Topic 2
Techniques of Integration

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Outline

- Integration by Substitution (Reverse Chain Rule)
- Integration by Substitution of Trigonometric Relations
- Products & Powers of Trigonometric Functions
- Using Partial Fractions in Integration
- Integration by Parts
- Numerical Integration
- Application of Integrals - Volume of Revolution

Problem with ‘Reversing’ Derivatives

In the last topic, we learnt that the fundamental theorem of calculus provides a convenient way to obtain some integrals by using the antiderivative of the integrand. However, there are some integrals that require further treatment, such as

$$F(x) = \int \tan x \, dx$$

because it is not immediately clear what is the antiderivative of **tan x**. Another example is

$$F(x) = \int \sqrt{2x+1} \, dx$$

where the integrand is a **composite function**. In many cases, it is not obvious what is the antiderivative F(x) being differentiated to get the integrand function. But we do have a few tricks which might work.

Integration by Substitution (Reverse Chain Rule)

Since a derivative can be obtained by applying the chain rule, it makes sense to assume that a 'reverse chain rule' would be needed to return the antiderivative. From the chain rule applied on a composite function F(x) = F(g(x)), we have

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = F'(g(x))\,g'(x) = f(g(x))\,g'(x)$$

where F'(x) = f(x). Integrating wrt x and making the **substitution u = g(x)** gives

$$F(x) = \int f(g(x))\,g'(x)\,dx = \int f(u)\,du \qquad \frac{du}{dx} = g'(x)$$

noting that

$$du = g'(x)dx$$

Integration by Substitution (Reverse Chain Rule)

Hence, to apply the 'reverse chain rule', we **identify the function g(x) and substitute it by u (change of variables)**, followed by **integrating wrt to u**. For example, to evaluate

$$F(x) = \int \underline{\sqrt{2x+1}}\, dx$$

we try letting **u = 2x + 1**, so **du = 2 dx → dx = ½ du**, which gives

$$F(x) = \int \sqrt{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int u^{1/2}\, du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} + c\right) = \frac{1}{3}u^{3/2} + c_1$$

Finally, substitute back **u = 2x + 1**, the integral is

$$F(x) = \frac{(2x+1)^{3/2}}{3} + c_1$$

You can verify that F'(x) equals the integrand.

Integration by Substitution (Reverse Chain Rule)

The integration by substitution method will sometimes require a bit of trial & error, because it might not be obvious what u = g(x) is.

If the variable x cannot be removed then try another u

Exercise: Evaluate the integrals below.

a) $H(x) = \int e^{\cos x} \sin x \, dx$

Let $u = \cos x \rightarrow du = -\sin x \, dx$

$dx = \frac{du}{-\sin x}$

$= \int e^u \cancel{\sin x} \frac{du}{-\cancel{\sin x}}$

$= -\int e^u \, du$

$= -e^u + C \quad = -e^{\cos x} + C$

b) $G(x) = \int 4x\left(x^2 - 1\right)^9 dx$

Let $u = x^2 - 1 \rightarrow du = 2x \, dx$

$dx = \frac{du}{2x}$

$\int \overset{2}{\cancel{4x}} (u)^9 \frac{du}{\cancel{2x}} = 2 \int (u)^9 \, du$

$= 2\left(\frac{u^{10}}{10} + C\right)$

$= \frac{(x^2-1)^{10}}{5} + C_1$

c) $F(x) = \int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

Let $u = \cos x \rightarrow du = -\sin x \, dx$

$F(x) = \int \frac{\cancel{\sin x}}{u} \left(-\frac{1}{\cancel{\sin x}} du\right)$

$= -\int \frac{1}{u} du$

$= -\ln|\cos x| + C$

or $= \ln|\cos x|^{-1} + C$

$= \ln|\sec x| + C$

ANS: a) $H(x) = -e^{\cos x} + c$. b) $G(x) = \frac{1}{5}\left(x^2 - 1\right)^{10} + c$. c) $F(x) = \ln|\sec x| + c$.

Integration by Substitution (Reverse Chain Rule)

Clearly, integration by substitution can also be applied definite integrals in the same way, but the **limits of integration have to be changed accordingly to u = g(x)**, i.e.

$$F(x) = \int_{x_1}^{x_2} f(g(x))\, g'(x)\, dx = \int_{u_1=g(x_1)}^{u_2=g(x_2)} f(u)\, du$$

because the integration has been changed to be **wrt to u, not x**.

Alternatively, one can also **back-substitute u = g(x)** after integration and **use the original limits for x**, i.e.

$$\int_{u_1=g(x_1)}^{u_2=g(x_2)} f(u)\, du = F(u)\Big|_{u_1}^{u_2} = F(g(x))\Big|_{x_1}^{x_2}$$

Integration by Substitution (Reverse Chain Rule)

Example: Evaluate the definite integral below. Is the answer expected?

$$\int_{-\pi/4}^{\pi/4} \tan x \, dx = -\ln|\cos x| \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left[-\ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(\frac{1}{\sqrt{2}}\right)\right] = 0$$

OR: $= -\ln|u| \Big|_{u=\cos(-\frac{\pi}{4})}^{u=\cos(\frac{\pi}{4})} = [\quad] = 0$

Expected, since $\tan x$ is odd, so integral over a symmetric interval is 0

ANS: 0.

Integration by Substitution (Reverse Chain Rule)

Exercise: Evaluate the following definite integrals.

a) $\displaystyle\int_{-1}^{1} \frac{t^2}{t^3+2}\, dt$

Let $u = t^3 + 2$ $\quad du = 3t^2\, dt$

$dt = \dfrac{du}{3t^2}$

$$= \int_{u=1}^{u=3} \frac{\cancel{t^2}}{u} \frac{du}{3\cancel{t^2}}$$

$$= \frac{1}{3}\int_{u=1}^{u=3} \frac{1}{u}\, du$$

$$= \frac{1}{3}\Big[\ln|u|\Big]\Big|_{u=1}^{u=3} = \frac{1}{3}\big[\ln(3) - \ln(1)\big]$$

$$= \frac{1}{3}\ln(3)$$

b) $\displaystyle\int_{0}^{1} x(1-x)^7\, dx$

Let $u = 1 - x \rightarrow du = -dx$

$dx = -du$

limits

At $x=1$, $u = 1-1 = 0$

At $x=0$, $u = 1$

$x = 1 - u$

$$I = \int_1^0 x u^7 (-du) = -\int_1^0 (1-u)u^7\, du$$

$$= \int_0^1 u^7 - u^8\, du = \left(\frac{u^8}{8} - \frac{u^9}{9}\right)\Bigg|_0^1$$

$$= \left(\frac{1}{8} - \frac{1}{9}\right) - 0$$

$$= \frac{1}{72}$$

ANS: a) ⅓ ln 3. b) 1/72.

