a) $\lim_{(x,y)\to(0,0)} \frac{4x^2+10y^2+4}{4x^2-10y^2+6} \xrightarrow{\text{direct Sub}} \frac{0+0+4}{0-0+6} = \frac{4}{6}$

$= \frac{2}{3}$

b) $\lim_{(x,y)\to(\pi/4,\,1)} \frac{y\tan x}{y+1} \xrightarrow{\text{direct sub}} \frac{\overbrace{\tan\left(\frac{\pi}{4}\right)}^{1}}{1+1} = \frac{1}{2}$

c) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \frac{0+0}{0}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} = \frac{x^2+y^2(\sqrt{x^2+y^2+1}+1)}{x^2+y^2+1-1}$$

$$= \frac{\cancel{x^2+y^2}(\sqrt{x^2+y^2+1}+1)}{\cancel{x^2+y^2}(1)}$$

$$= \sqrt{x^2+y^2+1}+1$$

$$= \sqrt{0+0+1}+1$$

$$= 2$$

d) $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2} \xrightarrow{\text{Polar coordinates}} \lim_{r\to 0} \frac{r^4\cos^4\theta - 4r^4\sin^4\theta}{r^2\cos^2\theta + 2r^2\sin^2\theta}$

$$= \lim_{r\to 0} r^2\cos^2\theta - 2r^2\sin^2\theta$$

$$= 0$$

e) $\lim_{(x,y)\to(0^+,0^+)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{0}{0}$

$$\lim_{(x,y)\to(0^+,0^+)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(0^+,0^+)} \frac{x^2-xy(\sqrt{x}+\sqrt{y})}{x-y}$$

$$= \lim_{(x,y)\to(0^+,0^+)} x(\sqrt{x}+\sqrt{y})$$

$$= 0$$

f) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - y^2 - z^2}{x^2 + y^2 - z^2}$

Try 2 path

Along $y=0, z=0$, $\lim_{x\to 0} \frac{x^2 - 0 - 0}{x^2 + 0 - 0} = 1$

Along $x=0, z=0$, $\lim_{y\to 0} \frac{0 - y^2 - 0}{0 + y^2 - 0} = -1$

$\therefore$ limits DNE

g) $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$

Along $x=0$, $L_1 = \lim_{y\to 0} \frac{0}{0+y^2} = 0$

Along $y=x^2$, $L_2 = \lim_{x\to 0} \frac{x^2(x^2)}{x^4 + x^4} = \frac{1}{2}$

Since $L_1 \neq L_2$, limits DNE

a) $\lim_{(x,y)\to(0,0)} \frac{2x}{x^2 + x + y^2} = \frac{0}{0}$

Try 2 path method:

Along $x=0$, $\lim_{y\to 0} \frac{0}{0+0+y^2} = 0 = L_1$

Along $y=x$, $\lim_{x\to 0} \frac{2x}{x^2+x+x^2} = \lim_{x\to 0} \frac{2x}{2x^2+x}$

$= \lim_{x\to 0} \frac{2}{2x+1}$

$= 2 = L_2$

$L_1 \neq L_2$ limits DNE

4. Determine the region where each function below is continuous. Sketch the region for (e).

b) $$f(x,y) = \begin{cases} \dfrac{1}{y(x-1)}, & (x,y) \neq (1,0) \\ 0, & (x,y) = (1,0) \end{cases}$$

Domain is $y \in \mathbb{R}, x \in \mathbb{R} \mid y \neq 0, x \neq 1$

$\therefore$ is Cont. in $\{\mathbb{R}^2 \mid x \neq 1, y \neq 0\}$

d) $$f(x,y,z) = \frac{y}{\underbrace{x^2+z^2-1}_{x^2+z^2 \neq 1}}$$

$\therefore f$ is Cont. in $\{\mathbb{R}^3 \mid x^2+z^2 \neq 1\}$

5 a) $z(x,y) = (x-1)^2 + (y-2)^2 - 3$

$$z_x = 2(x-1)$$
$$z_y = 2(y-2)$$
$$z_{xx} = 2$$
$$z_{yy} = 2$$
$$z_{xy} = 0$$

b) $f(x,y) = y/x^2$

$$f_x = \frac{-2y}{x^3} \rightarrow f_{xy} = \frac{-2}{x^3}$$
$$f_y = \frac{1}{x^2} \rightarrow f_{yx} = \frac{-2}{x^3}$$

