1. Evaluate the following integrals using integration by substitution.

a) $\int x\sqrt{1-x^2}\,dx$

b) $\int x^2 \sin(x^3)\,dx$

c) $\int x^5\sqrt{1+x^2}\,dx$

d) $\int_{-5/2}^{-2} x(2x+5)^8\,dx$

e) $\int_0^{\pi/4} \frac{\sin x}{\cos^3 x}\,dx$

f) $\int_1^{e^{\pi/4}} \frac{\sec^2(\ln x)}{x}\,dx$

g) $\int_1^2 \frac{2\ln x}{x}\,dx$

a) Let $u = 1-x^2$

$$\frac{du}{dx} = -2x$$

$$dx = -\frac{du}{2x}$$

$$\int \cancel{x}\,(u)^{\frac{1}{2}}\left(-\frac{du}{2\cancel{x}}\right) = \frac{(u)^{\frac{3}{2}}}{\frac{3}{\cancel{2}}} \cdot -\frac{1}{\cancel{2}}$$

$$= -\frac{(u)^{\frac{3}{2}}}{3} + C$$

$$= \frac{-(1-x^2)^{3/2}}{3} + C$$

b) $\int x^2 \sin(x^3)\,dx$ $= \int \cancel{x^2} \sin(u)\,\frac{du}{3\cancel{x^2}}$

Let $u = x^3$

$$\frac{du}{dx} = 3x^2$$

$$dx = \frac{du}{3x^2}$$

$$= \frac{-\cos(u)}{3} + C$$

$$= -\frac{\cos(x^3)}{3} + C$$

c) $\int x^5\sqrt{1+x^2}\,dx$ $= \int \cancel{x^5} (u)^{\frac{1}{2}} \frac{du}{2\cancel{x}} = \int x^4 (u)^{\frac{1}{2}} \frac{du}{2}$ $\quad x^2 = u-1$

let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$$= \frac{1}{2}\int (u-1)^2 (u)^{\frac{1}{2}}\, du$$

$$= \frac{1}{2}\int (u^2-2u+1)(u)^{\frac{1}{2}}\, du$$

$$= \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$$

$$= \frac{1}{\cancel{2}}\left[\frac{u^{7/2}}{7/\cancel{2}} - \frac{2u^{5/2}}{5/\cancel{2}} + \frac{u^{3/2}}{3/\cancel{2}}\right] + C$$

$$= \frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + C$$

d) $\int_{-5/2}^{-2} x(2x+5)^8\,dx$ $= \int_{-5/2}^{-2} \left(\frac{u-5}{2}\right)(u)^8 \frac{du}{2}$

let $u = 2x+5$ $\quad x = \frac{u-5}{2}$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$$= \frac{1}{4}\int_{-5/2}^{-2} (u-5)(u)^8\, du$$

$$= \frac{1}{4}\int_{-5/2}^{-2} (u^9 - 5u^8)\, du$$

$$= \frac{1}{4}\left[\frac{(2x+5)^{10}}{10} - \frac{5(2x+5)^9}{9}\right]\Bigg|_{-5/2}^{-2}$$

$$= -\frac{41}{360}$$

e) $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^3 x}\,dx$ $= \displaystyle\int_0^{\pi/4} \frac{\cancel{\sin x}}{u^3}\left(\frac{du}{-\cancel{\sin x}}\right) = -\int_0^{\pi/4} \frac{1}{u^3}\,du$

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

$= \left[\frac{1}{2u^2}\right]\Big|_0^{\pi/4}$

$= \left[\frac{1}{2\cos^2 x}\right]\Big|_0^{\pi/4}$

$= \frac{1}{2}$

f) $\displaystyle\int_1^{e^{\pi/4}} \frac{\sec^2(\ln x)}{x}\,dx$ $= \displaystyle\int_1^{e^{\pi/4}} \frac{\sec^2(u)}{\cancel{x}}(\cancel{x}\,du)$

let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$= \displaystyle\int_1^{e^{\pi/4}} \sec^2(u)\,du$

