Topic 5
Vector Calculus I

Dr Teh Yong Liang
Email: youliangzheng@gmail.com

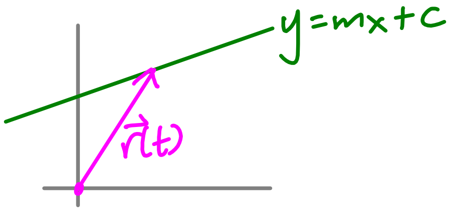
Outline

- A Vector Field
- A Gradient Field & its Scalar Potential
- Jacobian, Divergence & Curl
- Scalar & Vector Line Integrals
- Conservative Vector Fields
- Green's Theorem

Recall: Vector function of a line.

Concept of a Vector Function

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$



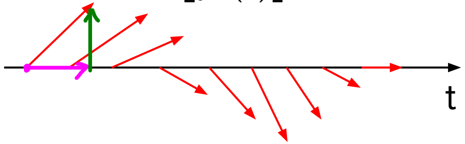
A vector function is one that takes in input/s and produces an output vector. The "machine" perspective of a multivariable vector function is shown below.

"Machine" or "Operation"

Input variables
x, y, z, ... → $$\mathbf{F}(x, y, z, \ldots) = \begin{bmatrix} f_1(x, y, z, \ldots) \\ f_2(x, y, z, \ldots) \\ \vdots \\ f_m(x, y, z, \ldots) \end{bmatrix}$$ → *Output vector, **F***

Conceptually, at each input coordinate **x** = (x, y, z, ...) in the domain of the function, there is an output vector of m dimensions. Hence, in the continuum of the input space, there exists a field of vectors. So a vector function is also called a vector field.

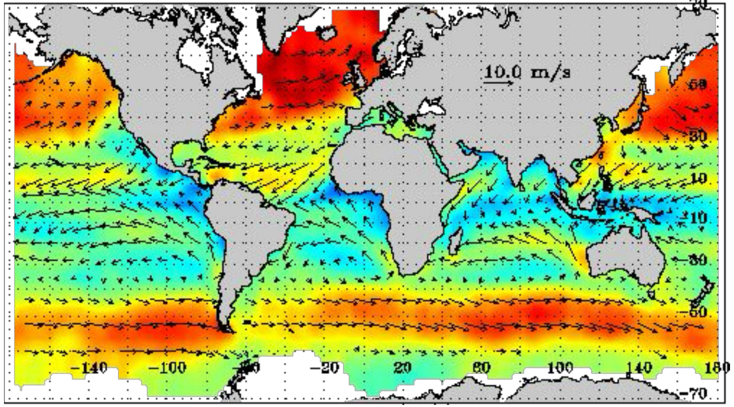
$$\mathbf{F}(t) = \begin{array}{cl} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} & \begin{array}{l} \text{x - component} \\ \text{y - component} \end{array} \end{array}$$



A Vector Field

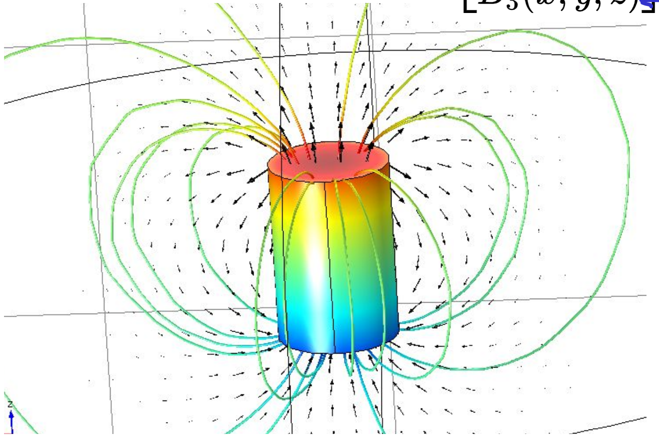
Graphically, a vector field can be represented as a field of vectors (duh). Some examples are shown.

$$\text{Wind velocity, } \mathbf{v}(x, y) = \begin{array}{cl} \begin{bmatrix} v_1(x, y) \\ v_2(x, y) \end{bmatrix} & \begin{array}{l} \leftarrow \text{x-comp.} \\ \leftarrow \text{y-comp.} \end{array} \end{array}$$



https://seos-project.eu/oceancurrents/oceancurrents-c02-p02.html

$$\text{Magnetic field, } \mathbf{B}(x, y, z) = \begin{array}{cl} \begin{bmatrix} B_1(x, y, z) \\ B_2(x, y, z) \\ B_3(x, y, z) \end{bmatrix} & \begin{array}{l} \leftarrow \text{x-comp.} \\ \leftarrow \text{y-comp.} \\ \leftarrow \text{z-comp.} \end{array} \end{array}$$



https://www.comsol.com/

A Vector Field

Example: Sketch the electric field for a negative point charge below. What happens to the electric field strength as the distance from the point charge increases?

$$E(x,y) = \frac{-1}{x^2+y^2}\begin{bmatrix} x \\ y \end{bmatrix}$$

At $(1,1)$,

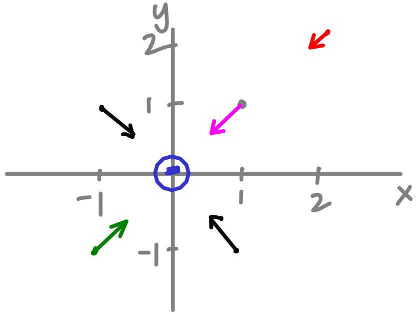
$$E(1,1) = \frac{-1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

At $(-1,-1)$,

$$E(-1,-1) = \frac{-1}{2}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

At $(2,2)$,

$$E(2,2) = \frac{-1}{8}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

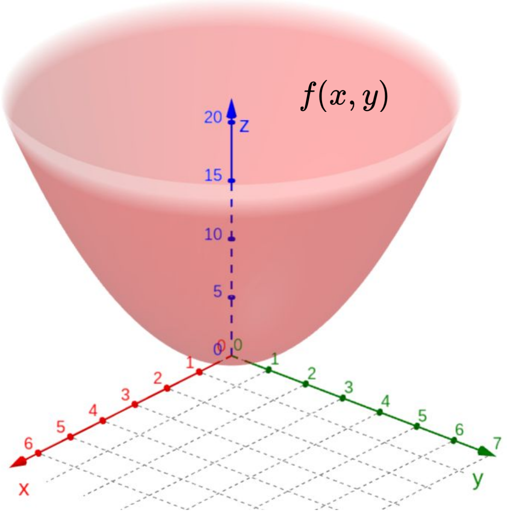


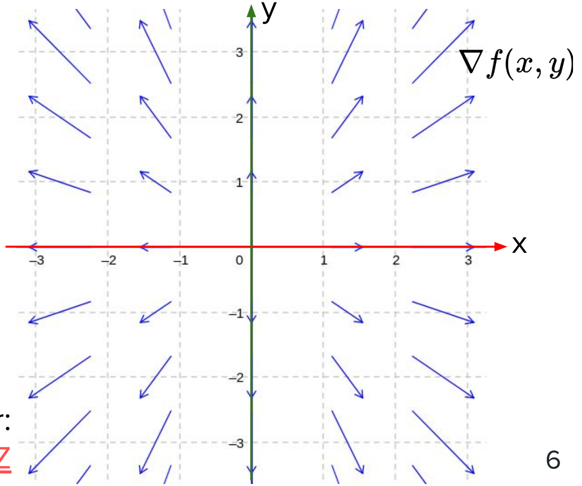
As the distance is increased, the electric field strength decreases.