Integration by Substitution of Trigonometric Relations

In some cases, an integral can be evaluated by substitution of trigonometric relations. Some examples are

$$\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4} + c$$

$$\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 dx = \int \frac{\sin^2 2x}{4} dx$$

$$= \frac{1}{4} \int \frac{1-\cos 4x}{2} dx = \frac{x}{8} - \frac{\sin 4x}{32} + c$$

where the double-angle formulas are used.

$2\sin(x)\cos(x) = \sin(2x) \leftarrow$

$\cos(2x) = 2\cos^2(x) - 1 \leftarrow$

$\cos(\underbrace{2x}_{2(2x)=4x}) = 1 - 2\sin^2(\underset{\substack{\uparrow \\ 2x}}{x}) \leftarrow$

Integration by Substitution of Trigonometric Relations

Exercise: Set up a definite integral representing the area of the semicircle of radius R below and evaluate it. Hence, verify the area formula of a circle.

Area of ▯, $dA = y(x)dx = \sqrt{R^2+x^2}\,dx$

$$A = \int dA = \int_{-R}^{R} \sqrt{R^2+x^2}\,dx$$

Let $x = R\cos\theta \rightarrow dx = -R\sin\theta\,d\theta$

↓ limits

At $x = R$, $\theta = 0$

At $x = -R$, $\theta = \pi$

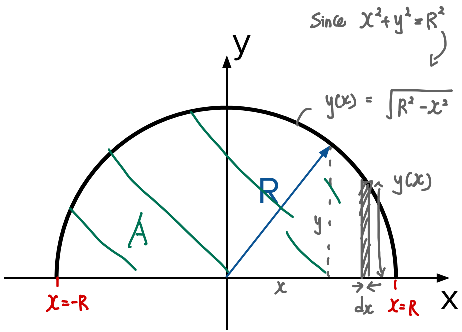
$$A = \int_{\pi}^{0} \sqrt{R^2 - R^2\cos^2\theta}\,(-R\sin\theta\,d\theta)$$

$$= -\int_{\pi}^{0} R^2 \underbrace{\sqrt{1-\cos^2\theta}}_{\sin^2\theta}\,\sin\theta\,d\theta = R^2\int_0^{\pi} \sin^2\theta\,d\theta$$

$$= R^2\int_0^{\pi} \tfrac{1}{2}\underbrace{(1-\cos 2\theta)}_{\text{check by } \frac{d}{d\theta}}\,d\theta$$

$$= \frac{R^2}{2}\left[\theta - \tfrac{1}{2}\sin(2\theta)\right]_0^{\pi}$$

$$= \frac{R^2}{2}\left[(\pi-0)-(0-0)\right] = \frac{\pi R^2}{2}$$



ANS: $\displaystyle\int_{-R}^{R} \sqrt{R^2-x^2}\,dx = \frac{\pi R^2}{2}$

Integration by Substitution of Trigonometric Relations

Notice that in the earlier exercise, the substitution of u = Rcos x is used to **rewrite the root expression for integration**. There are other root expressions where appropriate trigonometric relations can help, as tabulated below (k is a constant).

Root Expression	Substitution	Result
$\sqrt{k^2 - x^2}$	$x = k\sin\theta$ or $k\cos\theta$	$k\cos\theta$ or $k\sin\theta$
$\sqrt{k^2 + x^2}$	$x = k\tan\theta$	$k\sec\theta$
$\sqrt{x^2 - k^2}$	$x = k\sec\theta$	$k\tan\theta$

sub

$\sqrt{k^2(\underbrace{1+\tan^2\theta}_{\sec^2\theta})}$

$= k\sec\theta$

As one can observe, the selection of the substitution is such that the expression being rooted can be reduced to **a single trigonometric expression which removes the root** in the result.

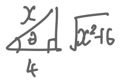
Integration by Substitution of Trigonometric Relations

Exercise: Evaluate the integral below. You may need (sec x)’ = sec x tan x.

$$\int \frac{1}{x^2\sqrt{x^2-\underbrace{16}_{k^2}}}\,dx$$

$x = \frac{4}{\cos\theta}$

$\cos\theta = \frac{4}{x}$



Let $x = 4\sec\theta \rightarrow dx = 4\sec\theta\tan\theta\,d\theta$

$$= \int \frac{1}{16\sec^2\theta\,(\cancel{4\tan\theta})}\left(\cancel{4\sec\theta}\,\cancel{\tan\theta}\,d\theta\right)$$

$$= \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int \cos\theta\,d\theta$$

$$= \frac{1}{16}\sin\theta + C = \frac{1}{16}\frac{\sqrt{x^2-16}}{x} + C$$

ANS: $\dfrac{\sqrt{x^2-16}}{16x} + c$

Products & Powers of Trigonometric Functions

Another type of integrands involves products & powers of trigonometric functions, such as

$$\int \sin^n x \cos x \, dx$$

where the substitution u = sin x (du = cos x dx) can be used to evaluate the integral, i.e.

$$\int \sin^n x \cos x \, dx = \int u^n \, du = \frac{u^{n+1}}{n+1} + c = \frac{\sin^{n+1} x}{n+1} + c$$

The same substitution approach can also be used for

$$\int \underbrace{\cos^n x}_{u^n} \underbrace{\sin x \, dx}_{-du} \quad \text{or} \quad \int \underbrace{\tan^n x}_{u^n} \underbrace{\sec^2 x \, dx}_{du}$$

where it should be clear that u = cos x & u = tan x should be used respectively.

