1.2 Logical Operators (Connectives)

Important Logical Operators (unary, binary)

Expression	Name	Parlance
$\neg\, \mathbf{a}$	Negation, NOT	it is not true, that **a**
$\mathbf{a} \wedge \mathbf{b}$	Conjunction, AND	**a** and **b**
$\mathbf{a} \vee \mathbf{b}$	Disjunction, OR	**a** or **b**
$\mathbf{a} \not\leftrightarrow \mathbf{b}$	Alternative, XOR	either **a** or **b**
$\mathbf{a} \longrightarrow \mathbf{b}$	Conditional, material implication, subjunction	if **a**, then **b**
$\mathbf{a} \longleftrightarrow \mathbf{b}$	Biconditional, material equivalence, bijunction	**a** if and only if **b**

Truth Table

Value assignment	truth values					
a **b**	$\neg\, \mathbf{a}$	$\mathbf{a} \wedge \mathbf{b}$	$\mathbf{a} \vee \mathbf{b}$	$\mathbf{a} \not\leftrightarrow \mathbf{b}$	$\mathbf{a} \longrightarrow \mathbf{b}$	$\mathbf{a} \longleftrightarrow \mathbf{b}$
t **t**	**f**	**t**	**t**	**f**	**t**	**t**
t **f**	**f**	**f**	**t**	**t**	**f**	**f**
f **t**	**t**	**f**	**t**	**t**	**t**	**f**
f **f**	**t**	**f**	**f**	**f**	**t**	**t**

Strength of operators: $\neg \quad \wedge\ \vee \quad \not\leftrightarrow \quad \longrightarrow\ \longleftrightarrow$

A) Laws (Tautologies, Equivalences)

of **Boolean propositional algebra** $(\{\mathbf{t}, \mathbf{f}\}; \wedge, \vee, \neg)$

Principle of duality:

(1)	$\mathbf{a} \wedge \mathbf{b} \Longleftrightarrow \mathbf{b} \wedge \mathbf{a}$; $\quad \mathbf{a} \vee \mathbf{b} \Longleftrightarrow \mathbf{b} \vee \mathbf{a}$	Commutativity
(2)	$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} \Longleftrightarrow \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ $(\mathbf{a} \vee \mathbf{b}) \vee \mathbf{c} \Longleftrightarrow \mathbf{a} \vee (\mathbf{b} \vee \mathbf{c})$	Associativity
(3)	$\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c}) \Longleftrightarrow (\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c})$ $\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) \Longleftrightarrow (\mathbf{a} \vee \mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{c})$	Distributivity
(4)	$\mathbf{a} \wedge \mathbf{a} \Longleftrightarrow \mathbf{a}$; $\quad \mathbf{a} \vee \mathbf{a} \Longleftrightarrow \mathbf{a}$	Idempotence
(5)	$\mathbf{a} \wedge (\mathbf{a} \vee \mathbf{b}) \Longleftrightarrow \mathbf{a}$ $\mathbf{a} \vee (\mathbf{a} \wedge \mathbf{b}) \Longleftrightarrow \mathbf{a}$	Absorption
(6)	$\mathbf{a} \wedge \mathbf{t} \Longleftrightarrow \mathbf{a}$; $\quad \mathbf{a} \vee \mathbf{f} \Longleftrightarrow \mathbf{a}$	Neutral element
(7)	$\mathbf{a} \wedge \mathbf{f} \Longleftrightarrow \mathbf{f}$; $\quad \mathbf{a} \vee \mathbf{t} \Longleftrightarrow \mathbf{t}$	Domination
(8)	$\mathbf{a} \wedge \neg \mathbf{a} \Longleftrightarrow \mathbf{f}$; $\quad \mathbf{a} \vee \neg \mathbf{a} \Longleftrightarrow \mathbf{t}$	Complementary element
(9)	$\neg (\neg \mathbf{a}) \Longleftrightarrow \mathbf{a}$	Double negation
(10)	$\neg (\mathbf{a} \wedge \mathbf{b}) \Longleftrightarrow \neg \mathbf{a} \vee \neg \mathbf{b}$ $\neg (\mathbf{a} \vee \mathbf{b}) \Longleftrightarrow \neg \mathbf{a} \wedge \neg \mathbf{b}$	De Morgan

B) Laws (Tautologies, Equivalences) of **Alternative** ($\not\leftrightarrow$, $\oplus$)

