Chapter 1 Kirchhoff's Laws

Reference Directions:

- Choose i and u directions
- can be +ve or −ve.

Ports: i_1, i_2

two terminals w/ $i_1 = i_2$ => port

KCL: $\sum_{j \in \text{branches of loop}} i_j(t) = 0$

e.g. $i_1 - i_2 - i_3 = 0$

super-node: $i_1 - i_2 = 0$

Two-terminal elements are one-ports! eg. 2 terminal = 1 port, 4 terminal = 2 port

2n terminals = n ports!

KVL: $\sum_{j \in \text{branches of loop}} u_j(t) = 0$

e.g. $-u_{k1} + u_{12} + u_{k2} = 0$

$u_{12} = u_{k1} - u_{k2}$

reference node always = 0

No. of Linearly Independent Kirchhoff's Equations

Given: n-nodes, b-branches

no. of KCL eqn: $n-1$

no. of KVL eqn: $b-(n-1)$

1 node; also 1 node!

Linearization

$$B_{lin}(a) = \frac{dB(a)}{a}\Big|_{a=A} \cdot (a-A) + B$$

$$\det\begin{bmatrix} 5 & -3 \\ x & y \end{bmatrix} = (5)(y) - (-3)(x)$$

Chap 3 for lossness

$$R = -R^{\mathrm{T}}$$

Multiplying Matrices
2x2 by 2x1

$$\begin{pmatrix} 7 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Given that $z = 3i$, find $|z|$.

$z = x + iy$ $\quad Re(z) = 0$

$|z| = \sqrt{x^2+y^2}$ $\quad Im(z) = 3$

$|z| = \sqrt{0^2+3^2} = \sqrt{9} = \boxed{3}$

Chap 8

State variables: Capacitors & inductors

state eqn: $u_L = L\dot{i}_L$

$i_C = C\dot{u}_C$

Chapter 2 Resistive One-Ports

(u,i) is an OP of $\mathcal{F}$.

$\mathcal{F}$ is characteristic of F

$\mathcal{F} \subseteq \mathcal{F}_{ui} \longrightarrow \mathcal{F}_{ui}$ is the whole plane

Explicit Repr. Form

volt. cont: $i = g(u)$ for G: $i = i_0 \arctan\frac{u}{u_0}$

curr. cont: $u = r(i)$ for G: $u = u_0 \tan\frac{i}{i_0}$

Properties of Resistive Two-Ports

- Bilateral $\longrightarrow g(-u) = -g(u)$, $r(-i) = -r(i)$
- Power $\longrightarrow p(t) = u(t)\cdot i(t)$

$\mathcal{F}$ is if	Active	Passive	Lossless	Lossy
	delivers P, $p<0$	consumes P, $p \geq 0$	$p = 0$	$p \neq 0$

- Source-Free $\longrightarrow$ origin $(0;0) \in \mathcal{F}$
- Duality $\longrightarrow$ exchanges role of u & i

$u \to R_d i^d$, $i \to \frac{1}{R_d}u^d$, $u^d \to R_d i$, $i^d \to \frac{1}{R_d}u$

$\mathcal{F}$	$\mathcal{F}^d$
volt	curr
curr	volt
passive lossy ⋮	same

if $\mathcal{F}$ is volt. cont, $\mathcal{F}^d$ is curr. cont

proof: $i = g(u)$ Apply duality transform $\frac{1}{R_d}u^d = g(R_d i^d) \to u^d = R_d g(R_d i^d) = r_d(i^d) \Rightarrow \mathcal{F}^d$ is curr. cont

Strictly Linear One-Ports

$\forall (u_1,i_1),(u_2,i_2) \in \mathcal{F}: (u_1+u_2),(i_1+i_2) \in \mathcal{F}$

- Nullator: $\mathcal{F} = (0,0)$
- Strictly Linear Resistor: G
- Open circuit: $i=0 \Leftrightarrow G=0$, $G = \frac{1}{R} \Rightarrow R = \infty$ — dual — short circuit: $u=0 \Leftrightarrow R=0$, $R = \frac{1}{G} \Rightarrow G = \infty$
- Ohmic Resistors $\binom{G>0}{R>0}$ consumes power: $p \geq 0$
- Negative Resistors $\binom{G<0}{R<0}$ delivers power: $p<0$
- Norator: u & i are arbitrary
- pn-Junction Diode: $i = I_S(e^{\frac{u}{U_T}} - 1)$, $u = U_T \ln(\frac{i}{I_S}+1)$

Connection of One-Ports

- Reversal $\longrightarrow$ if $\mathcal{F}$ is bilateral: $\mathcal{F}^r = \mathcal{F}$
- Parallel Connection $\longrightarrow$ sum of currents
- Series connection $\longrightarrow$ sum of voltages
- Connection of strictly Linear Resistors
- Parallel: current divider rule: eg. $\frac{i}{i_2} = \frac{G_1}{G_2}$
- Series: voltage divider rule: e.g. $\frac{u}{u_1} = \frac{R}{R_1}$

Linear Sources

$i' = G u' - i_0$ (volt. cont)

$u' = R i' + u_0$ (curr. cont)

u_0 = o.c volt, internal resistance $R = \frac{u_0}{i_0}$

i_0 = s.c curr, internal conductance $G = \frac{i_0}{u_0}$, $R = \frac{1}{G}$

- Independent Sources

current source — dual R_d — voltage source

if $i_s = 0$, $i' = 0 \Rightarrow$ o.c

if $u_s = 0$, $u' = 0 \Rightarrow$ s.c

source transform: $u_0 = \frac{i_0}{G}$, $R = \frac{1}{G}$; $i_0 = \frac{u_0}{R}$, $G = \frac{1}{R}$

Piecewise Linear Resistors

- Piecewise Linear Diodes

Ideal Diode: $i = 0$, $u \leq 0$ (I) cut-off; $u = 0$, $i \geq 0$ (II) conductive; pn-Junct. Diode

Concave Resistors (G,U)

Convex Resistor (R,I)

- Real Negative Resistors

Ideal Negative Resistor; N-type volt. cont; S-type curr. cont

One-Port Circuits

OPs: $I = g(U)$, $\mathcal{F}^x \cap \mathcal{L}$ $(U_{\mathcal{F}}, I_{\mathcal{F}}) \in$ OP

Large signal: $b(t) = b(a(t))$ $\quad \dim(S) = \left(\frac{1V}{1A}\right)$

Small signal: constant sources are set to zero!

Linearization: $i_{lin} = I + \frac{dg(u)}{du}\Big|_{u=U} \cdot (u-U)$

$u_{lin} = U + \frac{dr(i)}{di}\Big|_{i=I} \cdot (i-I)$

Chapter 3 Two-Ports

	R	G	H	H'	A	A'
R	$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22}\end{bmatrix}$	$\frac{1}{\det G}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{bmatrix}$	$\frac{1}{h_{22}}\begin{bmatrix} \det H & h_{12} \\ -h_{21} & 1\end{bmatrix}$	$\frac{1}{h'_{11}}\begin{bmatrix} 1 & -h'_{12} \\ h'_{21} & \det H'\end{bmatrix}$	$\frac{1}{a_{21}}\begin{bmatrix} a_{11} & \det A \\ 1 & a_{22}\end{bmatrix}$	$\frac{1}{a'_{21}}\begin{bmatrix} a'_{22} & 1 \\ \det A' & a'_{11}\end{bmatrix}$
G	$\frac{1}{\det R}\begin{bmatrix} r_{22} & -r_{12} \\ -r_{21} & r_{11}\end{bmatrix}$	$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix}$	$\frac{1}{h_{11}}\begin{bmatrix} 1 & -h_{12} \\ h_{21} & \det H\end{bmatrix}$	$\frac{1}{h'_{22}}\begin{bmatrix} \det H' & h'_{12} \\ -h'_{21} & 1\end{bmatrix}$	$\frac{1}{a_{12}}\begin{bmatrix} a_{22} & -\det A \\ -1 & a_{11}\end{bmatrix}$	$\frac{1}{a'_{12}}\begin{bmatrix} a'_{11} & -1 \\ -\det A' & a'_{22}\end{bmatrix}$
H	$\frac{1}{r_{22}}\begin{bmatrix} \det R & r_{12} \\ -r_{21} & 1\end{bmatrix}$	$\frac{1}{g_{11}}\begin{bmatrix} 1 & -g_{12} \\ g_{21} & \det G\end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22}\end{bmatrix}$	$\frac{1}{\det H'}\begin{bmatrix} h'_{22} & -h'_{12} \\ -h'_{21} & h'_{11}\end{bmatrix}$	$\frac{1}{a_{22}}\begin{bmatrix} a_{12} & \det A \\ -1 & a_{21}\end{bmatrix}$	$\frac{1}{a'_{11}}\begin{bmatrix} a'_{12} & 1 \\ -\det A' & a'_{21}\end{bmatrix}$
H'	$\frac{1}{r_{11}}\begin{bmatrix} 1 & -r_{12} \\ r_{21} & \det R\end{bmatrix}$	$\frac{1}{g_{22}}\begin{bmatrix} \det G & g_{12} \\ -g_{21} & 1\end{bmatrix}$	$\frac{1}{\det H}\begin{bmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11}\end{bmatrix}$	$\begin{bmatrix} h'_{11} & h'_{12} \\ h'_{21} & h'_{22}\end{bmatrix}$	$\frac{1}{a_{11}}\begin{bmatrix} a_{21} & -\det A \\ 1 & a_{12}\end{bmatrix}$	$\frac{1}{a'_{22}}\begin{bmatrix} a'_{21} & -1 \\ \det A' & a'_{12}\end{bmatrix}$
A	$\frac{1}{r_{21}}\begin{bmatrix} r_{11} & \det R \\ 1 & r_{22}\end{bmatrix}$	$\frac{1}{g_{21}}\begin{bmatrix} -g_{22} & -1 \\ -\det G & -g_{11}\end{bmatrix}$	$\frac{1}{h_{21}}\begin{bmatrix} -\det H & -h_{11} \\ -h_{22} & -1\end{bmatrix}$	$\frac{1}{h'_{21}}\begin{bmatrix} 1 & h'_{22} \\ h'_{11} & \det H'\end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$	$\frac{1}{\det A'}\begin{bmatrix} a'_{22} & a'_{12} \\ a'_{21} & a'_{11}\end{bmatrix}$
A'	$\frac{1}{r_{12}}\begin{bmatrix} r_{22} & \det R \\ 1 & r_{11}\end{bmatrix}$	$\frac{1}{g_{12}}\begin{bmatrix} -g_{11} & -1 \\ -\det G & -g_{22}\end{bmatrix}$	$\frac{1}{h_{12}}\begin{bmatrix} 1 & h_{11} \\ h_{22} & \det H\end{bmatrix}$	$\frac{1}{h'_{12}}\begin{bmatrix} -\det H' & -h'_{22} \\ -h'_{11} & -1\end{bmatrix}$	$\frac{1}{\det A}\begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11}\end{bmatrix}$	$\begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22}\end{bmatrix}$

