Assessments for EDE1012 Math 2

The assessment weightages are:

1) 7% Class Participation
2) 28% Quizzes (2 MCQ quizzes, 1 hr each) - Weeks 9 & 13
3) 30% Midterm Test - Week 11 (structured questions, 2.5 hrs)
4) 35% Final Exam - Week 14

Unless otherwise specified, all assessments are **openbook** (no internet access). So you have to **bring a laptop or tablet to do the tests**. Handphones are not allowed.

Topic 1
Integrals & The Fundamental Theorem of Calculus

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Outline

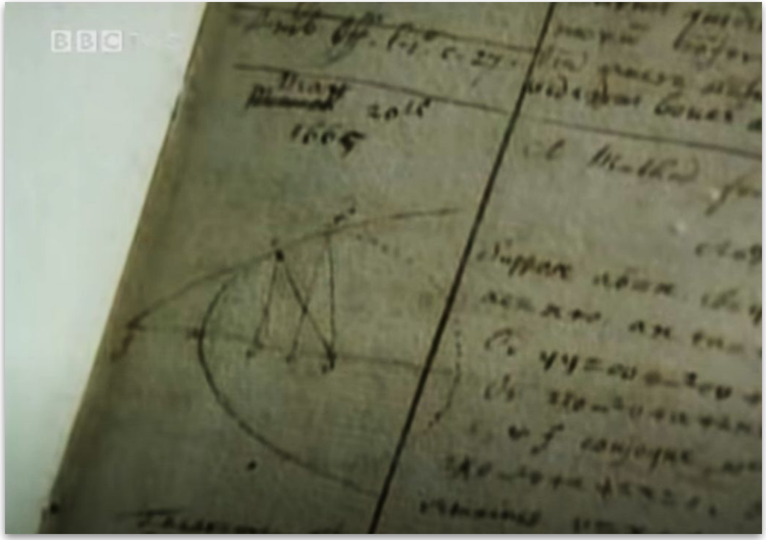
- The Area Problem
- Riemann Sum & Riemann Integral
- Fundamental Theorem of Calculus
- Physical Meaning of Integration
- Integrals of Elementary Functions
- Application of Integrals - Kinematics

Recap: Birth of Calculus: The Tangent Problem

In 20th May 1665, **Isaac Newton** wrote in his notebook (named 'Waste Book') the techniques of finding the equation of a tangent line (& normal line) at a point on a curve.

Using limits, the derivative is derived from the slope of the tangent which gives birth to differential calculus.

So, what about **integral calculus**?



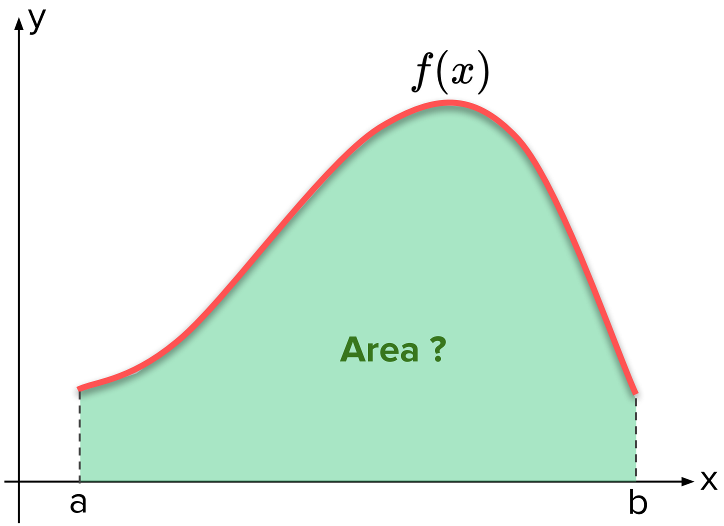
Source: https://youtu.be/ObPg3ki9GOI?t=213

The Area Problem

Besides the tangent problem, another problem that is also pivotal to the development of calculus is the **area problem**. That is, given a function f(x), what is the area bounded by f(x) above, x-axis below, x = a on the left and x = b on the right?

We can observe that this area cannot be calculated by the area formulas of a rectangle or a triangle etc because f(x) could be a curve.

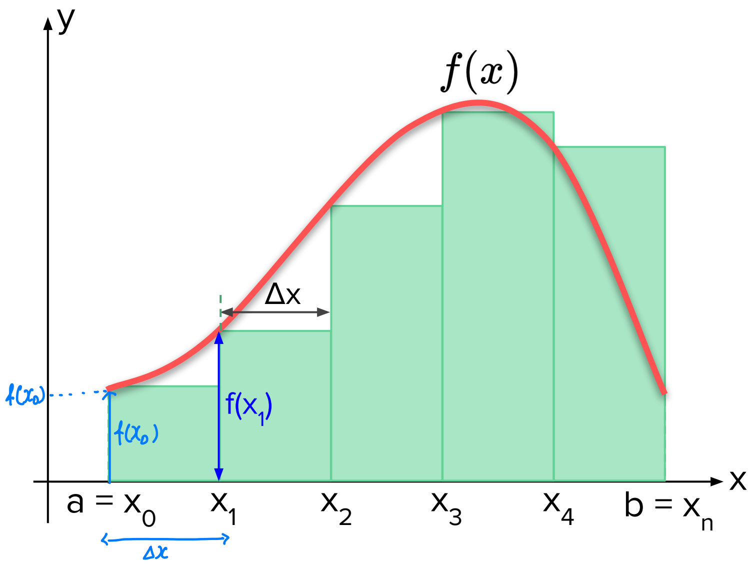
The motivation is then to develop a general method that gives the area bounded by any f(x).



Approximating the Area

For a start, it would be intuitive to approximate the area using rectangles of equal width. Dividing up total width, b - a, into n partitions, the area can be approximated by

$$Area \approx f(x_0)\Delta x + f(x_1)\Delta x + \ \ldots\ + f(x_{n-1})\Delta x$$
$$= \sum_{i=1}^{n} f(x_{i-1})\Delta x, \quad \Delta x = \frac{b-a}{n}$$

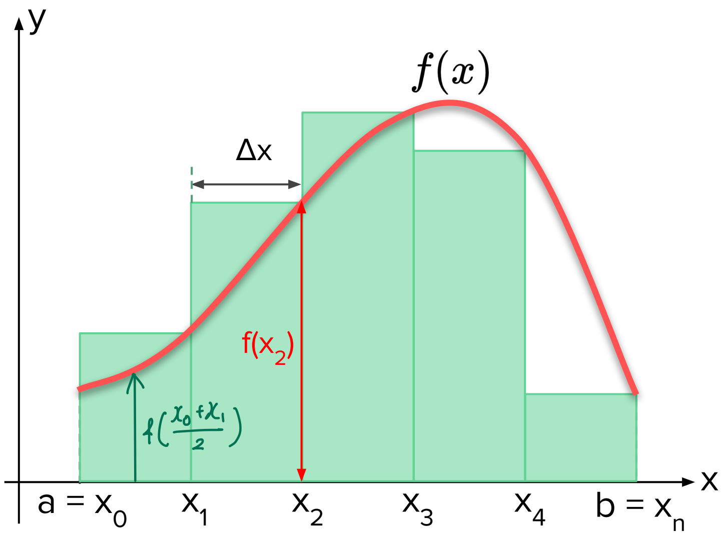


And, $f(x_{i-1})$ represents the height of each rectangle on the left endpoint. One can observe that depending on the curve of f(x), some rectangles may under- or over-approximate the area under each partition of the curve.

