EDE1012 MATHEMATICS 2

Tutorial 1

Integrals & The Fundamental Theorem of Calculus

1. Sketch the area represented by the integral below and identify the shape. Then, set up the Riemann sum using the right endpoint and evaluate the Riemann integral to derive the area formula. Repeat using the left endpoint.

$$\int_0^h \frac{b-a}{h}x + a\,dx$$

ANS: Trapezium. $\sum_{i=1}^{n}\left[\frac{(b-a)i}{n}+a\right]\frac{h}{n}$. $\text{Area} = \frac{(a+b)h}{2}$

2. Evaluate the definite integrals below by setting up the Riemann sum using the right endpoint and evaluating the Riemann integral.

a) $\int_0^1 x^3\,dx$

b) $\int_{-2}^2 -3x\,dx$

c) $\int_{-3}^0 2x^2\,dx$

d) $\int_{-1}^3 x^3\,dx$

ANS: **a)** ¼. **b)** 0. **c)** 18. **d)** 20

3. (https://openstax.org/books/calculus-volume-1/pages/5-2-the-definite-integral)
Without evaluating the Riemann integral, determine each area represented by each definite integral below using area formulas.

a) $\int_0^3 3 - x\,dx$

b) $\int_{-2}^6 3 - |x-3|\,dx$

c) $\int_{-2}^2 \sqrt{4-x^2}\,dx$

d) $\int_6^{12} \sqrt{36-(x-6)^2}\,dx$

ANS: **a)** 9/2. **b)** 7. **c)** 2π. **d)** 9π.

4. Express each limit below as a (right) Riemann integral.

a) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^5}{n^6}$

b) $\lim_{n\to\infty}\frac{5}{n}\sum_{i=1}^{n}\frac{1}{1-(2+\frac{5i}{n})}$

c) $\lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\ln\left(1+\frac{2i}{n}\right)$

d) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3n}{n^2+i^2}$

ANS: **a)** $\int_0^1 3x^5dx$. **b)** $\int_2^7 \frac{1}{1-x}dx$. **c)** $\int_1^3 2\ln x\,dx$. **d)** $\int_0^1 \frac{3}{1+x^2}\,dx$.

5. Prove the following integrals for even and odd functions over a symmetric interval. Hint: Use a change of variables, y = -x, where appropriate.

$$\int_{-a}^{a} f_e(x)dx = 2\int_0^a f_e(x)dx,$$
$$\int_{-a}^{a} f_o(x)dx = 0$$

6. Using the fundamental theorem of calculus, determine the derivative of each function below.

a) $F(x) = \int_0^x t^3 - 2\,dt$

b) $f(x) = \int_x^1 e^{2\sin t}\,dx$

c) $g(t) = \int_{-2}^{\ln t} e^{3x}\,dx$

d) $h(y) = \int_y^{2y} \sqrt{1-x^2}\,dx$

e) $p(t) = \int_{e^t}^{-2} \ln u^2\,du$

f) $G(z) = \int_{\sqrt{z}}^{\tan z} \frac{1-v}{1+v^2}\,dv$

g) $g(x) = \int_0^x xf(t)\,dt$

h) $h(x) = \int_3^{g(x)} xf(t)\,dt$

ANS: **a)** $F'(x) = x^3 - 2$. **b)** $f'(x) = -e^{2\sin x}$. **c)** $g'(t) = t^2$.
d) $h'(y) = 2\sqrt{1-4y^2} - \sqrt{1-y^2}$. **e)** $p'(t) = -2te^t$.

f) $G(z) = 1 - \tan z - \dfrac{1-\sqrt{z}}{2\sqrt{z}(1+z)}$. **g)** $g'(x) = xf(x) + \displaystyle\int_0^x f(t)\,dt$.

h) $h'(x) = xg'(x)f(g(x)) + \displaystyle\int_3^{g(x)} f(t)\,dt$.

7. By evaluating the physical meaning of an integral, answer the following questions.

   a) Given that I'(t) represents the new daily COVID19 cases, state the meaning of

   $$\int_0^{30} I'(t)\,dt$$

   b) If Q(t) represents the volumetric flow rate of water flowing into an empty tank in cubic meters per second, state the meaning of

   $$\rho \int_0^t Q(t)\,dt$$

   where ρ is the density of water in kg/m^3.

   c) A rod has length L and cross-sectional area A. If its material density function is ρ(x), where x is the position along the rod from one end, state an expression for determining the mass of the rod. What is the rate of change of mass wrt position?

8. Show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i+n} = \ln 2$$

9. For a linear spring with stiffness k, show that the work done in compressing or extending the spring over distance x is given by

$$W = \frac{1}{2}kx^2$$

   using integration. So what is the elastic potential energy stored in the spring?

ANS: Elastic potential energy = $kx^2/2$.

10. A force is applied on a mass m causing it to undergo an acceleration a(x) from position x_1 to x_2. Using integration, show that the change in kinetic energy of the mass is

$$\Delta KE = \frac{1}{2}m\left(v_2^2 - v_1^2\right).$$

Hint: Use a change of variables.

11. Evaluate each indefinite integral below.

a) $\int 4e^x - \frac{3}{x}\,dx$

b) $\int \frac{x^5 + 2x^3 + 3x + 4}{x^2}\,dx$

c) $\int \frac{1 - \cos^3 x}{\cos^2 x}\,dx$

d) $\int 7^x + \frac{\sin 2x}{\cos x}\,dx$

ANS: **a)** $4e^x - 3\ln x + c$. **b)** $\frac{x^4}{4} + x^2 + 3\ln x - \frac{4}{x} + c$. **c)** $\tan x - \sin x + c$. **d)** $\frac{7^x}{\ln 7} - 2\cos x + c$

12. **Mathematical Modelling:** Adaptive Cruise Control (ACC) (https://en.wikipedia.org/wiki/Adaptive_cruise_control)

A car activated with ACC is travelling along a straight road at velocity v_c. Suddenly, a truck travelling at a slower velocity v_T moved into the car's lane at distance d ahead. The ACC system senses the truck and activates the braking of the car. As a control engineer designing the ACC, answer the following questions.

a) Set up an integral that gives the braking distance, s, of the car from an initial velocity v_0 to the final velocity v_f. Hint: Use an appropriate change of variables.

b) If the car is to avoid a collision with the truck, determine the min constant deceleration that the ACC must activate on the car.

c) If the car is to maintain a safe distance d_s < d while cruising behind the truck, determine the min constant deceleration that the ACC must activate on the car. Is this value logically bigger or smaller than that in (b)?

ANS: **a)** $s = \int_{v_0}^{v_f} \frac{v}{a}\, dv$. **b)** $\text{min deceleration} = \left|\frac{v_T^2 - v_c^2}{2d}\right|$.

c) $\text{min deceleration} = \left|\frac{v_T^2 - v_c^2}{2(d - d_s)}\right|$.

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-2/pages/1-1-approximating-areas
2) https://openstax.org/books/calculus-volume-2/pages/1-2-the-definite-integral
3) https://openstax.org/books/calculus-volume-2/pages/1-3-the-fundamental-theorem-of-calculus
4) https://openstax.org/books/calculus-volume-2/pages/1-4-integration-formulas-and-the-net-change-theorem

End of Tutorial 1
(Email to youliangzheng@gmail.com for assistance.)