A Gradient Field

A gradient field is simply a field of gradient vectors, which means a gradient field is also a vector field. For example, for the scalar function $f(x, y) = x^2 + y^2$, its gradient field is

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$





For any gradient field $\nabla f(\mathbf{x})$ where **x** is the input vector, the scalar function $f(\mathbf{x})$ is also called a scalar potential.

Note that all gradient fields are vector fields but not vice-versa as elaborated later.

2D vector field plotter:
https://www.geogebra.org/m/QPE4PaDZ

Scalar Potential

A scalar potential can be obtained from a gradient field by integration, since the gradient field is obtained by differentiating the scalar potential function. For example, consider the gradient field

$$\nabla f(x,y) = \begin{bmatrix} 3x^2y + y\cos(xy) \\ x^3 + x\cos(xy) + 3 \end{bmatrix} \leftarrow f_y$$

To get f(x,y), we can integrate $f_x(x,y)$ w.r.t. x first, i.e.

$\frac{\partial}{\partial x} g(y) = 0.$

$$f(x,y) = \int 3x^2y + y\cos(xy)\,dx = x^3y + \sin(xy) + g(y)$$

Note that **g(y) has to been included in the antiderivative because differentiating g(y) w.r.t. x gives zero**. The constant of integration is embedded into g(y).

Scalar Potential

Then, to obtain g(y), we differentiate w.r.t. y to obtain $f_y(x,y)$ as

$$\underline{f_y(x,y) = x^3 + x\cos(xy) + g'(y)}$$

Comparing to $f_y(x, y)$ from the gradient field, i.e.

$$f_y(x,y) = x^3 + x\cos(xy) + 3$$

we can observe that g'(y) = 3. So we have

$$g(y) = \int 3\,dy = 3y + c$$

Hence, the scalar potential function is

$$f(x,y) = x^3y + \sin(xy) + 3y + c$$

Scalar Potential

Exercise: For the earlier example, show that the same scalar potential can be obtained by integrating $f_y(x,y)$ instead.

$$\nabla f(x,y) = \begin{bmatrix} \underline{3x^2y} + \underline{y\cos(xy)} \\ x^3 + x\cos(xy) + 3 \end{bmatrix} \leftarrow f_x$$

$$\frac{\partial}{\partial y} g(x) = 0$$

$$f(x,y) = \int x^3 + x\cos(xy) + 3\,dy = x^3y + \sin(xy) + 3y + g(x)$$

$$f_x(x,y) = \underline{3x^2y} + \underline{y\cos(xy)} + 0 + g'(x).$$

Comparing with 1st row of $\vec{\nabla} f$, $g'(x) = 0$.

$$\Rightarrow g(x) = C.$$

$$f(x,y) = x^3y + \sin(xy) + 3y + C \,//$$

Scalar Potential

Exercise: Evaluate the scalar potential for the gradient field below.

$$\nabla f(x,y,z) = \begin{bmatrix} e^x \sin y - yz \\ e^x \cos y - xz \\ z - xy \end{bmatrix} \begin{matrix} \leftarrow f_x \\ \leftarrow f_y \\ \leftarrow f_z . \end{matrix}$$

$\frac{\partial}{\partial z} g(x,y) = 0.$

$f(x,y,z) = \int z - xy \, dz = \frac{z^2}{2} - xyz + g(x,y)$

$f_x(x,y,z) = 0 - yz + g_x(x,y)$

$\frac{\partial}{\partial x} h(y) = 0.$

Comparing: $g_x(x,y) = e^x \sin y \longrightarrow g(x,y) = \int e^x \sin y \, dx = e^x \sin y + h(y).$

$\Rightarrow f(x,y,z) = \frac{z^2}{2} - xyz + e^x \sin y + h(y).$

$f_y(x,y,z) = 0 - xz + e^x \cos y + h'(y)$

ANS: $f(x,y,z) = e^x \sin y - xyz + \frac{z^2}{2} + c.$

Comparing: $h'(y) = 0 \rightarrow h(y) = C$.

$$\therefore f(x,y,z) = \frac{z^2}{2} - xyz + e^x \sin y + C$$

Jacobian of a Vector Field

Analogous to the gradient of a scalar field (function), the Jacobian of a vector field represents the rate of change (ROC) of the vector function w.r.t. each independent variable. Eg, for a vector field

$$\mathbf{F}(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$$

its Jacobian (matrix) is

$$\mathbf{J}_{\mathbf{F}}(x, y) = \begin{bmatrix} \dfrac{\partial \mathbf{F}}{\partial x} & \dfrac{\partial \mathbf{F}}{\partial y} \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} & \begin{array}{l} \leftarrow \text{ROC of x-component of } \mathbf{F} \\ \\ \leftarrow \text{ROC of y-component of } \mathbf{F} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{ROC w.r.t. x} & \text{ROC w.r.t. y} \end{array} & \end{array}$$

Jacobian of a Vector Field

Generally, for a vector function **F** of n inputs and m output vector components, its Jacobian is

$$\mathbf{J}_{\mathbf{F}}(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial x_1} & \frac{\partial \mathbf{F}}{\partial x_2} & \cdots & \frac{\partial \mathbf{F}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Hence, the Jacobian of a vector function is a matrix containing **rates of change of each output vector component w.r.t. each input variable**. It is the ‘gradient’ of a vector field.

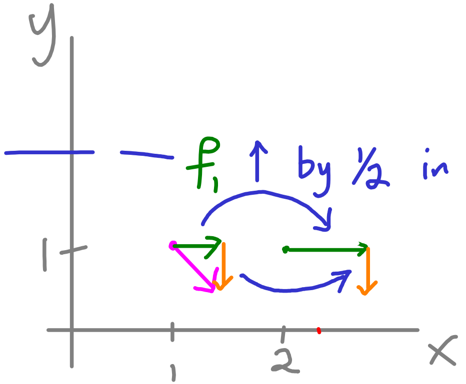
Jacobian of a Vector Field

Example: Determine the Jacobian of the vector field below and explain its meaning graphically with respect to the vector field at point (1, 1).

$$\mathbf{F}(x,y) = \begin{bmatrix} x/2 \\ -y/2 \end{bmatrix} \rightarrow \vec{F}(1,1) = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix},\ \vec{F}(2,1) = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

$$J_{\vec{F}}(x,y) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{array}$$



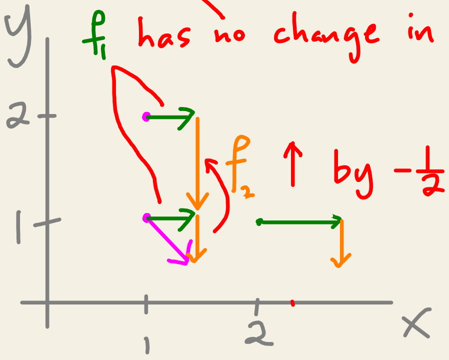
$f_1 \uparrow$ by ½ in x-dir.

f_2 has no change in x-dir.

$$\vec{F}(1,2) = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$$

$$J_{\vec{F}}(x,y) = \begin{array}{c} \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{array} \end{array}$$

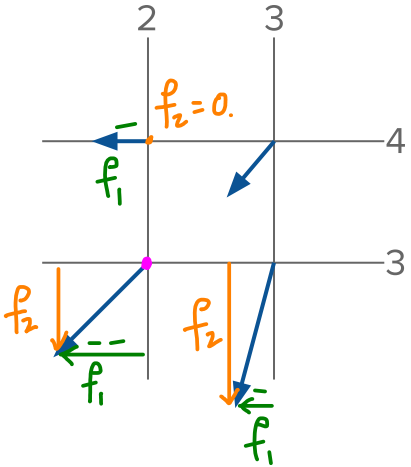
f_1 has no change in y-dir.



f_2 $\uparrow$ by $-\frac{1}{2}$ in y-dir.

