1. Using L'Hopital's rule, evaluate the limits below.

a) $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x}$

b) $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2}$

c) $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$

d) $\lim_{x\to 0^+} x^{\sqrt{x}}$

e) $\lim_{x\to 0^+} x \ln x$

f) $\lim_{x\to \pi/2} (1-\cos x)^{\tan x}$

g) $\lim_{x\to \infty} \left\{ x - \frac{x^2}{x+5} \right\}$

ANS: **a)** -π. **b)** 1. **c)** -1/2. **d)** 1. **e)** 0. f) 1/e. **g)** 5.

$L =$ d) $\lim_{x\to 0^+} x^{\sqrt{x}} = 0^0$

$e^{\ln \lim_{x\to 0^+} x^{\sqrt{x}}} = e^{\lim_{x\to 0^+} \ln x^{\sqrt{x}}} = e^{\lim_{x\to 0^+} \sqrt{x}\ln x}$

$\lim_{x\to 0^+} \sqrt{x}\ln x = 0(-\infty)$

$\lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} = \frac{-\infty}{\infty}$

L'H

$\lim_{x\to 0^+} \frac{1/x}{\frac{-1}{2x^{3/2}}} = \lim_{x\to 0^+} -2x^{1/2} = 0$

$\Rightarrow L = e^0 = 1$

g) $\lim_{x\to \infty} \left\{ x - \frac{x^2}{x+5} \right\} = \lim_{x\to\infty} \left\{ \frac{\cancel{x^2} + 5x - \cancel{x^2}}{x+5} \right\} = \frac{\infty}{\infty}$

L'H

$\lim_{x\to\infty} \left\{ \frac{5}{1} \right\} = 5$

$L =$ f) $\lim_{x\to \pi/2} (1-\cos x)^{\tan x} = (1-0)^{\infty} = 1^{\infty}$

$L = e^{\ln\left[\lim_{x\to\pi/2} (1-\cos x)^{\tan x}\right]}$

$\ln\left[\lim_{x\to\pi/2}(1-\cos x)^{\tan x}\right] = \lim_{x\to\pi/2} \tan x \ln(1-\cos x) = \infty(0)$

$= \lim_{x\to\pi/2} \frac{\ln(1-\cos x)}{1/\tan x} = \frac{0}{0}$

L'H

$\lim_{x\to\pi/2} \frac{\frac{\sin x}{1-\cos x}}{\frac{-\sec^2 x}{\tan^2 x}}$

$\frac{-\sec^2 x}{\tan^2 x} = -\frac{1}{\cos^2 x}\left(\frac{\cos^2 x}{\sin^2 x}\right) = -\frac{1}{\sin^2 x}$

$= \lim_{x\to\pi/2} \frac{-\sin^3 x}{1-\cos x} = \frac{-(1)^3}{1-0} = -1$

$\Rightarrow L = e^{-1} = \frac{1}{e}$

a) $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x} = \frac{0}{0}$

L'H

$\lim_{x\to 1} \frac{\cos(\pi x)\cdot \pi}{1/x} = (-1)(\pi)$

$= -\pi$

b) $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2} = \frac{0}{0}$

L'H

$\lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = \frac{0}{0}$

L'H

$\lim_{x\to 0} \frac{e^x + e^{-x}}{2} = \frac{2}{2}$

$= 1$

c) $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = \frac{0}{0}$

L'H

$\lim_{x\to 0} \frac{\cos x - \sec^2 x}{3x^2} = \frac{0}{0}$

L'H

$\lim_{x\to 0} \frac{-\sin x - 2\sec^2 x \tan x}{6x} = \frac{0}{0}$

L'H

$\lim_{x\to 0} \frac{-\cos x - 2\sec^4 x - 4\tan^2 x \sec^2 x}{6} = \frac{-1-2-0}{6}$

