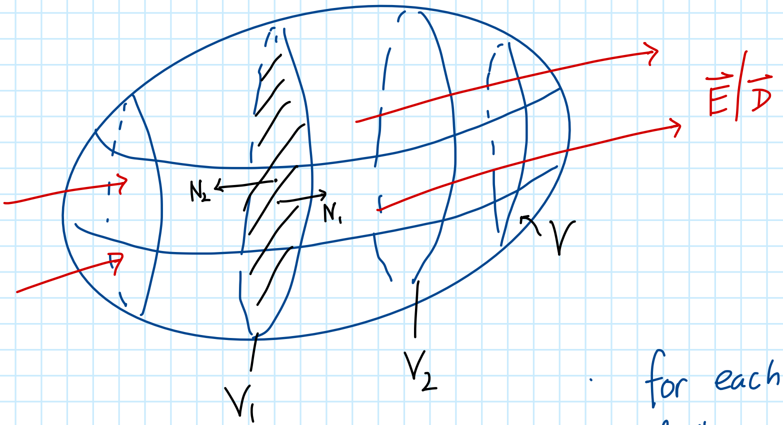
1.6.4. Gauss's law in differential form and Poisson's equation

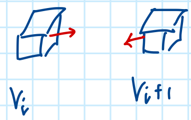
Integral form : $\int_{\partial V} \vec{D} \cdot d\vec{a} = Q(V(\vec{r})) = \int_V \rho(\vec{r})\, dV \Rightarrow$ 1st Maxwell's equation in integral form

(i) mathematical recall: divergence of a vector field

thought experiment: divide a control volume in many small subvolumes



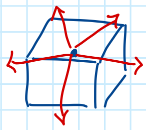
- Consider the flux of $\vec{D}$ through ∂V (enclosed surface of V)
- divide the control volume in small volumes V_i



- for each of the common surface (neighbouring surfaces) of the small volume V_i the flux cancels out

$\Rightarrow$ the total flux $\int_{\partial V} \vec{D}\, d\vec{a}$ is not change by division in small volumes V_i

$\operatorname{div} \vec{D} = \lim_{V_i \to 0} \frac{1}{V_i} \int_{\partial V_i} \vec{D} \cdot d\vec{a} \mathrel{\hat{=}}$ Source strength of a infinitesimal volume V_i



q = source of the local field

(ii) Gauss's law in differential form

(derived applying Gauss's integral theorem)

$\int_{\partial V} \vec{D} \cdot d\vec{a} = \int_V \rho(\vec{r})\, dV$ Gauss's Law in integral form (flux theorem for $\vec{D}$-field)

$$\int_{\partial V} \vec{D} \cdot d\vec{a} = \int_V \operatorname{div} \vec{D} \cdot dV = \int_V \rho(\vec{r})\, dV$$

$\downarrow$ very small volumes dV

$$\boxed{\operatorname{div} \vec{D} = \rho(\vec{r})}$$

Gauss's law in differential form; First Maxwell's equation

in words: The sources of $\vec{D}(\vec{E})$-fields are electric space charge densities / electric charges

1st Maxwell's equation tells how electrostatic fields are generated

$$\operatorname{div} \vec{D} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \partial/\partial x\, D_x + \partial/\partial y\, D_y + \partial/\partial z\, D_z$$

= Differential operator (div)

(iii) Poisson's equation

$\vec{E} = -\operatorname{grad} \phi \quad \vec{D} = \varepsilon \vec{E} \Rightarrow$ insert this into differential Gauss's Law

- General form of Possion's equation:

$$\operatorname{div}(\vec{D}) = -\operatorname{div}(\varepsilon \operatorname{grad} \phi) = \rho$$

$$\operatorname{div}(\varepsilon \operatorname{grad} \phi = -\rho$$

- Poisson's equation in simplified formulation:

if ε is not depending on position: $\varepsilon \operatorname{div}(\operatorname{grad} \phi) = -\rho$

$$\operatorname{div}(\operatorname{grad} \phi) = -\frac{\rho}{\varepsilon}$$

$$\operatorname{div}(\operatorname{grad}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$= \Delta =$ laplace operator

$$\boxed{\Delta \phi(\vec{r}) = -\frac{\rho}{\varepsilon}}$$

Annotations to zur Poisson's equation:

- is a partial differential equation
- allows for calculation of position-dependent electrostatic potential and, hence, elektric fields for given charge distributions
- for solution: boundary conditions needed (to determine the integration constants)
- there are systematic mathematical methods for solution (electrmagnetic field theory)
- allows for solution of general problems (numerical solution of equation by, e.g. Finite Element Methods - FEM)
- universally applicabel (in contrast to integral formulation of Gauss's law)

1.6.5. Coulomb potential and Coulomb field of charge distribution

↳ See printed lecture notes

(ii) Coulomb potential of a charge distribution

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(iii) Coulomb- field of a charge distribution