Jacobian of a Vector Field

Exercise: Given the vector field **F**(x, y) depicted at 4 points shown below, state the polarity (estimated) of each element in $\mathbf{J}_\mathbf{F}(2, 3)$.



$\frac{\partial f_1}{\partial x}$ $\frac{\partial f_1}{\partial y}$

$$J_{\vec{F}}(2,3) = \begin{pmatrix} + & + \\ - & + \end{pmatrix}$$

(polarity)

$\frac{\partial f_2}{\partial x}$ $\frac{\partial f_2}{\partial y}$

ANS: Polarity of $\mathbf{J}_\mathbf{F}(2,3) = \begin{bmatrix} + & + \\ - & + \end{bmatrix}$

Divergence of a Vector Field

The divergence of a vector field is a **scalar** quantity that measures the degree of 'outflow-ness' the vector field is **at a point**. For a 2D & 3D vector field, the divergence are respectively defined as

$\frac{\partial}{\partial x}$

$$\nabla \cdot \mathbf{F} = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}, \qquad \nabla \cdot \mathbf{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

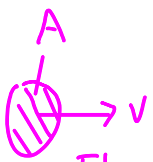
$\frac{\partial}{\partial y}$

Notice that the divergence is a sum of the **rate of change of each vector component in its own direction**.

But how does this scalar quantity measure 'outflow-ness'?

Divergence of a Vector Field

To understand the divergence intuitively, consider a 2D velocity field **V**(x,y) = $[v_1, v_2]^T$. At any point (x, y) in the field, consider an area element as shown below. We can see that the ‘net (volume) outflow’ from the area element is

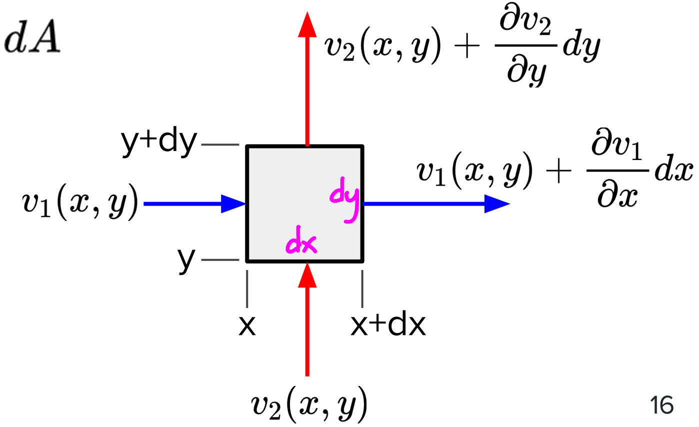


Flowrate = VA

$$\text{Net Outflow} = \left(v_1 + \frac{\partial v_1}{\partial x}dx\right)dy + \left(v_2 + \frac{\partial v_2}{\partial y}dy\right)dx - v_1 dy - v_2 dx$$

$$= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right)\underbrace{dx}_{+}\underbrace{dy}_{+} = (\nabla \cdot \mathbf{V})\, dA$$

Since dA > 0, the net outflow depends on the divergence of the velocity field. When the divergence is **positive**, it means there is **more outflow than inflow**, hence resulting in a **positive net outflow**.

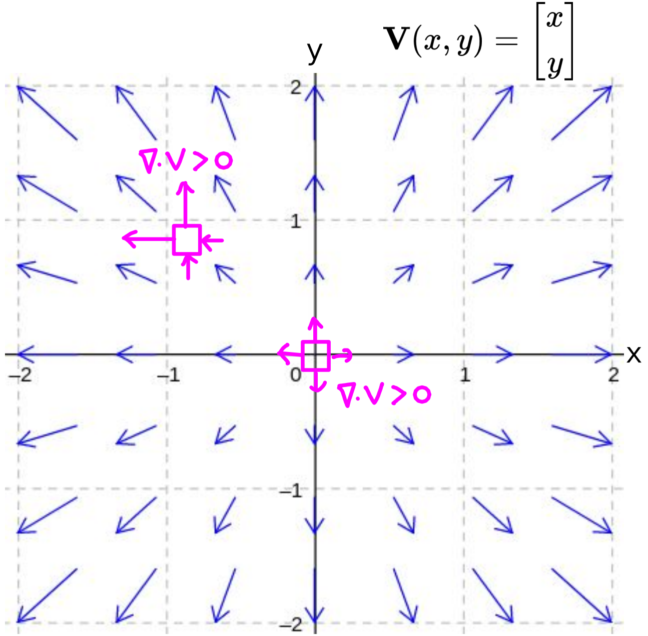


Divergence of a Vector Field

For example, consider the flow velocity field **V**(x, y) = [x, y]^T. The divergence is

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2$$

The **positive** divergence everywhere means at every point in the velocity field, there is **more outflow than inflow**, as can be verified by the **'expansionary' vector field** shown.



Draw an area element anywhere in this field and you can observe there is a positive net outflow across the element.

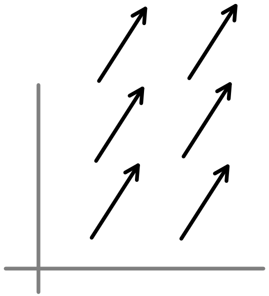
Divergence of a Vector Field

Exercise: For each vector field below, determine the divergence. For (a), explain the divergence with respect to the vector field.

a) $\mathbf{F}(x,y) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\nabla \cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$$

$$= 0 + 0 = 0.$$



Since the vector field is a constant vector field, the inflow to an element is always equal to the outflow, so the divergence is zero at all points.

b) $\mathbf{F}(x,y,z) = \begin{bmatrix} x^2y \\ xz \\ xyz \end{bmatrix}$

$$\nabla \cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 2xy + 0 + xy = 3xy.$$

ANS: a) 0. b) 3xy.

Curl of a Vector Field

The curl of a **3D** vector field is a **vector** quantity that measures the circulation (rotation effect) of the vector field **at a point**. It is defined by

$$\nabla \times \mathbf{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{bmatrix}$$

To understand the curl more intuitively, firstly consider the flow velocity field **V**(x, y, z) = $[y, 0, 0]^T$ as shown on the next slide. The curl is

$$\nabla \times \mathbf{V} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \leftarrow$$

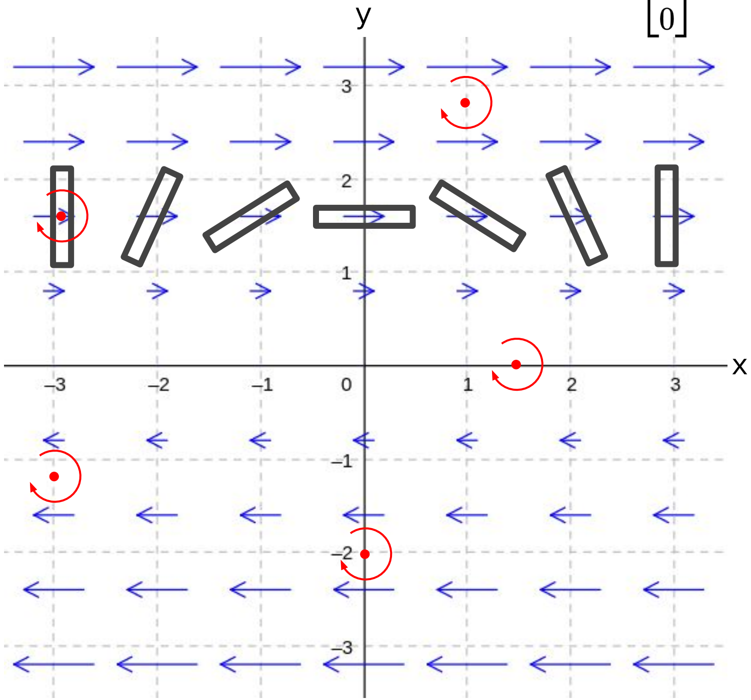
Curl of a Vector Field

Notice that the constant curl vector points in the negative z direction (into the screen). Using the right-hand rule, the circulation viewed from the top is clockwise.

