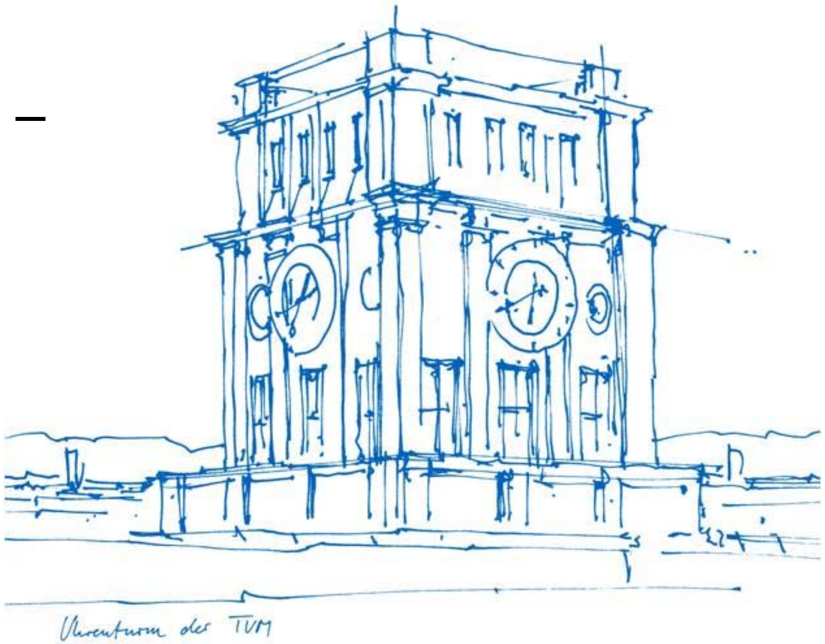
Lecture

Electricity and Magnetism

Chapter 1 Electrostatics

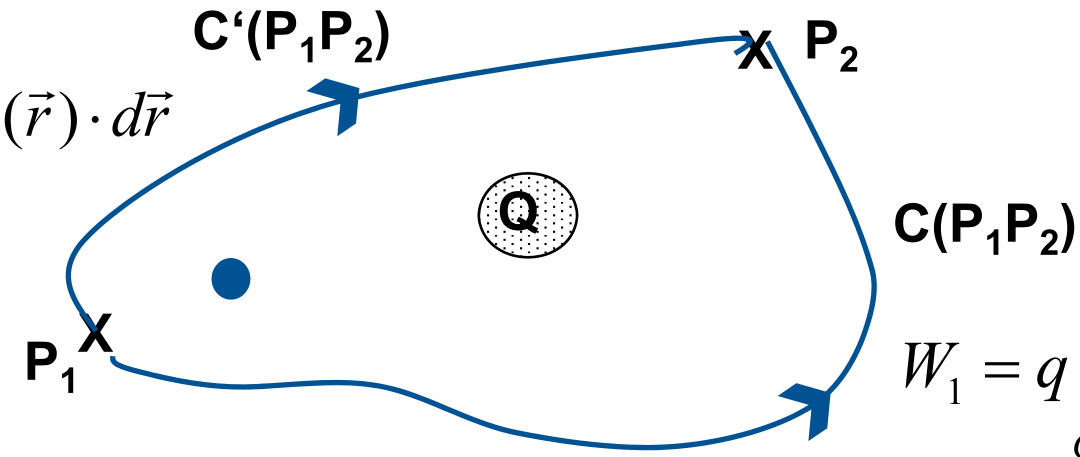
- Electric voltage
- Properties of electric field
- Fundamental law of electrostatics – conservative fields



Electric Work – Electric Voltage

Electric work depends only on starting and end location of the path, not on path length and/or shape of the path

$$W_2 = q \int_{C'(P_1P_2)} \vec{E}(\vec{r}) \cdot d\vec{r}$$



$$W_1 = q \int_{C(P_1P_2)} \vec{E}(\vec{r}) \cdot d\vec{r}$$

It applies: $W_1 = W_2$, since if : $W_1 \neq W_2$

$\Rightarrow$ there would be gain of energy each round which is passed through the closed loop $\Rightarrow$ this would be a perpetuum mobile („generates energy fom nothing“), which does not comply with basic principles in physics