(1)	$a \not\leftrightarrow b \iff b \not\leftrightarrow a$	Commutativity
(2)	$(a \not\leftrightarrow b) \not\leftrightarrow c \iff a \not\leftrightarrow (b \not\leftrightarrow c)$	Associativity
(3)	$a \wedge (b \not\leftrightarrow c) \iff a \wedge b \not\leftrightarrow a \wedge c$	Distributivity of $\wedge$ over $\not\leftrightarrow$
(4)	$a \not\leftrightarrow f \iff a$	Neutral element
(5)	$a \not\leftrightarrow t \iff \neg a$	Negation
	$\neg(a \not\leftrightarrow b) \iff a \not\leftrightarrow b \not\leftrightarrow t \iff \neg a \not\leftrightarrow b \iff a \not\leftrightarrow \neg b$	
(6)	$a \not\leftrightarrow \neg a \iff t$	Complementary element
	$a \not\leftrightarrow a \iff f$	
(7)	$a \not\leftrightarrow a \not\leftrightarrow a \iff a$	Idempotence
(8)	$a \not\leftrightarrow b \iff \neg a \not\leftrightarrow \neg b$	Contraposition
(9)	$a \not\leftrightarrow b \iff (a \wedge \neg b) \vee (\neg a \wedge b)$	
(10)	$a \not\leftrightarrow b \iff (a \vee b) \wedge (\neg a \vee \neg b)$	
(11)	$a \not\leftrightarrow b \iff \neg(a \longleftrightarrow b)$	
(12)	$a \not\leftrightarrow b \iff (a \longrightarrow b) \longrightarrow \neg(b \longrightarrow a)$	
(13)	$a \vee b \iff a \not\leftrightarrow b \not\leftrightarrow (a \wedge b)$	
(14)	$a \wedge b \iff a \not\leftrightarrow b \not\leftrightarrow (a \vee b)$	

C) Laws (Tautologies, Equivalences) of **Biconditional** $(\longleftrightarrow, \ominus)$

(1)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\iff$	$\mathbf{b} \longleftrightarrow \mathbf{a}$	Commutativity
(2)	$(\mathbf{a} \longleftrightarrow \mathbf{b}) \longleftrightarrow \mathbf{c}$	$\iff$	$\mathbf{a} \longleftrightarrow (\mathbf{b} \longleftrightarrow \mathbf{c})$	Associativity
(3)	$\mathbf{a} \vee (\mathbf{b} \longleftrightarrow \mathbf{c})$	$\iff$	$\mathbf{a} \vee \mathbf{b} \longleftrightarrow \mathbf{a} \vee \mathbf{c}$	Distributivity of $\vee$ over $\longleftrightarrow$
(4)	$\mathbf{a} \longleftrightarrow \mathbf{t}$	$\iff$	$\mathbf{a}$	Neutral element
(5)	$\mathbf{a} \longleftrightarrow \mathbf{f}$	$\iff$	$\neg \mathbf{a}$	Negation
	$\neg(\mathbf{a} \longleftrightarrow \mathbf{b})$	$\iff$	$\mathbf{a} \longleftrightarrow \mathbf{b} \longleftrightarrow \mathbf{f} \iff \neg \mathbf{a} \longleftrightarrow \mathbf{b} \iff \mathbf{a} \longleftrightarrow \neg \mathbf{b}$	
(6)	$\mathbf{a} \longleftrightarrow \neg \mathbf{a}$	$\iff$	$\mathbf{f}$	Complementary element
	$\mathbf{a} \longleftrightarrow \mathbf{a}$	$\iff$	$\mathbf{t}$	
(7)	$\mathbf{a} \longleftrightarrow \mathbf{a} \longleftrightarrow \mathbf{a}$	$\iff$	$\mathbf{a}$	Idempotence
(8)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\iff$	$\neg \mathbf{a} \longleftrightarrow \neg \mathbf{b}$	Contraposition
(9)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\iff$	$(\mathbf{a} \wedge \mathbf{b}) \vee (\neg \mathbf{a} \wedge \neg \mathbf{b})$	
(10)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\iff$	$(\mathbf{a} \vee \neg \mathbf{b}) \wedge (\neg \mathbf{a} \vee \mathbf{b})$	
(11)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\iff$	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{b} \longrightarrow \mathbf{a})$	
(12)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\iff$	$\neg(\mathbf{a} \not\longleftrightarrow \mathbf{b})$	
(13)	$\mathbf{a} \vee \mathbf{b}$	$\iff$	$\mathbf{a} \longleftrightarrow \mathbf{b} \longleftrightarrow (\mathbf{a} \wedge \mathbf{b})$	
(14)	$\mathbf{a} \wedge \mathbf{b}$	$\iff$	$\mathbf{a} \longleftrightarrow \mathbf{b} \longleftrightarrow (\mathbf{a} \vee \mathbf{b})$	

D) Laws (Tautologies, Equivalences) of **Conditional**
(antecedent $\longrightarrow$ consequent)

(1) $$\mathbf{a} \longrightarrow \mathbf{b} \iff \neg\, \mathbf{a} \vee \mathbf{b}$$

(2) $$\neg\,(\mathbf{a} \longrightarrow \mathbf{b}) \iff \mathbf{a} \wedge \neg\, \mathbf{b}$$ Negation

(3) $$\mathbf{a} \longrightarrow \mathbf{b} \iff \neg\, \mathbf{b} \longrightarrow \neg\, \mathbf{a}$$ Contraposition

(4) $$\mathbf{a} \longrightarrow \mathbf{t} \iff \mathbf{t}\,; \qquad \mathbf{f} \longrightarrow \mathbf{a} \iff \mathbf{t}$$

(5) $$\mathbf{t} \longrightarrow \mathbf{a} \iff \mathbf{a}\,; \qquad \mathbf{a} \longrightarrow \mathbf{f} \iff \neg\, \mathbf{a}$$

