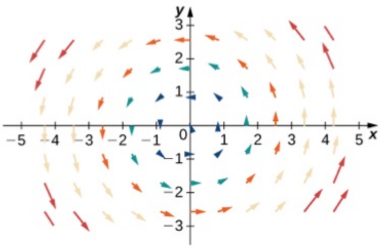
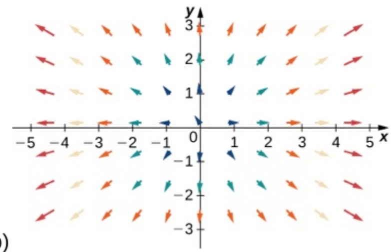
1. (https://openstax.org/books/calculus-volume-3/pages/6-1-vector-fields)
Without using a graphing tool, match the vector fields below with their graphs. Evaluate the Jocobian and divergence of each vector field.

b) a) c)

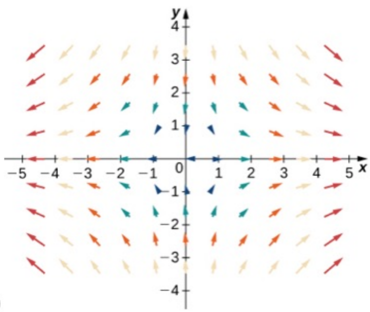
$$\mathbf{F}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{G}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}, \qquad \mathbf{H}(x,y) = \begin{bmatrix} x \\ -y \end{bmatrix}$$



a)



b)



c)

$$J_H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\nabla \cdot H = \begin{pmatrix} \partial x \\ \partial y \end{pmatrix} \cdot \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$= 1 - 1$$

$$= 0$$

ANS: **a)** $\mathbf{G}(x,y)$. $\mathbf{J_G} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $\nabla \cdot \mathbf{G} = 0$. **b)** $\mathbf{F}(x,y)$. $\mathbf{J_F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\nabla \cdot \mathbf{F} = 2$.

c) $\mathbf{H}(x,y)$. $\mathbf{J_H} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. $\nabla \cdot \mathbf{H} = 0$.

a) $J_G = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\nabla \cdot G = \begin{pmatrix} \partial x \\ \partial y \end{pmatrix} \cdot \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$= 0$$

b) $J_F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\nabla \cdot F = \begin{pmatrix} \partial x \\ \partial y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + 1$$

$$= 2$$

2. Evaluate the Jacobian, divergence and curl of the velocity field below. Then, determine them at the point (1, 2, 3).

$$\mathbf{V}(x, y, z) = \left[x^2 z,\ ye^{2z},\ xyz\right]^T$$

ANS: $\mathbf{J_V} = \begin{bmatrix} 2xz & 0 & x^2 \\ 0 & e^{2z} & 2ye^{2z} \\ yz & xz & xy \end{bmatrix}$. $\nabla \cdot \mathbf{V} = 2xz + e^{2z} + xy$. $\nabla \times \mathbf{V} = \begin{bmatrix} xz - 2ye^{2z} \\ x^2 - yz \\ 0 \end{bmatrix}$.

At $(1, 2, 3)$: $\mathbf{J_V} = \begin{bmatrix} 6 & 0 & 1 \\ 0 & e^6 & 4e^6 \\ 6 & 3 & 2 \end{bmatrix}$. $\nabla \cdot \mathbf{V} = 8 + e^6$. $\nabla \times \mathbf{V} = \begin{bmatrix} 3 - 4e^6 \\ -5 \\ 0 \end{bmatrix}$.

