BSc in Electrical Engineering and Information Technology

Module Examination on Electricity and Magnetism

First Exam 2016

Solution

Q1 (5 points)

*a)

$$\begin{aligned}\vec{F}_{el} &= \frac{q_3}{4\pi\varepsilon_0}\sum_{i=1}^{2}\frac{q_i}{|\vec{r}_3-\vec{r}_i|^3}(\vec{r}_3-\vec{r}_i)\\ &= \frac{q_3}{4\pi\varepsilon_0}[\frac{q_1}{|\vec{r}_3-\vec{r}_1|^3}(\vec{r}_3-\vec{r}_1)+\frac{q_2}{|\vec{r}_3-\vec{r}_2|^3}(\vec{r}_3-\vec{r}_2)]\\ &= -\frac{2q^2}{4\pi\varepsilon_0}(\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|^3}+\frac{\vec{b}}{|\vec{b}|^3})\\ &= -\frac{q^2}{2\pi\varepsilon_0}(\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|^3}+\frac{\vec{b}}{|\vec{b}|^3})\end{aligned}$$

*b)

$$\begin{aligned}\vec{E}(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{3}\frac{q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i)\\ &= \frac{1}{4\pi\varepsilon_0}(-q\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3}-q\frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|^3}+2q\frac{\vec{r}-\vec{r}_3}{|\vec{r}-\vec{r}_3|^3})\\ &= -\frac{q}{4\pi\varepsilon_0}(\frac{\vec{r}+\vec{a}}{|\vec{r}+\vec{a}|^3}+\frac{\vec{r}}{|\vec{r}|^3}-2\frac{\vec{r}-\vec{b}}{|\vec{r}-\vec{b}|^3})\end{aligned}$$

*c) $\omega_{el}(\vec{r})=\frac{1}{2}\varepsilon_0|\vec{E}(\vec{r})|^2$

Q2 (2 points)

☒ $\mathrm{div}\vec{D}=\rho$

☒ $\int\limits_{\partial V}\vec{D}\,\mathrm{d}\vec{a}=\int\limits_{V}\rho\,\mathrm{d}^3r$

☐ $\mathrm{rot}\vec{D}=\rho$

☐ $\int\limits_{V}\mathrm{div}\vec{D}\,\mathrm{d}^3r=\int\limits_{\partial V}\rho\,\mathrm{d}\vec{a}$

☐ $\int\limits_{V}\vec{D}\,\mathrm{d}^3r=\int\limits_{V}\rho\,\mathrm{d}^3r$

Q3 (5 points)

*a) $\vec{E}(x) = -\nabla\Phi(x) = -\dfrac{\partial\Phi(x)}{\partial x} \cdot \vec{e}_x$

$$\vec{E}(x) = \begin{cases} 0 & \text{for (1): } x < -x_0 \\ -(-\dfrac{\rho_0}{2\varepsilon}(2x + 2x_0))\vec{e}_x & \text{for (2): } -x_0 \leq x \leq 0 \\ -(\dfrac{\rho_0}{2\varepsilon}(2x - 2x_0))\vec{e}_x & \text{for (3): } 0 < x \leq +x_0 \\ 0 & \text{for (4): } +x_0 < x \end{cases}$$

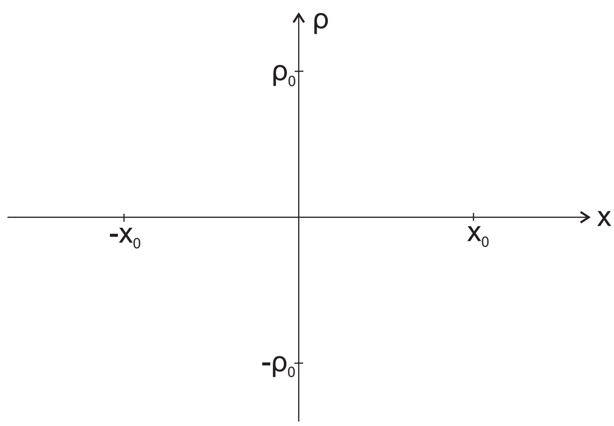
$$\vec{E}(x) = \begin{cases} 0 & \text{for (1): } x < -x_0 \\ \dfrac{\rho_0}{\varepsilon}(x + x_0)\vec{e}_x & \text{for (2): } -x_0 \leq x \leq 0 \\ -\dfrac{\rho_0}{\varepsilon}(x - x_0)\vec{e}_x & \text{for (3): } 0 < x \leq +x_0 \\ 0 & \text{for (4): } +x_0 < x \end{cases}$$