This means, at each point in the vector field **V**, there is a **tendency for an object to rotate clockwise about the axis of rotation** given by the curl vector.

Hence, if the curl of a vector field is **not the zero vector**, than an **object flowing along the field will rotate about the curl axis** as it moves.

$$\mathbf{V}(x,y,z) = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}$$



Top view of vector field at any z value.

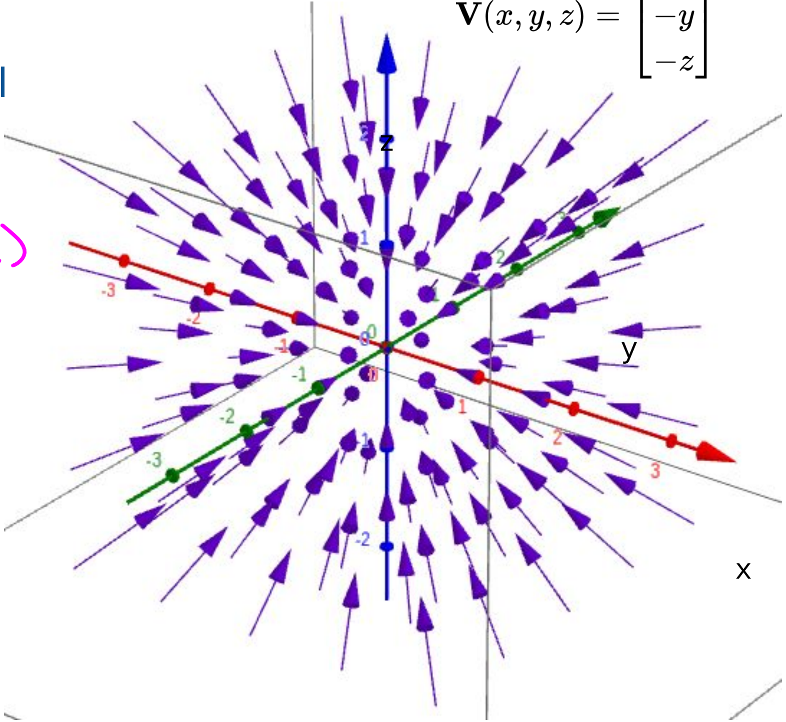
Curl of a Vector Field

Exercise: Using intuition, what do you think is the curl of the vector field shown? Compute it to verify.

$\vec{\nabla} \times \vec{V} = \vec{0}$ (since $\vec{V}$ is symmetric about the origin.)

$$\vec{\nabla} \times \vec{V} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} 0-0 \\ -(0-0) \\ 0-0 \end{pmatrix} = \vec{0} \; /\!/$$

$$\mathbf{V}(x,y,z) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$$



3D vector field plotter:
https://www.geogebra.org/m/u3xregNW

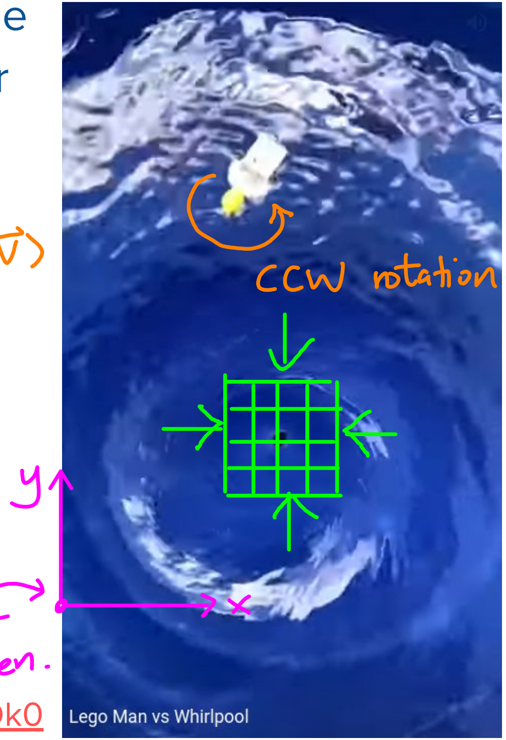
Divergence & Curl of a Whirlpool

Exercise: Watch the following video of a whirlpool. What is the average divergence and curl of the velocity field of the water flow (in the top-view 2D plane)?

Curl, $\vec{\nabla}\times\vec{V} \approx \begin{pmatrix}0\\0\\+\end{pmatrix}$ and the z-comp (or magnitude of $\vec{\nabla}\times\vec{V}$) increases towards the center of the whirlpool.

Avg $\vec{\nabla}\cdot\vec{V}$ for $V(x,y)$ is negative, since the water is $\underbrace{\text{flowing into a sink.}}_{\text{Net inflow}}$

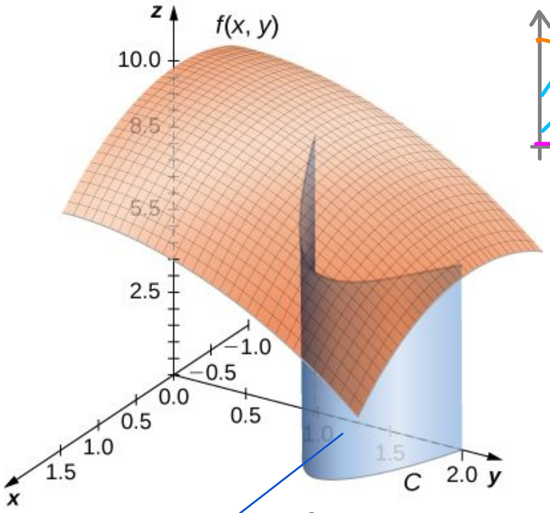
z-axis Pointing out of screen.

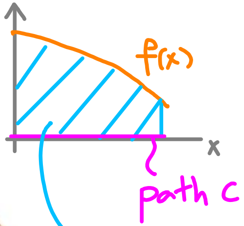


https://youtube.com/shorts/QwuC7f_HF3Y?si=wjDv-xqEoDB6IOk0

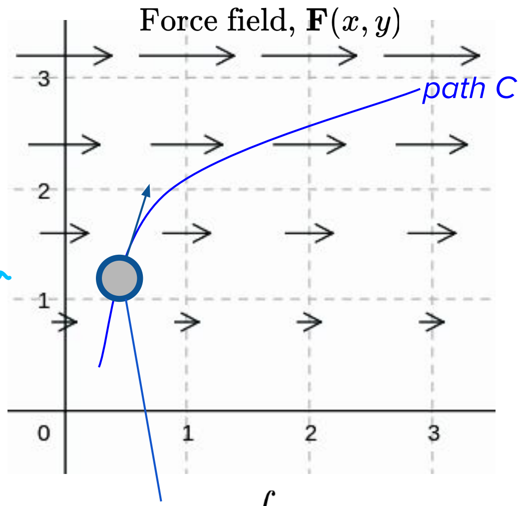
Line Integrals (aka Path integral.)