$= -\frac{1}{2}$

$2\left[\sec^2 x \cdot \sec^2 x + \tan x \cdot 2\sec^2 x \tan x\right]$

$= 2\sec^4 x + 4\tan^2 x \sec^2 x$

$\sec 0 = 1$

$\cos 0 = 1$

$\sin 0 = 0$

$\tan 0 = 0$

e) $\lim_{x\to 0^+} x \ln x$

$= \lim_{x\to 0^+} \frac{\ln x}{1/x} = \frac{\infty}{\infty}$

L'H

$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -\frac{x^2}{x}$

$= \lim_{x\to 0^+} -x$

$= 0$

2. (https://openstax.org/books/calculus-volume-1/pages/4-2-linear-approximations-and-differentials) Using linear approximation, estimate the following.

a) 2.001^6

b) $\sin(0.02)$

c) $1/0.98$

ANS: a) 64.192. b) 0.02. c) 1.02.

a) Let $f(x) = x^6$

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$
$$= a^6 + 6a^5(x-a)$$

Choose $a=2$

$$f(2.001) \approx 2^6 + 6(2)^5(2.001-2)$$
$$\approx 64.192$$

b) Let $f(x) = \sin x$

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$
$$= \sin a + \cos a\,(x-a)$$

Choose $a = 0$

$$f(0.02) \approx \sin 0 + \cos 0\,(0.02-0)$$
$$\approx 0.02$$

c) Let $f(x) = \frac{1}{x}$

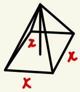
$$f(x) \approx \frac{1}{a} - \frac{1}{a^2}(x-a)$$

Choose $a=1$

$$f(0.98) \approx \frac{1}{1} - \frac{1}{1^2}(0.98-1)$$
$$\approx 1.02$$

3. A pyramid with a square base of side length equal to its height is required to have a volume of 3 m^3. Using linear approximation, estimate the $\underbrace{\text{height of the pyramid}}_{x}$.

ANS: $2\frac{1}{12}\,m$.



Vol, $V = \frac{1}{3}$ base area $\times$ height

$$= \frac{1}{3}x^2 \cdot x = \frac{1}{3}x^3$$

$$\Rightarrow \quad x = (3V)^{1/3} = f(V) \rightarrow f'(V) = \frac{1}{3}(3V)^{-2/3}(3)$$

$$x \approx L(V) = f(a) + f'(a)(V-a)$$

$$= (3a)^{1/3} + \frac{1}{\cancel{3}}(3a)^{-2/3}(\cancel{3})(V-a)$$

Choose $a = \frac{8}{3}$ (close to $3m^3$)

$$x \approx \left[\cancel{3}\left(\frac{8}{\cancel{3}}\right)\right]^{1/3} + \left[\cancel{3}\left(\frac{8}{\cancel{3}}\right)\right]^{-2/3}\left[3 - \frac{8}{3}\right]$$

$$\approx 2 + \frac{1}{4}\left(\frac{1}{3}\right) = \underline{\underline{2\frac{1}{12}\,m}}$$

4. Evaluate the Taylor series of each function below expanded about the given point a.

$= T(x)$ since $f(x)$ is already a power series

a) $f(x) = x^2 + x + 1,\ a = 0$

b) $f(x) = x^2 + x + 1,\ a = 2$

$= (x-2+2)^2 + (x-2+2) + 1$

$= (x-2)^2 + 4(x-2) + 4 + (x-2) + 3$

$= 7 + 5(x-2) + (x-2)^2 = T(x)$

c) $f(x) = \dfrac{x}{1-x},\ a = 0$

d) $f(x) = 1/x^2,\ a = 1$

e) $f(x) = \sin x,\ a = \pi$

f) $f(x) = \cos x,\ a = 0$

g) $f(x) = \ln(1+x),\ a = 0$

ANS: **a)** $T(x) = 1 + x + x^2$. **b)** $T(x) = 7 + 5(x-2) + (x-2)^2$. **c)** $T(x) = \sum_{n=1}^{\infty} x^n$. **d)** $T(x) = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$. **e)** $T(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(x-\pi)^{2k+1}$. **f)** $T(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$. **g)** $T(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$.

c) $f(x) = \dfrac{x}{1-x},\ a = 0$

$f'(x) = \frac{1}{(1-x)^2}$

$f''(x) = \frac{2}{(1-x)^3}$

$f'''(x) = \frac{6}{(1-x)^4}$

$f^{4}(x) = \frac{24}{(1-x)^5}$

$f(0) = 0,\ f'(0) = 1,\ f''(0) = 2$

$f'''(x) = 6,\ f^{4}(x) = 24$

$T(x) = 0 + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \frac{24x^4}{4!} + \dots$

