Relations

Relation : sub-set of a Cartesian product of multiple sets $A_1 \times A_2 \times \ldots \times A_n$

→ set of ordered n-tuples $(x_1, x_2) \neq (x_2, x_1)$

→ each tuple consists of n elements

We mostly consider $n = 2$

· Each relation is a set

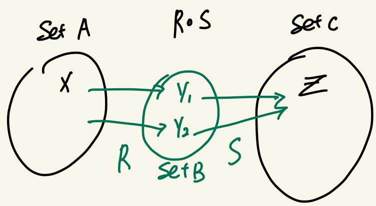
⇒ Relationship between predicate Logic and sets is also valid for relations

Two additional operations

1) Conversion : $(x, y) \in R^{-1} \iff xRy \iff yRx$, $R \subseteq A \times B$

2) Composition : $(x, y) \in R \circ S \iff x(R \circ S)z \iff \exists_{y \in B}\, xRy \wedge ySz$

with $R \subseteq A \times B$, $S \subseteq B \times C$, $x \in A$, $y \in B$, $z \in C$



Properties of Relations ; R is , $R \subseteq A^2$

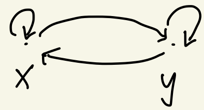
reflexive $\forall_{x \in A} xRx$

every node has a loop

irreflexive $\forall_{x \in A} \neg xRx$

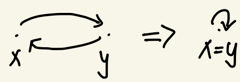
no node has a loop

Symmetric $\forall_{x \in A} xRy \rightarrow yRx$



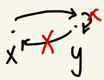
symmetric allow loops but are not neccessary

antisymmetric $\forall_{x,y \in A} [xRy \wedge yRx \rightarrow x=y]$



no arrow has a reverse arrow, loops are allowed

asymmetric $\forall_{x,y \in A} [xRy \rightarrow \neg yRx]$

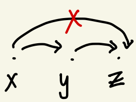


no arrow has a reverse arrow, no node has a loop

transitive $\forall_{x,y,z \in A} [xRy \wedge yRz \rightarrow xRz]$

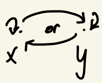
all consecutive arrows have a bypass arrow

intransitive $\forall_{x,y,z \in A} [xRy \wedge yRz \rightarrow \neg xRz]$



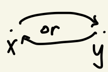
no consecutive arrows have a bypass arrow

Connex $\forall_{x,y \in R} [xRy \vee yRx]$



every node is connected to all other nodes as well as to itself (loops)

SemiConnex $\forall_{x,y \in A} [(x \neq y) \rightarrow xRy \vee yRx]$



X and y are different nodes

every node is connected to all other nodes, but loops are not necessary

Adjacency Matrix

Row x: Contains a 1 in column y, IFF y is a successor of x

Colum y: Contain a 1 in row x, IFF x is a predecessor of y

$$\begin{tikzcd} x \arrow[r, bend left] & y \end{tikzcd} \qquad \underset{\sim}{R} = \begin{array}{c c} & \cdots\, y \,\cdots \\ \vdots\; x \;\vdots & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{array} \iff (x,y) \in R \iff xRy$$

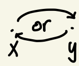
closure of relations

$c(R)$: Closure of R : smallest relation $c(R)$ that obtains property C
by adding edges/tuples to R

C : r(eflexive), s(ymmetric), t(ransitive)

Why is there no irreflexive closure?
↳ remove arrows

Why is there no connex closure?
↳ not unique



Laws N (p 68/69)

Warshall Algorithm to Compute R^+ in $G = (A, R)$

$i = 1, \ldots, n$, $n = |A|$

$i := 1$

$R_0 := R$

While $i \leq n$

$\Gamma^-(x_i) := (R_{i-1})^{-1}(x_i)$ (set of predecessors of x_i)

$\Gamma^+(x_i) := (R_{i-1})(x_i)$ (set of successors of x_i)

$R_i := R_{i-1} \cup \Gamma^-(x_i) \times \Gamma^+(x_i)$

$i := i + 1$

$R^+ := R_n$

Order Relations

order Relations

a) Partial order Relation

properties:

- reflexive
- antisymmetric
- transitive

$\Big\}$ directional order

b) Strict order relation:

- asymmetric
- trancitive

c) total order relation:

- partial order relation
- Connex

d) well order:

- total order relation
- every non-empty subset Contains a least element

Hasse - Diagram

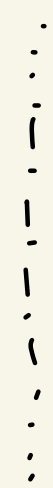
$G = (A, H)$

$H = (R \setminus I) \setminus (R \setminus I)^2$

$R = H^* = tr(H)$

$H = R \setminus R^2$

$R = H^+ = t(H)$



must not go on infinitely downwards

Equivalence Relations

- Properties :
  - reflexive : $r(B) = R$
  - symmetric : $S(B) = R$
  - transitive : $t(B) = R$
- Equivalence Closure $ECL(Q)$:
  - the smallest equivalence relation which contains Q

$$Q \subseteq ECL(Q) \subseteq A^2$$

$$ECL(Q) = tsr(Q) = (Q \cup Q^{-1})^*$$

- Equivalence core : $ECO(Q^*)$

the largest equivalence relation which is contained in Q^* : $ECO(Q^*) \subseteq Q^*$

Partition Π

Π is partition of $A \neq \emptyset$, $\Pi \in P(A)$

$$\Pi = \{ K \in P(A) \mid K \in \Pi \} \iff \begin{array}{l} \forall_{K \in \Pi} K \neq \emptyset \\ \forall_{K \in \Pi} K \neq L \to K \cap L = \emptyset \\ \left(\bigcup_{K \in \Pi} K \right) = A \end{array}$$

$A = \{2, 3, 4, 5, 6\}$, $\Pi_1 = \{ \{2,3\}, \{4,6\}, \{5\} \}$

$\Pi_2 = \{ \{2, 3, 4, 5, 6\} \}$; $\Pi_3 = \{ \{2\}, \{3\}, \{4\}, \{5\}, \{6\} \}$

4.1 Transform the following expressions given in component free notation into an expression in predicate logic component notation.