Products & Powers of Trigonometric Functions

In cases where there are powers of trigonometric functions multiplied together in the integrand, such as

$$\int \sin^2 x \cos^3 x \, dx$$

the strategy is to use trigonometric relations to convert the expression into the form shown in the earlier slide such that substitution can be used, i.e.

$$\begin{aligned}\int \sin^2 x \underline{\cos^2 x} \cos x \, dx &= \int \sin^2 x \underline{(1 - \sin^2 x)} \cos x \, dx \\ &= \int \sin^2 x \cos x \, dx - \int \sin^4 x \cos x \, dx\end{aligned}$$

The last two integrals can be solved by the substitution u = sin x. Notice that the **'trick'** is to **factorize cos x out** and **convert the rest into powers of sin x**.

Products & Powers of Trigonometric Functions

Naturally, the same strategy can be applied to

$$\int \sin^3 x \cos^2 x \, dx$$

where now the odd power is at sin x. Hence we can write

$$\begin{aligned} \int \underline{\sin^2 x} \cos^2 x \sin x \, dx &= \int \underline{\left(1 - \cos^2 x\right)} \cos^2 x \sin x \, dx \\ &= \int \cos^2 x \sin x \, dx - \int \cos^4 x \sin x \, dx \end{aligned}$$

where the last two integrals can be solved by the substitution u = cos x. Notice that the **'trick'** is to **factorize out the function with the odd power** and **convert the rest to the other function**. The same strategy works for higher odd powers.

Products & Powers of Trigonometric Functions

Exercise: Rewrite the integrals below such that substitution can be used to evaluate them.

a) $\displaystyle\int \sin^6 x \cos^{\overset{\text{Odd}}{5}} x \, dx$

$= \displaystyle\int \sin^6 x \cdot \cos^4 x \cdot \cos x \, dx = \int \sin^6 x \underbrace{(1-\sin^2 x)^2}_{(1-2\sin^2 x + \sin^4 x)} \cdot \cos x \, dx$

$= \displaystyle\int \sin^6 x \cos x \, dx - 2\int \sin^8 x \cos x \, dx + \int \sin^{10} x \cos x \, dx$

let $u = \sin x$ to solve

b) $\displaystyle\int \sin^5 x \underbrace{\cos^3 x}_{\cos^2 x \cdot \cos x} \, dx = \int \sin^5 x (1-\sin^2 x) \cdot \cos x \, dx$

Smaller power requires less expansion

$= \displaystyle\int \sin^5 x \cdot \cos x \, dx - \int \sin^7 x \cos x \, dx$

c) $\displaystyle\int \underbrace{\sin^5 x}_{\underbrace{\sin^4 x}_{(\sin^2 x)^2} \cdot \sin x} \, dx = \int (1-\cos^2 x)^2 \cdot \sin x \, dx$

$= \displaystyle\int \sin x \, dx - 2\int \cos^2 x \sin x \, dx$

$+ \displaystyle\int \cos^4 x \cdot \sin x \, dx$

Notice that in (b), you can chose to factorize either sin x or cos x out, since both have odd powers.

Products & Powers of Trigonometric Functions

So we now know how to integrate products & powers of trigonometric functions with odd powers. Other the other hand, for integrands with only even powers such as

$$\int \sin^2 x \cos^4 x \, dx$$

the double-angle formulas can be used to rewrite the integral as

$$\int \underbrace{\sin^2 x \cos^2 x}_{\left(\frac{\sin 2x}{2}\right)^2} \cos^2 x \, dx = \int \frac{\sin^2 2x}{4} \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{1}{8} \int \sin^2 2x + \sin^2 2x \ \cos 2x \, dx = \frac{1}{16} \int 1 - \cos 4x \, dx + \frac{1}{8} \int \sin^2 2x \ \cos 2x \, dx$$

where the first integral can be integrated directly and the second can be evaluated using the substitution u = sin 2x. Hence, the **‘trick’ is use the double-angle formulas to change the even powers to odd powers** and **use the ‘tricks’ earlier**.

Products & Powers of Trigonometric Functions

Exercise: Evaluate the following integral below.

$$\int \underbrace{\cos^4 x}_{(\cos^2 x)^2}\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int 1 + 2\cos 2x + \underbrace{\cos^2(2x)}\, dx$$

$$= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2}\, dx$$

$$= \frac{1}{4}\int \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\, dx$$

Let $u = 4x$ $\quad dx = \frac{1}{4}du$

$\int \cos u \left(\frac{1}{4}du\right)$

$= \frac{1}{4}\sin u$

$\downarrow \times\frac{1}{4}$

$$= \frac{1}{4}\left[\frac{3}{2}x + \cancel{\frac{2}{2}}\sin 2x + \frac{1}{8}\sin 4x\right]$$

ANS: $\dfrac{\sin(4x)}{32} + \dfrac{\sin(2x)}{4} + \dfrac{3x}{8} + c$

Products & Powers of Trigonometric Functions

Consolidating the approaches in earlier slides, the **general guideline** for integrals of the form

$$\int \sin^i x \cos^j x \, dx$$

is as follows.

1. If **i or j is odd**, then **factorize out the function with the odd power** and **convert the rest to the other function**.
2. If **i and j are both odd**, then **factorize out $\overbrace{\text{either function}}^{\text{preferrably the one smaller odd power}}$** and **convert the rest to the other function**.
3. If **both i and j are even**, use **double-angle formulas** to **convert the integrand to satisfy cases 1 or 2**.