Electric Voltage

- Electric work is defined as $W = q\int\limits_C \vec{E}(\vec{r})d\vec{r}$

  i.e., electric work depends on charge

$\Longrightarrow$ definition of electric voltage as electric work per unit charge

Electric voltage (definitions): $$\boxed{U_{12} := \frac{W_{12}}{q} = \int\limits_{C(P_1P_2)} \vec{E}\cdot d\vec{r}} \qquad (1.8.)$$

J = Nm = Joule → work

unit: dim[U] = 1J/As = 1 V

Ampere · second = Coulomb

- From consideration on electric work follows in analog way:

  If $\vec{E}(\vec{r}) \neq f(t)$: U depends only on location of points P1 and P2 ab and not on integration path!

$$\boxed{U_{12} := \int\limits_{P_1}^{P_2} \vec{E}\cdot d\vec{r}}$$

Path-independence of electric work – fundamental law of electrostatics

Electric work is independent of path, on which a charge is moved; depends only on starting and end point.

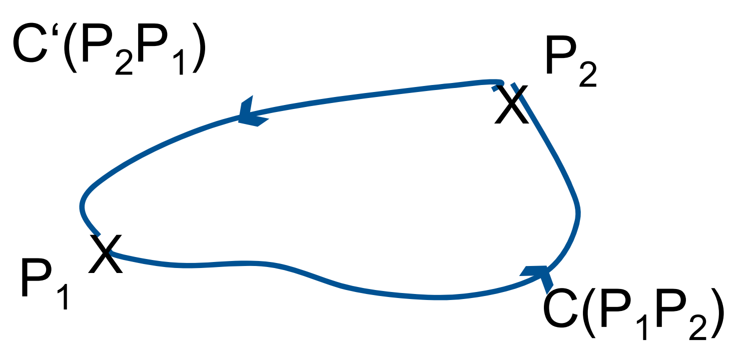
Electrostatic

Electric fields are conservative!

1) Path integral $\int\limits_{C(P_1P_2)} \vec{E}\cdot d\vec{r}$ is independent of integration path

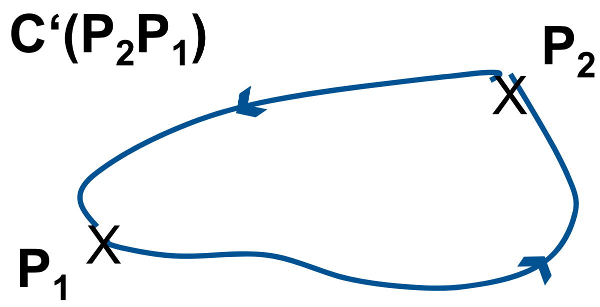
2) Follow the curve $C(P_1P_2)$ in reverse direction, then

$$\boxed{U_{12} = \int\limits_{C(P_1P_2)} \vec{E}\cdot d\vec{r} = -\int\limits_{C(P_2P_1)} \vec{E}\cdot d\vec{r} = -U_{21}}$$



Path-independence of electric work – fundamental law of electrostatics

$$\boxed{U_{12} = \int_{C(P_1P_2)} \vec{E}\cdot d\vec{r} = -\int_{C(P_2P_1)} \vec{E}\cdot d\vec{r} = -U_{21}}$$



This equation holds also for closed curves:

$$\Rightarrow \int_{C(P_1P_2)} \vec{E}\cdot d\vec{r} + \int_{C(P_2P_1)} \vec{E}\cdot d\vec{r} = \boxed{\oint \vec{E}\cdot d\vec{r} = 0}$$

The sum of these two integrals
is the same as a so-called
closed loop integral
(path goes one time round)