Approximating the Area

Besides using the left point of each rectangle to calculate its height, we can also use the right endpoint. In this case, the area will be approximated by

$$Area \approx f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$
$$= \sum_{i=1}^{n} f(x_i)\Delta x, \quad \Delta x = \frac{b-a}{n}$$



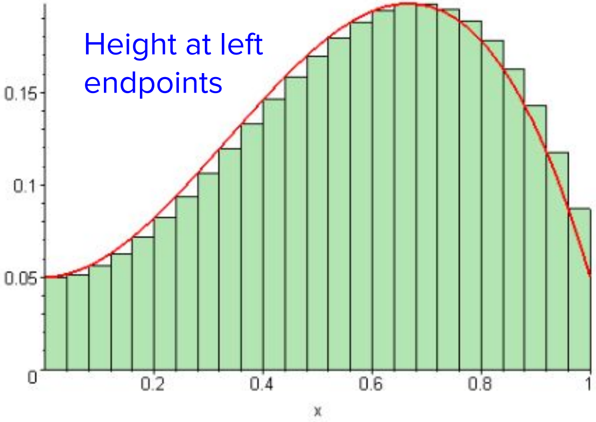
Clearly, It is also possible to use the midpoint of each rectangle to calculate its height, which then gives

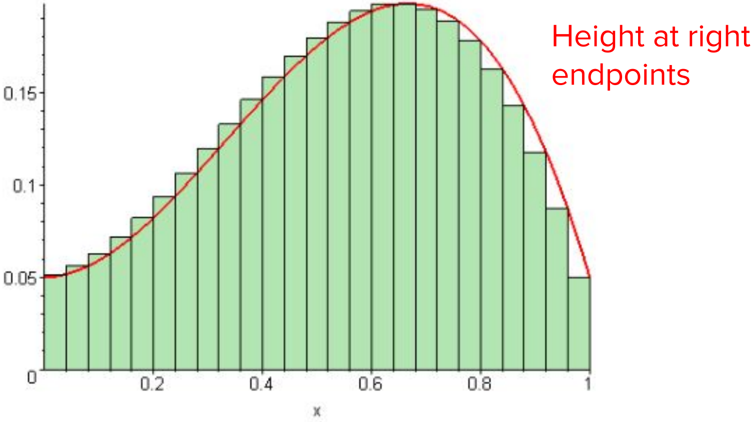
$$Area \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x, \quad \Delta x = \frac{b-a}{n}$$

The Exact Area

Now, we can observe that **as the number of partitions increases, the approximation of the area gets better**. So one can deduce the exact area to be given by

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$



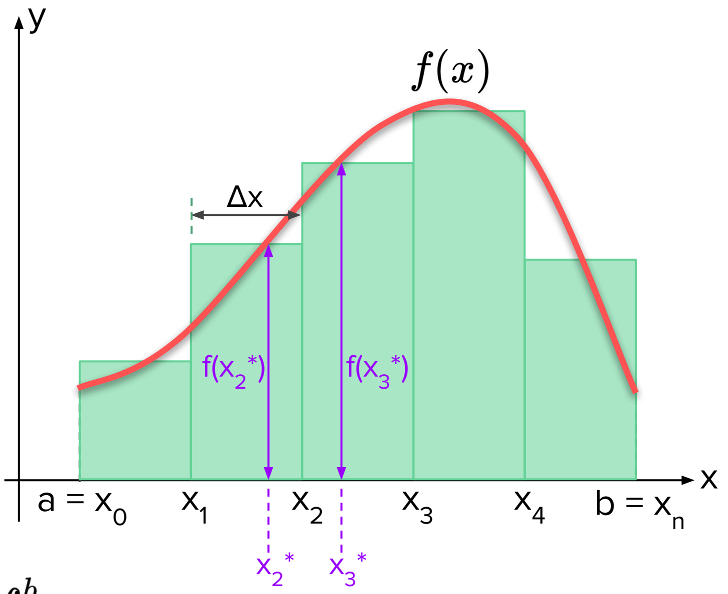


See animations at: https://en.wikipedia.org/wiki/Riemann_sum

Riemann Sum & Riemann Integral

Besides using the left or right endpoint for computing each rectangle's height, one can also approximate the area by using the height at any point x_i^* in $[x_{i-1}, x_i]$, i.e.

$$Area \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$



which is called the Riemann sum. Again, by having the **number of partitions approach infinity, the exact area** is given by

$$Area = \underbrace{\lim_{n\to\infty}}_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$

A function f(x) is called integrable in [a, b] if the limit exists.

which is called the Riemann integral. Hence, integration is about finding the exact area bounded by a function (the integrand).

Some Series Formulas

In the evaluation of the Riemann sum, the formulas for the **sum to n terms** of the following series will usually come in handy.

$$\sum_{i=1}^{n} 1 = 1 + 1 + \ldots + 1 = n$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Derivations can be found at https://brilliant.org/wiki/sum-of-n-n2-or-n3/.

Riemann Sum & Riemann Integral

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Example: Using the right endpoint, evaluate the integral below by setting up the (right) Riemann sum and evaluating the Riemann integral.

$$\int_0^1 x\,dx$$

eg) $x_2 = x_0 + 2\Delta x$
$x_3 = x_0 + 3\Delta x$

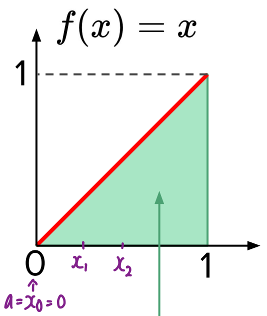
Firstly, define Δx and x_i^* in terms of i and n.