*b) $\rho(x) = div \cdot \varepsilon\vec{E}(x) = \varepsilon\dfrac{\partial E_x}{\partial x}$

$$\rho(x) = \begin{cases} 0 & \text{for (1): } x < -x_0 \\ \dfrac{\rho_0}{\varepsilon} \cdot \varepsilon & \text{for (2): } -x_0 \leq x \leq 0 \\ -\dfrac{\rho_0}{\varepsilon} \cdot \varepsilon & \text{for (3): } 0 < x \leq +x_0 \\ 0 & \text{for (4): } +x_0 < x \end{cases}$$

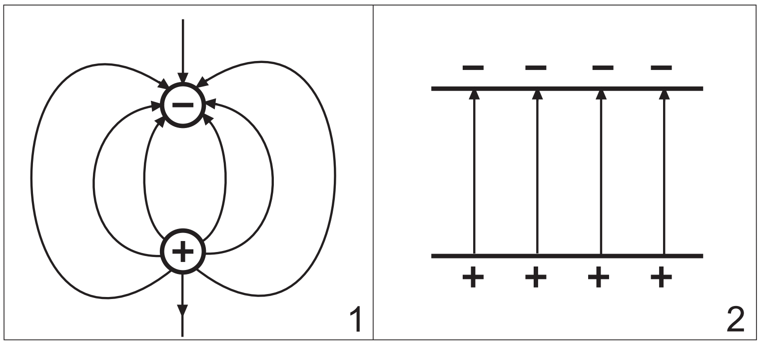
$$\rho(x) = \begin{cases} 0 & \text{for (1): } x < -x_0 \\ \rho_0 & \text{for (2): } -x_0 \leq x \leq 0 \\ -\rho_0 & \text{for (3): } 0 < x \leq +x_0 \\ 0 & \text{for (4): } +x_0 < x \end{cases}$$

c)

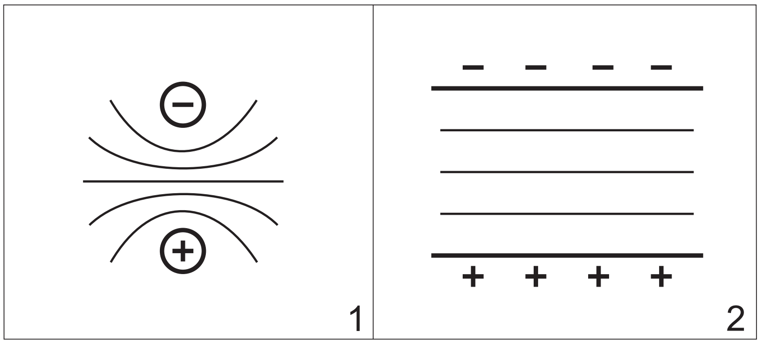


Q4 (4 points)

*a)



b)



Q5 (6 points)

*a)

$$
\begin{aligned}
C_{\mathrm{i}} &= \epsilon_{\mathrm{i}}\frac{A}{d} \qquad ; i = 1, 2 \\
\frac{1}{C_{\mathrm{tot}}} &= \frac{1}{C_1} + \frac{1}{C_2} \qquad ; C_{\mathrm{tot}} = \frac{C_1 C_2}{C_1 + C_2} \\
Q &= C_{\mathrm{tot}}\, U_0 = \frac{C_1 C_2}{C_1 + C_2}\, U_0 \\
\Rightarrow Q &= \frac{A}{d}\frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}\, U_0
\end{aligned}
$$

*b)

$$
\begin{aligned}
\vec{D}_1 &= \vec{D}_2 \\
\epsilon_1 \vec{E}_1 &= \epsilon_2 \vec{E}_2 \qquad \text{since } \epsilon_1 > \epsilon_2\text{: } |\vec{E}_1| < |\vec{E}_2|
\end{aligned}
$$

The electric field $\vec{E}$ of region 2 is larger than of region 1.