(6) $$\mathbf{a} \longrightarrow \mathbf{a} \iff \mathbf{t}\,; \qquad \neg\, \mathbf{a} \longrightarrow \mathbf{a} \iff \mathbf{a}\,; \qquad \mathbf{a} \longrightarrow \neg\, \mathbf{a} \iff \neg\, \mathbf{a}$$

(7) $$\mathbf{a} \longrightarrow (\mathbf{b} \wedge \mathbf{c}) \iff (\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{a} \longrightarrow \mathbf{c})$$ left-sided distributivity of $\longrightarrow$ over $\wedge$

$$(\mathbf{a} \wedge \mathbf{b}) \longrightarrow \mathbf{c} \iff (\mathbf{a} \longrightarrow \mathbf{c}) \vee (\mathbf{b} \longrightarrow \mathbf{c})$$

(8) $$\mathbf{a} \longrightarrow (\mathbf{b} \vee \mathbf{c}) \iff (\mathbf{a} \longrightarrow \mathbf{b}) \vee (\mathbf{a} \longrightarrow \mathbf{c})$$ left-sided distributivity of $\longrightarrow$ over $\vee$

$$(\mathbf{a} \vee \mathbf{b}) \longrightarrow \mathbf{c} \iff (\mathbf{a} \longrightarrow \mathbf{c}) \wedge (\mathbf{b} \longrightarrow \mathbf{c})$$

(9) $$\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \mathbf{c}) \iff \mathbf{b} \longrightarrow (\mathbf{a} \longrightarrow \mathbf{c})$$ interchange of premises

$$\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \mathbf{c}) \iff (\mathbf{a} \wedge \mathbf{b}) \longrightarrow \mathbf{c}$$ importation and exportation

(10) $$\mathbf{a} \longrightarrow (\mathbf{a} \longrightarrow \mathbf{b}) \iff \mathbf{a} \longrightarrow \mathbf{b}$$ reinforcement rule

(11) $$\mathbf{a} \longrightarrow \mathbf{b} \iff \neg\,(\mathbf{a} \not\longleftrightarrow (\mathbf{a} \wedge \mathbf{b}))$$

(12) $$\mathbf{a} \longrightarrow \mathbf{b} \iff \mathbf{a} \longleftrightarrow (\mathbf{a} \wedge \mathbf{b}) \iff \mathbf{b} \longleftrightarrow (\mathbf{a} \vee \mathbf{b})$$

(13) $$\mathbf{a} \vee \mathbf{b} \iff \neg\, \mathbf{a} \longrightarrow \mathbf{b} \iff (\mathbf{a} \longrightarrow \mathbf{b}) \longrightarrow \mathbf{b}$$

(14) $$\mathbf{a} \wedge \mathbf{b} \iff \neg\,(\mathbf{a} \longrightarrow \neg\, \mathbf{b})$$

E) Laws (Tautologies) using **Implication** $(\Longrightarrow)$

(1)	$\mathbf{f}$	$\Longrightarrow$	$\mathbf{a}$	ex falso quodlibet
(2)	$\mathbf{a}$	$\Longrightarrow$	$\mathbf{t}$	ex quodlibet verum
(3)	$\mathbf{a}$	$\Longrightarrow$	$\mathbf{a}$	identity law
(4)	$\mathbf{a} \wedge \mathbf{b}$	$\Longrightarrow$	$\mathbf{a}$	simplification
(5)	$\mathbf{a}$	$\Longrightarrow$	$\mathbf{a} \vee \mathbf{b}$	addition
(6)	$\mathbf{a} \wedge \mathbf{b}$	$\Longrightarrow$	$\mathbf{a} \vee \mathbf{b}$	conjunction implies disjunction
(7)	$\neg\,\mathbf{a}$	$\Longrightarrow$	$\mathbf{a} \longrightarrow \mathbf{b}$	denial of the antecedent
(8)	$\mathbf{b}$	$\Longrightarrow$	$\mathbf{a} \longrightarrow \mathbf{b}$	affirmation of the consequent
(9)	$\mathbf{a} \wedge (\mathbf{a} \longrightarrow \mathbf{b})$	$\Longrightarrow$	$\mathbf{b}$	modus ponens
	$\mathbf{a} \wedge (\mathbf{a} \vee \mathbf{b} \longrightarrow \mathbf{c})$	$\Longrightarrow$	$\mathbf{c}$	
(10)	$\neg\,\mathbf{b} \wedge (\mathbf{a} \longrightarrow \mathbf{b})$	$\Longrightarrow$	$\neg\,\mathbf{a}$	modus tollens
(11)	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{b} \longrightarrow \mathbf{c})$	$\Longrightarrow$	$\mathbf{a} \longrightarrow \mathbf{c}$	hypothetical syllogism (modus barbara)
	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{b} \vee \mathbf{c} \longrightarrow \mathbf{d})$	$\Longrightarrow$	$\mathbf{a} \longrightarrow \mathbf{d}$	
(12)	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\neg\mathbf{a} \longrightarrow \mathbf{b})$	$\Longrightarrow$	$\mathbf{b}$	constructive dilemma
	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\neg\mathbf{a} \longrightarrow \mathbf{b})$	$\Longleftrightarrow$	$\mathbf{b}$	
(13)	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{a} \longrightarrow \neg\mathbf{b})$	$\Longrightarrow$	$\neg\,\mathbf{a}$	destructive dilemma
	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{a} \longrightarrow \neg\mathbf{b})$	$\Longleftrightarrow$	$\neg\,\mathbf{a}$	(reductio ad absurdum)

RT1) Replacement theorem for tautologies

Let $\mathbf{B}(\underline{\mathbf{z}})$ be an arbitrary propositional form and $\mathbf{x_i} \in \text{set}(\underline{\mathbf{x}})$ of $\mathbf{A}(\underline{\mathbf{x}})$.

