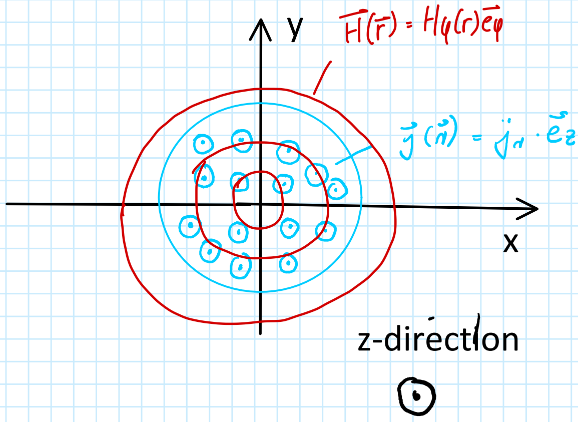
3.6.3 Magnetic field generated by cylindrically symmetric current distribution



- Current distribution $\vec{j}(\vec{r})$ in z-direction

$$\vec{j}(\vec{r}) = j(r)\cdot\vec{e}_z$$

- J varys over radius, but is radial-symmetric: $|\vec{J}| = J(r)$
- $\Rightarrow$ use cylindric coordinates $\vec{e}_r, \vec{e}_\varphi, \vec{e}_z$

* $\vec{H}(\vec{r})$ is Cylindric Symmetric as well

$$\vec{H}(\vec{r}) = H_\varphi(r)\cdot\vec{e}_\varphi$$

* Ampere's law: $\displaystyle\oint_{\partial A(r)} \vec{H}\,d\vec{r} = I(A(r)) = \int_{A(r)} \vec{J}(\vec{r})\cdot d\vec{a}$

$$d\vec{r} = r\,d\varphi\,\vec{e}_\varphi \qquad d\vec{a} = r\,dr\,d\varphi\,\vec{e}_z$$

$$\int_0^{2\pi} H_\varphi(r)\vec{e}_\varphi\cdot r\,d\varphi\,\vec{e}_\varphi = \int_0^{2\pi}\int_0^{r} J(r')\vec{e}_z\,r'dr'd\varphi\cdot\vec{e}_z$$

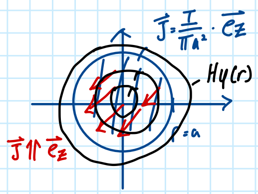
$$2\pi r H_\varphi(r) = 2\pi\int_0^r J(r')\cdot r'dr'$$

$$H_\varphi(r) = \frac{1}{r}\int_0^r J(r')\cdot r'dr'$$

$$\boxed{\vec{H}(\vec{r}) = H_\varphi(r)\cdot\vec{e}_\varphi = \frac{1}{r}\int_0^r J(r')r'dr'} \qquad (3.30)$$

Example:

$$\vec{J}(\vec{r}) = \begin{cases} \frac{I}{\pi a^2}\cdot\vec{e}_z & \text{for } 0\le r\le a \\ 0 & \text{for } r\ge a\end{cases}$$



Ampere's Law:

$$\oint_{\partial A}\vec{H}\cdot d\vec{r} = \int_{A(r)}\vec{J}\cdot d\vec{a} = I(A(r))$$

Cylindric Symmetric $\vec{J}$

$\Downarrow$

$\vec{H}$ field Cylindric Symmetric

$\Rightarrow$ Circles around z-axis

$$\oint_{\partial A}\vec{H}\cdot d\vec{r} = \int_0^{2\pi} H_\varphi\,\vec{e}_\varphi\cdot r\,d\varphi\,\vec{e}_\varphi = 2\pi r H_\varphi(r)$$

$\displaystyle\int_{A(r)}\vec{J}\cdot d\vec{a}$: inside wire $(0\le r\le a)$

$$\int_{A(r)}\frac{I}{\pi a^2}\cdot\vec{e}_z\cdot\underbrace{r\,dr\,d\varphi\cdot\vec{e}_z}_{d\vec{a}} = \int_0^{\pi}\int_0^{r}\frac{I}{\pi a^2}r'dr'd\varphi$$

$$= \frac{2\cancel{\pi}}{\cancel{\pi}a^2}I\int_0^r r'dr'$$

$$= \frac{2}{a^2}I\left[\tfrac{1}{2}r'^2\right]_0^r = \frac{2}{a^2}I\left(\tfrac{1}{2}r^2 - 0\right)$$

$$\int_{A(r)}\vec{J}\,d\vec{a} = \frac{Ir^2}{a^2}$$

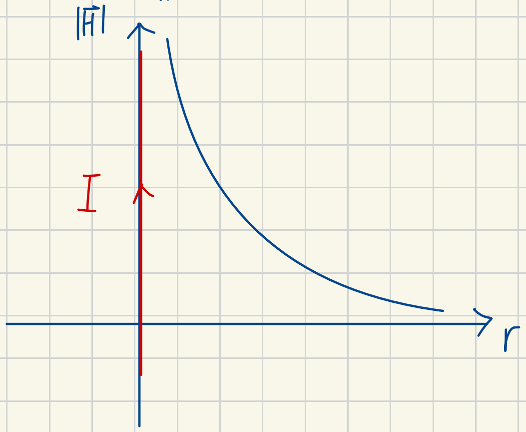
Outside wire: $r\ge a$

$$\int_{A(r)}\vec{J}\,d\vec{a} = \underbrace{\int_0^{2\pi}\int_0^{r=a}\vec{J}(r')r'dr'd\varphi}_{\frac{I\cdot a^2}{a^2} = I\ (r=a)} + \cancel{\int_0^{2\pi}\int_0^{r>a} 0\cdot r'dr'd\varphi}$$

Ampere's law: $2\pi r H_\varphi(r) = \begin{cases}\frac{Ir^2}{a^2} & \text{for } 0\le r\le a\\ I & \text{for } r>a\end{cases}$

$$H_\varphi(r) = \begin{cases}\frac{Ir}{2\pi a^2} & \text{for } 0\le r\le a\\ \frac{I}{2\pi r} & \text{for } r>a\end{cases}$$

for long, thin wire
(radius is approx. zero)



$$H_\varphi(r) = \frac{I}{2\pi r}$$

$$\vec{J} = J(r)\,\vec{e}_z = \frac{I}{\pi a^2}$$