Q6 (7 points)

a) $A_1(x) = A_0 - \frac{2}{3}\frac{A_0}{d}x \qquad$ for $0 < x < d$
$A_2(x) = \frac{1}{3}A_0 \qquad$ for $d < x < 2d$

b) $j_{x1} = \frac{I_0}{A_1(x)} = \frac{I_0}{A_0 - \frac{2}{3}\frac{A_0}{d}x} \qquad$ for $0 < x < d$
$j_{x2} = \frac{I_0}{A_2(x)} = \frac{3I_0}{A_0} \qquad$ for $d < x < 2d$

c) $E_1(x) = \frac{j_{x1}}{\sigma_1} = \frac{I_0}{(A_0 - \frac{2}{3}\frac{A_0}{d}x)\sigma_1} \qquad$ for $0 < x < d$
$E_2(x) = \frac{j_{x3}}{\sigma_2} = \frac{3I_0}{A_0\sigma_2} \qquad$ for $d < x < 2d$

d) There is a discontinuity (third answer is correct).

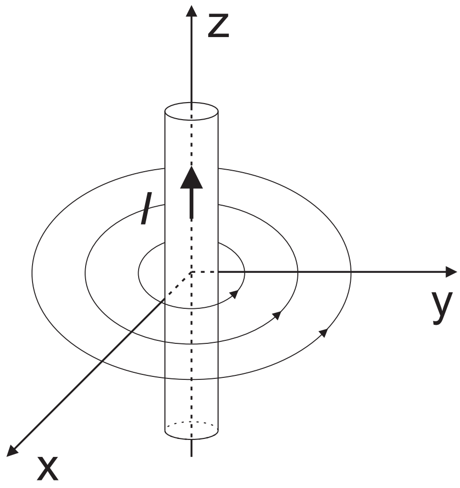
Q7 (5 points)

*a)
- $\vec{m} = I \cdot \vec{A} = I \cdot \frac{a \cdot b}{2} \cdot \vec{e}_x$
- $[\vec{m}] = Am^2$

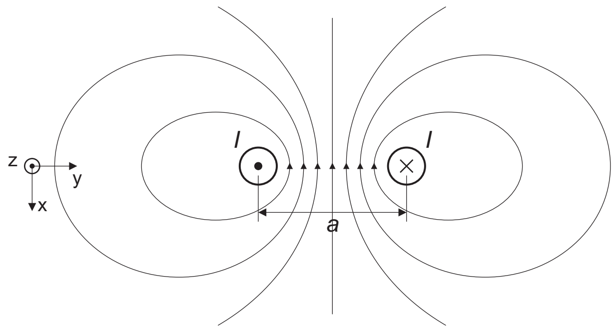
*b) $\vec{M} = \vec{m} \times \vec{B} = I \cdot \frac{a \cdot b}{2} \cdot B_0 \cdot (\vec{e}_x \times \vec{e}_z) = -IB_0\frac{a \cdot b}{2}\vec{e}_y$

Q8 (4 pointse)

*a)



*b)



*c) The direction of the force is perpendicular to the direction of the wires (parallel to distance vector $\vec{r}_{12}$)The force is repulsive.

Q9 (3 Punkte)

$U_{ind} = -\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{A(t)} \vec{B}(\vec{r}, t) \, \mathrm{d}\vec{a}$

oder

$U_{ind} = \int\limits_{\partial A(t)} \left(\vec{v} \times \vec{B}\right) \mathrm{d}\vec{r} - \int\limits_{A(t)} \frac{\partial \vec{B}}{\partial t} \mathrm{d}\vec{a}$

- motional induction, motional EMF $\frac{\mathrm{d}A}{\mathrm{d}t} \neq 0$
- motionless induction, motionless EMF $\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} \neq 0$