Using Partial Fractions in Integration

Although substitution can help evaluate integrals of rational functions, there are cases where substitution does not work. For example,

$$\int \frac{1}{x^2 - 1}\, dx$$

does not work with any substitution because **du = g'(x)dx is not present** in the integrand. Luckily, there is another technique of integration for rational functions, that is to use partial fractions. For example,

$$\int \frac{1}{x^2 - 1}\, dx = \int \frac{1}{(x-1)(x+1)}\, dx = \int \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\, dx$$
$$= \frac{1}{2}(\ln|x-1| - \ln|x+1|) + c$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$ ← Find A & B

sub x such that denom $=0$

linear factors

Method 1 : Cover - up

$$A = \frac{1}{\cancel{(x-1)}(x+1)} = \frac{1}{2}, \quad B = \frac{1}{\underbrace{x-1}_{-1}} = -\frac{1}{2}$$

* Only works for linear and highest degree of repeated factors

Method 2 : Comparing coeffs

$\times$ denom → $\underbrace{\quad}_{(x-1)(x+1)}$

$$1 = A(x+1) + B(x-1)$$
$$= (A+B)x + (A-B)$$

For LHS = RHS, comparing coeffs of :

x : $0 = A + B$ — R_1

Const : $1 = A - B$ — R_2

$R_1 - R_2 \Rightarrow 2B = -1 \Rightarrow B = -\frac{1}{2}$

$R_2 = A + 1 + B = \frac{1}{2}$

Method 3 : Substitution

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\left.\begin{array}{lll} \text{Sub } x=0 \rightarrow \dfrac{1}{(-1)(1)} = -A + B & \rightarrow & -1 = B - A \quad - R_1 \\ \text{sub } x=2 \rightarrow \dfrac{1}{(1)(3)} = A + \dfrac{B}{3} & \rightarrow & 1 = 3A + B \quad - R_2 \end{array}\right\} \text{ solve to get } A = \frac{1}{2},\ B = -\frac{1}{2}$$

Using Partial Fractions in Integration

Generally, given an integral of the form

$$\int \frac{p(x)}{q(x)}\,dx$$

where p(x) & q(x) are polynomials, we can try the following guideline.

1. Try to factorize first and simplify the rational function, if possible.
2. Try substitution, especially **if du = g'(x)dx is clearly present**.
3. If the **degree of p(x) ≥ degree of q(x)**, use **long division first** & check steps 1 & 3.
4. If the **degree of p(x) < degree of q(x)**, decompose the integrand into partial fractions.

It is possible that other techniques of integration are further required after partial fraction decomposition. We shall only deal with linear & repeated factors of partial fractions in this course.

Using Partial Fractions in Integration

Exercise: Evaluate the integral below.

$$\int \frac{x^4}{x^2 - 4}\,dx$$

$\leftarrow \deg(x^4) = 4 > \deg(x^2-4) = 2$

so use long division

$$\begin{array}{r} x^2+4 \\ x^2-4 \,\big)\, x^4 \\ -(x^4-4x^2) \\ \hline 0+4x^2 \\ -(4x^2-16) \\ \hline 16 \end{array}$$

$$= \int x^2 + 4 + \frac{16}{x^2-4}\,dx$$

$\downarrow$

Partial fractions

cover up method

$$\frac{16}{(x+2)(x-2)} = \frac{16/(-2-2)}{x+2} + \frac{16/(2+2)}{x-2}$$

$$= \int x^2 + 4 + \frac{-4}{x+2} + \frac{4}{x-2}\,dx = \frac{x^3}{3} + 4x - 4\ln|x+2| + 4\ln|x-2| + C$$

ANS: $4\ln|x-2| - 4\ln|x+2| + \frac{x^3}{3} + 4x + c$

Using Partial Fractions in Integration

Exercise: Evaluate the integral below.

$$\int \frac{x+1}{x^3 - 6x^2 + 9x}\, dx$$

$$\hookrightarrow \frac{x+1}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

Multiply by $x(x-3)^2$

$$x+1 = A\underbrace{(x-3)^2}_{x^2-6x+9} + Bx(x-3) + Cx$$

$$= x^2(A+B) + x(-6A-3B+C) + 9A$$

Compare coeff for :

Const : $1 = 9A \rightarrow A = \frac{1}{9}$

x^2 : $0 = A + B$

$B = -\frac{1}{9}$

x : $1 = -6A - 3B + C \rightarrow C = 1 + 6A + 3B$
$= 1 + \frac{6}{9} - \frac{3}{9} = \frac{12}{9} = \frac{4}{3}$

$$\Rightarrow I = \int \frac{1}{9x} - \frac{1}{9(x-3)} + \frac{4}{3(x-3)^2}\, dx$$

$$= \frac{1}{9}\ln|x| - \frac{1}{9}\ln|x-3| + \frac{4}{3}\left(\frac{-1}{x-3}\right) + C$$

ANS: $\frac{1}{9}\ln|x| - \frac{1}{9}\ln|x-3| - \frac{4}{3(x-3)} + c$

Integration By Parts

$\frac{dv}{dx} = v'$

Recall that when differentiating two functions, say u(x) & v(x), multiplied together, we use the product rule, i.e.

$$(uv)' = uv' + vu'$$

where the independent variable x has been omitted for notational simplicity. If we integrate the above, we get

$$uv = \int uv'dx + \int vu'dx$$

Since **v'dx = dv** and **u'dx = du**, the above can be simplified and rearranged as

$$uv = \int u\,dv + \int v\,du \rightarrow \int u\,dv = uv - \int v\,du$$

The last relation is called integration by parts, which can be used to integrate two functions multiplied together.

Integration By Parts

For example, evaluate

$$\int x \sin x \, dx$$

Firstly, integration by substitution will not work, because letting u = x results in the same integral, and u = sin x cannot provide a complete change of variables. Hence, we can try integration by parts since the integrand is a multiplication of two functions. We can let

$$\underline{u = x}, \ \underline{dv = \sin x \, dx}$$

and so

$$\underline{du = dx}, \ \underline{v = -\cos x}$$

where the constant of integration for v is omitted because it will be cancelled in the end as shown in the next slide.