$\frac{b-a}{n}$

$$\Delta x = \frac{1}{n},\ x_i^* = x_i = \underbrace{x_0}_{0} + i\Delta x = \underbrace{a}_{0} + \underbrace{i\Delta x}_{1/n} = \frac{i}{n}$$

Then, using f(x) = x at the right endpoint, set up the (right) Riemann sum.

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} x_i \Delta x = \sum_{i=1}^{n} \frac{i}{n}\left(\frac{1}{n}\right) = \sum_{i=1}^{n} \frac{i}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n} i$$



Obviously!

Evaluate the Riemann integral by applying the limit.

$$\int_0^1 x\,dx = \lim_{n\to\infty}\left\{\frac{1}{n^2}\sum_{i=1}^{n} i\right\} = \lim_{n\to\infty}\left\{\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]\right\} = \lim_{n\to\infty}\left\{\frac{1}{2} + \frac{1}{2n}\right\} = \frac{1}{2}$$

Riemann Sum & Riemann Integral

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Exercise: Using the left endpoint this time, evaluate the integral below by setting up the (left) Riemann sum and evaluating the Riemann integral. Is the area the same?

$$\int_0^1 x\,dx$$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$\underbrace{x_{i-1} = x_0 + (i-1)\Delta x}_{} = 0 + (i-1)\left(\frac{1}{n}\right) = \frac{i-1}{n}$$

Eg) $\underbrace{x_2}_{i=3} = x_0 + (3-1)\Delta x = x_0 + 2\Delta x$

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} x_{i-1}\,\Delta x = \sum_{i=1}^{n} x_0 + (i-1)\Delta x\,\Delta x$$

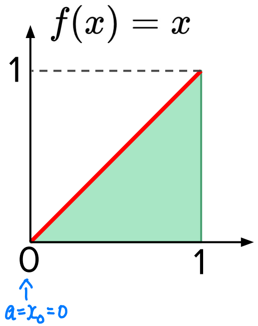
$$= \sum_{i=1}^{n}\left(\frac{i-1}{n}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(i-1) = \frac{1}{n^2}\left[\sum_{1}^{n} i - \sum_{1}^{n} 1\right]$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} - n\right] = \frac{1}{2} + \frac{1}{2n} - \frac{1}{n}$$

$$= \frac{1}{2} - \frac{1}{2n}$$

$$\therefore \int_0^1 x\,dx = \lim_{x\to\infty}\left(\frac{1}{2} - \frac{1}{2n}\right) = \frac{1}{2}$$

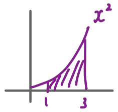


Steps: ① Define Δx & x_i → ② Sub into $\sum f(x_i)\Delta x$ → ③ take $\lim_{n\to\infty}\sum f(x_i)\Delta x$

Riemann Sum & Riemann Integral

Exercise: Using the right endpoint, evaluate the integral below by setting up the (right) Riemann sum and evaluating the Riemann integral. Verify the limit of the Riemann sum using a program or spreadsheet.

$$\int_1^3 \overbrace{x^2}^{f(x)}\,dx$$



$$\Delta x = \frac{3-1}{n} = \frac{2}{n} \qquad x_i = \underbrace{x_0}_{1} + i\Delta x$$

$$= 1 + \frac{2i}{n}$$

$$\sum_1^n \underbrace{f(x_i)}_{x_i^2}\Delta x = \sum_1^n \underbrace{\left(1+\frac{2i}{n}\right)^2}_{\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)}\left(\frac{2}{n}\right) = \sum_1^n \left[\frac{2}{n}+\frac{8i}{n^2}+\frac{8i^2}{n^3}\right]$$

$$= \frac{2}{n}\sum_1^n 1 + \frac{8}{n^2}\sum_1^n i + \frac{8}{n^3}\sum_1^n i^2$$

some polynomial of degree of n^2

$\underbrace{2n^3 + \underbrace{O(n^2)}_{\text{of order } n^2}}$

$$= \frac{2}{n}(n) + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 2 + 4 + \frac{4}{n} + \frac{8}{3} + \frac{O(n^2)}{n^3}$$

$$\lim_{n\to\infty}\left(2+4+\frac{4}{n}+\frac{8}{3}+\overbrace{\frac{O(n^2)}{n^3}}^{=0 \text{ as } n\to\infty}\right) = \int_1^3 x^2\,dx$$

$$= 8\tfrac{2}{3}$$

ANS: $\frac{2}{n}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2$, 8.666...

Step① : Indentify Δx and $a = x_0$
②: Work out b & x_i
③: Get $f(x_i) \to f(x)$ & $\int_a^b f(x)\,dx$

Riemann Sum & Riemann Integral

Identify $f(x_i)$ & Δx

Exercise: Express each of the following limits as a (right) Riemann integral. For (b), state another possible answer. (Identify Δx, x_i and $f(x_i)$ first.)

a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i}{n}}$

let $\Delta x = \frac{1}{n}$ & $x_0 = a = 0$

so $b = 1$ since $\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$\Rightarrow x_i = x_0 + i\Delta x = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$

$\therefore \sqrt{\frac{i}{n}} = f\left(\frac{i}{n}\right) = f(x_i) = \sqrt{x_i} \to f(x) = \sqrt{x}$

$\Rightarrow \int_0^1 \sqrt{x}\,dx$

b) $\displaystyle\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(2+\frac{3i}{n}\right)^4$

① let $\Delta x = \frac{3}{n}$ & $x_0 = a = 2$

$b = 5$

② $\Rightarrow x_i = 2 + i\left(\frac{3}{n}\right) = 2 + \frac{3i}{n}$

③ so $f(x_i) = \left(2+\frac{3i}{n}\right)^4 \to f(x) = x^4 = \int_2^5 x^4\,dx$

or

① let $\Delta x = \frac{3}{n}$ & $x_0 = a = 0$

$b = 3$

② $x_i = 0 + \frac{3i}{n} = \frac{3i}{n}$

③ $f(x_i) = \left(2+\frac{3i}{n}\right)^4 = (2+x_i)^4$

$f(x) = (2+x)^4$

$\Rightarrow \int_0^3 (2+x)^4\,dx$

ANS: a) $\displaystyle\int_0^1 \sqrt{x}\,dx.$ b) $\displaystyle\int_0^3 (2+x)^4\,dx, \int_2^5 x^4\,dx.$