Then:

IF $\mathbf{A}(\ldots,\mathbf{x_i},\ldots) \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{A}(\ldots,\mathbf{B}(\underline{\mathbf{z}}),\ldots) \Longleftrightarrow \mathbf{t}$

e.g. IF $\mathbf{a} \wedge (\mathbf{a} \longrightarrow \mathbf{b}) \Longrightarrow \mathbf{b}$, THEN $\mathbf{A}(\underline{\mathbf{x}}) \wedge (\mathbf{A}(\underline{\mathbf{x}}) \longrightarrow \mathbf{B}(\underline{\mathbf{x}})) \Longrightarrow \mathbf{B}(\underline{\mathbf{x}})$

e.g. IF $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{B}(\underline{\mathbf{x}})$, THEN $\text{dual}(\mathbf{A}(\underline{\mathbf{x}})) \Longleftrightarrow \text{dual}(\mathbf{B}(\underline{\mathbf{x}}))$

e.g. IF $\mathbf{A}(\underline{\mathbf{x}}) \Longrightarrow \mathbf{B}(\underline{\mathbf{x}})$, THEN $\text{dual}(\mathbf{B}(\underline{\mathbf{x}})) \Longrightarrow \text{dual}(\mathbf{A}(\underline{\mathbf{x}}))$

RT2) Replacement theorem for equivalent partial propositional forms

Let $\mathbf{A}(\underline{\mathbf{x}},\mathbf{B}(\underline{\mathbf{z}}))$ be an arbitrary propositional form and $\mathbf{B}$ a partial form of $\mathbf{A}$, with $\text{set}(\underline{\mathbf{z}}) \subseteq \text{set}(\underline{\mathbf{x}})$. Then:

IF $\mathbf{B}(\underline{\mathbf{z}}) \Longleftrightarrow \mathbf{C}(\underline{\mathbf{z}})$, THEN $\mathbf{A}(\underline{\mathbf{x}},\mathbf{B}(\underline{\mathbf{z}})) \Longleftrightarrow \mathbf{A}(\underline{\mathbf{x}},\mathbf{C}(\underline{\mathbf{z}}))$

e.g. IF $\mathbf{x} \longrightarrow \mathbf{a} \Longleftrightarrow \neg\mathbf{x} \vee \mathbf{a}$,

THEN $(\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \Longleftrightarrow (\neg\mathbf{x} \vee \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b})$

SR) Substitution rule

$\mathbf{x_i} \wedge \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{x_i} \wedge \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t})$; $\quad \neg\mathbf{x_i} \wedge \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \neg\mathbf{x_i} \wedge \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f})$

$\mathbf{x_i} \vee \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{x_i} \vee \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f})$; $\quad \neg\mathbf{x_i} \vee \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \neg\mathbf{x_i} \vee \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t})$

ER) Expansion rules for propositional logic functions

Let $\mathbf{A}(\underline{\mathbf{x}})$ be a propositional logic function with $\mathbf{x_i} \in \text{set}(\underline{\mathbf{x}})$.

(1) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow [\mathbf{x_i} \wedge \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t})] \vee [\neg\mathbf{x_i} \wedge \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f})]$ (Shannon)

(Boole's fundamental theorem)

(2) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow [\neg\mathbf{x_i} \vee \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t})] \wedge [\mathbf{x_i} \vee \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f})]$

(3) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow [\mathbf{x_i} \longrightarrow \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t})] \wedge [\neg\mathbf{x_i} \longrightarrow \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f})]$

(4) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow [\mathbf{x_i} \wedge (\mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t}) \not\longleftrightarrow \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f}))] \not\longleftrightarrow \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f})$ (Davio)

RS) Rule of specialization

$$\mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t}) \wedge \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f}) \Longrightarrow \mathbf{A}(\underline{\mathbf{x}}) \Longrightarrow \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t}) \vee \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f})$$

$$\mathbf{x_i} \in \text{set}(\underline{\mathbf{x}})$$

RRD) Resolution rule in disjunctive form

$$(\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{b}) \Longleftrightarrow (\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{b}) \vee \underbrace{(\mathbf{a} \wedge \mathbf{b})}_{\text{resolvent}}$$

$$(\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{a}) \Longleftrightarrow \mathbf{a} \qquad \text{(special case)}$$

$$(\mathbf{a} \wedge \mathbf{b}) \Longrightarrow (\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{b}) \qquad \text{(from RRD)}$$

RRC) Resolution rule in conjunctive form

$$(\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \Longleftrightarrow (\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \wedge \underbrace{(\mathbf{a} \vee \mathbf{b})}_{\text{resolvent}}$$

$$(\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{a}) \Longleftrightarrow \mathbf{a} \qquad \text{(special case)}$$

$$(\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \Longrightarrow \mathbf{a} \vee \mathbf{b} \qquad \text{(from RRC)}$$