Conversion Table of Two-Port matrices: Port one and two are rows one and two respectively.
Read from row to columns: e.g. A = 1/det A' […]

Strictly Linear Two-Ports

- Explicit Representation
- Parametric Representation (Always exist!)

Basis Matrix $(\underset{\sim}{U}, \underset{\sim}{I})^T$

- Two Measurements: $\begin{bmatrix} u^{(1)} \\ i^{(1)}\end{bmatrix}, \begin{bmatrix} u^{(2)} \\ i^{(2)}\end{bmatrix}$

$\underset{\sim}{U} = [u^{(1)}, u^{(2)}]$ $\quad \underset{\sim}{I} = [i^{(1)}, i^{(2)}]$

$\begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} \underset{\sim}{U} \\ \underset{\sim}{I} \end{bmatrix} \cdot \underset{\sim}{c}$, $c \in \mathbb{R}^2$ $\quad \underset{\sim}{R} = \underset{\sim}{U}\cdot\underset{\sim}{I}^{-1}$ $\quad \underset{\sim}{G} = \underset{\sim}{I}\cdot\underset{\sim}{U}^{-1}$

Linear Two-Ports

- Three Measurements: $\begin{bmatrix} u^{(1)} \\ i^{(1)}\end{bmatrix}, \begin{bmatrix} u^{(2)} \\ i^{(2)}\end{bmatrix}, \begin{bmatrix} u^{(3)} \\ i^{(3)}\end{bmatrix}$

$\begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} U \\ I \end{bmatrix}\cdot \underset{\sim}{c} + \begin{bmatrix} u_0 \\ i_0 \end{bmatrix}$ u_0, i_0 = shifting of the plane

Let $\begin{bmatrix} u_0 \\ i_0 \end{bmatrix} = \begin{bmatrix} u^{(1)} \\ i^{(1)} \end{bmatrix}$ Then, $\begin{bmatrix} U \\ I \end{bmatrix} = \begin{bmatrix} u^{(2)}-u^{(1)}, & u^{(3)}-u^{(1)} \\ i^{(2)}-i^{(1)}, & i^{(3)}-i^{(1)} \end{bmatrix}$

$u = \underset{\sim}{U}\underset{\sim}{c} + u_0$, $i = \underset{\sim}{I}\underset{\sim}{c} + i_0 \Rightarrow i = \underbrace{\underset{\sim}{I}\underset{\sim}{U}^{-1}}_{G} u + \underbrace{i_0 - \underset{\sim}{I}\underset{\sim}{U}^{-1}u_0}_{i_G}$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = G\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} i_{G1} \\ i_{G2} \end{bmatrix}$$

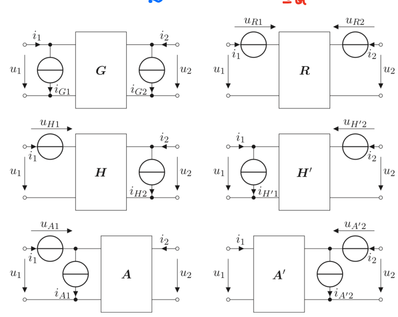
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = R\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} u_{R1} \\ u_{R2} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = H\begin{bmatrix} i_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} u_{H1} \\ i_{H2} \end{bmatrix}$$

$$\begin{bmatrix} i_1 \\ u_2 \end{bmatrix} = H'\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} i_{H'1} \\ u_{H'2} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = A\begin{bmatrix} u_2 \\ -i_2 \end{bmatrix} + \begin{bmatrix} u_{A1} \\ i_{A1} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ i_2 \end{bmatrix} = A'\begin{bmatrix} u_1 \\ -i_1 \end{bmatrix} + \begin{bmatrix} u_{A'2} \\ i_{A'2} \end{bmatrix}$$



Decomposition of Linear Two Ports

Properties of Two-Ports

- Power:
  - lossless: $\underset{\sim}{u}^T\underset{\sim}{I} + \underset{\sim}{I}^T\underset{\sim}{u} = 0$
  - passive: $\underset{\sim}{u}^T\underset{\sim}{I} + \underset{\sim}{I}^T\underset{\sim}{u}$ (non -ve definite)
  - active: $\underset{\sim}{u}^T\underset{\sim}{I} + \underset{\sim}{I}^T\underset{\sim}{u}$ (not non-ve definite)
- Duality: $G^d = \frac{1}{R_d^2}R$, $R^d = G\cdot R_d^2$
- Symmetry: $R = PRP$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $G = PGP$, $P = P^{-1}$; $A = A'$, $H = H'$
- Reciprocity: $r_{12} = r_{21}$, $R = R^T$; $g_{12} = g_{21}$, $G = G^T$; $h_{12} = -h_{21}$; $h'_{12} = -h'_{21}$; $\det A = 1$; $\det A' = 1$

parametric repr: $\underset{\sim}{U}^T\underset{\sim}{I} - \underset{\sim}{I}^T\underset{\sim}{U} = 0$

Special Two-Ports

- Controlled sources

$i_1 = 0$, $i_2 = g\cdot u_1$ $\quad G = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 0 & -\frac{1}{g} \\ 0 & 0 \end{bmatrix}$

voltage-controlled current source (VCCS)

$u_1 = 0$, $i_2 = \beta\cdot i_1$ $\quad H = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{\beta} \end{bmatrix}$

current-controlled current source (CCCS)

$i_1 = 0$, $u_2 = \mu\cdot u_1$ $\quad H' = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} \frac{1}{\mu} & 0 \\ 0 & 0 \end{bmatrix}$

voltage-controlled voltage source (VCVS)

$u_1 = 0$, $u_2 = r\cdot i_1$ $\quad R = \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 0 & 0 \\ \frac{1}{r} & 0 \end{bmatrix}$

current-controlled voltage source (CCVS)

$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$ $\quad u_1 = h_{11}i_1 + h_{12}u_2$ — series (sum of voltage); $i_2 = h_{21}i_1 + h_{22}u_2$ — parallel (sum of current)

example

Connection of Two-Ports

- Parallel: $G_G = G_{F_1} + G_{F_2}$
- Series: $R_G = R_{F_1} + R_{F_2}$

G consist of F_1 & F_2.

- Series-Parallel: $H_G = H_{F_1} + H_{F_2}$
- Parallel-Series: $H'_G = H'_{F_1} + H'_{F_2}$
- Cascade: $A_G = A_{F_1}\cdot A_{F_2}$, $A'_G = A'_{F_2}\cdot A'_{F_1}$

- Nullor: $\lim_{g\to\infty} A_{VCCS} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A_{Nullor}$

This means that nullor is the limit for all 4 controlled sources $\begin{bmatrix} g\to\infty & \beta\to\infty \\ \mu\to\infty & r\to\infty \end{bmatrix}$

- Transformer (lossless reciprocal): $H_{trans} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix}$, $A_{trans} = \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$
- Gyrator: $R_{gyr} = \begin{bmatrix} 0 & -R_d \\ R_d & 0 \end{bmatrix}$ $\quad G_{gyr} = \begin{bmatrix} 0 & G_d \\ -G_d & 0 \end{bmatrix}$

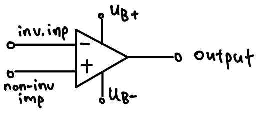
$A_{gyr} = \begin{bmatrix} 0 & R_d \\ \frac{1}{R_d} & 0 \end{bmatrix}$ $\quad H_{gyr}$ does not exist!