Integration By Parts

The integration by parts formula applied here gives

$$\int u\,dv = uv - \int v\,du$$

$$\rightarrow \int x \sin x\,dx = \underline{-}\underline{x}\,\underline{\cos x} - \int \underline{-\cos x}\,\underline{dx} = -x\cos x + \sin x + c$$

One can easily verify that the derivative of the above result gives the original integrand, so it is correct. Notice that if v = -cos x + c_1, we would have

$$\begin{aligned}\int x \sin x\,dx &= x(-\cos x + c_1) - \int -\cos x + c_1\,dx \\ &= -x\cos x + c_1 x + \sin x - c_1 x + c \\ &= -x\cos x + \sin x + c\end{aligned}$$

which is the same. So **omit the constant in v(x) for simplicity**.

Integration By Parts

Now, the obvious question is, **how do we choose u & dv**? In the last example, if we had swapped our selections, then we have

$$u = \sin x,\ dv = x\,dx$$

$$\rightarrow du = \cos x\,dx,\ v = \frac{x^2}{2}$$

And integration by parts gives

$$\int x \sin x\,dx = \frac{x^2 \sin x}{2} - \int \frac{x^2 \cos x}{2}\,dx$$

where the integral on the RHS looks more complicated than the original integral. Therefore, **some trial and error is usually required on the selection of u & v**, just like integration by substitution. However, there is a general guideline that might work.

Integration By Parts - LIATE

Generally, we can choose the type of elementary function for u according to the ranking (from left to right):

L (logarithmic) **I** (inverse trigo) **A** (algebraic) **T** (trigo) **E** (exponential)

For example, in the last integral, the selection u = x (algebraic) follows the above ranking because sin x is a trigonometric function which is behind algebraic functions. Another example shown below chooses u = ln x because log functions are higher in ranking compared to algebraic functions.

$$\int \underbrace{x \, \underbrace{\ln x}_{u} \, dx}_{dv} = \underbrace{\frac{x^2}{2}}_{v} \underbrace{\ln x}_{u} - \int \underbrace{\frac{x^2}{2}}_{v} \underbrace{\left(\frac{1}{x} dx\right)}_{du} = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

Integration By Parts - LIATE

The LIATE (or ILATE) ranking of choosing the function for u follows logically from the **'ease of integration'**. From the table of integrals in Topic 1, notice that we **do not have 'easy' antiderivatives defined for logarithmic and inverse trigonometric functions**, so LI (or IL) should be selected as u first (for differentiation).

And, also considering that **du should be simpler than u**, we should have algebraic functions ranked higher than trigonometric and exponential functions, because the **latter two will remain no matter how many times they are differentiated**.

So the ranking of LIATE (or ILATE) is logical. But, keep in mind that in a few cases it does not work, so some trial and error should still be expected (see following exercises).

Integration By Parts

Exercise: Evaluate the integrals below.

a) $\int xe^x \, dx$

(A) (E)

Let $u = x \to du = dx$

$dv = e^x dx \to v = e^x$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C$

b) $\int x \tan^{-1} x \, dx$

(A) (I)

Let $u = \tan^{-1}x \to du = \frac{1}{x^2+1} dx$

$dv = x dx \to v = \frac{x^2}{2}$

$\frac{x^2}{2}\tan^{-1}x - \int \underbrace{\frac{x^2}{2}\left(\frac{1}{x^2+1}\right)}_{\frac{1}{2}\frac{x^2+1-1}{x^2+1}} dx$

or by long division

$= \frac{1}{2}\left(1 - \frac{1}{x^2+1}\right)$

$\frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int 1 - \frac{1}{x^2+1} dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) + C$

ANS: a) $(x-1)e^x + c$. b) $\dfrac{(x^2+1)\tan^{-1}x - x}{2} + c$

Integration By Parts

No.

Exercise: Evaluate the integral below. Does the LIATE ranking work?

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

(E)

(A)

Let $u = \frac{x}{(1+x)^2}$, $dv = e^x dx \rightarrow \ldots$ Cannot integrate

Let $u = xe^x \rightarrow du = (e^x + xe^x)\,dx$

$= e^x(1+x)\,dx$

$dv = \frac{1}{(1+x)^2}dx \rightarrow v = \frac{-1}{1+x}$

$= \frac{-xe^x}{1+x} - \int \left(\frac{-1}{\cancel{1+x}}\right) e^x \cancel{(1+x)}\,dx$

$= \frac{-xe^x}{1+x} + e^x\left(\frac{1+x}{1+x}\right) + C$

$\Rightarrow \frac{-xe^x + e^x + xe^x}{1+x} + C$

$= \frac{e^x}{1+x} + C$

ANS: $\frac{e^x}{x+1} + c$

Integration By Parts

When the integrand is composed of **trigonometric and exponential functions multiplied** together, integration by parts **twice** might work as shown in the example below.

$$\int \underbrace{e^x}_{dv} \underbrace{\sin x}_{u} \, dx = e^x \sin x - \int e^x \underbrace{\cos x}_{u} \, dx$$

$$= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right)$$

Combining the original integral on both sides gives

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) + c$$

$$\rightarrow \int e^x \sin x \, dx = \frac{e^x (\sin x - \cos x)}{2} + c_1$$

Integration By Parts

Integration by parts can also be used to evaluate the antiderivative of some elementary functions, such as

$$\int \underbrace{\ln x}_{u}\, \underbrace{dx}_{dv} = x \ln x - \underbrace{\int 1\, dx}_{\int x\left(\frac{1}{x}\right)dx}$$

$$= x \ln x - x + c$$

noting that we choose $\underbrace{\textbf{u = ln x}}_{du = \frac{1}{x}dx}$ and $\underbrace{\text{dv = dx}}_{v = x}$ since we only **know how to differentiate ln x** before the result.