$= \left[\tan(u)\right]\Big|_1^{e^{\pi/4}}$

$= \left[\tan(\ln x)\right]_1^{e^{\pi/4}}$

$= 1$

g) $\displaystyle\int_1^2 \frac{2\ln x}{x}\,dx$ $= \displaystyle\int_1^2 \frac{2u}{\cancel{x}}(\cancel{x}\,du)$

let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$= \left[\frac{2u^2}{2}\right]_1^2$

$= (\ln 2)^2 - (\overset{0}{\ln 1})^2$

$= (\ln 2)^2$

2. Evaluate each integral below in terms of f(x). k is a constant.

a) $\int f'(x)\,dx = f(x) + C$

b) $\int f'(kx)\,dx$

c) $\int x f'(kx^2)\,dx$

ANS: **a)** $f(x)+c$. **b)** $\frac{f(kx)}{k}+c$. **c)** $\frac{f(kx^2)}{2k}+c$

b) let $u = kx$

$\frac{du}{dx} = k$

$dx = \frac{du}{k}$

$\Rightarrow \int f'(u)\frac{du}{k}$

$= \frac{f(u)}{k}$

$= \frac{f(kx)}{k} + C$

c) Let $u = kx^2$

$\frac{du}{dx} = 2kx$

$dx = \frac{du}{2kx}$

$\Rightarrow \int \cancel{x} f'(u)\frac{du}{2k\cancel{x}}$

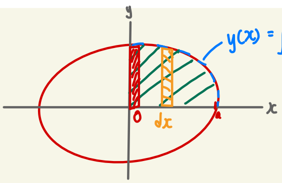
$= \frac{f(u)}{2} + C$

$= \frac{f(kx^2)}{2k} + C$

3. Set up a definite integral that represents the area of a quarter of the ellipse below. Evaluate it to obtain the area formula of the full ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ANS: $b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx = \frac{\pi ab}{4}$



From sub $y=0$ into eqn, solve for $x = \pm a$

$y(x) = \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} = b\sqrt{\left(1-\frac{x^2}{a^2}\right)}$

$dA = y(x)\cdot dx = b\sqrt{\left(1-\frac{x^2}{a^2}\right)}\cdot dx$

$A_{1/4} = \int_{x=0}^{x=a} b\sqrt{\left(1-\frac{x^2}{a^2}\right)}\cdot dx$

Let $x = a\cos\theta \rightarrow dx = -a\sin\theta\,d\theta$

At $x=a$, $\theta = 0$

At $x=0$, $\theta = \frac{\pi}{2}$

$A_{1/4} = b\int_{\frac{\pi}{2}}^{0}\sqrt{1-\frac{\cancel{a^2}\cos^2\theta}{\cancel{a^2}}}\,(-a\sin\theta\,d\theta) = ab\int_0^{\pi/2}\sin^2\theta\,d\theta$

$= ab\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{ab}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$

$= \frac{ab}{2}\left(\frac{\pi}{2}-0\right)$

$= \frac{\pi ab}{4}$

For whole ellipse, $A = 4A_{1/4} = \underline{\underline{\pi ab}}$

4. Using substitution of trigonometric relations, evaluate the following integrals.

a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

b) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$

c) $\displaystyle\int \frac{x^3}{\sqrt{x^2-4}}\,dx$

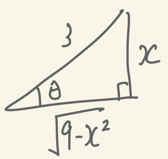
ANS: **a)** $\cos^{-1}\left(\frac{x}{3}\right)-\frac{\sqrt{9-x^2}}{x}+c$. **b)** $-\frac{\sqrt{1+x^2}}{x}+c$. **c)** $4\sqrt{x^2-4}+\frac{(x^2-4)^{3/2}}{3}+c$.

a) Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$\frac{\sqrt{9-x^2}}{x^2}\,dx \Rightarrow \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}(3\cos\theta\, d\theta) = \int \frac{\sqrt{9(1-\sin^2\theta)}}{9\sin^2\theta}(3\cos\theta\, d\theta)$$

$$= \int \frac{3\sqrt{\cos^2\theta}}{9\sin^2\theta}(3\cos\theta\, d\theta)$$



$\sin\theta = \frac{x}{3}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\cot\theta = \frac{a}{o}$

$= \frac{\sqrt{9-x^2}}{x}$

$$= \int \frac{\cancel{3}\cos\theta}{\cancel{9}\sin^2\theta}(\cancel{3}\cos\theta\, d\theta)$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

