1.1

Examples for propositions

1) $(3+4=7) \Leftrightarrow t$

equivalent sign

2) $(5<3) \Leftrightarrow f$

3) $(7 \in \mathbb{N}) \Leftrightarrow t$

natural number

4) g: the moon is made from green cheese

assign proposition to variable 'g'

5) s: the moon stinks

6) m: it snowed on the island that is now called Manhattan on the day the king of England signed the Magna Carta.

(cannot verify if true or false)

Examples which are not propositions:

1) Good morning!

2) $\underbrace{x+y}_{\text{unknown values}} > 2$

Logical operators

Not	AND	OR	EITHER, OR	IT FOLLOWS THAT / IT IMPLIES	IF AND ONLY IF
$\neg$	$\wedge$	$\vee$	$\leftarrow\!\!\!+\!\!\!\rightarrow$ (Alternative)	$\rightarrow$ (Conditional)	$\leftrightarrow$ (Biconditional)

Strength of operators

(order to evaluate logical propositions)

e.g. $(2<3) \wedge \neg(3<2) \Leftrightarrow t$

use paranthesis as much as possible

$(2<3) \rightarrow (2<4) \Leftrightarrow t$

Propositional variables: $a, b, c, \ldots, x, y, z$ (placeholders for statements or propositions)

x_1, x_2, x_3

Propositional form: e.g. $a \wedge (a \rightarrow b)$

$\underbrace{a \vee b}_{\text{before assigning value}}$

Ternary propositional form : $A(x_1, x_2, x_3) \triangleq A(\underline{x})$; $\underline{x} = (x_1, x_2, x_3)$

$\underbrace{\text{Ternary}}_{\text{3 variable}}$

e.g. $A(x_1, x_2, x_3) \Leftrightarrow (x_1 \vee x_2) \rightarrow x_3$

Truth value of a propositional form :

$\hat{A} \triangleq val(A) \triangleq \partial(A)$

Value assignment — Evaluation of propositional form — Interpretation

e.g. $\hat{a} \triangleq val(a)$

$A(\hat{x}_1, \hat{x}_2, \hat{x}_3) \triangleq A(\hat{\underline{x}}) \triangleq \hat{A}$

$\hat{A} \Leftrightarrow A(t, f, t) \Leftrightarrow (t \vee f) \rightarrow t \Leftrightarrow t \rightarrow t \Leftrightarrow t$

e.g. $\hat{g} \Leftrightarrow f$; $s \Leftrightarrow f$; $\hat{m} \in \{t, f\}$

Value assignment pair : $(\hat{a}, \hat{b})$; $(\hat{x}_1, \hat{x}_2)$

triple : $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$

n-tuple : $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$

e.g. (f, f) ; (f, t) ; (t, f) ; (t, t)

$(0,0)$; $(0,1)$; $(1,0)$; $(1,1)$

$\hat{\underline{x}}_0$; $\hat{\underline{x}}_1$; $\hat{\underline{x}}_2$; $\hat{\underline{x}}_3$

Value pattern of the propositional form

e.g. $A(x_1, x_2) \Leftrightarrow x_1 \vee x_2$

$$\underline{\hat{W}}[A] = (A(\hat{\underline{x}}_0), A(\hat{\underline{x}}_1), A(\hat{\underline{x}}_2), A(\hat{\underline{x}}_3))$$

$$= (f \vee f, f \vee t, t \vee f, t \vee t)$$

$$= (f, t, t, t)$$

$\underline{\hat{W}}[A]$ describes A completely

On-set of propositional form :

e.g. For $A \Leftrightarrow x_1 \vee x_2$

$$E[A] = \{(f,t), (t,f), (t,t)\}$$

$$= \{(0,1), (1,0), (t,t)\}$$ on-set, truth-set

$\overline{E}[A] = \{(0,0)\}$ off-set, falsity-set

$\underset{\text{Universal set,}}{E[A]} = G$; $G = \{(0,0), (0,1), (1,0), (1,1)\}$

$= \{1,0\} \times \{1,0\} = \{1,0\}^2$

Cross product

Cartesian product

$E[A]$ describes A completely

1.2 logical Operators (connectives)

Operators, Connectives, junctors : unary, binary

$a \to b \iff \neg a \vee b$

$a \leftrightarrow b \iff (a \to b) \wedge (b \to a)$

$\iff (\neg a \vee b) \wedge (\neg b \vee a)$

$a \not\leftrightarrow b \iff \neg(a \leftrightarrow b)$

(inverse of biconditonal)

$\iff \neg[(a \to b) \wedge (b \to a)]$

$\iff \neg[(\neg a \vee b) \wedge (\neg b \vee a)]$

$\iff (a \wedge \neg b) \vee (b \wedge \neg a)$

$(\{t, f\}; \wedge, \vee, \neg)$: Basis Boolean Algebra

1.3 Propositional forms

a) Representation by Normal forms

Disjunctive normal form (DNF) : disjunction of Conjunctive terms

Conjunctive normal form (CNF) : Conjunction of disjunctive terms

Canonical representation CDNF, CCNF : Unique representation of propositional form

Example : binary propositional form $(n=2)$

$A(a,b) \iff (a \to b) \wedge b$

Truth table :

i	$\hat{a}$	$\hat{b}$	$\hat{a} \to \hat{b}$	$\hat{A}$	$\hat{D}_1$	$\hat{D}_2$	$\hat{C}_1$	$\hat{C}_2$
0	0	0	1	0	0	1	0	0
1	0	1	1	1	1	1	1	0
2	1	0	0	0	1	0	0	0
3	1	1	1	1	1	1	0	1

$i \le 2^n - 1$ 2^n value assignment pairs

Value assignment pair $(\hat{a}, \hat{b})$; - triple ; n-tuple

set of all value assignment pairs : $\{(0,0), (0,1), (1,0), (1,1)\} = G$

universal set ; Basic set; $G = \{(1,0)\} \times \{1,0\} = \{1,0\}^2$

Cross product set ;

Universe of discourse

$G = \{(\hat{a}, \hat{b})\} \mid \hat{a} \in \{0,1\} \wedge \hat{b}$

value pattern $\hat{\underline{W}}[A] = (0,1,0,1)$

max term implement value 0 in value pattern

min term implement value 1 in value pattern

On-set $E[A] = \{(0,1), (1,1)\}$

off-set $\bar{E}[A] = \{(0,0), (1,0)\}$

$E[A] = \{(\hat{a}, \hat{b}) \mid \hat{A} \Leftrightarrow t\}$

$E[A] \subseteq G = \{1,0\}^2$

binary maxterm : $D_{(0,0)} \Leftrightarrow D_1(a,b)$

$D_{(1,0)} \Leftrightarrow D_2(a,b)$

$D_{(0,0)} \Leftrightarrow (a \not\leftrightarrow 0) \vee (b \not\leftrightarrow 0) \Leftrightarrow a \vee b \Leftrightarrow D_1$

$D_{(1,0)} \Leftrightarrow (a \not\leftrightarrow 1) \vee (b \not\leftrightarrow 0) \Leftrightarrow \neg a \vee b \Leftrightarrow D_2$

$\bar{E}[D_1] = \bar{E}[D_{(0,0)}] = \{(0,0)\} = \bar{E}[a \vee b]$

$\bar{E}[D_2] = \bar{E}[D_{(1,0)}] = \{(1,0)\} = \bar{E}[\neg a \vee b]$

Constructing the Canonical conjunctive normal form:

$CCNF[A] \Leftrightarrow D_1 \wedge D_2$

$\Leftrightarrow (a \vee b) \wedge (\neg a \vee b)$

$A \Leftrightarrow (a \to b) \wedge b \Leftrightarrow D_1 \wedge D_2 \Leftrightarrow (a \vee b) \wedge (\neg a \vee b) \Leftrightarrow b$

General structure of CCNF[A]:

$CCNF[A] : \Leftrightarrow \bigwedge_{\mu=1}^{m} D_\mu \Leftrightarrow \bigwedge_{\hat{\underline{x}} \in \bar{E}[A]} \bigvee_{\substack{x \in set(\underline{x}) \\ x \in \{x_1, x_2, \dots, x_n\}}} x \not\leftrightarrow \hat{x}$

multiple conjunction

$m = |\bar{E}[A]|$

e.g. $\underline{X} = (x_1, x_2, x_3, x_4)$

$set(\underline{X}) = \{x_1, x_2, x_3, x_4\}$

binary minterms : $C_{(0,1)} \Leftrightarrow C_1(a,b)$

$C_{(1,1)} \Leftrightarrow C_2(a,b)$

$C_{(0,1)} \Leftrightarrow (a \leftrightarrow 0) \wedge (b \leftrightarrow 1) \Leftrightarrow \neg a \wedge b \Leftrightarrow C_1$

$C_{(1,1)} \Leftrightarrow (a \leftrightarrow 1) \wedge (b \leftrightarrow 1) \Leftrightarrow a \wedge b \Leftrightarrow C_2$

$E[C_1] = E[C_{(0,1)}] = \{(0,1)\} = E[\neg a \wedge b]$

$E[C_2] = E[C_{(1,1)}] = \{(1,1)\} = E[a \wedge b]$

$\bar{E}[C] = G \setminus E[C]$

$$\boxed{\begin{aligned} CDNF[A] &\Leftrightarrow C_1 \vee C_2 \\ &\Leftrightarrow (\neg a \wedge b) \vee (a \wedge b) \end{aligned}}$$

$$A \Leftrightarrow (a \to b) \wedge b \Leftrightarrow (\neg a \wedge b) \vee (a \wedge b) \Leftrightarrow b$$

General structure of CDNF[A] :

$$CDNF[A] :\Leftrightarrow \bigvee_{\sigma=1}^{s=2^n-m} C_\sigma \Leftrightarrow \bigvee_{\underline{\hat{x}} \in E[A]} \quad \bigwedge_{x \in sef(\underline{x})} x \leftrightarrow \hat{x}$$

multiple disjunction

Example : turnary propositional form

$$A(a,b,c) \Leftrightarrow [a \to (b \wedge c)] \leftrightarrow b$$

i	$\hat{a}$ $\hat{b}$ $\hat{c}$	$\hat{b} \wedge \hat{c}$	$\hat{a} \to (\hat{b} \wedge \hat{c})$	$\hat{A}$	D_1	D_2	D_3
0	0 0 0	0	1	0	0	1	1
1	0 0 1	0	1	0	1	0	1
2	0 1 0	0	1	1	1	1	1
3	0 1 1	1	1	1	1	1	1
4	1 0 0	0	0	1	1	1	1
5	1 0 1	0	0	1	1	1	1
6	1 1 0	0	0	0	1	1	0
7	1 1 1	1	1	1	1	1	1

Value assignment triples : $(\hat{a}, \hat{b}, \hat{c})$

$$G = \{(\hat{a}, \hat{b}, \hat{c}) \mid \hat{a} \in \{0,1\} \wedge \hat{b} \in \{0,1\} \wedge \hat{c} \in \{0,1\}\} = \{1,0\}^3$$

$$= \{(0,0,0), (0,0,1), \ldots, (1,1,1)\}$$

$$|\{1,0\}^3| = |\{1,0\}|^3 = 2^3 = 8$$

Value pattern $\underline{w}^n[A] = (0,0,1,1,1,1,0,1)$

on-set : $E[A] = \{(\hat{a}, \hat{b}, \hat{c}) \mid \hat{A} \Leftrightarrow 1\} \subseteq G$

$$E[A] = \{(0,0,0), (0,0,1), (1,1,0)\}$$

Ternary maxterms :

$$D_1 \Leftrightarrow D_{(0,0,0)} \Leftrightarrow a \vee b \vee c$$

$$D_2 \Leftrightarrow D_{(0,0,1)} \Leftrightarrow a \vee b \vee \neg c$$

$$D_3 \Leftrightarrow D_{(1,1,0)} \Leftrightarrow \neg a \vee \neg b \vee c$$

$$CCNF[A] \Leftrightarrow D_1 \wedge D_2 \wedge D_3 \Leftrightarrow (a \vee b \vee c) \wedge (a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee c)$$

$$\Leftrightarrow (a \vee b) \wedge (\neg a \vee \neg b \vee c)$$

b) Properties of propositional forms

Propositional form as an (n-ary) discrete function

$A(\hat{\underline{x}}) : \{t,f\}^n \to \{t,f\}$

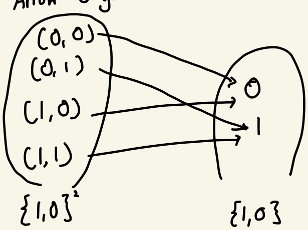
$\underline{X}$: variable n-tuple $(X_1, X_2, \dots, X_n)$

$\hat{\underline{x}}$: value assignment n-tuple $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$

$= (t, f, \dots, t)$

e.g. $A(a,b) \Leftrightarrow (a \to b) \wedge b \Leftrightarrow b$

Arrow diagram



$E[A] = \{\hat{\underline{x}} \mid \hat{A} \Leftrightarrow t\}_G = \{\hat{\underline{x}} \mid \hat{A}\}_G$

$G = \{t,f\}^n$

$|G| = |\{t,f\}^n| = |\{t,f\}|^n = 2^n$

Satisfiable propositional form : $E[A] \neq \emptyset$

$\hat{\underline{x}}$ verifies A, IF $A(\hat{\underline{x}}) \Leftrightarrow t$

$\hat{\underline{x}}$ falsifies A, IF $A(\hat{\underline{x}}) \Leftrightarrow f$

Contingency : $\emptyset \subset E[A] \subset G$

Notation for sets : $N \subset M$: N is proper subset of M

$N \subseteq M$: N is subset of M

$N \subseteq M \Leftrightarrow (N \subset M) \vee (N = M)$

$E[A] \neq \emptyset \wedge E[A] \neq G$

Contradiction : $E[A] = \emptyset$

e.g. $A(a) \Leftrightarrow a \wedge \neg a \Leftrightarrow f$

Tautology : $E[A] = G$

e.g. $A(a) \Leftrightarrow a \vee \neg a \Leftrightarrow t$

$A(\underline{X})$ and $B(\underline{X})$ are disjoint : $E[A \wedge B] = \emptyset$

$A(\underline{X})$ and $B(\underline{X})$ are complete : $E[A \vee B] = G$

e.g. $A(a,b) \Leftrightarrow (a \to b) \wedge b$

$C_1 \wedge C_2 \Leftrightarrow (\neg a \wedge b) \wedge (a \wedge b) \Leftrightarrow f$ C_1, C_2 are disjoint

$D_1 \vee D_2 \Leftrightarrow (a \vee b) \vee (\neg a \vee b) \Leftrightarrow t$ D_1, D_2 are complete

c) Equivalent Propositional Forms

Equivalence $A(\underline{x}) \Leftrightarrow B(\underline{x})$ "A is logically equivalent to B"

Universally valid biconditional $A(x) \leftrightarrow B(x) \Leftrightarrow t$
- biconditional
- evaluate to true with any value

Identical value patterns $\underline{\hat{w}}[A] = \underline{\hat{w}}[B]$

Identical on-set $E[A] = E[B]$

$E[A \leftrightarrow B] = G$

e.g. $\left.\begin{array}{l} A \Leftrightarrow a \rightarrow b \\ B \Leftrightarrow \neg a \vee b \end{array}\right\}$ $A \leftrightarrow B \Leftrightarrow t$; $A \Leftrightarrow B$

$$\underbrace{(a \rightarrow b) \leftrightarrow (\neg a \vee b)}_{\text{object level}} \underbrace{\Leftrightarrow t}_{\text{meta level}}$$

		$a \rightarrow b$	$\neg a \vee b$	
a	b	A	B	$A \leftrightarrow B$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	1	1	1

$(a \rightarrow b) \Leftrightarrow (\neg a \vee b)$

e.g. from $a \leftrightarrow b \leftrightarrow (a \rightarrow b) \wedge (b \rightarrow a) \Leftrightarrow t$

we get $a \leftrightarrow b \Leftrightarrow (a \rightarrow b) \wedge (b \rightarrow a)$

as well $a \Leftrightarrow b \leftrightarrow (a \rightarrow b) \wedge (b \rightarrow a)$

Proof of Equivalence (identical value patterns, tautology)

1) Truth table

2) Arrow diagram (to show tautology)

3) Canonical normal forms

4) Decision tree, decision diagram

5) Resolution method

6) Achieve structural equivalence through algebraic transformations

1.4 Laws and Rules

a) Laws of Boolean Propositional algebra $(\{t, f\}; \wedge, \vee, \neg)$

A_3, $a \wedge (b \vee c) \Leftrightarrow (a \wedge b) \vee (a \wedge c)$ Distributivity

$a \vee (b \wedge c) \Leftrightarrow (a \vee b) \wedge (a \vee c)$

$\boxed{a \wedge (b \vee c) \longleftrightarrow (a \wedge b) \vee (a \wedge c)}$ biconditonal is tautology

$a \Leftrightarrow t$ $\quad$ $a \Leftrightarrow f$

$\boxed{(b \vee c) \longleftrightarrow (b \vee c)}$ $\quad$ $\boxed{\begin{array}{l} f \longleftrightarrow (f \vee f) \\ f \longleftrightarrow f \end{array}}$ $\to$ $\boxed{t}$

$b \Leftrightarrow t$ $\quad$ $b \Leftrightarrow f$

$\boxed{t \longleftrightarrow t}$ $\to$ $\boxed{t}$ $\quad$ $\boxed{c \longleftrightarrow c}$

$c \Leftrightarrow t$ $\quad$ $c \Leftrightarrow f$

$\boxed{t \longleftrightarrow t}$ $\to$ $\boxed{t}$ $\quad$ $\boxed{f \longleftrightarrow f}$ $\to$ $\boxed{t}$

Structure of Law of distributivity :

$a * (b \circ c) \Leftrightarrow (a * b) \circ (a * c)$ $\quad$ $*$ is left sided distributive over $\circ$

$(a * b) \circ c \Leftrightarrow (a \circ c) * (b \circ c)$ $\quad$ $\circ$ is right sided distributive over $*$