A line integral is simply an integration of a function along a line, or a path. If the integrand function is a **scalar** function, then we have a **scalar** line integral. If it is a **vector** function, then we have a **vector** line integral. An example of each is shown below.





$\int_C f(x)dx = \text{Area}$

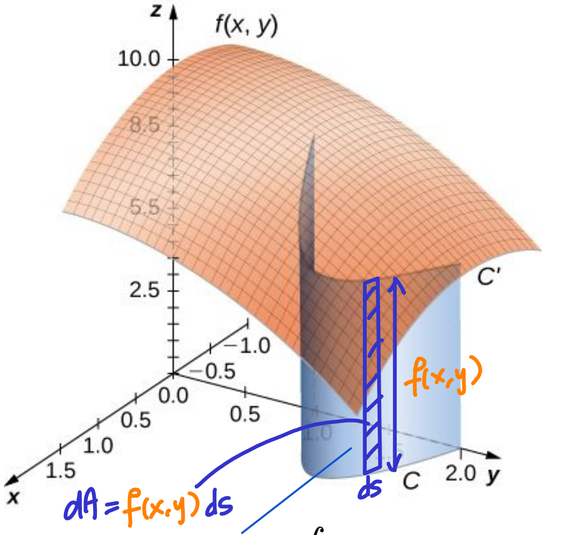


Scalar line integral $= \int_C f(x,y)\,ds =$ Area

Vector line integral $= \int_C \mathbf{F}(x,y) \cdot \mathbf{dr} =$ Work done

Scalar Line Integral

A scalar line integral is simply summing up the values of a function multiplied by an infinitesimal distance ds along a path C. As illustrated, for a function f(x, y), the line integral can be viewed as giving the area of a surface projected from path C towards the function surface.



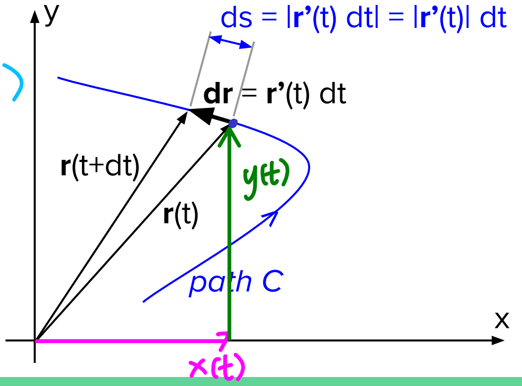
$$\text{Scalar line integral} = \int_C \underbrace{f(x,y)\,ds}_{dA} = \text{Area}$$

To evaluate the line integral more easily, **parameterization of the path C** can be applied. Let **r**(t) be a vector pointing to path C, we have

$\vec{r}\,'(t) = \frac{d\vec{r}}{dt}$ (velocity)

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \mathbf{r}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}, \; \mathbf{dr} = \mathbf{r}'(t)\,dt,$$

$$\Rightarrow ds = |\mathbf{dr}| = |\mathbf{r}'(t)\,dt| = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$



Scalar Line Integral

Hence, the scalar line integral of f(x, y) along path C parameterized by **r**(t) = [x(t), y(t)] is

$$\int_C f(x,y)\,ds = \int_C f(\mathbf{r}(t))\,\left|\mathbf{r}'(t)\right|\,dt = \int_C f(\mathbf{r}(t))\,\sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\,dt$$

Similarly , for a scalar line integral of f(x, y, z) along a path C in 3D space parameterized by **r**(t) = [x(t), y(t), z(t)], we have

$$\int_C f(x,y,z)\,ds = \int_C f(\mathbf{r}(t))\,\left|\mathbf{r}'(t)\right|\,dt = \int_C f(\mathbf{r}(t))\,\sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2 + \left[z'(t)\right]^2}\,dt$$

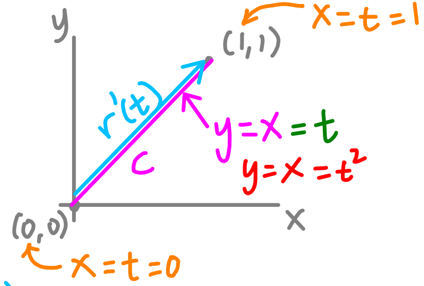
One can deduce that the total arc length of path C can be computed by the scalar line integral

$$\int_C ds = \int_C \left|\mathbf{r}'(t)\right|\,dt$$

Scalar Line Integral

Example: Evaluate the $\overbrace{\text{line integral}}^{L}$ of the function below along the straight line C on the plane from (0, 0) to (1, 1). Use the parameterizations x = t and x = t^2. What do you notice about line integral?

$$f(x, y) = x + y$$



Parameterize C_1:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} \rightarrow \vec{r}\,'(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$L = \int_C f(\vec{r}(t))\sqrt{(x')^2+(y')^2}\,dt = \int_{t=0}^{t=1} \underbrace{(t+t)}_{f(x(t),y(t)) = f(\vec{r}(t))}\sqrt{1^2+1^2}\,dt = 2\sqrt{2}\int_0^1 t\,dt = 2\sqrt{2}\left(\frac{t^2}{2}\right)\Big|_0^1$$

$$= \sqrt{2}\ //$$

ANS: $\sqrt{2}$.

Parameterize C_1:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t^2 \\ t^2 \end{pmatrix} \rightarrow \vec{r}\,'(t) = \begin{pmatrix} 2t \\ 2t \end{pmatrix}$$

$$L = \int_C f(\vec{r}(t))\sqrt{(x')^2 + (y')^2}\,dt = \int_{t=0}^{t=1} \underbrace{(t^2+t^2)}_{f(x(t),y(t)) = f(\vec{r}(t))} \sqrt{4t^2+4t^2}\,dt = 2\sqrt{8}\int_0^1 t^3\,dt = 2(2\sqrt{2})\left(\frac{t^4}{4}\right)\Big|_0^1$$

$$= \sqrt{2}$$

Notice that the line integrals are the same since the area is the same even when the parameterization is changed.

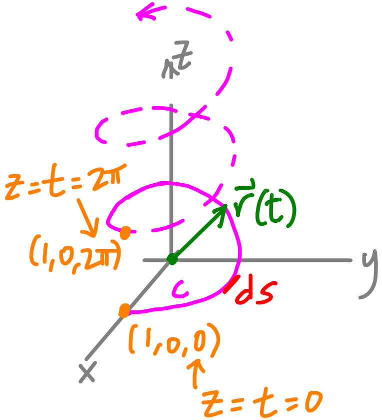
Scalar Line Integral

$\mathcal{L}$ C

Exercise: Evaluate the scalar line integral of the function below along the helix path parameterized by **r**(t) from (x, y, z) = (1, 0, 0) to (1, 0, 2π). What is the length of the helix path?

$$f(x,y,z) = xy + z, \quad \mathbf{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix} \leftarrow z$$

$$r'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$



$$\mathcal{L} = \int_C f(\vec{r}(t))\, ds = \int_0^{2\pi} (\cos t \sin t + t) \underbrace{\sqrt{\sin^2 t + \cos^2 t + 1}}_{\sqrt{2}}\, dt$$

ANS: $2\sqrt{2}\pi^2$. Length $= 2\sqrt{2}\pi$.