4.1.a $R \subseteq S^{-1} \cup (T \setminus S), \qquad R,\, S,\, T \subseteq A^2$

Pg 50

$$\Leftrightarrow \forall_{(x,y)\in A^2} (xRy \to ySx) \vee (T \setminus S)$$

J3

$$\Leftrightarrow \forall_{(x,y)\in A^2} (xRy \to ySx) \vee (T \cap \overline{S})$$

$$\Leftrightarrow \forall_{(x,y)\in A^2} (xRy \to ySx) \vee xTy \wedge ySx$$

$R \subseteq S^{-1} \cup (T \setminus S)$ $\qquad R, S, T \subseteq A^2 = A \times A$

$$\Leftrightarrow \forall_{(x,y)\in A^2} (x,y) \in R \to (x,y) \in (S^{-1} \cup (T \setminus S))$$

$$\Leftrightarrow \forall_{(x,y)\in A^2} xRy \to x(S^{-1} \cup T \setminus S)y$$

$$\Leftrightarrow \forall_{(x,y)\in A^2} xRy \to xS^{-1}y \vee x(T \cap \overline{S})y$$

$$\Leftrightarrow \forall_{(x,y)\in A^2} xRy \to ySx \vee (xTy \wedge x\overline{S}y)$$

$$\Leftrightarrow \forall_{(x,y)\in A^2} xRy \to ySx \vee (xTy \wedge \neg(xSy))$$

4.2 Transform the following expressions given in predicate logic component notation into component free notation without any predicate logic expressions.

Work from inner bracket to outer

4.2.a $\forall_{x,y \in A} \left[xRy \longleftrightarrow \left(\exists_{z \in A} (xSz \wedge zSy) \right) \wedge \neg xTy \right]$

$\forall_{x,y \in A} \Leftrightarrow x \in A,\ y \in A$

$(x,y) \in A^2 = A \times A$

$\forall_{(x,y) \in A^2} \left[xRy \longleftrightarrow \left(\exists_{z \in A} (xSz \wedge zSy) \right) \wedge \neg xTy \right]$

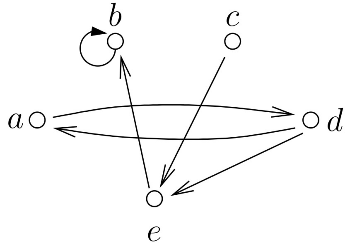
$\Leftrightarrow \forall_{(x,y) \in A^2} \left[xRy \longleftrightarrow x\, S \circ S\, y \wedge \neg xTy \right]$

$\Leftrightarrow \forall_{(x,y) \in A^2} \left[xRy \longleftrightarrow x S^2 y \wedge x \bar{T} y \right]$

$\Leftrightarrow \forall_{(x,y) \in A^2} \left[xRy \longleftrightarrow x (S^2 \cap \bar{T}) y \right.$

$\Leftrightarrow R = S^2 \setminus T$

4.3 Consider the arrow diagram of a homogeneous relation R:



	Set / number of successors		set / number of predecessors	
x	$\Gamma^+(x)$	$d^+(x)$	$\Gamma^-(x)$	$d^-(x)$
a	$\{d\}$	1	$\{d\}$	1
b	$\{b\}$	1	$\{b,e\}$	2
c	$\{e\}$	1	$\{\}$	0
d	$\{a,e\}$	2	$\{a\}$	1
e	$\{b\}$	1	$\{c,d\}$	2
	end-nodes of outgoing edges		start-nodes of incoming edges	

4.3.a Which of the nodes in R are initial, which are terminal?

x is initial $\Leftrightarrow \Gamma^-(x) = \emptyset$, c is initial no predecessors

x is terminal $\Leftrightarrow \Gamma^+(x) = \emptyset$, no node is terminal no successors

4.3.b Name all branch and reconvergence points.

b) x is branchpoint $\Leftrightarrow d^+(x) \geq 2$, d is branch point

c) x is reconvergence point $\Leftrightarrow d^-(x) \geq 2$, b,e are reconvergence point

4.3.c Does R have loops?

R has no loops $\Leftrightarrow R \subseteq \overline{I}$ (Complementary Identity Relation)

$$\underset{\sim}{I} = \begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 1 & & & & \\ & 1 & & \text{\Large 0} & \\ & & 1 & & \\ & \text{\Large 0} & & 1 & \\ & & & & 1 \end{bmatrix} \end{array}$$

$$\underset{\sim}{\overline{I}} = \begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & & & & \\ & 0 & & 1 & \\ & & 0 & & \\ & 1 & & 0 & \\ & & & & 0 \end{bmatrix} \end{array}$$

Counter example : $(b,b) \in R$ and $(b,b) \in I$

$\Rightarrow (b,b) \in \overline{I}$

Therefore $R \nsubseteq \overline{I}$ and R has a loop

$(b,b) \Leftrightarrow$ 

$\in R$

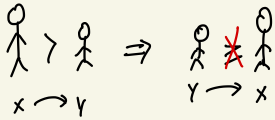
Graph of I

4.8 Examine the following relations and find out, which of the properties discussed in the manuscript they actually exhibit. The basic set is the set of all human beings.