$\frac{a}{o}$

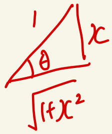
$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x^2} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

b) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx \Rightarrow \int \frac{1}{(\tan^2\theta)\cancel{\sec\theta}}\left(\sec^{\cancel{2}}\theta\, d\theta\right)$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$



$$= \int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

let $u = \sin\theta$

$\dfrac{du}{d\theta} = \cos\theta$

$d\theta = \dfrac{du}{\cos\theta}$

$$= \int \frac{\cancel{\cos\theta}}{u^2}\left(\frac{du}{\cancel{\cos\theta}}\right)$$

$$= \int \frac{1}{u^2}\, du$$

$$= -\frac{1}{u}$$

$$= -\frac{1}{\sin\theta} + C$$

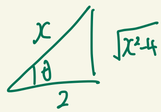
$$= -\frac{\sqrt{1+x^2}}{x} + C$$

c) $\displaystyle\int \frac{x^3}{\sqrt{x^2-4}}\,dx$

$$\Rightarrow \int \frac{8\sec^3\theta}{2\cancel{\tan\theta}}\left(2\sec\theta\cancel{\tan\theta}\,d\theta\right) = \int 8\sec^4\theta\, d\theta$$

let $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$



$$\sec\theta = \frac{x}{2} = \frac{1}{\cos\theta} = \frac{h}{a}$$

$$\tan\theta = \frac{\sqrt{x^2-4}}{2}$$

let $u = \tan\theta$

$$\frac{du}{d\theta} = \sec^2\theta$$

$$d\theta = \frac{du}{\sec^2\theta}$$

$$= 8\int (\sec^2\theta)^2\, d\theta$$

$$= 8\int (1-\tan^2\theta)^2\, d\theta$$

$$= 8\int (1-u^2)\cancel{\sec^2\theta}\left(\frac{du}{\cancel{\sec^2\theta}}\right)$$

$$= 8\int (1-u^2)\, du$$

$$= 8u - \frac{8u^3}{3} + C$$

$$= 8\tan\theta - \frac{8\tan^3\theta}{3} + C$$

$$= 4\sqrt{x^2-4} - \frac{(x^2-4)^{3/2}}{3} + C$$

a) $\int \frac{3x}{x^2+2x-8}\,dx$

$$\frac{3x}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$$

$$3x = A(x-2) + B(x+4)$$

$$3x = Ax - 2A + Bx + 4B$$

$$A + B = 3 \quad - (1)$$

$$-2A + 4B = 0 \quad - (2)$$

$$(2) + 2\times(1) : \quad -2A + 4B + 2A + 2B = 0 + 6$$

$$6B = 6$$

$$B = 1$$

$$A = 2$$

$$\Rightarrow \int \frac{2}{x+4} + \frac{1}{x-2}\,dx$$

$$= 2\ln|x+4| + \ln|x-2| + C$$

b) $\displaystyle\int \frac{x^2-1}{x^2+x-6}\,dx$ $\Rightarrow$ $\displaystyle\int 1 - \frac{x-5}{x^2+x-6}\,dx$

$$\begin{array}{r} 1 \\ x^2+x-6 \,\overline{\big)\, x^2-1} \\ -(x^2+x-6) \\ \hline -x+5 \end{array}$$

$$\frac{x-5}{(x+3)(x-2)} = \frac{A}{(x+3)} + \frac{B}{(x-2)}$$

$$x-5 = Ax - 2A + Bx + 3B$$

$$A+B = 1 \quad -(1)$$

$$3B-2A = -5 \quad -(2)$$

$(2) + 2\times(1)$: $3B - 2A + 2A + 2B = -5+2$

$$5B = -3$$

$$B = -\frac{3}{5}$$

$$A = \frac{8}{5}$$

$$\therefore \Rightarrow \int 1 - \frac{8}{5(x+3)} + \frac{3}{5(x-2)}\,dx$$

$$= x - \frac{8}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C$$

c) $\int \frac{x^5}{x^3 - 9x}\,dx = \int \frac{x^4}{x^2-9} = \int x^2+9+\frac{81}{x^2-9}$

$$\begin{array}{r} x^2+9 \\ x^2-9\,\overline{)\,x^4} \\ -(x^4-9x^2) \\ \hline 9x^2 \\ -\ (9x^2-81) \\ \hline 81 \end{array}$$

