Rotational Dynamics - Revision

Kinetic Energy of Rotation

- Refresher on angular quantities
  - Angular displacement θ
  - Angular velocity $\omega = \dfrac{d\theta}{dt}$
  - Angular acceleration $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$
- Relationship between linear and angular quantities
  - Displacement, velocity acceleration

$$s = r\theta$$

$$v = r\omega$$

$$a_{\mathrm{t}} = r\alpha \quad a_{\mathrm{c}} = \omega^2 r \quad a = \sqrt{a_{\mathrm{c}}^2 + a_{\mathrm{t}}^2}$$

Kinetic energy for rotation (point particle):

$$K = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m (r\omega)^2 = \tfrac{1}{2} m r^2 \omega^2$$

Kinetic Energy of Rotation

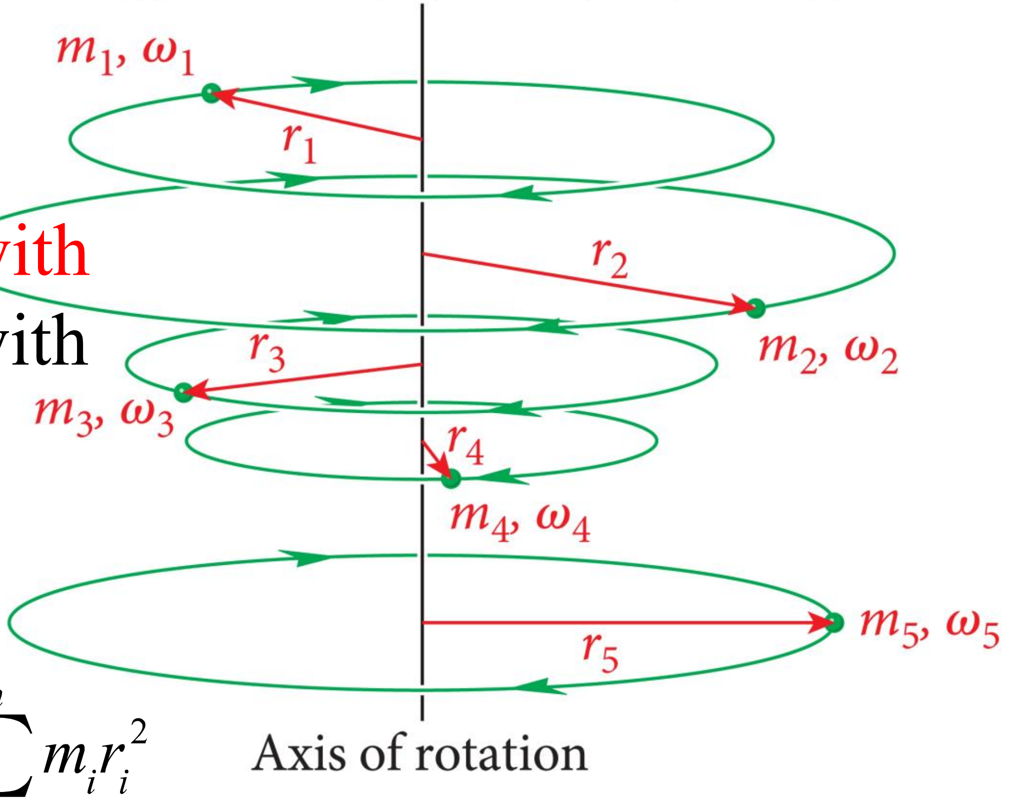
- Consider the kinetic energy of several rotating point particles:

$$K = \sum_{i=1}^{n} K_i = \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \tfrac{1}{2}\sum_{i=1}^{n} m_i r_i^2 \omega_i^2$$

- Assume all particles keep their distances fixed with respect to each other - solid object, all moving with the same angular velocity.

$$K = \tfrac{1}{2}\sum_{i=1}^{n} m_i r_i^2 \omega^2 = \tfrac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I\omega^2$$

- Where I is the moment of inertia given by: $I = \sum_{i=1}^{n} m_i r_i^2$



Compare $K_{\text{linear}} = \dfrac{1}{2} m v^2 \Leftrightarrow \boxed{K_{\text{rotation}} = \dfrac{1}{2} I\omega^2}$

Kinetic Energy of Rotation

WORKED EXAMPLE

A uniform solid cylinder of mass M = 5 kg is rolling without slipping along a horizontal surface. The velocity of its center of mass is 30 m/s. Calculate its energy.