RRP) Resolution rule in conditional clause form (premise form)

$$(\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \Longleftrightarrow (\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \wedge \underbrace{(\neg\mathbf{a} \longrightarrow \mathbf{b})}_{\text{resolvent}}$$

$$(\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{a}) \Longleftrightarrow \mathbf{a} \qquad \text{(special case)}$$

$$(\neg\mathbf{a} \longrightarrow \neg\mathbf{x}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \Longrightarrow (\neg\mathbf{a} \longrightarrow \mathbf{b}) \qquad \text{(from RRP)}$$

FRI) Fundamental rule for implication

Let $\mathbf{A} \Longrightarrow \mathbf{B}$. Then:

IF $\mathbf{A} \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{B} \Longleftrightarrow \mathbf{t}$

FRE) Fundamental rule for equivalence

Let $\mathbf{A} \Longleftrightarrow \mathbf{B}$. Then:

$\mathbf{A} \Longleftrightarrow \mathbf{t}$ IF AND ONLY IF $\mathbf{B} \Longleftrightarrow \mathbf{t}$

FRC) Fundamental rule for tautological conjunction

$\mathbf{A} \wedge \mathbf{B} \Longleftrightarrow \mathbf{t}$ IF AND ONLY IF $\mathbf{A} \Longleftrightarrow \mathbf{t}$ AND $\mathbf{B} \Longleftrightarrow \mathbf{t}$

RI) Rules of inference (meta rules) for tautologies

Equivalence and implication

(1) $\mathbf{A} \Longleftrightarrow \mathbf{B}$ IF AND ONLY IF $\mathbf{A} \Longrightarrow \mathbf{B}$ AND $\mathbf{B} \Longrightarrow \mathbf{A}$ C11

Contraposition

(2) $\mathbf{A} \Longrightarrow \mathbf{B}$ IF AND ONLY IF $\neg\mathbf{B} \Longrightarrow \neg\mathbf{A}$ D3

Enhancement rule

(3) $\mathbf{A} \Longrightarrow \mathbf{B}$ IF AND ONLY IF $\mathbf{A} \Longleftrightarrow \mathbf{A} \wedge \mathbf{B}$ D12

$\mathbf{A} \Longrightarrow \mathbf{B}$ IF AND ONLY IF $\mathbf{B} \Longleftrightarrow \mathbf{A} \vee \mathbf{B}$ D12

Expansion rule

(4) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{t}$ IF AND ONLY IF $\mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{t}) \Longleftrightarrow \mathbf{t}$ AND $\mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f}) \Longleftrightarrow \mathbf{t}$

$$\mathbf{x}_i \in \mathrm{set}(\underline{\mathbf{x}})$$

Affirmation of the consequent

(5) IF $\mathbf{B} \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{A} \Longrightarrow \mathbf{B}$ E8

Modus ponens

(6) IF $\mathbf{A} \Longleftrightarrow \mathbf{t}$ AND $\mathbf{A} \Longrightarrow \mathbf{B}$, THEN $\mathbf{B} \Longleftrightarrow \mathbf{t}$ E9

Transitivity

(7) IF $\mathbf{A} \Longrightarrow \mathbf{B}$, AND $\mathbf{B} \Longrightarrow \mathbf{C}$, THEN $\mathbf{A} \Longrightarrow \mathbf{C}$ E11

IF $\mathbf{A} \Longleftrightarrow \mathbf{B}$, AND $\mathbf{B} \Longleftrightarrow \mathbf{C}$, THEN $\mathbf{A} \Longleftrightarrow \mathbf{C}$

Compatibility

(8) IF $\mathbf{A} \Longrightarrow \mathbf{B}$, THEN $\mathbf{A} \wedge \mathbf{C} \Longrightarrow \mathbf{B} \wedge \mathbf{C}$ E14

IF $\mathbf{A} \Longrightarrow \mathbf{B}$, THEN $\mathbf{A} \vee \mathbf{C} \Longrightarrow \mathbf{B} \vee \mathbf{C}$ E15

IF $\mathbf{A} \Longleftrightarrow \mathbf{B}$, THEN $\mathbf{A} \wedge \mathbf{C} \Longleftrightarrow \mathbf{B} \wedge \mathbf{C}$

IF $\mathbf{A} \Longleftrightarrow \mathbf{B}$, THEN $\mathbf{A} \vee \mathbf{C} \Longleftrightarrow \mathbf{B} \vee \mathbf{C}$