A5) $a \wedge (a \vee b) \Leftrightarrow a$ $\quad$ absorption

$a \vee (a \wedge b) \Leftrightarrow a$

a b	$a \vee b$	$a \wedge (a \vee b)$	$a \wedge b$	$a \vee (a \wedge b)$
0 0	0	0	0	0
0 1	1	0	0	0
1 0	1	1	0	1
1 1	1	1	1	1

By algebraic transformation

1)	$a \wedge (a \vee b) \iff a$	
2)	$(a \vee f) \wedge (a \vee b)$	1) A6
3)	$(a \vee (b \wedge \neg b)) \wedge (a \vee b)$	2) A8
4)	$(a \vee b) \wedge (a \vee \neg b) \wedge \cancel{(a \vee b)}$ Idempotence	3) A3
5)	$(a \vee b) \wedge (a \vee \neg b)$	4) A4
6)	$a \vee (b \wedge \neg b)$	5) A3
7)	a	6) A8, A6

De Morgan : $\neg(a \wedge b) \iff \neg a \vee \neg b$

$\neg(a \vee b) \iff \neg a \wedge \neg b$

a	b	$a \wedge b$	$\neg(a \wedge b)$	$\neg a$	$\neg b$	$\neg a \vee \neg b$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

b) Laws of Alternative

a	b	$a \nleftrightarrow b$
0	0	0
0	1	1
1	0	1
1	1	0

$\triangleq a \oplus b \triangleq (a+b) \bmod 2$

B3, BDT

$\boxed{a \wedge (b \nleftrightarrow c) \longleftrightarrow a \wedge b \nleftrightarrow (a \wedge c)}$

$a \iff t$:

$\boxed{(b \nleftrightarrow c) \longleftrightarrow (b \nleftrightarrow c)}$

$\boxed{t}$

$a \iff f$:

$f \longleftrightarrow (f \nleftrightarrow f)$

$f \longleftrightarrow f$

$\boxed{t}$

B8) $a \not\leftrightarrow b \iff \neg a \not\leftrightarrow \neg b$

$\iff (a \not\leftrightarrow t) \not\leftrightarrow (b \not\leftrightarrow t)$

$\iff a \not\leftrightarrow b \not\leftrightarrow \underbrace{t \not\leftrightarrow t}_{f}$

$\iff a \not\leftrightarrow b \not\leftrightarrow f$

$\iff a \not\leftrightarrow b$ (RHS = LHS)

B9) 1) $a \not\leftrightarrow b \iff (a \wedge \neg b) \vee (\neg a \wedge b)$

2) $\neg(a \not\leftrightarrow b)$ — 1) B11

3) $\neg[(a \to b) \wedge (b \to a)]$ — 2) C11

4) $\neg[(\neg a \vee b) \wedge (\neg b \vee a)]$ — 3) D1

5) $\neg(\neg a \vee b) \vee \neg(\neg b \vee a)$ — 4) A10

6) $(a \wedge \neg b) \vee (b \wedge \neg a)$ — 5) A10

7) $(a \wedge \neg b) \vee (\neg a \wedge b)$ — 6) A1

CDNF[$a \not\leftrightarrow b$]

B10) 1) $a \not\leftrightarrow b \iff (a \vee b) \wedge (\neg a \vee \neg b)$

2) $(a \wedge \neg b) \vee (\neg a \wedge b)$ 1) B9

3) $[(a \wedge \neg b) \vee \neg a] \wedge [(a \wedge \neg b) \vee b]$ 2) A3

4) $\underbrace{(a \vee \neg a)}_{t} \wedge (\neg b \vee \neg a) \wedge (a \vee b) \wedge (\underbrace{\neg b \vee b}_{t})$ 3) A3

5) $t \wedge (\neg b \vee \neg a) \wedge (a \vee b) \wedge t$ 4) A8

6) $(a \vee b) \wedge (\neg a \vee \neg b)$ 5) A6, A1

CCNF $[a \not\leftrightarrow b]$

B12) 1) $a \not\leftrightarrow b \iff (a \rightarrow b) \rightarrow \neg(b \rightarrow a)$

2) $\neg(a \leftrightarrow b)$ 1) B11

3) $\neg[(a \rightarrow b) \wedge (b \rightarrow a)]$ 2) C11

4) $\neg(a \rightarrow b) \vee \neg(b \rightarrow a)$ 3) A10

5) $(a \rightarrow b) \rightarrow \neg(b \rightarrow a)$ 4) D1

B|3)

$$a \lor b \iff a \not\leftrightarrow b \not\leftrightarrow (a \land b)$$

case differential way

$$a \iff t : \quad t \iff t \not\leftrightarrow \underbrace{b \not\leftrightarrow b}_{f}$$

$$\iff t \not\leftrightarrow f$$

$$\iff t$$

$$a \iff f : \quad b \iff f \not\leftrightarrow b \not\leftrightarrow f$$

$$\iff b \not\leftrightarrow f \not\leftrightarrow f$$

$$\iff b \not\leftrightarrow f$$

$$\iff b$$

Proposition Logic

$A(a,b) \Leftrightarrow a \leftrightarrow b$

i	a	b	A
0	0	0	0
1	0	1	1
2	1	0	1
3	1	1	0

- Value Pattern

$\hat{W}[A] = (0,1,1,0)$

- Column A of the Table
- All assignments must be noted as ascending binary numbers
- Must Use Round Brackets

- Canonical disjunctive normal form (CDNF) and on set
- These represent the t(true)/1 assignments of A
- In the CDNF's conjunctive terms, variables assigned to f/0 will be negated, variables assigned to t/1 will be noted in positive form

$CDNF[A] \Leftrightarrow (\neg a \wedge b) \vee (a \wedge \neg b)$

$E[A] \Leftrightarrow \{(0,1), (1,0)\}$

Canonical Conjunctive Normal form (CCNF) and off-set

- These represent the f(false)/0 - assignments of A
- In the CCNF's disjunctive terms, variables assigned t/1 will be negated, variables assigned to f/0 will be noted in positive form

$CCNF[A] \Leftrightarrow (a \vee b) \wedge (\neg a \vee \neg b)$

$(\neg a \lor a \lor b) \land c$

$\Leftrightarrow \underbrace{(t \lor b)}\land c$ A8 Complementary Element

$\Leftrightarrow \underbrace{t \land c}$ A7 Domination

$\Leftrightarrow c$ A6 Neutral Element

Learn A Laws by heart

$(a \lor c) \Leftrightarrow (a \lor b \lor c) \land (a \lor \neg b \lor c)$

$\Leftrightarrow (a \lor c) \lor f$ A6

$\Leftrightarrow (a \lor c) \lor (b \land \neg b)$ A8

$\Leftrightarrow (a \lor b \lor c) \land (a \lor \neg b \lor c)$ A3

c) Laws Using Biconditional

a	b	$a \leftrightarrow b \triangleq a \ominus b$
0	0	1
0	1	0
1	0	0
1	1	1

(3) $a \vee (b \leftrightarrow c) \Leftrightarrow (a \vee b) \leftrightarrow (a \vee c)$

$b \vee f$

false cancels out

$a \Leftrightarrow t: \quad t \Leftrightarrow t \leftrightarrow t$

$t \Leftrightarrow t$

$a \Leftrightarrow f: \quad b \leftrightarrow c \Leftrightarrow b \leftrightarrow c$

(7) $a \leftrightarrow a \leftrightarrow a \Leftrightarrow a$

An equivalent is a biconditional which is always true

$a \leftrightarrow a \leftrightarrow a \leftrightarrow a \Leftrightarrow t$

$a \leftrightarrow a \Leftrightarrow a \leftrightarrow a$

$\underbrace{a \leftrightarrow a}_{t} \leftrightarrow \underbrace{a \leftrightarrow a}_{t} \Leftrightarrow t$

$t \leftrightarrow t \Leftrightarrow t$

$t \Leftrightarrow t$

(8) $a \leftrightarrow b \Leftrightarrow \neg a \leftrightarrow \neg b$ Contraposition

$a \leftrightarrow b \leftrightarrow \neg a \leftrightarrow \neg b \Leftrightarrow t$

$\underbrace{a \leftrightarrow \neg a}_{f} \leftrightarrow \underbrace{b \leftrightarrow \neg b}_{f} \Leftrightarrow t$

$f \leftrightarrow f \Leftrightarrow t$

$t \Leftrightarrow t$

C10) 1) $a \longleftrightarrow b \iff (\neg a \vee b) \wedge (a \vee \neg b)$

2) $(a \to b) \wedge (b \to a)$ 1) C11)

3) $(\neg a \vee b) \wedge (\neg b \vee a)$ 2) D1

4) $\underbrace{(\neg a \vee b) \wedge (a \vee \neg b)}_{CCNF[a \longleftrightarrow B]}$ 3) A1

C9) 1) $a \longleftrightarrow b \iff (a \wedge b) \vee (\neg a \wedge \neg b)$

2) $(\neg a \vee b) \wedge (a \vee \neg b)$ 1) C10

3) $[(\neg a \vee b) \wedge a] \vee [(\neg a \vee b) \wedge \neg b]$ 2) A3

4) $[\overset{f}{\cancel{(\neg a \wedge a)}} \vee (b \wedge a)] \vee [(\neg a \wedge \neg b) \vee \overset{f}{\cancel{(b \wedge \neg b)}}]$ 3) A3

5) $\underbrace{(a \wedge b) \vee (\neg a \wedge \neg b)}_{CDNF[a \longleftrightarrow b]}$ 4) A8, A6

C11) $a \longleftrightarrow b \iff (a \to b) \wedge (b \to a)$

$a \iff t$: $b \iff (t \to b) \wedge (b \to t)$

$\iff (f \vee b) \wedge (\neg b \vee t)$

$\iff b \wedge t$

$\iff b$

$a \iff f$: $\neg b \iff (f \to b) \wedge (b \to f)$

$\iff (t \vee b) \wedge (\neg b \vee f)$

$\iff t \wedge \neg b$

$\iff \neg b$

C13) $a \vee b \iff a \leftrightarrow b \leftrightarrow (a \wedge b)$

$a \iff t$: $t \iff t \leftrightarrow \underbrace{b \leftrightarrow b}_{t}$

$\iff t \leftrightarrow t$

$\iff t$

$a \iff f$: $b \iff f \leftrightarrow b \leftrightarrow f$

$\iff b \leftrightarrow t$

$\iff b$

d) Laws using conditional

a	b	$a \rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

a: antecedent
b: consequent

a: the power switch is off

b: the motor does not run

$a \rightarrow b$; $\cancel{b \rightarrow a}$

$a \rightarrow b \iff (a \rightarrow b) \wedge \underbrace{(\neg a \rightarrow \underbrace{(b \vee \neg b)}_{t})}_{t}$

D1) $a \rightarrow b \iff \underbrace{\neg a \vee b}$

DNF[$a \rightarrow b$]

CCNF[$a \rightarrow b$]

D3) $a \rightarrow b \iff \neg b \rightarrow \neg a$ Contraposition

$\neg a \vee b \iff b \vee \neg a$

D7) 1) $a \rightarrow (b \wedge c) \iff (a \rightarrow b) \wedge (a \rightarrow c)$

2) $\neg a \vee (b \wedge c)$ 1) D1

3) $(\neg a \vee b) \wedge (\neg a \vee c)$ 2) A3

4) $(a \rightarrow b) \wedge (a \rightarrow c)$ 3) D1

- - - - - - - - - - -

1) $(a \wedge b) \rightarrow c \iff (a \rightarrow c) \vee (b \rightarrow c)$

2) $\neg(a \wedge b) \vee c$ 1) D1

3) $\neg a \wedge \neg b \vee c$ 2) A10

4) $\neg a \vee \neg b \vee c \vee c$ 3) A4

5) $\neg a \vee c \vee \neg b \vee c$ 4) A1

6) $(a \to c) \vee (b \to c)$ 5) D1

e) Laws of Implication

Implication : $\underbrace{A(\underline{x})}_{\text{Premise}} \Rightarrow \underbrace{B(\underline{x})}_{\text{Conclusion}}$ " From A, if

"A implies B"

Universally valid Conditional:

$$A(\underline{x}) \to B(\underline{x}) \Leftrightarrow t$$

$$E[A \to B] = G$$

$$E[A] \subseteq E[B]$$

E1) $f \Rightarrow a$ "ex falso quodlibet"

$f \to a \Leftrightarrow t$

$t \vee a \Leftrightarrow t$

$t \Leftrightarrow t$

E2) $a \Rightarrow t$ "ex quodlibet verum"

$a \to t \Leftrightarrow t$

$\neg a \vee t \Leftrightarrow t$

$t \Leftrightarrow t$

E3) $a \Rightarrow a$ 'identity Law'

$a \to a \Leftrightarrow t$

$\neg a \vee a \Leftrightarrow t$

$t \Leftrightarrow t$

* E4) $a \wedge b \Rightarrow a$ 'simplification'

$a \wedge b \to a \Leftrightarrow t$

$\underbrace{\neg a \vee \neg b \vee a}_{} \Leftrightarrow t$

$t \Leftrightarrow t$

E5) $a \Rightarrow a \vee b$ "addition"

$a \to a \vee b \Leftrightarrow t$

$\underbrace{\neg a \vee a}_{t} \vee b \Leftrightarrow t$

$t \Leftrightarrow t$

E6) $a \wedge b \Rightarrow a \vee b$

$a \wedge b \rightarrow a \vee b \iff t$

$\underbrace{\neg a \vee \neg b \vee a \vee b}_{t \quad t} \iff t$

$t \iff t$

E9) $a \wedge (a \rightarrow b) \Rightarrow b$ 'modus tollens'

$a \wedge (a \rightarrow b) \rightarrow b \iff t$

$[a \wedge (\neg a \vee b)] \rightarrow b \leftrightarrow t$

$[\underbrace{(a \wedge \neg a)}_{f} \vee (a \wedge b)] \rightarrow b \iff t$

$a \wedge b \rightarrow b \iff t$

$\neg a \vee \underbrace{\neg b \vee b}_{t} \iff t$

$t \iff t$

E10) $\neg b \wedge (a \rightarrow b) \Rightarrow \neg a$ 'modus tollens'

$\neg b \wedge (a \rightarrow b) \rightarrow \neg a \iff t$

$\neg b \wedge (\neg a \vee b) \rightarrow \neg a \iff t$

$[(\neg b \wedge \neg a) \vee \cancel{(\neg b \wedge b)}] \rightarrow \neg a \iff t$

$(\neg a \wedge \neg b) \rightarrow \neg a \iff t$

$\overbrace{a \vee b \vee \neg a}^{t} \iff t$

$t \iff t$

e.g. a: the lectures gets cancelled

b: the students are happy

$V_1 \Leftrightarrow (a \to b) \wedge a$; $V_1 \Rightarrow b$ E9)

$V_2 \Leftrightarrow (a \to \neg b) \wedge b$; $V_2 \Rightarrow \neg a$ E10)

$V_3 \Leftrightarrow (a \to b) \wedge b$; $V_3 \Rightarrow a$??

ab	V_1	b	V_2	$\neg a$	V_3	a	$V_3 \to a$	S_1	S_2	S_3	$\cancel{S_4}$ trivial
0 0	0	0	0	1	0	0	1	0	1	0	1
0 1	0	1	1	1	1	0	0	1	1	1	1
1 0	0	0	0	0	0	1	1	1	0	0	1
1 1	1	1	0	0	1	1	1	1	1	1	1

$E[V_1] \subseteq E[b]$: $\{(1,1)\} \subseteq \{(0,1), (1,1)\}$

$E[V_2] \subseteq E[\neg a]$: $\{(0,1)\} \subseteq \{(0,0), (0,1)\}$

$E[V_3] \nsubseteq E[a]$: $\{(0,1), (1,1)\} \nsubseteq \{(1,0), (1,1)\}$

$V_3 \nRightarrow a$

$V_3 \Rightarrow S_\lambda$ $\lambda = 1, 2, 3, \cancel{4}$ trivial conclusion

$S_1 \Leftrightarrow D_1 \Leftrightarrow a \vee b$

$S_2 \Leftrightarrow D_2 \Leftrightarrow \neg a \vee b \Leftrightarrow a \to b$

$S_3 \Leftrightarrow D_1 \wedge D_2$

Propositional Rules of Inference

Implication $\underset{\text{Premise}}{V} \Rightarrow \underset{\text{Conclusion}}{S}$ $\quad (\text{IFF } E[V] \subseteq E[S])$

Special case

IF $V \Leftrightarrow t$, Then $S \Leftrightarrow t$

IF $S \Leftrightarrow f$, Then $V \Leftrightarrow f$

S is one of 2^m possible Conclusions from V

$2^m - 1$ non trival Conclusions

1 trival conclusion $S \Rightarrow t$

m: number of false assignments of V

E11) $(a \rightarrow b) \wedge (b \rightarrow c) \Rightarrow (a \rightarrow c)$ hypothetical syllogism (modus barbara) "Chain Conclusion"

$$(a \rightarrow b) \wedge (b \rightarrow c) \rightarrow (a \rightarrow c) \Leftrightarrow t$$

$$\neg[(a \rightarrow b) \wedge (b \rightarrow c)] \vee (a \rightarrow c) \Leftrightarrow t$$

D1 $\neg[(\neg a \vee b) \wedge (\neg b \vee c)] \vee (\neg a \vee c) \Leftrightarrow t$

A10 $\neg(\neg a \vee b) \vee \neg(\neg b \vee c) \vee (\neg a \vee c) \Leftrightarrow t$

A10 $(a \wedge \neg b) \vee (b \wedge \neg c) \vee (\neg a \vee c) \Leftrightarrow t$

$$[(a \wedge \neg b) \vee \neg a] \vee [(b \wedge \neg c) \vee c] \Leftrightarrow t$$

$$[\underbrace{\cancel{(a \vee \neg a)}}_{t} \wedge (\neg b \vee \neg a)] \vee [(b \vee c) \wedge \cancel{(\neg c \vee c)}] \Leftrightarrow t$$

$$\underbrace{\neg b \vee \neg a \vee b}_{t} \vee c \Leftrightarrow t$$

(Mathematical) Induction

Scheme : $\left.\begin{array}{l} a \land \\ (a \to b) \land \\ (b \to c) \land \\ (c \to d) \end{array}\right\} \begin{array}{ll} \Rightarrow (a \to d) & E11 \\ \Rightarrow d & E8 \end{array}$