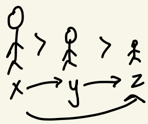
4.8.e $R_5 \iff$ "is at least as tall as"

$xRy \iff$ "x is at least as tall as y"

- reflexive: ✓ irreflexive: ✗
- symmetric: ✗



- anti symmetric: ✓ (Assumption: 2 persons have never exactly the same height)
- asymmetric: ✗ (because we have loops)
- transitive: ✓



- intransitive: ✗
- connex:
- semiconnex:

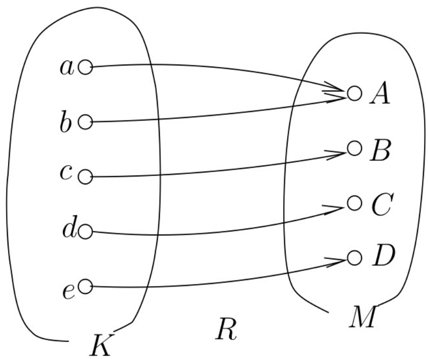
} You can always compare any 2 persons with respect to their height

$\boxed{\textbf{4.10}}$ Assume a set of children $K = \{a, b, c, d, e\}$, and a set of mothers $M = \{A, B, C, D\}$ and a relation $R \subseteq K \times M$ with

$$R = \{(x, y)|\ \ x \text{ is child of mother } y\}_{K\times M}.$$

The graph $G = (K \cup M, R)$ looks as follows:

$G = (K \cup M, R)$:



4.10.a Determine the adjacency matrices of R and R^{-1}.

$$\underset{\sim}{R} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

5 rows $\triangleq$ 5 childs domain

4 columns $\triangleq$ 4 mothers codomain

$$\underset{\sim}{R}^{-1} = \underset{\sim}{R}^{T} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

4.10.b Determine the adjacency matrices of RR^{-1} and of $R^{-1}R$.
How can these relations be interpreted?
Derive properties of R and R^{-1} from that.

$$\underset{\sim}{R}\circ R^{-1} = \underset{\sim}{R} \underset{\text{binary}}{\hat{\cdot}} \underset{\sim}{R}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Conduct the matrix multipliation as usual, but if result is ≥ 2, dip it to 1

$$I_K \subseteq RR^{-1}$$

Interpretation of $R \cdot R^{-1}$:

$$R \cdot R^{-1} = \{(x,y) \mid \exists_{z \in M} [(xRz) \wedge (zR^{-1}y)]\}_{K^2}$$

$$= \{(x,y) \mid \exists_{z \in M} [(xRz) \wedge (yRz)]\}_{K^2}$$

$$= \{(x,y) \mid \text{„}x \text{ and } y \text{ have a common successer in } R\text{"}\}_{K^2}$$

$$= \{(x,y) \mid \text{„}x \text{ and } y \text{ have a common mother"}\}_{K^2}$$

$$\underset{\sim}{R^{-1}\circ R} = \underset{\sim}{R^{T}} \cdot \underset{\sim}{R} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} A \\ B \\ C \\ D \end{matrix} \\ \begin{matrix} A & B & C & D \end{matrix} & \end{array} \iff I_M$$

Interpretation:

$$R^{-1}\circ R = \{(x,y) \mid \exists_{z\in K} [x R^{-1} z \wedge z R y]\}_{M^2} = \{(x,y) \mid \exists_{z\in K} [z R x \wedge z R y]\}_{M^2}$$

$$= \{(x,y) \mid \text{" x and y have a common predecessor } z \text{ "}\}_{M^2}$$

$$= \{(x,y) \mid \text{" x and y have common children "}\}_{M^2}$$

Properties (formulary p63/64 heterogeneous relations)

$I_K \subseteq RR^{-1} \iff R$ is <u>total</u>

(Law L6) $I_K \subseteq (RR^{-1})^{-1} = (R^{-1})^{-1}\circ R^{-1} \iff R^{-1}$ is surjective

· $I_M = R^{-1}\circ R \iff I_M \subseteq R^{-1}R \wedge R^{-1}R \subseteq I_M$

$\iff R$ is <u>surjective</u> AND R is <u>functional</u>

$\iff R^{-1}$ is total AND injective

R is <u>function</u>

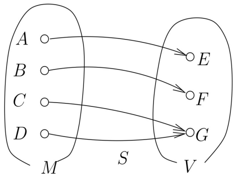
R is <u>surjective</u>

In addition, a relation

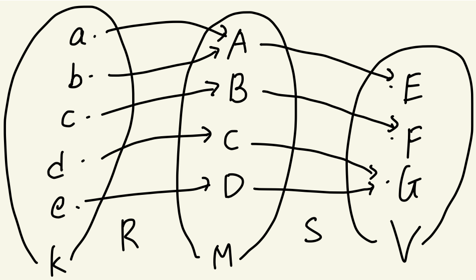
$$S = \{(x, y) | \text{ mother } x \text{ and father } y \text{ have common children } \}_{M\times V}$$

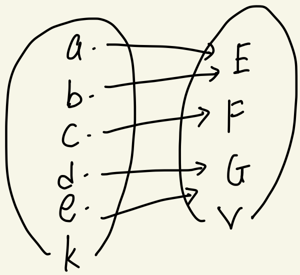
and its graph $G = (M \cup V, S)$ is given. $V = \{E, F, G\}$ is supposed to be a set of fathers.