- Negative Immittance Converter: $A_{NIC} = \begin{bmatrix} -k & 0 \\ 0 & 1 \end{bmatrix}$

By connecting the NIC to a strictly linear resistor, one obtains a negative resistor at port 1.

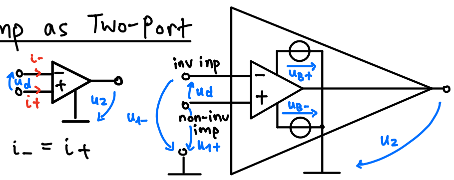
Chapter 4 Operational Amplifiers

• Op-Amps



Op-Amp as Two-Port

· Conventional model

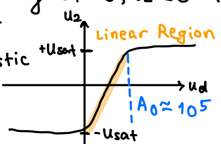


assumption: $i_- = i_+$

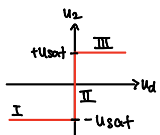
Modelling

$i_+ \approx 0,\ i_- \approx 0 \Rightarrow i_- = i_+ = 0$

· Realistic Transfer Characteristic



· Idealised Non-Linear Model



I: $u_2 = -U_{sat}$, $u_d < 0$ (negative saturation)

II: $u_d = 0$, $|u_2| \le U_{sat}$ (linear region)

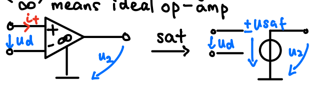
III: $u_d = +U_{sat}$, $u_d > 0$ (positive saturation)

· Saturation Regions (Regions I & III)

$u_d \neq 0$, $i_+ = i_- = 0 \Rightarrow$ O.C

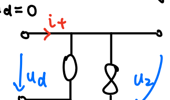
$u_2 = \pm U_{sat} \Rightarrow$ voltage source

'∞' means ideal op-amp



· Nullor Model (region II)

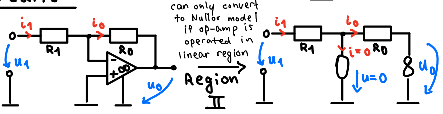
$u_d = 0$



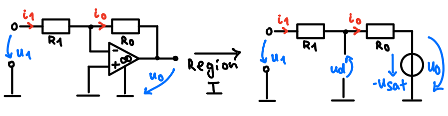
Op-Amp Circuits

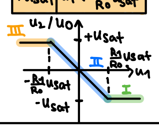
· Inverting Amplifier

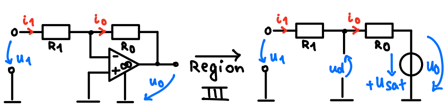
$u_0 = \left[-\frac{R_0}{R_1}\right] \cdot u_1$ (volt. gain μ)



u_2	condition
$-U_{sat}$	$u_1 > \frac{R_1}{R_0}U_{sat}$
$\frac{-R_0}{R_1}\cdot u_1$	$\lvert u_1\rvert \le \frac{R_1}{R_0}U_{sat}$
$+U_{sat}$	$u_1 < -\frac{R_1}{R_0}U_{sat}$

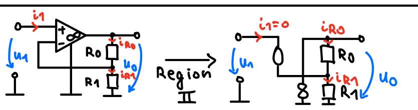


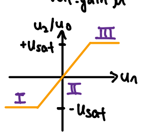




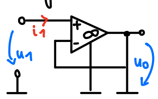
· Non-Inverting Amplifier

$u_0 = \left[1 + \frac{R_0}{R_1}\right]\cdot u_1$ (volt. gain μ)



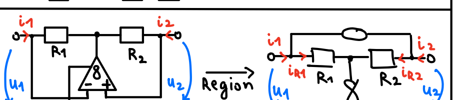


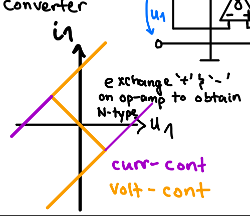
- Voltage Follower: $i_1 = 0$, $u_0 = u_1$

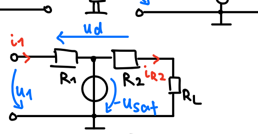


$\mu_{volt.fol} = 1 + \frac{R_0}{R_1} = 1$ if $R_0 = 0$ (SC) $R_1 = \infty$ (OC)

· Negative Immittance Converter:

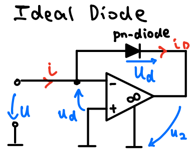




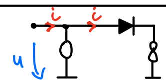


· Piecewise Linear Resistor

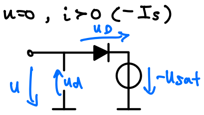
- Ideal Diode



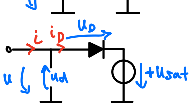
Region II (region II of ideal diode) $u = 0$, $i > 0$ $(-I_s)$



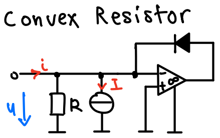
Region I (unrealistic)



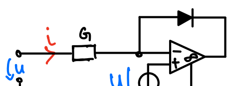
Region III (region I of ideal diode)



Convex Resistor



Concave Resistor



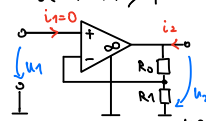
Linear Op-Amp Circuits

· Controlled Sources

VCVS

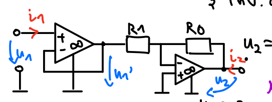
$i_1 = 0$, $u_2 = \mu \cdot u_1$

Case 1: $\mu \ge 1$: use non.inv amp



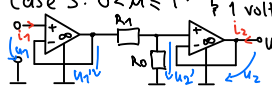
$u_2 = \left(1 + \frac{R_0}{R_1}\right)u_1$, $\mu > 1$

Case 2: $\mu < 0$: use volt.fol & inv.amp



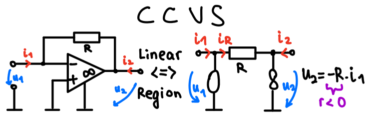
$u_2 = -\frac{R_0}{R_1}u_1$, $\mu < 0$

Case 3: $0 < \mu \le 1$: use 2 volt.fol & 1 volt.div

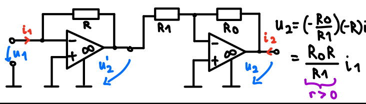


$u_2 = \frac{R_0}{R_1 + R_0}u_1$, $0 < \mu \le 1$

CCVS



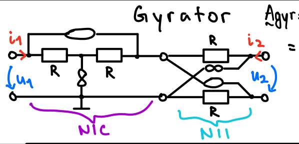
$u_2 = -R\cdot i_1$, $r < 0$



$u_2 = \left(-\frac{R_0}{R_1}\right)(-R)i_1 = \frac{R_0 R}{R_1} i_1$

Gyrator

$A_{gyr} = \begin{bmatrix}0 & R\\ \frac{1}{R} & 0\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \times \begin{bmatrix}0 & R\\ -\frac{1}{R} & 0\end{bmatrix}$ (NIC, NII)



Chapter 5 General Circuit Analysis

Incidence Matrices

· form K-Law for every node except ref. node

KCL: $A i = 0$ KVL: $u = A^T u_k$ (node volt. repre)

Tableau Equations

KCL: $A i = 0$ $e \in$ linear sources

KVL: $B u = 0$ e.g. u_0, i_0 ...

Elements: $Mu + Ni = e$

· Node Analysis

$(2b+n-1)^2$

$$\begin{array}{l} KVL \\ KCL \\ elements \end{array} \begin{bmatrix} -A^T & 1 & 0 \\ 0 & 0 & A \\ 0 & M & N \end{bmatrix} \begin{bmatrix} u_k \\ u \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e \end{bmatrix}$$

Nodal Analysis

Existence of N^{-1} means all elements must be voltage controlled

vol. cont. rep: $G_k \cdot u_k - i_k = i_{0k}$

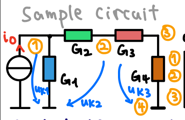
· Tableau Equation System

EXAMPLE

$$\begin{array}{l} KVL \\ \\ \\ KCL \\ \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & & & \\ 0 & -1 & 0 & 1 & 0 & & & \\ 1 & -1 & 0 & 0 & 1 & & & \\ & & & & & 1 & 0 & -1 \\ & & & & & 0 & 1 & 1 \\ & & G_1 & & & -1 & & \\ & & & G_2 & & & -1 & \\ & & & & G_3 & & & -1 \end{bmatrix} \begin{bmatrix} u_{k1} \\ u_{k2} \\ u_1 \\ u_2 \\ u_3 \\ i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i_{01} \\ i_{02} \\ i_{03} \end{bmatrix}$$

$-A^T$, A, M, $N = -1 \rightarrow N^{-1} = -1$

Sample circuit



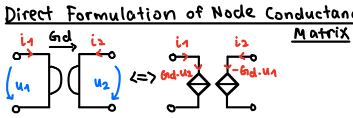
Identify no. of branches and nodes. Set up K'Laws.

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} G_1 + G_2 & -G_2 & \\ -G_2 & G_2 + G_3 & -G_3 \\ & -G_3 & G_3 + G_4 \end{bmatrix} \cdot \begin{bmatrix} u_{k1} \\ u_{k2} \\ u_{k3} \end{bmatrix} = \begin{bmatrix} i_0 \\ 0 \\ 0 \end{bmatrix}$$

Conductor/Resistor in btw node α & β => +ve on main diagonal, -ve on off di

Current flowing in is negative, flowing out is positive for L.H.S. For R.H.S, i.e. $i_{01} \dots i_{0k}$, flowing into node = positive, flowing out of node = negative.

