta = \sqrt[3]{\frac{Q^2}{16\pi\varepsilon_0 l^2 m g}}$$

Given:
$Q = 10^{-8}\,C$
$l = 1m$
$m = 0.5g$
$g = 9.81\,m/s^2$

$$\theta = \left(\frac{(10^{-8})^2 \cdot (As)^2}{16\pi \cdot 8.85 \cdot 10^{-12} \frac{As}{Vm} \cdot 1m^2 \cdot 5 \cdot 10^{-4} kg \cdot 9.81\, m/s^2} \right)^{1/3}$$

$$= 0.0358 \text{ rad} \qquad \left(\cdot \frac{360^\circ}{2\pi} \right)$$

$$= 2.05^\circ$$

Problem 2 Twin Point Charges

We consider two different configurations of two point charges Q_1 and Q_2:

A) Two opposite charges with equal magnitude (Fig. 1):

$$Q_1 = +Q \quad \text{and} \quad Q_2 = -Q$$

(forming an electric dipole)

B) Two equal charges (Fig. 2):

$$Q_1 = Q_2 = Q$$

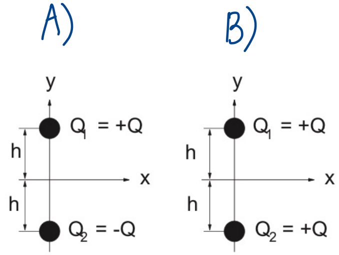


Fig. 1 Fig. 2

The charges are located on the y-axis of a Cartesian coordinate system at $y = h$ and $y = -h$, respectively. The following subtasks shall be solved for both configurations A and B:

a) Calculate the electric field (magnitude and direction) generated by the two charges at an arbitrary point on the x-axis?

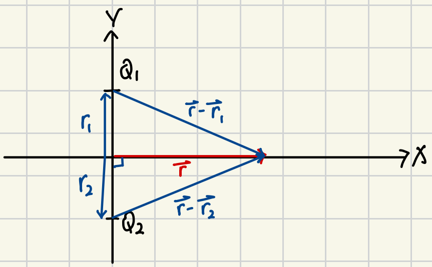
b) What are the Cartesian components E_x and E_y of the electric field at the point $H(x = h, y = h, z = 0)$?

c) Calculate the voltage U between the origin (0,0,0) and an arbitrary point $P_1(x, 0, 0)$ on the x-axis.

d) What work is necessary to move a test charge q from infinity to the origin?

a) $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \sum_{i=1}^{2} \frac{q_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$ (Superposition)

for Q_1: $\vec{r}-\vec{r}_1 = x\cdot\vec{e}_x - h\cdot\vec{e}_y = \begin{pmatrix} x \\ -h \\ 0 \end{pmatrix}$



for Q: $\vec{r}-\vec{r}_2 = x\cdot\vec{e}_x + h\cdot\vec{e}_y = \begin{pmatrix} x \\ h \\ 0 \end{pmatrix}$

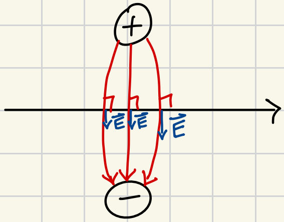
for Case A) opposite Charges:

$$\vec{E}(\vec{r}_x) = \vec{E}(x,0,0) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} - \frac{Q(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}\right]$$

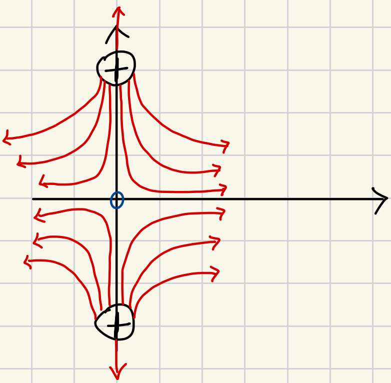
$|\vec{r}-\vec{r}_1| = |\vec{r}-\vec{r}_2| = \sqrt{x^2+h^2}$

$$\hookrightarrow \vec{E}(x,0,0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{(\sqrt{x^2+h^2})^3}\left[\begin{pmatrix} x \\ -h \\ 0 \end{pmatrix} - \begin{pmatrix} x \\ h \\ 0 \end{pmatrix}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{1}{(\sqrt{x^2+h^2})^3} \underbrace{\begin{pmatrix} 0 \\ -2h \\ 0 \end{pmatrix}}_{-2h\cdot\vec{e}_y}$$

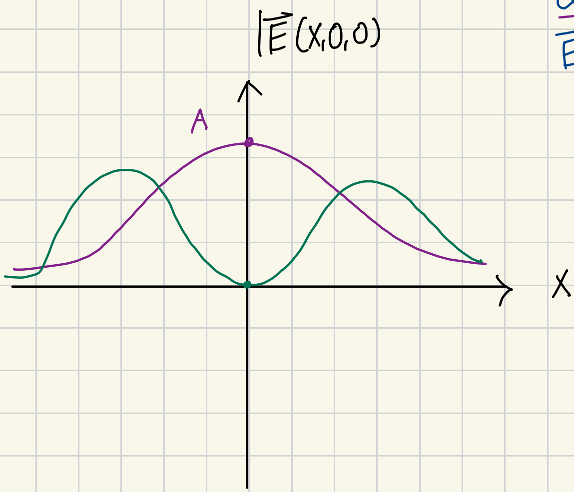


Case B: Equal Charges



$$\vec{E}(x,0,0) = \frac{Q}{4\pi\varepsilon_0 (x^2+h^2)^{3/2}}\left(\begin{pmatrix} x \\ -h \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ h \\ 0 \end{pmatrix}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0 (x^2+h^2)^{3/2}} \underbrace{\begin{pmatrix} 2x \\ 0 \\ 0 \end{pmatrix}}_{2x \cdot \vec{e}_x}$$



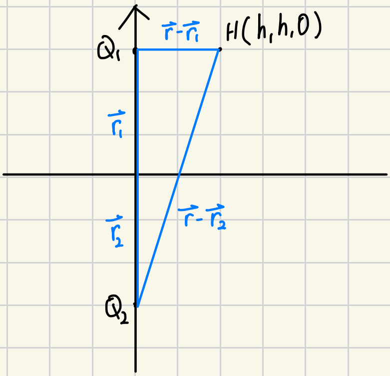
Case A

$$\vec{E}(x,0,0) = \frac{Q}{4\pi\varepsilon_0 (x^2+h^2)}\begin{pmatrix} 0 \\ -2h \\ 0 \end{pmatrix}$$

Case B

$$\vec{E}(x,0,0) = \frac{Q}{4\pi\varepsilon_0 (x^2+h^2)^{3/2}}\begin{pmatrix} 2x \\ 0 \\ 0 \end{pmatrix}$$

b) What are the Cartesian components E_x and E_y of the electric field at the point $H(x=h, y=h, z=0)$?



$$\vec{r}-\vec{r}_1 = h \cdot \vec{e}_x \rightarrow |\vec{r}-\vec{r}_1| = h$$

$$\vec{r}-\vec{r}_2 = h \cdot \vec{e}_x + 2h\vec{e}_y \rightarrow |\vec{r}-\vec{r}_2| = \sqrt{h^2 + 4h^2} = h \cdot \sqrt{5}$$

$$\vec{E}(H) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3} \overset{A}{\underset{B}{\ominus \oplus}} \frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|^3}\right]$$

Case A (opposite Charges) :

$$\vec{E}(H) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{h\vec{e}_x}{h^3} - \frac{h\vec{e}_x + 2h\vec{e}_y}{h^3 \cdot \sqrt{5}^3}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(h\vec{e}_x\left(\frac{1}{h^3} - \frac{1}{h^3\sqrt{5}^3}\right) - 2h\vec{e}_y \cdot \frac{1}{h^3\sqrt{5}^3}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{h^2}\left[\underbrace{\left(1 - \frac{1}{\sqrt{5}^3}\right)}_{E_x}\vec{e}_x - \underbrace{\frac{2}{\sqrt{5}^3}}_{E_y}\vec{e}_y\right]$$

Case B (Equal Charges) :

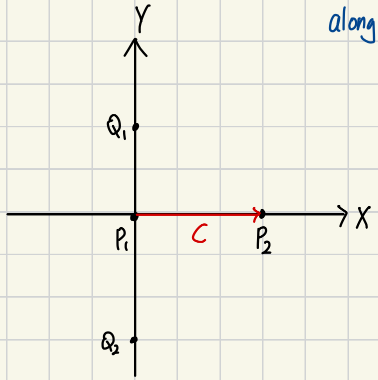
$$\vec{E}(H) = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{h^2}\left[\overbrace{\left(1 + \frac{1}{\sqrt{5}^3}\right)}^{E_x}\vec{e}_x + \overbrace{\frac{2}{\sqrt{5}^3}}^{E_y}\vec{e}_y\right]$$

c) Calculate the voltage U between the origin (0,0,0) and an arbitrary point $P_1(x,0,0)$ on the x-axis.

U between $P_1(0,0,0)$ & $P_2(x,0,0)$

↳ we solve $\boxed{V_{12} = \int_{P_1 C}^{P_2} \vec{E}\, d\vec{r}}$ over Curve C

Evaluate the Potential V along Curve C



· Parameterize $\vec{r}(\lambda)$

$\vec{r}(0) = \vec{r}_1 \quad (\hat{=} P_1)$

$\vec{r}(x) = \vec{r}_2 \quad (\hat{=} P_2)$

$\vec{r}(\lambda) = \lambda \vec{e}_x$

$\lambda \in [0, x]$

tangent vector : $\vec{t}(\lambda) = \frac{d\vec{r}}{d\lambda} = \vec{e}_x \longrightarrow d\vec{r} = \vec{e}_x \cdot d\lambda$

· Calculate $V_{12} = \int_{\lambda_1}^{\lambda_2} E(\vec{r}(\lambda)) \frac{d\vec{r}}{d\lambda}\, d\lambda$

$$= \int_{\lambda_1 = 0}^{\lambda_2 = x} \vec{E}(\vec{r}(\lambda)) \cdot \vec{e}_x \cdot d\lambda$$

Case A)

$$\vec{E}(x,0,0) = \frac{Q}{4\pi\varepsilon_0 (x^2+h^2)^{3/2}} \begin{pmatrix} 0 \\ -2h \\ 0 \end{pmatrix}$$

B)

$$\vec{E}(x,0,0) = \frac{Q}{4\pi\varepsilon_0 (x^2+h^2)^{3/2}} \begin{pmatrix} 2x \\ 0 \\ 0 \end{pmatrix}$$

B: equal Charges

$$\vec{E}(\vec{r}(\lambda)) = \frac{Q}{\cancel{4}\pi\varepsilon_0 (\lambda^2+h^2)^{3/2}} \cdot \cancel{2}\lambda \cdot \vec{e}_x \quad , \quad \vec{r}(\lambda) = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} = \lambda \vec{e}_x$$

$$\Rightarrow V_{12} = \int_0^x \frac{Q}{2\pi\varepsilon_0 (\lambda^2+h^2)^{3/2}} \cdot \lambda \cdot \underbrace{\vec{e}_x \cdot \vec{e}_x}_{=1} \cdot d\lambda = \int_0^x \frac{Q}{2\pi\varepsilon_0} \frac{\lambda}{\sqrt{\lambda^2+h^2}^{\,3}} \, d\lambda$$

Substitution: $u = \lambda^2+h^2 \Rightarrow \sqrt{\lambda^2+h^2}^{\,3} = \sqrt{u}^{\,3} = u^{3/2}$

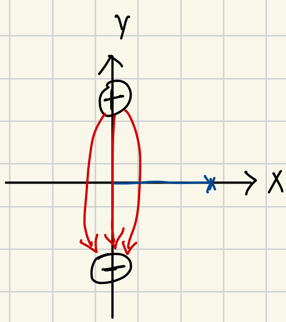
$$\frac{du}{d\lambda} = 2\lambda \rightarrow d\lambda = \frac{du}{2\lambda}$$

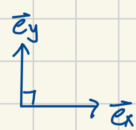
$$V_{12} = \frac{Q}{2\pi\varepsilon_0} \int_{u(\lambda=0)}^{u(\lambda=x)} \frac{\cancel{\lambda}}{\sqrt{u}^{\,3}} \frac{du}{2\cancel{\lambda}} = \frac{Q}{2\pi\varepsilon_0} \int_{h^2}^{x^2+h^2} \frac{1}{2} \cdot \frac{1}{\sqrt{u}^{\,3}} \, du = \frac{Q}{2\pi\varepsilon_0} \left[-\frac{1}{\sqrt{u}} \right]_{h^2}^{x^2+h^2}$$

$$\hookrightarrow u^{-\frac{3}{2}} \underset{\left(\frac{d}{du}\right)}{\overset{\int}{\rightleftarrows}} -2u^{-\frac{1}{2}}$$

$$\Rightarrow V_{12} = \frac{Q}{2\pi\varepsilon_0} \left(-\frac{1}{\sqrt{x^2+h^2}} + \frac{1}{h} \right)$$

Case A) opposite Charges





$$V_{12} = \int_0^x \frac{Q}{2\pi\varepsilon_0 \sqrt{\lambda^2+h^2}^{\,3}} (-h) \overbrace{\vec{e}_y \cdot \vec{e}_x}^{0} \, d\lambda$$

$$= 0$$

d) What work is necessary to move a test charge q from infinity to the origin?

