EDE1011 ENGINEERING MATHEMATICS 1

Tutorial 3

Vectors & Geometry

1. The complex conjugate of a complex number z = x + iy is

$$\bar{z} = x - iy$$

   Sketch the vector of z and its conjugate on the complex plane and hence derive the polar and exponential form of $\bar{z}$.

   ANS: $\bar{z} = r(\cos\varphi - i\sin\varphi) = re^{-i\varphi}$.

2. Given the vectors **u** = (u_1, u_2, u_3) and **v** = (v_1, v_2, v_3), by considering a triangle formed by vectors **u**, **v** and **u - v**, prove the following dot-product equality by using the cosine rule.

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{m} u_i v_i = |\mathbf{u}||\mathbf{v}|\cos\theta$$

3. Using the dot product, determine the projected vector (aka vector projection) of **u** onto **v**, called proj_v **u**, for the following vectors. For (b) and (c), explain your answer.

   a. **u** = $[1\ \ 2\ \ 3]^T$, **v** = $[-2\ \ 1\ \ -1]^T$
   b. **u** = $[-3\ \ 1\ \ -2]^T$, **v** = $[3\ \ -3\ \ -6]^T$
   c. **u** = $[2\ \ 4\ \ 3]^T$, **v** = $[2\ \ 4\ \ 0]^T$

   ANS: **a)** proj_v **u** = $[1\ \ -½\ \ ½]^T$. **b)** proj_v **u** = **0**. **c)** proj_v **u** = $[2\ \ 4\ \ 0]^T$.

4. Given **u** = $[1\ \ 2\ \ 3]^T$, **v** = $[-2\ \ 1\ \ -1]^T$ and **w** = $[5\ \ 5\ \ -5]^T$, determine the following:

   a. Angle between vectors **u** and **v**,
   b. Cross product **u** × **v**
   c. Angle between vectors **u** & **w**
   d. Angle between vectors **v** & **w**

   How is **w** related to **u** and **v**?

   ANS: **a)** 1.9 rads. **b)** $[-5\ \ -5\ \ 5]^T$. **c)** π/2. **d)** π/2.

W is the normal vector of the plane spanned by U and V

5. Compute the area of the parallelogram formed by the following pairs of vectors. For (b), show that the parallelogram is also a rectangle using the cross product.

   a. **u** = [-1 -1 1]^T, **v** = [2 1 5]^T
   b. **u** = [3 1 -4]^T, **v** = [-2 2 -1]^T

ANS: **a)** Area = √86. **b)** Area = √234.

6. Determine the vector equation of a line in $\mathbb{R}^3$ given by the Cartesian equations below. (Note that there is no unique answer.)

$$\frac{x-1}{3} = y+2 = \frac{5-z}{4}$$

ANS: $\mathbf{r}(t) = \begin{bmatrix} 7 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$.

7. Evaluate the vector equation of the line passing through the point (9, 3, 1) and (7, 13, 9) in $\mathbb{R}^3$. (Note that there is no unique answer.)

ANS: $\mathbf{r}(t) = \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$.

8. Determine the Cartesian and vector equations of a line formed by the intersection of two planes defined below. (Note that there is no unique answer.)

$$P_1 : 2x + z = 4, \qquad P_2 : x - y + z = 3$$

ANS: $z = 2 + 2y = 4 - 2x$. $\mathbf{r}(t) = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

9. Determine the Cartesian equation of a plane that contains the point (5, 1, 3) and has a normal vector **n** = [1 −4 2]^T.

ANS: x - 4y + 2z = 7.

10. Find the Cartesian equation of the plane given by

$$\mathbf{r_1}(s,t) = \begin{bmatrix}1\\1\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\2\end{bmatrix} + t\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

Show that the equation

$$\mathbf{r_2}(s,t) = \begin{bmatrix}1\\2\\3\end{bmatrix} + s\begin{bmatrix}2\\-2\\-7\end{bmatrix} + t\begin{bmatrix}6\\2\\-5\end{bmatrix}$$

represents the same plane.

ANS: 3x - 4y + 2z = 1.

11. Show that the line given by

$$\mathbf{X}(t) = \begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

intersects the line given by

$$x = 5,\ \ y - 4 = \frac{z-1}{2}$$

Determine the point of intersection.

ANS: (5, 3 -1).

12. Show that the line given by:

$$\mathbf{X}(t) = \begin{bmatrix}2\\3\\1\end{bmatrix} + t\begin{bmatrix}-1\\4\\2\end{bmatrix}$$

does not intersect the plane 2x + z = 9. Then, determine the equation of a line through the point (2,3,1) which is parallel to the normal vector of the plane and determine the point where it intersects the plane.

ANS: $\mathbf{r}(t) = \begin{bmatrix}2\\3\\1\end{bmatrix} + t\begin{bmatrix}2\\0\\1\end{bmatrix}$. Intersect at (18/5, 3, 9/5).

13. A linear combination of vectors, **b**, is defined by

$$\mathbf{b} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3}$$

where c_i are scalars.

a. Draw a graphical representation of this linear combination.
b. Given that the vectors $\mathbf{u_i}$ and **b** are prescribed, show that finding the unknown scalars is equivalent to solving an SLE in the form below. Define matrix A and vector **v**.

$$A\mathbf{v} = \mathbf{b}$$

c. For the SLE in (b), what is the condition necessary of vectors $\mathbf{u_i}$ if there is to be a solution given any constant vector **b**? Explain.

ANS: **b)** $A = [\mathbf{u_1} \quad \mathbf{u_2} \quad \mathbf{u_3}]$, $\mathbf{v} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. **c)** Vectors $\mathbf{u_i}$ must be linearly independent.

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-3/pages/2-1-vectors-in-the-plane
2) https://openstax.org/books/calculus-volume-3/pages/2-2-vectors-in-three-dimensions
3) https://openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product
4) https://openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
5) https://tutorial.math.lamar.edu/Problems/CalcIII/EqnsOfLines.aspx
6) https://tutorial.math.lamar.edu/Problems/CalcIII/EqnsOfPlanes.aspx

End of Tutorial 3

(Email to youliangzheng@gmail.com for assistance.)