BB) The $\beta - \mathbf{t} - \mathbf{f}$–basis

$$
\begin{array}{lllll}
(1) & \mathbf{a} \iff \beta(\mathbf{a},\mathbf{t},\mathbf{f}); & \neg\mathbf{a} & \iff \beta(\mathbf{a},\mathbf{f},\mathbf{t}) \\
 & \iff \beta(\mathbf{a},\mathbf{a},\mathbf{f}); & & \iff \beta(\mathbf{a},\neg\mathbf{a},\mathbf{t}) \\
 & \iff \beta(\mathbf{a},\mathbf{t},\mathbf{a}); & & \iff \beta(\mathbf{a},\mathbf{f},\neg\mathbf{a}) \\
 & \iff \beta(\mathbf{t},\mathbf{a},\mathbf{b}); & & \iff \beta(\mathbf{t},\neg\mathbf{a},\neg\mathbf{b}) \\
 & \iff \beta(\mathbf{f},\mathbf{b},\mathbf{a}); & & \iff \beta(\mathbf{f},\neg\mathbf{b},\neg\mathbf{a}) \\
 & \iff \beta(\mathbf{x},\mathbf{a},\mathbf{a}); & & \iff \beta(\mathbf{x},\neg\mathbf{a},\neg\mathbf{a})
\end{array}
$$

$$
\begin{array}{lllll}
(2) & \mathbf{t} \iff \beta(\mathbf{a},\mathbf{t},\mathbf{t}); & \mathbf{f} & \iff \beta(\mathbf{a},\mathbf{f},\mathbf{f}) \\
 & \iff \beta(\mathbf{a},\mathbf{a},\mathbf{t}); & & \iff \beta(\mathbf{a},\neg\mathbf{a},\mathbf{f}) \\
 & \iff \beta(\mathbf{a},\mathbf{a},\neg\mathbf{a}); & & \iff \beta(\mathbf{a},\neg\mathbf{a},\mathbf{a}) \\
 & \iff \beta(\mathbf{a},\mathbf{t},\neg\mathbf{a}); & & \iff \beta(\mathbf{a},\mathbf{f},\mathbf{a})
\end{array}
$$

$$
\begin{array}{lllll}
(3) & \mathbf{a}\wedge\mathbf{b} \iff \beta(\mathbf{a},\mathbf{b},\mathbf{f}); & \mathbf{a}\vee\mathbf{b} & \iff \beta(\neg\mathbf{a},\mathbf{b},\mathbf{t}) \\
 & \iff \beta(\mathbf{a},\mathbf{b},\mathbf{a}); & & \iff \beta(\neg\mathbf{a},\mathbf{b},\mathbf{a}) \\
 & \iff \beta(\neg\mathbf{a},\mathbf{f},\mathbf{b}); & & \iff \beta(\mathbf{a},\mathbf{t},\mathbf{b}) \\
 & \iff \beta(\neg\mathbf{a},\mathbf{a},\mathbf{b}); & & \iff \beta(\mathbf{a},\mathbf{a},\mathbf{b})
\end{array}
$$

$$
\begin{array}{lll}
(4)\ \mathbf{a}\longrightarrow\mathbf{b} \iff \beta(\mathbf{a},\mathbf{b},\mathbf{t}) & \iff \beta(\mathbf{b},\mathbf{t},\neg\mathbf{a}) \\
\qquad\iff \beta(\neg\mathbf{a},\mathbf{t},\mathbf{b}) & \iff \beta(\mathbf{a},\mathbf{b},\neg\mathbf{a})
\end{array}
$$

$$
\begin{array}{l}
(5)\ \mathbf{a}\longleftrightarrow\mathbf{b} \iff \beta(\mathbf{a},\mathbf{b},\neg\mathbf{b}) \\
\qquad \mathbf{a}\not\longleftrightarrow\mathbf{b} \iff \beta(\mathbf{a},\neg\mathbf{b},\mathbf{b})
\end{array}
$$

BO) Laws using the β-operation

$$
\begin{array}{llll}
(1) & \beta(\mathbf{x},\mathbf{t},\mathbf{t}) \Longleftrightarrow \mathbf{t}; & & \beta(\mathbf{x},\mathbf{f},\mathbf{f}) \Longleftrightarrow \mathbf{f} \\
(2) & \beta(\mathbf{x},\mathbf{t},\mathbf{f}) \Longleftrightarrow \mathbf{x}; & & \beta(\mathbf{x},\mathbf{f},\mathbf{t}) \Longleftrightarrow \neg\mathbf{x} \\
(3) & \beta(\mathbf{t},\mathbf{a},\mathbf{b}) \Longleftrightarrow \mathbf{a}; & & \beta(\mathbf{f},\mathbf{a},\mathbf{b}) \Longleftrightarrow \mathbf{b}
\end{array}
$$

$$
\left.
\begin{array}{lrcl}
(4) & \neg\beta(\mathbf{x},\mathbf{a},\mathbf{b}) & \Longleftrightarrow & \beta(\mathbf{x},\neg\mathbf{a},\neg\mathbf{b}) \\
(5) & \beta(\neg\mathbf{x},\mathbf{a},\mathbf{b}) & \Longleftrightarrow & \beta(\mathbf{x},\mathbf{b},\mathbf{a}) \\
(6) & \text{dual } \beta(\mathbf{x},\mathbf{a},\mathbf{b}) & \Longleftrightarrow & \beta(\mathbf{x},\mathbf{b},\mathbf{a})
\end{array}
\right\}\text{laws of duality}
$$