$P_1(0,0,0)$

$W(\infty \to 0)$

$W(\infty \to 0): W_{12} = q \cdot U_{12}$

Case A) opposite Charges : $W = q \cdot U_{0,\infty} = 0$

Case B) equal Charges :

$$W = q \cdot U_{0,\infty} = \lim_{x \to \infty} \left(q \cdot \frac{Q}{2\pi \varepsilon_0} \left(-\underbrace{\frac{1}{\sqrt{x^2+h^2}}}_{\to 0 \text{ for } x \to \infty} + \frac{1}{h} \right) \right)$$

$$\Rightarrow W_{0,\infty} = \frac{q \cdot Q}{2\pi \varepsilon_0} \cdot \frac{1}{h}$$

electrical work

$W_{\infty,0} = -W_{0,\infty}$

$W_{12} = q \cdot U_{12} = q \cdot \int_{P_1}^{P_2} \vec{E} \cdot d\vec{r} \rightarrow$ the work the electric field performs on the charge to move it from P_1 to P_2

blem 3 Electrostatic Field

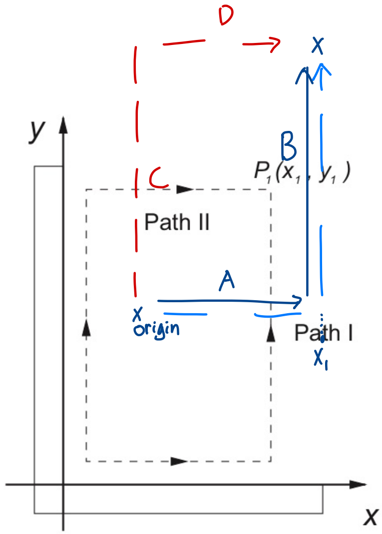
region $\Omega = \{(x,y,z) \in \mathbb{R}^3 \,|\, x \geq 0 \text{ and } y \geq 0\}$ in a Carte- coordinate system is confined by two thin conductive es, which cover the two half-planes

$$H_1 = \{(x,y,z) \in \mathbb{R}^3 \,|\, y = 0 \text{ and } x \geq 0\}$$

$$H_2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x = 0 \text{ and } y \geq 0\}$$

figure).
the region Ω the following electric field is defined:

$$\vec{E}(\vec{r}) = -Ay\vec{e}_x - Ax\vec{e}_y \quad \text{with } A = \text{constant} \neq 0$$



Calculate the voltage U_{01} between the origin and an arbitrary position $P_1(x_1, y_1, 0)$ as a function of A, x_1 and y_1. To this end, calculate the path integral

$$\int_0^{P_1} \vec{E}\, d\vec{r}$$

along the path I and then, alternatively, also along path II (see figure).

What is the value of U_{01}, if P_1 is located on one of the plates H_1 or H_2?

Is the field $\vec{E}(\vec{r})$ conservative?

Sketch the intersection of the equipotential surfaces with the x-y-plane (equipotential lines) and the field lines of the electric field.

a) V_{01} between $P_1(0,0,0)$ and $P_2(x_1, y_1, 0)$ $= U_{01} \hat{=} \int_0^{P_2} \vec{E}\, d\vec{r}$

$\vec{E}(\vec{r}) = \begin{pmatrix} -Ay \\ -Ax \end{pmatrix}$

Path I $\quad V_{01} = \int_A \vec{E}\, d\vec{r} + \int_B \vec{E}\, d\vec{r}$

Curve A: $\quad r(\lambda) = \lambda \vec{e}_x \qquad \lambda \in [0, x]$

$\dot{\vec{r}}(\lambda) = \frac{d\vec{r}}{d\lambda} = \vec{e}_x \quad \left(\hat{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$\vec{E}(\lambda) = \begin{pmatrix} -A \cdot 0 \\ -A \cdot \lambda \end{pmatrix}$

$$\int_0^{x_1} \vec{E} \frac{d\vec{r}}{d\lambda} d\lambda = \int_0^{x_1} \vec{E}(x=\lambda, y=0) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot d\lambda$$

$$= \int_0^{x_1} (\underbrace{-A \cdot 0 \cdot \vec{e}_x}_{0} - A \cdot \lambda \cdot \underbrace{\vec{e}_y) \cdot \vec{e}_x}_{0}\, d\lambda = 0$$

Curve B: $r(\mu) = X_1 \cdot \vec{e}_x \quad \mu \cdot \vec{e}_y$ $\qquad \mu \in [0, y_1]$

$$\vec{E} = \begin{pmatrix} -Ay \\ -Ax \end{pmatrix}$$

$$\frac{d\vec{r}}{d\mu} = \vec{e}_y$$

$$\vec{E}(\vec{r}(\mu)) = (-A\mu \cdot \vec{e}_x - A \cdot x_1 \vec{e}_y)$$

$$\int_B \vec{E}\, d\vec{r} = \int_{\mu_0}^{\mu_1} \vec{E} \frac{d\vec{r}}{d\mu}\, d\mu = \int_0^{Y_1} (-A \cdot \mu \underbrace{\vec{e}_x - Ax_1 \overbrace{\vec{e}_y) \cdot \vec{e}_y}^{1}}_{0} \cdot d\mu$$

$$= \int_0^{Y_1} -Ax_1\, d\mu = -Ax_1 [\mu]_0^{Y_1}$$

$$= -AX_1Y_1$$

$$\Rightarrow V_{01} = -AX_1Y_1$$

Path II

Curve C: $\vec{r}(\lambda) = \lambda \vec{e}_y = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}$; $\lambda \in [0, Y_1]$

$$\frac{d\vec{r}}{d\lambda} = \vec{e}_y \qquad \vec{E} = \begin{pmatrix} -Ay_1 \\ -Ax \end{pmatrix}$$

$$\vec{E}(\vec{r}(\lambda)) = \begin{pmatrix} -A\lambda \\ -A0 \end{pmatrix} = \begin{pmatrix} -A\lambda \\ 0 \end{pmatrix}$$

Curve D: $\vec{r}(\mu) = Y_1 \vec{e}_y + \mu \vec{e}_x$; $\mu \in [0, X_1]$

$$\frac{d\vec{r}}{d\mu} = \vec{e}_x$$

$$\vec{E}(\vec{r}(\mu)) = \begin{pmatrix} -Ay_1 \\ -A\mu \end{pmatrix}$$

$$\rightarrow V_{01} = \int_0^{Y_1} (-A\cdot\lambda\, \underbrace{\vec{e}_x)\cdot \vec{e}_y}_{0} \cdot d\lambda \; + \int_0^{X_1} \overbrace{\left(-AY_1\cdot \vec{e}_x - A\mu \cdot \underbrace{\vec{e}_y\right)\cdot \vec{e}_x}_{0}}^{1} \cdot d\mu$$

$$= \int_0^{X_1} -AY_1 \, d\mu = \underline{\underline{-AY_1 X_1}}$$

$\Rightarrow$ Value of V_{01} is independant of the chosen path !

c) Is the field $\vec{E}(\vec{r})$ conservative?

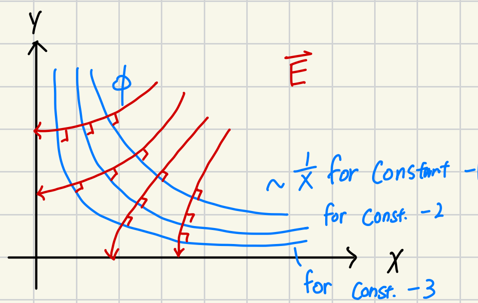
$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \qquad \text{with} \qquad E = -Ay\vec{e}_x - Ax\vec{e}_y$$

$$\frac{\partial}{\partial y} -Ay = -A = \frac{\partial}{\partial x} -Ax = -A \qquad (\text{true}) \Rightarrow \vec{E} \text{ is Conservative}$$

d) Sketch the intersection of the equipotential surfaces with the x-y-plane (= equipotential lines) and the field lines of the electric field.

$$V_{01} = \text{Constant} = -AX_1Y_1 \quad \Rightarrow \quad Y_1 = \frac{\text{Constant}}{-A} \cdot \frac{1}{X_1}$$

$(y = b\cdot x + c)$



Q3 What is the "fundamental law of electrostatics"?

Answer : Electrostatic fields are "Conservative" !

$\vec{E}$ is a gradient field

$(\vec{E} = -\nabla\phi)$

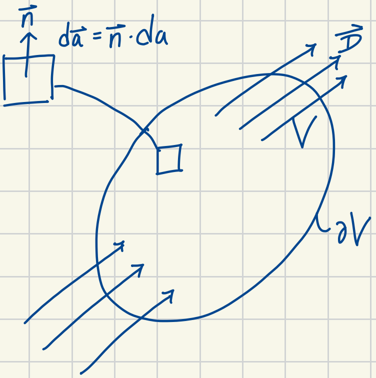
or: $\frac{\partial E_i}{\partial x_k} = \frac{\partial E_k}{\partial x_i}$ for x-y : $\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$

$(i, k = 1, 2, 3)$

in Cartesian Coordinates

Q5 Derive the differential form of Gauss's law $\mathrm{div}\vec{D} = \rho$ from the integral form of Gauss's law $\int_{\partial V} \vec{D} \mathrm{d}\vec{a} = Q(V)$.

Gauss flux theorem : $\int_{\partial V} \vec{D}\, d\vec{a} = \int_V \mathrm{div}\,\vec{D}\, dV$



charge density

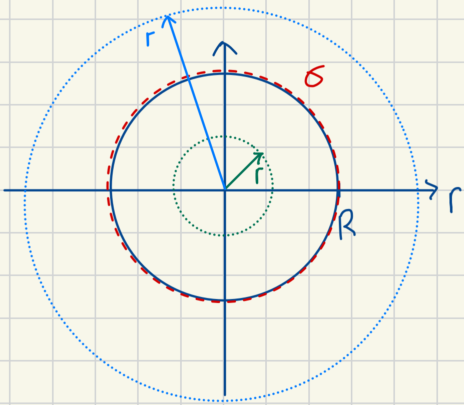
① $Q(V) = \int_V \rho\, dV$

② $\int_{\partial V} \vec{D}\, d\vec{a} = \int_V \mathrm{div}\,\vec{D}\, dV$

$$\hookrightarrow \int_{\partial V} \vec{D}\, d\vec{a} = \int_V \mathrm{div}\,\vec{D}\, dV \underset{\uparrow}{=} \int_V \rho\, dV = Q(V)$$

$\mathrm{div}\,\vec{D} = \rho$

Q6 Imagine a sphere with radius R centered in the origin. A uniform surface charge density σ is placed on the surface of the sphere. Determine the electric field $\vec{E}(\vec{r})$ in the interior of the sphere $0 \leq r < R$ (Reason!).



Apply Gauss's Law

$$\int_{\partial V} \vec{D}\, d\vec{a} = Q(V)$$

Use Spherical Coordinate System :

Because uniform symmetric surface charge density σ

$\Rightarrow$ Symmetry : $\vec{D}(r) = D_r(r) \cdot \vec{e}_r$ $\quad (r = r_0)$

surface element : $d\vec{a} = \vec{e}_r \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi$

$$\int_{\partial V} \vec{D}\, d\vec{a} = \int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi} D_r \underbrace{\vec{e_r}\cdot\vec{e_r}}_{1} \cdot r^2 \sin\theta \, d\theta \, d\varphi$$

$$= D_r\, r^2 \int_0^{2\pi}\int_0^{\pi} \sin(\vartheta)\, d\theta\, d\varphi$$

$$= D_r\, r^2 \int_0^{2\pi} [-\cos\theta]_0^{\pi}\, d\varphi$$

$$= D_r\, r^2 \int_0^{2\pi} [\underbrace{-\cos(\pi)}_{+1} - \underbrace{(-\cos(0))}_{+1}]\, d\varphi$$

$$= D_r\, r^2 \int_0^{2\pi} 2\, d\varphi$$

$$= 4\pi D_r r^2$$

$$\overset{!}{=} Q(r) \qquad \Rightarrow \quad 4\pi D_r r^2 = Q(r)$$

$$D_r = \frac{Q(r)}{4\pi r^2}$$

for $0 \le r < R \Rightarrow Q(r) = 0$

$$\Rightarrow D_r = \frac{0}{4\pi r^2} = 0$$

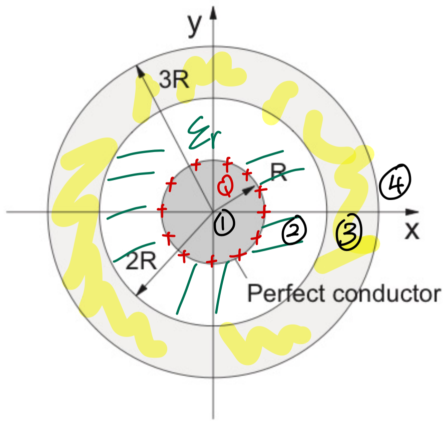
Outside : $R < r$

$$A = 4\pi R^2$$

$$D_r(r) = \frac{Q(r)}{4\pi r^2} = \frac{\sigma \cdot A}{4\pi r^2} = \frac{\sigma \cdot 4\pi R^2}{4\pi r^2} = \sigma\left(\frac{R}{r}\right)^2$$

$$\vec{D} = \varepsilon \vec{E} \Rightarrow \vec{E_r} = \frac{R^2 \sigma}{\varepsilon r^2}$$

Problem 7 Charge Distribution



Consider a perfectly conducting sphere with radius R centered at the origin. A charge $Q > 0$ is stored on the sphere (see figure). All space around the sphere is filled with a homogeneous dielectric with $\varepsilon = \varepsilon_0\varepsilon_r$. In addition, there exists a space charge density:

$$\rho(r) = \begin{cases} \boxed{\dfrac{\alpha}{4\pi r R}} & \text{for} \quad 2R < r < 3R \\ 0 & \text{otherwise} \end{cases}$$

Choose spherical coordinates to solve the following subtasks.