Jacob: $J_V = \begin{bmatrix} 2xz & 0 & x^2 \\ 0 & e^{2z} & 2ye^{2z} \\ yz & xz & xy \end{bmatrix}$

Divergence: $\nabla \cdot V = \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \cdot \begin{pmatrix} x^2 z \\ ye^{2z} \\ xyz \end{pmatrix} = 2xz + e^{2z} + xy$

Curl: $\nabla \times V = \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} x^2 z \\ ye^{2z} \\ xyz \end{pmatrix} = \begin{pmatrix} xz - 2ye^{2z} \\ -(yz - x^2) \\ 0 - 0 \end{pmatrix}$

$= \begin{pmatrix} xz - 2ye^{2z} \\ x^2 - yz \\ 0 \end{pmatrix}$

At $(1, 2, 3)$

$J_V = \begin{bmatrix} 6 & 0 & 1 \\ 0 & e^6 & 4e^6 \\ 6 & 3 & 2 \end{bmatrix}$

Divergence $= 6 + e^6 + 2$

$= 8 + e^6$

Curl $= \begin{pmatrix} 3 - 4e^6 \\ -5 \\ 0 \end{pmatrix}$

4. Determine if each vector field below is a gradient field (conservative). If so, evaluate the scalar potential function E such that **F** = ∇E.

a) $\mathbf{F}(x,y) = \left[2x\sin y,\ x^2\cos y\right]^T$

$$F(x,y) = \begin{pmatrix} 2x\sin y \\ x^2\cos y \end{pmatrix}$$

Let $\vec{F} = \nabla \vec{E}$

$$E = \int 2x\sin y\, dx = x^2\sin y + g(y)$$

$$E_y = x^2\cos y + g'(y)$$

Compare : $g'(y) = 0$

$\Rightarrow g(y) = C$

$$E(x,y) = x^2\sin y + C$$

Check : $$\vec{\nabla} \times \vec{F} = \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} 2x\sin y \\ x^2\cos y \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2x\cos y - 2x\cos y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

b) $\mathbf{F}(x,y,z) = \left[e^x\sin y,\ e^x\cos y,\ 3z^2+2\right]^T$

Check : $$\begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} e^x\sin y \\ e^x\cos y \\ 3z^2+2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^x\cos y - e^x\sin y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $\vec{F} = \nabla \vec{E}$

$$E = \int e^x\sin y\, dx = e^x\sin y + h(y,z)$$

$$E_y = e^x\cos y + h_y(y,z)$$

Compare : $h_y(y,z) = 0$

$$h(y,z) = C + g(z)$$

$$E = e^x \sin y + C + g(z)$$

$$E_z = g'(z) . \quad \text{Comparing} \quad g'(z) = 3z^2 + 2$$

$$g(z) = z^3 + 2z$$

$$\therefore \quad E = e^x \sin y + z^3 + 2z + C$$

c) $\mathbf{F}(x,y,z) = \left[e^{-yz},\, e^{xyz},\, 2xy\right]^T$

$$\text{Check :} \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} e^{-yz} \\ e^{xyz} \\ 2xy \end{pmatrix} = \begin{pmatrix} 2x - xye^{xyz} \\ -(2y + ye^{-yz}) \\ yze^{xyz} + ze^{-yz} \end{pmatrix}$$

$$\neq 0$$

5. A force field (in Newtons) is defined by $\frac{-x^2-1}{y^3}$

$$\mathbf{F}(x,y) = \left[\frac{x}{y^2} + x,\ -\frac{x^2+1}{y^3}\right]^T,\ y \neq 0$$

where coordinates x and y are in meters.

a) Show that the force field is conservative.
b) Evaluate a scalar potential E(x,y) such that **F** = ∇E.
c) Calculate the work done required to move an object subjected to the force field from point (0, 1) to point (1, 1), along the curve $y = 1 + x - x^2$.

ANS: **a)** Yes. **b)** $E(x,y) = \frac{x^2}{2y^2} + \frac{x^2}{2} + \frac{1}{2y^2} + c.$ **c)** Work done = 1 J.