Eg. $\mathbb{N}_0 = \{0, 1, 2, 3 \ldots\}$

$\mathbb{E} = \{0, 0, 4, 6, \ldots\}$

$\mathbb{O} = \{1, 3, 5, \ldots\}$

$n, k \in \mathbb{N}_0$

$(n \in \mathbb{E} \to n = 2 \cdot k) \land$

$(n = 2 \cdot k \to n^2 = 2.2k^2) \land$

$n^2 = 2 \cdot (2k^2) \to n^2 \in \mathbb{E})$

$\Rightarrow n \in \mathbb{E} \Rightarrow n^2 \in \mathbb{E}$

$CDNF[A] \Leftrightarrow CCNF[\neg A]$

Structural

$CCNF[A] \Leftrightarrow \neg CDNF[\neg A]$

Struct.

maxterm 0

minterm 1

e.g. $A \Leftrightarrow (a \to b) \land b$

a b	A	¬A
0 0	0	1
0 1	1	0
1 0	0	1
1 1	1	0

$CCNF[\neg A] \Leftrightarrow D_{(0,1)} \land D_{(1,1)}$

$\Leftrightarrow (a \lor \neg b) \land (\neg a \lor \neg b)$

$CDNF[A] \Leftrightarrow C_{(0,1)} \lor C_{(1,1)}$

$\Leftrightarrow (\neg a \land b) \lor (a \land b)$

$\neg CCNF[\neg A] \Leftrightarrow (\neg a \land b) \lor (a \land b)$

$\text{dual}(A(a,b,c,\dots,t,f)) :\Longleftrightarrow \neg A(\neg a,\neg b,\neg c,\dots,t,f)$

$\text{dual}(\text{dual}(A)) \Longleftrightarrow A$

IF $A(\underline{x})$ is a propositional form on $(\{t,f\}; \wedge, \vee, \neg)$

THEN $\text{dual}(A(\underline{x}))$: $(\{f,t\}; \vee, \wedge, \neg)$

e.g. $\text{dual}(a \wedge (a \vee b)) \Longleftrightarrow \neg(\neg a \wedge (\neg a \vee \neg b))$

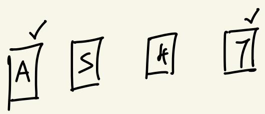
$\Longleftrightarrow a \vee (a \wedge b)$

$\text{dual}(a) \Longleftrightarrow \neg(\neg a)$

$\Longleftrightarrow a$

IF $A(\underline{x}) \Longleftrightarrow B(\underline{x})$ THEN $\text{dual}(A(\underline{x})) \Longleftrightarrow \text{dual}(B(\underline{x}))$

$a \wedge (a \vee b) \Longleftrightarrow a$ $\quad a \vee (a \wedge b) \Longleftrightarrow a$ $\quad$ A5



Question : which cards do you need to turn over to verify if they fulfill the rule?

Logic variables : v : Vowel is on front side of card

e : even digit is on the reverse side of card

Rule : $v \Rightarrow e$

Contraposition : $\neg e \Rightarrow \neg v$

RT1) Replacement theorem for tautologies

IF $A(\ldots, x_i, \ldots) \Leftrightarrow t$, THEN $A(\ldots, B(\underline{z}), \ldots) \Leftrightarrow t$

e.g. IF $a \wedge (a \to b) \Rightarrow b$ Eq

THEN $A(\underline{x}) \wedge (A(\underline{x}) \to B(\underline{x})) \Rightarrow B(\underline{x})$

(IF $a \wedge (a \to b) \to b \Leftrightarrow t$

THEN $A(\underline{x}) \wedge (A(\underline{x}) \to B(\underline{x})) \to B(\underline{x}) \Leftrightarrow t$)

$a \wedge (a \to b) \Rightarrow b$ IFF $a \wedge (a \to b) \to b \Leftrightarrow t$

e.g. IF $A(a, b, c, \ldots, t, f) \Leftrightarrow B(a, b, c, \ldots t, f)$

THEN $A(\neg a, \neg b, \neg c, \ldots t, f) \Leftrightarrow B(\neg a, \neg b, \neg c, \ldots, t, f)$

$\neg A(\neg a, \neg b, \neg c, \ldots, t, f) \Leftrightarrow \neg B(\neg a, \neg b, \neg c, \ldots, t, f)$

$\text{dual}(A(\underline{x})) \Leftrightarrow \text{dual}(B(\underline{x}))$

RT2) Replacement theorem for equivalent partial propositional forms

Let $A(\underline{x}, B(\underline{z}))$ be an arbitary propositional form
with B being a partial form of A with
$\text{set}(\underline{z}) \subseteq \text{set}(\underline{x})$

IF $B(\underline{z}) \Leftrightarrow C(\underline{z})$, THEN $A(x, B(\underline{z})) \Leftrightarrow A(\underline{x}, C(\underline{z}))$

SR: Substitution Rule

$x_i \wedge A(\underline{x}) \Leftrightarrow x_i \wedge A(x_i \Leftrightarrow t)$

Proof : $x_i \Leftrightarrow f$: $f \wedge A(\underline{x}) \Leftrightarrow f \wedge A(x_i \Leftrightarrow t)$

$f \Leftrightarrow f$

$x_i \Leftrightarrow t$: $t \wedge A(x_i \Leftrightarrow t) \Leftrightarrow t \wedge A(x_i \Leftrightarrow t)$

$A(x_i \Leftrightarrow t) \Leftrightarrow A(x_i \Leftrightarrow t)$

$\neg x_i \wedge A(\underline{x}) \Leftrightarrow \neg x_i \wedge A(x_i \Leftrightarrow f)$

$x_i \vee A(\underline{x}) \Leftrightarrow x_i \vee A(x_i \Leftrightarrow f)$

$\neg x_i \vee A(\underline{x}) \Leftrightarrow \neg x_i \vee A(x_i \Leftrightarrow t)$

ER1) Expansion rules for propositional logic functions

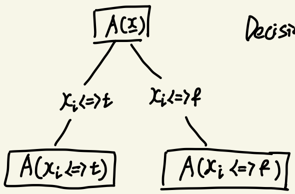
used more*

$$A(\underline{x}) \Leftrightarrow [x_i \wedge A(x_i \Leftrightarrow t)] \vee [\neg x_i \wedge A(x_i \Leftrightarrow f)]$$

Proof : $A(\underline{x}) \Leftrightarrow A(\underline{x}) \wedge (x_i \vee \neg x_i)$

$\Leftrightarrow (x_i \wedge A(\underline{x})) \vee (\neg x_i \wedge A(\underline{x}))$

$\Leftrightarrow [x_i \wedge A(x_i \Leftrightarrow t)] \vee [\neg x_i \wedge A(x_i \Leftrightarrow f)]$



Decision rule apply expansion rule

e.g. $\underbrace{a \leftrightarrow b}_{A(\underline{x})} \Leftrightarrow [a \wedge (t \leftrightarrow b)] \vee [\neg a \wedge (f \leftrightarrow b)]$

$\Leftrightarrow (a \wedge b) \vee (\neg a \wedge \neg b)$

$\Leftrightarrow CDNF[a \leftrightarrow b]$

ER2) $A(\underline{x}) \Leftrightarrow [\neg x_i \vee A(x_i \Leftrightarrow t)] \wedge [x_i \vee A(x_i \Leftrightarrow f)]$

e.g. $a \leftrightarrow b \Leftrightarrow [\neg a \vee (t \leftrightarrow b)] \wedge [a \vee (f \leftrightarrow b)]$

$\Leftrightarrow (\neg a \vee b) \wedge (a \vee \neg b)$

$\Leftrightarrow CCNF[a \leftrightarrow b]$

ER3) $A(\underline{x}) \Leftrightarrow [x_i \rightarrow A(x_i \Leftrightarrow t)] \wedge [\neg x_i \rightarrow A(x_i \Leftrightarrow f)]$

ER4) $A(\underline{x}) \Leftrightarrow [x_i \wedge (A(x_i \Leftrightarrow t) \not\leftrightarrow A(x_i \Leftrightarrow f))] \not\leftrightarrow A(x_i \Leftrightarrow f)$

RRD) $(x \wedge a) \vee (\neg x \wedge b) \iff (x \wedge a) \vee (\neg x \wedge b) \vee \underbrace{(a \wedge b)}_{\text{Resolvent}}$

RRC) $(\neg x \vee a) \wedge (x \vee b) \iff (\neg x \vee a) \wedge (x \vee b) \wedge (a \vee b)$

RRP) $(x \to a) \wedge (\neg x \to b) \iff (x \to a) \wedge (\neg x \to b) \wedge (\neg a \to b)$

$x \iff t$: $a \iff a \vee (a \wedge b)$ RRD
$a \iff a \wedge (a \vee b)$ RRC
} A5 (Absorption)

$x \iff f$: $b \iff b \vee (a \wedge b)$ RRD
$b \iff b \wedge (a \vee b)$ RRC
} A5 (Absorption)

e.g. $(x \wedge y \wedge z) \vee (x \wedge \neg y) \iff \overset{A5}{\cancel{(x \wedge y \wedge z)}} \vee (x \wedge \neg y) \vee \underbrace{(x \wedge z)}_{\text{Resolvent}}$

$\iff (x \wedge \neg y) \vee (x \wedge z)$

Allows to simplify boolean

FRE) Fundamental Rule for equivalence

Let $A \iff B$, THEN

$A \iff t$ IFF $B \iff t$

FRC
Fundamental rule for tautological conjunction

$A \wedge B \iff t$ IFF

$A \iff t$ AND $B \iff t$

FRI
Fundamental rule for implication

Let $A \Rightarrow B$. Then

IF $A \iff t$, THEN $B \iff t$

Using CI1 $A \leftrightarrow B \iff (A \to B) \wedge (B \to A)$

FRE and FRC, we get

$A \iff B$ IFF $A \Rightarrow B$ AND $B \Rightarrow A$

Using D12 $A \to B \Leftrightarrow A \leftrightarrow A \wedge B$

and FRE, we get

$A \Rightarrow B$ IFF $A \Leftrightarrow A \wedge B$ (RI3 Enhancement rule)

$E[A] \subseteq E[B]$ IFF $E[A] = E[A \wedge B]$

IFF $E[A] = E[A] \wedge E[B]$

1.5)

$r \Rightarrow s$

$r_1 \wedge r_2 \wedge r_3 \Rightarrow s$

Premises Conclusion

Logically equivalent statements

1) $r_1 \wedge r_2 \wedge r_3 \to s \Leftrightarrow t$

2) $\neg r_1 \vee \neg r_2 \vee \neg r_3 \vee s \Leftrightarrow t$ (D1, A10)

$s \vee \neg r_1 \vee \neg r_2 \vee \neg r_3 \Leftrightarrow t$

3) $\neg s \wedge r_1 \wedge r_2 \Rightarrow \neg r_3 \Leftrightarrow t$

$r_1 \wedge r_2 \wedge r_3 \Rightarrow s$ IFF

$\neg s \wedge r_1 \wedge r_2 \Rightarrow \neg r_3$ indirect conclusion

4) $\neg s \to \neg r_1 \vee \neg r_2 \vee \neg r_3 \Leftrightarrow t$

$\neg s \to \neg(r_1 \wedge r_2 \wedge r_3) \Leftrightarrow t$

$\neg s \to \neg(r_1 \wedge r_2 \wedge r_3)$ indirect conclusion

5) $V_1 \wedge V_2 \wedge V_3 \wedge \neg S \Longleftrightarrow f$

Possible conclusions

$V \Rightarrow S_\lambda \ ; \ \lambda = 1, 2, \ldots, l$

$CCNF[V] = D_1 \wedge D_2 \wedge \ldots \wedge D_m \, , \quad m = |E[V]|$

$V \Rightarrow D_4$ (E4, simplification)

$V \Rightarrow D_4 \wedge D_6$

$V \Rightarrow D_4 \wedge D_6 \wedge D_7$

$\vdots$

$V \Rightarrow D_1 \wedge D_2 \wedge D_3 \wedge \ldots D_m$

$l = \sum_{i=1}^{m} \binom{m}{i} = 2^m - 1$ # of non-trivial conclusions

Deduction scheme (for drawing logical conclusion, for logical analysis)

Assertion : $s \Leftrightarrow t$

proof : $V_1 \wedge V_2 \Rightarrow s$

Enhancement rule : $V \Rightarrow V \Rightarrow U_r$ IFF $V \Leftrightarrow V \wedge U_r$
(RI3, from D12 using FRE)

Scheme

1)	V_1	
2)	$\wedge V_2$	$\Rightarrow s$
3)	$\wedge U_3$	Comments (used laws)
	$\vdots$	
r)	$\wedge U_r$	
	$\vdots$	
n)	$\wedge \underline{s}$	

↑ "$\wedge$": usually not written

e.g. $[a \to (b \to \neg c)] \wedge a \wedge c \Rightarrow \neg b$ deduction scheme

1)	$a \to (b \to \neg c)$	
2)	$a \wedge c$	$\Rightarrow \neg b$
3)	$\neg a \vee \neg b \vee \neg c$	1) D1
4)	$\neg(\neg a \vee \neg c) \to \neg b$	3) D1
5)	$(a \wedge c) \to \neg b$	4) A10
6)	$\neg b$	line 2, 5, E9

1) Proof of $A \iff f$

e.g. $r_1 \land r_2 \Rightarrow s$ IFF

$r_1 \land r_2 \land \neg s \iff f$

1) $a \to (b \to \neg c)$

2) $a \land c$

3) b $\Rightarrow f$

---

4) $\neg a \lor \neg b \lor \neg c$ 1) D1

5) $(a \land c) \to \neg b$ 4) D1, A10

6) $\neg b$ line 2, 5, E9

7) f line 3, 6, A8

Resolution method to show $CNF[A] \Leftrightarrow f$

RRC : $(\neg x \vee a) \wedge (x \vee b) \Leftrightarrow (\neg x \vee a) \wedge (x \vee b) \wedge (a \vee b)$

$(\neg x \vee a) \wedge (x \vee b) \Rightarrow a \vee b$ (E4)

Deduction Scheme for resolution (Layer algorithm)

CNF[A]	Layer
.... $\wedge(\neg x \vee a) \wedge (x \vee b) \wedge$ ...	0
.... $\wedge(a \vee b)\wedge$	1
⋮	
$\wedge f$	n

Generation of Layer k+1

- Compare every term in Layer k with every other term in Layer k
- Compare every term in Layer k with every other term in previous layers
- Absorption to reduce the number of terms
- $r := k+1$

$[a \rightarrow (b \rightarrow \neg c)] \wedge a \wedge c \wedge b \Leftrightarrow f$

$\underbrace{(\neg a \vee \neg b \vee \neg c) \wedge a \wedge c \wedge b}_{CNF} \Leftrightarrow f$

$(\neg a \vee \cancel{\neg b \vee \neg c}) \wedge a \wedge c \wedge b$	0
$\wedge(\cancel{\neg b \vee \neg c}) \wedge (\cancel{\neg a \vee \neg b}) \wedge (\cancel{\neg a \vee \neg c})$	1
$\wedge \neg b \wedge \neg c \wedge \neg a$	2
$\wedge f$	3

Deduction Proof scheme

Task : Show $V \Rightarrow S$ with $V \Leftrightarrow V_1 \wedge V_2 \wedge V_3$

Basis : $V \Rightarrow S$ IFF $V \Leftrightarrow V \wedge S$

Approach : Derive further premises V_1 from $V_1, V_2, \dots, V_3$ until conclusion $V_1 \Leftrightarrow S$ is obtained

Result : $V_1 \wedge V_2 \Leftrightarrow V_1 \wedge V_2 \wedge V_3$

$\Leftrightarrow V_1 \wedge V_2 \wedge V_3 \wedge V_4$

$\vdots$

$\Leftrightarrow V_1 \wedge V_2 \wedge V_3 \wedge \dots \wedge S$

$\Rightarrow S$

Resolution Rules

	disjunctive form (RRD)	Conjunctive form (RRC)
general case	$(x \land a) \lor (\neg x \land b) \Leftrightarrow$ $(x \land a) \lor (\neg x \land b) \lor (a \land b)$	$(x \lor a) \land (\neg x \lor b) \Leftrightarrow$ $(x \lor a) \land (\neg x \lor b) \land (a \lor b)$
special case 1 $a \Leftrightarrow b$	$(x \land a) \lor (\neg x \land a) \Leftrightarrow a$	$(x \lor a) \land (\neg x \lor a) \Leftrightarrow a$
special case 2 a is neutral element	$a \Leftrightarrow t$ $x \lor (\neg x \land b) \Leftrightarrow x \lor b$	$a \Leftrightarrow f$ $x \land (\neg x \lor b) \Leftrightarrow x \land b$

Resolution Method

Objective	Disjunctive form	Conjunctive form	Stop Criteria
Proof of Tautology	✓		resolvent: ... $\lor t$
Proof of Contradiction		✓	resolvent: .... $\lor f$
Simplification / Proof of Contingency	✓	✓	no further resolvent are possible

Layer Algorithm schemo

	Layer
original proposition form A	0
All resolvents from layer 0	1
All resolvents from Layer 1 and between Layers 0 and 1	2
All resolvents from Layer 2 and between Layer 2 and all layers above	3

Premise Normal Form PNF

Base $(\{t, f\}, B)$

Form : $PNF[A] \Longleftrightarrow B(X, Y, Z)$

X : selector variable

- atomic variable
- not negated

Y : selected if $X \Longleftrightarrow t$

Z : selected if $X \Longleftrightarrow f$

Y and Z are either

- B. operation
- t
- f

Calculation by recursive expansion using law B017 : $A(\underline{x}) \Longleftrightarrow B(x, A(x, \Longleftrightarrow t), A(x, \Longleftrightarrow f))$