*a) How is the charge Q distributed over the perfectly conducting sphere? Give a reason for your answer. Calculate the surface charge density on the sphere.

*b) What direction has the electric field $\vec{E}(r, \vartheta, \varphi)$? Calculate the electric field $\vec{E}(r, \vartheta, \varphi)$ in each of the four regions $0 \le r < R$, $R < r < 2R$, $2R < r < 3R$, and $3R < r$.

*c) Determine the factor α so that the electric field vanishes in the outer region $r > 3R$.

Use the value of α determined in c) for the following subtasks.

d) Plot the radial component of the electric field $\vec{E}$ as a function of r over all four regions. Don't forget to label the coordinate axes sufficiently.

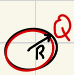
e) A test charge q is moved from infinity onto the surface of the conducting sphere. Calculate the mechanical work.

a) Charge is distributed uniformly on the surface of the sphere

due to: - electrostatic repulsion

- spherical symmetry

Surface charge density ?



$$\sigma = \frac{Q}{A} = \frac{Q}{4\pi R^2}$$

*b) What direction has the electric field $\vec{E}(r, \vartheta, \varphi)$? Calculate the electric field $\vec{E}(r, \vartheta, \varphi)$ in each of the four regions $0 \le r < R$, $R < r < 2R$, $2R < r < 3R$, and $3R < r$.

Symmetry $\rightarrow \vec{E} = E_r(r) \cdot \vec{e_r}$ (independent of θ, φ)

Gauss's Law : $\int_{\partial V} \vec{D}\, d\vec{a} = Q(r)$ (enclosed Charges)

With $\vec{D} = D_r(r) \cdot \vec{e_r}$ & $d\vec{a} = r^2 \sin\theta \, d\theta \, d\varphi \, \vec{e_r}$

$$\int_{\partial V} \vec{D}\, d\vec{a} = \int_0^{2\pi}\int_0^{\pi} D_r \underbrace{\vec{e_r}\,\vec{e_r}}_{1} r^2 \sin\theta \, d\theta \, d\varphi$$

$$= D_r r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\varphi$$

$$= 4\pi D_r r^2$$

$$\stackrel{!}{=} Q(r)$$

$$\Rightarrow D_r(r) = \frac{Q(r)}{4\pi r^2} \qquad E_r(r) = \frac{Q(r)}{4\pi r^2 \varepsilon_0 \varepsilon_r}$$

① $0 \le r \le R$ $\quad Q(r) = 0 \rightarrow D_r = 0$, $E_r = 0$

② $R \le r < 2R$ $\quad Q(r) = Q \rightarrow D_r = \frac{Q}{4\pi r^2}$, $E_r = \frac{Q}{4\pi r^2 \varepsilon_0 \varepsilon_r}$

③ $2R \le r < 3R$ $\quad Q(r) = Q + \int_V \rho \, dV = Q + \int_0^{2\pi}\int_0^{\pi}\int_{2R}^{r} \frac{\alpha}{4\pi R} \cdot \frac{1}{r} \, dV$

$dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$

$$Q(r) = Q + \frac{\alpha}{4\pi R} \int_0^{2\pi} \int_0^{\pi} \int_{2R}^{r} \underbrace{\frac{1}{r'}}_{r} r'^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$= Q + \frac{\alpha}{4\pi R} 4\pi \int_{2R}^{r} r' dr'$$

$$= Q + \frac{\alpha}{R} \left[\frac{1}{2} r'^2\right]_{2R}^{r}$$

$$= \underset{\substack{\uparrow \\ \text{from inner sphere}}}{Q} + \underbrace{\frac{\alpha}{R}\left(\frac{r^2}{2} - 2R^2\right)}_{\text{from Charge density } \rho}$$

$$\rightarrow D_r = \frac{Q(r)}{4\pi r^2} \quad , \quad E_r = \frac{D_r}{\varepsilon_0 \varepsilon_r}$$

④ $r > 3R$

$$Q(r) = Q + \frac{\alpha}{R}\left(\frac{(3R)^2}{2} - 2R^2\right)$$

$$= Q + \frac{5}{2}\alpha R$$

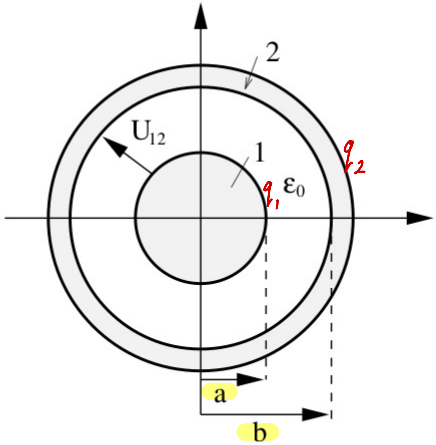
$$D_r = \frac{Q + \frac{5}{2}\alpha R}{4\pi r^2} \quad , \quad E_r = \frac{Q + \frac{5}{2}\alpha R}{4\pi r^2 \varepsilon_0 \varepsilon_r}$$

*c) Determine the factor α so that the electric field vanishes in the outer region $r > 3R$.

$$\Rightarrow \frac{Q + \frac{5}{2}\alpha R}{4\pi r^2 \varepsilon_0 \varepsilon_r} = 0 \quad \Rightarrow \quad \alpha = -\frac{2}{5}\frac{Q}{R}$$

Problem 8 Coaxial Tube

Consider two coaxial metallic cylinders, which extend to infinity in both directions perpendicular to the plane of projection (see figure). The inner conductor 1 has the radius a and carries the charge $q_1 = +q$ per unit length. The outer conductor 2 has the (inner) radius b and carries the opposite charge $q_2 = -q$. The space in between $a < r < b$ is filled with air, i.e., $\varepsilon = \varepsilon_0$.



a) Determine the radial components $E_r(r)$ and $D_r(r)$ of the electric field and the dielectric displacement, respectively, as a function of r for $a \leq r \leq b$.

b) Find the surface charge densities σ_1 and σ_2 on the inner and the outer conductor.

c) Determine the voltage U_{12} between the inner and the outer conductor as a function of q, a and b.

d) Calculate the quantity $c := q/U_{12}$ (= capacitance of the tube per unit length). How does this quantity change when, instead of air, a dielectric with relative permittivity $\varepsilon_r > 1$ is filled into the space between the conductors?

Now, the following numerical parameters are given:

$U_{12} = 1000\,\mathrm{V}$, $a = 2\,\mathrm{mm}$, $b = 5.44\,\mathrm{mm}$, $\varepsilon_r = 1$ and $\varepsilon_r = 2.5$, respectively, $\varepsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{As/Vm}$

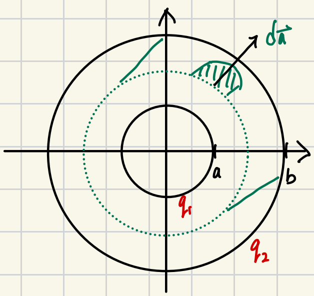
e) Find the numerical value of $c := q/U_{12}$.

f) Find the radial component of the electric field $E_r(a)$ on the surface of the inner conductor.

a) E_r, D_r

$$\int_{\partial V} \vec{D}\, d\vec{a} = Q(V) = q \cdot L$$

$$d\vec{a} = r \cdot d\varphi\, dz\, \vec{e_r}$$



$$\Rightarrow \int_0^L \int_0^{2\pi} D_r \underbrace{\vec{e_r} \cdot \vec{e_r}}_{1}\, r\, d\varphi\, dz = \int_0^L \int_0^{2\pi} D_r\, r\, d\varphi\, dz$$

"must be equal"

$$= D_r r L \cdot 2\pi \overset{!}{=} q \cdot L$$

$$\Rightarrow D_r = \frac{qL}{2\pi L r} = \frac{q}{2\pi r}$$

$$E_r = \frac{1}{\varepsilon} D_r = \frac{q}{2\pi \varepsilon_0 r}$$

b) σ_1 , σ_2

$$\sigma = \frac{Q}{A} = \frac{q \cdot L}{2\pi r \cdot L} = \frac{q}{2\pi r}$$

$$\sigma_1 = \frac{Q_1}{A_1} = \frac{q \cdot L}{2\pi a \cdot L} = \frac{q}{2\pi a} \longrightarrow = D_r(a)$$

$$\sigma_2 = \frac{Q_2}{A_2} = \frac{-q \cdot L}{2\pi b \cdot L} = \frac{-q}{2\pi b}$$

c) $$V_{12} = \int_C \vec{E} d\vec{r} = \int_a^b E_r \overbrace{\vec{e_r} \cdot \vec{e_r}}^{1} dr = \int_a^b \frac{q}{2\pi \varepsilon_0 r} dr$$

$$= \left[\frac{q}{2\pi\varepsilon_0} \ln(r) \right]_a^b = \frac{q}{2\pi\varepsilon_0} \left(\ln(b) - \ln(a) \right)$$

$$= \frac{q}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

d) $$C := \frac{q}{U_{12}} = \frac{q}{\frac{q}{2\pi\varepsilon_0\varepsilon_r} \ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)}$$

$$C = \frac{Q}{U}$$

for air : $\varepsilon_r = 1$

for dielectric : $\varepsilon_r > 1$

→ C would increase in case of dielectric

e) $\varepsilon_{r,\text{air}} = 1$ $\qquad$ $\varepsilon_{r,\text{dielectric}} = 2.5$

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} = \begin{cases} 55.6 \frac{pF}{m}, & \varepsilon_r = 1 \\ 139 \frac{pF}{m}, & \varepsilon_r = 2.5 \end{cases}$$

f) $E_r(a) = \frac{q}{2\pi\varepsilon_0\varepsilon_r} \cdot \frac{1}{a}$

$C = \frac{q}{V_{12}}$

$\rightarrow q = C \cdot V_{12}$

$\Rightarrow E_r(a) = \frac{\overbrace{\frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)}}^{C} \cdot U_{12}}{2\pi\varepsilon_0\varepsilon_r} \cdot \frac{1}{a}$

$= \frac{U_{12}}{\ln\left(\frac{b}{a}\right)} \cdot \frac{1}{a} = 500\,\frac{kV}{m}$

Q4 What is the definition of the gradient of a scalar field? What is the geometric interpretation of the gradient?

(eg $\vec{E} = -\nabla\phi$ ϕ = electrostatic potential)
(scalar field)

① Vector assigned to total derivative of $d\phi$

$d\phi = \langle \text{grad } \phi, n \rangle = \text{grad } \phi \cdot \vec{n}$

in Cartesian Coordinates:

$$\text{grad } \phi(x, y, z) = \frac{\partial \phi}{\partial x} \vec{e_x} + \frac{\partial \phi}{\partial y} \vec{e_y} + \frac{\partial \phi}{\partial z} \vec{e_z}$$

② math operator applied to a scalar field ϕ

→ gradient field (describes spatial variation of the field)

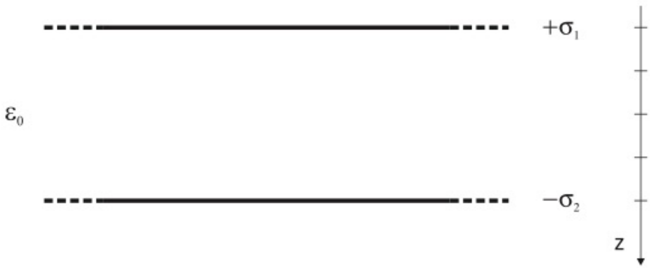
③ gradient always points in the direction of the steepest increase of ϕ

④ perpendicular to "Iso - Surfaces"

$\vec{E} = -\nabla\phi$

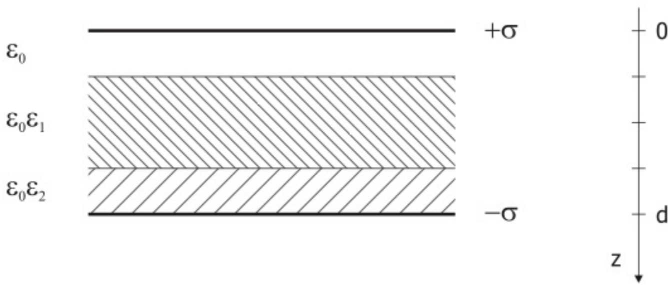
Problem 10 Plate Capacitor

Consider the following set-up of perfectly conducting, infinitely large plates. The plate at $z = 0$ carries the surface charge density $\sigma_1 > 0$, whereas the plate at $z = d$ carries the surface charge $-\sigma_2 < 0$.



a) Calculate the magnitude and the direction of the electric field in all regions outside and inside the capacitor under the assumpation that $\sigma_1 > \sigma_2$ holds.

In the following, we assume that $\sigma := \sigma_1 = \sigma_2$ holds. Both plates now have the same area A. The distance d between the plates is so small that stray fields are negligible. The space between the plates is filled with two different dielectrics ($\varepsilon_1 > 1$ and $\varepsilon_2 > 1$), as can be seen in the figure below.



