Institute for Physics of Electrotechnology
Munich University of Technology

Electricity and Magnetism
Plenary Tutorial Notes

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Contents

1. Short Questions

Q1 What is the definition of the electric field $\vec{E}(\vec{r})$? What is the physical unit of the electric field?

Q2 How are electrostatic fields generated? What is the link between the Coulomb force and the electric field?

Q3 What is the "fundamental law of electrostatics"?

Q4 What is the definition of the gradient of a scalar field? What is the geometric interpretation of the gradient?

Q5 Derive the differential form of Gauss's law $\mathrm{div}\vec{D} = \rho$ from the integral form of Gauss's law $\int_{\partial V} \vec{D}\mathrm{d}\vec{a} = Q(V)$.

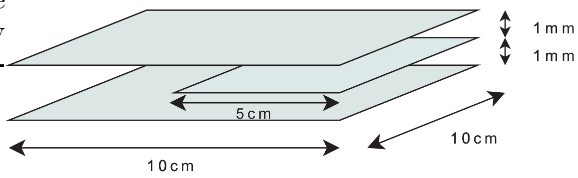
Q6 Imagine a sphere with radius R centered in the origin. A uniform surface charge density σ is placed on the surface of the sphere. Determine the electric field $\vec{E}(\vec{r})$ in the interior of the sphere $0 \leq r < R$ (Reason!).

Q7 A neutral conducting body is exposed to an external electrostatic field. Derive an expression of the induced surface charge density σ from the integral form of Gauss's law.

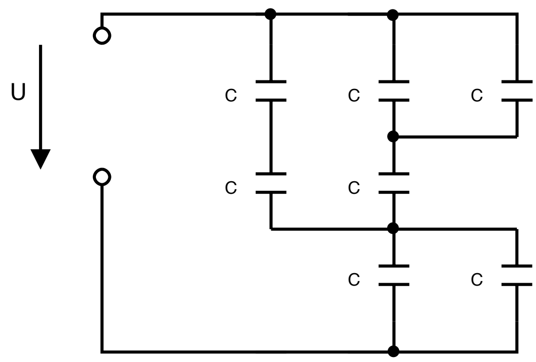
Q8 Compare the electric potential of a point charge with that of an infinite, straight line charge. How does it depend on the distance to the point charge or the distance to the line charge, respectively.

Q9 What is the definition of the capacitance of a two-electrode capacitor?

Q10 Calculate the capacitance between the upper and the lower metal plate of the given configuration. You may neglect boundary effects (stray fields). The permittivity between the plates is $\varepsilon = \varepsilon_0 = 8.85 \cdot 10^{-12}\,\mathrm{As/Vm}$.



Q11 Determine the total capacitance of the following capacitor circuit.



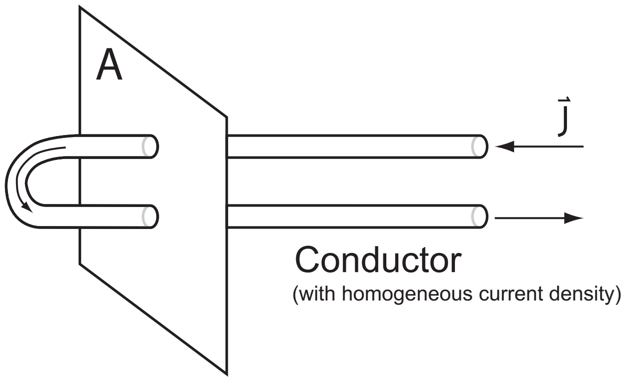
Q12 A perfect plate capacitor has been charged in free space with a charge Q at a voltage U. Then a dielectric material with relative permittivity $\varepsilon_\mathrm{r} = 2$ is filled between the capacitor plates. What is the change of the electric energy stored in the capacitor, if

a) the voltage U between the plates is kept constant

b) the charge Q on the plates is constant.

