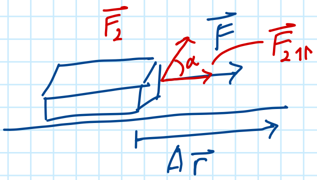
1.4. Electric Work and Electric Voltage

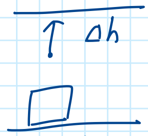
(i) Mechanical work - calculation on an arbitrary path in a position-dependent force field

- Electric field = force field: if you move charged particles in an electric field, you have to perform mechanical work

- Simple case: force is not depending on location (constant force field), path is straight line



mechanical work performed : $W_{mech} = \vec{F} \cdot \underbrace{\Delta \vec{r}}_{\text{distance}}$



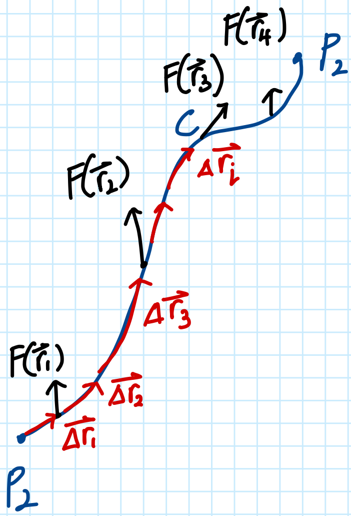
$W = |\vec{F}| \cdot \Delta h$
$= m \cdot g \cdot \Delta h$

$\vec{F}$ is parallel $\Delta\vec{r}$ $\Rightarrow W_{mech} = |\vec{F}_1| \cdot |\Delta\vec{r}|$

Case 2 : $W_{mech} = |\vec{F}_2| \cdot |\Delta\vec{r}| \cdot \cos\alpha$ $\quad \vec{F}_2 \cdot \cos\alpha \rightarrow$ projection on the direction of the path

$\Rightarrow$ $\boxed{\text{only component of } \vec{F} \text{ parallel to } \Delta\vec{r} \text{ contributes to the } W_{mech}.}$

- General case: magnitude and direction of force depend on location (force field not constant), moving something along an arbitrary path/curve C



- Curve arbitrary $C(P_1, P_2)$

- Force field position-dependent $\vec{F}(\vec{r})$

- Piece-wise summation along curve C(P1,P2)

  $\Delta W_{mech\,i} = \Delta\vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$

- Calculate line integral in vector field along curve C

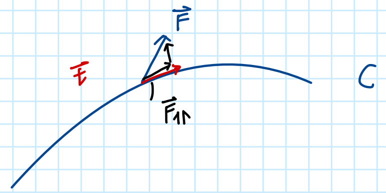
  Sum up different contributions $\Delta W_{mech\,i}$ along the curve, if we move from P_1 to P_2

  $\Rightarrow$ total work along $C(P_1, P_2)$

  $$W_{mech} = \sum_i \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$$

  $\Delta\vec{r}_i$ infinitesimal small $\Rightarrow d\vec{r}$ $\Rightarrow$ Replace sum by integral

- To solve: $\boxed{W = \int_{C(P_1,P_2)} \vec{F}(\vec{r})\, d\vec{r}}$ ← Projection of $\vec{F}$ onto Curve C when following the Curve



For determining work performed in a vector field, we have to solve a path integral in the vector field:

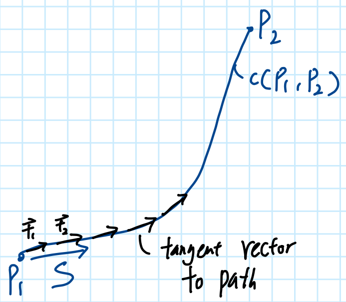
Basically we have to do three step

(i) Find a Suitable parameterization of the Curve $C(P_1, P_2)$ and the limits of integration

(ii) Parameterize vector field accordingly (= force field)

(iii) adjust / parameterize differential line element $d\vec{r}$ in a way that we follow the path $C, C(P_1, P_2)$

$$\boxed{W = \int_{C(P_1,P_2)} \vec{F}(\vec{r})\, d\vec{r}}$$



(i) introduce parameters S, which follows the path

$$S \in [0, l) \qquad C: S \mapsto \vec{r}(S)$$

(ii) vector field $\vec{F}(\vec{r}) \rightarrow \vec{F}(\vec{r}(s))$

(iii) adjust differential line element $d\vec{r}$ and the integration limits $s \in [0, l)$

C can be represented by a sequence of tangent vectors $\vec{t}$

$$\vec{t} = \frac{d\vec{r}(s)}{ds} \Rightarrow d\vec{r} = \vec{t} \cdot ds$$

⇒ Mechanical work along Curve C

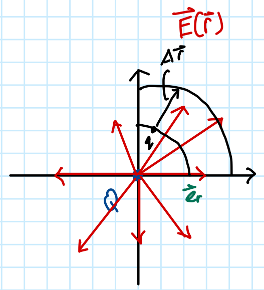
$$\boxed{W = \int_{C(P_1,P_2)} \vec{F}(\vec{r})\, d\vec{r} = \int_{\vec{r}_1(s)=0}^{\vec{r}_2(s)=l} \vec{F}(\vec{r}(s)) \cdot \underbrace{\vec{t} \cdot ds}_{d\vec{r}} = \int_0^l \vec{F}(\vec{r}(s)) \cdot \frac{d\vec{r}(s)}{ds} \cdot ds}$$

(ii) Electric work:

$$\vec{F}_{el} = q \cdot \vec{E}(\vec{r}) \Rightarrow \boxed{W_{el} = \int_{C(P_1,P_2)} q \cdot \vec{E}(\vec{r}) \cdot d\vec{r}}$$

electric work performed on charge q when moving along curve C

Example: electric work in the field of a point charge:



- Point charge Q, $Q > 0$, located at $\vec{r}_Q = \begin{pmatrix}0\\0\\0\end{pmatrix} = \vec{0}$
- Test charge q is moved from radius r_1 to radius r_2
- Calculate work, which is performed along a path from r_1 to r_2
- monopole field; radial symmetric

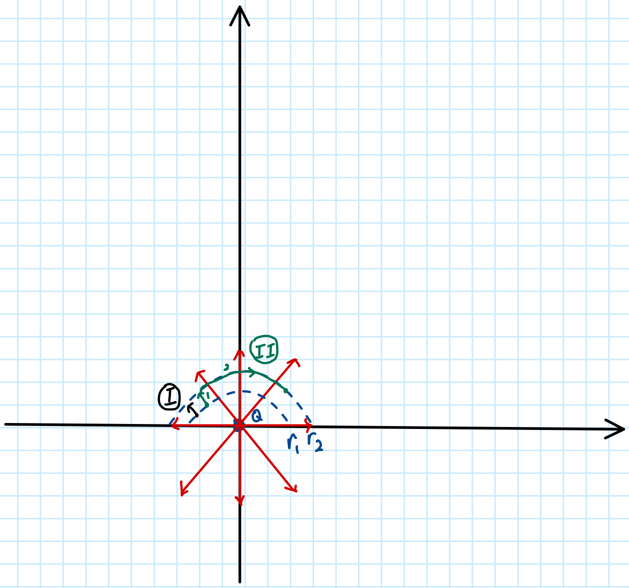
To solve: $W = \int_C \vec{F} d\vec{r} = \int_C q \cdot \vec{E}(\vec{r}) d\vec{r}$ with $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q(\vec{r}-\vec{r}_Q)}{|\vec{r}-\vec{r}_Q|^3}$

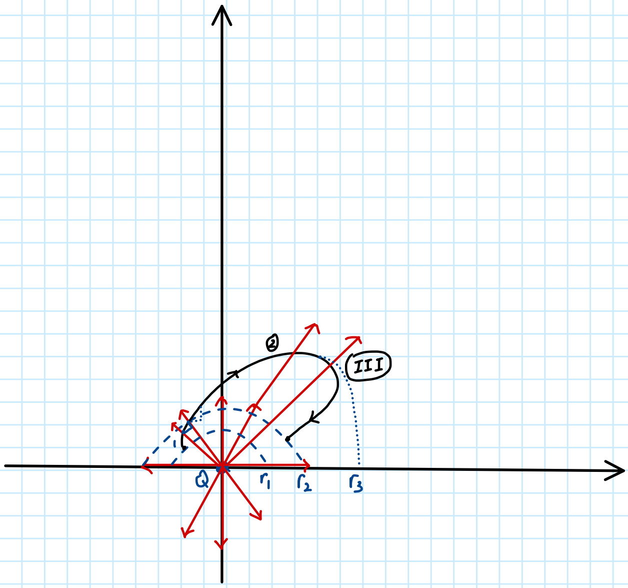
- spherical symmetric problem → integrate along radius ⇒ introduce a unit vector in r-direction $\vec{e}_r$ $(|\vec{e}_r| = 1) \Rightarrow d\vec{r} = \vec{e}_r \cdot dr$

$$W = \int q \cdot \vec{E}(\vec{r}) d\vec{r} = \frac{1}{4\pi\varepsilon_0} \int \frac{q \cdot Q}{|\vec{r} - \cancel{\vec{r}_Q}_{\,0}|^3} (\overset{r \cdot \vec{e}_r}{\vec{r}} - \cancel{\vec{r}_Q}_{\,0}) \cdot \vec{e}_r \cdot d\vec{r} = \frac{1}{4\pi\varepsilon_0} \int_{r_1}^{r_2} \frac{q \cdot Q}{|r \cdot \vec{e}_r|^3} \cdot r \cdot \underbrace{\vec{e}_r \cdot \vec{e}_r}_{1} \cdot dr = \frac{1}{4\pi\varepsilon_0} qQ \int_{r_1}^{r_2} \frac{1}{r^2} dr = \frac{1}{4\pi\varepsilon_0} qQ \left[-\frac{1}{r}\right]_{r_1}^{r_2}$$

$$W = \frac{qQ}{4\pi\varepsilon_0}\left[-\frac{1}{r_2} + \frac{1}{r_1}\right] = -\frac{qQ}{4\pi\varepsilon_0}\underbrace{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}_{<0}$$

Consideration on electric work:





① $W = -\frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$

(II) $W = W_1 + W_2$

$= -\frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) + 0$

since $d\vec{r} \perp \vec{E}$

(III) $W = \underbrace{\frac{-qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}_{①}$

$+ \cancel{\left(\frac{-qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_3} - \frac{1}{r_2}\right)\right)} + \cancel{\left(\frac{-qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_3}\right)\right)}$

$W = \frac{-qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$

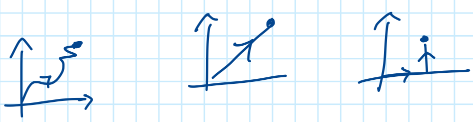
$\Rightarrow W_I = W_{II} = W_{III}$

⇒ Electric work performed in a electrostatic field does not depend on the path, but only the starting point and the ending point

$\Rightarrow \int \vec{E} d\vec{r}$ is path independent

↳ Electrostatic fields are conservative

(energy is conserved)



1.4.2 Electric voltage see ppt presentation