Path-independence of electric work – fundamental law of electrostatics

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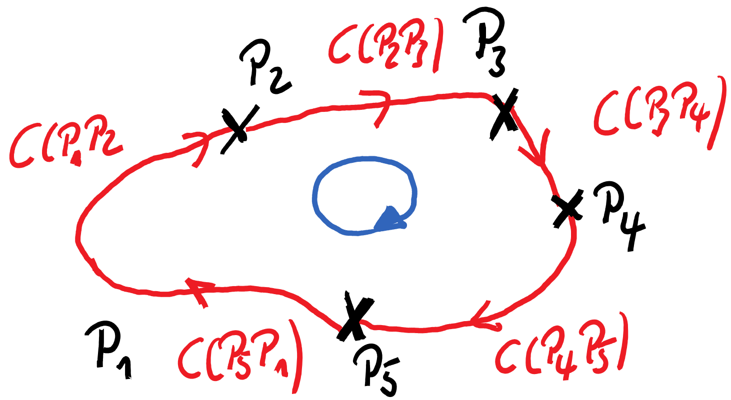

This equation holds also for closed curves:

$$\Rightarrow \underbrace{\int\limits_{C(P_1P_2)} \vec{E}\cdot d\vec{r} + \int\limits_{C(P_2P_1)} \vec{E}\cdot d\vec{r}}_{} = \boxed{\oint \vec{E}\cdot d\vec{r} = 0}$$

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Path-independence of electric work – fundamental law of electrostatics

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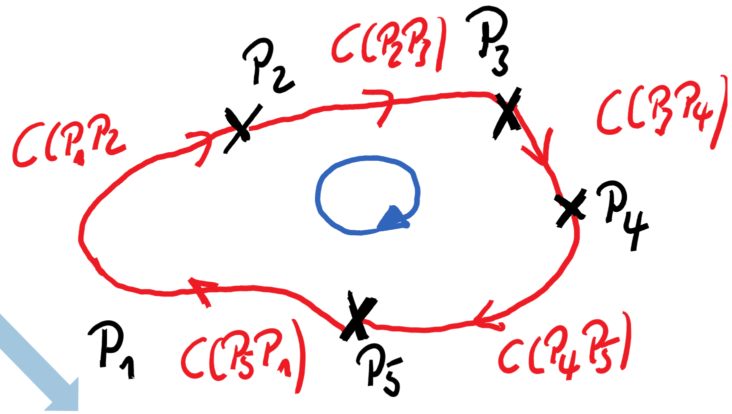


$$\underbrace{\int_{P_1}^{P_2}\vec{E}\,d\vec{r}}_{U_{12}} + \underbrace{\int_{P_2}^{P_3}\vec{E}\,d\vec{r}}_{U_{23}} + \underbrace{\int_{P_3}^{P_4}\vec{E}\,d\vec{r}}_{U_{34}} + \underbrace{\int_{P_4}^{P_5}\vec{E}\,d\vec{r}}_{U_{45}} + \underbrace{\int_{P_5}^{P_1}\vec{E}\,d\vec{r}}_{U_{51}} = 0$$

$$U_{12} + U_{23} + U_{34} + U_{45} + U_{51} = 0$$

$$\boxed{\sum_i U_i = 0}$$

Kirchhoff‘s voltage law: in a closed loop in an electric circuit, all electric voltages add up to zero.

Circuit Theory;
stationary currents

Kirchhoff‘s voltage law is a consequence of the conservative character of the electrostatic field (path-independence of electric work)

Path-independence of electric work – fundamental law of electrostatics

3) Equivalent formulation: differential form of (1.11)

$$\boxed{curl\left\{\vec{E}(\vec{r})\right\} = 0}$$

electrostatic

electric fields are curl-fress

Only valid in cartesian coordinate system(!):

integrability condition $$\boxed{\frac{\partial E_k}{\partial x_j} = \frac{\partial E_j}{\partial x_k}}$$ (j,k= 1,2,3)

x, y, z

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$$

1)– 3) are equivalent mathematical formulations of the fundamental law of electrostatics: electrostatic fields are conservative!