$G = (M \cup V, S)$:



4.10.c Draw the graph of the composition of R with S and determine the adjacency matrix of RS.
Which properties has RS?





$$\underset{\sim}{R\circ S} = \underset{\sim}{R} \cdot \underset{\sim}{S} = \begin{array}{c c} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{c c} & \begin{matrix} E & F & G \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c c} & \begin{matrix} E & F & G \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Properties of $R\circ S$

- total (at least 1 successor)
- Surjective (at least 1 predecessor)
- functional (at most 1 successor)
- not injective (G & E have more than one predecessor)

$\Rightarrow$ $R\circ S$ is total and functional $\Rightarrow$ $R\circ S$ is a function

$\Rightarrow$ $R\circ S$ is a function and surjective $\Rightarrow$ $R\circ S$ is a surjection

$\Rightarrow$ $R\circ S$ is not bijective because not injective

4.10.d Prove that generally, the composition of two functional relations produces a functional relation again.

V_1: $R \subseteq K \times M$ with $R^{-1} \cdot R \subseteq I_M$ (functional)

V_2: $S \subseteq M \times V$ with $S^{-1} \circ S \subseteq I_V$ (functional)

Task: show that $\underbrace{(R^{-1} \circ R \subseteq I_M)}_{V_1} \wedge \underbrace{(S^{-1} \circ S \subseteq I_V)}_{V_2} \Rightarrow \underbrace{(R \circ S)^{-1} \circ (R \circ S) \subseteq I_V}_{S}$

Option 1: Deduction

Option 2: Start with $(R \circ S)^{-1} \circ (R \circ S)$

- transform the expressions in set notation while using preconditions V_1 and V_2 until we arrive at $\subseteq I_V$

$$(R \circ S)^{-1} \circ (R \circ S) \overset{M6}{=} S^{-1} \circ R^{-1} \circ R \circ S \overset{M1}{=} (R^{-1} \circ R) \circ S$$

auxiliary calculation (a.c)

$$(B \subseteq C) \wedge \underbrace{(A \subseteq A)}_{t} \overset{M7}{\Rightarrow} A \circ B \subseteq A \circ C \Leftrightarrow (A \circ B \subseteq A \circ C) \wedge \underbrace{(D \subseteq D)}_{t} \overset{M7}{\Rightarrow} A \circ B \circ D \subseteq A \circ C \circ D$$

$$B \subseteq C \Rightarrow A \circ B \circ D \subseteq A \circ C \circ D$$

$$\underbrace{R^{-1} \circ R \subseteq I_M}_{V_1} \overset{a.c}{\Rightarrow} S^{-1} \circ (R^{-1} \circ R) \circ S \subseteq S^{-1} \circ I_M \circ S$$

$$(R \circ S)^{-1} \circ (R \circ S) \subseteq S^{-1} \circ I_M \circ S \overset{M10}{=} S^{-1} \circ S \overset{V_2}{\subseteq} I_V$$

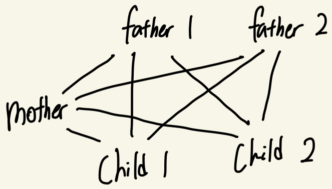
4.10.e How can the composition RS be interpreted, considering that RS is functional?

$x(R \circ S)y \iff$ "x is child of a mother who has common children with father y"

$R \circ S$ is functional

$\iff$ "x is a child of father y"

Counter example if $R \circ S$ is not functional

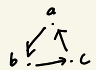


don't make sense
for child to
be related to
2 fathers

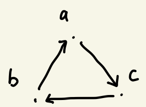
4.12 Consider the set $A_1 = \{a, b, c\}$ and the relation $R_1 = \{(a,b), (b,c), (c,a)\}_{A_1^2}$.

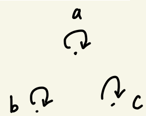
4.12.a Draw the arrow diagram of $G_1 = (A_1, R_1)$.

$G_1 = (A_1, R_1)$



4.12.b Determine the relations R_1^2 and R_1^3 and state the graph. Determine $p_1 \in \mathcal{N}$, such that $R_1^{s+p_1} = R_1^s$, $s \in \mathcal{N}_0, p_1 \in \mathcal{N}$ holds.





$R_1^2 = \{(b,a), (a,c), (c,b)\}$ $\quad R_1^3 = \{(a,a), (b,b), (c,c)\}$

$R_1^3 = I_{A_1}$

$p_1 \in \mathbb{N}, \quad R_1^{s+p_1} = R_1^s, \quad s \in \mathbb{N}_0$

$R_1^s = R_1^s \circ I_{A_1} = R_1^s \circ R_1^3 = R_1^{s+3} \quad \Rightarrow P_1 = 3$

4.12.c Compose the adjacency matrix M_1 of relation R_1.

$$\underset{\sim}{R_1} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 0 \\ b & 0 & 0 & 1 \\ c & 1 & 0 & 0 \end{array}$$

4.12.d Determine the adjaceny matrices M_1^2 and M_1^3 of relations R_1^2 and R_1^3 using matrix multiplication.

$$\underset{\sim}{R_1^2} = \underset{\sim}{R_1} \cdot \underset{\sim}{R_1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad \underset{\sim}{R_1^3} = \underset{\sim}{R_1} \cdot \underset{\sim}{R_1^2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.12.g Now the relation $R = R_1 \cup R_2$ over the basic set $A = A_1 \cup A_2$ is to be analyzed. Determine $p \in \mathcal{N}$ such that $R^{s+p} = R^s$, $s \in \mathcal{N}_0$ holds. (Think before doing!)