BDT is direct graphical representation of a PNF

Ring normal Form RNF

(Canonical Ring Normal Form) (Reed Muller expansion)

Base : $(\{t, f\}, \wedge, \not\leftrightarrow)$

Form : $RNF[A] \Leftrightarrow M_1 \not\leftrightarrow M_2 \not\leftrightarrow \dots$

with $M_1 \Leftrightarrow a \wedge b \wedge \dots$ (Conjunctive term)

Contains only $\wedge$ and $\not\leftrightarrow$

no negation

ER4 : $A(\underline{x}) \Leftrightarrow [x_i \wedge (A(x_i \Leftrightarrow t) \not\leftrightarrow A(x_i \Leftrightarrow f))] \not\leftrightarrow A(x_i \Leftrightarrow f)$

$A(a, b, c, d) \Leftrightarrow a \vee (b \wedge c) \vee \neg d$

$A \Leftrightarrow [d \wedge (A(d \Leftrightarrow t) \not\leftrightarrow A(d \Leftrightarrow f))] \not\leftrightarrow A(d \Leftrightarrow f)$

$\Leftrightarrow [d \wedge (a \vee (b \vee c)) \not\leftrightarrow t] \not\leftrightarrow t$

$[a \vee (b \wedge c)] \overset{B13}{\Leftrightarrow} a \wedge b \wedge c \not\leftrightarrow b \wedge c \not\leftrightarrow a$

$RNF[A] \Leftrightarrow \underbrace{a \wedge b \wedge c \wedge d}_{\text{monomes}} \not\leftrightarrow b \wedge c \wedge d \not\leftrightarrow a \wedge d \not\leftrightarrow d \not\leftrightarrow t$

(product terms containing only possible variables)

- Always canonical
- No further minimization possible

Calculation by recursive expansion using Law ER4

$$A(\underline{x}) \Leftrightarrow [x_i \wedge (A(x_i \Leftrightarrow t) \not\leftrightarrow A(x_i \Leftrightarrow f))] \not\leftrightarrow A(x_i \Leftrightarrow f)$$

Decision - Trees and Diagrams

Binary Decision

- Nodes represent variables
- each node has exactly two output edges (t - and f - assignment of variable)
  - * except terminal node

BDT / BDD (SPR3)
- T: Tree: each node (except root node) has exactly one input edge
- D: Diagram: equivalent sub-BDTs may be merged into one sub-BDT
  - ↳ the first node of the new sub-BDT has multiple input edges

Ordered : OBDT / OBDD

- Full substitution of variables (SPR1)

- In each path from root to terminal node, each variable appears at most once

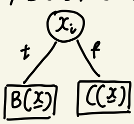
- the variable order is the same in all paths

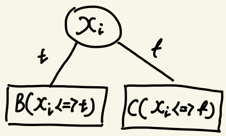
Reduced : ROBDT / ROBDD

- Complete resolution of variables with SPR2
  ↳ no node has two equivalent sub-BDTs

Simplification rule 1

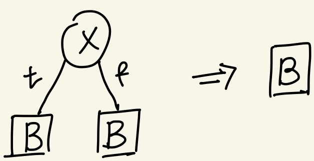
$$B(x_i, B(\underline{x}), C(\underline{x})) \Leftrightarrow B(x_i, B(x_i \Leftrightarrow t), C(x_i \Leftrightarrow f))$$





Simplification rule 2

$$B(X, B, B) \Leftrightarrow B$$



How to read CDNF and CCNF from BDT

CDNF : t-assignments of A , paths direct from BDT

CCNF : f-assignments of A

variables assigned to f will be positives

variables assigned to t will be negated

B014) Substitution rule

$$x_i \wedge A(\underline{x}) \iff x_i \wedge A(x_i \iff t)$$

$$\neg x_i \wedge A(\underline{x}) \iff \neg x_i \wedge A(x_i \iff f)$$

$$\beta(x_i, A, B) \iff (x_i \wedge A(\underline{x})) \vee (\neg x_i \wedge B(\underline{x}))$$

$$\iff (x_i \wedge A(x_i \iff t)) \vee (\neg x_i \wedge B(x_i \iff f))$$

$$\beta(x_i, A, B) \iff \beta(x_i, A(x_i \iff t), B(x_i \iff f))$$

B017) Expansion Rule

$$\boxed{A(\underline{x})} \iff A(\underline{x}) \wedge (x_i \vee \neg x_i)$$

$$\iff (x_i \wedge A(\underline{x})) \vee (\neg x_i \wedge A(\underline{x}))$$

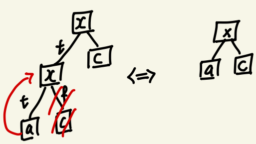
$$\iff \beta(x_i, A(\underline{x}), A(\underline{x}))$$

$$\iff \boxed{\beta(x_i, A(x_i \iff t), A(x_i \iff f))}$$

B015) $\beta(x, \beta(x, a, b), c)$

$\iff \beta(x, \beta(t, a, b), c)$

$\iff \beta(x, a, c)$



B016) $\beta(x, a, \beta(x, b, c))$

$\iff \beta(x, a, \beta(f, b, c))$

$\iff \beta(x, a, c)$

B020) $a \Leftrightarrow \beta(a, t, f)$

$$\Leftrightarrow \underbrace{(a \wedge t)}_{a} \vee \underbrace{(\neg a \wedge f)}_{f} \Leftrightarrow a$$

$$\text{RTI} \downarrow$$

$$A \Leftrightarrow \beta(A, t, f)$$

B021) $\neg a \Leftrightarrow \beta(a, f, t)$

$$\downarrow \text{RTI}$$

$$\neg A \Leftrightarrow \beta(A, f, t)$$

$$\Leftrightarrow A(t := f, f := t) \quad \text{IF } A \text{ is in PNF}$$

B022) $a \wedge b \Leftrightarrow \beta(a, t \wedge b, f \wedge b) \Leftrightarrow \beta(a, b, f)$

$$A \wedge B \Leftrightarrow \beta(A, B, f)$$

$$\Leftrightarrow A(t := B) \quad \text{IF } A \text{ is in PNF}$$

B023) $a \vee b \Leftrightarrow \beta(a, t \vee b, f \vee b) \Leftrightarrow \beta(a, t, b)$

$$A \vee B \Leftrightarrow \beta(A, t, B)$$

$$\Leftrightarrow A(f := B) \quad \text{IF } A \text{ is in PNF}$$

e.g. $A(a, b, c, d) \Leftrightarrow a \vee (b \wedge c) \vee \neg d$

$$\Leftrightarrow \underbrace{(a \vee \neg d)}_{G(a,d)} \vee \underbrace{(b \wedge c)}_{H(b,c)}$$

$$A \Leftrightarrow G \vee H$$

$$\text{PNF}[G] \Leftrightarrow \beta(a, t, \beta(d, f, t))$$

$$A \Leftrightarrow G \vee H$$

$$\Leftrightarrow G(f := H)$$

$$\Leftrightarrow \beta(a, t, \beta(d, b \wedge c, t))$$

B024) $A \rightarrow B \Leftrightarrow \beta(A, B, t) \Leftrightarrow A(t := B, f := t)$

B018) Composition Rule

$$\beta(A, B, C) \Leftrightarrow A(t := B, f := C) \quad \text{IF } A \text{ is in PNF}$$

IF B, C are PNF, THEN $\beta(A, B, C)$ is a PNF

"All metals conduct current" a

$\wedge$ "Copper is a metal" b

$\Rightarrow$ "Copper conducts current" c

$c \wedge b \Rightarrow c$ IFF $(a \wedge b) \rightarrow c \Leftrightarrow t$

2. Predicate Logic

2.1 Predicate Logic Propositional Forms

Predicates : - Unary predicates : properties of individual variables

- binary predicates : relations between individual variables
- n-nary "
- Examples ">", "=", "≠"

e.g. Qx : X is valuable

Qx : propositional form

X_1 : the golden coin

X_2 : the CDMA patent

Qx_1, Qx_2 : propositions

e.g. Qxy: x divides y without remainder

$x_1 = 2$ $\quad y_1 = 6$ $\quad Qx_1y_1 \Leftrightarrow t$

$x_2 = 4$ $\quad y_2 = 9$ $\quad Qx_2y_1 \Leftrightarrow f$

$x_3 = 6$

$\neg Qx_2y_1 \Leftrightarrow t$

Quantifiers

- Universal quantifier : $\forall Px \Leftrightarrow Px_1 \wedge Px_2 \wedge \ldots \wedge Px_n$

$x \in \{x_1, x_2, \ldots x_n\}$

- Existential quantifier : $\exists Px \Leftrightarrow Px_1 \vee Px_2 \vee \ldots \vee Px_n$

$x \in \{x_1, x_2, \ldots, x_n\}$

Example : Formalization using quantifiers

a) $x \circ y = y \circ x$ $\quad Qxyz : z = x \circ y$

(Law of Commutativity) $\quad \forall (Qxyz \rightarrow Qyxz)$

$x, y, z \in \mathbb{R}$

b) There exists only one $x \in M$, for which P_x is true

$$\exists_{x \in M} \left[P_x \wedge \forall_{y \in M} P_y \rightarrow (x = y) \right]$$

c) A sequence of real numbers x_n, $n \in \mathbb{N}_0$, has a limit g, if for every $\varepsilon > 0$ there exists an index $n_0 \in \mathbb{N}_0$ such that for all $n \geq n_0$ $|x_n - g| < \varepsilon$ holds ($\varepsilon, x_n \in \mathbb{R}$)

$$\forall_{\varepsilon > 0} \ \exists_{n_0 \in \mathbb{N}_0} \ \forall_{n \geq n_0} \ |x_n - g| < \varepsilon$$

$$\forall_{x \in M} P_{xy} \Longleftrightarrow P_{x_1 y} \wedge P_{x_2 y} \wedge P_{x_3 y} \wedge \dots \ P_{x_m y}$$

$$\forall_{x \in M} P_{x y_1} \Longleftrightarrow P_{x_1 y_1} \wedge P_{x_2 y_1} \wedge P_{x_3 y_1} \wedge \dots \wedge P_{x_m y_1}$$

$$\exists_{y \in N} \forall_{x \in M} P_{xy} \Longleftrightarrow (P_{x_1 y_1} \wedge P_{x_2 y_1} \wedge \dots \wedge P_{x_m y_1}) \vee$$

$$(P_{x_1 y_2} \wedge P_{x_2 y_2} \wedge \dots \wedge P_{x_m y_2}) \vee$$

$$\vdots$$

$$(P_{x_1 y_n} \wedge P_{x_2 y_n} \wedge \dots \ P_{x_m y_n})$$

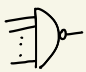
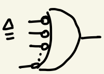
2.2 Laws

F1) $\neg \forall_{x \in M} P_x \iff \neg(P_{x_1} \wedge P_{x_2} \wedge P_{x_3} \wedge \dots)$

$\iff \neg P_{x_1} \vee \neg P_{x_2} \vee \neg P_{x_3} \vee \dots$

$\iff \exists_{x \in M} \neg P_x$

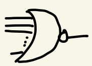
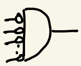
$\neg(a \wedge b) = \neg a \vee \neg b$

 $\hat{=}$ 

F2) $\neg \exists_{x \in M} P_x \iff \neg(P_{x_1} \vee P_{x_2} \vee P_{x_3} \vee \dots)$

$\iff \neg P_{x_1} \wedge \neg P_{x_2} \wedge \neg P_{x_3} \wedge \dots$

$\iff \forall_{x \in M} \neg P_x$

 $\hat{=}$ 

$\exists_{x \in M} P_x \iff \neg \forall_{x \in M} \neg P_x$

$\forall_{x \in M} P_x \iff \neg \exists_{x \in M} \neg P_x$

Formalization

$$\forall_{\varepsilon > 0} \ \exists_{n_0 \in \mathbb{N}_0} \ \forall_{n \geq n_0} \ |x_n - g| < \varepsilon$$

Negation:

$$\neg \left[\forall_{\varepsilon > 0} \ \exists_{n_0 \in \mathbb{N}_0} \ \forall_{n \geq n_0} \ |x_n - g| < \varepsilon \right] \iff$$

$$\exists_{\varepsilon > 0} \ \forall_{n_0 \in \mathbb{N}_0} \ \exists_{n \geq n_0} \ |x_n - g| \geq \varepsilon$$

F3) $\forall_{x \in M} \forall_{y \in N} P_{xy} \Leftrightarrow \forall_{y \in N} \forall_{x \in M} P_{xy}$

F4) $\exists_{x \in M} \exists_{y \in N} P_{xy} \Leftrightarrow \exists_{y \in N} \exists_{x \in M} P_{xy}$

M: set of all married people

1) $\forall_{x \in M} \exists_{y \in M}$ [x is married to y]

2) $\exists_{y \in M} \forall_{x \in M}$ [x is married to y] - False statement

only can switch if quantifiers are same

F5) $\forall_{x \in M} (P_x \wedge Q_x) \Leftrightarrow (\forall_{x \in M} P_x) \wedge (\forall_{x \in M} Q_x)$

$(P_{x_1} \wedge Q_{x_1}) \wedge (P_{x_2} \wedge Q_{x_2}) \wedge (P_{x_3} \wedge Q_{x_3}) \wedge \dots \Leftrightarrow$

$(P_{x_1} \wedge P_{x_2} \wedge P_{x_3} \wedge \dots) \wedge (Q_{x_1} \wedge Q_{x_2} \wedge Q_{x_3} \wedge \dots) \Leftrightarrow$

$(\forall_{x \in M} P_x) \wedge (\forall_{x \in M} Q_x)$

F6) $\exists_{x \in M} (P_x \vee Q_x) \Leftrightarrow (\exists_{x \in M} P_x) \vee (\exists_{x \in M} Q_x)$

F7) $\forall_{x \in M} (S \vee Q_x) \Leftrightarrow S \vee (\forall_{x \in M} Q_x)$

$(S \vee Q_{x_1}) \wedge (S \vee Q_{x_2}) \wedge (S \vee Q_{x_3}) \wedge \dots \Leftrightarrow$

$S \vee (Q_{x_1} \wedge Q_{x_2} \wedge Q_{x_3} \wedge \dots) \Leftrightarrow$

$S \vee \forall_{x \in M} Q_x$

F8) $\exists_{x\in M}(S \wedge Q_x) \Leftrightarrow S \wedge \exists_{x\in M} Q_x$

F9) $\forall_{x\in M}\exists_{y\in M}(P_x \wedge Q_y) \Leftrightarrow \exists_{y\in N}\forall_{x\in M}(P_x \wedge Q_y)$ Changable due to 2 unary predicate

F10) $\Leftrightarrow (\forall_{x\in M} P_x) \wedge (\exists_{y\in M} Q_y)$

1) $\forall_{x\in M}\exists_{y\in M}(P_x \wedge Q_y)$

2) $\forall_{x\in M}[P_x \wedge \exists_{y\in N} Q_y]$ 1) F8

3) $(\forall_{x\in M} P_x) \wedge (\forall_{x\in M}\exists_{y\in N} Q_y)$ 2) F5

$\exists_{y\in N} Q_y \wedge \exists_{y\in N} Q_y \wedge \exists_{y\in N} Q_y \wedge \dots \Leftrightarrow \exists_{y\in N} Q_y$

$(A4: \forall_{x\in M} S \Leftrightarrow S \wedge S \wedge S \wedge S \dots \Leftrightarrow S)$

4) $(\forall_{x\in M} P_x) \wedge (\exists_{y\in N} Q_y)$ 3, A4 (F10!)

