EDE1012 MATHEMATICS 2

Tutorial 6

Vector Calculus II

1. A surface is defined by the vector function

$$\mathbf{r}(s,t) = [s^2\cos t \quad s^2\sin t \quad s]^T$$

   a) Evaluate the normal vectors to the surface at (1, 0, -1).

   b) Determine the Cartesian equation of the tangent plane at (1, 0, -1).

   c) Determine the Cartesian equation of the surface in the form F(x, y, z) = 0.

ANS: **a)** $\mathbf{N} = \pm[1 \quad 0 \quad 2]^T$. **b)** $x + 2z = -1$. **c)** $x^2 + y^2 - z^4 = 0$.

2. Evaluate the flux of the vector field below across the triangular surface S that is the plane 2x - 2y + z = 2 cut out by the coordinate planes. The surface is orientated with an upward-pointing normal.

$$\mathbf{F}(x,y,z) = [x \quad y \quad z]^T$$

ANS: Flux = 1.

3. A surface S is the closed cylinder with its top and bottom at z = 4 and z = 0 respectively and a cylindrical surface $x^2 + y^2 = 9$. A vector field **F** is defined below.

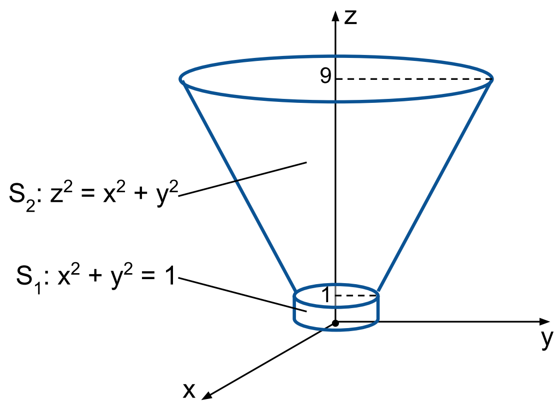
$$\mathbf{F}(x,y,z) = [-y \quad x \quad 0]^T$$

   a) Determine the flux of **F** across S and explain why it is zero. The surface is orientated with outward-pointing normals.

   b) Verify the flux again using the divergence theorem.

ANS: **a)** Flux = 0. **b)** Flux = 0.

4. A surface S = $S_1 + S_2$ that looks like a funnel is shown below.



a) Determine the outward-pointing normals of surfaces S_1 and S_2.

b) Evaluate the flux of **F** below through S, which is orientated by outward-pointing normals.

$$\mathbf{F}(x, y, z) = [-y \quad x \quad z]^T$$

ANS: **a)** $S_1 : \mathbf{N} = [x \quad y \quad 0]^T$. $S_2 : \mathbf{N} = [x \quad y \quad -z]^T$. **b)** -1456π/3.

5. Use the divergence theorem to evaluate the flux of the vector field below through surface S of the unit cube in the domain [0, 1] × [0, 1] × [0, 1],

$$\mathbf{V}(x, y, z) = \left[ze^{x^2} \quad 3y \quad 2 - yz\right]^T$$

ANS: Flux = e/2 + 2.

6. A vector field **F** and surface S is defined by the functions

$$\mathbf{F}(x, y, z) = [x \quad y \quad 2 - 2z]^T, \qquad S : z = e^{1-x^2-y^2}, z \geq 1$$

Given that surface S is oriented by upward normal vectors, use Gauss's theorem to calculate the flux of F across S.

ANS: Flux = 0.

7. (https://openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem)
Evaluate the flux of **F** below across the surface S consisting of all faces of the tetrahedron bounded by plane x + y + z = 1 and the coordinate planes, with outward normal vectors

$$\mathbf{F}(x, y, z) = [x^2 \quad xy \quad x^3y^3]^T$$

ANS: 1/8.

8. Consider a cylinder of height H with a base of radius R on the xy-plane.

   a) Using a surface integral, show that the area of the cylinder mantle is 2πRH.

   b) Evaluate the flux of the vector field defined below through the cylinder mantle using Gauss's theorem. Orientate the cylinder with outward normals.

$$\mathbf{F}(x, y, z) = [xz + y \quad yz - x \quad z]^T$$

ANS: **b)** Flux = $\pi R^2 H^2$.

9. Verify Stokes' theorem for a conservative vector field **F**(x, y, z) over a closed curve C that is the boundary of surface S.

10. (https://openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem)
Use Stokes' theorem to evaluate the line integral below, where C is the curve given by x = cos t, y = sin t, z = sin t, 0 ≤ t ≤ 2π, traversed in the direction of increasing t.

$$\int_C [2xy^2 z\, dx + 2x^2 yz\, dy + (x^2y^2 - 2z)\, dz]$$

ANS: 0.

11. Use Stokes' theorem to evaluate the line integral below, where C is the intersection curve between the plane $x + y + z = 8$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise.

$$\int_C \mathbf{F} \cdot \mathbf{dr}, \quad \mathbf{F}(x, y, z) = \begin{bmatrix} x^2 z \\ xy^2 \\ z^2 \end{bmatrix}$$

ANS: 81π/2.

12. Using Stokes' theorem, evaluate the circulation of F over surface S defined below.

$$\mathbf{F}(x,y,z) = \begin{bmatrix} e^{y+z} - 2y \\ xe^{y+z} + y \\ e^{x+y} \end{bmatrix}, \quad S : \left\{ (x,y,z) \,\middle|\, z = e^{-(x^2+y^2)}, z \geq 1/e \right\}$$

ANS: 2π.

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals
2) https://openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem
3) https://openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem

End of Tutorial 6

(Email to youliangzheng@gmail.com for assistance.)