The Net Signed Area

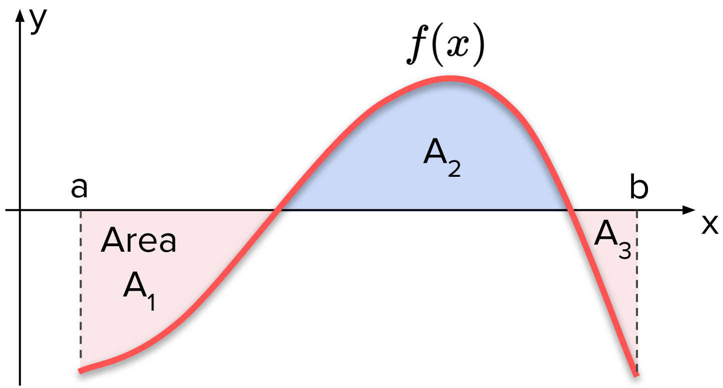
From the Riemann integral representing the area bounded by f(x), it can be deduced that in [a, b],

$$\underline{f(x) < 0 \rightarrow \int_a^b f(x)dx < 0,}\quad \underline{f(x) > 0 \rightarrow \int_a^b f(x)dx > 0,}$$

$$f(x) = 0 \rightarrow \int_a^b f(x)dx = 0$$

And if f(x) takes on both positive and values in [a, b], then the Riemann integral gives the **net signed area**, such as

$$\int_a^b f(x)dx = \underline{-A_1} + \underline{A_2} \underline{- A_3}$$



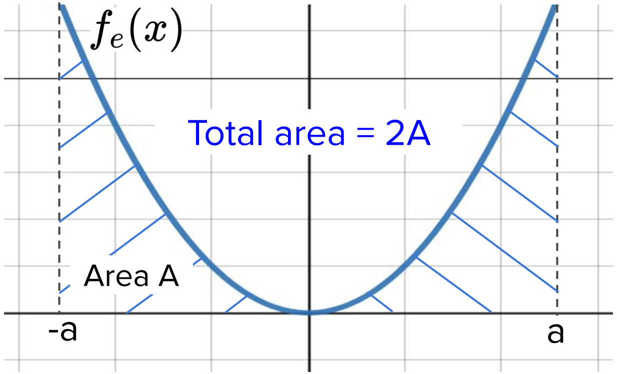
Integral of Even & Odd Functions over Symmetric Interval

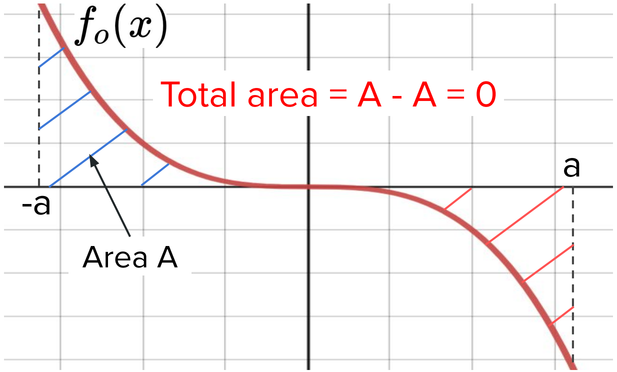
From the symmetries of even and odd functions, we can deduce that their integrals over a symmetric interval [-a, a] respectively are

$$\int_{-a}^{a} f_e(x)dx = 2\int_{0}^{a} f_e(x)dx,$$

$$\int_{-a}^{a} f_o(x)dx = 0$$

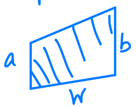
which are evident from the graphs. The proofs are left as an exercise in the tutorial.





The Net Signed Area

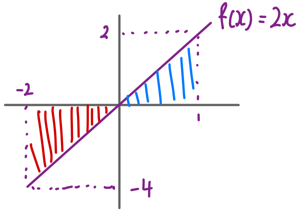
Trapezium



Area $= \frac{1}{2}(a+b)\,w$

Example: Without using the Riemann sum, determine each Riemann integral below using area formulas.

a) $\displaystyle\int_{-2}^{1} 2x\,dx = I$



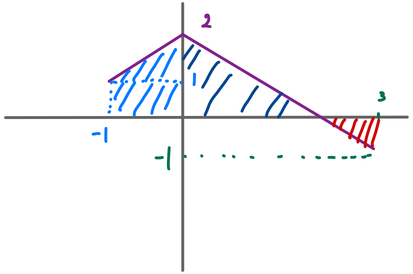
$I = \frac{1}{2}(1)(2) - \frac{1}{2}(2)(4)$

$= -3$

b) $\displaystyle\int_{-1}^{3} 2 - |x|\,dx = I$

At $x=3$, $f(3) = 2-|3|$
$= -1$

At $x=-1$, $f(x) = 2-|-1| = 2-1$



$I = \frac{1}{2}(1+2)(1) + \frac{1}{2}(2)(2) - \frac{1}{2}(1)(1)$

$= \frac{3}{2} + 2 - \frac{1}{2}$

$= 3$

c) $\displaystyle\int_{-7}^{7} \underbrace{x^5 - \sin x}_{f_o(x)}\,dx = 0$

symmetric interval

ANS: a) -3. b) 3. c) 0

The Definite Integral

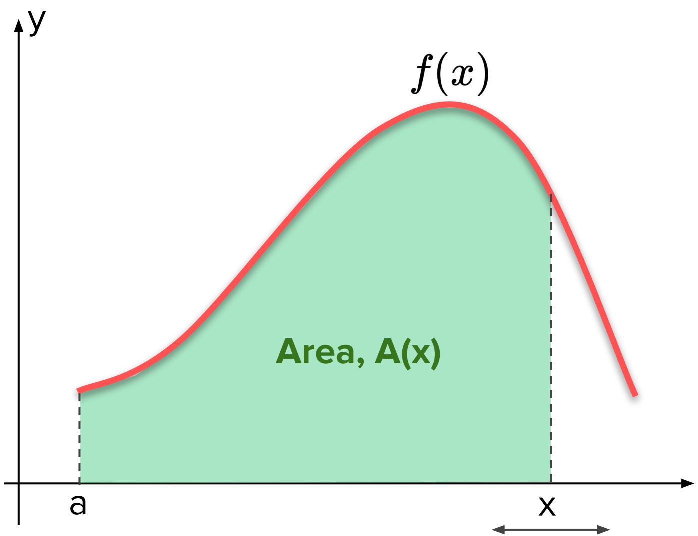
When limits are specified in an integral, it is called a definite integral, i.e.

$$\int_a^b f(x)dx$$

It is also possible that the limits are variables (or even functions), such as

$$A(x) = \int_a^x f(x)dx$$

which can be viewed as finding the area bounded by f(x) from x = a to an arbitrary x. We can observe that shifting x will change the area, hence resulting in the area being a function of x.



Shifting x will change the area.