We have

$$\varepsilon = \varepsilon_0 \varepsilon_\mathrm{r} = \begin{cases} \varepsilon_0 & \text{for} \quad 0 < z \leq \frac{d}{4} \\ \varepsilon_0 \varepsilon_1 & \text{for} \quad \frac{d}{4} < z \leq \frac{3d}{4} \\ \varepsilon_0 \varepsilon_2 & \text{for} \quad \frac{3d}{4} < z < d \end{cases}$$

b) Calculate the electric displacement field $\vec{D}$ in all regions (magnitude and direction).

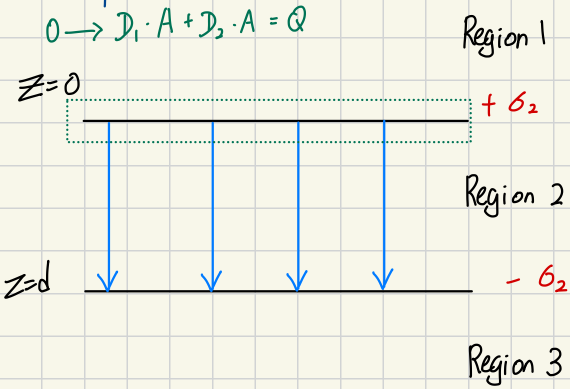
c) Calculate the electric field $\vec{E}$ in all regions (magnitude and direction).

a) $\vec{E}$

- sub problem 1

two plates with balanced surface charge density σ_2 and $-\sigma_2$:

$0 \rightarrow D_1 \cdot A + D_2 \cdot A = Q$



$$\vec{D} = \begin{cases} 0 & , z<0 \\ \sigma_2 \vec{I}_z & , 0<z<d \\ 0 & , d<z \end{cases}$$

$\iint \vec{D} da = Q(v)$

$\hookrightarrow D \cdot A = Q(v) \Rightarrow D_1 \cdot A + D_2 \cdot A = Q$

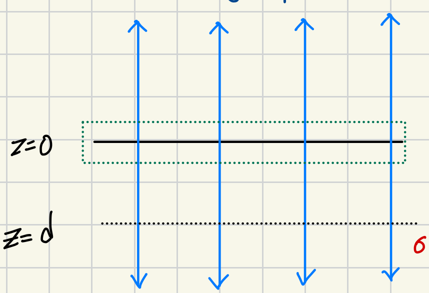
$\Rightarrow D_2 \cdot A = Q$

$\Rightarrow D_2 = \sigma_2$

- subproblem 2

one single plate with the charge density $(\sigma_x + \sigma_2 = \sigma_1)$

$\sigma_x = \sigma_1 - \sigma_2$



$D \cdot A = Q$

$\sigma_1 - \sigma_2$

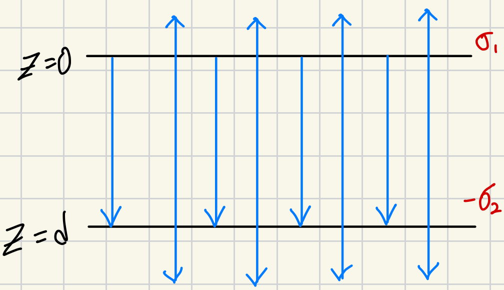
$D_1 \cdot A + D_2 \cdot A = Q$

$\sigma = 0 \quad D \cdot 2A = Q$

$\rightarrow D = \frac{Q}{2A} = \frac{\sigma}{2}$

$$\vec{D} = \begin{cases} -\frac{(\sigma_1 - \sigma_2)}{2} \vec{e}_z & , z<0 \\ \frac{\sigma_1 - \sigma_2}{2} \vec{e}_z & , 0<z<d \\ \frac{\sigma_1 - \sigma_2}{2} \vec{e}_z & , d<z \end{cases}$$

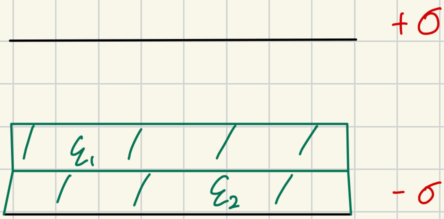
By superposition of electric fields we find :



$$\vec{D} = \begin{cases} -\dfrac{\sigma_1 - \sigma_2}{2}\vec{e_z} \\ \left(\dfrac{\sigma_1 - \sigma_2}{2} + \sigma_2\right)\vec{e_z} = \dfrac{\sigma_1 + \sigma_2}{2}\vec{e_z}, & 0<z<d \\ \dfrac{\sigma_1 - \sigma_2}{2}\vec{e_z} \end{cases}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon}$$

b)



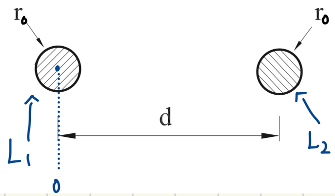
$$\vec{D} = \begin{cases} \sigma \cdot \vec{e_z} & , \ 0<z<d \\ 0 & , \ \text{otherwise} \end{cases}$$

c) Electric field

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \begin{cases} \dfrac{\sigma}{\varepsilon_0}\vec{e_z} & , \ 0<z<\frac{d}{4} \\ \dfrac{\sigma}{\varepsilon_0\varepsilon_1}\vec{e_z} & , \ \frac{d}{4}<z \leq \frac{3}{4}d \\ \dfrac{\sigma}{\varepsilon_0\varepsilon_2}\vec{e_z} & , \ \frac{3d}{4}<z<d \\ 0 & , \ \text{elsewhere} \end{cases}$$

Problem 12 Two-Wire Line

Calculate the capacitance per unit length of an infinitely long two-wire line consisting of two thin wires with radius r_0 and parallel distance d. Assume that $r_0 << d$.



- Electric line 1 (L1) with $+q$ charge per unit length

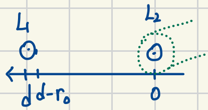
$$\int_{\partial V} \vec{D}\, d\vec{a} = 2\pi r \cdot L \cdot D_{1r}(r) = q \cdot L$$

$$\Rightarrow D_{1r}(r) = \frac{q}{2\pi r} \Rightarrow E_{1r} = \frac{D}{\varepsilon} = \frac{q}{2\pi\varepsilon r}$$

$$U_1 = \phi_1(r_0) - \phi_1(d-r_0) = \int_{r_0}^{d-r_0} E_{1r}\, dr = \frac{q}{2\pi\varepsilon} \ln\left(\frac{d-r_0}{r_0}\right)$$

- Electric line 2 (L2) with charge per unit length $-q$

$$\int_{\partial V_2} \vec{D}_2\, d\vec{a}_2 = 2\pi r L D_{2r}(r) = -q \cdot L$$



$$\Rightarrow E_{2r} = \frac{-q}{2\pi\varepsilon r}$$

$$U_2 = \phi_2(d-r_0) - \phi_2(r_0) = \int_{d-r_0}^{r_0} E_{2r}(r)\, dr = \frac{-q}{2\pi\varepsilon} \ln\left(\frac{r_0}{d-r_0}\right)$$

$$= \frac{q}{2\pi\varepsilon} \ln\left(\frac{d-r_0}{r_0}\right)$$

Difference potential between L_1 and L_2

$$U = U_1 + U_2 = \frac{q}{\pi\varepsilon} \ln\left(\frac{d-r_0}{r_0}\right) \overset{r_0 << d}{\approx} \frac{q}{\pi\varepsilon} \ln\left(\frac{d}{r_0}\right)$$

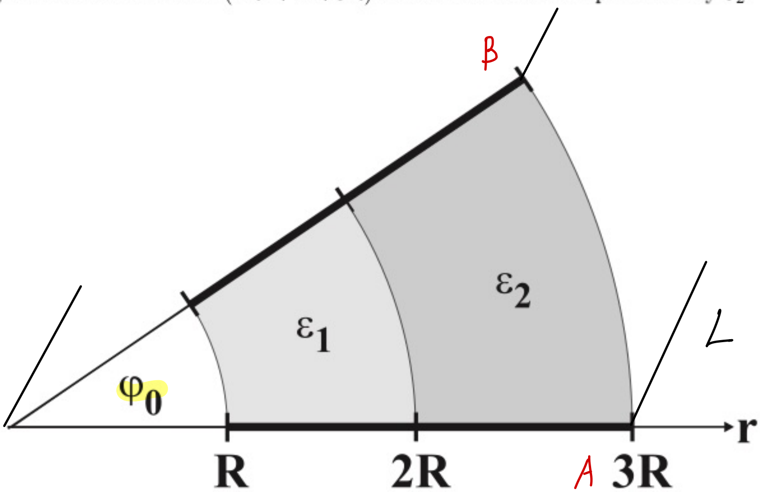
$$\hookrightarrow \underbrace{\frac{d}{r_0}}_{\hookrightarrow \gg 1} - \underbrace{\frac{r_0}{r_0}}_{\hookrightarrow 1}$$

Capacitance per unit length

$$C = \frac{q}{U} = \frac{q}{\frac{q}{\pi\varepsilon} \ln\left(\frac{d}{r_0}\right)} = \frac{\pi\varepsilon}{\ln\left(\frac{d}{r_0}\right)}$$

Problem 13 (Exam Problem 2009I-1)

A plate capacitor has the shape of a cylindrical segment with the angle φ_0 (see figure). Two perfectly conducting electrodes A and B, which extend in r-direction from R to $3R$, confine the segment. In z-direction the capacitor has the length L, which is very large compared to R. Hence, the problem can be reduced to the two dimensions (r, φ) in cylindrical coordinates. Stray fields at the boundary of the capacitor are negligible.
There are two dielectrics placed between the electrodes. The first dielectric ($R < r < 2R$) has the constant permittivity ε_1, the second dielectric ($2R < r < 3R$) has the constant the permittivity ε_2.



The electric potential $\Phi(\varphi)$ in the interior of the capacitor ($R < r < 3R$) reads in cylindrical coordinates:

$$\Phi(\varphi) = -K \cdot \varphi\,, \quad 0 \leq \varphi \leq \varphi_0\,,$$

where K is a positive constant.

*a) Sketch the equipotential lines and the electric field lines between the two electrodes (draw two separate plots). On which electrode is the positive charge located, and on which one the negative charge?

*b) Calculate the electric field $\vec{E}$ in the capacitor.

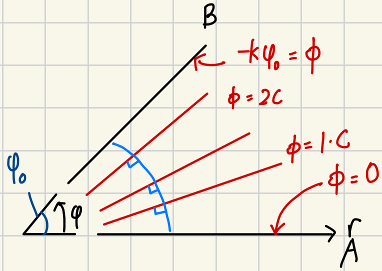
*c) Calculate the voltage U between the two electrodes.

d) Calculate (e.g., via Gauss's law) the charge Q on the lower electrode ($\varphi = 0$). To this end, determine the electric displacement field $\vec{D}(r)$ in the interior of the capacitor. Differentiate the regions of different permittivity.

e) Calculate the capacitance C of the configuration.

f) Calculate the electric energy W_{el} stored in the capacitor.

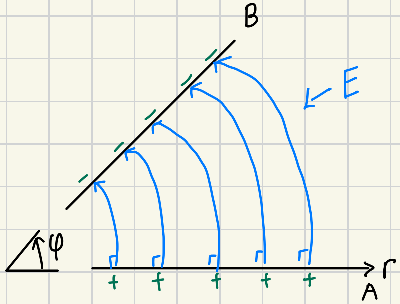
a)



$$\phi(\varphi) = -k\varphi$$

$$\text{Const.} = \phi(\varphi_c) = -k\varphi_c$$

equipotential lines



$$\vec{E} = -\nabla\phi$$

electric field lines

b) $\phi(\varphi) = -k\varphi$

$$\vec{E} = -\nabla\phi$$

$$\vec{E} = -\left(-\frac{k}{r}\right)\vec{e_\varphi}$$

$$= \frac{k}{r}\vec{e_\varphi}$$

Cylindrical Coordinates

$$\nabla\phi = \frac{\partial\phi}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\vec{e_\varphi} + \frac{\partial\phi}{\partial z}\vec{e_z}$$

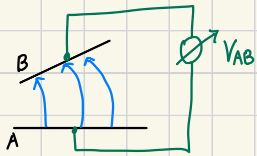
$$= 0\cdot\vec{e_r} + \frac{1}{r}\cdot(-k)\vec{e_\varphi} + 0\cdot\vec{e_z}$$

$$= -\frac{k}{r}\cdot\vec{e_\varphi}$$

c)

$$V_{AB} = \int_{P_A}^{P_B} \vec{E}\, d\vec{r} = \int_0^{\varphi_0} \frac{k}{r}\, \overbrace{\vec{e_\varphi}\, \vec{e_\varphi}}^{1}\, r\, d\varphi \qquad \underbrace{\vec{e_\varphi}\, r\, d\varphi}_{d\vec{r}}$$