$= 0 + x + x^2 + x^3 + x^4 + \dots$

$= \sum_{n=1}^{\infty} x^n$

d) $f(x) = 1/x^2,\ a = 1$

$f'(x) = \frac{-2}{x^3}$

$f''(x) = \frac{6}{x^4}$

$f'''(x) = \frac{-24}{x^5}$

$f^{(4)}(x) = \frac{120}{x^6}$

$f(1) = 1,\ f'(1) = -2,\ f''(1) = 6$

$f'''(x) = -24,\ f^{4}(x) = 120$

$T(x) = 1 - \frac{2(x-1)}{1!} + \frac{6(x-1)^2}{2!} - \frac{24(x-1)^3}{3!} + \frac{120(x-1)^4}{4!} + \dots$

$= \underbrace{1}_{n=0} - \underbrace{2(x-1)}_{n=1} + \underbrace{3(x-1)^2}_{n=2} - \underbrace{4(x-1)^3}_{n=3} + \underbrace{5(x-1)^4}_{n=4} + \dots$

$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

e) $f(x) = \sin x,\ a = \pi$

$f(\pi) = 0$

$f'(\pi) = \cos(\pi) = -1$

$f''(\pi) = -\sin(\pi) = 0$

$f'''(\pi) = -\cos(\pi) = -1$

$f^4(\pi) = \sin(\pi) = 0$

$$T(x) = 0 - \underbrace{(x-\pi)}_{\substack{n=1 \\ k=0}} + 0 - \underbrace{\frac{(x-\pi)^3}{3!}}_{\substack{n=3 \\ k=1}} + 0 \ldots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!}(x-\pi)^{2k+1}$$

$$\underbrace{n = 1, 3, 5, 7}_{2k+1}$$

f) $f(x) = \cos x,\ a = 0$

$$f(0) = \cos(0) = 1$$

$$f'(0) = -\sin(0) = 0$$

$$f''(0) = -\cos(0) = -1$$

$$f'''(0) = \sin(0) = 0$$

$$f''''(0) = \cos(0) = 1$$

sub into $T_N(x)$:

$$T_N(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{1}{N!}x^N \text{ (N is even)}$$

$$= 1 + 0 + \frac{(-1)x^2}{2!} + 0 + \frac{(1)x^4}{4!} + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$$= \sum_{n=0}^{N} (-1)\frac{x^n}{n!}, \quad n = \underbrace{0, 2, 4, 6, 8}_{\text{let } n = 2k}$$

$$\cos(x) \approx T_N(x) = \sum_{k=0}^{N} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\Rightarrow \cos(x) = T_N(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

g) $f(x) = \ln(1 + x),\ a = 0$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$$f^{4}(x) = \frac{-6}{(1+x)^4}$$

$f(0) = 0,\ f'(0) = 1,\ f''(0) = -1$

$f'''(0) = 2,\ f^{4}(0) = -6$

$$T(x) = 0 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \dots$$

$$= \underbrace{0}_{n=0} + \underbrace{x}_{n=1} - \underbrace{\frac{x^2}{2}}_{n=2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

5. From the Maclaurin series of other functions, determine the Maclaurin series of each function below.

From 4c), replace x by $3x^2$

a) $f(x) = \dfrac{3x^2}{3x^2-1} = -\left(\dfrac{3x^2}{1-3x^2}\right)$

Replace x by $2x$ from $T(x)$ for $\sin x$

b) $f(x) = \sin x \cos x = \frac{1}{2}\sin(2x)$

c) $f(x) = x^2 \cos(2x)$

ANS: **a)** $T(x) = -\sum_{n=1}^{\infty} 3^n x^{2n}$. **b)** $T(x) = \dfrac{1}{2}\sum_{k=0}^{\infty} \dfrac{(-1)^k 2^{2k+1} x^{2k+1}}{(2k+1)!}$.

c) $T(x) = \sum_{k=0}^{\infty} \dfrac{(-1)^k 2^{2k}}{(2k)!} x^{2(k+1)}$.