$$
\left.
\begin{array}{lrcl}
(7) & \beta(\mathbf{x},\mathbf{a},\mathbf{b}) & \Longleftrightarrow & \beta(\mathbf{x},\mathbf{a},\mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{b}) \\
(8) & \beta(\mathbf{x},\mathbf{a},\mathbf{b}) & \Longleftrightarrow & \beta(\mathbf{x},\mathbf{a},\mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{b}) \\
(9) & \beta(\mathbf{x},\mathbf{a},\mathbf{a}) & \Longleftrightarrow & \mathbf{a}
\end{array}
\right\}\text{resolution rules}
$$

$$
(10)\quad \beta(\mathbf{x},\mathbf{A},\mathbf{B}) \quad\Longleftrightarrow\quad [\mathbf{x} \wedge (\mathbf{A} \wedge \neg\mathbf{B})] \vee [\neg\mathbf{x} \wedge (\neg\mathbf{A} \wedge \mathbf{B})] \vee (\mathbf{A} \wedge \mathbf{B})
$$

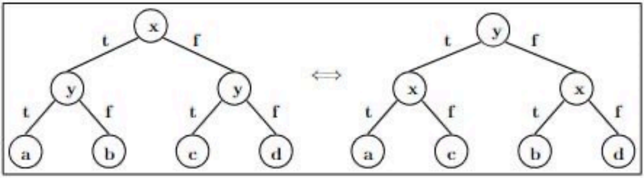
$\mathbf{A} \wedge \neg\mathbf{B},\ \neg\mathbf{A} \wedge \mathbf{B},\ \mathbf{A} \wedge \mathbf{B}$ are pairwise disjoint

$$
(11)\quad \beta(\mathbf{x},\mathbf{A},\mathbf{B}) \quad\Longleftrightarrow\quad [\neg\mathbf{x} \vee (\mathbf{A} \vee \neg\mathbf{B})] \wedge [\mathbf{x} \vee (\neg\mathbf{A} \vee \mathbf{B})] \wedge (\mathbf{A} \vee \mathbf{B})
$$

$$
(12)\quad \mathbf{f} \Longrightarrow \mathbf{a} \wedge \mathbf{b} \Longrightarrow \beta(\mathbf{x},\mathbf{a},\mathbf{b}) \Longrightarrow \mathbf{a} \vee \mathbf{b} \Longrightarrow \mathbf{t}
$$

enclosure law using $\beta(\mathbf{x},\mathbf{a},\mathbf{b})$

$$
(13)\quad \beta(\mathbf{x},\beta(\mathbf{y},\mathbf{a},\mathbf{b}),\beta(\mathbf{y},\mathbf{c},\mathbf{d})) \Longleftrightarrow \beta(\mathbf{y},\beta(\mathbf{x},\mathbf{a},\mathbf{c}),\beta(\mathbf{x},\mathbf{b},\mathbf{d}))
$$



$$(14)\quad \beta(\mathbf{x_i}, \mathbf{A}(\underline{\mathbf{x}}), \mathbf{B}(\underline{\mathbf{x}})) \Longleftrightarrow \beta(\mathbf{x_i}, \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t}), \mathbf{B}(\mathbf{x_i} \Leftrightarrow \mathbf{f}))$$

$\mathbf{x_i} \in \mathrm{set}(\underline{\mathbf{x}})$
substitution rule

$$\left.\begin{array}{ll} (15) & \beta(\mathbf{x}, \beta(\mathbf{x}, \mathbf{a}, \mathbf{b}), \mathbf{c}) \Longleftrightarrow \beta(\mathbf{x}, \mathbf{a}, \mathbf{c}) \\ (16) & \beta(\mathbf{x}, \mathbf{a}, \beta(\mathbf{x}, \mathbf{b}, \mathbf{c})) \Longleftrightarrow \beta(\mathbf{x}, \mathbf{a}, \mathbf{c}) \end{array}\right\} \text{special cases of substitution rules}$$

$$(17)\quad \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \beta(\mathbf{x_i}, \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{t}), \mathbf{A}(\mathbf{x_i} \Leftrightarrow \mathbf{f}))$$

$\mathbf{x_i} \in \mathrm{set}(\underline{\mathbf{x}})$;
From BO14 with $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{B}(\underline{\mathbf{x}})$
Expansion rules for propositional logic functions

$$(18)\quad \beta(\mathbf{A}(\underline{\mathbf{x}}), \mathbf{B}(\underline{\mathbf{x}}), \mathbf{C}(\underline{\mathbf{x}})) \Longleftrightarrow \mathbf{A}(\mathbf{t} := \mathbf{B}, \mathbf{f} := \mathbf{C})$$

composition rule
(if $\mathbf{A}$ in PNF)

$$(19)\quad \beta(\beta(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{a}, \mathbf{b}) \Longleftrightarrow \beta(\mathbf{x}, \beta(\mathbf{y}, \mathbf{a}, \mathbf{b}), \beta(\mathbf{z}, \mathbf{a}, \mathbf{b}))$$