$\frac{1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$

$1 = Ax-3A+Bx+3B$

$A+B=0$ -①

$-3A+3B=1$ -②

② + 3×① : $6B=1$

$B=\frac{1}{6}$

$A=-\frac{1}{6}$

$= \int x^2+9-81\left[\frac{1}{6(x+3)} - \frac{1}{6(x-3)}\right] dx$

$= \frac{x^3}{3} + 9x - \frac{27}{2}\ln|x+3|$

$+ \frac{27}{2}\ln|x-3| + C$

$= \frac{x^3}{3} + 9x - \frac{27}{2}\,\frac{\ln|x-3|}{\ln|x+3|} + C$

d) $\int \frac{x^2-1}{x^2+5x-6}\,dx \Rightarrow \int 1+\frac{-5x+5}{x^2+5x-6}\,dx$

$$\begin{array}{r} 1 \\ x^2+5x-6\,\overline{)\,x^2-1} \\ -(x^2+5x-6) \\ \hline -5x+5 \end{array}$$

$= \int 1 + \frac{-5}{x+6}\,dx$

$= x - 5\ln|x+6| + C$

$\frac{-5x+5}{(x+6)(x-1)} = \frac{A}{x+6} + \frac{B}{x-1}$

$-5x+5 = A(x-1) + B(x+6)$

$= Ax - A + Bx + 6B$

$A+B=-5$ -①

$-A+6B=5$ -②

① + ② : $7B=0$

$B=0$

$A=-5$

e) $\int \frac{x}{x^2+2x+1}\,dx \Rightarrow \int \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx$

$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$

$x = A(x+1) + B$

$= Ax + A + B$

$A=1$

$A+B=0$

$B=-1$

$= \ln|x+1| + \frac{1}{x+1} + C$

6 a) $\displaystyle\int x^4 \ln(2x)\, dx$

$$\Rightarrow \frac{x^5}{5}\ln(2x) - \int \frac{x^5}{5}\left(\frac{1}{x}dx\right)$$

$$= \frac{x^5}{5}\ln(2x) - \frac{x^5}{25} + C$$

Let $u = \ln(2x)$

$\frac{du}{dx} = \frac{2}{2x}$

$du = \frac{1}{x}dx$

$dv = x^4 dx$

$v = \frac{x^5}{5}$

$$\int u\,dv = uv - \int v\,du$$

b) $\displaystyle\int x^2 \cos x\, dx$

$$= x^2 \sin x - \int \sin x\,(2x\,dx)$$

$$= x^2 \sin x - \int 2x \sin x\, dx$$

Let $u = x^2$

$du = 2x\,dx$

$dv = \cos x\,dx$

$v = \sin x$

Let $u = 2x$

$du = 2dx$

$dv = \sin x\,dx$

$v = -\cos x$

$$= x^2 \sin x + 2x\cos x + \int -\cos x\, 2dx$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + C$$

$$= (x^2 - 2)\sin x + 2x\cos x + C$$

c) $\displaystyle\int \overbrace{\underbrace{e^{2x}}_{dv}\underbrace{\cos(4x)}_{u}\underbrace{dx}_{dv}}^{I}$

	u	dv
$+$	$\cos(4x)$	e^{2x}
$-$	$-4\sin(4x)$	$\frac{1}{2}e^{2x}$
$+$	$-16\cos(4x)$	$\frac{1}{4}e^{2x}$

$$= \frac{1}{2}\cos(4x)e^{2x} + \sin(4x)e^{2x} \overbrace{-4\int e^{2x}\cos(4x)\,dx}^{-4I}$$

$$\Rightarrow 5I = \frac{1}{2}\cos(4x)e^{2x} + \sin(4x)e^{2x}$$

$$I = \frac{e^{2x}}{5}\left[\frac{\cos(4x)}{2} + \sin(4x)\right]$$

Conditions for stopping in tabular form:
1) u(x) get differentiated to 0
2) udv (or vdu) can be integrated
3) udv gives the same integrand in the original integral