5) $\exists_{y\in N}[(\forall_{x\in M} P_x) \wedge Q_y]$ 4, F8

6) $\exists_{y\in N}[(\forall_{x\in M} P_x) \wedge (\forall_{x\in M} Q_y)]$ 5, A4

A4: $Q_y \Leftrightarrow Q_y \wedge Q_y \wedge \dots \Leftrightarrow \forall_{x\in M} Q_y$

7) $\exists_{y\in N}\forall_{x\in M}(P_x \wedge Q_y)$ 6, F5

F11) 1) $\forall_{x\in M}\exists_{y\in N}(P_x \vee Q_y) \Leftrightarrow \exists_{y\in N}\forall_{x\in M}(P_x \vee Q_y)$

2) $\forall_{x\in M}[\exists_{y\in N} P_x \vee \exists_{y\in N} Q_y]$ 1, F6

3) $\forall_{x \in M} [P_x \vee \exists_{y \in N} Q_y]$ 2, A4

4) $(\forall_{x \in M} P_x) \vee (\exists_{y \in N} Q_y)$ 3, F7 (F12)

5) $(\exists_{y \in N} \forall_{x \in M} P_x) \vee (\exists_{y \in N} Q_y)$ 4, A4

6) $\exists_{y \in N} [(\forall_{x \in M} P_x) \vee Q_y]$ 5, F6

7) $\exists_{y \in N} \forall_{x \in M} (P_x \vee Q_y)$ 6, F7

F13) 1) $\exists_{x \in M} (P_x \rightarrow Q_x) \Leftrightarrow (\forall_{x \in M} P_x) \rightarrow (\exists_{x \in M} Q_x)$

2) $\exists_{x \in M} (\neg P_x \vee Q_x)$ 1, D1

3) $(\exists_{x \in M} \neg P_x) \vee \exists_{x \in M} Q_x$ 2, F6

4) $(\neg \forall_{x \in M} P_x) \vee \exists_{x \in M} Q_x$ 3, F1

5) $\forall_{x \in M} P_x \rightarrow \exists_{x \in M} Q_x$ 4, D

G1) $\forall P_x \Rightarrow P_{x_1}$ $\qquad x_1 \in M$

$P_{x_1} \wedge P_{x_2} \wedge P_{x_3} \wedge \ldots \Rightarrow P_{x_1}$ $\qquad E_4 \ (a \wedge b \Rightarrow a)$

$P_{x_1} \Rightarrow \underset{x \in M}{\exists} P_x$

$P_{x_1} \Rightarrow P_{x_1} \vee P_{x_2} \vee P_{x_3} \vee P_{x_4} \vee \ldots$ $\qquad E5 \ (a \Rightarrow a \vee b)$

G2) $\underset{x \in M}{\forall} P_x \Rightarrow \underset{x \in M}{\exists} P_x$ $\qquad E6 \ (a \wedge b \rightarrow a \vee b)$

G3) $V \Rightarrow S_1 \vee S_2$ IFF

$\neg S_1 \wedge \neg S_2 \Rightarrow \neg V$ IFF $\qquad$ (RI2, Contraposition)

$V \wedge \neg S_2 \Rightarrow S_1$

$\underset{x \in M}{\forall} (P_x \vee Q_x) \Rightarrow (\underset{x \in M}{\forall} P_x) \vee (\underset{x \in M}{\exists} Q_x)$ IFF

$\neg \underset{x \in M}{\exists} Q_x \wedge \underset{x \in M}{\forall} (P_x \vee Q_x) \Rightarrow \underset{x \in M}{\forall} P_x$

1) $\neg \underset{x \in M}{\exists} Q_x$

2) $\underset{x \in M}{\forall} (P_x \vee Q_x)$ $\qquad \Rightarrow \underset{x \in M}{\forall} P_x$

---

3) $\underset{x \in M}{\forall} \neg Q_x$ $\qquad$ 1, F2

4) $\underset{x \in M}{\forall} [\neg Q_x \wedge (P_x \vee Q_x)]$ $\qquad$ 2,3, F5

5) $\underset{x \in M}{\forall} [\overset{f}{\cancel{(\neg Q_x \wedge Q_x)}} \vee (\neg Q_x \wedge P_x)]$ $\qquad$ 4, A3

6) $\underset{x \in M}{\forall} (P_x \wedge \neg Q_x)$ $\qquad$ 5, A8, A6

7) $\underset{x \in M}{\forall} P_x \wedge \underset{x \in M}{\forall} Q_x$ $\qquad$ 6, F5

8) $\forall_{x \in M} P_x$ 7, E4

(G4) 1) $\exists_{x\in M}(P_x \wedge Q_x) \Rightarrow (\exists_{x\in M} P_x) \wedge (\exists_{x\in M} Q_x)$

2) $\exists_{x\in M}(P_x \wedge Q_x) \vee \exists_{x\in M} P_x$ 1, E5

3) $\exists_{x\in M}((P_x \wedge Q_x) \vee P_x)$ 2, F6

4) $\exists_{x\in M} P_x$ 3, A5

5) $\exists_{x\in M}(P_x \wedge Q_x) \vee \exists_{x\in M} Q_x$ 1, E5

6) $\exists_{x\in M}[(P_x \wedge Q_x) \vee Q_x]$ 5, F6

7) $\exists_{x\in M} Q_x$ 6, A5

8) $(\exists_{x\in M} P_x) \wedge (\exists_{x\in M} Q_x)$ 4, 7

G5) $\exists_{x\in M}(\neg P_x \wedge \neg Q_x) \Rightarrow \exists_{x\in M}\neg P_x \wedge \exists_{x\in M}\neg Q_x$ (G4)

$\neg\left[\exists_{x\in M}\neg P_x \wedge \exists_{x\in M}\neg Q_x\right] \Rightarrow \neg\exists_{x\in M}(\neg P_x \wedge \neg Q_x)$ RI2, Contraposition

$\forall_{x\in M} P_x \vee \forall_{x\in M} Q_x \Rightarrow \forall_{x\in M} P_x \vee Q_x$ F2, A10

G6) $\forall_{x\in M}(\neg P_x \vee \neg Q_x) \Rightarrow \forall_{x\in M}\neg Q_x \vee \exists_{x\in M}\neg P_x$ (G3)

$\neg\left[\forall_{x\in M}\neg Q_x \vee \exists_{x\in M}\neg P_x\right] \Rightarrow \neg\forall_{x\in M}(\neg P_x \vee \neg Q_x)$ RI2

$$\exists_{x\in M} Qx \wedge \forall_{x\in M} Px \Rightarrow \exists_{x\in M} Px \wedge Qx$$ F2, A10

$$\forall_{x\in M} Px \wedge \exists_{x\in M} Qx \Rightarrow \exists_{x\in M} Px \wedge Qx$$ G6

G7) $\forall_{x\in M} \forall_{y\in M} Pxy \Rightarrow \forall_{x\in M} Pxx$

$$(Px_1y_1 \wedge Px_1y_2 \wedge Px_1y_3 \wedge \dots) \wedge (Px_2y_1 \wedge Px_2y_2 \wedge \dots)$$
$$\wedge (Px_1 y_2 \wedge Px_2y_2 \wedge \dots)$$

$$\Rightarrow Px_1x_1 \wedge Px_2x_2 \wedge Px_3x_3$$

$$\Leftrightarrow \forall_{x\in M} Pxx$$

G8) $\exists_{x\in M} \forall_{x\in M} Pxy \Rightarrow \forall_{x\in M} \exists_{x\in E} Pxy$

$$\exists_{x\in M} (Pxy_1 \wedge Pxy_2 \wedge Pxy_3 \wedge \dots) \Rightarrow$$
$$\exists_{x\in M} Pxy_1 \wedge \exists_{x\in M} Pxy_2 \wedge \exists_{x\in M} Pxy_3 \wedge \dots$$ (G4)

$$\Leftrightarrow \forall_{y\in N} \exists_{x\in M} Pxy$$

G9) 1) $\forall_{x\in M} (Px \rightarrow Qx) \rightarrow (\forall_{x\in M} Px) \rightarrow (\forall_{x\in M} Qx)$

2) $\forall_{x\in M} (\neg Px \vee Qx)$ 1, D1

3) $\forall_{x\in M} Qx \vee \exists_{x\in M} \neg Px$ 2, G3

4) $\neg\forall_{x\in M} Px \vee \forall_{x\in M} Qx$ 3, F1

5) $(\forall_{x\in M} Px) \rightarrow (\forall_{x\in M} Qx)$ 4, D1

G10) 1) $\forall_{x\in M} (Px \rightarrow Qx) \Rightarrow (\exists_{x\in M} Px) \rightarrow (\exists_{x\in M} Qx)$

2) $\forall_{x\in M} (\neg Px \vee Qx)$ 1, D1

3) $\forall_{x\in M} \neg Px \vee \exists_{x\in M} Qx$ 2, G3

4) $\neg\exists_{x\in M} Px \vee \exists_{x\in M} Qx$ 3, F1

5) $\exists_{x\in M} Px \rightarrow \exists_{x\in M} Qx$ 4, D1

G11) 1) $\forall_{x \in M} P_x$

2) $\forall_{x \in M} (P_x \to Q_x) \quad \Rightarrow \forall_{x \in M} Q_x$

3) $\forall_{x \in M} P_x \to \forall_{x \in M} Q_x$ 2, G9

4) $\forall_{x \in M} Q_x$ 1, 3, E9

B12) 1) $\exists_{x \in M} P_x$

2) $\forall_{x \in M} (P_x \to Q_x) \quad \Rightarrow \exists_{x \in M} Q_x$

3) $(\exists_{x \in M} P_x) \to (\exists_{x \in M} Q_x)$ 2, G10

4) $\exists_{x \in M} Q_x$ 1, 3, E9

2.4 Mathematical Induction

Basis step 1) P_0

Induction step 2) $\forall_{n \in \mathbb{N}_0} [P_n \to P_{n+1}] \Rightarrow \forall_{n \in \mathbb{N}_0} P_n$

3) $P_0 \to P_1$ 2, G1

4) P_1 1, 3, E9

5) $P_1 \to P_2$ 2, G1

6) P_2 4, 5, E9

$\vdots$

n) $\forall_{n \in \mathbb{N}_0} P_n$ 1, 4, 6, 8....

eg $P_n \Longleftrightarrow \left[\sum_{r=0}^{n} r = \frac{n}{2}(n+1)\right]$

1) $P_0 \Longleftrightarrow \left[\sum_{r=0}^{0} r = \frac{0}{2}(0+1)\right] \Longleftrightarrow 0 = 0 \Longleftrightarrow t$

2) $P_n \to P_{n+1} \Longleftrightarrow$

$\left[\sum_{r=0}^{n} r = \frac{n}{2}(n+1)\right] \longrightarrow \left[\sum_{r=0}^{n+1} r = \frac{n+1}{2}(n+2)\right] \Longleftrightarrow$

$\quad \text{||} \longrightarrow \left[\sum_{r=0}^{n} r + (n+1) = \frac{n+1}{2}(n+2)\right] \Longleftrightarrow$

$\quad \text{||} \to \left[\sum_{r=0}^{n} r + \cancel{(n+1)} = \underbrace{\frac{1}{2}(n+1) + \frac{n}{2} + \frac{1}{2}}_{\cancel{n+1}} + \frac{n}{2}(n+1)\right] \Longleftrightarrow$

$\quad \text{||} \to \left[\sum_{r=0}^{n} r = \frac{n}{2}(n+1)\right] \Longleftrightarrow$

$\left[\sum_{r=0}^{n} r = \frac{n}{2}(n+1)\right] \to \left[\sum_{r=0}^{n} r = \frac{n}{2}(n+1)\right] \Longleftrightarrow$

$a \to a \Longleftrightarrow t$

$$\forall_{n \in \mathbb{N}_0} \boxed{\sum_{r=0}^{n} r = \frac{n}{2}(n+1)}$$

3 sets

3.1 Notation

e.g. $\{x \mid 1 \leq x \leq 3\}_{\mathbb{N}} = \{1,2,3\} = \{3,1,2\} = \{1,3,2\}$

no concept of an order

$\neq \{1,2,2,1,1,3,3,2,1\}$

objects can only appear in set once

e.g. $M = \{1,2,3\}$

$P(M) = \{X \mid X \subseteq M\}$

$= \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \emptyset\}$

$|M| = 3$

$|P(M)| = 2^{|M|} = 2^3 = 8$

$M \subseteq G \iff M \in P(G)$

Power sets always contain other sets

$M = \{x \mid x \in G \wedge P_x\}$

$= \{x \in G \mid P_x\}$

$= \{x \mid P_x\}_G$

$= \{x \mid x \in M\}_G$

$x \in M \underset{G}{\iff} P_x$

$x \notin M \underset{G}{\iff} \neg(x \in M) \underset{G}{\iff} \neg P_x$

3.2 operations and Definition

$$A = \{x \mid P_x\}_G \;;\; B = \{x \mid Q_x\}_G \;;\; A, B \in P(G) \Leftrightarrow A, B \subseteq G$$

$$A \cap B := \{x \mid P_x \wedge Q_x\}_G = \{x \mid x \in A \wedge x \in B\}_G$$ (Logic operation $\wedge$)

$$x \in A \cap B \Leftrightarrow P_x \wedge Q_x \Leftrightarrow x \in A \wedge x \in B$$

$$A \cup B := \{x \mid P_x \vee Q_x\}_G = \{x \mid x \in A \wedge x \in B\}_G$$ (Logic operation $\vee$)

$$\bar{A} := \{x \mid \neg P_x\}_G = \{x \mid x \notin A\}_G$$

Set representation and predicate logic representation

$$\forall_{x \in G}\, x \in A \Leftrightarrow A = G \;;\; \exists_{x \in G}\, x \in A \Leftrightarrow A \neq \emptyset$$

At least one element is in A

$$\forall_{x \in G}\, x \notin A \Leftrightarrow A = \emptyset \qquad \exists_{x \in G}\, x \notin A \Leftrightarrow A \neq G$$

$$\neg \forall_{x \in G}\, x \notin A \Leftrightarrow \exists_{x \in G} \neg (x \notin A) \Leftrightarrow \exists_{x \in G}\, x \in A$$

$$\neg (A \neq \emptyset) \Leftrightarrow A = \emptyset$$

$$\forall_{x \in G}\, x \in A \Leftrightarrow A = G \Leftrightarrow \bar{A} = \emptyset \Leftrightarrow \forall_{x \in G}\, x \notin \bar{A}$$

$\bar{G} = \emptyset$

$$\exists_{x \in G}\, x \in A \Leftrightarrow A \neq \emptyset \Leftrightarrow \bar{A} \neq G \Leftrightarrow \exists_{x \in G}\, x \notin \bar{A}$$

$$\forall_{x \in G}\, x \notin A \Leftrightarrow A = \emptyset \Leftrightarrow \bar{A} = G \Leftrightarrow \forall_{x \in G}\, x \in \bar{A}$$

$$\exists_{x \in G}\, x \notin A \Leftrightarrow A \neq G \Leftrightarrow \bar{A} \neq \emptyset \Leftrightarrow \exists_{x \in G}\, x \in \bar{A}$$

Propositional Forms and on-sets

$$E[A] = \{\hat{x} \mid A(\hat{x}) \Leftrightarrow t\}_G \;;\; G = \{t, f\}^n$$

$$E[A], E[B] \subseteq G$$

$$E[A \wedge B] = \{\hat{x} \mid A \wedge B\}_G = \{\hat{x} \mid A\} \cap \{\hat{x} \mid B\}$$

$$= E[A] \cap E[B]$$

$$E[\neg A] = \{\hat{\underline{x}} \mid \neg A\}_G = \overline{E[A]}$$

$$E[A \to B] = \{\hat{\underline{x}} \mid A \to B\}_G$$

$$= \{\hat{\underline{x}} \mid \neg A \vee B\}_G$$

$$= \{\hat{\underline{x}} \mid \neg A\}_G \cup \{\hat{\underline{x}} \mid B\}_G$$

$$= \overline{E[A]} \cup E[B]$$

$$= \overline{E}[A] \cup E[B]$$

3.3 Relations between sets

$$A = B \iff \forall_{x \in G} \, (x \in A \longleftrightarrow x \in B)$$

$$\iff \forall_{x \in G} \, \neg (x \in A \not\leftrightarrow x \in B)$$

$$\iff \forall_{x \in G} \, x \notin (A \triangle B)$$

$$\iff A \triangle B = \emptyset$$

$$A \subset B \iff (A \subseteq B) \wedge (A \neq B)$$

$$\iff (\forall_{x \in G} \, x \notin A \setminus B) \wedge (\exists_{x \in G} \, x \in A \triangle B)$$

G6

$$\iff \exists_{x \in G} \left[(x \notin A \setminus B) \wedge (x \in A \triangle B) \right]$$

$$\iff \exists_{x \in G} \left[\neg (\underbrace{x \in A}_{a} \wedge x \notin B) \wedge (x \in A \not\leftrightarrow \underbrace{x \in B}_{b}) \right]$$

$$\iff \exists_{x \in G} \left[\neg (a \wedge \neg b) \wedge (a \not\leftrightarrow b) \right]$$

$$\iff \exists_{x \in G} \left[(\neg a \vee b) \wedge (a \vee b) \wedge \neg (a \wedge b) \right] \qquad \text{B10}$$

$$\iff \exists_{x \in G} \left[\underbrace{(\neg a \vee b) \wedge (a \vee b) \wedge (\neg a \vee \neg b)}_{\text{CCNF}} \right]$$

$$\iff \exists_{x \in G} (\neg a \wedge b) \qquad \text{CDNF}$$

$$\iff \exists_{x \in G} (x \notin A \wedge x \in B)$$

$$\iff \exists_{x \in G} (x \in B \setminus A)$$

$$\iff B \setminus A \neq \emptyset$$

$$\boxed{A \subset B \Rightarrow B \setminus A \neq \emptyset}$$

3.4 Set Law

$A, B, C \subseteq G \Leftrightarrow A, B, C \in P(G)$

a) Set Equality

$A = B \Leftrightarrow t$

$\{x \in G \mid x \in A\} = \{x \in G \mid x \in B\} \Leftrightarrow t$

$\forall_{x \in G} [x \in A \leftrightarrow x \in B] \Leftrightarrow t$

$\forall_{x \in G} [P_x \longleftrightarrow Q_x] \Leftrightarrow t$

$x \in A \underset{G}{\Leftrightarrow} x \in B$

5) $A \cap (A \cup B) = A$

$x \in [A \cap (A \cup B)] \underset{G}{\Leftrightarrow} x \in A$

$\underbrace{x \in A}_{a} \wedge (x \in A \vee \underbrace{x \in B}_{b}) \underset{G}{\Leftrightarrow} x \in A$

$\boxed{a \wedge (a \vee b) \Leftrightarrow a}$ (A5)

H6) $\boxed{A \cap G = A}$

$x \in (A \cap G) \Leftrightarrow x \in A$

$\underbrace{x \in A}_{a} \wedge \underbrace{x \in G}_{t} \Leftrightarrow x \in A$

$\boxed{a \wedge t \Leftrightarrow a}$ (A6)

J7) $A \cap (B \setminus A) = \phi$

$x \in (A \cap (B \setminus A)) \Leftrightarrow x \in \phi$

$x \in A \wedge (x \in B \wedge x \notin A) \Leftrightarrow x \in \phi$

$\boxed{a \wedge (b \wedge \neg a) \Leftrightarrow f}$

B3) $\boxed{a \wedge (b \not\leftrightarrow c) \Leftrightarrow (a \wedge b) \not\leftrightarrow (a \wedge c)}$

$x \in A \wedge (x \in B \not\leftrightarrow x \in C) \Leftrightarrow (x \in A \wedge x \in B)$
$\not\leftrightarrow (x \in A \wedge x \in C)$

$\boxed{A \cap (B \triangle C) \Leftrightarrow (A \cap B) \triangle (A \cap C)}$ (J14)

b) Subset Relations

$\boxed{A \subseteq B} \Leftrightarrow t$

$\{x \in G \mid x \in A\} \subseteq \{x \in A \mid x \in B\} \Leftrightarrow t$

$\forall_{x \in G} [x \in A \rightarrow x \in B] \Leftrightarrow t$

$x \in A \underset{G}{\Rightarrow} x \in B$

$a \Rightarrow b$

J21) $\boxed{A \subseteq A \cup B}$

$x \in A \Rightarrow x \in (A \cup B)$

$\underbrace{x \in A}_{a} \Rightarrow x \in A \vee \underbrace{x \in B}_{b}$

$\boxed{a \Rightarrow a \vee b}$ (E5)