Properties of the Definite Integral

The definite integral follows the properties below. Each property is intuitive or can be proven easily.

width = 0 → Area = 0

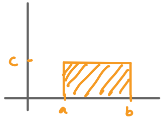
a) $$\int_a^a f(x)dx = 0$$

$$F(x)\Big|_b^a = F(a) - F(b) = -\left[F(b) - F(a)\right]$$

b) $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Constant

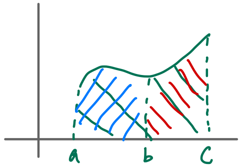
c) $$\int_a^b c\,dx = c(b-a) \longrightarrow$$



d) $$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

e) $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

f) $$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$



g) $$\underbrace{\int_a^b f(t)dt}_{F(t)\big|_a^b} = \underbrace{\int_a^b f(x)dx}_{F(x)\big|_a^b}$$

$F(b) - F(a)$ $\quad$ $F(a) - F(b)$

so the integration (dummy) variable does not matter in definite integrals

The Area Function

Example: Using the right endpoint, evaluate the area function by setting up the (right) Riemann sum and evaluating the Riemann integral.

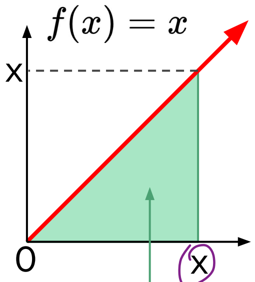
$$A(x) = \int_0^x x\,dx$$

Firstly, define Δx and x_i in terms of i and n.

$$\Delta x = \frac{x}{n},\ x_i^* = x_i = x_0 + i\Delta x = a + i\Delta x = \frac{ix}{n}$$

Then, using the f(x) = x at the right endpoint, set up the (right) Riemann sum.

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} x_i \Delta x = \sum_{i=1}^{n} \frac{ix}{n}\left(\frac{x}{n}\right) = \sum_{i=1}^{n} \frac{ix^2}{n^2} = \frac{x^2}{n^2}\sum_{i=1}^{n} i$$



Obviously!

Evaluate the Riemann integral by applying the limit.

$$A(x) = \lim_{n\to\infty}\left\{\frac{x^2}{n^2}\sum_{i=1}^{n} i\right\} = x^2 \lim_{n\to\infty}\left\{\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]\right\} = x^2 \lim_{n\to\infty}\left\{\frac{1}{2} + \frac{1}{2n}\right\} = \frac{x^2}{2}$$

The Area Function

Example: Using the right endpoint, evaluate the area function by setting up the (right) Riemann sum and evaluating the Riemann integral.

$$A(x) = \int_0^x x^2 dx$$

$x_0 = a = 0$

$\int_0^x t^2 dt$

$\Delta x = \frac{x-0}{n} = \frac{x}{n}$ $\quad x_i = x_0 + i\Delta x = \frac{ix}{n}$

$$\sum_1^n f(x_i)\Delta x = \sum_1^n \left(\frac{ix}{n}\right)^2 \left(\frac{x}{n}\right) = \frac{x^3}{n^3}\sum_1^n i^2 = \frac{x^3}{n^3}\left[\frac{n\overbrace{(n+1)(2n+1)}^{2n^2+3n+1}}{6}\right]$$

$$= \frac{x^3}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]$$

$$= \frac{x^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\Rightarrow A(x) = \lim_{n\to\infty} \frac{x^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= \frac{x^3}{3}$$

ANS: $x^3/3$

Area Function VS Gradient Function

Notice that obtaining the area function of f(x) is rather tedious, unlike getting the gradient function by simply differentiating f(x).

But in the last two examples, we notice a possible relationship.

$$A(x) = \int_0^x f(x)dx = \int_0^x x dx = \frac{x^2}{2}$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2}\right) = x = f(x)$$

$$A(x) = \int_0^x f(x)dx = \int_0^x x^2 dx = \frac{x^3}{3}$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3}{3}\right) = x^2 = f(x)$$

It seems that the finding area (integration) and finding gradients (differentiation) are inverse operations of each other. **Could this be true for all functions?**

Area Function VS Gradient Function

In order to answer the question, consider the graph shown below. We can state the relationship between the coloured areas as

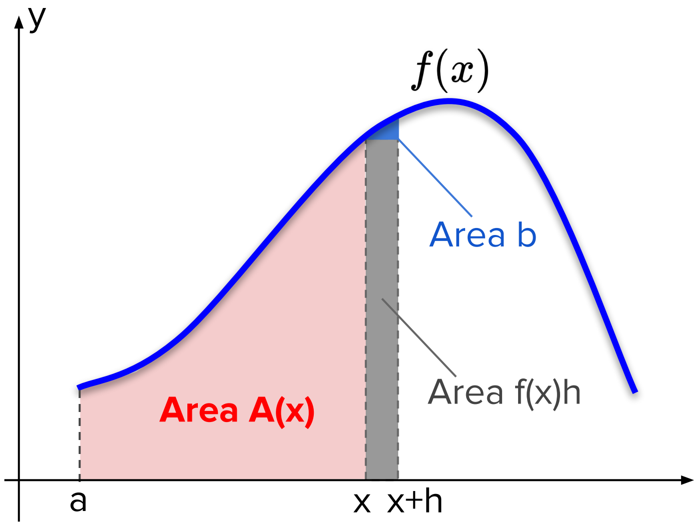
$$A(x+h) - A(x) = \underline{f(x)h} + \underline{b}$$

Rearranging, we have

$$f(x) = \frac{A(x+h) - A(x)}{h} - \frac{b}{h}$$

Letting h → 0 gives

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} - \lim_{h \to 0} \frac{b}{h}$$



Area Function VS Gradient Function

The second limit on the RHS can be evaluated to be

$$\lim_{h\to 0}\frac{b}{h} \le \lim_{h\to 0}\frac{\overbrace{h[f(x+h_1)-f(x+h_2)]}^{\text{Area of black rectangle}}}{h}$$