d) $$\int \underbrace{(1-x^3)}_{u}\underbrace{e^{x/2}}_{dv}\,dx$$

	u	dv
$+$	$1-x^3$	$e^{x/2}$
$-$	$-3x^2$	$2e^{x/2}$
$+$	$-6x$	$4e^{x/2}$
$-$	-6	$8e^{x/2}$
$+$	0	$16e^{x/2}$

$$= (1-x^3)(2e^{x/2}) - (-3x^2)(4e^{x/2}) + (-6x)(8e^{x/2}) - (-6)(16e^{x/2})$$

$$= e^{x/2}\left[2-2x^3+12x^2-48x+96\right]$$

$$= -2e^{x/2}\left[x^3-6x^2+24x-49\right] + C$$

$$\int \underbrace{\sin^{-1}x}_{u}\,\underbrace{dx}_{dv} = x\sin^{-1}x + \sqrt{1-x^2} + c$$

	u	dv
$+$	$\sin^{-1}x$	1
$-$	$\frac{1}{\sqrt{1-x^2}}$	x

$$= x\sin^{-1}x - \int \frac{x}{\underbrace{\sqrt{1-x^2}}}\,dx \Rightarrow \int \frac{x}{\sqrt{u}}\left(\frac{du}{-2x}\right)$$

let $u = 1-x^2$

$du = -2x\,dx$

$dx = \frac{du}{-2x}$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$= -\frac{1}{2}\cdot\frac{(u)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= -\frac{1}{\cancel{2}}\cdot\frac{\sqrt{u}}{\frac{1}{\cancel{2}}}$$

$$\Rightarrow x\sin^{-1}x + \sqrt{1-x^2} + C$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \rightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$$
$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \rightarrow \int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}x + c$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \rightarrow \int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$$

$$\int \underbrace{\cos^{-1} x}_{u}\, \underbrace{dx}_{dv} = x\cos^{-1} x - \sqrt{1-x^2} + c$$

u	dv
$\cos^{-1} x$	1
$\frac{-1}{\sqrt{1-x^2}}$	x

$$\Rightarrow x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}}\, dx$$

let $u = 1-x^2$

$du = -2x\, dx$

$dx = \frac{du}{-2x}$

$$\int \frac{\cancel{-x}}{\sqrt{u}} \frac{du}{-2\cancel{x}}$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du$$

$$= \frac{\cancel{1}}{\cancel{2}} \cdot \frac{(u)^{1/2}}{\frac{1}{\cancel{2}}}$$

$$= \sqrt{1-x^2}$$

$$\Rightarrow x\cos^{-1} x - \sqrt{1-x^2} + C$$

$$\int \underbrace{\tan^{-1} x}_{u}\, \underbrace{dx}_{dv} = x\tan^{-1} x - \frac{\ln(x^2+1)}{2} + c$$

u	dv
$\tan^{-1} x$	1
$\frac{1}{1+x^2}$	x

$$= x\tan^{-1} x - \int \frac{x}{\underbrace{1+x^2}}\, dx$$

let $u = 1+x^2$

$du = 2x\, dx$

$dx = \frac{du}{2x}$

$$\Rightarrow \int \frac{\cancel{x}}{u} \frac{du}{2\cancel{x}}$$

$$= \int \frac{1}{2u}\, du$$

$$= \frac{1}{2}\ln|u| + C$$

$$\therefore \Rightarrow x\tan^{-1} x - \frac{\ln(x^2+1)}{2} + C$$

a) $\displaystyle\int \frac{\sin^{-1}(\ln x)}{x}\,dx$ $\Rightarrow$ $\displaystyle\int \underbrace{\sin^{-1}(w)}_{u}\,\underbrace{dw}_{dv}$

u	dv
$\sin^{-1}(w)$	dw
$\frac{1}{\sqrt{1-w^2}}$	w

Let $w = \ln x$
$dw = \frac{1}{x}dx$

$$= w\sin^{-1}(w) - \int \underbrace{\frac{w}{\sqrt{1-w^2}}\,dw}$$

Let $u = 1-w^2$
$dw = \frac{du}{-2w}$

$$\Rightarrow \int \frac{\cancel{w}}{\sqrt{u}}\,\frac{du}{-2\cancel{w}}$$

$$= -\frac{1}{\cancel{2}}\cdot\frac{(u)^{1/2}}{\frac{1}{\cancel{2}}}$$

$$= -\sqrt{u}$$

$$= w\sin^{-1}(w) - \sqrt{u} + C$$

$$= \ln x\,\sin^{-1}(\ln x) - \sqrt{1-(\ln x)^2} + C$$

b) $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$ $= \displaystyle\underbrace{\int \frac{1}{1+\cos x}}\; + \underbrace{\frac{\sin x}{1+\cos x}}_{u = 1+\cos x} dx$