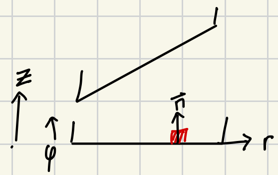
$$= \int_0^{\varphi_0} k\, d\varphi = k\varphi_0$$



d) Gauss law : $\int_{\partial V} \vec{D}\, d\vec{a} = Q(r)$

$d\vec{a} = dr\, dz\, \vec{e_\varphi}$

$\vec{D} = \varepsilon \vec{E}$



$$Q = \int_0^L \int_R^{2R} \varepsilon_1 \frac{k}{r} \vec{e_\varphi}\, \vec{e_\varphi}\, dr\, dz$$

$$+ \int_0^L \int_{2R}^{3R} \varepsilon_2 \frac{k}{r} \vec{e_\varphi}\, \vec{e_\varphi}\, dr\, dz$$

$$= L \cdot k \left(\varepsilon_1 [\ln(r)]_R^{2R} + \varepsilon_2 [\ln(r)]_{2R}^{3R} \right) \qquad \Big| \; \ln\left(\frac{2R}{R}\right) = \ln 2$$

$$= L \cdot k \left(\varepsilon_1 \ln 2 + \varepsilon_2 \ln\left(\frac{3}{2}\right) \right) \qquad \ln\left(\frac{3R}{2R}\right) = \ln\left(\frac{3}{2}\right)$$

e)

$$C = \frac{Q}{V} = \frac{L \cdot \cancel{k} \left(\varepsilon_1 \ln(2) + \varepsilon_2 \ln\left(\frac{3}{2}\right) \right)}{\cancel{k} \cdot \varphi_0}$$

$$= \frac{L}{\varphi_0} \left(\varepsilon \ln(2) + \varepsilon_2 \left(\ln \frac{3}{2} \right) \right)$$

f) $W_{el} = \frac{1}{2} C U^2$

$$= \frac{1}{2} \frac{L}{\varphi_0} \left(\varepsilon_1 \ln(2) + \varepsilon_2 \ln\left(\frac{3}{2}\right) \right) \cdot K^2 \cdot \varphi_0^2$$

alternative

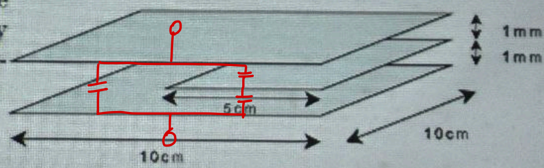
$$w_{el} = \frac{1}{2} \vec{E} \vec{D}$$

$$[w_{el}] = \left[\frac{1}{2} \vec{E} \vec{D}\right] = \frac{V}{m} \cdot \frac{As}{m}$$

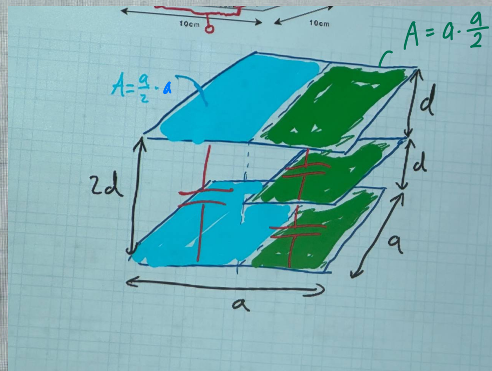
$$= \frac{J}{m^3}$$

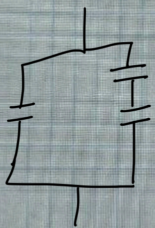
$$W_{el} = \int_V w_{el} \, dV = \int_V \frac{1}{2} \vec{E} \cdot \vec{D} \, dV$$

Q10 Calculate the capacitance between the upper and the lower metal plate of the given configuration. You may neglect boundary effects (stray fields). The permittivity between the plates is $\varepsilon = \varepsilon_0 = 8.85 \cdot 10^{-12}$ As/Vm.

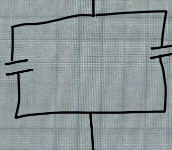


$$C = \varepsilon \cdot \frac{A}{d}$$





$\hat{=}$



C_1

$C_{23} = C_2$ series C_3

$$C_{23} = \frac{1}{\frac{1}{C_2} + \frac{1}{C_3}} = \frac{1}{\frac{2}{C_2}} = \frac{C_2}{2} = \varepsilon \frac{a^2}{4d}$$

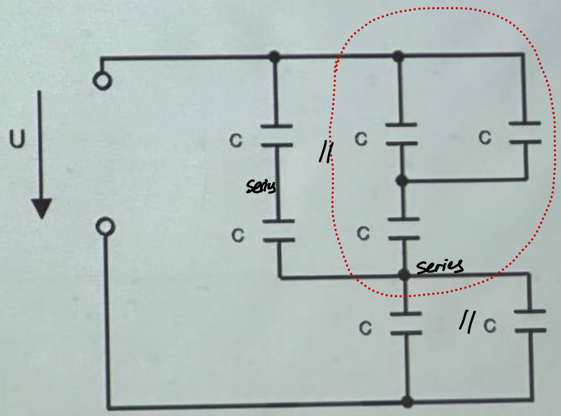
$$C_1 = \varepsilon \cdot \frac{A}{2d} = \varepsilon \frac{a^2/2}{2d} = \varepsilon \frac{a^2}{4d}$$

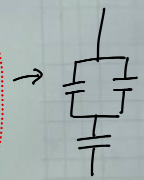
$$C_2 = \varepsilon \cdot \frac{a^2/2}{d} = \varepsilon \frac{a^2}{2d} = C_3$$

$$C_{total} = C_1 // C_{23} = C_1 + C_{23}$$

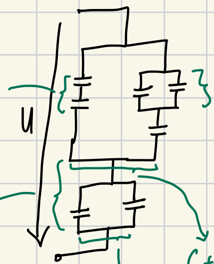
$$= \frac{1}{4} \varepsilon \frac{a^2}{d} + \frac{1}{4} \varepsilon \frac{a^2}{d} = \frac{1}{2} \varepsilon \frac{a^2}{d} = 44.3\,pF$$

Q11 Determine the total capacitance of the following capacitor circuit.





$$C_{tot} = \frac{1}{\frac{1}{C}+\frac{1}{C}} = \frac{C}{2}$$



$$C_{tot} = C + C = 2C$$

$$C_{tot} = \frac{1}{\frac{1}{2C}+\frac{1}{C}} = \frac{2C}{3}$$

$$C_{tot} = \frac{C}{2} + \frac{2C}{3} = \frac{7C}{6}$$

$2C$

$$C_{tot} = \frac{1}{\frac{6}{7C}+\frac{1}{2C}} = \frac{1}{\frac{12}{14C}+\frac{7}{14C}} = \frac{14C}{19}$$

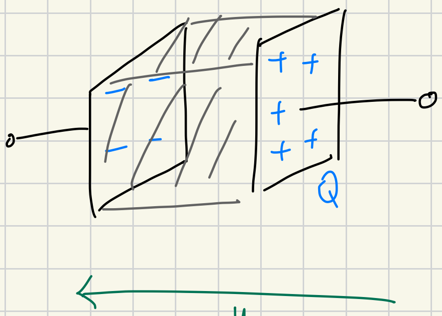
Q12 A perfect plate capacitor has been charged in free space with a charge Q at a voltage U. Then a dielectric material with relative permittivity $\varepsilon_r = 2$ is filled between the capacitor plates. What is the change of the electric energy stored in the capacitor, if

a) the voltage U between the plates is kept constant

b) the charge Q on the plates is constant.

① Plate Capacitor is Charged with Q at the voltage U

② Dielectric material With $\varepsilon_r = 2$



$W_{el} = ?$

$$W_{el} = \frac{1}{2} C U^2 \underset{\text{with } C = \frac{Q}{U}}{=} \frac{1}{2} Q U \underset{U = \frac{Q}{C}}{=} \frac{1}{2}\frac{Q^2}{C}$$

$C = \varepsilon_0 \varepsilon_r \frac{A}{d}$ only Change from ① → ②

$\varepsilon_r = 1 \rightarrow \varepsilon_r = 2$

a) $U = \text{Constant} = U_0$

$C_1 = \varepsilon_0 \varepsilon_1 \frac{A}{d}$ $\quad C_2 = \varepsilon_0 \varepsilon_2 \frac{A}{d}$

$C_1 = \varepsilon_0 \frac{A}{d}$ $\quad C_2 = 2 \varepsilon_0 \frac{A}{d}$

$W_{el_2} = \frac{1}{2} C_2 U_0^2$

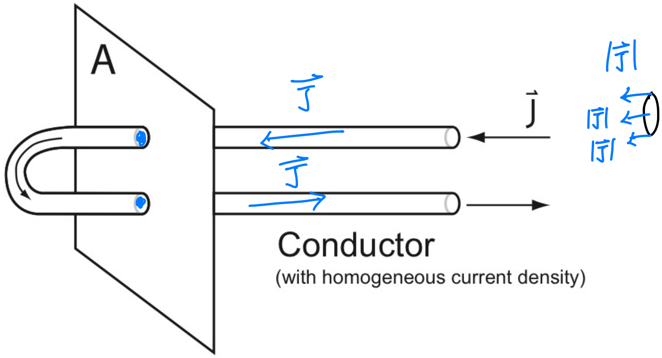
$\Rightarrow W_{el_2} = 2 \cdot W_{el_1}$

$W_{el_1} = \frac{1}{2} C_1 U_0^2$

b) $Q = \text{Const. } Q_0$

$$W_{el_2} = \frac{1}{2}\frac{Q_0^2}{C_2} = \frac{1}{2}\frac{Q_0^2}{2 \cdot C_1} = \frac{1}{2} W_{el_1}$$

Q14 A curved U-shaped conductor carries a uniform current density (see figure). There is no current outside the conductor.



a) What is the meaning of the term "uniform current density"?

b) What is the total current I flowing through the area A?

c) What will change (w.r.t. the total current I), if the current density in the conductor is not uniform, but still stationary?

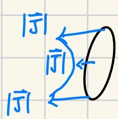
a) Uniform current density : $|\vec{J}|$ is Constant in space

b) Total Current through A : $I = 0$

(inflow = outflow)

c) Stationary but not uniform

(t = Const.)



still, inflow = outflow

$|\vec{J}|_{upper} = |\vec{J}|_{lower}$

Q15 Assume that several species of charge carriers contribute to the current flow in a conductor. What is the impact of the specific charge of the carries on the electric conductivity?

a) none

b) only electrons contribute to the electric conductivity

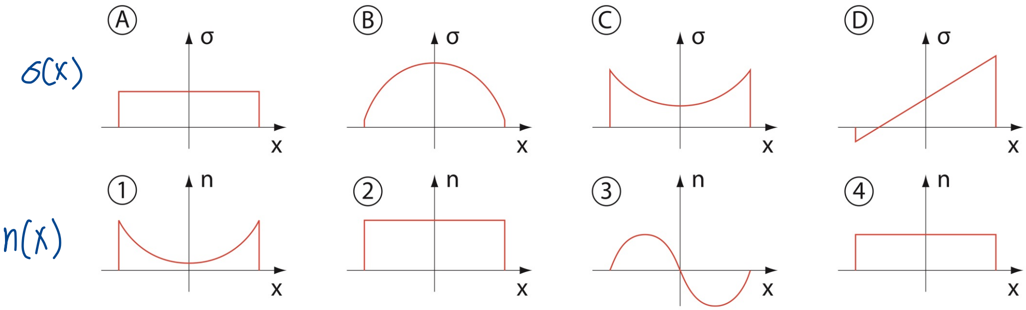
c) the algebraic sign of the specific charge is important

d) only the absolute value of the specific charge is relevant

$$\sigma = \underbrace{\sum_{\alpha=1}^{k}}_{\text{Charge Carrier}} |q_\alpha|\, n_\alpha \cdot \mu_\alpha$$

$\Rightarrow$ d) is Correct

Q16 Which combinations of (y, z), $y \in \{A, B, C, D\}$, $z \in \{1, 2, 3, 4\}$ of charge carrier density $n(x)$ and conductivity $\sigma(x)$ are physically reasonable? Assume that only one species of charge causes the electric current, and assume a constant carrier mobility. Give a reason for your answer.



- (A,4)
- (C,1)
- (A,2)

$$\sigma(x) = \cancel{\sum_{\alpha=1}^{k}} |q_\alpha| \cdot n_\alpha(x) \cdot \underbrace{\mu_\alpha}_{\text{Const.}}$$

$$\sigma(x) \sim n(x)$$

Problem 14 Gas Discharge

In an electric gas discharge, an electron and an ion density of $n_e = n_i = 10^{15}\,\mathrm{1/cm^3}$ are measured. The ions have a positive charge (single-charged particles). In the interior of the discharge region, there is an electric field $E = 1\,\mathrm{V/cm}$. The mobilities of electrons and ions are:

$$\mu_e = \frac{v_e}{E} = 10^4\,\mathrm{cm^2/Vs} \qquad\qquad \mu_i = \frac{v_i}{E} = 1\,\mathrm{cm^2/Vs}$$

a) Find the velocity (magnitude and direction) of electrons and ions with reference to the electric field $\vec{E}$.

b) Find the current density $\vec{j}$. In which direction is $\vec{j}$ oriented?

c) Calculate the electric conductivity σ.