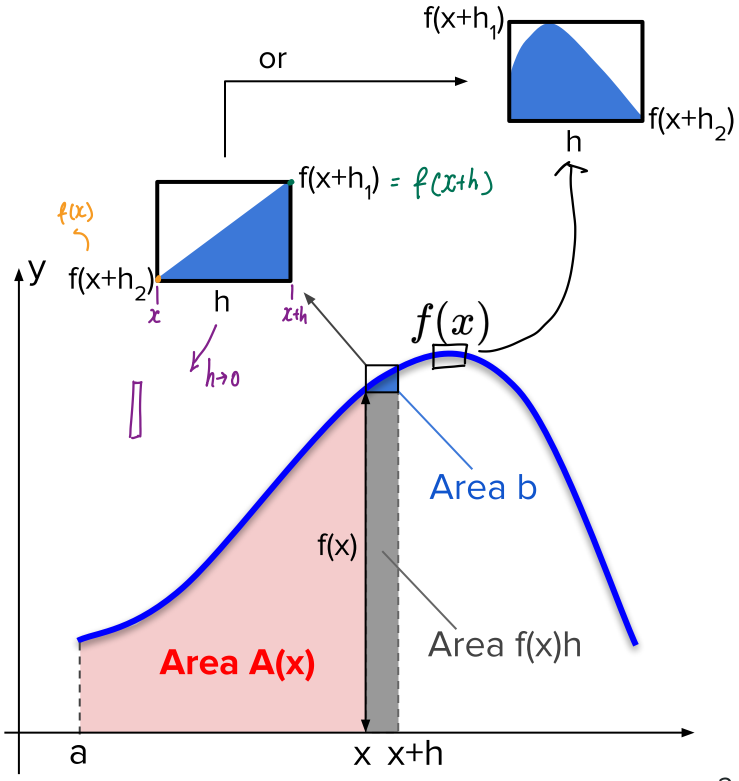
$$= \lim_{h\to 0}\{f(x+h_1)-f(x+h_2)\} = 0$$

$f(x)$ $f(x)$ $h_1 \to 0$, $h_2 \to 0$

where f(x+h_1) and f(x+h_2) are max and min in [x, x+h] respectively. Hence we have

$$f(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = \frac{\mathrm{d}A}{\mathrm{d}x}$$

Indeed, if A(x) is the area function of f(x), then f(x) is the gradient function of A(x)!



The Fundamental Theorem of Calculus (FTC)

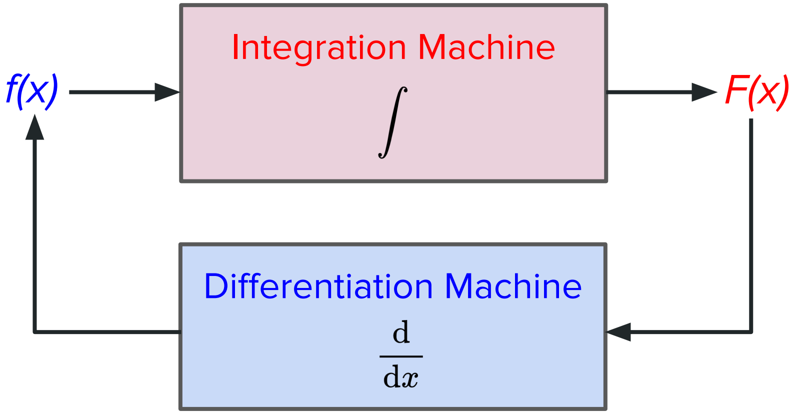
The discovery of the inverse relationship between differentiation and integration is one of the most pivotal in calculus, so much so it is called the **Fundamental Theorem of Calculus (FTC)**. There are two parts to this theorem. **Part I** states that for a continuous function f(x) in [a, b], and

$$F(x) = \int_a^x f(t)dt$$

then for all x in (a, b) we have

$$F'(x) = f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)dt$$

And F(x) is called an antiderivative of f(x).



It is interesting to know that the discovery of FTC happened hundreds of years after differentiation and integration are invented, prior to which they are thought of be unrelated operations.

The Fundamental Theorem of Calculus Part 1 (FTC1)

OR : $-\int_9^t 7e^{-2x} - \frac{1}{x}\,dx$

Example: By FTC1, determine the derivative of each expression below.

a) $y(x) = \int_0^x (t-3)^5 dt$

b) $G(r) = \int_{-2}^{r} \underbrace{\sqrt{1+3t^2}}_{g(t)}\, dt$

c) $f(t) = \underbrace{\int_t^9 \underbrace{7e^{-2x} - \frac{1}{x}}_{h(x)}\, dx}_{H(9)-H(t)}$

$\underbrace{y'(x)}_{f(x)} = \frac{d}{dx} \underbrace{\int_0^x (t-3)^5 dt}_{y(x) = F(x)}$

$= (x-3)^5$

$G'(r) = g(r) = \sqrt{1+3r^2}$

$f'(t) = -H'(t) = -h(t)$

$= -\left[7e^{-2t} - \frac{1}{t}\right]$

$= \frac{1}{t} - 7e^{-2t}$

ANS: a) $y'(x) = (x-3)^5$. b) $G'(r) = \sqrt{1+3r^2}$. c) $f'(t) = \frac{1}{t} - 7e^{-2t}$.

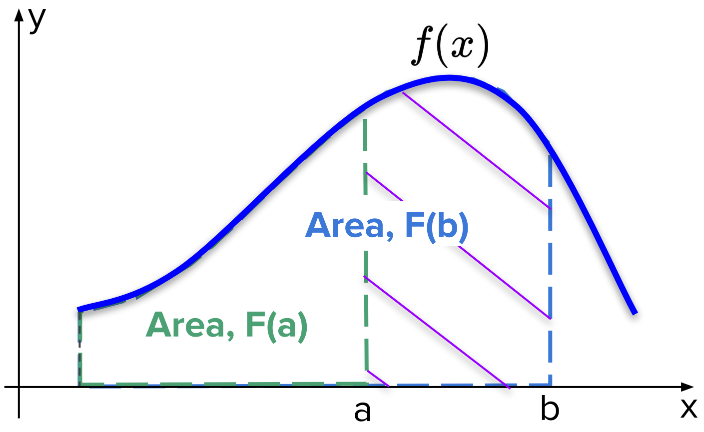
The Fundamental Theorem of Calculus Part 2 (FTC2)

Part 2 of the **Fundamental Theorem of Calculus** states that for a continuous function f(x) in [a, b], and if F(x) is an antiderivative of f(x) in (a, b), i.e.

$$F'(x) = f(x)$$

then we have

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$



Net Area = **Area F(b)** - **Area F(a)**

This means that if the antiderivative F(x) of a function f(x) is known, then a definite integral of f(x) can be evaluated by substituting the limits of integration into F(x) and subtracting as shown.