J23) $\boxed{A \cap (\bar{A} \cup B) \subseteq B}$

$x \in A \wedge (x \notin A \vee x \in B) \Rightarrow x \in B$

$x \in A \wedge (x \in A \rightarrow x \in B) \Rightarrow x \in B$

$\boxed{a \wedge (a \rightarrow b) \Rightarrow b}$ (E9)

E11) $\boxed{(a \rightarrow b) \wedge (b \rightarrow c) \Rightarrow a \rightarrow c}$

$(\neg a \vee b) \wedge (\neg b \vee c) \Rightarrow \neg a \vee c$

$\boxed{(\bar{A} \cup B) \cap (\bar{B} \cup C) \subseteq (\bar{A} \cup C)}$

$\overline{\bar{A} \cup C} \subseteq \overline{(\bar{A} \cup B) \cap (\bar{B} \cup C)}$

$A \cap \bar{C} \subseteq (A \cap \bar{B}) \cup (B \cap \bar{C})$ (H10)

$\boxed{A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)}$

c) Rules of Inference

$$\boxed{\begin{array}{l} a \rightarrow b \Longleftrightarrow \neg a \vee b \Longleftrightarrow \neg(a \vee \neg b) \\ \quad \Longleftrightarrow a \longleftrightarrow (a \wedge b) \Longleftrightarrow b \longleftrightarrow a \vee b \end{array}}$$

D1, A10

D12

$a \Rightarrow b$ IFF $\neg a \vee b \Longleftrightarrow t$ IFF $a \wedge \neg b \Longleftrightarrow f$ (RI3)

IFF $a \Longleftrightarrow a \wedge b$ IFF $b \Longleftrightarrow a \vee b$

$$\boxed{\begin{array}{l} A \subseteq B \Longleftrightarrow (\bar{A} \cup B) = G \Longleftrightarrow A \cap \bar{B} = \phi \\ \quad \Longleftrightarrow A = A \cap B \Longleftrightarrow B = A \cup B \end{array}}$$

(J29)

J32)

$$\boxed{\begin{array}{l} A \subseteq B \cup C \Longleftrightarrow \bar{A} \cup B \cup C = G \\ \quad \Longleftrightarrow A \cap \bar{B} \subseteq C \end{array}}$$

$$\boxed{\begin{array}{l} a \Rightarrow b \vee c \text{ IFF } \neg a \vee b \vee c \Longleftrightarrow t \\ \quad \text{IFF } a \wedge \neg b \Rightarrow c \end{array}}$$

J31) $\boxed{A \subseteq B \Longleftrightarrow \bar{B} \subseteq \bar{A}}$

$\boxed{a \Rightarrow \text{ IFF } \neg b \Longleftrightarrow \neg a}$ RI2

$$(\forall_{x \in M} P_x) \wedge (\exists_{x \in M} Q_x) \Rightarrow (\forall_{x \in M} P_x) \vee (\forall_{x \in M} Q_x)$$

E4, E5

$a \wedge b \Rightarrow a$ $\quad a \Rightarrow a \vee b$

$$(\forall_{x \in M} P_x) \wedge (\exists_{x \in M} Q_x) \underset{E4}{\Rightarrow} (\forall_{x \in M} P_x)$$

$$\underset{E5}{\Rightarrow} (\forall_{x \in M} P_x) \vee (\forall_{x \in M} Q_x)$$

Set Relations are illustrative

$P_x \Leftrightarrow x \in A \ ; \ Q_x \Leftrightarrow x \in B \ ; \ A, B \in P(M)$

e.g. F1) $\neg \forall_{x \in M} x \in A \Leftrightarrow \exists_{x \in M} x \notin A$

$\neg (A = M) \Leftrightarrow A \neq M$

F5) $\forall_{x \in M} (x \in A \wedge x \in B) \Leftrightarrow (\forall_{x \in M} x \in A) \wedge (\forall_{x \in M} x \in B)$

$A \cap B = M \Leftrightarrow (A = M) \wedge (B = M)$

F6) $\exists_{x \in M} (x \in A \vee x \in B) \Leftrightarrow (\exists_{x \in M} x \in A) \vee (\exists_{x \in M} x \in B)$

$A \cup B \neq \emptyset \Leftrightarrow (A \neq \emptyset) \vee (B \neq \emptyset)$

G1) $x_1 \in A \Rightarrow \exists_{x \in M} x \in A \ ; \ x_1 \in A \Rightarrow A \neq \emptyset$

$\forall_{x \in M} x \in A \Rightarrow x_1 \in A \ ; \ A = M \Rightarrow x_1 \in A$

G2) $\forall_{x \in M} x \in A \Rightarrow \exists_{x \in M} x \in A \ ; \ A = M \Rightarrow A \neq \emptyset$

G4) $\exists_{x \in M} (x \in A \wedge x \in B) \Rightarrow (\exists_{x \in M} x \in A) \wedge (\exists_{x \in M} x \in B)$

$A \cap B \neq \emptyset \Rightarrow (A \neq \emptyset) \wedge (B \neq \emptyset)$

4) Relations

$R = \{(x,y)\} \in M \times N \,/\, P_{xy}\}$ Relation is a set

R: set of all (x,y) which fulfill P_{xy}

x, y : individual numbers

P_{xy} : binary predicate

Examples for P_{xy}

1) $x \geq y$

2) $x \in y$

3) x is a divisor of y

4) x has more inhabitants than y

5) $f(x) = y$

(x,y) : ordered pair, tuple

$x \neq y \Rightarrow \boxed{(x,y) \neq (y,x)}$

$(x,y) = (y,x) \Rightarrow x = y$

$A \times B$: Cross product, Cartesian product

$A \times B = \{(x,y) \,|\, x \in A \wedge y \in B\}$

$(x,y) \in A \times B \Leftrightarrow x \in A \wedge y \in B \Leftrightarrow (y,x) \in B \times A$

$|A \times B| = |A| \cdot |B|$

e.g. $A = \{\text{Anne}, \text{Betty}, \text{Catherine}\} = \{a,b,c\}$

$B = \{m,n\}$

$A \times B = \{(a,m), (a,n), (b,m), (b,n), (c,m), (c,n)\}$

$|A \times B| = |A| \cdot |B| = 3 \cdot 2$
$= 6$

$B \times B = \{(m,m), (m,n), (n,m), (n,n)\}$

$|B \times B| = |B|^2 = 2 \cdot 2 = 4$

$S = \{(x,y) \in A \times B \,/\, x \text{ studies in } y\}$

e.g. $S = \{(a,n), (b,n), (c,m)\}$

$R = \{(x,y) \in B^2 \,/\, x \text{ has more inhabitants than } y\}$

e.g. $R = \{(m,n)\}$

$A = \emptyset \vee B = \emptyset \Rightarrow A \times B = \emptyset$

$A \neq B \wedge A \neq \emptyset \wedge B \neq \emptyset \Rightarrow A \times B \neq B \times A$

k1) $(A \times B) \times C = A \times (B \times C)$ Associativity

$((x, y), z) \in (A \times B) \times C \Leftrightarrow (x, (y, z)) \in A \times (B \times C)$

$x \in A \wedge y \in B \wedge z \in C \Leftrightarrow x \in A \wedge y \in B \wedge z \in C$

K2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ Distribufivity

1) $(x, y) \in A \times (B \cup C) \Leftrightarrow (x, y) \in [(A \times B) \cup (A \times C)]$

2) $x \in A \wedge y \in (B \cup C)$

3) $x \in A \wedge (y \in B \vee y \in C)$

4) $(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$ 3, A3

5) $(x, y) \in A \times B \vee (x, y) \in A \times C$

6) $(x, y) \in [(A \times B) \cup (A \times C)]$

$R = \{(x, y) \in A \times B \mid P_{xy}\}$

$(x, y) \in R \Leftrightarrow P_{xy}$

$(x, y) \in R \Leftrightarrow R_{xy} \Leftrightarrow x R_y$

"x is in relation to y"

Homogeneous relation : $R \subseteq A^2 = A \times A$

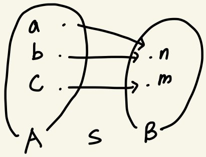
$R = \{(x, y) \mid x R_y\}_{A^2}$ binary relation within A^2

Heterogeneous relation : $R \subseteq \underset{\text{domain}}{A} \times \underset{\text{codomain}}{B}$

(Inhomogeneous)

$R = \{(x, y) \mid x R_y\}_{A \times B}$ binary relation from A to B / over $A \times B$

e.g.



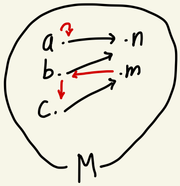
$S \subseteq A \times B$

$|A \times B| = |A| \times |B|$

$= 3 \cdot 2$

$= 6$

Inhomogeneous / Heterogeneous relation



Red arrow arent possible in Heterogeneous

$S \subseteq M^2 = (A \cup B)^2$

$|M^2| = 25$

Homogeneous Relation

4.1 Operations on Relations (unary, binary)

A) Set operations : $\overline{R}$, $R \cap S$, $R \cup S$, $R \setminus S$, $R \Delta S$

$R, S \subseteq A \times B$ or $R, S \subseteq A^2$

eg $\overline{R} \subseteq A^2$, $\overline{R} = A^2 \setminus R$ Complimentary Relation $\overline{R}$

$$\overline{R} = \{(x,y) \mid \neg x R y\}_{A \times B} \; ; \; \overline{R} \subseteq A \times B$$

$$\neg(xRy) \iff (x,y) \notin R \iff (x,y) \in \overline{R} \iff x\overline{R}y$$

e.g. $(x,y) \notin R \cup S \iff x(\overline{R \cup S})y \iff x(\overline{R} \cap \overline{S})y$

B) Conversion (Transposition, Reciprocal)

$R^{-1} \triangleq R^T \triangleq R^c$

$R \subseteq A \times B$

$R^{-1} = \{(y,x) \mid (x,y) \in R\}_{B \times A} \subseteq B \times A$

$(y,x) \in R^{-1} \iff yR^{-1}x \iff xRy$

$\iff (x,y) \in R$

Reverse arrow direction

L1) : $(A \times B)^{-1} = B \times A$

$\emptyset^{-1} = \emptyset$

$I^{-1} = I$

L2) : $(\overline{R})^{-1} = \overline{(R^{-1})}$

$(R^{-1})^{-1} = R$

1) $(x,y) \in (\overline{R})^{-1} \Leftrightarrow (x,y) \in \overline{(R^{-1})}$

2) $(y,x) \in \overline{R}$

3) $(y,x) \notin R$

4) $(x,y) \notin R^{-1}$

5) $(x,y) \in \overline{R^{-1}}$

1) $(x,y) \in (R^{-1})^{-1} \Leftrightarrow (x,y) \in R$

2) $(y,x) \in R^{-1}$

3) $(x,y) \in R$

L3) $(R \cup Q)^{-1} = R^{-1} \cup Q^{-1}$, $R,Q \subseteq A \times B$

1) $(x,y) \in (R \cup Q)^{-1} \Leftrightarrow (x,y) \in R^{-1} \cup Q^{-1}$

2) $(y,x) \in (R \cup Q)$

3) $(y,x) \in R \lor (y,x) \in Q$

4) $(x,y) \in R^{-1} \lor (x,y) \in Q^{-1}$

5) $(x,y) \in (R^{-1} \cup Q^{-1})$

L4) $(R \cap Q)^{-1} = R^{-1} \cap Q^{-1}$

L5) $(R \setminus Q)^{-1} = R^{-1} \setminus Q^{-1}$

Compliment vs Converse

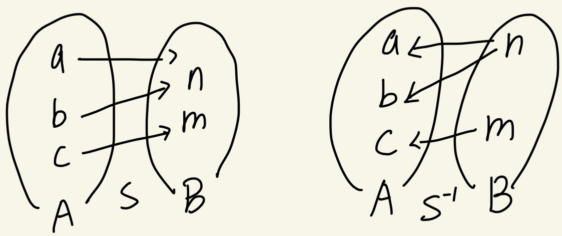
$A = \{a, b, c\}$

$B = \{m, n\}$

$A \times B = \{(a,n), (b,n), (c,n), (a,m), (b,m), (c,m)\}$

$S = \{(a,n), (b,n), (c,m)\} \subseteq A \times B$

$S^{-1} = \{(n,a), (n,b), (m,c)\} \subseteq B \times A$



$\overline{S} = (A \times B) \setminus S = \{(a,m), (b,m), (c,n)\}$

L6) $R \subseteq Q \iff R^{-1} \subseteq Q^{-1}$

1) $\forall_{(x,y) \in A \times B} \; [(x,y) \in R \rightarrow (x,y) \in Q]$

2) $\forall_{(x,y) \in A \times B} \; [(y,x) \in R^{-1} \rightarrow (y,x) \in Q^{-1}]$

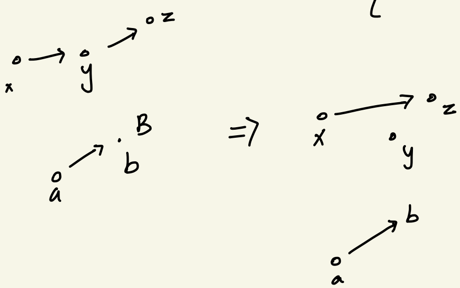
3) $R^{-1} \subseteq Q^{-1}$

C) Composition (Product)

$R \subseteq A \times B \ ; \ S \subseteq A \times B$

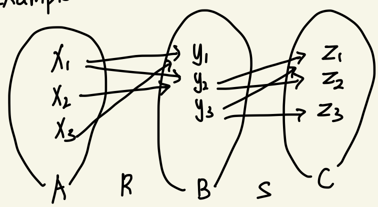
$$R \circ S = RS = \{ (x,z) \mid \underset{y \in B}{\exists} \ xRy \wedge ySz \}_{A \times C}$$

$$\{ \ldots, (x,y), \ldots, (a,b) \ldots \} \circ \{ \ldots, (y,z), \ldots, (b,b), \ldots \}$$
$$= \{ \ldots, (x,z), (a,b), \ldots \}$$



$$(x,z) \in RS \Leftrightarrow xRS \Leftrightarrow \underset{y \in B}{\exists} [xRy \wedge ySz]$$

Example



$R = \{ (x_1, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_1) \}$

$S = \{ (y_2, z_1), (y_2, z_2), (y_3, z_1), (y_3, z_3) \}$

$RS = \{ (x_2, z_1), (x_2, z_2), (x_1, z_1), (x_1, z_2) \}$

Options for performing Composition

- "reachability" in arrow diagram
- "pairwise" chaining
- binary matrix multiplication of adjacency matrices

Adjacency matrices of R, S: $\underset{\sim}{R}$, $\underset{\sim}{S}$

$$\underset{\sim}{R} = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 1 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 1 & 0 & 0 \end{array} \circ \underset{\sim}{S} = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0 & 0 & 0 \\ y_2 & 1 & 1 & 0 \\ y_3 & 1 & 0 & 1 \end{array} = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & 1 & 1 & 0 \\ x_2 & 1 & 1 & 0 \\ x_3 & 0 & 0 & 0 \end{array}$$

Application of $\circ$ for $(A \times B) \circ (B \times C) = (A \times C)$, if $B \neq \emptyset$

↑ Compose

1) $(x,z) \in (A \times B) \circ (B \times C) \iff (x,z) \in (A \times C) \wedge B \neq \emptyset$

2) $\underset{y \in B}{\exists} (x,y) \in A \times B \wedge (y,z) \in (B \times C)$

3) $\underset{y \in B}{\exists} x \in A \wedge y \in B \wedge \cancel{y \in B} \wedge z \in C$

4) $x \in A \wedge z \in C \wedge \underset{y \in B}{\exists} y \in B$ (F8)

5) $(x,z) \in (A \times C) \wedge B \neq \emptyset$

Laws of Composition

$Q, R \subseteq A \times B$; $S, T \subseteq B \times C$; $V \subseteq A \times C$; $W \subseteq C \times D$

$R \circ S \neq S \circ R$ IF $R \neq S$ AND $R, S \neq \emptyset, I$

~~Commutativity~~

M1) $(R \circ S) \circ W = R \circ (S \circ W)$ Associativity

$(x, z) \in (R \circ S) \circ W \iff (x, z) \in R \circ (S \circ W)$

$\exists_{y \in C} [(x, y) \in R \circ S \wedge (y, z) \in W] \iff$

$\exists_{y \in C} [x R S y \wedge y W z] \iff$

$\exists_{y \in C} [\exists_{r \in B} [x R r \wedge r S y] \wedge y W z] \iff$ F8

$\exists_{y \in C} [\exists_{r \in B} [x R r \wedge r S y \wedge y W z]] \iff$ F4

$\exists_{r \in B} [\exists_{y \in C} [x R r \wedge r S y \wedge y W z]] \iff$ F8

$\exists_{r \in B} [x R r \wedge \exists_{y \in C} [r S y \wedge y W z]] \iff$

$\exists_{r \in B} [x R r \wedge r S W z] \iff$

$(x, z) \in R \circ (S \circ W)$

M2) $R \circ (S \cup T) = R \cup S \cup R \circ T$ Distributivity of $\circ$ over $\cup$

$(x,z) \in R \circ (S \cup T) \iff (x,z) \in (R \circ S \cup R \circ T)$

$\exists_{y \in B} [(x,y) \in R \wedge (y,z) \in (S \cup T)] \iff$

$\exists_{y \in B} [xRy \wedge (ySz \vee yTz)] \iff$ A3

$\exists_{y \in B} [(xRy \wedge ySz) \vee (xRy \wedge yTz)] \iff$ F6

$\exists_{y \in B} (xRy \wedge ySz) \vee \exists_{y \in B} (xRy \wedge yTz) \iff$