Please note: There are also electric fields, which are not conservative! (see chapter 4 of this lecture)

Summary: Electric work and voltage

- Electrostatic fields are conservative (which means the energy is preserved).

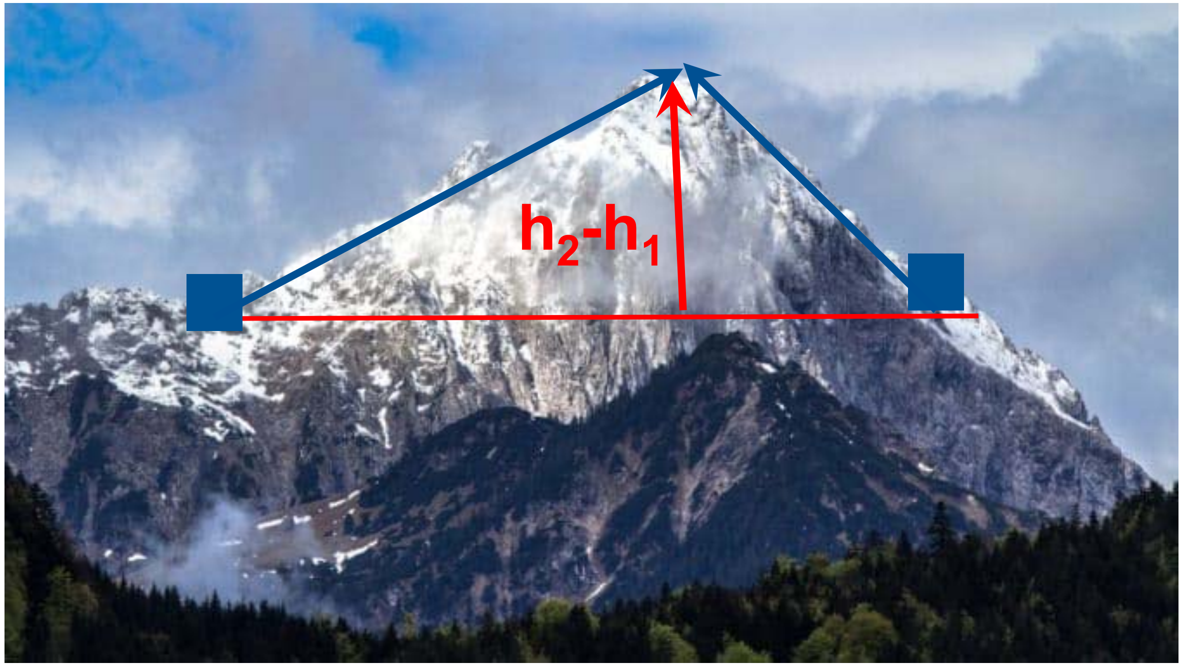
- The work performed along any trajectory in an electrostatic field
  - does not depend on the specific path, but only
  - depends on the starting and ending positions.

- As a result, the energy obtained and/or released for closed curves is zero.

$$\text{For closed curve C with } \vec{r}_a = \vec{r}(t_a) = \vec{r}_b = \vec{r}(t_b) \text{ it is}$$
$$W = \oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \mathrm{q} \cdot \oint_C \vec{\mathrm{E}}(\vec{\mathrm{r}}) \cdot d\vec{r} = 0$$

- Work performed per charge between two points is defined as electric voltage.
- Kirchhoff's voltage law is a consequence of the conservative character of the electrostatic field.

Other conservative fields



Potential energy: $W=mg(h_2-h_1)$

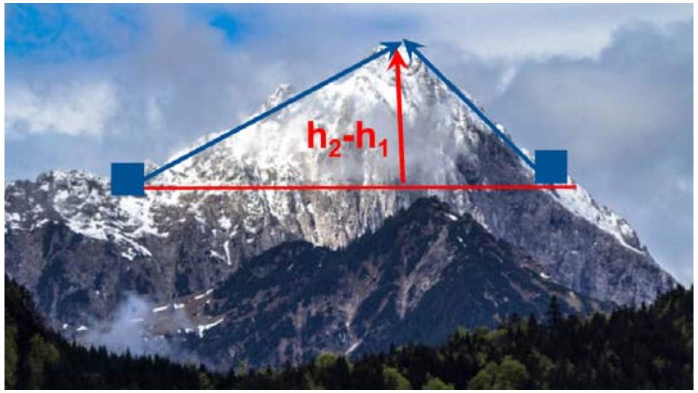
If dissipative forces are neglegted, then $\int_{P_1}^{P_2} \vec{F}_g \cdot d\vec{r}$ is independent of integration path.