Direct Formulation of Node Conductance Matrix

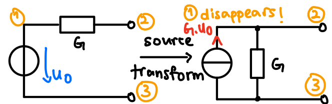


$i_1 = G_d \cdot u_2$ $i_1 = G_d \cdot u_2$

$i_2 = -G_d \cdot u_1$ $i_2 = -G_d \cdot u_1$

Non-Voltage Cont. Elements

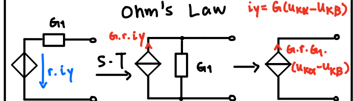
Source Transform (S.T)



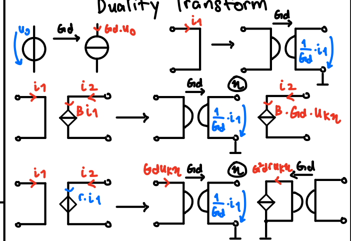
Same for controlled sources

Note the change in direction of u & i

Ohm's Law $i_y = G_i(u_{kx} - u_{kB})$

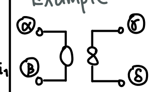


Duality Transform



Inclusion of Nullors

Example



Nullator btw ⓐ & ⓑ, add up col of ⓐ & ⓑ of ⓐ & k and drop either u_{ka} or u_{kB}. If nullator is between node x & ref node, drop col x & u_{kx}

For norator btw ⓨ & ⓢ, add up KCL of ⓨ & ⓢ and drop either ⓨ or ⓢ. If the norator is btw node x & ref node, drop KCL x

Chapter 6 Reactive Circuit Elements

Mainly 2 reactances: (inductor) and (capacitor)

Formulas: (initial state)

$q(t) = q(t_0) + \int_{t_0}^{t} i(\tau)\,d\tau$

$\Phi(t) = \Phi(t_0) + \int_{t_0}^{t} u(\tau)\,d\tau$

Capacitor	Inductor
$q(t) = C \cdot u(t)$	$\Phi(t) = L \cdot i(t)$
$\dot q(t) = C \cdot \dot u(t)$	$\dot\Phi(t) = L \cdot \dot i(t)$
$i(t) = C \cdot \dot u(t)$	$u(t) = L \cdot \dot i(t)$

Duality $q^d = C^d \cdot u^d$, $\Phi^d = L^d \cdot i^d$

$C^d = \frac{1}{R_d^2} L$, $L^d = R_d^2 C$

Properties of Reactances

· Continuity: State Variables = $\begin{cases} u \text{ \& } q & \text{cap.} \\ i \text{ \& } \Phi & \text{ind.} \end{cases}$

· Energy: $p(t) = u(t) \cdot i(t)$

$E = \int_{t_1}^{t_2} p(t)\,dt = \frac{C}{2}(u_2^2 - u_1^2) = \frac{1}{2C}(q_2^2 - q_1^2)$

For uncharged capacitor/inductor: $E_C(u) = \frac{C}{2}u^2$, $E_L(i) = \frac{L}{2}i^2$

Dimensions/Units:

$q = it$ $L = \frac{\Phi}{i}$ (H) $R = \frac{u}{i}$ (Ω)

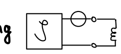
$\Phi = ut$ $C = \frac{q}{u}$ (F) $S = \frac{i}{u}$ $\left(\frac{1}{\Omega}\right)$

· Connection of Reactances

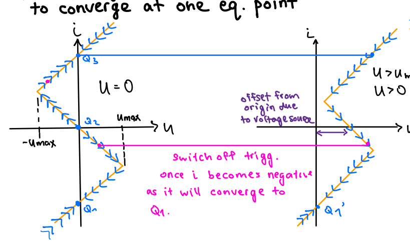
Capacitor	Inductor
$C_{parallel} = C_1 + C_2$	$L_{parallel} = L_1 \parallel L_2$
$C_{series} = C_1 \parallel C_2$	$L_{series} = L_1 + L_2$

Bistable circuit can be used to store a bit. $+u \Rightarrow Q_1'$, $-u \Rightarrow Q_3'$

· Triggering



By triggering, the characteristic is forced to converge at one eq. point



Chapter 7 First Order Circuits

FOCs: Resistive circuit + one reactance

Linear FOCs

State Equations

General Form: $\dot x(t) = -\frac{1}{\tau}x(t) + \frac{1}{\tau}x_0(t)$

$\tau = RC$ time variable $\tau = GL$

$x = u_C$ state variable $x = i_L$

$x_0 = u_0$ excitation $x_0 = i_0$

$\dot u_C = -\frac{1}{RC}u_C + \frac{1}{RC}u_0$

$\dot i_L = -\frac{1}{GL}i_L + \frac{1}{GL}i_0$

· General Solution

$x(t) = x(t_0) \cdot e^{-\frac{t-t_0}{\tau}} + e^{-\frac{t}{\tau}} \int_{t_0}^{t} \frac{1}{\tau} e^{\frac{t'}{\tau}} x_0(t')\,dt'$

0-input response

0-state response

$u_C(t) = u_C(t_0) \cdot e^{-\frac{t-t_0}{RC}} + e^{-\frac{t}{RC}} \int_{t_0}^{t} \frac{1}{RC} e^{\frac{t'}{RC}} u_0(t')\,dt'$

$i_L(t) = i_L(t_0) \cdot e^{-\frac{t-t_0}{GL}} + e^{-\frac{t}{GL}} \int_{t_0}^{t} \frac{1}{GL} e^{\frac{t'}{GL}} i_0(t')\,dt'$

· Constant Excitation

$(u_0(t) \longrightarrow u_0,\ i_0(t) \longrightarrow i_0)$

$x(t) = x_\infty + (x_0 - x_\infty)e^{-\frac{t-t_0}{\tau}}$

Stable case: $x_\infty - (x_0 - x_\infty)\frac{2}{3}$

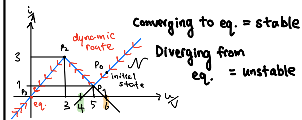
Unstable case: $x_0 + (x_0 - x_\infty)\frac{5}{3}$

Piecewise Linear FOCs

$\dot u = 0 \Rightarrow i = 0$ Equilibrium $\dot i = 0 \Rightarrow u = 0$

$i > 0 \Rightarrow u\downarrow$ $u > 0 \Rightarrow i\downarrow$

$i < 0 \Rightarrow u\uparrow$ $u < 0 \Rightarrow i\uparrow$

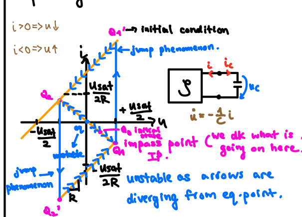


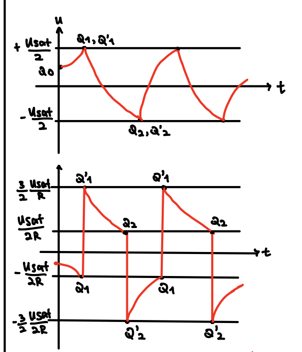
Converging to eq. = stable

Diverging from eq. = unstable

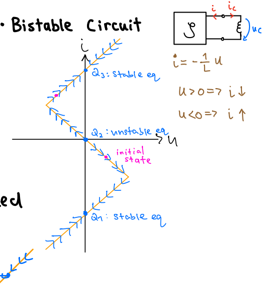
· Jump Phenomenon

- state var must be constant, depending on which reactance is used





· Bistable Circuit



Chapter 8 Second Order Circuits

SOCs: two port + two reactances

Formulation of State Equations

$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{B}\,\underset{\sim}{v}$ $\underset{\sim}{v}$ = excitation vector, $\underset{\sim}{x}$ = state vector

$y = \underset{\sim}{c}^T \underset{\sim}{x} + d\,v$ $\underset{\sim}{A}$ = state matrix, $\underset{\sim}{B}$ = input matrix / vector

Solution of State Equations

- General Solution $A = -\frac{1}{\tau}$ $B = \frac{1}{\tau}$

First-order: $\dot{x}(t) = A x(t) + B v(t)$

$x(t) = e^{A(t-t_0)} \cdot x(t_0) + \int_{t_0}^{t} e^{A(t-t')} B v(t')dt'$

Second-Order: $\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\,\underset{\sim}{x}(t) + \underset{\sim}{B}\,\underset{\sim}{v}(t)$

$\underset{\sim}{x}(t) = e^{\underset{\sim}{A}(t-t_0)} \cdot \underset{\sim}{x}(t_0) + \int_{t_0}^{t} e^{\underset{\sim}{A}(t-t')} \underset{\sim}{B}\,\underset{\sim}{v}(t')dt'$

- Autonomous Case $\underset{\sim}{v}(t) = \underset{\sim}{v}_0$ (constant)

$\underset{\sim}{x}(t) = \underset{\sim}{x}_\infty - e^{\underset{\sim}{A}(t-t_0)}(\underset{\sim}{x}_\infty - \underset{\sim}{x}_0)$ $\underset{\sim}{x}_\infty = -\underset{\sim}{A}^{-1}\underset{\sim}{B}\,\underset{\sim}{v}_0$, $\underset{\sim}{x}_0 = \underset{\sim}{x}(t_0)$