$\cos 2\underbrace{\theta}_{x} = 2\cos^2\underbrace{\theta}_{x/2} - 1$

$\cos x + 1 = 2\cos^2 x/2$

$\displaystyle\int \frac{1}{2\cos^2 x/2}\,dx = \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right) dx$

$= \frac{1}{2}\left[2\tan\left(\frac{x}{2}\right)\right]$

$= \tan\left(\frac{x}{2}\right)$

$dx = \frac{du}{-\sin x}$

$\displaystyle\int \frac{\cancel{\sin x}}{u} \cdot \frac{du}{\cancel{-\sin x}}$

$= -\displaystyle\int \frac{1}{u}\,du$

$= -\ln|u|$

$= -\ln|1+\cos x|$

$\Rightarrow \tan\left(\frac{x}{2}\right) - \ln|1+\cos x| + C$

c) $\displaystyle\int \frac{3e^x + 4e^{-x} + 2}{1 - e^{2x}}\,dx$

Trial : let $u = e^x$, $du = e^x dx$

$dx = e^{-x} du = \frac{1}{u} du$

$\displaystyle\int \frac{3u + \frac{4}{u} + 2}{1 - u^2}\left(\frac{1}{u}du\right) = \int \frac{\frac{1}{u}(3u^2 + 4 + 2u)}{u(1-u^2)}\,du$

$= \displaystyle\int \underbrace{\frac{3u^2 + 4 + 2u}{u^2(1-u)(1+u)}} du$

cover up

$\frac{A}{u} + \frac{4/1}{u^2} + \frac{\frac{3+4+2}{2}}{1-u} + \frac{\frac{3+4-2}{2}}{1+u} = \frac{A}{u} + \frac{4}{u^2} + \frac{9}{2(1-u)} + \frac{5}{2(1+u)}$

To find A, sub $u = 2$,

$\frac{3(2)^2 + 4 + 2(2)}{2^2(1-2)(1+2)} = \frac{A}{2} + \frac{4}{4} - \frac{9}{2} + \frac{5}{6}$

$\frac{20}{-12} = \frac{A}{2} - \frac{8}{3}$

$\frac{A}{2} = 1$

$A = 2$

$$\int \frac{2}{u} + \frac{4}{u^2} + \frac{9}{2(1-u)} + \frac{5}{2(1+u)}\,du$$

$$= 2\ln|u| - \frac{4}{u} + \frac{9}{2}\ln|1-u| + \frac{5}{2}\ln|1+u| + C$$

$$= 2x - 4e^{-x} + \frac{9}{2}\ln|1-e^{x}| + \frac{5}{2}\ln|1+e^{x}| + C$$

d) $\displaystyle\int \frac{a^x}{a^x + a^{-x}}\,dx$

$$= \int \frac{a^x}{a^x + \frac{1}{a^x}}\,dx$$

$$= \int \frac{a^x}{\frac{1}{a^x}(a^{2x}+1)}\,dx$$

$$= \int \frac{a^{2x}}{a^{2x}+1}\,dx \Rightarrow \int \frac{\cancel{a^{2x}}}{u}\left(\frac{du}{2\cancel{a^{2x}}\ln a}\right)$$