a) From 4c)

$$f(x) = \frac{x}{1-x} = T(x) = \sum_{n=1}^{\infty} x^n$$

$$f(x) = \frac{3x^2}{3x^2-1} = -\left(\frac{3x^2}{1-3x^2}\right)$$

$$T(x) = 0 + 3x^2 + 9x^4 + 27x^6 + 81x^8 + \dots$$

$$= -\sum_{n=1}^{\infty} 3^n x^{2n}$$

b)

$$f(x) = \sin x \cos x = \frac{1}{2}\sin(2x)$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k+1}}{(2k+1)!}$$

c)

$$f(x) = x^2 \cos(2x)$$

$$= x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2x)^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2^{2k}) x^{2k+2}$$

6. Evaluate all critical points and local extrema of each function below.

a) $f(x) = 3x^5 - 5x^3$

b) $f(x) = x^2 - 2|x| + 1$

c) $f(x) = x\sqrt{4 - x^2}$

d) $f(x) = \ln\left(x^2 + 2x + 2\right)$

ANS: **a)** (-1, 2) max, (1, -2) min, (0, 0). **b)** (-1, 0) min, (1, 0) min, (0, 1) max.
c) (-√2, -2) min, (√2, 2) max, (-2, 0), (2, 0). **d)** (-1, 0) min.

b) $f(x) = x^2 - 2|x| + 1$ $= \begin{cases} x^2 - 2x + 1, & x \geq 0 \\ x^2 + 2x + 1, & x < 0 \end{cases}$

$$f'(x) = \begin{cases} 2x - 2, & x > 0 \\ 2x + 2, & x < 0 \end{cases}$$

$f'(x) = 0$

$2x - 2 = 0$

$x = 1$ in $x > 0$

$f(1) = 0$.

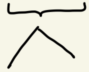
$2x + 2 = 0$

$x = -1$ in $x < 0$

$f(-1) = 0$

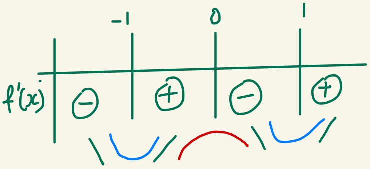
f' DNE Check at $x = 0$

$$\left.\begin{array}{l} \lim_{x \to 0^-} 2x + 2 = 0 \\ \lim_{x \to 0^+} 2x - 2 = -2 \end{array}\right\} \neq \rightarrow x = 0 \text{ is a } \underbrace{\text{cusp point}}, \text{ also a local max}$$



$f(0) = 1$

Classify:



$\Rightarrow$ Local min at $(-1, 0)$ & $(1, 0)$

Local max at $(0, 1)$

a) $f(x) = 3x^5 - 5x^3$

solve for $f'(x)$ and $f'(x)$ DNE

$$f'(x) = 15x^4 - 15x^2 = 0$$
$$15x^2(x^2 - 1) = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

$f'(x)$ DNE

$x = 0, \quad f(0) = 0$

$f(1) = -2$

$f(-1) = 2$

Classify :

	$-\infty$ to -1	-1 to 0	0 to 1	1 to $+\infty$
$f'(x)$	+	−	−	+

(0,0)

max $(-1, 2)$

min $(1, -2)$

c) $f(x) = x\sqrt{4 - x^2}$ $= x(4-x^2)^{\frac{1}{2}}$

$$f'(x) = x \cdot \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x) + (4-x^2)^{\frac{1}{2}}$$

$$= \frac{-\cancel{2}x^2}{\cancel{2}(4-x^2)^{\frac{1}{2}}} + (4-x^2)^{\frac{1}{2}}$$

$$= \frac{-x^2 + 4 - x^2}{(4-x^2)^{\frac{1}{2}}} = \frac{-2x^2+4}{(4-x^2)^{\frac{1}{2}}} = 0$$

$x^2 = 2$

$x = \pm\sqrt{2}$

$f(\sqrt{2}) = 2$

$f(-\sqrt{2}) = -2$

$f'(x)$ DNE

$(4-x^2)^{\frac{1}{2}} = 0$

$x^2 = 4$

$x = \pm 2$

$f(2) = 0$

$f(-2) = 0$

Classify:

	-2 to $-\sqrt{2}$	$-\sqrt{2}$ to 0	0 to $\sqrt{2}$	$\sqrt{2}$ to 2
$-2x^2+4$	$-$	$+$	$+$	$-$
$(4-x^2)^{\frac{1}{2}}$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$+$	$-$

$(-\sqrt{2}, -2)$ min

$(\sqrt{2}, 2)$ max

$(2, 0)$

$(-2, 0)$

d) $f(x) = \ln\left(\underbrace{x^2 + 2x + 2}\right)$

> 0, since $b^2 - 4ac = 2^2 - 4(1)(2) < 0$, so no real roots

$\therefore$, Domain is $\mathbb{R}$

$f'(x) = 0$	$f'(x)$ DNE
$\dfrac{2x+2}{x^2+2x+2} = 0$	$x^2 + 2x + 2 = 0 \rightarrow$ no sol$^{\underline{n}}$
$2x + 2 = 0$	
$x = -1$	
$f(-1) = \ln(1 - 2 + 2)$	
$= 0$	

Classify: $f'(-1^-) = \dfrac{\ominus}{\oplus} < 0$, $f'(-1^+) = \dfrac{\oplus}{\oplus} > 0$

$\therefore (-1, 0)$ is a local min

7. For a quadratic function $f(x) = ax^2 + bx + c$, determine the vertex (stationary point) from the root formula. Then, verify the stationary point of the function from calculus.

Stationary point found at $x = \frac{-b}{2a}$

ANS: $x = -\frac{b}{2a}$

$$f'(x) = 2ax + b = 0$$

$$x = \frac{-b}{2a}$$

8. Evaluate the concavity and points of inflection, if any, of each function below.

a) $f(x) = 3x^5 - 10x^3$

b) $f(x) = x^2 - 2|x| + 1$

c) $f(x) = x\sqrt{4 - x^2}$ $\rightarrow$ Domain is $[-2, 2]$

d) $f(x) = \ln\left(x^2 + 2x + 2\right)$

ANS: **a)** Convex in (-1, 0) & (1, ∞). Concave in (-∞, -1) & (0, 1). Inflection points at (-1, 7), (0, 0) & (1, -7). **b)** Convex in (-∞, 0) & (0, ∞). No inflection point. **c)** Convex in (-2, 0). Concave in (0, 2). Inflection point at (0, 0). **d)** Convex in (-2, 0). Concave in (-∞, -2) & (0, ∞). Inflection points at (-2, ln(2)) & (0, ln(2)).

b) $f(x) = x^2 - 2|x| + 1 = \begin{cases} x^2 - 2x + 1, & x \geq 0 \\ x^2 + 2x + 1, & x < 0 \end{cases}$

$f'(x) = \begin{cases} 2x - 2, & x > 0 \\ 2x + 2, & x < 0 \end{cases} \rightarrow f''(x) = \begin{cases} 2, & x > 0 \\ 2, & x < 0 \end{cases}$ In each interval, there is no inflection point & f(x) is $\underbrace{\text{concave up}}_{\text{(convex)}}$

Check at $x = 0$ since $f''(x)$ DNE. Since $f''(0^-) > 0$ & $f''(0^+) > 0$, so no change of concavity. So $x = 0$ is not a point of inflection, so f(x) has no inflection pt.

a) $f(x) = 3x^5 - 10x^3$

$f'(x) = 15x^4 - 30x^2$

$f''(x) = 60x^3 - 60x = 0$

$= 60x(x^2 - 1) = 0$

$x = 0,\ x = 1,\ x = -1$

$f(0) = 0 \quad f(1) = -7 \quad f(-1) = 7$

f'' DNE $\rightarrow$ no sol$^{\underline{n}}$

	$-\infty$	$x=-1$	$x=0$	$x=1$	∞
$f''(x)$	⊖	⊕	⊖	⊕	

point of inflection = (0,0) (1,−7) (−1,7)