Proof of FTC2

Dividing the interval [a, b] into n partitions of equal width $\Delta x = x_i - x_{i-1}$, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \end{aligned}$$

By the mean value theorem,

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x, \quad c_i \in [x_{i-1}, x_i]$$

Combining the above relations gives

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x$$

Finally, taking limits gives

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx \qquad \textbf{(FTC2)}$$

The Fundamental Theorem of Calculus

Example: By FTC1 & FTC2, determine the derivative of each expression below.

a) $y(x) = \underbrace{\int_0^{2x} \underbrace{(t-3)^5}_{f(t)}\, dt}$

$F(t)\Big|_0^{2x} = F(2x) - F(0)$

$y'(x) = \frac{d}{dx}\left[F(2x) - \overbrace{F(0)}^{\text{constant}}\right]$

$= 2F'(2x) - 0$

$= 2f(2x)$

$= 2(2x-3)^5$

b) $G(r) = \underbrace{\int_{r^2}^{-2} \underbrace{\sqrt{1+3t^2}}_{g(t)}\, dt}$

$G(t)\Big|_{r^2}^{-2} = G(-2) - G(r^2)$

$G'(r) = \frac{d}{dr}\left[G(-2) - G(r^2)\right]$

$= 0 - g(r^2)\cdot(2r)$

$= -2r\sqrt{1+3(r^2)^2}$

$= -2r\sqrt{1+3r^4}$

c) $f(t) = \int_{1/t}^{\sin t} \underbrace{7e^{-2x} - \frac{1}{x}}_{h(x)}\, dx$

$= H(x)\Big|_{1/t}^{\sin t} = H(\sin t) - H(1/t)$

$\frac{d}{dt} \qquad \frac{d}{dt}$

$f'(t) = h(\sin t)\cdot\cos t - h(1/t)\cdot\left(-\frac{1}{t^2}\right)$

$\Rightarrow \left(7e^{-2\sin t} - \frac{1}{\sin t}\right)\cos t - \left(7e^{-2/t} - t\right)\left(-\frac{1}{t^2}\right)$

$= 7e^{-2\sin t}\cdot\cos t - \cot t + \frac{7e^{-2/t}}{t^2} - \frac{1}{t}$

ANS: a) $y'(x) = 2(2x-3)^5$. b) $G'(r) = -2r\sqrt{1+3r^4}$. c) $f'(t) = 7e^{-2\sin t}\cos t - \cot t + \frac{7e^{-2/t}}{t^2} - \frac{1}{t}$.

Meaning of an Integral in Applications

While the integral represents the area bounded by the integrand function graphically, it can take on a more physical meaning in various applications. For example, if the velocity of an object is v(t), then its net displacement over a time interval [t_1, t_2] is

$$s = \int_{t_1}^{t_2} \underbrace{v(t)\,dt}_{ds}$$

infinitesimal displacement ds

We can understand the above integral better by looking at its corresponding Riemann sum, i.e.

$$s \approx \sum_{i=1}^{n} v(t_i)\Delta t = \sum_{i=1}^{n} \Delta s_i$$

which means that the total net displacement from t_1 to t_2 can be approximated by adding the incremental displacements Δs_i moved by the object at velocity $v(t_i)$ over a time-step during step i. The velocity $v(t_i)$ is treated as constant over the duration of Δt.

Meaning of an Integral in Applications

Then, by taking the limit of n ➔ ∞, which means Δt ➔ 0, we get

$$\lim_{\Delta t \to 0} \Delta s_i = \lim_{\Delta t \to 0} v(t_i)\Delta t = v(t_i) \lim_{\Delta t \to 0} \Delta t$$

$$ds = v(t)\,dt \longrightarrow v(t) = \frac{ds}{dt}$$

This means that the **infinitesimal displacement ds is caused by the object moving at velocity v(t) over an infinitesimal duration dt**. Logically, if we ‘sum up’ (integrate) all the infinitesimal displacement ds from t_1 to t_2, we must get the total net displacement, i.e.

sum (infinite)

$$s = \int_{t_1}^{t_2} \underline{ds} = \int_{t_1}^{t_2} \underline{v(t)\,dt}$$

Hence, when identifying the physical meaning of an integral, firstly evaluate what the **integrand multiplied by a change in the independent variable** represents.

Essence of Integration

Evaluating an integral, called integration, is finding the sum of infinitesimal changes of the (integral) function.

$$F(x) = \int f(x)\, dx = \int dF$$

Leibniz's notation
(He probably wrote S (sum) in a real hurry and got this. Try it.)

Meaning of an Integral in Applications

Exercise: Explain the physical meaning of each integral below.

a) $I = \int_{t_1}^{t_2} \underbrace{a(t)\, dt}_{dv}, \quad a(t) = \text{acceleration function of time}$

$dv \rightarrow$ small change in velocity

so I is the total change of velocity from t_1 to t_2 $\rightarrow I = V(t_2) - V(t_1)$

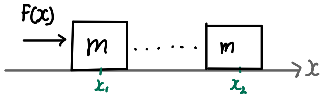
b) $I = \int_{t_1}^{t_2} \underbrace{r(t)\, dt}_{dg}, \quad r(t) = \underbrace{\text{growth}}_{g(t)} \text{ rate of a tree}$

$dg \rightarrow$ small growth over dt

so I is the total tree's growth from t_1 to t_2

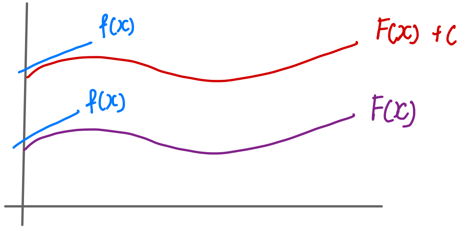
c) $I = \int_{x_1}^{x_2} \underbrace{F(x)\, dx}_{dW}, \quad F(x) = \text{force function of position acting on an object}$

$dW \rightarrow$ small work done over distance dx



so I is the total work done from moving the object from x_1 to x_2

Indefinite Integral & Antiderivative



With the FTC, we can define the indefinite integral (without limits) as

$$\int f(x)\,dx = F(x) + c$$

where **c is called the constant of integration**. F(x) + c is also an antiderivative of f(x), since

$$\frac{\mathrm{d}}{\mathrm{d}x}[F(x)+c] = f(x),$$

$$\int_a^x f(t)\,dt = F(x) + c - F(a) - c = F(x) - F(a)$$

which satisfies both FTC1 & FTC2. Keep in mind that when **evaluating an indefinite integral, the constant c must be added**.

Integrals of Elementary Functions

Since now we know that integration and differentiation are inverse operations, the integrals of elementary functions can be deduced from their derivatives by ‘reverse engineering’. Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}x}c = 0 \to \int 0\,dx = c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}kx = k \to \int k\,dx = kx + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1} \to \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \to \int e^x\,dx = e^x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a \to \int a^x\,dx = \frac{a^x}{\ln a} + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x} \to \int \frac{1}{x}\,dx = \ln x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \to \int \cos x\,dx = \sin x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \to \int \sin x\,dx = -\cos x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \to \int \sec^2 x\,dx = \tan x + c$$

Integrals of other elementary functions not shown here will be derived in the next topic.