Let $u = a^{2x}+1$

$$\frac{du}{dx} = 2a^{2x}\ln a$$

$$dx = \frac{du}{2a^{2x}\ln a}$$

$$= \frac{1}{2\ln a}\int \frac{1}{u}\,du$$

$$= \frac{1}{2\ln a}\cdot \ln u + C$$

$$= \frac{\ln|a^{2x}+1|}{2\ln a} + C$$

e) $\displaystyle\int \frac{x \sin^{-1}(x^2)}{\sqrt{1-x^4}}\,dx$

Let $u = \sin^{-1}(x^2)$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$dx = \frac{\sqrt{1-x^4}\,du}{2x}$$

$$\Rightarrow \int \frac{\cancel{x}\, u}{\cancel{\sqrt{1-x^4}}} \left(\frac{\cancel{\sqrt{1-x^4}}}{2\cancel{x}} \right) du$$

$$= \frac{1}{2} \int u \, du$$

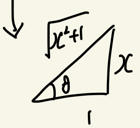
$$= \frac{u^2}{4} + C$$

$$= \frac{[\sin^{-1}(x^2)]^2}{4} + C$$

f) $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

Let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$



$$= \int \frac{\tan^3\theta}{\cancel{\sec\theta}} (\sec^{\cancel{2}}\theta \, d\theta)$$

$$= \int \tan^3\theta \sec\theta \, d\theta$$

$$= \int \frac{\sin^3\theta}{\cos^4\theta} \, d\theta$$

$$= \underbrace{\int \frac{\sin\theta(1-\cos^2\theta)}{\cos^4\theta} \, d\theta}$$

Let $u = \cos\theta$

$$\frac{du}{d\theta} = -\sin\theta$$

$$d\theta = \frac{du}{-\sin\theta}$$

$$\Rightarrow \int \frac{\cancel{\sin\theta}(1-u^2)}{u^4} \left(\frac{du}{-\cancel{\sin\theta}} \right)$$

$$= -\int \frac{(1-u^2)}{u^4} \, du$$

$$= -\int \frac{1}{u^4} - \frac{1}{u^2}$$

$$= -\left[-\frac{1}{3u^3} + \frac{1}{u} \right] + C$$

$$= \frac{1}{3u^3} - \frac{1}{u} + C$$

$$= \frac{1}{3\cos^3\theta} - \frac{1}{\cos\theta} + C$$

$$= \frac{1}{3}(x^2+1)^{3/2} - \sqrt{x^2+1} + C$$

g) $\int \sin^4 x \cos^2 x \, dx$ $= \int \underbrace{\sin^2 x \ \sin^2 x \cos^2 x}_{\frac{1}{2}[1-\cos 2x]\left[\frac{1}{2}\sin 2x\right]^2} \, dx$

$$\underbrace{\frac{1}{2}\sin 2x}_{\frac{1}{4}\sin^2(2x)}$$

$$\underbrace{\sin^2(2x)}_{\frac{1}{2}[1-\cos(4x)]}$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(2\theta) = 1 - 2\sin^2\theta$

$\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Product to Sum of Two Angles

$\sin\theta \sin\phi = \dfrac{[\cos(\theta-\phi) - \cos(\theta+\phi)]}{2}$

$\cos\theta \cos\phi = \dfrac{[\cos(\theta-\phi) + \cos(\theta+\phi)]}{2}$

$\sin\theta \cos\phi = \dfrac{[\sin(\theta+\phi) + \sin(\theta-\phi)]}{2}$

$\cos\theta \sin\phi = \dfrac{[\sin(\theta+\phi) - \sin(\theta-\phi)]}{2}$

$$= \frac{1}{2}[1-\cos 2x]\left[\frac{1}{4}\left\{\frac{1}{2}[1-\cos(4x)]\right\}\right]$$

$$= \frac{1}{16}[1-\cos(2x)][1-\cos(4x)]$$

$$= \frac{1}{16}\left[1 - \cos(4x) - \cos(2x) + \underbrace{\cos(2x)\cos(4x)}_{\cos\theta\cos\phi}\right]$$

$$\Rightarrow \frac{1}{2}[\cos(2x) + \cos(6x)]$$

$$= \frac{1}{16}\left[1 - \cos(4x) - \cos(2x) + \frac{1}{2}\cos(2x) + \frac{1}{2}\cos(6x)\right]$$

$$= \frac{1}{16}\left[1 - \frac{1}{2}\cos(2x) - \cos(4x) + \frac{1}{2}\cos(6x)\right]$$

$$\Rightarrow \frac{1}{16}\int 1 - \frac{1}{2}\cos(2x) - \cos(4x) + \frac{1}{2}\cos(6x) \, dx$$