$$= \sqrt{2} \int_0^{2\pi} \frac{1}{2}\sin(2t) + t \, dt = \sqrt{2} \left[\frac{-1}{4}\cos(2t) + \frac{t^2}{2} \right] \Big|_0^{2\pi}$$

$$= \sqrt{2} \left[\frac{4\pi^2}{2} \right] = 2\sqrt{2}\,\pi^2 \;//$$

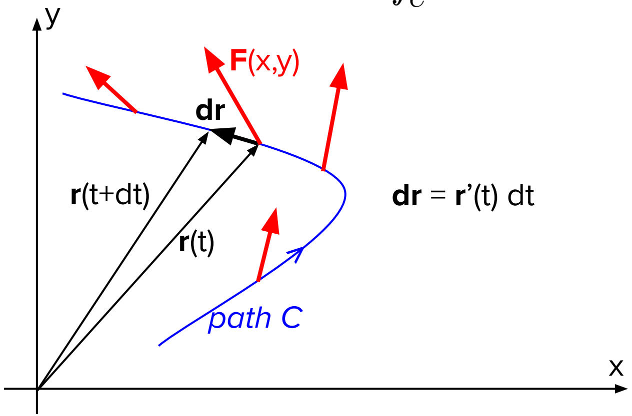
$$\text{Path length} = \int_C ds = \int_0^{2\pi} \underbrace{\sqrt{\sin^2 t + \cos^2 t + 1}}_{\sqrt{2}} \, dt = \sqrt{2}\, t \Big|_0^{2\pi} = 2\sqrt{2}\,\pi \;//$$

Vector Line Integral

Analogous to a scalar line integral, the vector line integral sums up the dot-product of a vector function with an infinitesimal displacement **dr** along a path C. Using parameterization **r**(t) of the path, the vector line integral of **F**(x, y) is

$$\begin{aligned}\int_C \mathbf{F}(x,y)\cdot d\mathbf{r} &= \int_C \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \\ &= \int_C \begin{bmatrix} f_1(\mathbf{r}(t)) \\ f_2(\mathbf{r}(t)) \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} dt \\ &= \int_C f_1(\mathbf{r}(t))x'(t) + f_2(\mathbf{r}(t))y'(t)\,dt\end{aligned}$$

$$\text{Vector line integral} = \int_C \mathbf{F}(x,y)\cdot d\mathbf{r}$$



dr: infinitesimal displacement vector.
r'(t): tangential velocity vector to path.

Vector Line Integral

Similarly, the vector line integral of **F**(x, y, z) over a path C in 3D space parameterized by **r**(t) = [x(t), y(t), z(t)] is

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_C \begin{bmatrix} f_1(\mathbf{r}(t)) \\ f_2(\mathbf{r}(t)) \\ f_3(\mathbf{r}(t)) \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} dt$$

$$= \int_C f_1(\mathbf{r}(t))x'(t) + f_2(\mathbf{r}(t))y'(t) + f_3(\mathbf{r}(t))z'(t)\,dt$$

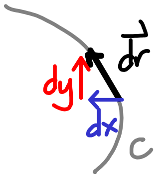
In the case where the vector field is a force field **F**, then the vector line integral gives the **work done** by the vector field on an object when it traversed path C. This is because the infinitesimal work done over a small displacement **dr** is dW = **F.dr**, so

$$\text{Work done, } W = \int_C dW = \int_C \mathbf{F}\cdot d\mathbf{r}$$

Vector Line Integral

Another commonly used way to express a vector line integral is by recognizing that

$$\mathbf{dr} = \mathbf{r}'(t)\,dt = \begin{bmatrix} x'(t)\,dt \\ y'(t)\,dt \\ z'(t)\,dt \end{bmatrix} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$



which gives

$$\vec{dr} = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$\int_C \mathbf{F}(x,y) \cdot \mathbf{dr} = \int_C \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \int_C f_1\,dx + f_2\,dy$$

$$\int_C \mathbf{F}(x,y,z) \cdot \mathbf{dr} = \int_C \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \int_C f_1\,dx + f_2\,dy + f_3\,dz$$

The above is called the differential form of a vector line integral.

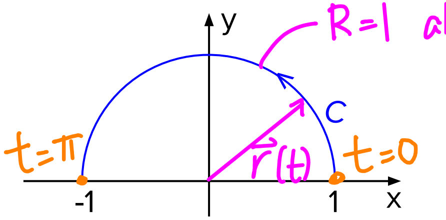
Vector Line Integral

Example: Evaluate the $\overbrace{\text{vector line integral}}^{L}$ of the function below over a semi-circular path C shown.

$$\mathbf{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Parameterize C by $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$

$$r'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$



R=1 along C.

$$\vec{F}(\vec{r}(t)) \cdot r'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \sin^2 t + \cos^2 t = 1.$$

$$L = \int_0^{\pi} 1 \, dt = t \Big|_0^{\pi} = \pi \quad //$$

ANS: π.

Vector Line Integral

Exercise: Evaluate the work done on a object subjected to the radial force field (in Newtons) below over a path parameterized by **r**(t) (in meters) from (x, y, z) = (0, 0, 0) to (1, 3, 2). What is the work done if the path is traversed by the object in the reverse direction?

x = t from 0 to 1.

$$\mathbf{F}(x,y,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{r}(t) = \begin{bmatrix} t \\ 3t^2 \\ 2t^3 \end{bmatrix} \rightarrow r'(t) = \begin{pmatrix} 1 \\ 6t \\ 6t^2 \end{pmatrix}$$

$$\vec{F} \cdot \vec{r}' = \begin{pmatrix} t \\ 3t^2 \\ 2t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 6t \\ 6t^2 \end{pmatrix} = t + 18t^3 + 12t^5$$

$$W = \int_0^1 \vec{F} \cdot \vec{r}' \, dt = \int_0^1 t + 18t^3 + 12t^5 \, dt = \cdots = 7\,J$$

In reverse dir,

$$W = -7J$$

since the path is reversed, r'(t) would be multiplied by -1, so the line integral become multiplied by a -1.

ANS: 7 J. -7 J.

A Conservative Vector Field

A conservative vector field is one where its **line integral is independent of the path** taken, but **only depends on the start and end points of the path**. For example, a gravitational force field **F**(x, y, z) (near Earth's surface) given by

$$\mathbf{F}(x, y, z) = [0, 0, -mg]^T = \vec{\nabla E}$$

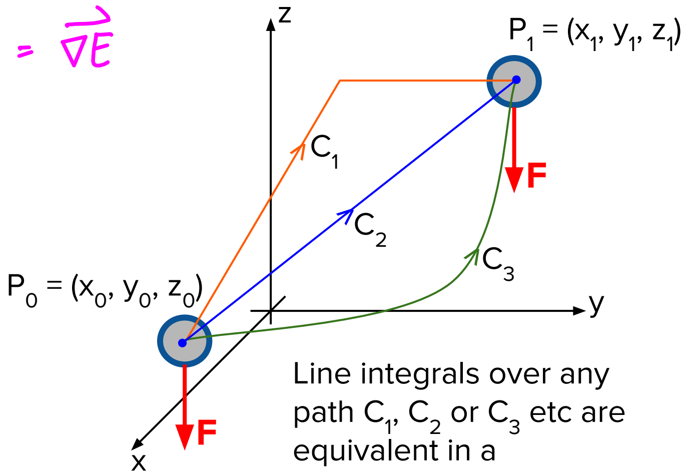
E_x E_y E_z

is conservative, because

$$\int_C \mathbf{F}(x, y, z) \cdot \mathbf{dr} = \int_{t_0}^{t_1} -mgz'(t)\,dt$$

$$= -mg \int_{z_0}^{z_1} dz = mg(z_0 - z_1)$$

which **only depends on the height z at the start and end points**.



Line integrals over any path C_1, C_2 or C_3 etc are equivalent in a conservative vector field.