[Ans: 3375 J]

Moment of Inertia

- Extend expression for the moment of inertia to continuous objects. Approximate our extended object as a collection of small, identically sized cubes of volume *V* of density *ρ*:

$$m_i = V\rho(\vec{r}_i) \Rightarrow I = \sum_{i=1}^{n} \rho(\vec{r}_i) r_i^2 V$$

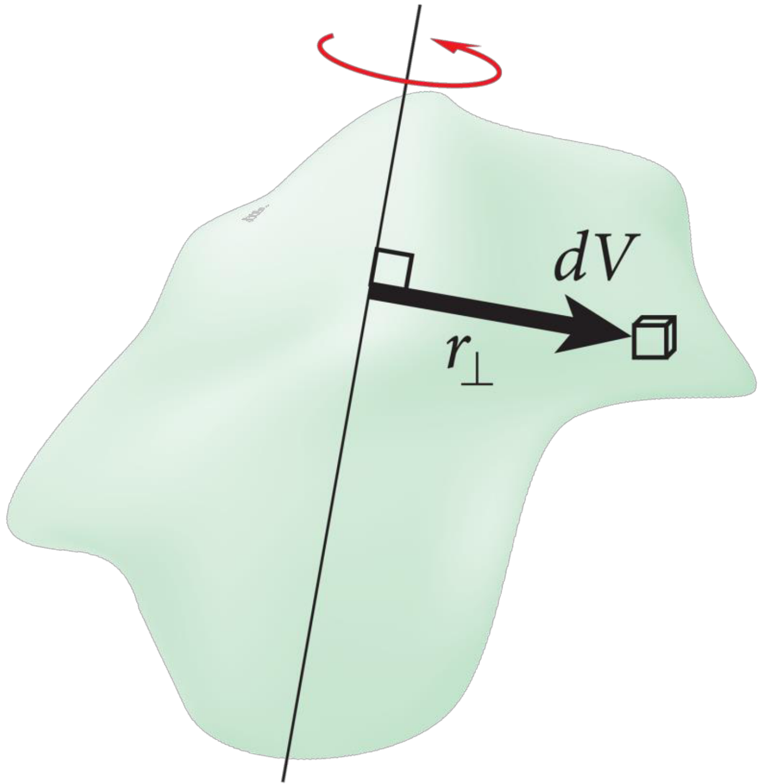
- Letting the volume of the cubes go to 0:

$$\boxed{I = \int_V r_\perp^2 \rho(\vec{r})\, dV}$$

- The total mass of the object is: $M = \int_V \rho(\vec{r})\, dV$

- Assume density is constant:

$$\boxed{I = \frac{M}{V}\int_V r_\perp^2\, dV}$$



Moment of Inertia

WORKED EXAMPLE

Calculate the moment of inertia of a hollow cylinder of uniform mass density r with length L, inner radius R_1, and outer radius R_2 about its axis of symmetry

[Ans: $I = \frac{1}{2}M(R_1^2 + R_2^2)$]

Moment of Inertia

Parallel Axis Theorem

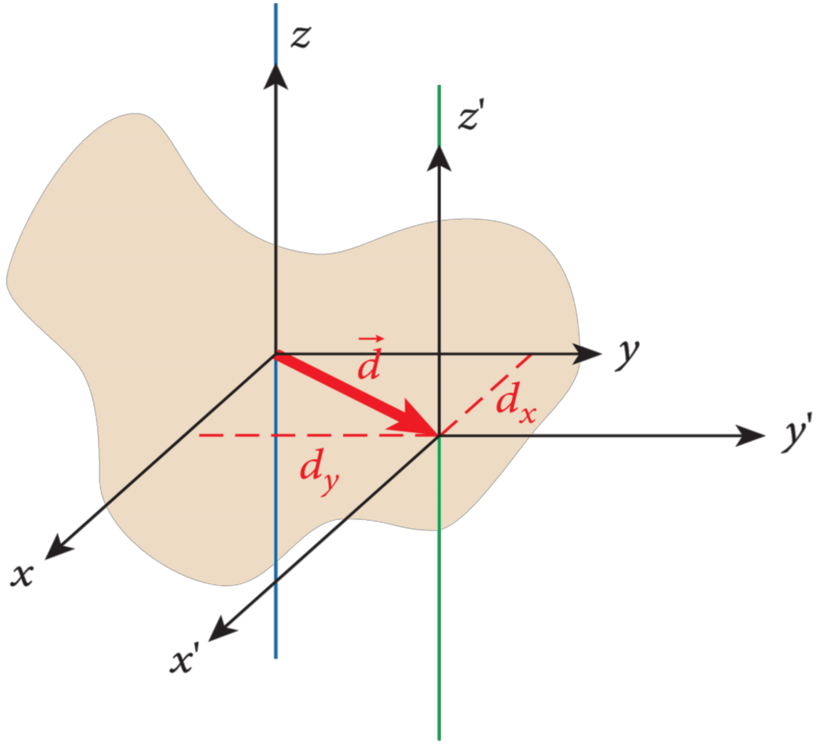
- Moment of inertia for bodies rotating on an axis that does not go through the center of mass.
- Relationship between old and new coordinates: $x' = x - d_x;\ \ y' = y - d_y;\ \ z' = z$
- Moment of inertia about new axis:

$$I_{\parallel} = \int_V (r'_{\perp})^2 \rho\, dV$$

$$= \int_V r_{\perp}^2 \rho\, dV + d^2 \int_V \rho\, dV - 2d_x \int_V x\rho\, dV - 2d_y \int_V y\rho\, dV$$

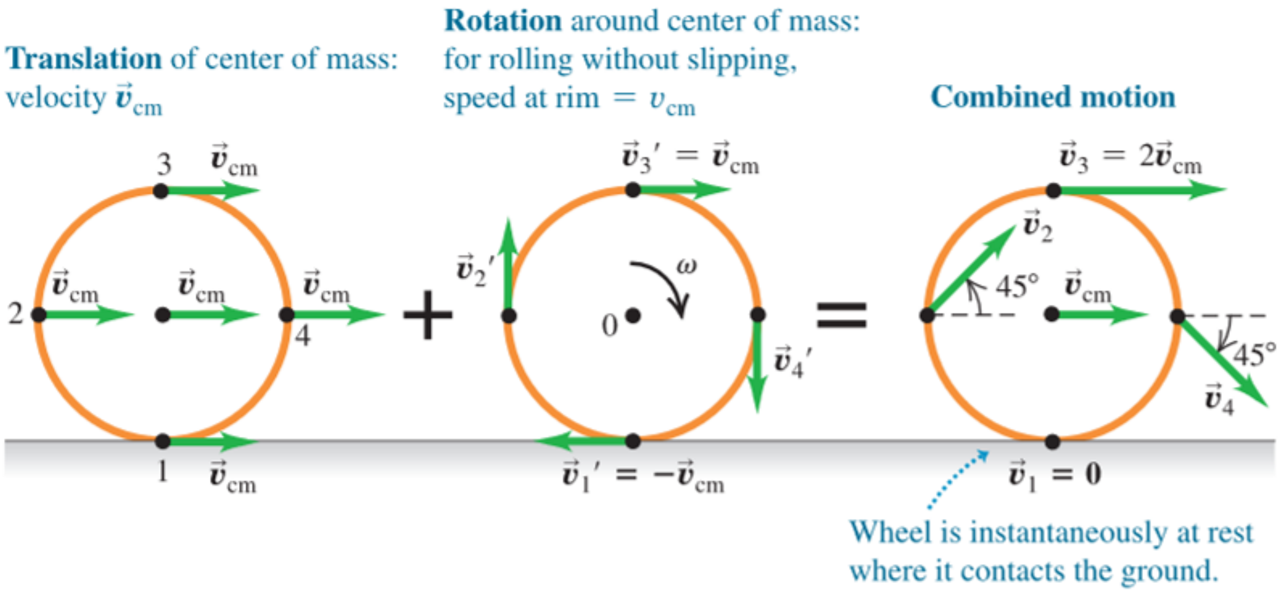
$$\boxed{I_{\parallel} = I_{\mathrm{cm}} + Md^2}$$

$$I = (cR^2 + d^2)M \quad (0 < c \le 1,\ \text{depending on } I_{\mathrm{cm}})$$



Rolling Without Slipping

- An important case of combined translation and rotation is rolling without slipping. The point on the wheel that contacts the surface must be instantaneously at rest so that it does not slip.
- The velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward twice as fast as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