Exercise: Making use of the above result, state the antiderivative of $\log_a x$.

$$\int \log_a x\, dx = \int \frac{\ln x}{\ln a}\, dx = \frac{1}{\ln a}\int \ln x\, dx = \frac{1}{\ln a}(x \ln x - x) + C$$

Integration By Parts - Tabular Method

As seen earlier, in some integrals, integration by parts needs to be **applied multiple times**. For example, we can expect that

$$\int x^3 \sin x \, dx$$

would require integration by parts **thrice** in order for u = x^3 to be differentiated to a constant such that the integration of the standalone trigonometric function can be performed directly, i.e.

$$\begin{aligned}\int x^3 \sin x \, dx &= -x^3 \cos x - \int -3x^2 \cos x \, dx \\ &= -x^3 \cos x + 3x^2 \sin x - \int 6x \sin x \, dx \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - \int 6 \cos x \, dx\end{aligned}$$

Int. by parts 3 times

Integration By Parts - Tabular Method

The **tabular method** of performing integration by parts typically speeds up the process and reduces potential mistakes. In the earlier example, we let u = x^3 and dv = sin x dx. So we create a **table**:

1. Differentiate u until you get 0 or vdu can be integrated directly.

	u	dv
$+$	x^3	$\sin x$
$-$	$3x^2$	$-\cos x$
$+$	$6x$	$-\sin x$
$-$	6	$\cos x$
$+$	0	$\sin x$

2. Integrate dv until the same row as the u-column.

3. Multiply the functions along the diagonal with the sign and sum them to get the integral.

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$$

Integration By Parts - Tabular Method

Another example shown below is $\int (4x^3 - 2x) \ln(3x)\, dx$. Using the **tabular method** gives:

$$\begin{array}{ccc} & u & dv \\ + & \ln(3x) & 4x^3 - 2x \\ - & \underbrace{\frac{1}{x}}_{\textbf{du}} & \underbrace{x^4 - x^2}_{\textbf{v}} \end{array}$$

Stop differentiating further because vdu can be integrated directly.

$$vdu = \frac{x^4 - x^2}{x} = x^3 - x$$

$$\begin{aligned} \int (4x^3 - 2x) \ln(3x)\, dx &= (x^4 - x^2) \ln(3x) - \int x^3 - x\, dx \\ &= (x^4 - x^2) \ln(3x) - \frac{x^4}{4} + \frac{x^2}{2} + c \end{aligned}$$

Most of the time, using the **tabular method works well and is recommended**.

Integration By Parts - Tabular Method

Exercise: Evaluate the integral below.

$$\int \underbrace{(2x-1)^2}_{u} \underbrace{e^{-x}\,dx}_{dv}$$

	u	dv
$+$	$(2x-1)^2$	e^{-x}
$-$	$2(2x-1)(2)$	$-e^{-x}$
$+$	8	e^{-x}
	0	$-e^{-x}$

$\frac{d}{dx}$

$= (2x-1)^2 e^{-x} - 4(2x-1)e^{-x} - 8e^{-x} + C$

$= -e^{-x}(4x^2 + 4x + 5) + C$

ANS: $-\left(4x^2+4x+5\right)\mathrm{e}^{-x}+c$

Numerical Integration

Despite all the techniques of integration being covered (which is not exhaustive), there are functions that cannot be integrated analytically and expressed in elementary functions. Some examples are

$$\int \frac{1}{\ln x} dx, \quad \int \sin x \ln x \, dx, \quad \int e^{-x^2} dx$$

The last integral is in fact an important one that is related to **evaluating probabilities under a normal distribution**, which is used to **describe many phenomena statistically**.

Although an analytical antiderivative is not available, we can always depend on the **first-principles of integration, the Riemann sum**. The process of computing an integral numerically is called numerical integration.

Numerical Integration

Recall that the **Riemann sum approximation** of a definite integral is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x, \quad \Delta x = \frac{b-a}{n}$$

where x_i^* is any point in $[x_{i-1}, x_i]$. The algorithm for computing the **Riemann sum** (RS) using a **constant Δx** is

1. Define a, b and n. Initialize x_i = a.
2. Define function f(x_i*).
3. Loop from i = 1 to n to sum up f(x_i*).
4. Return RS = f(x_i*)(b-a)/n.

Clearly, a **larger n will result in a more accurate approximation of the integral**, albeit at a higher computational cost.

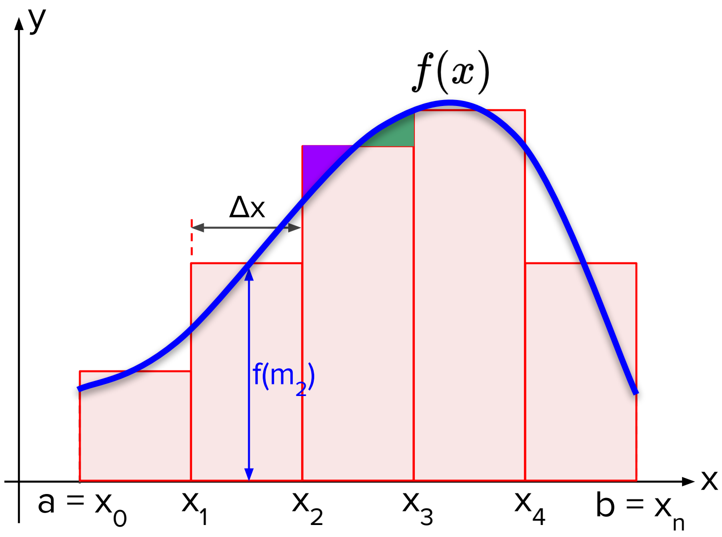
Numerical Integration - Midpoint Rule

Besides using the left endpoint x_{i-1} or the right endpoint x_i as x_i^*, a **better approach is to use the midpoint**

$$m_i = x_i - \frac{\Delta x}{2}$$

which gives the integral approximation as

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(m_i)\Delta x$$



As observed from the graph, using **m_i generally gives a better approximation** (especially for smaller n) because the 'extra' area on one side will tend to balance off the 'deficit' area on the other.

Numerical Integration - Midpoint Rule

Exercise: From the last example, evaluate the exact integral below to 3 decimals.

Compute the Riemann sum using n = 100 using the $\underbrace{\text{right endpoint}}_{4.764}$ and the $\underbrace{\text{midpoint rule}}_{4.790}$.

Which is more accurate?

$$\int_0^7 (2x-1)^2 e^{-x}\,dx = -e^{-x}(4x^2+4x+5)\Big|_0^7 = 4.791$$

ANS: Exact 4.791. Right 4.764. Midpoint 4.790.