given $R_2^{s+P_2} = R_2^s$ for $P_2 = 5$

$$\underset{\underset{\sim}{R}}{\begin{array}{c|cc} & A_1 & A_2 \\ \hline A_1 & \underset{\sim}{R_1} & \underset{\sim}{0} \\ A_2 & \underset{\sim}{0} & \underset{\sim}{R_2} \end{array}} \cdot \underset{\underset{\sim}{R}}{\begin{bmatrix} \underset{\sim}{R_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{R_2} \end{bmatrix}} = \begin{bmatrix} \underset{\sim}{R_1^2} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{R_2^2} \end{bmatrix}$$

$p =$ least common multiple $(3,5) = 15$

4.12.h For a relation R over the basic set A with $|A| = n$ it's usually sufficient to calculate the union of the relations R^ν and $\nu = 1, \ldots, n$ to determine the transitive closure. Until which ν do you have to calculate the union if you want to build the transitive closure of $R = R_1 \cup R_2$? Justify your answer.

precondition : $A_1 \cap A_2 = \emptyset$ => no edge between A_1 and A_2

$$\underset{\sim}{R} = \begin{bmatrix} \underset{\sim}{R_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{R_2} \end{bmatrix} \; ; \; t(\underset{\sim}{R}) = \begin{bmatrix} \underset{\sim}{R_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{R_2} \end{bmatrix} \cup \begin{bmatrix} \underset{\sim}{R_1^2} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{R_2^2} \end{bmatrix} \cup \begin{bmatrix} \underset{\sim}{R_1^3} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{R_2^3} \end{bmatrix}$$

$$V_{max} = \max(|A_1|, |A_2|) = \max(3, 5) = 5$$

4.13 Determine the closures of the following relations depicted by their adjacency matrices.

4.13.a

$$R : \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ Determine } r(R),\ s(R) \text{ and } t(R)$$

$$\underset{\sim}{r(R)} = \underset{\sim}{R} \cup \underset{\sim}{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{s(R)} = \underset{\sim}{R} \overset{\substack{\text{binary} \\ \downarrow}}{+} \underset{\sim}{R^T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

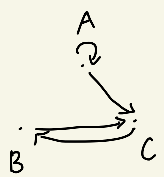
$$\underset{\sim}{t(R)} = \bigcup_{v=1}^{|A|} R^v \quad ; \quad \underset{\sim}{R^2} = \underset{\sim}{R} \cdot \underset{\sim}{R} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R^3} = \underset{\sim}{R^2} \cdot \underset{\sim}{R} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} ;$$

$$\underset{\sim}{t(R)} = \underset{\sim}{R} \cup \underset{\sim}{R^2} \cup \underset{\sim}{R^3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

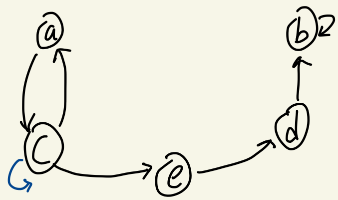
Graph:



$$\underset{\sim}{t(R)} = \begin{array}{c|ccc} & A & B & C \\ \hline A & 1 & 1 & 1 \\ B & 0 & 1 & 1 \\ C & 0 & 1 & 1 \end{array}$$

4.15 Relation $R \subseteq A^2$ with $A = \{a, b, c, d, e\}$ is given as list of ordered pairs:
$R = \{(a, c), (b, b), (c, a), (c, e), (d, b), (e, d)\}$

Carry out the Warshall-Algorithm step by step to build the transitive closure of R. Determine the cross product $\Gamma^-(x) \times \Gamma^+(x)$ for every node.
Write down R^+ as set of ordered pairs.



x_i	$\Gamma^-(x_i)$	$\Gamma^+(x_i)$	$\Gamma^-(x_i) \times \Gamma^+(x_i)$	R_i
a	$\{c\}$	$\{c\}$	$\{(c,c)\}$	$\{(a,c),(b,b),(c,a),(c,c),(c,e),(d,b),(e,d)\}$
b	$\{b,d\}$	$\{b\}$	$\{(b,b),(d,b)\}$	no change
c	$\{a,c\}$	$\{a,c,e\}$	$\{(a,a),(a,c),(a,e),(c,a),(c,c),(c,e)\}$	$\{(a,a),(a,c),(a,e),(b,b),(c,a),(c,c),(c,e),(d,b),(e,d)\}$
d	$\{e\}$	$\{b\}$	$\{(e,b)\}$	$\{(a,a),(a,c),(a,e),(b,b),(c,a),(c,c),(c,e),(d,b),(e,d),(e,b)\}$
e	$\{a,c\}$	$\{b,d\}$	$\{(a,b),(a,d),(c,b),(c,d)\}$	$\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(c,a),(c,b),(c,c),(c,d),(c,e),(d,b),(e,d),(e,b)\} = R^+$

4.18 Consider relation $R = \{(a,e),(b,a),(b,c),(c,d),(e,b)\}$ with $R \subseteq A^2$ and $A = \{(a,b,c,d,e)\}$.

Draw the arrow diagram of relation R.

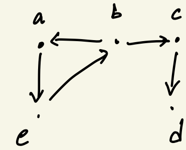
Are the pairs $\{a,c\}$ and $\{a,e\}$

- connected?
- indirectly accessible?
- mutually accessible?

Determine the set of roots in $G = (A,R)$.

Justify your answers with the aid of the arrow diagram.