Problem 1 (15 points)

a) **(3 points)**
space charge densities of W_1 and W_2:

$$\rho_1 = \frac{Q_1}{V_1} = +\frac{3Q}{4\pi(R_2^3 - R_1^3)}$$
$$\rho_2 = \frac{Q_2}{V_2} = -\frac{3Q}{4\pi(R_4^3 - R_3^3)}$$

b) **(6 points)**
Gauß' law in integral formulation:

$$\int_{\partial V} \vec{D}(\vec{r}) \cdot \, \mathrm{d}\vec{a} = \int_V \rho(\vec{r}) \, \mathrm{d}^3 r = Q_{\mathrm{encl}}(V)$$

Due to spherical symmetry:

$$\vec{D}(\vec{r}) = D_r(r) \cdot \vec{e}_r.$$

Inserting this into Gauss' law

$$\int_0^{2\pi} \int_0^{\pi} D_r(r) \cdot \vec{e}_r \cdot r^2 \cdot \sin(\vartheta) \cdot \vec{e}_r \, \mathrm{d}\vartheta \mathrm{d}\varphi = 2\pi \cdot 2 \cdot r^2 \cdot D_r(r) = 4\pi r^2 D_r(r)$$

hence (for constant space charge density):

$$4\pi r^2 D_r(r) = \int_V \rho(\vec{r}) \, \mathrm{d}^3 r = 4\pi \int_0^r \rho(\xi) \, \xi^2 \, \mathrm{d}\xi$$

and: $\vec{D}(\vec{r}) = \varepsilon \vec{E}(\vec{r})$

$$\vec{E}(\vec{r}) = \frac{1}{\varepsilon r^2} \int_0^r \rho(\xi) \, \xi^2 \, \mathrm{d}\xi \cdot \vec{e}_r.$$

Region 1 ($R_1 \leq r \leq R_2$):
Space charge density in this region: $\rho = \rho_1 = 3Q/(4\pi(R_2^3 - R_1^3))$ and permittivity $\varepsilon = \varepsilon_1$.

$$\begin{aligned}\vec{E}_1(\vec{r}) &= \frac{1}{\varepsilon_1 r^2} \int_{R_1}^r \rho_1 \xi^2 \, \mathrm{d}\xi \cdot \vec{e}_r = \frac{\rho_1}{\varepsilon_1 r^2} \left[\frac{1}{3}\xi^3\right]_{R_1}^r \cdot \vec{e}_r = \rho_1 \cdot \frac{r^3 - R_1^3}{3\varepsilon_1 r^2} \cdot \vec{e}_r \\ &= \frac{Q(r^3 - R_1^3)}{4\pi\varepsilon_1 r^2 (R_2^3 - R_1^3)} \cdot \vec{e}_r.\end{aligned}$$

Region 2 ($R_2 \leq r \leq R_3$):
Space charge density in this region $\rho = 0$ and permittivity $\varepsilon = \varepsilon_0$. Enclosed charge is $+Q$, hence:

$$4\pi r^2 D_r(r) = \int_V \rho(\vec{r}) \, \mathrm{d}^3 r = Q_{\mathrm{ein}}(V) = +Q$$

and

$$\vec{E}_2(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r^2} \cdot \vec{e}_r.$$

c) **(2 points)**
For $r > R_4$ the enclosed charge amounts to $Q_{\text{ein}} = +Q - Q = 0$. Therefore: magnitude of electric field for $r > R_4$ is zero.

d) **(2 points)**
From sub-problem b) we know the electric field in the respective region. The electric voltage U can now be calculated according to

$$U = \int_{R_2}^{R_3} \vec{E}_2(\vec{r})\, \mathrm{d}\vec{r} = \int_{R_2}^{R_3} \frac{Q}{4\pi\varepsilon_0 r^2} \cdot \vec{e}_r \cdot \vec{e}_r\, \mathrm{d}r = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{R_2} - \frac{1}{R_3} \right).$$

e) **(2 points)**
Electric energy density: $w_{\text{el}} = \vec{E}(\vec{r}) \cdot \vec{D}(\vec{r})/2$
Hence:

$$w_{\text{el}} = \frac{\varepsilon_0}{2} \vec{E}_2^2 = \frac{\varepsilon_0}{2} \frac{Q^2}{(4\pi\varepsilon_0)^2\, r^4}.$$