$\dot{\underset{\sim}{x}}' = \underset{\sim}{A}\,\underset{\sim}{x}'$, $\dot{\underset{\sim}{x}}' = \dot{\underset{\sim}{x}}$, $\underset{\sim}{x}' = \underset{\sim}{x} - \underset{\sim}{x}_\infty$

- Homogenous Differential Equation

$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x}$ $(\underset{\sim}{v} = 0)$

$\underset{\sim}{x}(t) = e^{\underset{\sim}{A}(t-t_0)} \underset{\sim}{x}(t_0)$ $\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\, e^{\underset{\sim}{A}(t-t_0)} \underset{\sim}{x}(t_0)$

- Characteristic Polynomial of $\underset{\sim}{A}$:

roots of $\det(\underset{\sim}{A} - \underset{\sim}{1}\lambda) = 0$ $\lambda_1, \lambda_2 \ldots$

$(\underset{\sim}{A} - \lambda \underset{\sim}{1})\underset{\sim}{q} = \underset{\sim}{0}$, $\underset{\sim}{A}\,\underset{\sim}{q}_1 = \lambda \underset{\sim}{q}_1$, $\underset{\sim}{A}\,\underset{\sim}{q}_2 = \lambda \underset{\sim}{q}_2$

- Normal Form: $\underset{\sim}{\xi}(t) = \underset{\sim}{Q}^{-1}\underset{\sim}{x}(t)$ $(\lambda_1 \neq \lambda_2)$

$\underset{\sim}{x}(t) = \underset{\sim}{q}_1 e^{\lambda_1(t-t_0)} \xi_1(t_0) + \underset{\sim}{q}_2 e^{\lambda_2(t-t_0)} \xi_2(t_0)$

- Jordan Normal Form $(\lambda_1 = \lambda_2)$

$\xi'_1(t) = e^{\lambda(t-t_0)} \xi'_1(t_0) + (t-t_0) e^{\lambda(t-t_0)} \xi'_2(t_0)$

$\xi'_2(t) = e^{\lambda(t-t_0)} \xi'_2(t_0)$

reverse sub: $\underset{\sim}{x}(t) = \underset{\sim}{q}_1 \xi'_1(t) + \underset{\sim}{q}_2 \xi'_2(t)$

- Real Valued Normal Form $(\lambda_1, \lambda_2 = \pm\sqrt{\ldots})$

complex normal form:

$\lambda_1 = \alpha + j\beta$, $\lambda_2 = \lambda_1^* = \alpha - j\beta$

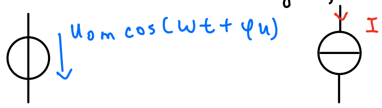
$\underset{\sim}{Q}_{real} = [\underset{\sim}{q}_r, -\underset{\sim}{q}_i]$ $\underset{\sim}{x}(t) = \underset{\sim}{Q}_{real}\, \underset{\sim}{\xi}_{real}(t)$

$\underset{\sim}{\xi}_{real} = \begin{bmatrix} \xi_{real1} \\ \xi_{real2} \end{bmatrix}$ $= 2.\operatorname{Re}\{\xi\}$, $= 2.\operatorname{Im}\{\xi\}$

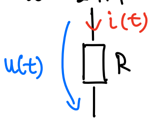
Sinusoidal Excitation

gives steady-state response

linear (no diodes e.g.), sinusoidal excitations



$\omega = 2\pi f$ $i(t) = I_m \cos(\omega t)$



Ohm: $u(t) = R\,i(t) = \underbrace{R I_m}_{u_m} \cos(\omega t)$

$= U_m \cos(\omega t)$

$a(t) = A_m \cos(\omega t + \alpha)$

$\dot{a}(t) = A_m \frac{d}{dt}\cos(\omega t + \alpha)$

$= -\omega A_m \sin(\omega t + \alpha)$

$= \omega A_m \cos(\omega t + \gamma)$ phase shift $\gamma = \alpha + \frac{\pi}{2}$

Complex Phasors

$a(t) = A_m \cos(\omega t + \alpha)$

$= \operatorname{Re}\{A_m \cos(\omega t + \alpha)\}$

$= \operatorname{Re}\{A_m \cos(\omega t + \alpha) + j A_m \sin(\omega t + \alpha)\}$

$= \operatorname{Re}\{A_m e^{j(\omega t + \alpha)}\}$

$a(t) = \operatorname{Re}\{A \cdot e^{j\omega t}\}$

phasor $A = A_m e^{j\alpha} \in \mathbb{C}$

phasors has power of phase shift

$A \cdot A^* = A_m e^{j\alpha} \cdot A_m e^{-j\alpha} = A_m^2$

$A_m = |A|$ $\tan\alpha = \frac{\operatorname{Im}\{A\}}{\operatorname{Re}\{A\}}$

$a(t) = \operatorname{Re}\{A e^{j\omega t}\}$

$= \operatorname{Re}\{A_m e^{j\alpha} e^{j\omega t}\} = A_m \cos(\omega t + \alpha)$

Energy & Power

instantaneous power: $p(t) = u(t) \cdot i(t)$

$u(t) = U_m \cos(\omega t + \alpha_u)$
$i(t) = I_m \cos(\omega t + \alpha_I)$ — sinusoidal function

period: $T = \frac{2\pi}{\omega}$ frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$

Chapter 9 Complex Phasor Analysis

$\boxed{\begin{array}{ll} I_0 = I_{om}, & i_0(t) = \operatorname{Re}\{I_0 e^{j\omega t}\} \\ U_0 = U_{om} e^{j\pi} = -U_{om}, & u_0(t) = \operatorname{Re}\{U_0 e^{j\omega t}\} \\ i_c = C.\dot{u}_c & I_c = j\omega.CU_c \\ u_L = L.\dot{i}_L & U_L = j\omega.LI_L \end{array}}$

phasors

Network Analysis with Phasors

- Kirchhoff's Laws

KCL: $\underset{\sim}{A}\,\underset{\sim}{i}(t) = 0 \xrightarrow{\text{Lemma}} \underset{\sim}{A}.\underset{\sim}{I} = \underset{\sim}{0}$ $\underset{\sim}{i}(t) = \operatorname{Re}\{\underset{\sim}{I} e^{j\omega t}\}$

KVL: $\underset{\sim}{B}\,\underset{\sim}{u}(t) = \underset{\sim}{0} \longrightarrow \underset{\sim}{B}.\underset{\sim}{U} = \underset{\sim}{0}$ $\underset{\sim}{u}(t) = \operatorname{Re}\{\underset{\sim}{U} e^{j\omega t}\}$

$\underset{\sim}{u}(t) = \underset{\sim}{A}^T \underset{\sim}{u}_k(t) \longrightarrow \underset{\sim}{U} = \underset{\sim}{A}^T.\underset{\sim}{U}_k$ $\underset{\sim}{u}_k(t) = \operatorname{Re}\{\underset{\sim}{U}_k e^{j\omega t}\}$

for time-dependent: small letters
for phasors: big letters

- Network Elements

$\underset{\sim}{M}\,\underset{\sim}{u} + \underset{\sim}{N}\,\underset{\sim}{i} = \underset{\sim}{e}$ (elements)

$i_c(t) = C.\dot{u}_c(t)$, $u_L(t) = L.\dot{i}_L(t)$

$\frac{d}{dt}\underset{\sim}{M}_1 \underset{\sim}{u} + \underset{\sim}{M}_0 \underset{\sim}{u} + \frac{d}{dt}\underset{\sim}{N}_1 \underset{\sim}{i} + \underset{\sim}{N}_0 \underset{\sim}{i} = \underset{\sim}{e}$

$\underbrace{j\omega \underset{\sim}{M}_1 \underset{\sim}{U} + \underset{\sim}{M}_0 \underset{\sim}{U}}_{\underset{\sim}{M}(j\omega)\underset{\sim}{U}} + \underbrace{j\omega \underset{\sim}{N}_1 \underset{\sim}{I} + \underset{\sim}{N}_0 \underset{\sim}{I}}_{\underset{\sim}{N}(j\omega)\underset{\sim}{I}} = \underset{\sim}{E}$

Tableau: $\begin{bmatrix} \underset{\sim}{B} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A} \\ \underset{\sim}{M}(j\omega) & \underset{\sim}{N}(j\omega) \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{U} \\ \underset{\sim}{I} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ \underset{\sim}{E} \end{bmatrix}$ $\underset{\sim}{M}(j\omega)\underset{\sim}{U} + \underset{\sim}{N}(j\omega)\underset{\sim}{I} = \underset{\sim}{E}$

define complex power:

$P = \frac{1}{2} U . I^*$ complex conjugate

$\boxed{\operatorname{Re}\{P\} = \frac{1}{2} U_m . I_m \cos(\alpha_u - \alpha_I) = P_W}$ ← average pow

$\boxed{\operatorname{Im}\{P\} = P_B = \frac{1}{2} U_m I_m \sin(\alpha_u - \alpha_I)}$ ← one tries to avoid the blind power as it creates losses

Blind Power

$\boxed{P = P_W + jP_B = \frac{1}{2} U I^*}$

apparent power:

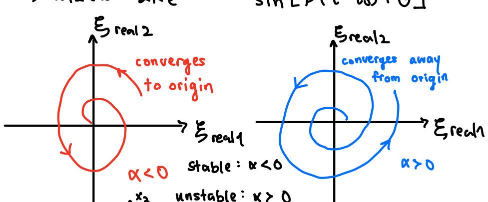
$S = |P| = \sqrt{P_W^2 + P_B^2}$

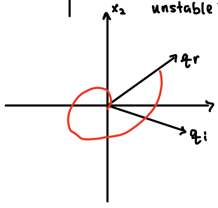
Phase Portraits

- Focus [Case 1 & 3] $\lambda_1 = \alpha + j\beta$, $\alpha \neq 0$, $\beta > 0$

$\xi_{real1}(t) = 2k e^{\alpha(t-t_0)} \cos[\beta(t-t_0) + \theta]$

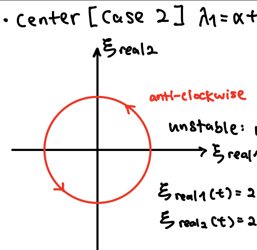
$\xi_{real2}(t) = 2k e^{\alpha(t-t_0)} \sin[\beta(t-t_0) + \theta]$





$\underset{\sim}{x}(t) = \underset{\sim}{Q}_{real}\, \underset{\sim}{\xi}_{real}(t)$

- Center [Case 2] $\lambda_1 = \alpha + j\beta$, $\alpha = 0$, $\beta > 0$



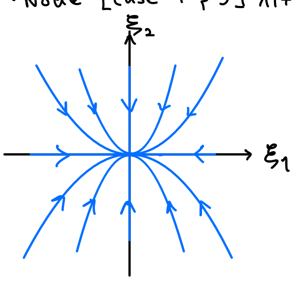
unstable: $\alpha \geq 0$ This is now a circle because $\alpha = 0$

$\xi_{real1}(t) = 2k\, e^{\alpha(t-t_0)} \cos[\beta(t-t_0) + \theta]$

$\xi_{real2}(t) = 2k\, e^{\alpha(t-t_0)} \sin[\beta(t-t_0) + \theta]$

phase portray of a center

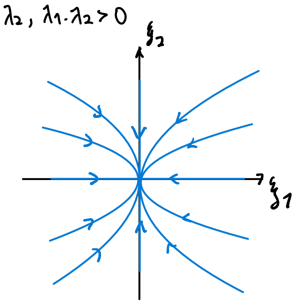
- Node [Case 4 & 5] $\lambda_1 \neq \lambda_2$, $\lambda_1 . \lambda_2 > 0$



both λ are negative

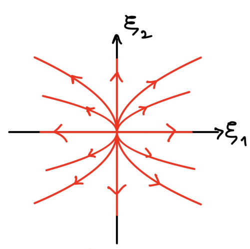
$\lambda_2 < \lambda_1 < 0$

$|\lambda_2| > |\lambda_1|$



$\lambda_1 < \lambda_2 < 0$

$|\lambda_1| > |\lambda_2|$



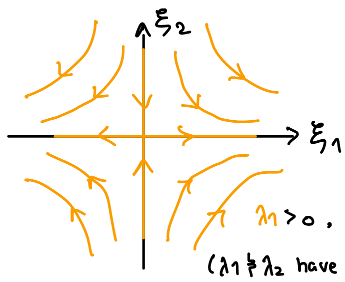
both λ are positive

$\lambda_1 > \lambda_2 > 0$

$|\lambda_1| > |\lambda_2|$

- Saddle Point [Case 6] $\lambda_1 \neq \lambda_2$, $\lambda_1 . \lambda_2 < 0$

Stable
↳ -ve
↳ toward 0

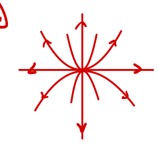


$\lambda_1 > 0$, $\lambda_2 < 0$

(λ_1 & λ_2 have opp. signs)

$\lambda_2 > \lambda_1 > 0$

$|\lambda_2| > |\lambda_1|$



$\lambda^2 - T\lambda + \Delta = 0$

$\lambda_{1/2} = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - \Delta}$

size of node incidence matrix

$\underset{\sim}{A} \in \{0,1,-1\}^{n-1 \times b}$

$\underset{\sim}{B} \in \{0,1,-1\}^{n-1 \times b}$

$\omega = 0 : |\;\;| = 1$

$\omega = \frac{1}{T} : |\;\;| = R$

$\omega \to \infty : |\;\;| \to 0$

Lowpass

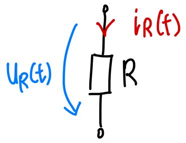
$\omega = 0 : |\;\;| = 0$

$\omega = \frac{1}{T} : |\;\;| = R$

$\omega \to \infty : |\;\;| \to 1$

Highpass

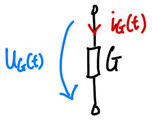
Phasor



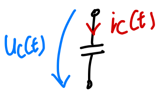
$U_R(t) = R i_R(t) \overset{L_1, L_2}{\Longleftrightarrow} U_R = R I_R$

$U_R(t) = Re\{U_R e^{j\omega t}\}$

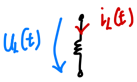
$i_R(t) = Re\{I_R e^{j\omega t}\}$



$i_G(t) = G U_G(t) \overset{L_1, L_2}{\Longleftrightarrow} I_G = G U_G$

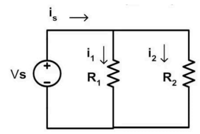


$i_C = C \cdot \dot{u}_C(t) \overset{L_1, L_2, L_3}{\Longleftrightarrow} I_C = j\omega C U_C$



$U_L(t) = L \cdot \dot{i}_L(t) \overset{L_1, L_2, L_3}{\Longleftrightarrow} U_L = j\omega L I_L$

Current divider rule :



$$i_1 = \frac{R_2}{R_1 + R_2} i_s$$

$$i_2 = \frac{R_1}{R_1 + R_2} i_s$$

Voltage divider rule :

$$\boxed{v_1 = \frac{R_1}{R_1 + R_2} v}$$

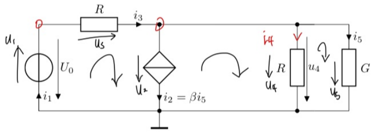
$\omega = 0 : |\;\;| = 0$

$\omega = \frac{1}{T} : |\;\;| = R$

$\omega \to \infty : |\;\;| \to 0$

bandpass

Consider the following circuit with a CCCS, three resistors, and a voltage source.



a)* What is the number of nodes n and the number of branches b of this circuit?

$n = 3$
$b = 5$

b) Give the number of node voltages for above circuit.

$n - 1 = 2$

c) What is the number of linearly independent KCL equations in nullspace representation?

$n - 1 = 2$

d) Give the number of linearly independent KVL equations in nullspace representation.

$b - (n-1) = 3$

e)* Define the branch voltage vector $\boldsymbol{u}$ and the branch current vector $\boldsymbol{i}$.

$\underset{\sim}{i} = [i_1, i_2, i_3, i_4, i_5]^T$

$\underset{\sim}{u} = [u_1, u_2, u_3, u_4, u_5]^T$

f)* Give the number of equations for the KVL in rangespace representation.

$b = 5$

g) Find the loop incidence matrix $\boldsymbol{B}$.

$u_1 + u_2 + u_3 = 0$
$-u_2 + u_4 = 0$
$-u_4 + u_5 = 0$

$\underset{\sim}{B} u = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} u$

h)* Determine the node incidence matrix $\boldsymbol{A}$.

$-i_1 + i_3 = 0$
$-i_3 + i_2 + i_4 + i_5 = 0$

$\underset{\sim}{A} \underset{\sim}{i} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix} \underset{\sim}{i}$

i) Based on $\boldsymbol{A}$ and $\boldsymbol{B}$, show that the Tellegen's theorem is fulfilled.

$\underset{\sim}{A} \cdot \underset{\sim}{B}^T = \underset{\sim}{0}$

$\overset{2\times5}{\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix}} \overset{5\times3}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underset{\sim}{0}$

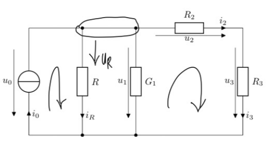
j) Find i_3 depending on u_4.

$i_3 = i_2 + i_4 + i_5$ $\quad i_3 = \beta G u_4 + \frac{u_4}{R} + G u_4$

$u_4 = u_5$

$i_5 = G u_4$

$i_4 = \frac{u_4}{R}$



a)* Give the current i_R depending on u_0.

$-U_0 + R i_R = 0 \quad i_R = \frac{U_0}{R}$

b)* Determine the overall conductance G_{overall} of the parallel connection of R with G_1.

$G_{overall} = \frac{1}{R} + G_1$

c)* Find the voltage u_2 depending on i_2.

$U_2 = R_2 i_2$

d)* What is the overall resistance R_{overall} of the series connection of R_2 and R_3?

$R_2 + R_3$

e) Hence, what are the voltages u_2 and u_3 depending on u_0?

$U_R = U_1 = U_0$

$U_2 = \frac{R_2}{R_2 + R_3} U_0$

$U_3 = \frac{R_3}{R_2 + R_3} U_0$

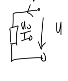
f) Find i_0 depending on u_0.

$i_0 = \frac{U_0}{R} + U_0 G_1 + \frac{1}{R_2 + R_3} U_0 \qquad i_0 = i_R + i_1 + i_2$

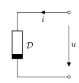
g) Determine i_2 depending on i_0.