A Conservative Vector Field

And, the scalar potential E(x, y, z) of the gravitational field can be inspected to be

$$E(x, y, z) = -mgz + c$$

which means the line integral can be evaluated by

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = mg(z_0 - z_1) = -mgz_1 - (-mgz_0) = E(x_1, y_1, z_1) - E(x_0, y_0, z_0)$$

This implies that the **scalar potential can be used to evaluate the line integral** in a conservative vector field. In fact, the proof is

FTC

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \overbrace{\nabla E(\mathbf{r}(t)) \cdot \mathbf{r}'(t)}^{\frac{dE}{dt} = E_x \cdot x' + E_y \cdot y' + E_z \cdot z'} \, dt = \int_{t_0}^{t_1} \frac{dE(\mathbf{r}(t))}{dt} \, dt = E(\mathbf{r}(t_1)) - E(\mathbf{r}(t_0))$$

Chain rule

A Conservative Vector Field

Hence, **all conservative vector fields are gradient fields (& vice-versa)**.

But, how do we know if a vector field is conservative in the first place before the scalar potential is used to compute the line integral? It turns out that all conservative vector fields (gradient fields) have **zero curl**, because

$$\nabla \times \nabla E = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_{zy} - E_{yz} \\ -(E_{zx} - E_{xz}) \\ E_{yx} - E_{xy} \end{bmatrix} = \mathbf{0}$$

by symmetry of mixed partials. So one can **check for zero curl of a vector field before using the scalar potential** to evaluate a line integral.

A Conservative Vector Field

Example: From the last exercise, is the force field **F** conservative? If so, evaluate the line integral from (x, y, z) = (0, 0, 0) to (1, 3, 2) over any path and reconcile with the value obtained earlier. E_x

$$\mathbf{F}(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

E_y

E_z.

Check if F is conservative:

$$\vec{\nabla} \times \vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0-0 \\ -(0-0) \\ 0-0 \end{pmatrix} = \vec{0} \rightarrow \vec{F} \text{ is conservative.}$$

Let $\vec{F} = \vec{\nabla} E$. Get $E(x,y,z)$.

$\frac{\partial}{\partial x} = 0$

$$E = \int x \, dx = \frac{x^2}{2} + g(y,z).$$

$E_y = g_y(y,z)$. Comparing, $g_y(y,z) = y. \rightarrow g(y,z) = \int y \, dy = \frac{y^2}{2} + h(z).$

$$E = \frac{x^2}{2} + \frac{y^2}{2} + h(z).$$

ANS: 7 J.

$E_z = h'(z)$. Comparing, $h'(z) = z \rightarrow h(z) = z^2/2 + C$.

$\therefore E(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + z^2/2 + C$

Over any path from $(0,0,0)$ to $(1,3,2)$,

$$W = E(1,3,2) - \underbrace{E(0,0,0)}$$

$$= \frac{1}{2} + \frac{9}{2} + \frac{4}{2} - 0 = 7\,J$$ //

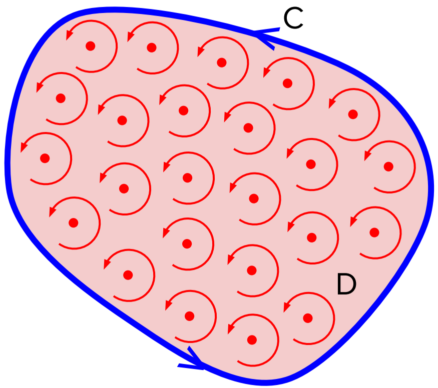
(Same as $\int \vec{F} \cdot \vec{dr}$ in earlier example! ✓)

Green's Theorem

The Green's theorem relates the curl of a vector field **F**(x, y) **inside** a closed curve to the line integral **along** the curve, defined by

closed curve C.

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



Imagine curve C behaves like a 'conveyor belt' being moved by circulative flow inside it.

Intuitively, one can imagine that the sum of circulation (rotation effect) of a vector field **within a region (D)** has a net circulative effect on the **boundary (C) of the region**, that is the line integral.

Note that the Green's theorem applies **only to a closed curve C** (counter-clockwise).

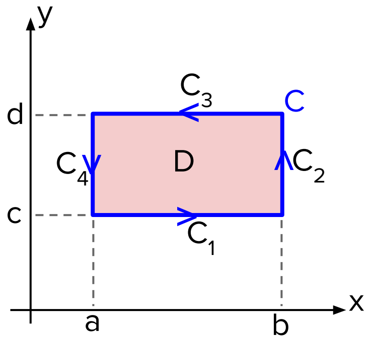
Proof of Green's Theorem

The proof of the Green's theorem is divided into 3 parts. Firstly, consider a rectangular region D bounded by closed curve C as shown. The line integral of a vector field **F**(x,y) is

$$\oint_C \mathbf{F}(x,y)\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}(x,y)\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}(x,y)\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}(x,y)\cdot d\mathbf{r} + \int_{C_4} \mathbf{F}(x,y)\cdot d\mathbf{r}$$

The four paths C_1 to C_4 can be parameterized by

$$C_1 : x = t, y = c,\ C_2 : x = b, y = s,\ C_3 : x = t, y = d,\ C_4 : x = a, y = s.$$



Hence, the **F.dr** integrands are

$\vec{r}_1(t) = \begin{pmatrix} t \\ c \end{pmatrix}$

$\vec{r}_1{}'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$C_1 : \mathbf{F}\cdot d\mathbf{r} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt = f_1(t,c)\,dt,$$

$$C_2 : f_2(b,s)\,ds, \quad C_3 : f_1(t,d)\,dt, \quad C_4 : f_2(a,s)\,ds.$$

Proof of Green’s Theorem

The Green theorem is then proven for the rectangular region D as follows.

$$\int_C \mathbf{F}(x,y)\cdot d\mathbf{r} = \underline{\int_a^b f_1(t,c)\,dt} + \underline{\int_c^d f_2(b,s)\,ds} \;\cancel{+}\,- \underline{\int_{\cancel{b}\,a}^{\cancel{a}\,b} f_1(t,d)\,dt} \;\cancel{+}\,- \underline{\int_{\cancel{d}\,c}^{\cancel{c}\,d} f_2(a,s)\,ds}$$

By FTC

$$= \underline{\int_a^b f_1(t,\underline{c}) - f_1(t,\underline{d})\,dt} + \underline{\int_c^d f_2(\underline{b},s) - f_2(\underline{a},s)\,ds}$$

$$= \int_a^b \int_{\underline{d}}^{\underline{c}} \frac{\partial f_1(t,y)}{\partial y}\,dydt + \int_c^d \int_{\underline{a}}^{\underline{b}} \frac{\partial f_2(x,s)}{\partial x}\,dxds$$

By Fubini’s theorem

$$= -\int_a^b \int_c^d \frac{\partial f_1(x,y)}{\partial y}\,dydx + \int_c^d \int_a^b \frac{\partial f_2(x,y)}{\partial x}\,dxdy$$

$$= \int_a^b \int_c^d \frac{\partial f_2(x,y)}{\partial x} - \frac{\partial f_1(x,y)}{\partial y}\,\cancel{dxdy}\; dydx \leftarrow \text{correction.}$$

$$= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\,dA = \iint_D \underline{(\nabla\times\mathbf{F})}\cdot\mathbf{k}\,dA$$

$$\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix}$$

Proof of Green's Theorem

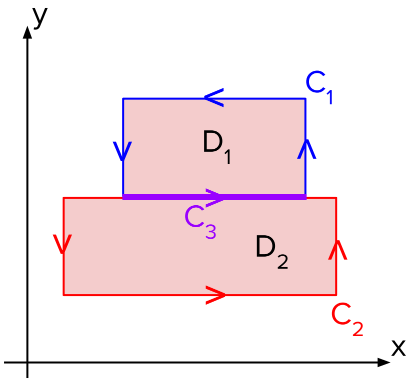
The second part of the proof uses the fact that **line integrals over non-overlapping connected regions is equal to that of the overall region**, i.e.

$$\underbrace{\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_3}\mathbf{F}\cdot d\mathbf{r}}_{\iint_{D_1}(\nabla\times F)\cdot k\,dA}+\overbrace{\int_{C_2}\mathbf{F}\cdot d\mathbf{r}-\int_{C_3}\mathbf{F}\cdot d\mathbf{r}}^{\iint_{D_2}(\nabla\times F)\cdot k\,dA}=\overbrace{\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}}^{C_1+C_2\text{ enclose }D_1+D_2}$$

$$=\oint_{C_1+C_2}\mathbf{F}\cdot d\mathbf{r}=\iint_{D_1+D_2}(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA$$

Clearly, the above can be extended to a region that is composed of any number of non-overlapping rectangles

Hence, the Green's theorem is proven over a general region composed of rectangles as shown in the next slide.