Numerical Integration - Trapezoidal Rule

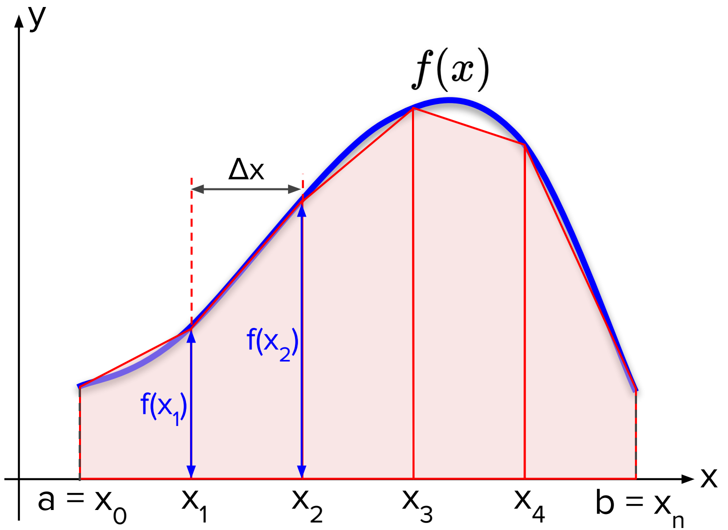
$\frac{1}{2}(a+b)h$

Another variation of numerical integration uses **trapeziums** instead of rectangles to approximate the area as shown. The integral approximation is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x$$

$$= \frac{f(x_0) + f(x_n)}{2} + \sum_{i=1}^{n-1} f(x_i) \Delta x$$

right endpoint



As observed from the graph, the **trapezoidal rule** overestimates the area where f(x) is convex and underestimates the area where f(x) is concave. In contrast, the **midpoint rule balances out the over- & under-estimations** regardless whether f(x) is convex or concave, and therefore is **generally more accurate**.

Numerical Integration - Trapezoidal Rule

Exercise: Continuing from the last exercise, compute the integral approximation using n = 100 using the $\underbrace{\text{trapezoidal rule}}_{4.793}$ and compare its accuracy to the $\underbrace{\text{midpoint rule}}_{4.790}$.

more accurate

$$\int_0^7 (2x-1)^2 e^{-x}\,dx$$

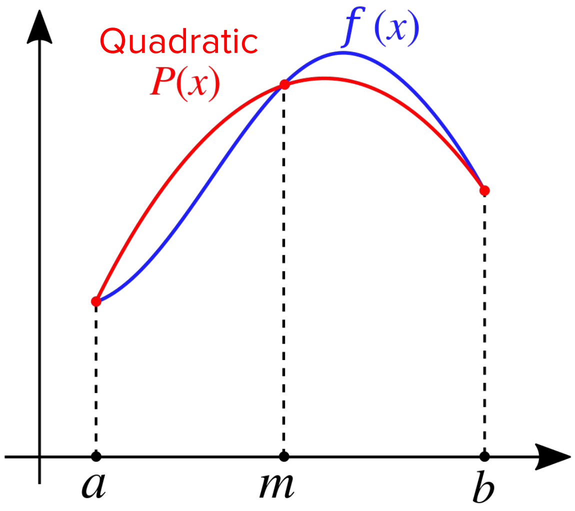
ANS: Exact 4.791. Right 4.764. Midpoint 4.790. Trapezoidal 4.793.

Numerical Integration

Besides the two methods of numerical integration that use rectangles (midpoint rule) and trapeziums (trapezoidal rule) for area approximation, there are a **myriad of other 'shapes'** that can be considered.

For example, the **Simpson's rule(s)** uses a parabola (quadratic function) to approximate f(x) as shown. The accuracy is improved, but at the expense of complexity.

We shall leave the student to self-explore further if interested.

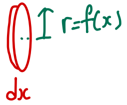


https://en.wikipedia.org/wiki/Simpson%27s_rule

Application of Integrals - Volume of Revolution

Integration can be used to evaluate the volumes of solids formed by revolution about an axis. From the figure, we can see that a disc of infinitesimal thickness dx has the volume

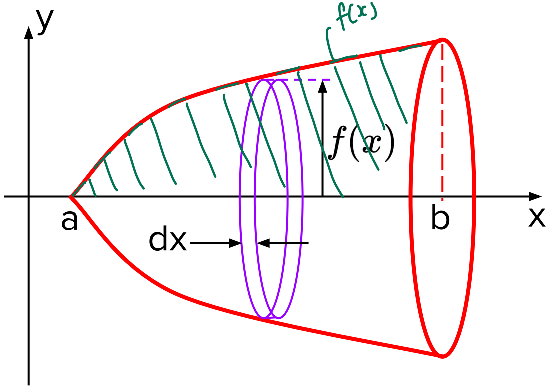
$$dV = \pi \underbrace{f^2(x)}_{r^2}\, \underbrace{dx}_{\text{thickness}}$$



where **f(x) represents the radius of the disc**. Hence, the volume of the solid from x = a to x = b is

$$V = \int_0^V dV = \pi \int_a^b f^2(x)\, dx$$

which represents a sum (integral) of the volumes of all discs in the interval [a, b].



Application of Integrals - Volume of Revolution

Example: By integration of 'discs', show that the volume of a cone with base radius r and height h is

$$V = \frac{1}{3}\pi r^2 h$$

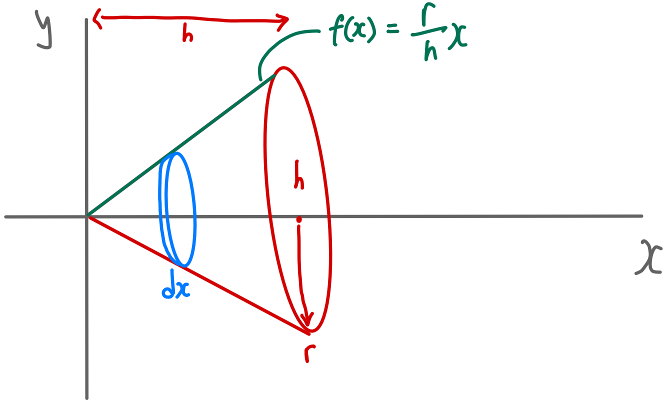
For one disc,

$$dV = \pi f^2(x)\,dx = \pi \frac{r^2}{h^2} x^2\,dx$$

$$\therefore \text{Vol of cone, } V = \int_0^V dV = \int_{x=0}^{x=h} \pi \frac{r^2}{h^2} x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\int_0^h x^2\,dx = \frac{\pi r^2}{h^2}\left(\frac{x^3}{3}\right)\Big|_0^h = \frac{\pi r^2}{h^2}\left(\frac{h^3}{3}\right)$$