$G(A,R)$



- Connected : $x\,(\underbrace{R \cup R^{-1}}_{})^{*}\,y$

"arrow direction does not matter"

· as many steps as we want

e.g. $x \leftrightarrow z \leftrightarrow y$

$\{a,c\}$: ✓ $aR^{-1}b \wedge bRc$

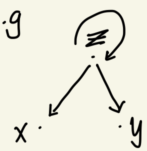
$\{a,e\}$: ✓ aRe

- indirectly accessible : $x\,(R^*)^{-1} \circ R^*\,y$

first take steps against arrow direction

second take steps in arrow direction

e.g.



"z and y is also indirectly accessible"

$\{a,c\}$: ✓ $aR^{-1}b \wedge bRc$

$\{a,e\}$: ✓ $aR^{-1}a \wedge aRe$

- mutually accessible : $xR^*y \wedge yR^*x$

e.g. $x \rightleftarrows z \rightleftarrows y$

$\{a,c\}$: ✗ no cR^*a

$\{a,e\}$: ✓ $aR^*e \wedge eR^*a$

set of roots in G : $W(G) = \{r \mid \forall_{x \in A}\, rR^*x\} = \{a,b,e\}$

$$R^* = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

4.22 Consider the following relation $R \subseteq A^2$ with $A = \{2,3,4,5,6\}$:

$$(a,b) \in R \iff (a \text{ is factor of } b) \vee [(a \text{ is a prime number}) \wedge (a < b)]$$

4.22.a Write down relation R as set of ordered pairs and draw its Hasse diagram $G_H = (A, H)$.

4.22.b Now consider set $B \subseteq A$ with $B = \{3,4,5\}$.
Determine ub(B), grt(B), lub(B), lb(B), lst(B), glb(B), max(B) and min(B) by conception or by calculation.

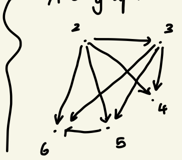
a) $R = \{(2,4),(2,6),(2,2),(3,3),(3,6),(4,4),(5,5),(6,6),$
$(2,3),(2,5),(3,4),(3,5),(5,6)\}$

$H = (R\setminus I) \setminus (R \setminus I)^2$

$(R\setminus I)^2 = \{(2,4),(2,5),(2,6),(3,6)\}$

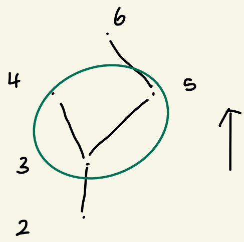
must take 2 steps

Aid graph of $R\setminus I$



Remove all loops, remove $(R\setminus I)^2$

$H = \{(2,3),(3,4),(3,5),(5,6)\}$



b) $B = \{3,4,5\}$

$ub(B) = \bigcap_{x\in B} \Gamma^+(x) = \{3,4,5,6\} \cap \{4\} \cap \{5,6\} = \emptyset$

$grt(B) = B \cap ub(B) = B \cap \emptyset = \emptyset$

$lub(B) = ub(B) \cap lb(ub(B)) = \emptyset \cap lb(\emptyset) = \emptyset$

$lb(B) = \bigcap_{x\in B} \Gamma^-(x) = \{2,3\} \cap \{2,3,4\} \cap \{2,3,5\} = \{2,3\}$

$$\mathrm{lst}(B) = B \cap \mathrm{lb}(B) = \{3,4,5\} \cap \{2,3\} = \{3\}$$

$$\mathrm{glb}(B) = \mathrm{lb}(B) \cap \mathrm{ub}(\mathrm{lb}(B)) = \{2,3\} \cap \mathrm{ub}(\{2,3\})$$

$$= \{2,3\} \cap \{2,3,4,5,6\} \cap \{3,4,5,6\} = \{3\}$$

$$\max(B) = B \cap \Big[\bigcap_{x \in B} \overline{\Gamma(x) \setminus \{x\}} \Big] = B \cap \overline{\{2\}} \cap \overline{\{2,3\}} \cap \overline{\{2,3\}}$$

$$\left\{ \begin{array}{l} A \cap \overline{B} \\ = A \setminus B \end{array} \right.$$

$$= \{3,4,5\} \setminus \{2\} \setminus \{2,3\} \setminus \{2,3\} = \{4,5\}$$

$$\min(B) = B \cap \Big[\bigcap_{x \in B} \overline{\Gamma^{-1}(x) \setminus \{x\}} \Big] = B \cap \overline{\{4,5,6\}} \cap \overline{\{\}} \cap \overline{\{6\}}$$

$$= \{3,4,5\} \setminus \{4,5,6\} \setminus \emptyset \setminus \{6\} = \{3\}$$

$\boxed{4.24}$ Assume a relation T over the basic set $A = \mathcal{R} \times \mathcal{R}$ which is defined as follows:

$$(x_1, y_1)T(x_2, y_2) \iff (x_1 \leq x_2) \wedge (y_1 \leq y_2)$$

Which of the following assertions are true? Justify your answers.

4.24.a T is an order relation

4.24.b T is a total order

4.24.c T is a well order

4.24.d Every subset B of $\mathcal{R} \times \mathcal{R}$, which has a lower bound, has a greatest lower bound concerning T.