special case
of composition rule

$$\begin{array}{lrcl} (20) & \mathbf{A} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{t}, \mathbf{f}) \\ (21) & \neg\mathbf{A} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{f}, \mathbf{t}) \quad \Longleftrightarrow \mathbf{A}(\mathbf{t} := \mathbf{f}, \mathbf{f} := \mathbf{t}) \\ (22) & \mathbf{A} \wedge \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{B}, \mathbf{f}) \quad \Longleftrightarrow \mathbf{A}(\mathbf{t} := \mathbf{B}) \\ (23) & \mathbf{A} \vee \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{t}, \mathbf{B}) \quad \Longleftrightarrow \mathbf{A}(\mathbf{f} := \mathbf{B}) \\ (24) & \mathbf{A} \longrightarrow \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{B}, \mathbf{t}) \quad \Longleftrightarrow \mathbf{A}(\mathbf{t} := \mathbf{B}, \mathbf{f} := \mathbf{t}) \\ (25) & \mathbf{A} \longleftrightarrow \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{B}, \neg\mathbf{B}) \Longleftrightarrow \mathbf{A}(\mathbf{t} := \mathbf{B}, \mathbf{f} := \neg\mathbf{B}) \\ (26) & \mathbf{A} \not\longleftrightarrow \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \neg\mathbf{B}, \mathbf{B}) \Longleftrightarrow \mathbf{A}(\mathbf{t} := \neg\mathbf{B}, \mathbf{f} := \mathbf{B}) \end{array}$$

F) Logical Equivalence $(\Longleftrightarrow)$

$$(1) \qquad \neg \bigvee_{x \in M} Px \iff \exists_{x \in M} \neg Px$$

$$(2) \qquad \neg \exists_{x \in M} Px \iff \bigvee_{x \in M} \neg Px$$

$$(3) \qquad \bigvee_{x \in M} \bigvee_{y \in N} Pxy \iff \bigvee_{y \in N} \bigvee_{x \in M} Pxy$$

$$(4) \qquad \exists_{x \in M} \exists_{y \in N} Pxy \iff \exists_{y \in N} \exists_{x \in M} Pxy$$

$$(5) \qquad \bigvee_{x \in M} (Px \wedge Qx) \iff (\bigvee_{x \in M} Px) \wedge (\bigvee_{x \in M} Qx)$$

$$(6) \qquad \exists_{x \in M} (Px \vee Qx) \iff (\exists_{x \in M} Px) \vee (\exists_{x \in M} Qx)$$

$$(7) \qquad \bigvee_{x \in M} (S \vee Qx) \iff S \vee \bigvee_{x \in M} Qx$$

$$(8) \qquad \exists_{x \in M} (S \wedge Qx) \iff S \wedge \exists_{x \in M} Qx$$

$$(9) \qquad \bigvee_{x \in M} \exists_{y \in N} (Px \wedge Qy) \iff \exists_{y \in N} \bigvee_{x \in M} (Px \wedge Qy)$$

$$(10) \qquad \iff (\bigvee_{x \in M} Px) \wedge (\exists_{y \in N} Qy)$$

$$(11) \qquad \bigvee_{x \in M} \exists_{y \in N} (Px \vee Qy) \iff \exists_{y \in N} \bigvee_{x \in M} (Px \vee Qy)$$

$$(12) \qquad \iff (\bigvee_{x \in M} Px) \vee (\exists_{y \in N} Qy)$$

$$(13) \qquad \exists_{x \in M} (Px \longrightarrow Qx) \iff (\bigvee_{x \in M} Px) \longrightarrow (\exists_{x \in M} Qx)$$

G) (Logical) Implication $(\Longrightarrow)$

(1) $\forall_{x \in M} Px \Longrightarrow Px_1 , \quad Px_1 \Longrightarrow \exists_{x \in M} Px , \quad x_1 \in M$

(2) $\forall_{x \in M} Px \Longrightarrow \exists_{x \in M} Px$

(3) $\forall_{x \in M} (Px \vee Qx) \Longrightarrow (\forall_{x \in M} Px) \vee (\exists_{x \in M} Qx)$

(4) $\exists_{x \in M} (Px \wedge Qx) \Longrightarrow (\exists_{x \in M} Px) \wedge (\exists_{x \in M} Qx)$

(5) $(\forall_{x \in M} Px) \vee (\forall_{x \in M} Qx) \Longrightarrow \forall_{x \in M} (Px \vee Qx)$

(6) $(\forall_{x \in M} Px) \wedge (\exists_{x \in M} Qx) \Longrightarrow \exists_{x \in M} (Px \wedge Qx)$

(7) $\forall_{x \in M} \forall_{y \in M} Pxy \Longrightarrow \forall_{x \in M} Pxx$

(8) $\exists_{x \in M} \forall_{y \in N} Pxy \Longrightarrow \forall_{y \in N} \exists_{x \in M} Pxy$

(9) $\forall_{x \in M} (Px \longrightarrow Qx) \Longrightarrow \forall_{x \in M} Px \longrightarrow \forall_{x \in M} Qx$

(10) $\forall_{x \in M} (Px \longrightarrow Qx) \Longrightarrow \exists_{x \in M} Px \longrightarrow \exists_{x \in M} Qx$

(11) $(\forall_{x \in M} Px) \wedge \forall_{x \in M} (Px \longrightarrow Qx) \Longrightarrow \forall_{x \in M} Qx$

(12) $(\exists_{x \in M} Px) \wedge \forall_{x \in M} (Px \longrightarrow Qx) \Longrightarrow \exists_{x \in M} Qx$