Concave in $(-\infty, -1)$ & $(0, 1)$

Convex in $(-1, 0)$ & $(1, \infty)$

d) $f(x) = \ln\left(x^2 + 2x + 2\right)$

$$f'(x) = \frac{2x+2}{x^2+2x+2} \rightarrow f''(x) = \frac{\overbrace{(x^2+2x+2)(2)}^{2x^2+4x+4} - \overbrace{(2x+2)(2x+2)}^{4x^2+8x+4}}{(x^2+2x+2)^2}$$

$$= \frac{-2x^2-4x}{(x^2+2x+2)^2} = \frac{-2x(x+2)}{(x^2+2x+2)^2}$$

$f''(x) = 0$

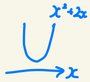
$$\frac{-2x(x+2)}{\cancel{(x^2+2x+2)^2}} = 0$$

$x = 0$ or $x = -2$

$f''(x)$ DNE $\rightarrow (\underbrace{x^2+2x+2}_{b^2-4ac<0 \rightarrow \text{no real roots}})^2 = 0$

no sol$^{\underline{n}}$.

$x^2+2x+2>0$



		$x=-2$	$x=0$
$-2x$	+	+	−
$x+2$	−	+	+
$(x^2+2x+2)^2$	+	+	+
$f''(x)$	−	+	−

Inflection pts at $(-2, f(-2) = \ln 2)$ & $(0, \ln 2)$

$f(x)$ is concave in $(-\infty, -2)$ & $(0, \infty)$, and

convex in $(-2, 0)$

c) $f(x) = x\sqrt{4 - x^2}$, $x \in [-2, 2]$

$$f'(x) = \frac{-2x^2 + 4}{(4-x^2)^{\frac{1}{2}}} = \frac{4-2x^2}{\sqrt{4-x^2}}$$

$$f''(x) = \frac{\sqrt{4-x^2}(-4x) - \frac{1(-2x)}{2\sqrt{4-x^2}}(4-2x^2)}{4-x^2}$$

$$= \frac{\frac{(4-x^2)(-4x) + 4x - 2x^3}{\sqrt{4-x^2}}}{4-x^2}$$

$$= \frac{-16x + 4x^3 + 4x - 2x^3}{(4-x^2)^{3/2}}$$

$$= \frac{2x(x^2-6)}{(4-x^2)^{3/2}}$$

$f''(x) = 0$

$2x(x^2-6) = 0$

$\underline{x = 0}$, $x = \pm\sqrt{6}$ (NA, since outside domain of $f(x)$)

$f''(x)$ DNE

$(4-x^2)^{3/2} = 0$

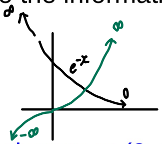
$4 - x^2 = 0$

$\underline{x = \pm 2}$

	$x=-2$ to $x=0$	$x=0$ to $x=2$
$2x$	$-$	$+$
(x^2-6)	$-$	$-$
$(4-x^2)^{3/2}$	$+$	$+$
$f''(x)$	$+$	$-$

$x=0$ inflection point

9. Evaluate the stationary points, absolute extrema, concavity and inflection points of the function below. Use the information to sketch the function.



$$f(x) = x^3 \underbrace{e^{-x}}_{>0}$$

$\lim_{x \to -\infty} f(x) = -\infty$ (no abs min)

$\lim_{x \to \infty} f(x) = 0$

ANS: Stationary at (0, 0) & (3, $27/e^3$) abs max. Convex in (0, 3-√3) & (3+√3, ∞). Concave in (-∞, 0) & (3-√3, 3+√3). Inflection at (0, 0), (3-√3, 0.574) & (3+√3, 0.933).