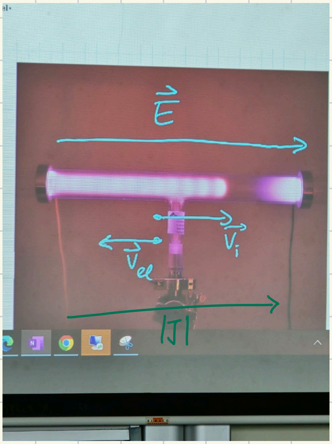
d) Calculate the electric power density p_{el}.

a) $\vec{V_e} = -\mu_e \vec{E}$

$\vec{V_i} = \mu_i \vec{E}$

$\vec{V_e} = -10^4 \frac{\mathrm{cm^2}}{\mathrm{Vs}} \cdot 1 \frac{\mathrm{V}}{\mathrm{cm}} \vec{e_x}$ $\qquad$ $\vec{V_i} = 1 \frac{\mathrm{cm^2}}{\mathrm{Vs}} \cdot 1 \frac{\mathrm{V}}{\mathrm{cm}} \vec{e_x}$

$= -10^4\,\mathrm{cm/s}\,\vec{e_x}$ $\qquad$ $= 1 \frac{\mathrm{cm}}{\mathrm{s}} \vec{e_x}$



b) $\vec{J} = n_e \cdot \vec{V_e} \cdot (-e) + n_i \cdot \vec{V_i} \cdot (+e)$

$|\vec{J}| = n_e |\vec{E}| \mu_e \cdot e + n_i |\vec{E}| \mu_i \cdot e$ $\qquad | \; n_e = n_i = n$

$= n |\vec{E}| e (\mu_e + \mu_i)$

$= 10^{15} \frac{1}{\mathrm{cm^3}} \, 1 \frac{\mathrm{V}}{\mathrm{cm}} \cdot 1.6 \cdot 10^{-19}\,\mathrm{As} \cdot 10001 \frac{\mathrm{cm^2}}{\mathrm{Vs}}$

$= 1.6 \frac{\mathrm{A}}{\mathrm{cm^2}}$

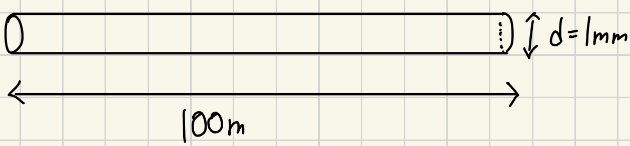
c) $\vec{J} = \sigma \vec{E} \Rightarrow \sigma = \frac{|\vec{J}|}{|\vec{E}|} = 1.6 \frac{A}{Vcm} = 160 \frac{S}{m}$

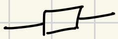
$S = \frac{1}{\Omega}$

d) Electric power loss density

$P_{el} = \vec{J} \cdot \vec{E} = \sigma \cdot |\vec{E}|^2 = \frac{|\vec{J}|^2}{\sigma} = 1.6 \frac{W}{cm^3}$

Q17 An aluminum wire with length 100 m and diameter 1 mm has an electric resistance of 3.5 Ω. The mobility of the mobile electrons in aluminum is $\mu_n = 1.26 \cdot 10^{-3}\,\mathrm{m^2/Vs}$. What is the density of mobile electrons in aluminum? What is the number of mobile electrons per atom? (Density of aluminum is $2.7 \cdot 10^3\,\mathrm{kg/m^3}$; there are $N_A = 6 \cdot 10^{23}$ atoms in 27 g aluminum, elementary charge $e_0 = 1.6 \cdot 10^{-19}$ As)





Density of electrons: $\left[\frac{1}{cm^3}\right]$

$R = \frac{1}{\sigma} \frac{l}{A}$

Conductivity

$\sigma = e_0 n \mu_n$ elementary Charge $e_0 = 1.6 \cdot 10^{-19}$ C

$\Rightarrow R = \frac{1}{e_0 n \mu_n} \frac{l}{A} \Rightarrow n = \frac{l}{R A e_0 \mu_n}$

With $R = 3.5\,\Omega$, $A = \pi\left(\frac{d}{2}\right)^2$, $\mu_n = 1.26 \times 10^{-3} \frac{m^2}{Vs}$

$$\Rightarrow n = \frac{100m}{3.5\frac{V}{A} \cdot \pi \cdot \left(\frac{10^{-3}}{2}\right)^2 m^2 \cdot 1.6 \cdot 10^{-19} As \cdot 1.26 \cdot 10^{-3} \frac{m^2}{Vs}}$$

$= 1.8 \cdot 10^{23} \frac{1}{cm^3}$

Number of mobile (free) electrons per atom:

Volume

$$N_e = \frac{n \cdot V}{N_A} = \frac{n \cdot m}{N_A \cdot \rho} = 3$$

$m = \rho \cdot V$ mass density

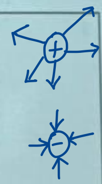
$\Rightarrow V = \frac{m}{\rho}$

Q1 (2 points)

State Gauss' law in differential formulation. Give a short physical interpretation of this relation.

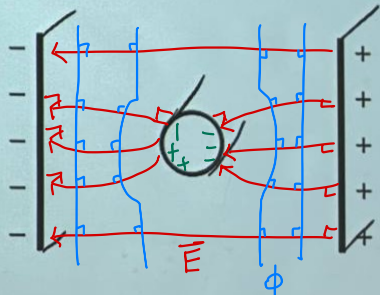
$$\boxed{\operatorname{div} \vec{D} = \rho}$$

source density of the displacement field $\vec{D}$ is equal to space charge density ρ (→ "source of the $\vec{D}$ field = space charge density")



Q2 (4 points) (... field = space charge density")

Imagine a plate capacitor with a cylindrical, ideal conductor located in its center. The capacitor is charged as shown in the figure. The total charge on the conductor inside is zero.



*a) Draw a qualitative sketch of the electrical field lines inside the plate capacitor (use figure above).

*b) What is the angle of intersection between the electric field lines and the surface of the ideal conductor?

90°

*c) Draw a qualitative sketch of the charge distribution on the surface of the cylindrical conductor (use figure above).

Q3 (3 points)

Consider a point charge q in an electrostatic field $\vec{E}$. Calculate the work W_{el} performed by the electric field, while the charge is moved along a curve C from position P1 to position P2. Which property of the electrostatic field ensures that the work is independent of the choice of the path connecting P1 and P2?

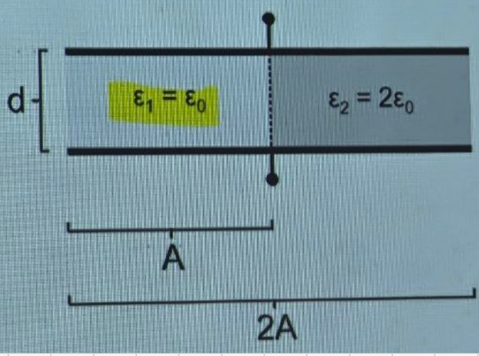
$$W = \int_{P_1}^{P_2} \vec{F}_{el}\, d\vec{r} = \int_{P_1}^{P_2} q \cdot \vec{E} \cdot d\vec{r} = q \int_{P_1}^{P_2} \vec{E}\, dr$$

Electro static field is conservative

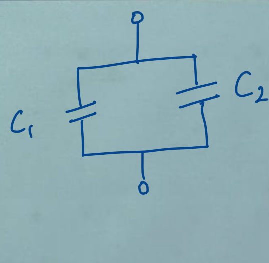
$\hat{=}$ work is independant of the path

Q4 (5 points)

Imagine a plate capacitor consisting of two equally-sized regions with different permittivities $\varepsilon_1 = \varepsilon_0$ and $\varepsilon_2 = 2\varepsilon_0$ as shown in the figure below. The area of each of the two regions is A and their thickness is d. Stray fields may be neglected.



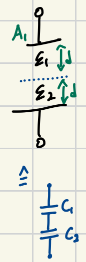
*a) This configuration can be described by an equivalent circuit of two capacitors. Draw a proper equivalent circuit of the total plate capacitor.



b) Calculate the total Capacitance C_{tot} of the plate capacitor

$C = \varepsilon \frac{A}{d}$ $\rightarrow$ $C_1 = \varepsilon_1 \frac{A}{d}$

$C_2 = \varepsilon_2 \frac{A}{d}$



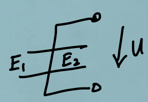
$C_{tot} = C_1 + C_2 = (\varepsilon_1 + \varepsilon_2) \frac{A}{d}$

$= 3\varepsilon_0 \frac{A}{d}$

$C_{total} = \frac{1}{C_1 + C_2}$ serial Connection

*c) Now a voltage U is applied to the capacitor plates. In which of the two regions is the stored electric energy W_{el} larger? Give a reason for your answer.

$W_{el} = \frac{1}{2} C U^2 = \frac{1}{2} \varepsilon \frac{A}{d} U^2$

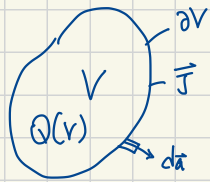


$W_{el} \sim \varepsilon$

Since $\varepsilon_2 > \varepsilon_1 \rightarrow W_{el,2} > W_{el,1}$

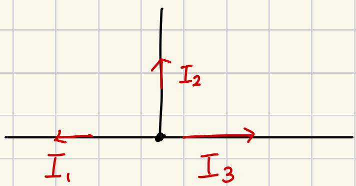
Q18 Formulate the "law of charge conservation" as an integral balance equation. Write a mathematical equation which relates the electric current flux through a closed surface ∂V to the charge $Q(V)$ in the enclosed volume V.

$\vec{J}$ $\quad Q(V)$

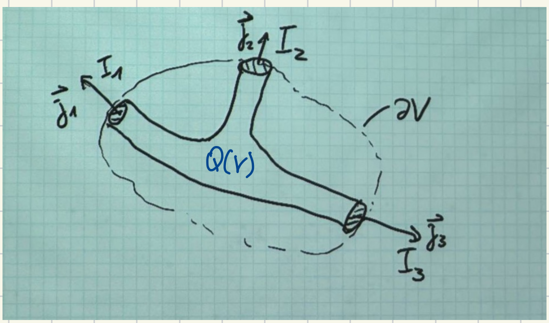


$$\Rightarrow \underbrace{\int_{\partial V} \vec{J}\, d\vec{a}}_{\text{net flux of Charges flowing out the Volume}} = \underbrace{-\frac{dQ(V)}{dt}}_{\text{Change in charge } Q \text{ over time in } V}$$

Q19 Give a physical interpretation of Kirchhoff's current law (referring to charge conservation in a physical conductor node in the stationary case).



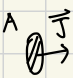
Kirchhoff : $I_3 = -I_1 - I_2$



Stationary

$$\boxed{\frac{dQ(V)}{dt} = 0}$$

$$\Rightarrow \int_{\partial V} \vec{J} \cdot d\vec{a} = 0$$

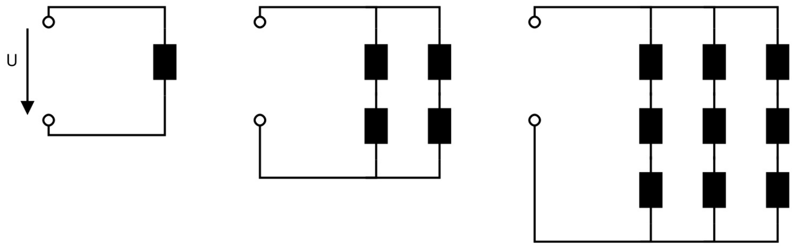


$\rightarrow I = \vec{J} \cdot \vec{A}$

$$\Rightarrow \int_{\partial V} \vec{J}\, d\vec{a} = \sum_i \int_{A_i} \vec{J}\, d\vec{a}$$

$$= \underbrace{\sum_i I_i = 0}_{\text{Kirchhof's Current Law}}$$

Q20 The resistance of a circuit with applied voltage U is stepwise extended as shown in the figure below.



Which statement is true? The power consumption of the circuit is stepwise

a) halved b) doubled c) increased by $\sqrt{2}$ d) unchanged

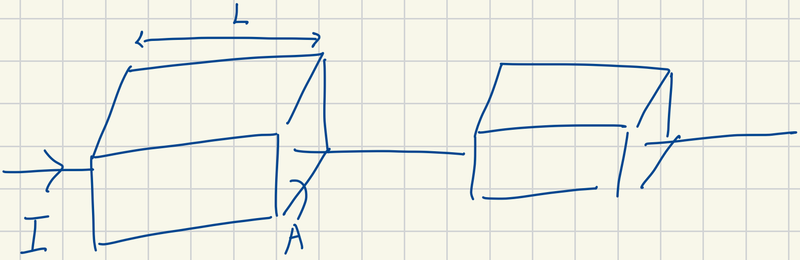
① $R_{tot} = R$

② $R_{tot} = \left(\frac{1}{2R} + \frac{1}{2R}\right)^{-1} = R$

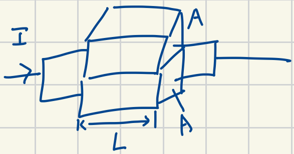
③ $R_{tot} = \left(\frac{1}{3R} + \frac{1}{3R} + \frac{1}{3R}\right)^{-1} = R$

$P = I \cdot V = R \cdot I^2$ (with $U = RI$

↳ $P \sim R$



$R = \frac{1}{\sigma} \cdot \frac{L}{A}$ $\Rightarrow$ $R_{tot} = \frac{1}{\sigma} \cdot \frac{2L}{A} = 2R$



$$R_{tot} = \frac{1}{\sigma}\cdot\frac{L}{2A} = \frac{1}{2}R$$
$$= \left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-1}$$

Q21 What is the differential work performed on a charge carrier in an electric field $\vec{E}$? Derive from this expression the electric power density rendered by an electric system with current density $\vec{j}$.