Check :

$$\vec{\nabla} \times \vec{F} = \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} \frac{x}{y^2} + x \\ -\frac{x^2+1}{y^3} \\ 0 \end{pmatrix}$$

$x(y)^{-2} + 0$

$\frac{-2x}{y^3}$

$$= \begin{pmatrix} 0 \\ -(0-0) \\ \frac{-2x}{y^3} - \left(-\frac{2x}{y^3}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\ \text{yes}$$

b)

$$E = \int \frac{x}{y^2} + x \, dx$$

$$= \frac{x^2}{2y^2} + \frac{x^2}{2} + g(y)$$

$$E_y = \frac{1}{\cancel{2}}\left(\frac{-\cancel{2}x^2}{y^3}\right) + g'(y)$$

$$= \frac{-x^2}{y^3} + g'(y)$$

Comparing : $g'(y) = -\frac{1}{y^3}$

$$g(y) = \frac{1}{2y^2} + C$$

$$\therefore E = \frac{x}{2y^2} + \frac{x^2}{2} + \frac{1}{2y^2} + C$$

$$W = E(1,1) - E(0,1)$$

$$= \left[\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right] - \left[\frac{1}{2}\right]$$

$$= 1\,J$$

6. A vector field shown below contains scalar functions f(x), g(y) and h(z) that are differentiable.

$$\mathbf{V}(x,y,z) = \begin{bmatrix} f(x)+y+z \\ g(y)+x+z \\ h(z)+x+y \end{bmatrix}$$

a) Determine if the vector field is conservative. If so, evaluate the scalar potential E such that ∇E = **V**.

b) Evaluate the line integral below, where C is any path from (x_0, y_0, z_0) to (x_1, y_1, z_1).

$$L = \int_C \mathbf{V} \cdot \mathbf{dr}$$

ANS: **a)** Yes. $E(x,y,z) = xy + xz + yz + F(x) + G(y) + H(z)$, where F(x), G(y) and H(z) are antiderivatives of f(x), g(y) and h(z).

b) $L = x_1y_1 + x_1z_1 + y_1z_1 + F(x_1) + G(y_1) + H(z_1) - x_0y_0 - x_0z_0 - y_0z_0 - F(x_0) - G(y_0) - H(z_0)$.

a) Check: $\begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} f(x)+y+z \\ g(y)+x+z \\ h(z)+x+y \end{pmatrix}$

$= \begin{pmatrix} 1-1 \\ -(1-1) \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Yes

b)

$E = \int f(x)+y+z \, dx$

$= F(x) + xy + zx + p(y,z)$

$E_y = 0 + x + z + p_y(y,z)$

Compare : $p_y(y,z) = g(y)$

$p(y,z) = G(y) + q(z)$

$E = F(x) + xy + zx + G(y) + q(z)$

$$E_z = x + q'(z) \quad \text{Compare: } q'(z) = h(z) + y$$

$$q(z) = H(z) + zy + C$$

$$\therefore \; E = F(x) + G(y) + H(z) + xy + zx + zy + C$$

c) $E(x_1, y_1, z_1) - E(x_0, y_0, z_0)$

7. (https://openstax.org/books/calculus-volume-3/pages/6-2-line-integrals)
Evaluate the line integral of each function defined below over the path given.

a) $F(x,y) = xy^4$ over the curve C that is the right half of circle $x^2+y^2=16$ and traversed in the clockwise direction.

Parameterize by letting

$x^2 + y^2 = 16$

$\therefore r = 4$

$x(t) = 4\cos t$

$y(t) = 4\sin t$

since right half of circle $t=0$ to $t=\pi$, however this is for CCW

$r(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4\cos t \\ 4\sin t \end{pmatrix}$ $\therefore$ for CW $t=\pi$ to $t=0$

$r'(t) = \begin{pmatrix} -4\sin t \\ 4\cos t \end{pmatrix}$

$$L = \int_\pi^0 \underbrace{(4\cos t)(4\sin t)^4}_{1024\cos t\sin^4 t} \cdot \sqrt{\underbrace{(-4\sin t)^2 + (4\cos t)^2}_{16\sin^2 t + 16\cos^2 t}}\, dt$$

Let $\sin t = u$

$du = \cos t\, dt$

$dt = \frac{du}{\cos t}$

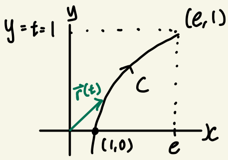
$$= \int_\pi^0 (4\cos t)(4\sin t)^4 \cdot \sqrt{16}\, dt$$

$$= 4096\left[-\frac{\cos^5 t}{5}\right]\Bigg|_\pi^0$$

$$= 4096\left[-\frac{\cos^5 0}{5} - \left(-\frac{\cos^5 \pi}{5}\right)\right]$$

$$= 4096\left[-\frac{2}{5}\right] = -\frac{8192}{5}$$

b) $F(x,y) = xe^y$ over the curve C that is the arc of curve $x = e^y$ from (1, 0) to (e, 1).