$$f(x) = x^3 e^{-x}$$

$$f'(x) = 3x^2 e^{-x} + (e^{-x} \cdot -1)(x^3)$$

$$= \frac{3x^2 - x^3}{e^x} = \frac{x^2(3-x)}{e^x}$$

$f'(x) = 0$

$$\frac{x^2(3-x)}{e^x} = 0$$

$$x^2(3-x) = 0$$

$x = 0$ or $x = 3$

$f(0) = 0$ $\quad f(3) = \frac{27}{e^3}$

$f'(x)$ DNE

$\to e^x = 0 \to$ no sol^n

$$f''(x) = \frac{(e^x)(6x - 3x^2) - (e^x)(3x^2 - x^3)}{e^{2x}}$$

$$= \frac{(e^x)(6x - 6x^2 + x^3)}{e^{2x}}$$

$$= \frac{(x\cancel{e^x})(x^2 - 6x + 6)}{e^{\cancel{2x}}}$$

$$= \frac{x(x^2 - 6x + 6)}{e^x}$$

$$f''(x) = 0$$

$$\frac{x(x^2-6x+6)}{e^x} = 0$$

$$x(x^2-6x+6) = 0$$

$x = 0$, $x = 3+\sqrt{3}$, $x = 3-\sqrt{3}$

$f(0) = 0$ $\quad f(3+\sqrt{3}) = 0.933$ $\quad f(3-\sqrt{3}) = 0.573$

$f''(x)$ DNE

$e^x = 0 \rightarrow$ no sol^n

		$x=0$	$x=3-\sqrt{3}$	$x=3+\sqrt{3}$
$f''(x)$	$-$	$+$	$-$	$+$

inflection at $(0,0)$, $(3-\sqrt{3}, 0.573)$, $(3+\sqrt{3}, 0.933)$

Concave in $(-\infty, 0)$, $(3-\sqrt{3}, 3+\sqrt{3})$

Convex in $(0, 3-\sqrt{3})$, $(3+\sqrt{3}, \infty)$

10. (https://openstax.org/books/calculus-volume-1/pages/4-3-maxima-and-minima)
Evaluate the global extrema, if any, for each function in their prescribed interval below. Use the extreme value theorem if possible.

a) $f(x) = x^2 + \frac{2}{x}, \; 1 \le x \le 4$

b) $f(x) = x + \sin x, \; 0 \le x \le 2\pi$

ANS: **a)** Max at (4, 16.5). Min at (1, 3). **b)** Max at (2π, 2π).

b) $f(x) = x + \sin x, \; \underbrace{0 \le x \le 2\pi}_{\text{Close interval } [0, 2\pi]}$

since $f(x)$ is Continuous in the Closed interval $[0, 2\pi]$, EVT applies

Step 1) Get $f'(x)$ & solve for 'Critical' points

$f'(x) = 1 + \cos x$

$f'(x) = 0$

$\cos x = -1$

$x = \pi$

$f'(x)$ DNE → no sol$^{\underline{n}}$

$f(\pi) = \pi + 0$
$= \pi$

Step 2) Get $f(x)$ at endpoints

$f(0) = 0 + \sin 0$
$= 0$

$f(2\pi) = 2\pi + 0$
$= 2\pi$

Step 3) Compare

Abs max at $(2\pi, 2\pi)$

Abs min at $(0, 0)$

a) $f(x) = x^2 + \frac{2}{x},\ 1 \le x \le 4$

$$f'(x) = 2x - \frac{2}{x^2} = \frac{2x^3 - 2}{x^2}$$

$$f'(x) = 0$$

$$\frac{2x^3 - 2}{x^2} = 0$$

$$2x^3 = 2$$

$$x = 1$$

$$f(1) = 3$$

$f'(x)$ DNE

$x^2 = 0 \rightarrow$ no sol^{n}

Get $f(x)$ at end points

$$f(4) = 16.5$$

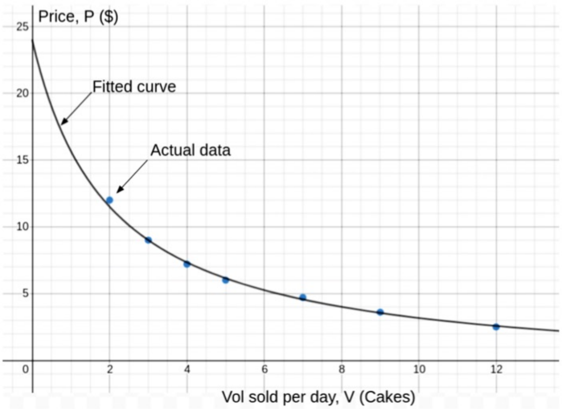
Compare

min $(1, 3)$

max $(4, 16.5)$

11. **Mathematical Modelling:** Optimizing Business Returns

Valerie, an online seller of home baked cakes decided to do some data analytics in order to improve sales and maximize her income / profit. She recorded the price-volume relationship of her pandan cake sales and plotted the graph below.