$i_2 = \dfrac{\frac{R_3}{R_2 + R_3}}{\frac{1}{R} + G_1 + \frac{R_3}{R_2 + R_3}} i_0 = \dfrac{\frac{1}{R_2 + R_3}}{\frac{1}{R} + G + \frac{1}{R_2 + R_3}} i_0$

d) Draw the equivalent circuit diagram of $r_{\mathcal{D},\text{lin}}(i)$.



←



The characteristic of the one-port $\mathcal{D}$ is given by

$$u = r_{\mathcal{D}}(i) = U_0 + U_0 \sin\left(\frac{i - I_0}{I_0}\right)$$

with the constants U_0 and I_0.

a)* Give the port-quantities that control $\mathcal{D}$.

i

In the operating point, the current is $I = 2I_0$.

b)* Find the operating point value U of the voltage across $\mathcal{D}$.

$U = r_D(2I_0) = U_0 + U_0 \sin\left(\frac{2I_0 - I_0}{I_0}\right) = U_0 + U_0 \sin(1)$

In following, consider the different operating point $(U,I) = (U_0,I_0)$.

c)* Determine the linearization $r_{\mathcal{D},\text{lin}}(i)$ of $\mathcal{D}$ in the given operating point $(U,I) = (U_0,I_0)$.

$r_{D\,lin}(i) = \left.\frac{d r_D(i)}{di}\right|_{i = I_0} \cdot (i - I_0) + U_0$

$= \frac{U_0}{I_0} \cos\left(\cancel{\frac{I_0 - I_0}{I_0}}^{\,0}\right) \cdot (i - I_0) + U_0$

$= \frac{U_0}{I_0} (i - I_0) + U_0$

$= \frac{U_0}{I_0} i$

The resistance matrix of the two-port $\mathcal{S}$ reads as

$$\boldsymbol{R}_{\mathcal{S}} = \begin{bmatrix} 0 & -\frac{1}{G_d} \\ \frac{1}{G_d} & 0 \end{bmatrix}$$

with finite G_d.

a)* Based on the given resistance matrix $\boldsymbol{R}_{\mathcal{S}}$, show that the two-port $\mathcal{S}$ is not reciprocal.

$r_{12} \neq r_{21}$ $\therefore$ not reciprocal

b)* What special two-port is $\mathcal{S}$?

gyrator

c)* Determine the conductance matrix $\boldsymbol{G}_{\mathcal{S}}$ of $\mathcal{S}$.

$G_S = \frac{1}{\det R}\begin{bmatrix} 0 & \frac{1}{G_d} \\ -\frac{1}{G_d} & 0 \end{bmatrix} = G_d^2 \begin{bmatrix} 0 & \frac{1}{G_d} \\ -\frac{1}{G_d} & 0 \end{bmatrix} = \begin{bmatrix} 0 & G_d \\ -G_d & 0 \end{bmatrix}$ $\quad \det R = (0)(0) - (-\frac{1}{G_d})(\frac{1}{G_d}) = \frac{1}{G_d^2}$

A different two-port $\mathcal{Z}$ has got the conductance matrix

$$\boldsymbol{G}_{\mathcal{Z}} = \begin{bmatrix} 0 & -G_d \\ 2G_d & 0 \end{bmatrix}.$$

d)* Why is $\mathcal{Z}$ lossy?

$-G_Z^T = \begin{bmatrix} 0 & -2G_d \\ G_d & 0 \end{bmatrix} \neq G_Z$

The two-ports $\mathcal{S}$ and $\mathcal{Z}$ are connected in parallel to get the two-port $\mathcal{X}$.

e) Find a two-port matrix for the two-port $\mathcal{X}$.

$G_X = G_S + G_Z$
$= \begin{bmatrix} 0 & 0 \\ G_d & 0 \end{bmatrix}$

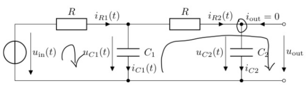
f) Give expressions for u_i and i_i of $\mathcal{X}$.

$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = G_X \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$
$i_1 = 0$
$i_2 = G_d u_1$
$u_1 = \frac{1}{G_d} i_2$

g) What special two-port is $\mathcal{X}$?

VCCS with gain in G_d

Given is the following circuit with two capacitors.
The angular frequency of the sinusoidal excitation $u_{\text{in}}(t)$ is $\omega > 0$.



a)* Give $i_{C1}(t)$ depending on $u_{C1}(t)$.

$i_{C_1}(t) = C_1 \cdot \dot{U}_{C_1}(t)$

Let I_{C1} denote the phasor corresponding to $i_{C1}(t)$ and U_{C1} the phasor for $u_{C1}(t)$.

b) What is the current phasor I_{C1} depending on the voltage phasor U_{C1}?

$I_{C_1} = j\omega \cdot C_1 U_{C_1}$

c)* Find the voltage phasor U_{C1} depending on the voltage phasor U_{out} (independent of other current or voltage phasors) taking into account that $I_{\text{out}} = 0$.

KVL: $-U_{C1} + R I_{R2} + U_{out} = 0$
$i_{C2} = i_{R2}(t)$ $\quad \therefore U_{C1} = j\omega \cdot R C_2 U_{out} + U_{out}$
$U_{C_2}(t) = U_{out}$
$I_{C2} = j\omega \cdot C_2 U_{out}$

d) Determine the current phasor I_{R1} depending on the voltage phasor U_{out}.

KCL: $i_{R1} = i_{R2} + i_{C1}$ $\quad I_{R1} = j^2\omega^2 \cdot R C_1 C_2 U_{out}$
$I_{C_1} = j\omega \cdot C_1 U_{C_1}$ $\quad + j\omega C_1 U_{out}$
$I_{R2} = j\omega \cdot C_2 U_{out}$ $\quad + j\omega \cdot C_2 U_{out}$

Let the output phasor be given by

$$U_{\text{out}} = 2\,\text{V}\,e^{j\frac{\pi}{3}}.$$

j)* Give the time-signal $u_{\text{out}}(t)$ corresponding to the phasor U_{out}.

$U_{out}(t) = Re\{2V e^{j\frac{\pi}{3}}\} = 2V \cos(\omega t + \frac{\pi}{3})$

e)* Express the voltage phasor U_{C1} depending on the voltage phasor U_{in} and the current phasor I_{R1}.

KVL: $-U_{in} + R I_{R1} + U_{C1} = 0$
$U_{C1} = U_{in} - R I_{R1}$

With the time constant τ, the transfer function of another circuit can be written as

$$H(j\omega) = \frac{U_{\text{out}}(j\omega)}{U_{\text{in}}(j\omega)} = \frac{1}{(j\omega\tau)^2 + \frac{3}{2}j\omega\tau + 1}.$$

f)* Investigate $|H(j\omega)|$ at $\omega = 0$, $\omega = \frac{1}{\tau}$, and $\omega \to \infty$.

$\omega = 0$: $|H(j\omega)| = 1$
$\omega = \frac{1}{\tau}$: $|H(j\omega)| = \frac{1}{|-1 + \frac{3}{2}j + 1|} = \frac{2}{3}$
$\omega \to \infty$: $|H(j\omega)| = 0$

g) What filter type (lowpass, highpass, bandpass, bandstop, allpass) is the transfer function $H(j\omega)$? Justify your answer based on above results.

Lowpass

The input voltage is given by $u_{\text{in}}(t) = 6\,\text{V}\cos(\frac{2}{\tau}t + \frac{\pi}{4})$.

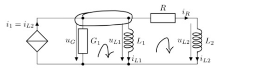
h)* Give the phasor U_{in} corresponding to the given $u_{\text{in}}(t)$.

$U_{in} = 6V e^{j\pi/4}$

i) Find the output phasor U_{out} in polar form, i.e., as the product of magnitude and the exponential depending on the phase.

$U_{out} = H(j\frac{2}{\tau}) U_{in}$
$H(j\frac{2}{\tau}) = \frac{1}{(j2)^2 + 3j + 1} = \frac{1}{|3j - 3|} = \frac{1}{\sqrt{9+9}} = \frac{1}{3\sqrt{2}}$
$H(j\frac{2}{\tau}) = \frac{1}{3\sqrt{2}} e^{j\frac{-3\pi}{4}}$
$U_{out} = \sqrt{2}\,V e^{-j\frac{\pi}{2}}$

Given is the following second-order circuit with the two inductors L_1 and L_2 connected to
CCCS and two resistors.



a)* Give the state variables of the circuit.

i_{L1}, i_{L2}

b) Define the state vector $\boldsymbol{x}$.

$X = \begin{bmatrix} i_{L1} \\ i_{L2} \end{bmatrix}$

c)* Find u_{L1} depending on u_{L2} and i_R.

$U_{L1} = R i_R + U_{L2}$

d)* Give i_{L2} depending on i_R.

$i_R = i_{L2}$

e)* Determine u_{L1} depending on u_G.

$U_G = U_{L1}$

f) Express u_{L1} depending on i_{L1} without using any time derivatives.

KCL: $-i_{L2} + U_G G_1 + i_{L1} + i_R = 0$ $\quad U_G = U_{L1}$
$U_{L1} = \frac{1}{G_1} i_{L1}$

g) Find u_{L2} depending on i_{L1} and i_{L2}.