1) $$W_{el} = \int \vec{F}_{el}\, d\vec{r} = q\cdot\int \vec{E}\, d\vec{r}$$
$$= \int dW_{el}$$

$$dW_{el} = q\vec{E}\, d\vec{r}$$

2) el. power density

$$p_{el} = \frac{P_{el}}{Vol}$$

- $$P_{el} = \frac{dW_{el}}{dt} = q\vec{E}\underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}\ \text{velocity}} = q\vec{E}\vec{v}$$

$$[P_{el}] = W = \frac{J}{s}$$

- $$p_{el} = \underbrace{n}_{\text{Charge carrier density } [\frac{1}{m^3}]}\cdot P_{el} = n\cdot q\,\vec{E}\vec{v} = \vec{J}\cdot\vec{E}$$

$$\vec{J} = n\, q\cdot\vec{v}$$

$$[\vec{J}] = \frac{A}{m^2}$$

$$[p_{el}] = \frac{VA}{m^3} = \frac{W}{m^3}$$

$$[\vec{E}] = \frac{V}{m}$$

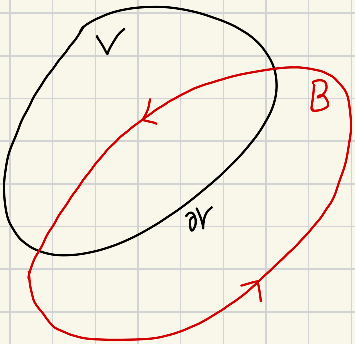
Q24 Do magnetic monopoles exist? Justify your answer by referring to Maxwell's equations.

No: $\boxed{\int_{\partial V} \vec{B}\, d\vec{a} = 0} \xrightarrow{} \int_V \operatorname{div} \vec{B}\, dv = 0$

↑ Integral Gauss Theorem

$\int_{\partial V} \vec{X}\, d\vec{a} = \int_V \operatorname{div} \vec{X}\, dv$

↑ can be electric field / magnetic field



$\Rightarrow \boxed{\operatorname{div} \vec{B} = 0}$

$\hat{=}$ $\vec{B}$-field is solenoid

→ no magnetic monopole

Q25 A voltage U_{ind} is induced by a time-variant magnetic field $\vec{B}(\vec{r}, t)$ in an almost closed time-variant conductor loop $C(t) = \partial A(t)$. Which of the following statements are true?

false a) $U_{\text{ind}} = 0$, if the magnetic field does not vary with time. → B const., $A = A(t)$

false b) $U_{\text{ind}} = 0$, if the magnetic field varies with time, but the conductor loop is at rest. → $B = B(t)$, $A =$ const.

true c) $U_{\text{ind}} = -\frac{d}{dt} \int_{A(t)} \vec{B}(\vec{r}, t) d\vec{a}$ $\quad U_{ind} = -\frac{d\phi}{dt} \quad \phi = \int_{A(t)} \vec{B}(t)\, d\vec{a}$

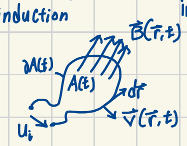
false d) $U_{\text{ind}} = -\int_{A(t)} \frac{\partial \vec{B}}{\partial t}(\vec{r}, t) d\vec{a}$ $\quad \forall \int_{\partial A(t)} (\vec{v} \times \vec{B})\, d\vec{r}$

true e) $U_{\text{ind}} = -\int_{A(t)} \frac{\partial \vec{B}}{\partial t}(\vec{r}, t) d\vec{a} + \int_{\partial A(t)} \vec{v}(\vec{r}, t) \times \vec{B}(\vec{r}, t) d\vec{r}$

Formulary: $V_{ind} = -\frac{d\Phi_{mag}}{dt}$

$\Phi_{mag} = \int_{A(t)} \vec{B}(t) . d\vec{a}$

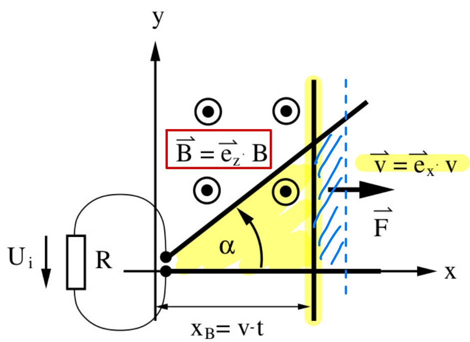
$$U_{ind} = \underbrace{-\int_{A(t)} \frac{\partial B}{\partial t} \cdot d\vec{a}}_{\text{motionless induction}} + \underbrace{\int_{\partial A(t)} (\vec{v} \times \vec{B})\, d\vec{r}}_{\text{motional induction}}$$



Velocity of point $\vec{r}$ on the wire loop

Problem 22 Slide-Wire

Two straight wires are placed in the x-y-plane as shown in the figure. One of them is aligned along the x-axis, the other one is inclined by an angle α, so that both wires enclose an angular segment in the x-y-plane. At the origin they nearly meet and are connected to an external resistor with ohmic resistance R. A third straight wire slides on them; it is aligned parallel to the y-axis and is moved with constant velocity v in the x-direction. Its momentary position is $x_B(t) = v\cdot t$. The x-y-plane is penetrated by a uniform, constant magnetic field $\vec{B} = B \cdot \vec{e}_z$. All wires are perfect conductors with no resistance.



The magnetic field generated by the current through the wire loop is negligible. Determine as a function of time:

a) the voltage U_i on the resistor,

b) the current I flowing through wire loop and resistor,

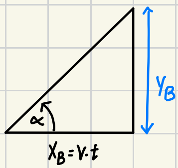
c) the magnetic force $\vec{F}_M$ acting on the moving slide-wire.

a) $U_i = -\dfrac{d\phi}{dt}$

$$\phi = \int_{A(t)} \vec{B}\cdot d\vec{a} = \int_{A(t)} B \cdot \underbrace{\vec{e}_z \cdot \vec{e}_z}_{1}\, da$$

$$= B \cdot A(t)$$

$$\Rightarrow \phi(t) = B \cdot \frac{1}{2} v^2 t^2 \tan\alpha$$



$$\tan\alpha = \frac{Y_B}{v\cdot t}$$

$$\Rightarrow Y_B = v\cdot t \tan\alpha$$

$$\Rightarrow U_i = -B\cdot \frac{1}{2} v^2\, 2\cdot t \tan\alpha$$

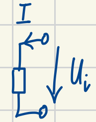
$$= -B\cdot v^2 \cdot t \cdot \tan\alpha$$

$$A(t) = \frac{1}{2} X_B \cdot Y_B$$

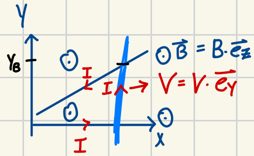
$$= \frac{1}{2} v^2 t^2 \tan\alpha$$

b) Current:

$$I = \frac{U_i}{R} = -\frac{V^2 t\, B \tan\alpha}{R}$$



c) Magnetic force:



$$d\vec{F_M} = I\cdot d\vec{l} \times \vec{B}$$

$$\vec{F_M} = \int_{C(t)} d\vec{F_M} = \int_0^{Y_B} I\cdot B\underbrace{(\vec{e_Y}\times\vec{e_Z})}_{\vec{e_X}}\,dy$$

$$= I\cdot B\vec{e_X}\int_0^{Y_B} dy$$

$$Y_B = V\cdot t\cdot \tan\alpha$$

$$I = -\frac{V^2 t\, B\tan\alpha}{R}$$

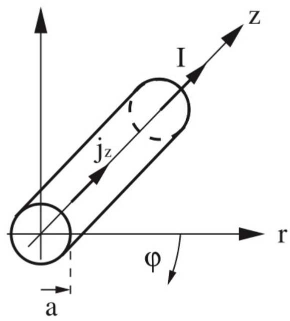
$$= I\cdot B\, Y_B\vec{e_X}$$

$$= \left(-\frac{V^2 t\, B\tan\alpha}{R}\right)\cdot B\cdot Vt\tan\alpha\,\vec{e_X}$$

$$= \frac{-B^2\cdot V^3 t^2 \tan^2\alpha}{R}\,\vec{e_X}$$

Problem 20 Gas Discharge Tube

The interior of a cylindrically symmetric gas discharge tube (radius $a = 1\,\text{cm}$) is filled with a conductive plasma of charge carries, which flow parallel to the cylinder axis ($=$ z-axis). The electric current density has only a z-component which depends on the radial coordinate r as follows:



$$j_z(r) = \begin{cases} j_0\left(1 - \dfrac{r^2}{a^2}\right) & \text{for} \quad r \le a \\ 0 & \text{for} \quad r > a \end{cases} \qquad \text{with} \quad j_0 = 20\,\frac{\text{A}}{\text{cm}^2}$$

a) What is the total current I flowing through the discharge tube?

b) Calculate the magnetic field strength $H_\varphi(r)$ inside the discharge tube, i.e., for $0 < r \le a$, and outside, i.e., for $r > a$.

c) At which distance r_m from the cylindrical axis does $H_\varphi(r)$ attain its maximum value?

d) Find the value of $H_\varphi(a)$ and $H_\varphi(r_m)$.

e) Plot $H_\varphi(r)$ as a function of r.

a) $I = \int_A \vec{j}\, d\vec{a} = \int_A \vec{j}\, \vec{e}_z\, r\, d\varphi\, dr$

$I = \int_0^a \int_0^{2\pi} j_0\left(1 - \frac{r^2}{a^2}\right) \underbrace{\vec{e}_z \cdot \vec{e}_z}_{1}\, r\, d\varphi\, dr$

$= 2\pi j_0 \int_0^a \left(r - \frac{r^3}{a^2}\right) dr$

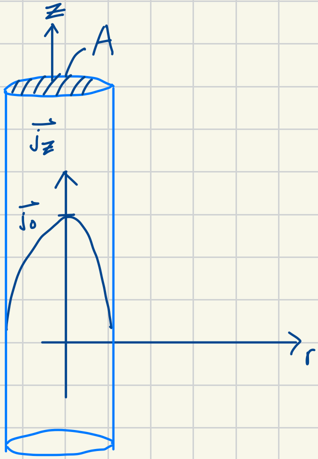
$= 2\pi j_0 \left[\frac{1}{2}r^2 - \frac{r^4}{4a^2}\right]_0^a$

$= 2\pi j_0 \left[\frac{a^2}{2} - \frac{a^2}{4}\right]$

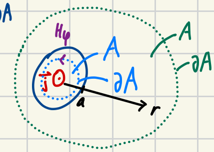
$= \frac{\pi}{2} j_0 a^2$

$= \frac{\pi}{2} 20 \frac{A}{cm^2} \cdot 1\, cm^2$

$= 31.4\, A$



b) $\vec{H} = H_\varphi(r)\,\vec{e}_\varphi$: $\int_{\partial A} \vec{H}\,d\vec{r} = I(A)$ enclosed current



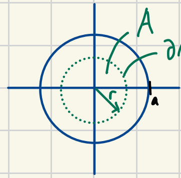
$$\int_{\partial A} \vec{H}\,d\vec{r} = \int_0^{2\pi} H_\varphi(r)\,\vec{e}_\varphi\,\vec{e}_\varphi\,r\,d\varphi$$
$$= 2\pi r\,H_\varphi(r)$$
$$= I_{encl}$$
$$\Rightarrow H_\varphi(r) = \frac{I_{encl}}{2\pi r}$$

region 2:

$$I_{encl} = I_{tot} = \frac{\pi}{2} j_0 a^2$$
$$\Rightarrow H_\varphi(r) = \frac{\frac{\pi}{2} j_0 a^2}{2\pi r}$$
$$= \frac{j_0 a^2}{4r}$$
$$H_\varphi(r) \sim \frac{1}{r}$$

region 1 : inside the tube

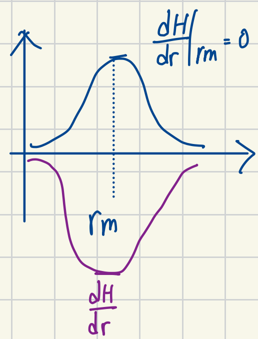
$$0 \leq r \leq a$$



$$I(r) = \int_0^{2\pi}\int_0^{r} j_0\left(1 - \frac{r'^2}{a^2}\right) r'\,dr'\,d\varphi$$
$$= 2\pi j_0 \int_0^r \left(r' - \frac{r'^3}{a^2}\right) dr'$$
$$= 2\pi j_0 \left[\frac{r'^2}{2} - \frac{r'^4}{4a^2}\right]_0^r$$
$$= 2\pi j_0 \left(\frac{r^2}{2} - \frac{r^4}{4a^2}\right)$$
$$\rightarrow H_\varphi(r) = \frac{I(r)}{2\pi r} = j_0\left(\frac{r}{2} - \frac{r^3}{4a^2}\right)$$

c) Max value for $H(r)$

$$\frac{dH}{dr} = j_0\left(\frac{1}{2} - \frac{3r^2}{4a^2}\right) \overset{!}{=} 0$$



$$\rightarrow r_m^2 = \frac{2}{3}a^2$$
$$\Rightarrow r_m = \sqrt{\frac{2}{3}}\cdot a$$

$\frac{d^2H}{dr^2} < 0$ for maximum

$$\frac{d^2H}{dr^2} = j_0\left(0 - \frac{3r}{2a^2}\right) = -j_0\frac{3r}{2a^2}$$
$$\left.\frac{d^2H}{dr^2}\right|_{r_m} = -j_0\frac{3}{2}\frac{\sqrt{\frac{2}{3}}a}{a^2} < 0 \quad \Rightarrow \text{Maximum} \checkmark$$

d) $H_\varphi(r=a) = 5\,\frac{A}{cm}$

$H_\varphi(r=r_m) = 5.44\,\frac{A}{cm} = \dot{j}_0\left(\frac{\sqrt{\frac{2}{3}}\cdot a}{2} - \frac{\left(\sqrt{\frac{2}{3}}\cdot a\right)^3}{4a^2}\right)$