Other conservative fields

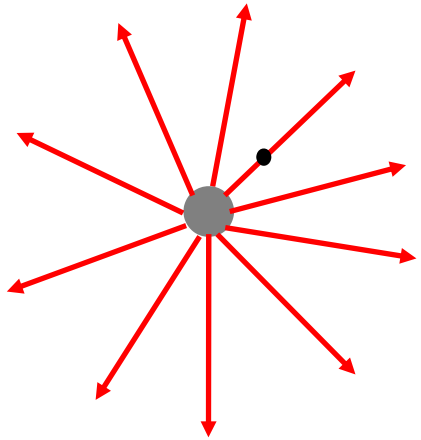
Both, electrostatic fields and gravitational fields are conservative.

This means: Energy is conserved, no work is done on closed loop paths.

Gravitational Forces



Electrostatic Forces



If dissipative forces are neglected, then $\int_{P_1}^{P_2} \vec{F}_g \cdot d\vec{r}$ is independent of path.

$W_{el} = \int \vec{F}_{el} d\vec{r}$ is independent of path.

And is „regained“, if q is back to initial position.

Potential energy: $W=mg(h_2-h_1)$

Electrostatic potential energy $W_{el} = \frac{-qQ}{4\pi\varepsilon}(\frac{1}{r_2} - \frac{1}{r_1})$

Electric Potential

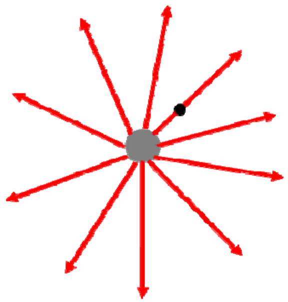
Electric potential φ represents potential energy W per charge q.

$$\boxed{\varphi(\vec{r}) := \frac{W(\vec{r})}{q} = \frac{q \cdot \int_C \vec{E}(\vec{r}) \cdot d\vec{r}}{q} = \int_C \vec{E}(\vec{r}) \cdot d\vec{r}}$$

- Electric potential does not depend on q.
- The electrostatic potential φ is a scalar field (not a vector!) To any location in space electrical potential value is attributed.
- The physical unit for the physical quantity potential is Volt (short: V), same as for voltage, named after the Italian scientist A. Volta (1745-1827).

Electric Potential

What is the meaning of “electric potential”: see analogy to gravitational potential:

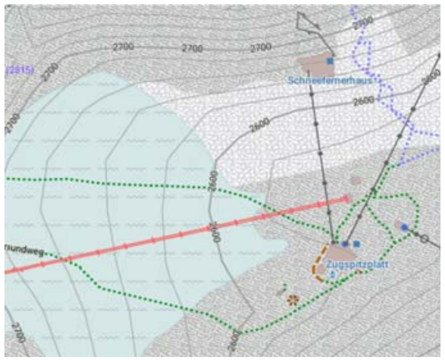


$$\varphi(\vec{r}) := \frac{W(\vec{r})}{q} = \frac{q \cdot \int_C \vec{E}(\vec{r}) \cdot d\vec{r}}{q} = \int_C \vec{E}(\vec{r}) \cdot d\vec{r}$$