$(x,z) \in R \circ S \vee (x,z) \in R \circ T \iff$

$(x,z) \in (R \circ S \cup R \circ T$

M3) $(S \cup T) \circ W = S \circ W \cup S \circ T$

M4) $R \circ (S \cap T) \subseteq R \circ S \cap R \circ T$ Distributivity of $\circ$ over $\cap$

1) $(x,z) \in R \circ (S \cap T) \Rightarrow (x,z) \in (R \circ S \cap R \circ T)$

2) $\exists_{y \in B}\, xRy \wedge y(S \cap T)z$

3) $\exists_{y \in B}\, xRy \wedge ySz \wedge yTz$

4) $\exists_{y \in B}\, xRy \wedge ySz \wedge xRy \wedge yTz$ 3) A4

5) $\exists_{y \in B} (xRy \wedge ySz) \wedge \exists_{y \in B} (xRy \wedge yTz)$ 4) G4

6) $(x,z) \in R \circ S \wedge (x,z) \in R \circ T$

7) $(x,z) \in (R \circ S \cap R \circ T)$

M5) $(S \cap T) \circ W \subseteq S \circ W \cap T \circ W$

M6) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

$(x,z) \in (R \circ S)^{-1} \Leftrightarrow (x,z) \in (S^{-1} \circ R^{-1})$

$(x,z) \in (R \circ S)^{-1}$

$(z,x) \in R \circ S$

$\exists_{y \in B} (z,y) \in R \wedge (y,x) \in S \Leftrightarrow$

$\exists_{y \in B} (y,z) \in R^{-1} \wedge (x,y) \in S^{-1} \Leftrightarrow$

$\exists_{y \in B} (x,y) \in S^{-1} \wedge (y,z) \in R^{-1} \iff$

$(x,z) \in S^{-1} \circ R^{-1}$

M8) Schröder rule

1) $R \circ S \subseteq V \iff R^{-1} \circ \bar{V} \subseteq \bar{S}$

2) $\forall_{x,y} [x R \circ S y \to x V y]$

3) $\forall_{x,y} [\exists_z (x R z \wedge z S y) \to x V y]$

4) $\forall_{x,y} [\neg \exists_z (x R z \wedge z S y) \vee x V y]$ 3, D1

5) $\forall_{x,y} [\forall_z (x \bar{R} z \vee z \bar{S} y) \vee x V y]$ 4, F2, A10

6) $\forall_{x,y} [\forall_z (x \bar{R} z \vee z \bar{S} y \vee x V y)]$ 5, F7

7) $\forall_{x,y,z} [x \bar{R} z \vee z \bar{S} y \vee x V y]$ 6,

8) $\forall_{y,z} [\forall_x (x \bar{R} z \vee x V y) \vee z \bar{S} y]$ 7, F7

9) $\forall_{y,z} [\forall_x (z (\bar{R})^{-1} x \vee x V y) \vee z \bar{S} y]$ 8, ref converse

10) $\forall_{y,z} [\neg \forall_x (z (\bar{R})^{-1} x \vee x V y) \to z \bar{S} y]$ 9, D1

11) $\forall_{y,z} [\exists_x (z R^{-1} x \wedge x \bar{V} y) \to z \bar{S} y]$ 10, F2, A10, L2

12) $\forall_{y,z} [z R^{-1} \circ \bar{V} y \to z \bar{S} y]$ 11, Def. Composition

(2) $R^{-1} \circ \overline{V} \subseteq \overline{S}$ 12

M7) Monotony

$$Q \subseteq R \wedge S \subseteq T \Rightarrow Q \circ S \subseteq R \circ T$$

1) $Q \subseteq R$
2) $S \subseteq T$ $\Rightarrow QS \subseteq RT$

3) $QT \subseteq QT$ J20

4) $Q^{-1}\overline{QT} \subseteq \overline{T}$ 3, M8

5) $\overline{T} \subseteq \overline{S}$ 2) J31

6) $Q^{-1}\overline{QT} \subseteq \overline{S}$ 4, 5, J34

7) $QS \subseteq QT$ 6, M8

8) $RT \subseteq RT$ J20

9) $\overline{RT}\, T^{-1} \subseteq \overline{R}$ 8, M8

10) $\overline{R} \subseteq \overline{Q}$ 1, J31

11) $\overline{RT}\, T^{-1} \subseteq \overline{Q}$ 9, 10, J34

12) $QT \subseteq RT$ 11, M8

13) $QS \subseteq RT$ 7, 12, J34

4.2 Binary Graphs

$G = (M, S)$; (M, S) is a graph

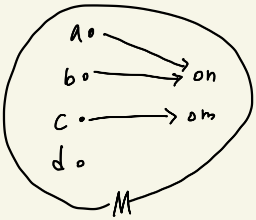
M : set

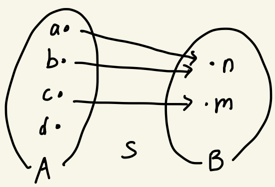
S : relation

e.g. $M = \{a, b, c, d\}$

$S = \{(a,n), (b,n), (c,m)\}$

Arrow diagram





$S \subseteq M^2$

$|M^2| = |M|^2 = 36$

S is homogeneous

$G_p\,(A \cup B, S)$

$S \subseteq A \times B$

$|A \times B| = |A| \cdot |B| = 8$

S is heterogeneous

x is predecessor of y : $(x, y) \in S \Leftrightarrow xSy$

y is successor of x $\Leftrightarrow ySx$

gamma

$(x,y) \in S \Leftrightarrow y \in \Gamma^{+}(x)$ successor

$\Leftrightarrow x \in \Gamma^{-}(y)$

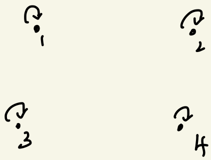
predecessor

$$I = \{(x,y) \in M^2 \mid x = y\}$$

$$(x,y) \in I \iff x I y \iff x = y$$

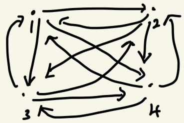
e.g.

$G = (\{1,2,3,4\}, I)$



e.g.

$G = (\{1,2,3,4\}\ \bar{I})$



4.3 Properties of relations

x has a successor $\iff (x,x)\in RR^{-1}$

$$\exists_{y\in Y}\, xRy \iff \exists_{y\in Y}\, xRy \wedge xRy \iff \exists_{y\in Y}\, xRy \wedge yR^{-1}x$$

$$\iff xR\circ R^{-1}x \iff (x,x)\in RR^{-1}$$

Alternative representation:

x has a successor $\iff \forall_{y\in Y}\, xRY^2y$

$$\exists_{z\in Y}\, xRz \iff \forall_{y\in Y}\exists_{z\in Y}\, xRz \wedge \underbrace{z(\underbrace{Y\times Y}_{Y^2 \text{ universal relation}})y}_{t}$$

$$\iff \forall_{y\in Y}\, xRY^2y$$

R is total $\iff \forall_{x\in X}\exists_{y\in Y}\, xRy$

$$\iff \forall_{x\in X}\,(x,x)\in RR^{-1}$$

$$\iff \forall_{(x,z)\in X^2}\,(x,z)\in I_X \to (x,z)\in RR^{-1}$$

$$\iff I_X \subseteq RR^{-1}$$

R is total $\iff \forall_{x\in X}\exists_{z\in Y}\, xRz$

$$\iff \forall_{x\in X}\forall_{y\in Y}\, xRY^2y$$

$$\iff \forall_{(x,y)\in X\times Y}\,(x,y)\in RY^2$$

$$\iff RY^2 = X\times Y$$

$$\iff R\circ(I_Y \cup \overline{I_Y}) = X\times Y$$

$$\iff R\cup R\circ\overline{I_Y} = X\times Y$$

$$\iff \overline{R}\subseteq R\overline{I_Y}$$

J29

1) R is functional $\iff R^{-1}R \subseteq I_Y$

2) $\forall_{y_1, y_2 \in Y} \forall_{x \in X} [xRy_1 \wedge xRy_2 \rightarrow y_1 = y_2]$

3) $\forall_{(y_1, y_2) \in Y^2} [[\forall_{x \in X} \neg(xRy_1 \wedge xRy_2)] \vee y_1 = y_2]$ 2, F2, D1

4) $\forall_{(y_1, y_2) \in Y^2} [[\neg\exists_{x \in X} (y_1 R^{-1} x \wedge xRy_2)] \vee y_1 = y_2]$ 3, F2

5) $\forall_{(y_1, y_2) \in Y^2} [\exists_{x \in X} (y_1 R^{-1} x \wedge xRy_2) \rightarrow y_1 = y_2]$

6) $\forall_{(y_1, y_2) \in Y^2} [y_1 R^{-1} R y_2 \rightarrow y_1 = y_2]$

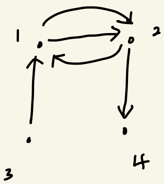
7) $R^{-1} \circ R \subseteq I_Y$

Composition of a relation with itself

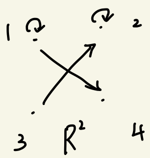
$R \subseteq A^2$

e.g. $A = \{1, 2, 3, 4\}$

$R = \{(1,2), (2,1), (2,4), (3,1)\}$



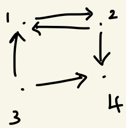
$(x,y) \in R \iff P_1(x,y)$ exists in $G = (A,R)$



$R \circ R = RR = R^2 \subseteq A^2$

$R^2 = \{(1,1), (1,4), (2,2), (3,2)\}$

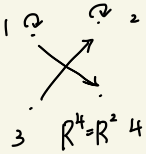
$(x,y) \in R^2 \iff P_2(x,z,y)$ exists in G



$R \circ R \circ R = R^2 \circ R = R \circ R^2 = R^3 \subseteq A^2$

$R^3 = \{(3,4), (1,2), (2,1), (2,4), (3,1)\}$

$(x,y) \in R^3 \iff P_3(x,z,w,y)$ exists in G



$R \circ R \circ R \circ R = R^2 \circ R^2 = R^3 \circ R = R \circ R^3 = R^4 \subseteq A^2$

$R^4 = \{(1,1), (1,4), (2,2), (3,2)\}$

$(x,y) \in R^4 \iff P_4(x,\dots,y)$ exists in G

$R^5 = R^4 \circ R = R^2 \circ R = R^3$

$R^6 = R^5 \circ R = R^3 \circ R = R^4$

$\vdots$

For $s \in \mathbb{N}_0$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, we have

$$R^{s+p} = R^s \Rightarrow \forall_{n \in \mathbb{N}_0} \left[R^n \in \{R^0, R^1, R^2, \dots, R^{s+p-1}\} \right]$$

In our example: $s=2$, $p=2$, $s+p-1=3$:

$$R^4 = R^2 \Rightarrow \forall_{n \in \mathbb{N}_0} \left[R^n \in \{R^0, R^1, R^2, R^4\} \right]$$

$$R^0 = I$$

$$R^m \circ R^n = R^{m+n}$$

$$(R^m)^n = R^{m+n}$$

1) $R^{s+p} = R^s \Rightarrow R^{s+k \cdot p+r} = R^{s+r}$, $k \in \mathbb{N}_0$, $0 \le r \le p-1$

2) $R^s \circ R^p = R^s$ 1, M12

3) $(R^s \circ R^p) \circ R^p = R^s$ 2

4) $((R^s \circ R^p) \circ R^p) \circ R^p = R^s$ 3, 2

5) $R^{s+k \cdot p} = R^s$

6) $R^{s+kp+r} = R^{s+r}$

Composition of a relation with itself

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1,2), (2,1), (2,4), (3,1)\}$$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{R} \qquad \underset{\sim}{R} \qquad \underset{\sim}{R^2}$$

$$\underset{\sim}{R^2} = \{(1,1), (2,2), (3,2), (1,4)\}$$

$$\underset{\underset{\sim}{R}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \circ \underset{\text{"2"}}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} = \underset{\Gamma^-(2)}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\underset{\text{"2"}}{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}} \circ \underset{\underset{\sim}{R}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} = \underset{\Gamma^+(2)}{\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}}$$

4.4 Closure operations on relations

$R \subseteq A^2 \iff R \in P(A^2)$

$r(R), s(R), t(R)$: reflextive, Symmetric, transitive
Closure of R

$r(R) = \text{inflimum}\{H \in P(A^2) \mid (R \subseteq H) \wedge (I \subseteq H)\}$

$= R \cup I$

R is reflexive $\iff I \subseteq R \iff R = r(R)$

$s(R) = \inf\{H \in P(A^2) \mid (R \subseteq H) \wedge (H = H^{-1})\}$

$= R \cup R^{-1}$

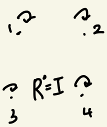
R is symmetric $\iff R = R^{-1} \iff R = s(R)$

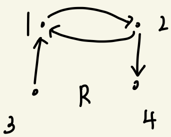
$t(R) = \inf\{H \in P(A^2) \mid (R \subseteq H) \wedge H^2 \subseteq H\}$

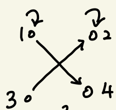
$= \bigcup_{\nu=1}^{\infty} R^{\nu} = R^{+}$

R is transitive $\iff R^2 \subseteq R \iff R = t(R)$
$\iff R = R^{+}$

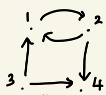
Example : $R \subseteq A^2$; $G = (A, R)$



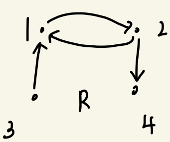




R^2
R^4
R^6
$\vdots$

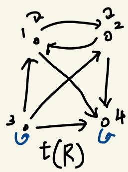


R^3
R^5
R^7
$\vdots$



$\Rightarrow t(R) = R^{+}$

$rt(R) = R^{*}$



$t(R)$

since $\bigcup$ of all R

$t(R) = R^{+} = \bigcup_{\nu=1}^{|A|} R^{\nu} = R \cup R^2 \cup R^3 \cup R^4$
"plus Closure"

$rt(R) = R^{*} \quad \bigcup_{\nu=0}^{|A|} R^{\nu} = R^0 \cup R \cup R^2 \cup R^3 \cup R^4$ "Star Closure"

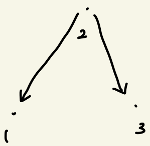
$= I \cup R^{+}$

$= (I \cup R)^{+}$

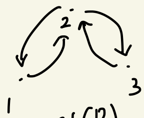
$rs(R) = sr(R)$

$rt(R) = tr(R)$

$st(R) \subseteq ts(R)$

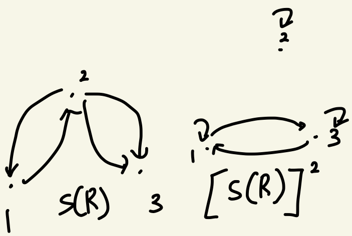


$R^2 = R^3 = \dots = \phi$



st(R)

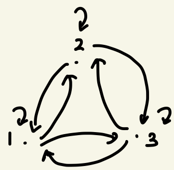
Symmetric
intransitive



s(R)

$[S(R)]^2$

$[S(R)^3]$



reflexive
symmetric
transitive

ts(R)

$$t(R) = \bigcup_{v=1}^{\infty} R^v$$

Lower bound for $t(R)$ is desired

$$R^n \subseteq \bigcup_{v=1}^{\infty} R^v \quad , \quad \forall_{n \in \mathbb{N}}$$

$$(x,y) \in t(R) \iff \exists_{n \in \mathbb{N}} (x,y) \in R^n$$

$$\iff \exists_{n \in \mathbb{N}} P_n(x, \dots, y) \text{ exists in } G = (A, R)$$

Let $n \in \mathbb{N}$, and $|A| < \infty$. Then

$P_n(x, \ldots, y)$ exists in G $\Rightarrow \underset{1 \le v \le |A|}{\exists} P_v(x, \ldots, y)$ exists in G

possibly with cycles

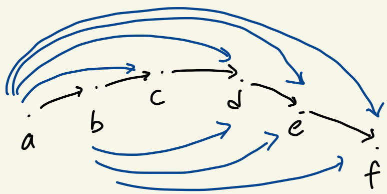
possibly as an elementary path (No cycles)

$$(x,y) \in R^n \Rightarrow \underset{1 \le v \le |A|}{\exists} (x,y) \in R^v$$

$$\Rightarrow R^n \subseteq \bigcup_{1 \le v \le |A|} R^v$$

$$t(R) = \bigcup_{v=1}^{|A|} R^v$$

Transitivity: why is $R^2 \subseteq R$ sufficient?



$$R^2 \subseteq R \Leftrightarrow \underset{n \in \mathbb{N}}{\forall} R^n \subseteq R$$

Proof sketch:

"$\Leftarrow$": trivial

"$\Rightarrow$": Induction

a) $R \subseteq R$ J20 (Basis, $n=1$)

b) $R^n \subseteq R \Rightarrow R^{n+1} \subseteq R$

1) $R^n \subseteq R$ Premise

2) $R \subseteq R$ Basis

3) $R^n \circ R \subseteq R \circ R$ 1, 2, M7

4) $R^{n+1} \subseteq R$

5) $R^2 \subseteq R$ 1

6) $R^{n+1} \subseteq R$ 4, 5, J34

$t(R) = R^+$ 'plus Closure'

$rt(R) = R^*$ 'star Closure'

$$rt(R) = r(t(R))$$
$$= I \cup R^+$$
$$= (I \cup R)^+$$

$$R^+ = R \circ R^* = R^* \circ R$$

$$sr(R) = rs(R)$$

$$\boxed{sr(R)} = r(R) \cup (r(R))^{-1}$$
$$= (R \cup I) \cup (R \cup I)^{-1}$$
$$= R \cup I \cup R^{-1} \cup I^{-1} \qquad L3$$
$$= R \cup I \cup R^{-1} \cup I \qquad L1$$
$$= \underbrace{R \cup R^{-1}} \cup I$$
$$= s(R) \cup I$$
$$= \boxed{rs(R)}$$

$$tr(R) = rt(R) = R^*$$

$$tr(R) = t(R \cup I) = (R \cup I)^*$$
$$= \bigcup_{v=1}^{\infty} (R \cup I)^v$$
$$= (R \cup I) \cup (R \cup I)^2 \cup (R \cup I)^3 \cup (R \cup I)^4 \cup \dots$$

$$(R \cup I)^v = R^v \cup R^{v-1} \cup R^{v-2} \cup \dots \cup R \cup I, \quad v \geq 1$$