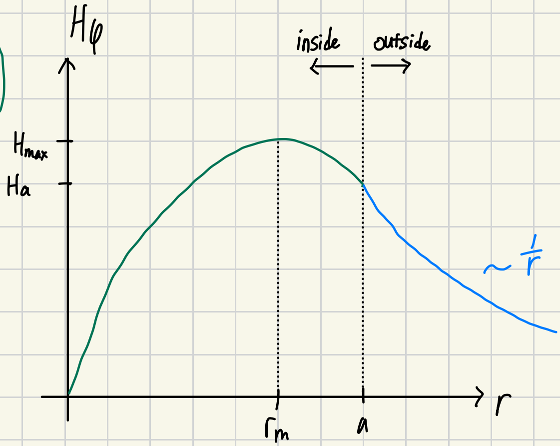
region 1:

e) $\boxed{H_\varphi(r) = \dot{j}_0\left(\frac{r}{2} - \frac{r^3}{4a^2}\right)}$

for $0 < r < a$

$\boxed{H_\varphi(r) = \dot{j}_0\,\frac{a^2}{4}\,\frac{1}{r}}$

for $r > a$

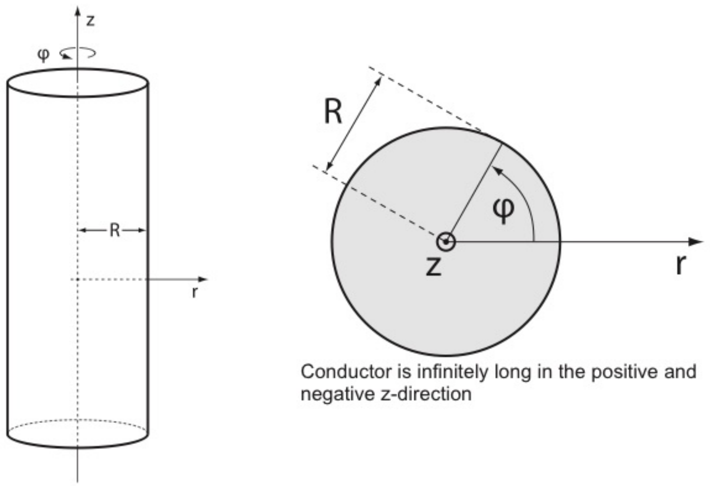


Problem 24 (Exam Problem 2009I-3)

A cylindrical conductor with radius R has an infinite long extension along the $\pm z$-direction (see figure). The conductor carries a time-variant current in $+\vec{e}_z$-direction. The current density $\vec{j}(r,t)$ in the conductor is given in cylindrical coordinates:

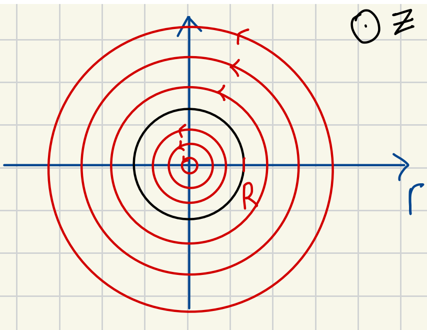
$$\vec{j}(r,t) = \begin{cases} j_0 \cdot \left(\frac{r}{R}\right)^2 \cdot \sin(2\pi t/T) \cdot \vec{e}_z & \text{for } 0 \le r < R \\ 0 & \text{for } r \ge R \end{cases}$$

Assume for all calculations that the conductor is placed in free space (i.e., $\mu = \mu_0$ = vacuum permeability).



Conductor is infinitely long in the positive and negative z-direction

*a) Sketch the magnetic field lines in a plane perpendicular to the z-direction. Also sketch the direction of the current. Differentiate between the regions inside and outside of the conductor.



*b) Calculate the total current I flowing through the conductor by integration over its cross section.

$$\vec{j}(r,t) = \begin{cases} j_0 \cdot \left(\frac{r}{R}\right)^2 \cdot \sin(2\pi t/T) \cdot \vec{e}_z & \text{for } 0 \le r < R \\ 0 & \text{for } r \ge R \end{cases}$$

$$I = \int_A \vec{J} \cdot d\vec{a}$$

$$I = \int_0^{2\pi}\int_0^R j_0 \left(\frac{r}{R}\right)^2 \cdot \sin(2\pi t/T) \cdot \vec{e}_z \cdot r\,dr\,d\varphi \cdot \vec{e}_z = 2\pi \int_0^R j_0 \left(\frac{r}{R}\right)^2 \cdot \sin(2\pi t/T) \cdot r\,dr$$

$$= 2\pi j_0 \sin\left(\frac{2\pi t}{T}\right) \int_0^R \frac{r^3}{R^2}\,dr = 2\pi j_0 \sin\left(\frac{2\pi t}{T}\right)\left[\frac{1}{4}\frac{r^4}{R^2}\right]_0^R$$

$$= 2\pi j_0 \sin\left(\frac{2\pi t}{T}\right)\left(\frac{1}{4}\frac{R^4}{R^2}\right) = \frac{\pi}{2} j_0 R^2 \sin\left(\frac{2\pi t}{T}\right)$$

c) Determine the magnitude and the direction of the magnetic field strength $\vec{H}(r,\varphi,t)$ by using Ampère's circuital law.

$$\int_{\partial A = C} \vec{H}\,d\vec{r} = \int_A \vec{j}(\vec{r})\,d\vec{a} = I_{encl}(A)$$

Take into account 2 Regions : ① $0 \le r \le R$; ② $r \ge R$

$\boxed{\text{Region 2}}$:

$$\int_{\partial A = C} \vec{H}\,d\vec{r} = I_{tot} \quad \text{for } r \ge R$$

$$\int_0^{2\pi} H_\varphi \cdot \vec{e}_\varphi \cdot r\,d\varphi = 2\pi r H_\varphi = \frac{\pi}{2} j_0 R^2 \cdot \sin\left(\frac{2\pi t}{T}\right)$$

$$H_\varphi = \frac{1}{4} \cdot j_0 \cdot R^2 \cdot \frac{1}{r} \cdot \sin\left(\frac{2\pi t}{T}\right) \quad \text{Outside Conductor}$$

$\boxed{\text{region 1}}$: $0 \le r \le R$ $\qquad 2\pi r H_\varphi = I_{encl.}(A)$

$$H_\varphi = \frac{I_{encl.}(A(r))}{2\pi r}$$

$$I_{encl}(A(r)) = \int_0^{2\pi}\int_0^r j_0 \left(\frac{r'}{R}\right)^2 \sin\left(\frac{2\pi t}{T}\right) \vec{e}_z \cdot r'\,dr'\,d\varphi \cdot \vec{e}_z$$

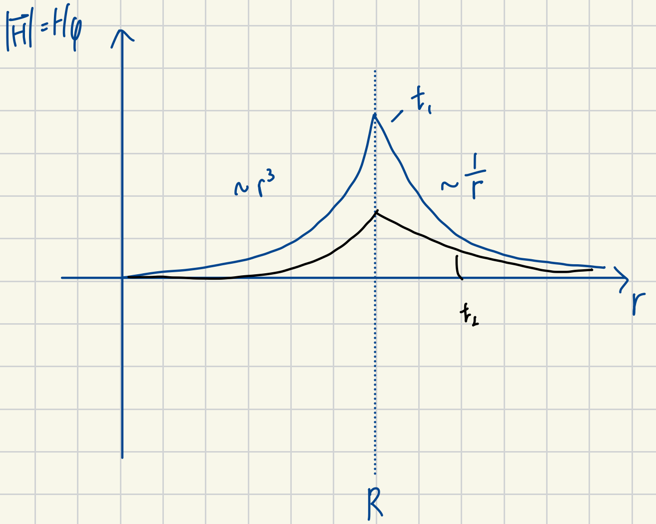
$$= 2\pi j_0 \cdot \frac{r^4}{4R^2} \sin\left(\frac{2\pi t}{T}\right) = \frac{1}{2}\pi j_0 \frac{r^4}{R^2} \sin\left(\frac{2\pi t}{T}\right)$$

$$H_\varphi = \frac{1}{4} j_0 \frac{r^3}{R^2} \sin\left(\frac{2\pi t}{T}\right) \quad \text{for } 0 \le r \le R$$

$$H_\varphi = \frac{1}{4} j_0 \frac{r^3}{R^2} \sin\left(\frac{2\pi t}{T}\right) \quad \text{for} \quad 0 \leq r \leq R$$

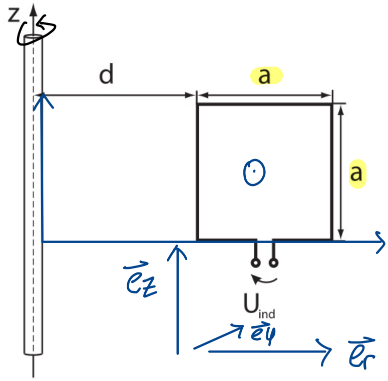
$$H_\varphi = \frac{1}{4} \cdot j_0 \cdot R^2 \cdot \frac{1}{r} \cdot \sin\left(\frac{2\pi t}{T}\right) \quad \text{for} \quad r > R$$

d) Plot the magnitude of the magnetic field strength $\vec{H}(r, \varphi, t)$ as a function of r at the time $t_1 = T/4$ and $t_2 = T/8$.



$$t_1 = \frac{T}{4} \Rightarrow \sin\left(\frac{2\pi T}{4T}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$t_2 = \frac{T}{8} \Rightarrow \sin\left(\frac{2\pi T}{8T}\right) = \sin\left(\pi/4\right) < 1$$



In a distance $d \gg R$, there is a quadratic conductor loop with edge length a (see figure). Assume for the following calculations that the magnetic field is given as

$$\vec{H}(r,\varphi,t) = \frac{H_0 R}{r} \cdot \sin(2\pi t/T) \cdot \vec{e}_\varphi(\varphi) \qquad \text{for } r \geq R$$

*e) Calculate the voltage $U_{\text{ind}}(t)$ induced in the motionless conductor loop in the period $0 < t < T$ and draw the result in an appropriate sketch.

$$U_{ind} = -\frac{d\Phi(A)}{dt}$$ motionless induction : $\frac{\partial B}{\partial T}$

$$\Phi(t) = \int_A \vec{B}(\vec{r},t)\, d\vec{a} = \mu_0 \int_0^a \int_d^{d+a} \frac{H_0 R}{r} \sin\left(\frac{2\pi t}{T}\right) \cdot \vec{e}_\varphi \, dr\, dz \cdot \vec{e}_\varphi$$

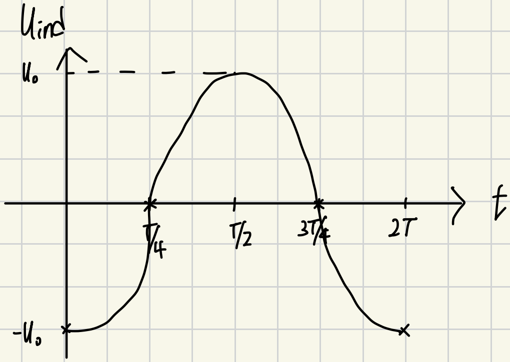
$$= \mu_0 H_0 R \cdot \sin\left(\frac{2\pi t}{T}\right) \cdot a \cdot \int_d^{d+a} \frac{1}{r} dr$$

$$= \mu_0 H_0 R \cdot \sin\left(\frac{2\pi t}{T}\right) \cdot a \left[\ln r\right]_d^{d+a}$$

$$\Phi(t) = \mu_0 H_0 R \cdot \sin\left(\frac{2\pi t}{T}\right) \cdot a \cdot \ln\left(\frac{d+a}{d}\right)$$

$$U_{ind} = -\frac{d\Phi}{dt} = -\mu_0 H_0 R \cdot a \cdot \ln\left(\frac{d+a}{d}\right) \frac{2\pi}{T} \cos\left(\frac{2\pi t}{T}\right)$$

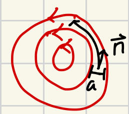
$$U_{ind} = \underbrace{-\mu_0 H_0 R\, a\, \ln\left(\frac{d+a}{d}\right) \cdot \frac{2\pi}{T}}_{\text{Const.}} \cos\left(\frac{2\pi t}{T}\right) \sim -\cos\left(\frac{2\pi t}{T}\right)$$



*f) Does the induced voltage $U_{\text{ind}}(t)$ change when the conductor loop is rotating around the z-axis? (Note that the normal vector of the area surrounded by the conductor loop is always directed in φ-direction.) How do you include this motion in your calculation?

no, since $\vec{B}$ cylindric symmetric

$\Rightarrow \phi$ is constant



$\vec{n}$ is moving along $\vec{e}_\varphi$ $\quad A(t) = A \cdot \vec{n} = A \cdot \vec{e}_\varphi$

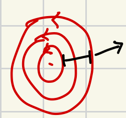
Projection of $\vec{B} = B_0 \cdot \vec{e}_\varphi$ on $\vec{n}$ is always constant

*g) Does the induced voltage change, when the conductor loop is parallel-shifted in radial direction with constant velocity $\vec{V}$? How do you account for this effect in the calculation of the magnetic flux?

$\vec{B}$ field decreases with $\frac{1}{r}$

$\Rightarrow \phi(r)$ is also decreasing

$\Rightarrow U_{ind} \neq 0$



motional induction