To Check: Partial order relation

- T is reflexive $\iff \forall_{(x,y)\in A} (x,y)T(x,y) \iff \forall_{(x,y)\in A} (x \leq x) \wedge (y \leq y) \iff \forall_{(x,y)\in A} t \iff t$

- T is transitive $\iff \forall_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in A \\ (x_3,y_3)\in A}} (x_1,y_1)T(x_2,y_2) \wedge (x_2 \wedge y_2)T(x_3,y_3) \rightarrow (x_1,y_1)T(x_3,y_3)$

$$\iff \forall \underbrace{(x_1 \leq x_2)}_{a_1} \wedge \underbrace{(y_1 \leq y_2)}_{c_1} \wedge \underbrace{(x_2 \leq x_3)}_{a_2} \wedge \underbrace{(y_2 \leq y_3)}_{c_2} \rightarrow \underbrace{(x_1 \leq x_3)}_{b} \wedge \underbrace{(y_1 \leq y_3)}_{d} \overset{E14}{\iff} t$$

$a \iff a_1 \wedge a_2$; $c \iff c_1 \wedge c_2$

$$\underbrace{(a \rightarrow b)}_{t} \wedge \underbrace{(c \rightarrow d)}_{t} \overset{E14}{\Rightarrow} (a \wedge c) \rightarrow (b \wedge d)$$

$(x_1 \leq x_2) \wedge (x_2 \leq x_3) \rightarrow (x_1 \leq x_3) \iff t$

- T is antisymmetric $\iff \forall_{\substack{(x_1,y_1)\in A \\ (x_1,y_2)\in A}} (x_1,y_1)T(x_2,y_2) \wedge (x_2,y_2)T(x_1,y_1) \rightarrow (x_1,y_2) = (x_2,y_2)$

$$\iff \forall \underbrace{(x_1 \leq x_2)}_{a_1} \wedge \underbrace{(y_1 \leq y_2)}_{c_1} \wedge \underbrace{(x_2 \leq x_1)}_{a_2} \wedge \underbrace{(y_2 \leq y_1)}_{c_2} \rightarrow \underbrace{(x_1 = x_2)}_{b} \wedge \underbrace{(y_1 = y_2)}_{d} \iff d$$

$(x_1 \leq x_2) \wedge (x_2 \leq x_1) \rightarrow (x_1 = x_2) \iff t$

$\Rightarrow T$ is partial order relation

b) T is total order $\iff$ T is partial order $\wedge$ T is connex

- T is connex $\iff \forall_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in A}} (x_1,y_1)T(x_2,y_2) \vee (x_2,y_2)T(x_1,y_1)$

$$\iff \forall \left[(x_1 \le x_2) \wedge (y_1 \le y_2)\right] \vee \left[(x_2 \le x_1) \wedge (y_2 \le y_1)\right]$$

Counter example : $(x_1, y_1) = (0,1)$; $(x_2, y_2) = (1,0)$

$\hookrightarrow \left[(0 \le 1) \wedge (1 \le 0)\right] \vee \left[(1 \le 0) \wedge (0 \le 1)\right]$

$\iff [t \wedge f] \quad \vee [f \wedge t] \iff f$

$\Rightarrow$ T is not total order

c) T is well order $\iff$ T is total order $\wedge$ every non-empty sub set contains a least element

T is not total order $\Rightarrow$ T is no well order

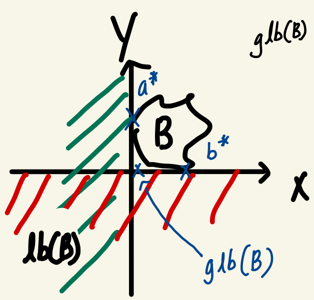
d) $B \subseteq \mathbb{R} \times \mathbb{R} = A$

$lb(B) = \{ a \in A \mid \forall_{x \in A} x \in B \rightarrow a \le x \}$

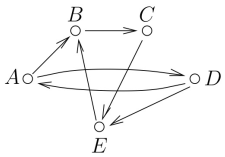
$= \{ (a,b) \in A \mid \forall_{(x,y)\in A} (x,y) \in B \rightarrow \underbrace{(a,b)T(x,y)}_{} \}$

means : $a \le x \wedge b \le y$

$glb(B) = \{ (a^*, b^*) \in lb(B) \mid \forall_{(a,b)\in A} (a,b) \in lb(B) \rightarrow \underbrace{(a,b)T(a^*, b^*)}_{\substack{a \le a^* \\ \wedge b \le b^*}} \}$



4.27 Over the basic set $M = \{A, B, C, D, E\}$, relation R is defined by the following graph G:



4.27.a Determine the adjacency matrices R and R^*

4.27.b Compose the equivalence core $\mathrm{EC}(R^*)$. Re-sort the resulting adjacency matrix so that the connectivities become apparent.

4.27.c Draw the graph of the equivalence core.

$$R^* = tr(R)$$

$$t(R) = \bigcup_{v=1}^{|A|=5} R^v$$

$$R^1 \cup R^2 \cup R^3 \cup \ldots$$

a) $$\underset{\sim}{R} = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 1 & 0 & 1 & 0 \\ B & 0 & 0 & 1 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & 1 \\ D & 1 & 0 & 0 & 0 & 1 \\ E & 0 & 1 & 0 & 0 & 0 \end{array} \quad ; \quad \underset{\sim}{r(R)} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$(r(R))^2 = \overset{r(R)}{\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{r(R)}{\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R^*} = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 1 & 1 & 1 & 1 & 1 \\ B & 0 & 1 & 1 & 0 & 1 \\ C & 0 & 1 & 1 & 0 & 1 \\ D & 1 & 1 & 1 & 1 & 1 \\ E & 0 & 1 & 1 & 0 & 1 \end{array}$$

$$(r(R))^3 = r(R) \cdot (r(R))^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