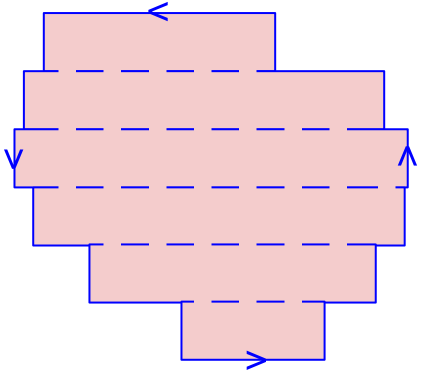


Proof of Green’s Theorem

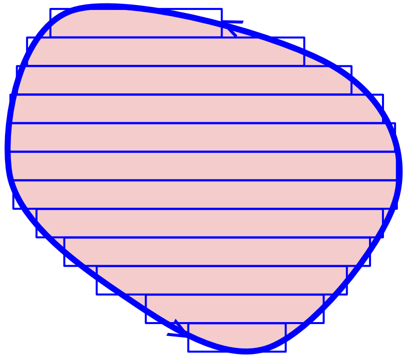
The last part of the proof uses the fact that any general region can be ‘fitted’ **exactly** by an **infinite** number of rectangles. This is the same concept when a Riemann sum becomes a Riemann integral in finding the exact area under a curve.

Hence, the Green’s theorem is proven for a general region D bounded by **closed curve** C oriented **counterclockwise**.

The following examples will demonstrate the use of the Green’s theorem.



As the rectangles get **infinitesimally thin**, a general region can be fitted.



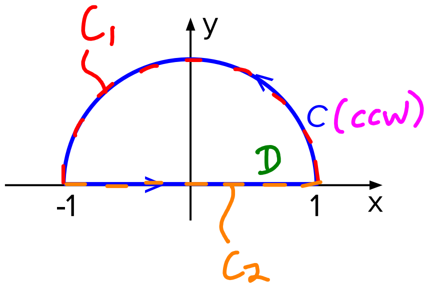
Green's Theorem

For 2D vector fields:

$$\vec{\nabla}\times\vec{F} = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

Example: Continuing from an earlier example, use Green's theorem to evaluate the line integral of **F** over the closed curve C. Then, calculate the line integral directly to verify.

$$\mathbf{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix} \longrightarrow \vec{\nabla}\times\vec{F} = [1-(-1)]\hat{k} = 2\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$



$*$ preferred since more efficient.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \overbrace{(\vec{\nabla}\times\vec{F})\cdot\hat{k}}^{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2}\, dA = \boxed{\iint_D \overbrace{\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}}^{2}\, dA}$$

$$= \underbrace{2\iint_D dA}_{\text{Area of semi-circle.}} = 2\left(\frac{\pi(1)^2}{2}\right) = \pi \,//$$

ANS: π.

From previous example:
(Slide 31)

$$L_{C_1} = \int_0^{\pi} 1\, dt = t\Big|_0^{\pi} = \pi \quad //$$

For C_2, $\vec{r}_2(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \to \vec{r}_2{}'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$L_{C_2} = \int_{-1}^{1} \vec{F} \cdot \vec{r}_2{}'\, dt = \int_{-1}^{1} \begin{pmatrix} 0 \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt$$

$$= \int_{-1}^{1} 0\, dt = 0.$$

$$L_C = L_{C_1} + L_{C_2}$$

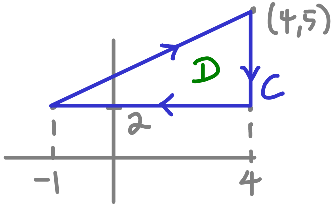
$$= \pi + 0 = \pi \checkmark$$ same as

Green's Theorem

Exercise: Use Green's theorem to evaluate the line integral below over the closed curve C which is a triangle with vertices (-1, 2), (4, 2) and (4, 5), oriented clockwise.

$$L = \int_C \underbrace{\sin\left(x^2\right) dx + (3x - y)\, dy}$$

$$\begin{array}{l} f_1 \rightarrow \\ f_2 \rightarrow \end{array} \underbrace{\begin{pmatrix} \sin(x^2) \\ 3x-y \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}}_{\vec{F} \cdot \vec{dr}}$$



Since curve C is CW,

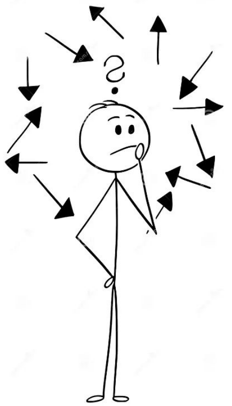
$$L = -\iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dA = -\iint_D 3 - 0\, dA = -3\iint_D dA = -3\left(\tfrac{1}{2}(5)(3)\right)$$

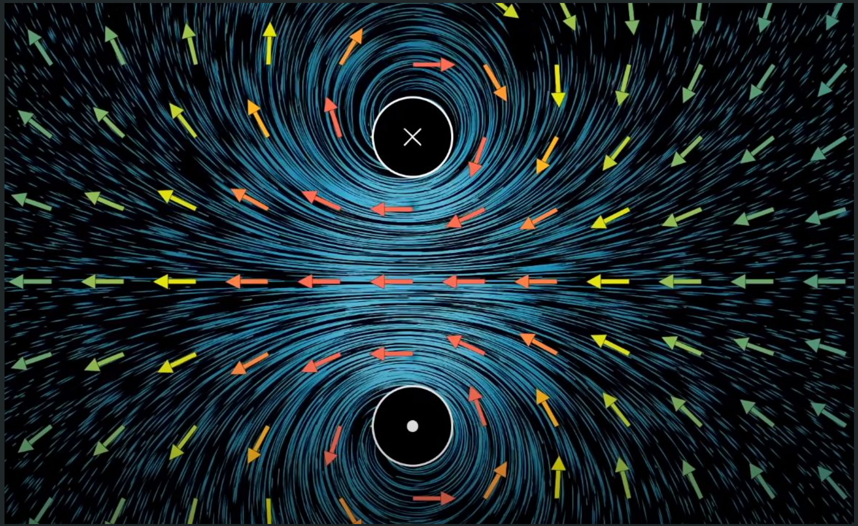
$$= -45/2$$

ANS: -45/2.

End of Topic 5

If you thought vectors and calculus are hard, vector calculus probably just brought it to a whole new level.





Source: 3Blue1Brown

Excellent Visualization of Vector Fields, Divergence & Curl
https://youtu.be/rB83DpBJQsE