Using a curve-fitting method, she found that the actual price-volume relationship can be modelled by the price function of volume sold, P(V), below.

$$P(V) = \frac{50}{V+2} - 1$$

a) Given that sales revenue is price multiplied by volume sold, state the optimization problem for Valerie to maximize her sales revenue (R) and solve for the optimal price of one pandan cake.

b) After some thought, Valerie realizes that it is her profit (revenue minus cost) that she should maximize instead. Hence, she estimated that her daily cost of making the pandan cakes, C(V), to be:

$$C(V) = V + 5$$

where \$5 is the fixed costs (for web hosting etc) and V represents the variable costs (for ingredients etc). In this case, restate the optimization problem for Valerie to maximize her profit (Q) and solve for the optimal price of one pandan cake. What is the loss of profit if she maximizes revenue instead?

ANS: a) P = \$4, Max R = \$32 per day. b) P ≈ \$6.10, Max Q ≈ \$21 per day.

↳ $V = 8$

$Q(8) = 32 - 13 = 19$ → Lower profit of \$2 less per day

a)

Max : Revenue , $R(v) = P(v)V$

$$= \frac{50v}{v+2} - V$$

$$R'(v) = \frac{(v+2)50 - 50v(1)}{(v+2)^2} - 1$$

$$= \frac{100 - (v+2)^2}{(v+2)^2}$$

$$R'(v) = 0$$

$$\frac{100-(v+2)^2}{(v+2)^2} = 0$$

$$(v+2)^2 = 100$$

$$v+2 = \pm 10$$

$$v = 8 \quad \text{or} \quad v = -12$$

(NA) since $v > 0$

$R'(v)$ DNE

$$(v+2)^2 = 0$$

$$v = -2$$

(NA since $v > 0$)

Notice that at $v = 0$, $R = 0$ And at $P = 0$, $R = 0$

$$P(V) = \frac{50}{V+2} - 1 = 0$$

$$v + 2 = 50$$

$$v = 48$$

In $v \in [0, 48]$, $R(v)$ is continuous and hence EVT applies

$$R(8) = \frac{50(8)}{8+2} - 8 = 32$$

$$P(8) = \frac{50}{8+2} - 1 = 4$$

Comparing, abs max revenue at \$32 at a price of \$4

b) Max:

$$Q(v) = R(v) - C(v)$$

$$= \frac{50v}{v+2} - v - (v+5)$$

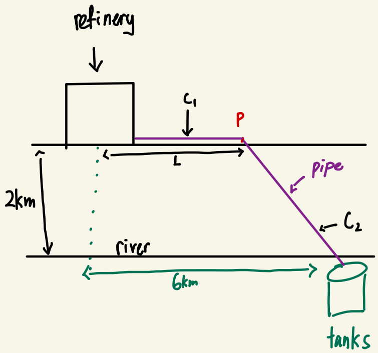
$$= \frac{50v}{v+2} - 2v - 5$$

$$Q'(v) =$$

12. **Mathematical Modelling:** Minimizing Cost of Construction

An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000 / km over land to a point P on the north bank and \$800,000 / km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

ANS: At 4.845 km east of the refinery. Min cost = \$3,785,641.



Min : Cost, $C = C_1 + C_2$

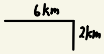
$= 400000L + 800000\sqrt{(6-L)^2+2^2} = C(L)$

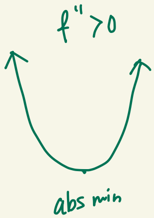
$C(L)$ is continuous in $L \in [0,6]$, so EVT applies

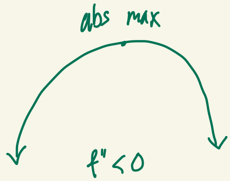
no other constraints

At $\underbrace{L=0}_{\text{All pipe under river}}$, $C = 800000\sqrt{2^2+6^2}$

At $\underbrace{L=6}_{6\text{km}}$, $C = 800000(2) + 400000(6)$




f'' > 0
abs min


abs max
f'' < 0