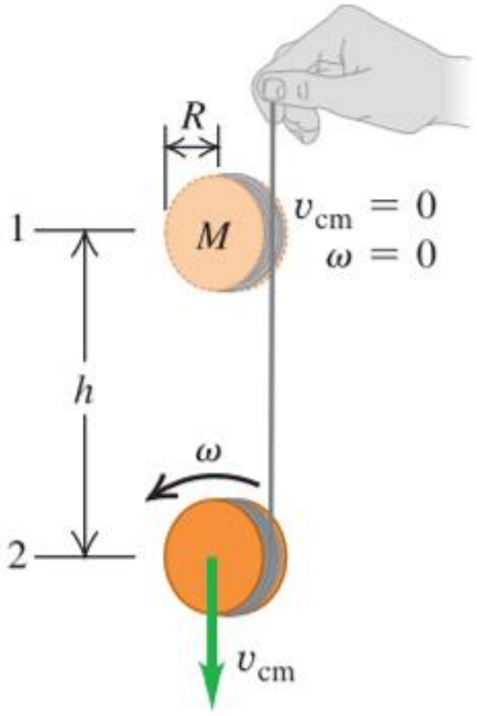


$$K = K_{\text{trans}} + K_{\text{rot}} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

Rolling Without Slipping

WORKED EXAMPLE

A yo-yo has a massless string wrapped around a solid cylinder with mass M and radius R. The string unwinds but does not slip or stretch as the cylinder descends and rotates. Find the speed v_{cm} of the cylinder's center of mass after it has descended a distance h.



[Ans: $\sqrt{\frac{4}{3}gh}$]

Torque

- Can exert a force on an extended object at a point away from its center of mass, which can cause the extended object to rotate as well as move linearly.
- Define torque as the vector cross product of the force and the moment arm:

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad\qquad \tau = rF\sin\theta$$

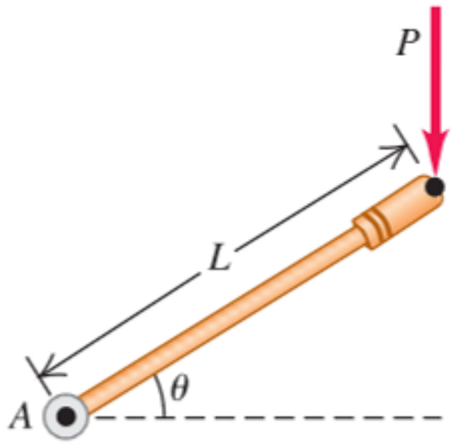
- Force, direction, and moment arm are important.
- Torque is an axial vector.
- Define the **net torque** as the difference between the sum of all clockwise torques and the sum of all counter-clockwise torques

$$\tau_{\text{net}} = \sum_i \tau_{\text{counter-clockwise},i} - \sum_j \tau_{\text{clockwise},j}$$