$U_{L2} = U_{L1} - R i_R = \frac{1}{G_1} i_{L1} - R i_{L2}$

h) Give the state equations for the given circuit.

$U_L = L \dot{i}_L$
$\dot{i}_{L_1} = \frac{1}{G_1 L_1} i_{L1}$ $\quad \dot{i}_{L_2} = \frac{1}{G_1 L_1} i_{L1} - \frac{R}{L_1} i_{L2}$

The state equations for another second-order system read as

$$\dot{x}_1 = -2x_1$$
$$\dot{x}_2 = -3x_1 + x_2$$

with the state vector $\boldsymbol{x} = [x_1, x_2]^{\text{T}}$.

i)* Find the state matrix $\boldsymbol{A}$.

$\dot{\underline{x}} = \underset{\sim}{A}\,\underline{x}$ $\quad A = \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix}$

j) Determine the eigenvalues of $\boldsymbol{A}$.

$\det(A - \underset{\sim}{1}\lambda) = (-2-\lambda)(1-\lambda) = 0$
$\lambda_1 = -2$ or $\lambda_2 = 1$

k) Investigate **both** eigenvalues and explain why the circuit is unstable.

$\lambda_1 = -2$: stable eigenvalue
$\lambda_2 = 1$: unstable eigenvalue
Due to λ_2, the circuit is unstable

The current-controlled characteristic of the non-linear two-port $\mathcal{T}$ can be written as

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \boldsymbol{r}_{\mathcal{T}}(\boldsymbol{i}) = \begin{bmatrix} U_0(\frac{i_1}{I_0} + \frac{i_2^2}{I_0^2}) \\ R_0 i_1 + 2R_0 i_2 \end{bmatrix}$$

with the constants U_0, I_0, and $R_0 = \frac{U_0}{I_0}$.

a)* Find u_1 and u_2 of the two-port $\mathcal{T}$ for $i_1 = 0$ and $i_2 = 0$.

$u_1 = 0$ and $u_2 = 0$ ✓

b) Argue why the two-port is sourcefree.

The origin is part of the characteristic as $\boldsymbol{r}_{\mathcal{T}}(\boldsymbol{0}) = \boldsymbol{0}$. ✓

c)* Determine $\boldsymbol{r}_{\mathcal{T}}([i_2, i_1]^{\text{T}})$.

$\boldsymbol{r}_{\mathcal{T}}([i_2, i_1]^{\text{T}}) = \begin{bmatrix} R_0 i_2 + R_0 \frac{i_1^2}{I_0} \\ R_0 i_2 + 2R_0 i_1 \end{bmatrix}$ ✓

d) Argue why the two-port $\mathcal{T}$ is not symmetric.

$\boldsymbol{r}_{\mathcal{T}}([i_2, i_1]^{\text{T}}) \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{r}_{\mathcal{T}}(\boldsymbol{i})$ ✓

e)* Why does the hybrid-controlled characteristic of $\mathcal{T}$ not exist?

u_1 and i_1 depend on i_2^2. ✓
It is impossible to find a unique expression for i_2 depending on the i_1 and u_2. ✓

f)* Find the inverse hybrid representation of $\mathcal{T}$.

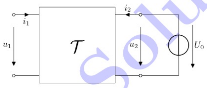
$i_1 = \frac{u_1}{R_0} - \frac{i_2^2}{I_0}$ ✓
$u_2 = u_1 + 2R_0 i_2 - U_0 \frac{i_2^2}{I_0^2}$ ✓

Assume that the two-port $\mathcal{T}$ is connected to a short circuit at port one and to the current source I_0 at port two.

g) Determine the corresponding operating point (U_1,U_2,I_1,I_2).

short circuit: $U_1 = 0$ ✓
current source: $I_2 = I_0$ ✓
$I_1 = -\frac{I_0^2}{I_0} = -I_0$ ✓
$U_2 = U_0$ ✓

Now the two-port is connected to the voltage source U_0 at port two as illustrated below.



h)* What is u_2 depending on U_0?

$u_2 = U_0$ ✓

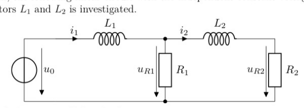
i) Express i_2 depending on i_1 and I_0.
Remember that $R_0 = \frac{U_0}{I_0}$.

$i_2 = -\frac{1}{2} i_1 + \frac{I_0}{2}$ ✓

j) What is the power p_2 at port two?

$p_2 = u_2 i_2 = -\frac{1}{2} U_0 i_1 + \frac{U_0 I_0}{2}$ ✓

In this problem, the following linear circuit with the independent constant voltage source u_0 and the two inductors L_1 and L_2 is investigated.



Note that u_{R1} is the output of the circuit.

a)* Give the state variables of the circuit.

i_1, i_2 ✓

b)* Determine the state equations of the circuit.

KVL: $u_0 = u_1 + u_{R1}$ ✓
KCL: $\frac{1}{R_1} u_{R1} = i_1 - i_2$ ✓
KVL: $u_{R1} = u_2 + R_2 i_2$ ✓
$u_1 = -R_1(i_1 - i_2) + u_0$
$u_2 = R_1(i_1 - i_2) - R_2 i_2$
$\dot{i}_1 = -\frac{R_1}{L_1} i_1 + \frac{R_1}{L_1} i_2 + \frac{1}{L_1} u_0$ ✓
$\dot{i}_2 = \frac{R_1}{L_2} i_1 - \frac{R_1 + R_2}{L_2} i_2$ ✓

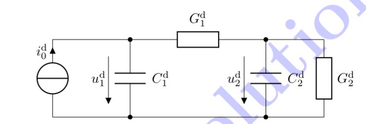
c) What are the state matrix and the input vector for the circuit?

$\boldsymbol{A} = \begin{bmatrix} -\frac{R_1}{L_1} & \frac{R_1}{L_1} \\ \frac{R_1}{L_2} & -\frac{R_1 + R_2}{L_2} \end{bmatrix}$ ✓ $\quad \boldsymbol{b} = \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix}$ ✓

d) Give the output vector and the feedthrough if u_{R1} is the output.

$\boldsymbol{c}^{\text{T}} = [R_1, -R_1]$ ✓
$d = 0$ ✓

e)* Draw the circuit which is dual to the given circuit.



For a particular choice of the element values, the following normalized state matrix can be obtained

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 \\ -3 & -5 \end{bmatrix}.$$

f)* Determine the eigenvalues of this state matrix.

$\det(\boldsymbol{A} - \lambda\boldsymbol{1}) = \det\begin{bmatrix} -1-\lambda & 1 \\ -3 & -5-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 8 = 0$ ✓
$\lambda_{1/2} = -3 \pm \sqrt{9 - 8} = -3 \pm 1$
$\lambda_1 = -2$ und $\lambda_2 = -4$ ✓✓

g) Give the name of the equilibrium.

stable node ✓

h) Find the eigenvectors of the given state matrix.

$[\boldsymbol{A} - \lambda_1\boldsymbol{1}]\boldsymbol{q}_1 = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix}\boldsymbol{q}_1 = \boldsymbol{0}$ $\quad \boldsymbol{q}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓
$[\boldsymbol{A} - \lambda_2\boldsymbol{1}]\boldsymbol{q}_2 = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix}\boldsymbol{q}_2 = \boldsymbol{0}$ $\quad \boldsymbol{q}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ ✓

From now on, $u_0 = 0$. Correspondingly, the state equations read as

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -3x_1 - 5x_2.$$

i)* What is the equilibrium $\boldsymbol{x}_\infty$ of the circuit for $u_0 = 0$?

$\dot{x}_1 = \dot{x}_2 = 0$ ✓
$\boldsymbol{x}_\infty = \boldsymbol{0}$ ✓

For a different second-order circuit, the eigenvalues and eigenvectors can be written as

$$\lambda_1 = -4 \qquad \lambda_2 = 2$$
$$\boldsymbol{q}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \boldsymbol{q}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The excitation is zero. In other words, $v(t) = 0$.

j)* Why is this circuit unstable?

Since $\lambda_2 = 2 > 0$. ✓

k)* Give the general expression of the state vector $\boldsymbol{x}(t)$ of the circuit with the given eigenvalues and eigenvectors.

$\boldsymbol{x}(t) = \xi_{01}\,e^{-4t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \xi_{02}\,e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓✓

Due to $v(t) = 0$, the output equation is given by

$$y(t) = \boldsymbol{c}^{\text{T}}\boldsymbol{x}(t).$$

l)* Determine a non-trivial output vector $\boldsymbol{c}^{\text{T}}$, such that $y(t)$ converges to zero.

The eigenmode with the eigenvalue $\lambda_2 = 2$ must be suppressed by $\boldsymbol{c}^{\text{T}}$. Therefore, $\boldsymbol{c}$ must be orthogonal to $\boldsymbol{q}_2$. ✓
e.g., $\boldsymbol{c}^{\text{T}} = [1, -1]$ ✓

m) What is $y(t)$ for this particular choice for $\boldsymbol{c}^{\text{T}}$?

$y(t) = \boldsymbol{c}^{\text{T}}\boldsymbol{q}_1\xi_{01}\,e^{\lambda_1 t} = 4\xi_{01}\,e^{-4t}$ ✓

Sample Solution