Parameterize by letting $y = t$

$$\vec{r}(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$\vec{r}\,'(t) = \begin{pmatrix} e^t \\ 1 \end{pmatrix}$$

$$L = \int_0^1 f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$= \int_0^1 e^t \cdot e^t \cdot \sqrt{e^{2t}+1}\;dt$$

Let $e^{2t}+1 = u$

$du = 2e^{2t}\,dt$

$$= \int_0^1 e^{2t} \cdot \sqrt{e^{2t}+1}\;dt$$

$$= \frac{1}{2}\int_2^{e^2+1} (u)^{\frac{1}{2}}\,du$$

$$= \frac{1}{\cancel{2}}\left[\frac{(u)^{3/2}}{\frac{3}{\cancel{2}}}\right]\Bigg|_2^{e^2+1}$$

$$= \frac{1}{3}\left[(e^2+1)^{3/2} - (2)^{3/2}\right]$$

$$= 7.16$$

c) $F(x,y,z) = 1/(x^2+y^2+z^2)$ over the curve C that is the helix $x = \cos t$, $y = \sin t$, $z = t$, from t = 0 to t = T.

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} \qquad \vec{r}\,'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

$$|\vec{r}\,'(t)| = \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{1} + 1}$$

$$= \sqrt{2}$$

$$L = \int_0^T F(\vec{r}(t))|\vec{r}\,'(t)|\,dt$$

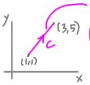
$$= \int_0^T \frac{1}{\underbrace{(\cos t)^2 + (\sin^2 t)}_{1} + (t)^2} |\vec{r}\,'(t)|\,dt$$

$$= \int_0^T \frac{1}{1+t^2}\sqrt{2}\,dt$$

$$= \sqrt{2}\left[\tan^{-1} t\right]\Big|_0^T$$

$$= \sqrt{2}\tan^{-1} T$$

8. Evaluate the line integral of the vector field below over the straight line path from (1, 1) to (3, 5).



$y - y_1 = m(x - x_1)$

$m = \frac{5-1}{3-1} = \frac{4}{2} = 2.$

$y - 1 = 2(x-1)$

$y = 2x - 1.$

$$\mathbf{F}(x,y) = \begin{bmatrix} x^2 - y \\ x - y^2 \end{bmatrix}$$

Parametrize : let $x = t$, $\vec{r}(t) = \begin{pmatrix} t \\ 2t-1 \end{pmatrix}$.

DIY.

ANS: -92/3.

$$\vec{r}\,'(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{F} \cdot \vec{r}\,' = \begin{pmatrix} t^2 - (2t-1) \\ t - (2t-1)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$t^2 - 2t + 1$

$t - (4t^2 - 4t + 1)$

$-4t^2 + 5t - 1$

$$= \begin{pmatrix} t^2 - 2t + 1 \\ -4t^2 + 5t - 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= t^2 - 2t + 1 - 8t^2 + 10t - 2$$

$$= -7t^2 + 8t - 1$$

$$L = \int_1^3 (-7t^2 + 8t - 1)\,dt$$

$$= \left[-\frac{7}{3}t^3 + 4t^2 - t \right] \Big|_1^3$$

$$= -30 - \left(\frac{2}{3} \right)$$

$$= -\frac{92}{3}$$

9. Consider a vector field **F**(x,y) over two paths **r**(t) and **s**(t) given below.

$$\mathbf{F}(x,y) = \begin{bmatrix} x^2+y \\ y-x \end{bmatrix}, \quad \mathbf{r}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}, \quad 0 \le t \le 1,$$
$$\mathbf{s}(t) = \begin{bmatrix} 1-2t \\ 4t^2-4t+1 \end{bmatrix}, \quad 0 \le t \le 1/2$$

a) Evaluate the line integrals of **F** over each of the two paths.

b) What are the Cartesian equations and directions of path **r** and **s**? Hence explain the values obtained in (a).