WORKED EXAMPLE

What is the magnitude of the torque about point A

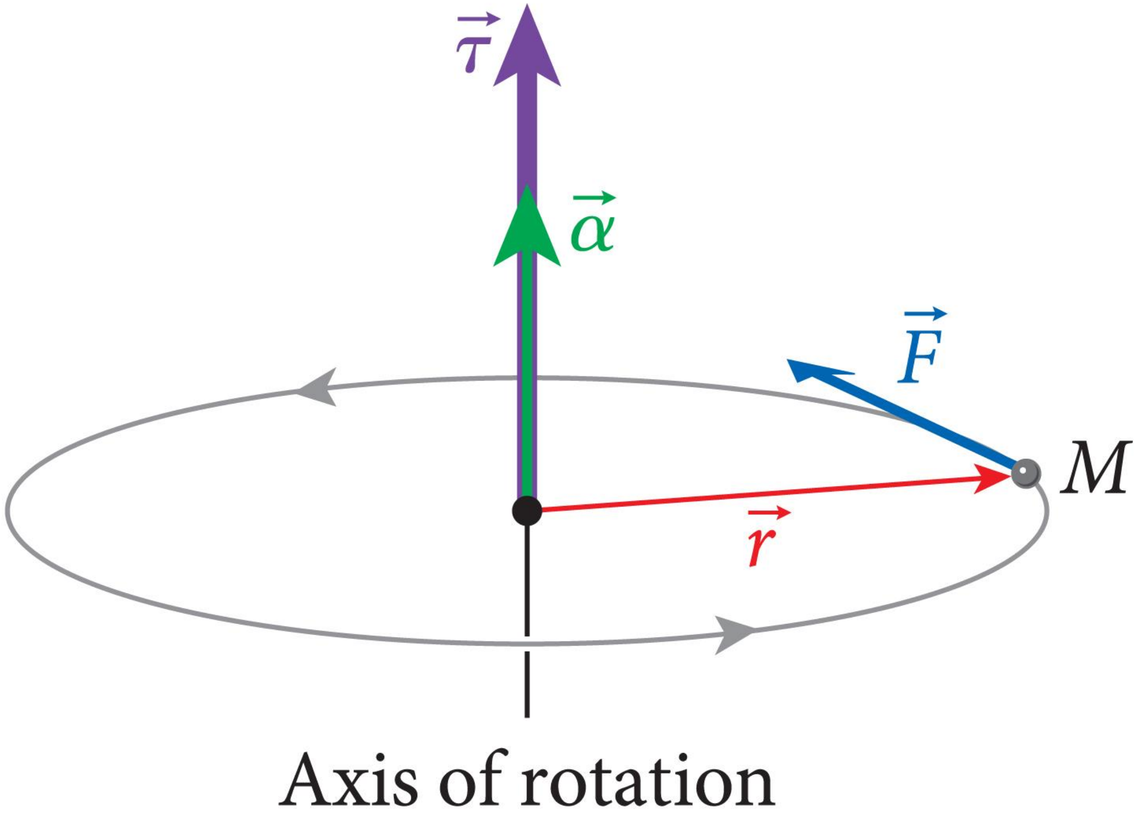
[Ans: $PLcos\theta$]



Newton's Second Law for Rotation

- Consider a point particle of mass M moving around an axis at a distance R.
- Multiply the moment of inertia by the angular acceleration:

$$I a = (R^2 M) a = RM(Ra) = RMa = RF_{\text{net}}$$



- Can see that: $t = I a$

- Combine these results to get:

$$\vec{\tau} = \vec{r} \times \vec{F}_{\text{net}} = I\vec{\alpha}$$

Newton's Second Law for Rotation

WORKED EXAMPLE

A 100 kg barrel with a radius 50 cm has two ropes wrapped around it. The barrel is released from rest, causing the ropes to unwind and the barrel to fall spinning toward the ground. Assuming that the barrel's mass is uniformly distributed and that the barrel rotates as a solid cylinder determine the speed of the barrel after it has fallen a distance of 10 m and the tension in each rope?

[Ans: $11.4\frac{m}{s}$; $163.5\ N$]

Work Done by Torque

- The work done by a torque is:

$$W = \int_{\theta_0}^{\theta} \tau(\theta')\,d\theta'$$

- For the special case of constant torque:

$$W = \tau(\theta - \theta_0)$$

- The work-kinetic energy theorem is: $\Delta K \equiv K - K_0 = W$
- The angular equivalent of work-kinetic energy theorem is:

$$\Delta K = K - K_0 = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = W$$

- For constant torque we can write: $\frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = \tau(\theta - \theta_0)$

Work Done by Torque

WORKED EXAMPLE

The turbine and associated rotating parts of a jet engine have a total moment of inertia of 25 kg m^2. The turbine is accelerated uniformly from rest to an angular speed of 150 rad/s in a time of 25 s. Find:

a) the angular acceleration

b) the net torque required

c) the angle turned through in 25 s

d) the work done by the net torque

e) the kinetic energy of the turbine at the end of the 25 s

[Ans: $6\frac{rad}{s^2}$; $150\ Nm$; $1875\ rad$; $281\ kJ$]

Angular Momentum

- Define the angular momentum of a point particle: $\boxed{\vec{L} = \vec{r} \times \vec{p}}$ $\boxed{L = rp\sin\theta}$
- Take the time derivative of the angular momentum:

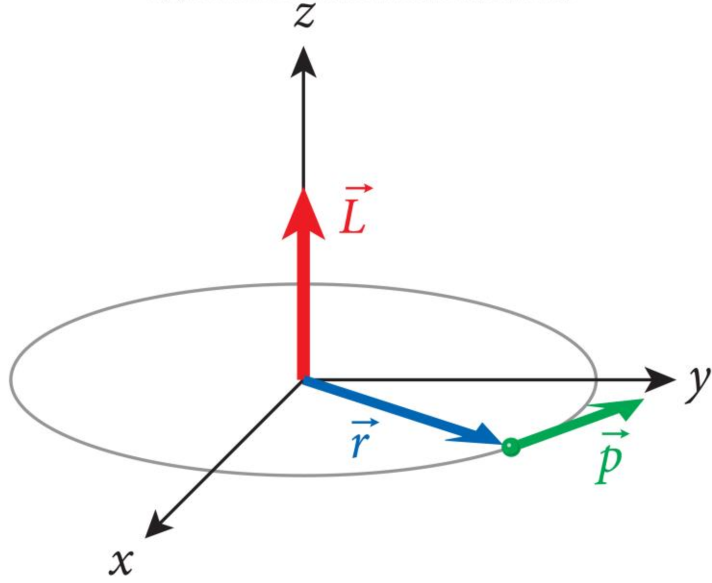
$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \left(\left(\frac{d}{dt}\vec{r}\right) \times \vec{p}\right) + \left(\vec{r} \times \frac{d}{dt}\vec{p}\right) = (\vec{v} \times \vec{p}) + (\vec{r} \times \vec{F})$$

$$\frac{d}{dt}\vec{L} = \vec{\tau}$$

(reminds you of $\frac{d\vec{p}}{dt} = \vec{F}$)

- Representing a rigid body as a collection of point particles which maintain their relative distance constant

$$\mathbf{L} = \sum_{i=1}^{n} m_i r_{i\perp}^2 \boldsymbol{\omega} = \boldsymbol{\omega} \sum_{i=1}^{n} m_i r_{i\perp}^2 = I\boldsymbol{\omega}$$



- If net torque is zero, then:

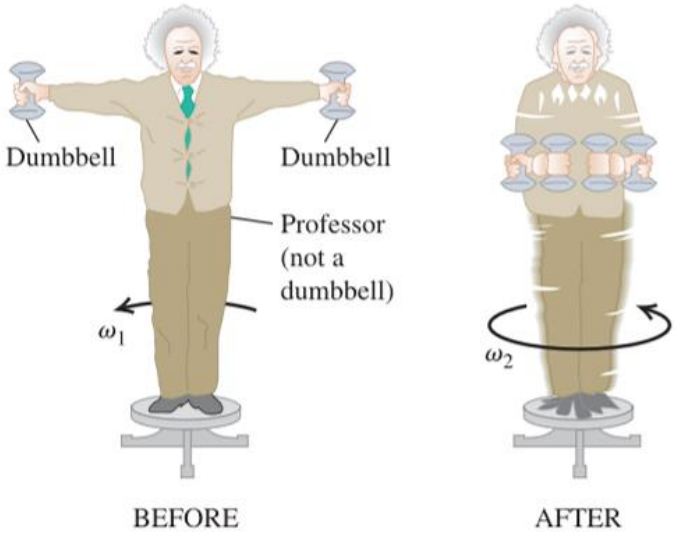
$$\boxed{\text{if } \vec{\tau}_{net} = 0 \Rightarrow \vec{L} = \text{constant} \Rightarrow \vec{L}(t) = \vec{L}(t_0) \equiv \vec{L}_0}$$

Angular Momentum

WORKED EXAMPLE

A physics professor stands at the center of a frictionless turntable with arms outstretched and a 5 kg dumbbell in each hand. He is set rotating about the vertical axis, making one revolution in 2 s. Find his final angular velocity if he pulls the dumbbells inward to his stomach. His moment of inertia (without the dumbbells) is 3 kgm^2 with arms outstretched and 2.2 kgm^2 with his hands at his stomach. The dumbbells are 1 m from the axis initially and 0.2 m at the end. Further determine the change in kinetic energy and what caused it.

[Ans: $2.5\frac{rev}{s}$; $256\ J$]



Angular Momentum

WORKED EXAMPLE

A door 1 m wide, of mass 15 kg, can rotate freely about a vertical axis through its hinges. A bullet with a mass of 10 g and a speed of 400 m/s strikes the center of the door in a direction perpendicular to the plane of the door and embeds itself there. Find the door's angular speed. Is kinetic energy conserved?

[Ans: 0.4 rad/s]