Problem 2 (14points)

a) **(4 points)**
Orientation of conductor loop has been chosen in clockwise direction, hence: $\mathrm{d}\vec{a} = (-\vec{e}_x)\mathrm{d}y\mathrm{d}z$

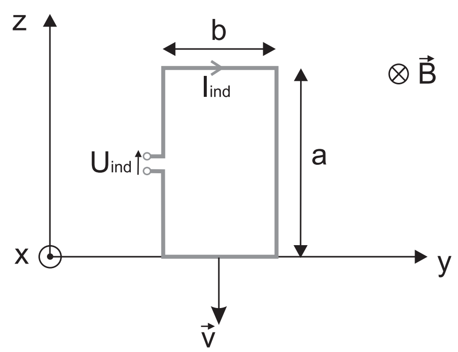
$$\begin{aligned}
\Phi(t) &= \int\limits_{A(t)} \vec{B}(z) \cdot \mathrm{d}\vec{a} = \int\limits_{-v_0t}^{a-v_0t} \int\limits_{0}^{b} (B_0 + k_0 z)(-\vec{e}_x)(-\vec{e}_x)\mathrm{d}y\mathrm{d}z \\
&= \int\limits_{-v_0t}^{a-v_0t} b(B_0 + k_0 z)\mathrm{d}z = \left[bB_0 z + bk_0 \frac{z^2}{2} \right]_{-v_0t}^{a-v_0t} \\
&= bB_0(a - v_0t) + bk_0\frac{1}{2}(a - v_0t)^2 - bB_0(-v_0t) - bk_0\frac{1}{2}(v_0t)^2 \\
&= bB_0 a + \frac{1}{2}a^2 bk_0 - abk_0 v_0 t
\end{aligned}$$

b) **(2 points)**

$$U_{ind} = -\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = bk_0 a v_0$$

c) **(2 pints)**

$$I_{ind} = \frac{U_{ind}}{R} = \frac{bk_0 a v_0}{R}$$



d) **(6 points)** $\mathrm{d}\vec{F}_m = I_{ind}(\mathrm{d}\vec{s} \times \vec{B})$

Left part of conductor loop: $\mathrm{d}\vec{s} = \vec{e}_z\mathrm{d}s$

$$\mathrm{d}\vec{F}_{m,l} = I_{ind}(\mathrm{d}s\,\vec{e}_z \times B(z)(-\vec{e}_x)) = -I_{ind}B(z)\mathrm{d}s\vec{e}_y$$

Right part of conductor loop: $\mathrm{d}\vec{s} = -\vec{e}_z\mathrm{d}s$

$$\mathrm{d}\vec{F}_{m,r} = I_{ind}(\mathrm{d}s\,(-\vec{e}_z) \times B(z)(-\vec{e}_x)) = I_{ind}B(z)\mathrm{d}s\vec{e}_y$$

Therefore, the contributions of these two segments of the loop to the total force compensate each other, no impact on total force.

For the two remaining parts of the loop, B is constant, the relation given above simplifies to $\vec{F}_m = I_{ind}(\vec{l} \times \vec{B})$

upper part:

$$\vec{F}_{m,o} = I_{ind}\left[b\vec{e}_y \times (B_0 + k_0 z(t))(-\vec{e}_x)\right] = I_{ind} b(B_0 + k_0(a - v_0 t))\vec{e}_z$$

lower part:

$$\vec{F}_{m,u} = I_{ind}\left[b(-\vec{e}_y) \times (B_0 + k_0 z(t))(-\vec{e}_x)\right] = I_{ind} b(B_0 - k_0 v_0 t)(-\vec{e}_z)$$

total force:

$$\vec{F}_m = \vec{F}_{m,o} + \vec{F}_{m,u} = I_{ind} b k_0 a \vec{e}_z = \frac{(abk_0)^2 v_0}{R}\vec{e}_z$$