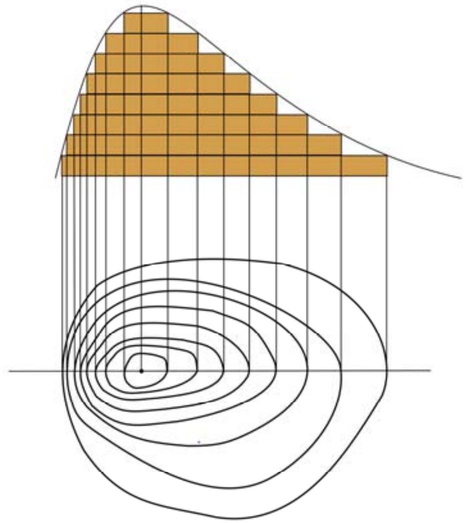
Potential energy: $W = mg(h_2 - h_1)$

Gravitational potential: $\varphi = W/m = \int_{P_1}^{P_2} \vec{F}_g \cdot d\vec{r}$

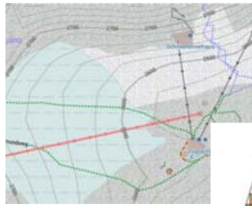
Hiking map of Zugspitze

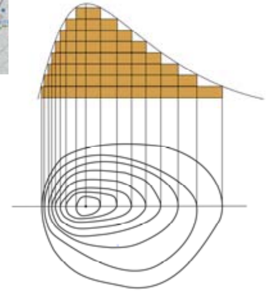


Projection of height profile



Electric Potential





What is the meaning of “electric potential”:
see analogy to gravitational potential:

- Surfaces at constant height: same gravitational potential

  -> along a path contained in this surface, no potential work is performed

- These planes/lines are called equipotential surfaces/lines or iso-surfaces/iso-lines
- The steepest increase is, where these lines are very dense. Here the change in the potential energy ($\frac{dW_{pot}}{dh}$) is very large.
- Change in the height is perpendicularly to the iso-lines (lines at constant height)
- The potential function is defined with respect to a reference point and reference potential (What is this in the case of the gravitational potential on earth?)
- From a potential function the respective force field can be calculated and vice versa
- Iso-surfaces are perpendicular to force field

Electric Potential

Electric potential vs. electric voltage vs. electric work:

- Electric work: $W(\vec{r}) = \int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \mathrm{q} \cdot \int_C \vec{\mathrm{E}}(\vec{\mathrm{r}}) \cdot d\vec{r}$ with $C = C(P_1,P_2)$

  work performed in an electric field $\vec{E}(\vec{r})$, when moving a charge q from P_1 to P_2; does not depend on path itself, but only on starting and end position.

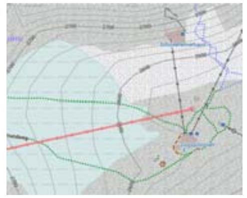
- Electric potential: $\varphi(\vec{r}) = \frac{\Delta W(\vec{r})}{q} = -\int_{\vec{r}}^{\infty} \vec{E} \cdot d\vec{r} = -\{\varphi(\infty) - \varphi(\vec{r})\} = \varphi(\vec{r})$

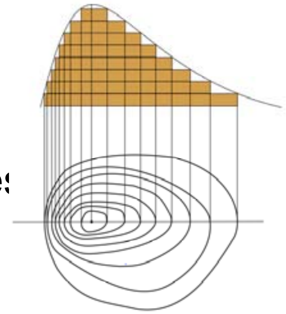
  Potential function related to $\vec{E}(\vec{r})$. Gives the potential energy of a charge q in the electric field $\vec{E}(\vec{r})$ at any location in space w.r.t to a reference location (in this case $r \to \infty$); this is why an arbitrary position vector $\vec{r}$ is contained in the relation. Minus sign is convention.

- Electric voltage: $U_{12} := \frac{W_{12}}{q} = \int_{C(P_1 P_2)} \vec{E} \cdot d\vec{r}$ with $C = C(P_1,P_2)$

  - work per charge performed in an electric field $\vec{E}(\vec{r})$
  - potential difference between two fixed points P_1 and P_2 (reference potential cancels out); this is what we measure in technical applications.

Electric Potential – Iso/equipotential Surfaces/Lines





- Surfaces at constant height: same gravitational potential

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See OneNote lecture notes