$$= \frac{1}{16}\left[x - \frac{1}{2}\sin(2x)\cdot\frac{1}{2} - \sin(4x)\cdot\frac{1}{4} + \frac{1}{2}\sin(6x)\cdot\frac{1}{6}\right] + C$$

$$= \frac{x}{16} - \frac{\sin(2x)}{64} - \frac{\sin(4x)}{64} + \frac{\sin(6x)}{192} + C$$

h) $\int \sin^3 x \cos^4 x \, dx$ $= \int \sin^2 x \cos^4 x \sin x \, dx$

$= \int (1-\cos^2 x) \cos^4 x \sin x \, dx$

let $u = \cos x$

$dx = \frac{du}{-\sin x}$

$\Rightarrow \int (1-u^2) \cdot u^4 \cdot -du$

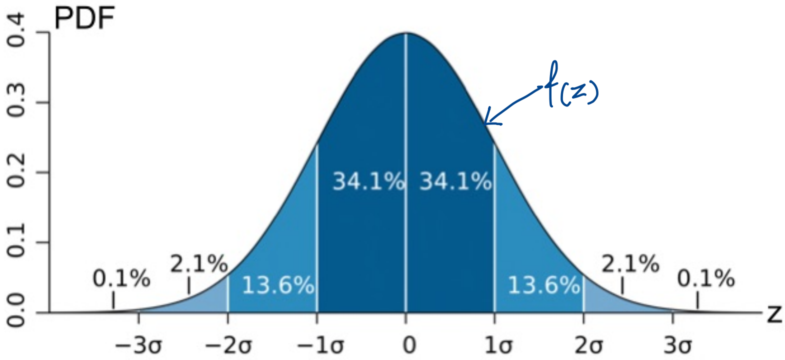
$= -\int (u^4 - u^6) \, du$

$= -\frac{u^5}{5} + \frac{u^7}{7} + C$

$= \frac{\cos^7 x}{7} - \frac{\cos^5}{5} + C$

9. **Use of Calculus in Statistics**

According to the probability density function (PDF) of the standard normal distribution shown below, there is a 68.27% chance that a sample's reading, z, will lie within one standard deviation from the mean, as represented by the area under the PDF.



Source: https://en.wikipedia.org/wiki/Normal_distribution#Standard_normal_distribution

Generally, the probability P that z will lie in between a and b in a normal distribution is given by the integral

f(z)

$$P(a < z < b) = \int_a^b \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

a) Using the midpoint rule, verify that $P(-1 < z < 1) \approx 0.6827$.

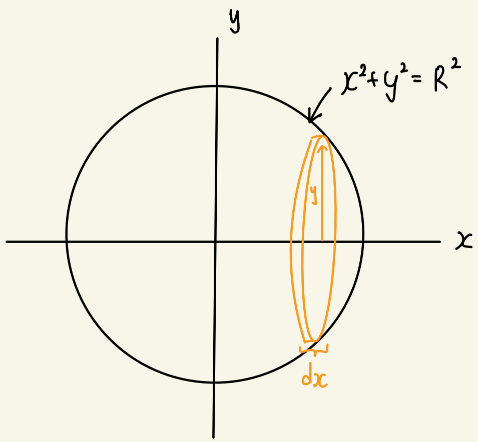
b) Using the trapezoidal rule, verify that $P(-2 < z < 2) \approx 0.9545$.

From program

c) What do you think $\underbrace{P(-\infty < z < \infty)}_{=1}$ logically is? Verify it using numerical integration.

10. Using integration, evaluate the volume of each object described below. Use integration of discs and hollow cylinders.

a) A sphere with radius R.



Vol of one disk $= 2\pi y^2 \, dx$

$y^2 = R^2 - x^2$

Vol of sphere $= \int 2\pi (R^2 - x^2) \, dx$

$$= 2\pi \int_0^R (R^2 - x^2) \, dx$$

$$= 2\pi \left[R^2 x - \frac{x^3}{3} \right] \Big|_0^R$$

$$= 2\pi \left[R^3 - \frac{R^3}{3} \right]$$

$$= \frac{4\pi R^3}{3}$$