same

$$f_{xx} = \frac{6y}{x^4}$$
$$f_{yy} = 0$$

c) $g(x,y) = x^y$

$$g_x = yx^{y-1} \qquad g_{xy} = x^{y-1} + y[x^{y-1}\ln x]$$
$$g_y = x^y \ln x \qquad g_{yx} = x^y\left(\frac{1}{x}\right) + \ln x[yx^{y-1}]$$
$$g_{xx} = (y-1)yx^{y-2} \qquad = x^{y-1} + yx^{y-1}\ln x$$
$$g_{yy} = x^y(\ln x)^2$$

d) $w(x,t) = e^{xt}\ln(xt)$

$$w_x = te^{xt}\ln(xt) + e^{xt}\left(\frac{t}{xt}\right)$$
$$= e^{xt}\left[t\ln(xt) + \frac{1}{x}\right] = te^{xt}\ln(xt) + \frac{e^{xt}}{x}$$

$$w_t = xe^{xt}\ln(xt) + e^{xt}\left(\frac{x}{xt}\right)$$
$$= e^{xt}\left[x\ln(xt) + \frac{1}{t}\right] = xe^{xt}\ln(xt) + \frac{e^{xt}}{t}$$

$$w_{xx} = t^2e^{xt}\ln(xt) + te^{xt}\left(\frac{1}{x}\right) + \frac{xte^{xt} - e^{xt}}{x^2}$$
$$= e^{xt}\left[t^2\ln(xt) + \frac{t}{x} + \frac{t}{x} - \frac{1}{x^2}\right]$$
$$= e^{xt}\left[t^2\ln(xt) + \frac{2t}{x} - \frac{1}{x^2}\right]$$

$$w_{tt} = x^2e^{xt}\ln(xt) + xe^{xt}\left(\frac{1}{t}\right) + \frac{txe^{xt} - e^{xt}}{t^2}$$
$$= e^{xt}\left[x^2\ln(xt) + \frac{x}{t} + \frac{x}{t} - \frac{1}{t^2}\right]$$
$$= e^{xt}\left[x^2\ln(xt) + \frac{2x}{t} - \frac{1}{t^2}\right]$$

$$w_{xt} = e^{xt}\left\{\left[\ln(xt) + t\left(\frac{1}{t}\right)\right] + 0\right\} + xe^{xt}\left[t\ln(xt) + \frac{1}{x}\right]$$
$$= e^{xt}\left[\ln(xt) + 1 + xt\ln(xt) + 1\right]$$
$$= e^{xt}\left\{\ln(xt)\left[1 + xt\right] + 2\right\}$$

8. Evaluate ∂z/∂x at the point (1, 1, 1) from the equation below where z is an implicit function of x & y.

$$xy + xz^3 - 2yz = 0$$

ANS: ~~-1/5.~~
-2

$$\frac{\partial}{\partial x}\{Eq^n\} \rightarrow \quad y + [z^3 + 3xz^2 \cdot z_x] - 2yz_x = 0$$

$$z_x(3xz^2 - 2y) = -z^3 - y$$

$$z_x = \frac{-z^3 - y}{(3xz^2 - 2y)}$$

$$\text{sub}(1,1,1) = \frac{-(1) - 1}{3 - 2}$$

$$= -2$$

a) $f(x,y) = x^2 + y^2,\ x(t) = t,\ y(t) = t^2$

$$f_t = f_x \cdot x_t + f_y \cdot y_t$$
$$= 2x \cdot 1 + 2y \cdot 2t$$
$$= 2x + 4yt$$
$$= 2t + 4t^3$$

b) $g(x,y,z) = \sin(xyz),\ x(t) = 1 - 3t,\ y(t) = e^{-t},\ z(t) = 2t$

$$g_t = \cos(xyz) \cdot yz \cdot (-3) + \cos(xyz) \cdot xz \cdot (-e^{-t})$$
$$+ \cos(xyz) \cdot xy \cdot (2)$$
$$= \cos[2te^{-t}(1-3t)] \cdot 2te^{-t} \cdot -3 + \cos[2te^{-t}(1-3t)] \cdot 2t(1-3t) \cdot -e^{-t}$$
$$+ \cos[2te^{-t}(1-3t)] \cdot e^{-t}(1-3t) \cdot 2$$
$$= 2e^{-t}[-3t - t + 3t^2 + 1 - 3t]\cos[2te^{-t}(1-3t)]$$
$$= 2e^{-t}[3t^2 - 7t + 1]\cos[2te^{-t}(1-3t)]$$

b) $g'(t) = 2e^{-t}(3t^2 - 7t + 1)\cos[2te^{-t}(1-3t)]$.