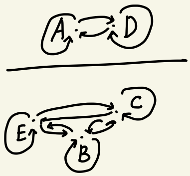
$$(r(R))^4 = r(R) \cdot (r(R))^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R^*} = r(R) \cup (r(R))^2 \cup (r(R))^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

b) $ECO(R^*) = R^* \cap (R^*)^{-1}$

$$\underset{\sim}{R^*} \cap \underset{\sim}{(R^*)^T} = \begin{bmatrix} 1&1&1&1&1 \\ 0&1&1&0&1 \\ 0&1&1&0&1 \\ 1&1&1&1&1 \\ 0&1&1&0&1 \end{bmatrix} \cap \begin{bmatrix} 1&0&0&1&0 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&0&0&1&0 \\ 1&1&1&1&1 \end{bmatrix} = \begin{array}{c c} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1&0&0&1&0 \\ 0&1&1&0&1 \\ 0&1&1&0&1 \\ 1&0&0&1&0 \\ 0&1&1&0&1 \end{bmatrix} \end{array}$$

$$ECO(R^*) = \begin{array}{c c} & \begin{matrix} A & D & B & C & E \end{matrix} \\ \begin{matrix} A \\ D \\ B \\ C \\ E \end{matrix} & \begin{bmatrix} 1&1& & 0 & \\ 1&1& & & \\ & & 1&1&1 \\ 0 & & 1&1&1 \\ & & 1&1&1 \end{bmatrix} \end{array}$$



$[A]_{ECO(R^*)} = [D]_{ECO(R^*)} = \{A, D\}$

$[B]_{ECO(R^*)} = \{B, C, E\}$

$\boxed{\textbf{4.31}}$ Prove that the following holds for arbitrary relations U, V, W:

$$\overline{U^{-1}\overline{V}} \subseteq \overline{(WU)^{-1}\overline{WV}}$$

Name all laws you use and do not use any laws of predicate logic.
Hint: Start with $WV \subseteq WV$ and use the Schröder rule.

To proof this relationship, we will use this approach

IF $t \Rightarrow X$, THEN $X \Leftrightarrow t$ (FRI)

Schröder Rule (M8)

$RS \subseteq V \Leftrightarrow R^{-1}\overline{V} \subseteq \overline{S} \Leftrightarrow \overline{V}S^{-1} \subseteq \overline{R}$

1) $WV \subseteq WV \Rightarrow \overline{U^{-1}\overline{V}} \subseteq \overline{(WU)^{-1}\overline{WV}} \overset{J31}{\Leftrightarrow} (WU)^{-1}(\overline{WV}) \subseteq U^{-1}\overline{V}$

---

2) $W^{-1}\overline{WV} \subseteq \overline{V}$ 1, M8

3) $U^{-1} \subseteq U^{-1}$ J20

4) $U^{-1} \circ W^{-1} \circ \overline{WV} \subseteq U^{-1} \circ \overline{V}$ M7, 2, 3

5) $(WV)^{-1}\overline{WV} \subseteq U^{-1} \circ \overline{V}$ 4, M6

6) $\overline{U^{-1}\overline{V}} \subseteq \overline{(WU)^{-1}\overline{WV}}$ 5, J31

4.33 Prove without using any laws of predicate logic that for a transitive and reflexive relation $V \subseteq A^2$, the following is fulfilled:

$$(V = V^2) \wedge (V \cap \overline{V^2} = \emptyset)$$

Name all laws you use.

$V \subseteq V^2$
$V^2 \subseteq V$

1) $I \subseteq V$ (reflexive)

2) $V^2 \subseteq V$ (transitive) $\Rightarrow (V = V^2) \wedge (V \cap \overline{V^2} = \emptyset)$

---

3) $V \subseteq V$ J20

4) $V \circ I \subseteq V \circ V$ 1, 3, M7

5) $V \subseteq V^2$ 4, M10

6) $V = V^2$ 2, 5, J28

7) $V \cap \overline{V^2} = \emptyset$ 5, J29

4.35 In the manuscript you'll find an overview of different possibilities of representing relations. If you want to automate a task using a computer program, an important step to take is to think about the structure of your task and to choose an appropriate data structure.

Determine the required amount of memory space for each of the possible representations named in the subsequent list. The basic set of relation R is set A. Use the following variables:

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- $a = |A|$: number of nodes
- $r = |R|$: number of (relations) edges
- b = required number of bits for storing one element of A
- i = number of bits required for storing a pointer

Representation	Memory Space				
List of ordered pairs	· 2 elements per tuple · b bits per element · r tuples	$2 \cdot b \cdot r$	$2a^2$	$2r^2$ (+)	$2ra$
Adjacency Matrix	1 bit per entry in matrix a^2 entries in matrix	a^2	a^2 (+)	a^2 (-)	a^2 (+)
Incident Matrix	2 bits per entry in matrix a rows ; r column	$2ar$	$2a^2$	$a2r$	$2ra$
Chained successor list	b+i bits per element a+r entries a start nodes + 1 entry per edge	$(r+a)(b+i)$	$4a^2$ (-)	$2ar+2r^2$	$2ra+2a^2$ (-)
Successor Table	2 rows ; a+r columns $\Rightarrow 2(a+r)$ entries pointer & nodes are shared $\Rightarrow \max(b,i)$ per entry	$2(a+r)\cdot\max(b,i)$	$4a^2$ (-)	$2ar+2r$	$2ra+2a^2$ (-)

Compare the results in a table and mark those representations which are very memory efficient (+) or inefficient (-) if used in the three subsequent cases.

4.35.f $a = r = b = i$ blue

4.35.g $a \gg r = b = i$ purple

4.35.h $r \gg a = b = i$ red