Some Other Integrals

And we also have

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \to \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \to \int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{1+x^2} \to \int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$$

Integrals of Elementary Functions

Exercise: Evaluate the integrals below.

a) $\int 4x\left(x^2+1\right)dx$

$= \int 4x^3 + 4x \, dx$

$= \frac{4x^4}{4} + \frac{4x^2}{2} + C$

$= x^4 + 2x^2 + C$

b) $\int \frac{2-x}{x^2} + 5\sin x \, dx$

$= \int \frac{2}{x^2} - \frac{1}{x} + 5\sin x \, dx$

$\frac{d}{dx} \rightarrow -\sin x$ Check

$= -\frac{2}{x} - \ln|x| - 5\cos x + C$

c) $\int 3^x - \frac{2}{\sqrt{x}} \, dx$

$= \frac{3^x}{\ln 3} - \frac{2}{1/2}\sqrt{x} + C$

$= \frac{3^x}{\ln 3} - 4\sqrt{x} + C$

ANS: a) x^4+2x^2+c. b) $-\frac{2}{x}-\ln|x|-5\cos x+c$. c) $\frac{3^x}{\ln 3}-4\sqrt{x}+c$.

Application of Integrals - Kinematics

Integrals are commonly applied to analyze the motion characteristics of an object, called kinematics. Given that an object has displacement function s(t), we know that its velocity and acceleration functions respectively are

$$v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}, \quad a(t) = \frac{\mathrm{d}v}{\mathrm{d}t}$$

We can manipulate the above as

$$ds = v(t)\,dt, \quad dv = a(t)\,dt$$

which represents the infinitesimal changes in displacement and velocity respectively. Hence, the displacement and velocity functions of time are

$$s(t) = \int ds = \int v(t)\,dt, \quad v(t) = \int dv = \int a(t)\,dt$$

Application of Integrals - Kinematics

Therefore, if the velocity and acceleration functions of time are known, one can integrate to obtain the displacement and velocity functions respectively.

Example: Considering a constant acceleration a, evaluate the velocity function v(t). The velocity at t = 0 is v_0, or $v(0) = v_0$. This is called an initial condition. What is the average velocity from t = 0 to any time t?

IC

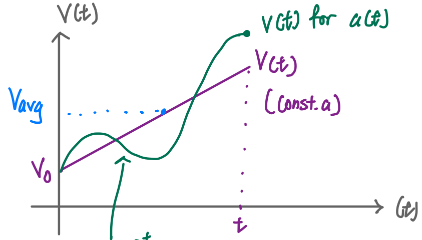
$$V(t) = \int a\,dt = at + C$$

Apply IC: $V(0) = a(0) + C = V_0$

$\Rightarrow C = V_0$

$$V(t) = at + V_0$$

$$V_{avg} = \frac{V(t) + V_0}{2}$$



$$V_{avg} = \frac{\int_0^t V(t)\,dt}{t}$$

ANS: $v(t) = v_0 + at, \quad v_{avg} = \dfrac{v_0 + v(t)}{2}$

Application of Integrals - Kinematics

Exercise: Considering a constant acceleration a, evaluate the displacement function s(t). The initial displacement is s(0) = s_0. Then, combine v(t) and s(t) to obtain a relationship between them.

$s(t) = \int v(t)\,dt$

$= \int at + V_0\,dt$

$= \frac{at^2}{2} + V_0 t + C$

Apply IC :

$S(0) = \frac{a}{2}(0) + V_0(0) + C = S_0$

$C = S_0$

$S(t) = \frac{at^2}{2} + V_0 t + S_0$

$V = V_0 + at \rightarrow t = \frac{V - V_0}{a}$

$S = \frac{\cancel{a}}{2}\left(\frac{(V-V_0)^2}{a^2}\right) + V_0 \frac{V-V_0}{a} + S_0$

$\times 2a$

$2aS = V^2 - \cancel{2VV_0} + V_0^2 + \cancel{2VV_0} - 2V_0^2 + 2aS_0$

$2a(S - S_0) = V^2 - V_0^2$

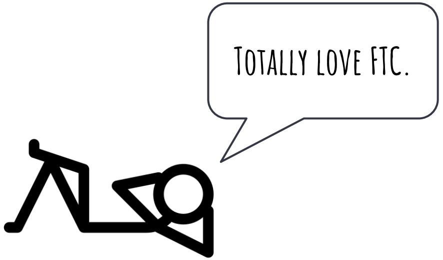
$\Rightarrow V^2 = V_0^2 + 2a(S - S_0)$

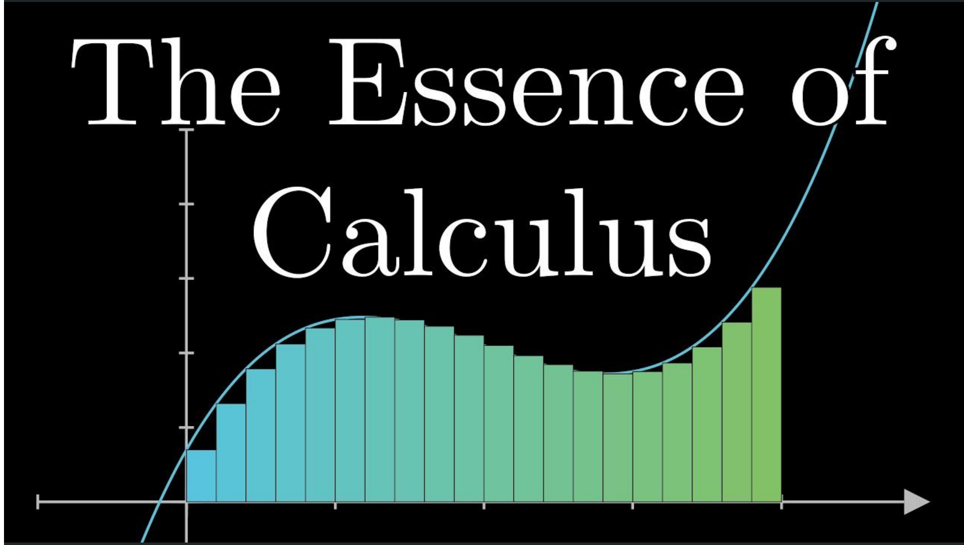
The 4 equations derived are called the basic kinematic equations of motion for constant acceleration.

ANS: $s(t) = s_0 + v_0 t + \frac{1}{2}at^2,\ v^2 = v_0^2 + 2a(s - s_0)$

End of Topic 1

The fundamental theorem of calculus practically means doing nothing.





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