$= I \cup R \cup R^2 \cup R^3 \cup \dots$

$= I \cup \bigcup_{v=1}^{\infty} R^v$

$= I \cup t(R)$

$= rt(R)$

4.5 Accessibility in binary graphs

$R \subseteq A^2$

$G = (A, R)$

x is ancestor of y : $(x, y) \in R^+$
(predecessor)

y is descendant of x :
(successor)

Set of descendant : $des(x) = \{y \mid x R^+ y\}_A$

Set of ancestors : $anc(x) = \{y \mid y R^+ x\}_A$

$(x, y) \in R^+ \iff x R^+ y \iff y (R^+)^{-1} x$

$\iff y \in des(x) \iff x \in anc(y)$

$\iff P(x, \dots, y)$ exists in G

$(x, x) \in R^+$ There exists a cycle through x in G

$\exists_{(x,y) \in A^2} \; x I y \wedge x R^+ y$ There exists a cycle through x in G

$\neg \left[\exists_{(x,y) \in A^2} \; x I y \wedge x R^+ y \right]$ G has no cycles

$\iff \forall_{(x,y) \in A^2} (\neg (x I y) \vee \neg (x R^+ y))$

$\iff \forall_{(x,y) \in A^2} (x I y \rightarrow x \overline{R^+} y)$

$\Leftrightarrow I \subseteq \overline{R^+} \Leftrightarrow R^+ \subseteq \overline{I}$ Could be loops

J31

$\exists_{(x,y)\in A^2} (x\overline{I}y) \wedge (xR^+y) \wedge (yR^+x)$ There exists a proper cycle in G

exclude loop

$\neg\left[\exists_{(x,y)\in A^2} (x\overline{I}y) \wedge (xR^+y) \wedge (yR^+x)\right] \Leftrightarrow$

$\forall_{(x,y)\in A^2} xIy \vee \neg\left[(xR^+y) \wedge (yR^+x)\right] \Leftrightarrow$

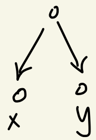
$\forall_{(x,y)\in A^2}\left[\left[(xR^+y) \wedge (yR^+x)\right] \to xIy\right] \Leftrightarrow$

$\forall_{(x,y)\in A^2}\left[\left[xR^+y \wedge x(R^+)^{-1}y\right] \to xIy\right] \Leftrightarrow$

$R^+ \cap (R^+)^{-1} \subseteq I$ There exists no proper Cycle in G

$(x,y) \in R^* \Leftrightarrow xR^*y$ y is accessible from x in G

$\forall_{(x,y)\in A^2} (x,y) \in R^* \Leftrightarrow R^* = A^2$ G is strongly Connected (every node is accessible from every node)

Mutually Accessible	accessible	at least one sided accessible	indirectly accessible
	$\circ \to \circ \to \circ$	$\circ \to \circ \to \circ$ or $\circ \leftarrow \circ \leftarrow \circ$	
$xR^*y \wedge yR^*x$	xR^*y	$xR^*y \vee yR^*x$	$x(R^*)^{-1} \circ R^*y$

Can be connected

$\underset{x}{\circ} \to \circ \leftarrow \circ \leftarrow \underset{y}{\circ}$

$x(R \cup R^{-1})^* y$

4.6 order relations

Generalization of "$<$", "$\leq$", "$\subset$", "$\subseteq$"
(composition of objects)

R is a partial order (relation): $\Leftrightarrow$

$R \subseteq A^2$, R is reflexive; antisymmetric; transitive

$I \subseteq R$ $R \cap R^{-1} \subseteq I$ $R^2 \subseteq R$

(A, R) is a partially ordered set (poset): $\Leftrightarrow$

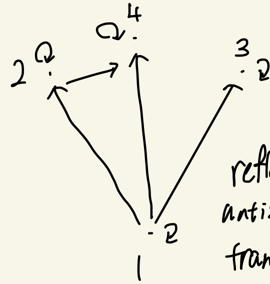
$G = (A, R)$ is a binary graph (digraph) with
R as partial order

e.g. $A = \{1, 2, 3, 4\}$

$x \leq y \Leftrightarrow x$ divide y w/o remainder

$\preceq$: order relation

$(A, \preceq)$: partially ordered set



reflexive
antisymmetric
transitive

R is strict (quasi) order (relation) : $\Leftrightarrow$

$R \subseteq A^2$, R is asymmetric and transitive

$R \cap R^{-1} = \emptyset$ $\qquad$ $R^2 \subseteq R$

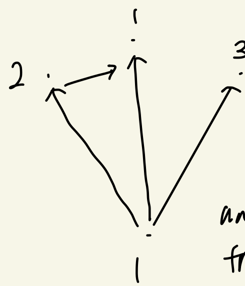
(A, R) is a strictly ordered set : $\Leftrightarrow$ $G = (A, R)$ is a binary digraph, with R as strict order (quasi)

e.g. $A = \{1, 2, 3, 4\}$

$x \preceq y \Leftrightarrow x$ divide y w/o remainder

$\preceq$: order relation

$(A, \preceq)$: partially ordered set



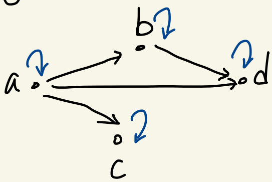
antisymmetric
transitive

R is a strict order $\Rightarrow$ $R \cup I = r(R)$ is a partial order

R is a partial order $\Rightarrow$ $R \setminus I$ is a strict order

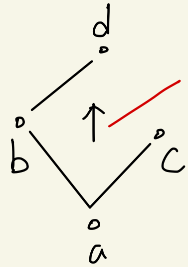
Simplified representation of ordered sets : Hasse diagram

e.g.



Arrow Diagram

$G = (A, R)$



take all edge as arrow up

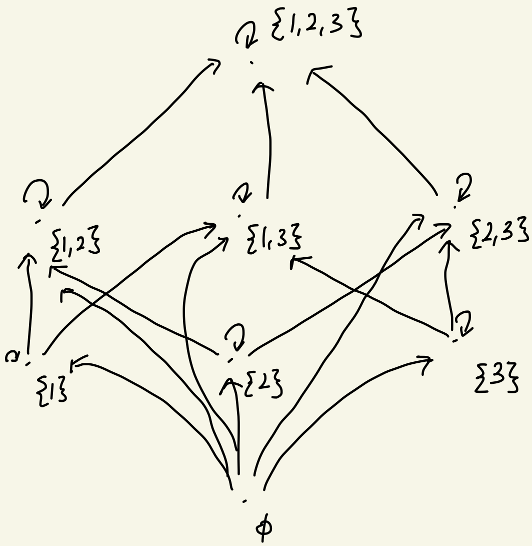
Hasse diagram

$G_H = (A, H)$

$H = R \setminus R^2$

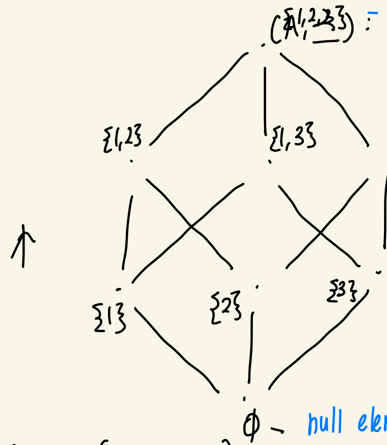
e.g. $A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\},$



$\succeq$: order relation

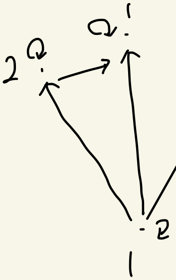
$(A, \subseteq)$: partially ordered set



- Universal element of P(A)

- Atomic element of P(A)

- null element of P(A)



reflexive
antisymmetric
transitive

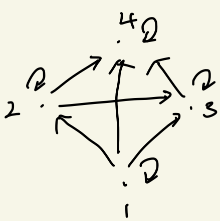
$G_H = (P(A), H)$

R is a total (linear, script) order : <=>
R is an order R is connex

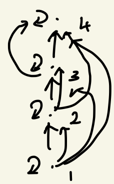
e.g. $A = \{1,2,3,4\}$

$\leq$: is a linear order

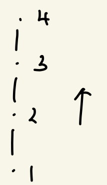
$(A, \leq)$: linearly ordered set (chain)



$G = (A, \leq)$



$G(A, \leq)$



Hasse diagram

R is $\left\{\begin{matrix}\text{strict}\\\text{partial}\\\text{total}\end{matrix}\right\}$ order $\Rightarrow R^{-1}$ is $\left\{\begin{matrix}\text{strict}\\\text{partial}\\\text{total}\end{matrix}\right\}$ order

order $\leq$ on A is well order $(A, \leq)$ is a well ordered set $:\Longleftrightarrow \forall_{B \in P(A) \setminus \{\emptyset\}} \ lst(B) \neq \emptyset$

$\leq$ is a well order on $A \Rightarrow \leq$ is total order on A

Bounds and Extrema

e.g. $A = \{p, q, r, s, t, u, v, w\}$

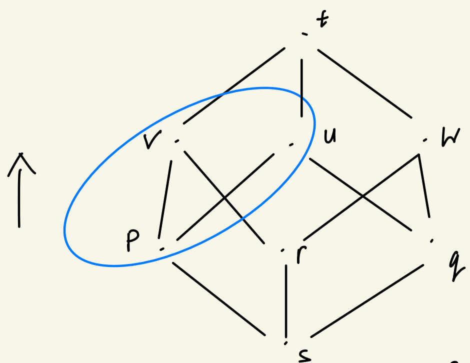
$B = \{p, u, v\}$

$$ub(B) = \bigcap_{x \in B} \Gamma^{+}(x)$$

$$= \Gamma^{+}(p) \cap \Gamma^{+}(v) \cap \Gamma^{+}(u)$$

$$= \{u, v, t, p\} \cap \{v, t\} \cap \{u, t\}$$

$$= \{t\}$$



Hasse diagram $G_H = (A, H)$

$grt(B) = B \cap ub(B) = \{\}$

$lst(B) = B \cap lb(B) = \{p\}$

$$lb(B) = \bigcap_{x \in B} \Gamma^{-}(x)$$

$$= \Gamma^{-}(p) \cap \Gamma^{-}(v) \cap \Gamma^{-}(u)$$

$$= \{s, p\} \cap \{p, r, s, v\} \cap \{p, q, s, u\}$$

$$= \{p, s\}$$

$$
\begin{aligned}
lub(B) &= lst(ub(B)) \\
&= ub(B) \cap lb(ub(B)) \\
&= \{t\} \cap lb(\{t\}) \\
&= \{t\} \cap \Gamma^{-}(t) \\
&= \{t\} \cap \{p, q, r, s, t, u, v, w\}
\end{aligned}
$$

$$
\begin{aligned}
glb(B) &= grt(lb(B)) \\
&= lb(B) \cap ub(lb(B)) \\
&= \{p, s\} \cap ub(\{p, s\}) \\
&= \{p, s\} \cap \bigcap_{x \in \{p,s\}} \Gamma^{+}(x) \\
&= \{p, s\} \cap \{p, u, v, t\} \\
&\quad \cap \{p, q, r, s, t, u, v, w\} \\
&= \{p\}
\end{aligned}
$$

$$
\begin{aligned}
max(B) &= B \cap \left[\bigcap_{x \in B} \overline{\Gamma^{-}(x) \setminus \{x\}} \right] \\
&= B \cap \left[\overline{\{s\}} \cap \overline{\{p, r, s\}} \cap \overline{\{p, q, s\}} \right] \\
&= B \cap \left[\{p, q, r, t, u, v, w\} \cap \{q, t, u, v, w\} \cap \{r, t, u, v, w\} \right] \\
&= B \cap \left[\{t, u, v, w\} \right] \\
&= \{u, v\}
\end{aligned}
$$

$$min(B) = \{p\}$$

4.7 Equivalence relations

Generalization of "="

R is equivalent relation : $\Leftrightarrow$

$R = r(R) \wedge R = s(R) \wedge R = t(R) \Leftrightarrow$

$R = tsr(R) = trs(R) \Leftrightarrow$

$I \subseteq R \wedge R = R^{-1} \wedge R^2 \subseteq R \Leftrightarrow$

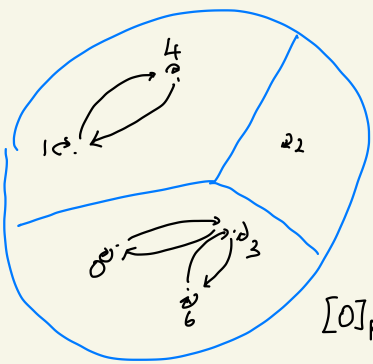
$R = (R \cup R^{-1})^*$

Example : $R = \{(x,y) \mid (x \bmod 3) = (y \bmod 3)\}$

$A = \{0,1,2,3,4,6\}$

$xRy = \exists_{n \in \mathbb{Z}} (y = x + 3 \cdot n)$

$R = \{(0,0), (1,1), (2,2), (3,3), (4,4), (6,6),$
$(3,6), (6,3), (0,3), (3,0), (0,6), (6,0), (1,4), (4,1)\}$



$G = (A, R)$

$[2]_R = \{2\}$

$[0]_R = [3]_R = [6]_R$

$= \{0, 3, 6\}$

Let $G=(A,R)$ with $R=tsr(R)$

Then it holds: the components of G are Cliques (PG1)

Equivalent Class of x in A, induced by R $\quad [x]_R = \{y \mid yRx\}$, $R=tsr(R)$

$$[x]_R = \Gamma^-(x) = \Gamma^+(x) = \Gamma(x)$$

$$(x,y) \in R \iff xRy \iff x \sim y \iff x \in [y]_R$$
$$\iff y \in [x]_R$$
$$\iff [x]_R = [y]_R$$

Properties of $R=tsr(R)$:

1) $[x] \neq [y] \iff [x] \cap [y] = \emptyset$

2) $[x] \neq \emptyset$, $\forall x \in A$

3) $\bigcup_{x\in A} [x] = A$

$A \backslash R = \{[x]_R \in P(A) \mid x \in A\}$

Quotient set ; Partition of A induced by relation R

Example : $A \backslash R = \{\{2\}, \{1,4\}, \{0,3,6\}\}$

$$= \{[2], [1], [0]\}$$

$R=tsr(R)$ induces a partition of A

$$\boxed{A \backslash R = \pi \text{ (Partition)}}$$

Partition ; $\pi \subseteq P(A)$

$\pi = \{k \in P(A) \mid k \in \pi\}$ $\quad$ k: block, Class

Properties:

1) $\forall_{k \in \Pi} \; k \neq \emptyset$

2) $\forall_{k \in \Pi} \; k \neq L \longleftrightarrow k \cap L \neq \emptyset$

3) $\bigcup_{k \in \Pi} k = A$

$$xRy \iff x \sim y \iff [x] = [y] \iff \exists_{k \in \Pi} (x \in k \wedge y \in k)$$

$Q \subseteq A^2$

1) $ECL(Q) = tsr(Q) = trs(Q) = (Q \cup Q^{-1})^*$

$Q \subseteq (Q \cup Q^{-1})^* = \inf \{ R \subseteq A^2 \mid (Q \subseteq R) \wedge (R = tsr(R)) \}$

"Equivalence Closure"

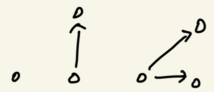
2) $ECO(Q^*) = Q^* \wedge (Q^*)^{-1}$

$\sup \{ R \subseteq A^2 / (R \subseteq Q^*) \wedge (R = tsr(R) \} = Q^* \wedge (Q^*)^{-1} \subseteq Q^*$

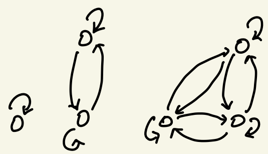
"Equivalence Core"

Example:

$G = (A, R)$



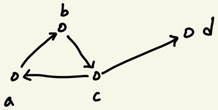
$ECL(R) =$
$(R \cup R^{-1})^*$



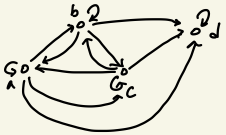
The components of G correspond to equivalence closure of $ECL(R)$

Example

$G = (A, R)$

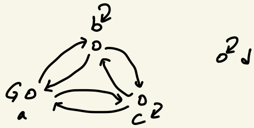


$G^* = (A, R^*)$



$R \subseteq R^*$

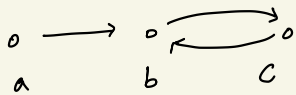
$ECO(R^*) = R^* \wedge (R^*)^{-1}$



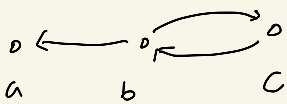
$(R^* \wedge (R^*)^{-1}) \subseteq R^*$

Cycles in G (represented in R) correspond to cliques in G^* (rep. in R^*) and equivalence closure of $ECO(R^*)$

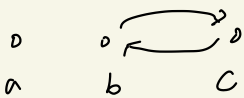
R



R^{-1}

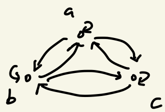


$R \wedge R^{-1}$

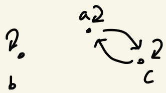


Refinement

$G = (A, R)$:



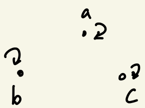
$G' = (A, R')$



$R' \subseteq R$

G' is a refinement of G

$G'' = (A, R'')$



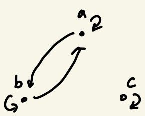
$R'' \subseteq R' \subseteq R$

G'' is a refinement of G'

G'' is a refinement of G

G'' is a refinement of G'''

$G''' = (A, R''')$



$R''' \subseteq R$

G'' is a refinement of G