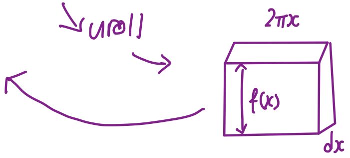
$$= \frac{1}{3}\pi r^2 h$$



Application of Integrals - Volume of Revolution

Besides 'summing discs' along the axis of the solid, we can also sum up hollow cylinders instead as shown in the figure. The volume of a hollow cylinder with infinitesimal thickness dx is

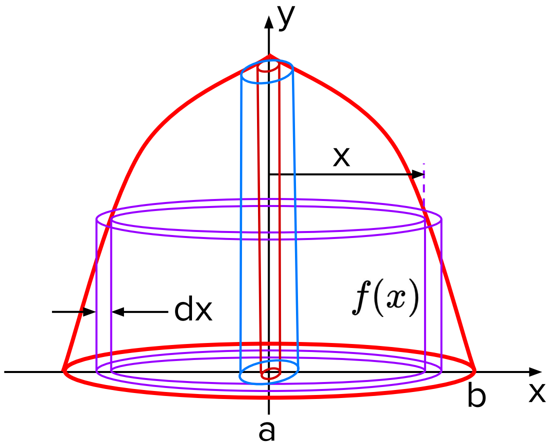
$$dV = 2\pi x f(x)\, dx$$



where **x & f(x) represent the (inner) radius & height of the hollow cylinder** respectively. The volume of the solid is then

$$V = \int_0^V dV = 2\pi \int_a^b x f(x)\, dx$$

which represents a sum (integral) of the volumes of all hollow cylinders in the interval [a, b].



Application of Integrals - Volume of Revolution

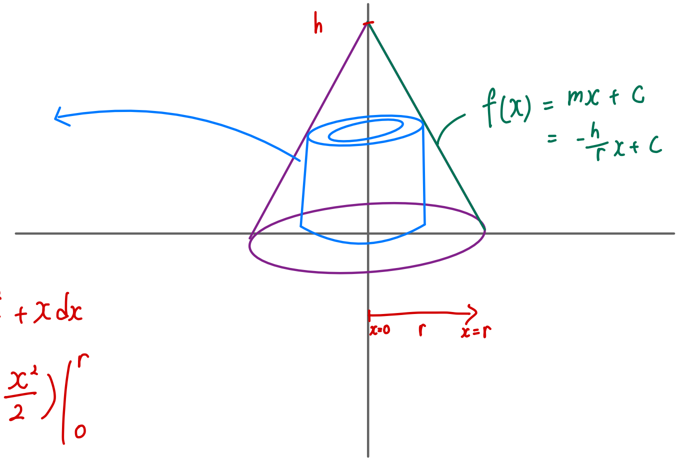
Example: By integration of 'hollow cylinders', show that the volume of a cone with base radius r and height gives the same formula as before. What does this similarity imply about using integration to find volumes?

$$V = \frac{1}{3}\pi r^2 h$$

For one cylinder,

$dV = 2\pi x f(x) dx$

$= 2\pi x\left(-\frac{h}{r}x + h\right) dx$



$\Rightarrow$ Vol of cone, $V = \int_0^V dV = \int_{x=0}^{x=r} 2\pi x\left(-\frac{h}{r}x+h\right)dx = 2\pi h\int_0^r -\frac{x^2}{r} + x\,dx$

$= 2\pi h\left(\frac{-x^3}{3r} + \frac{x^2}{2}\right)\Big|_0^r$

$= 2\pi h\left(\frac{-r^3}{3r} + \frac{r^2}{2}\right)$

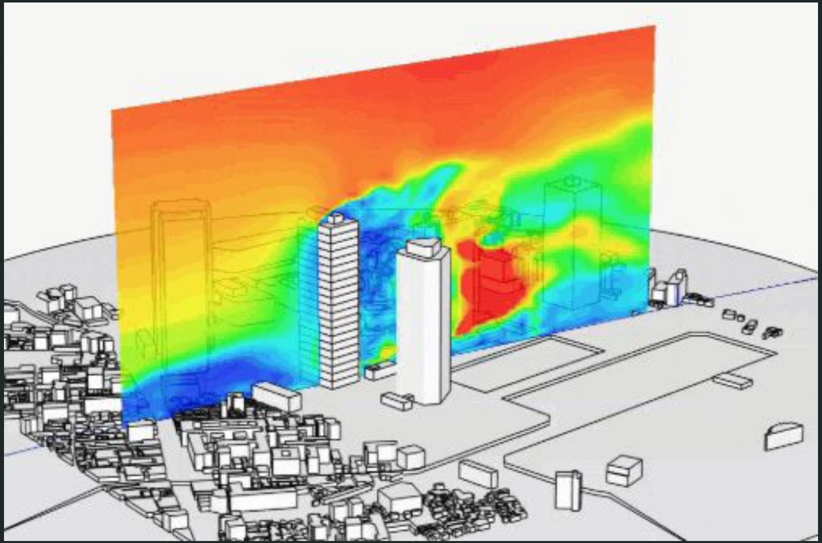
$= 2\pi h\left(\frac{r^2}{6}\right) = \frac{1}{3}\pi r^2 h$

End of Topic 2

God does not care about our mathematical difficulties.
He integrates empirically.

Albert Einstein, 1942.

Airflow Simulation across Buildings for Passive Cooling.
Flow Velocities Solved by **Numerical Integration**.



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