ANS: **a)** Path **r**: ½. Path **s**: -½. **b)** Path **r**: y = x^2, (0, 0) to (1, 1).
Path **s**: y = x^2, (1, 1) to (0, 0).

$$r'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$s'(t) = \begin{pmatrix} -2 \\ 8t-4 \end{pmatrix}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \begin{pmatrix} 2t^2 \\ t^2-t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$(1-2t)(1-2t)$$
$$= 1-4t+4t^2$$

$$= 2t^2 + 2t^3 - 2t^2$$
$$= 2t^3$$

$$\mathcal{L} = \int_0^1 2t^3 \, dt = \left[\frac{2t^4}{4}\right]\Big|_0^1$$
$$= \frac{1}{2}$$

$$\vec{F}(\vec{s}(t)) \cdot \vec{s}'(t) = \begin{pmatrix} 8t^2-8t+2 \\ 4t^2-2t \end{pmatrix} \begin{pmatrix} -2 \\ 8t-4 \end{pmatrix}$$

$$= -16t^2 + 16t - 4 + 32t^3 - 16t^2 - 16t^2 + 8t$$

$$= 32t^3 - 48t^2 + 24t - 4$$

$$\mathcal{L} = \int_0^{1/2} 32t^3 - 48t^2 + 24t - 4 \, dt$$

$$= \left[8t^4 - 16t^3 + 12t^2 - 4t\right]\Big|_0^{1/2} = -\frac{1}{2}$$

11. Verify Green's theorem for **F**(x, y) below over the semicircular region D given by $x^2 + y^2 \leq R^2$, y ≥ 0.

$$\mathbf{F}(x,y) = \begin{bmatrix} 2x \\ y \end{bmatrix} \begin{array}{l} \leftarrow F_1 \\ \leftarrow F_2 \end{array}$$

ANS: 0.

$$\iint_D \frac{F_2}{\partial x} - \frac{F_1}{\partial y} \, dA$$

$$= \iint_D 0 - 0 \, dA$$

$$= 0$$

13. Given that C is any closed path in $\mathbb{R}^2$ oriented counterclockwise, show that the line integral below is independent of the path C and only dependent on the area enclosed by C.

$$\int_C \underbrace{(x^2y^3 - 3y)\,dx + x^3y^2\,dy}_{\begin{pmatrix} x^2y^3 - 3y \\ x^3y^2 \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}}$$

$$= \iint_D 3x^2y^2 - (3x^2y^2 - 3) \, dA$$

$$= \iint_D 3 \, dA$$

$$= 3 \iint_D dA$$

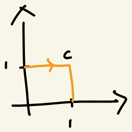
= 3 x Area enclosed by C

12. Using Green's theorem, evaluate the line integral for **F**(x,y) below, C is the boundary of the square with vertices (0, 0), (1, 0), (0, 1) and (1, 1), oriented clockwise.

$$\mathbf{F}(x,y) = \begin{bmatrix} x - y^2 \\ x + y^2 \end{bmatrix} \begin{matrix} \leftarrow F_1 \\ \leftarrow F_2 \end{matrix}$$

ANS: -2.

CW



$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= (1 - (-2y))$$

$$= 1 + 2y$$

$$\oint_C \vec{F} \cdot d\vec{r} = -\iint \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$

$$= -\int_0^1 \int_0^1 (1+2y)\, dy\, dx$$

$$= -\int_0^1 dx \cdot \int_0^1 (1+2y)\, dy$$

$$= -(1) \cdot \left[y + y^2\right]\Big|_0^1$$

$$= -(1) \cdot (2)$$

$$= -2$$