9 c) $z(x,y) = \tan^{-1}\frac{x}{y},\ x(r,\theta) = r\cos\theta,\ y(r,\theta) = r\sin\theta$

$$\frac{\partial z}{\partial r} = \underbrace{\frac{1}{1+x^2/y^2}\cdot\frac{1}{y}}_{\frac{y^{\cancel{2}}}{y^2+x^2}\cdot\frac{1}{\cancel{y}}}\cdot\cos\theta + \underbrace{\frac{1}{1+x^2/y^2}\cdot-\frac{x}{y^2}}_{\frac{\cancel{y^2}}{y^2+x^2}\cdot\frac{-x}{\cancel{y^2}}}\cdot\sin\theta$$

$$= \frac{y}{y^2+x^2}\cdot r\cos\theta - \frac{x}{y^2+x^2}\cdot r\sin\theta$$

$$= \frac{\cancel{r}\sin\theta}{\cancel{r^2}}\cdot\cancel{r}\cos\theta - \frac{\cancel{r}\cos\theta}{\cancel{r^2}}\cdot\cancel{r}\sin\theta$$

$$= 0$$

$$\frac{\partial z}{\partial\theta} = \frac{y}{y^2+x^2}\cdot -r\sin\theta - \frac{x}{y^2+x}\cdot r\cos\theta$$

$$= \frac{\cancel{r}\sin\theta}{\cancel{r^2}}\cdot -\cancel{r}\sin\theta - \frac{\cancel{r}\cos\theta}{\cancel{r^2}}\cdot\cancel{r}\cos\theta$$

$$= -1$$

9 d) $w(x,y,z) = xy + xz + yz,\ x(u,v) = u+v,\ y(u,v) = u-v,\ z(u,v) = uv$

$$w_u = (y+z)(1) + (x+z)(1) + (x+y)(v)$$

$$= \underbrace{(u-v+uv) + (u+v+uv)}_{2u+2uv} + \underbrace{(u+v+u-v)(v)}_{2uv}$$

$$= 2u + 4uv$$

$$= 2u(1+2v)$$

$$w_v = (y+z)(1) + (x+z)(-1) + (x+y)(u)$$

$$= \underbrace{(u-v+uv) - (u+v+uv)}_{-2v} + \underbrace{(u+v+u-v)(u)}_{2u^2}$$

$$= -2v + 2u^2$$

$$= 2(u^2-v)$$

10. Given the following information, determine $w_s(0, 0)$ and $w_t(0, 0)$.

$$w(s,t) = F(x(s,t),\ y(s),\ z(2\sin t))$$

$$x(0,0) = 2,\ y(0) = 4,\ z(0) = 1,\ x_s(0,0) = -1,\ x_t(0,0) = 3,\ y'(0) = 1,\ z'(0) = 8,$$
$$F_x(0,0,2) = 2,\ F_y(0,0,2) = -9,\ F_z(0,0,2) = 2,\ F_x(2,4,1) = 6,\ F_y(2,4,1) = 0,$$
$$F_z(2,4,1) = 5, F_x(2,4,2) = -1,\ F_y(2,4,2) = 2,\ F_z(2,4,2) = 3$$

$$W_s = \overset{6}{F_x}(x,y,z)\cdot \overset{-1}{x_s(s,t)} + \overset{0}{F_y}(x,y,z)\cdot \overset{1}{y'(s)} + \overset{0}{\cancel{F_z \cdot z_s}}$$

$x(0,0) = 2$ $\quad y(0) = 4$ $\quad z(0) = 1$

$$= 6(-1) + 0 + 0$$

$$= -6$$

$$W_t = \overset{6}{F_x}(x,y,z)\cdot \overset{3}{x_t(s,t)} + \overset{0}{F_y}\cdot 0 + \overset{5}{F_z}(x,y,z)\cdot \overset{8}{z'(2\sin t)}\cdot \overset{2}{(2\cos t)}$$

$$= (6)(3) + 0 + (5)(8)(2)$$

$$= 98$$

11. Determine the region where each function below is differentiable.

a) $f(x,y) = x^2\sqrt{y-2}$

b) $$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

ANS: a) $\mathbb{R}^2$ | y > 2. b) $\mathbb{R}^2$ | (x, y) ≠ (0,0).

a)

1 Check Continuity: $y > 2$