Q13 Two identical molecules, both with a mass of 50 carbon atoms, are at rest in a distance of 10 nm. By means of a short laser pulse one electron is removed from each of the molecules, so that they are repelling each other and move apart. What is the final speed of the two molecules in the limit of an infinite distance? You may assume that the molecules move in free space and the electron mass is negligible. (Hint: Kinetic energy: $E_K = \frac{1}{2}mv^2$, mass of a carbon atom: 12 u with $\mathrm{u} \approx 1.67 \cdot 10^{-27}$ kg, elementary charge $e_0 = 1.6 \cdot 10^{-19}$ As, permittivity of the vacuum $\varepsilon_0 = 8.85 \cdot 10^{-12}\,\mathrm{As/Vm}$)

Q14 A curved U-shaped conductor carries a uniform current density (see figure). There is no current outside the conductor.



a) What is the meaning of the term "uniform current density"?

b) What is the total current I flowing through the area A?

c) What will change (w.r.t. the total current I), if the current density in the conductor is not uniform, but still stationary?

Q15 Assume that several species of charge carriers contribute to the current flow in a conductor. What is the impact of the specific charge of the carries on the electric conductivity?

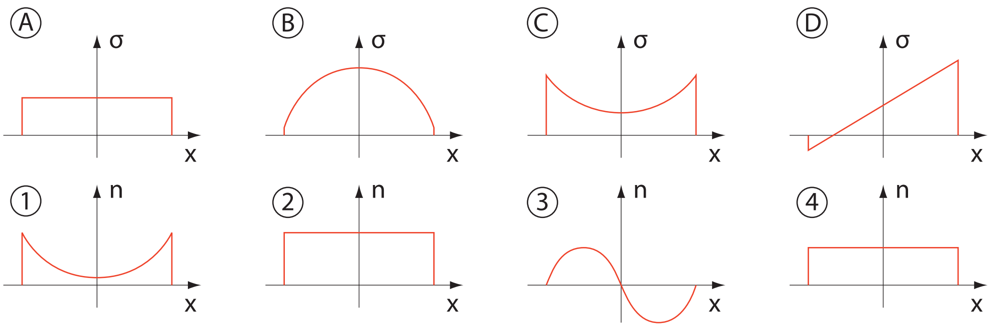
a) none

b) only electrons contribute to the electric conductivity

c) the algebraic sign of the specific charge is important

d) only the absolute value of the specific charge is relevant

Q16 Which combinations of (y, z), $y \in \{A, B, C, D\}$, $z \in \{1, 2, 3, 4\}$ of charge carrier density $n(x)$ and conductivity $\sigma(x)$ are physically reasonable? Assume that only one species of charge causes the electric current, and assume a constant carrier mobility. Give a reason for your answer.

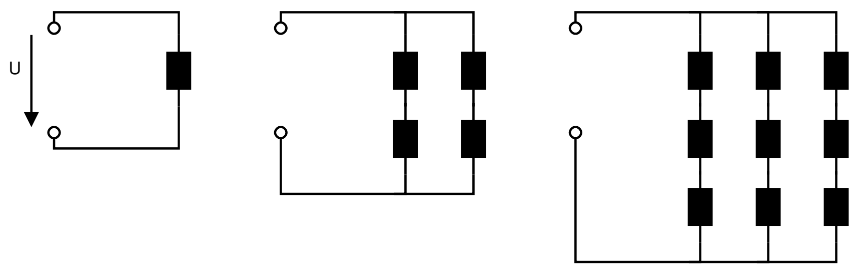


Q17 An aluminum wire with length 100 m and diameter 1 mm has an electric resistance of $3.5\,\Omega$. The mobility of the mobile electrons in aluminum is $\mu_n = 1.26 \cdot 10^{-3}\,\mathrm{m^2/Vs}$. What is the density of mobile electrons in aluminum? What is the number of mobile electrons per atom? (Density of aluminum is $2.7 \cdot 10^3\,\mathrm{kg/m^3}$; there are $N_A = 6 \cdot 10^{23}$ atoms in 27 g aluminum, elementary charge $e_0 = 1.6 \cdot 10^{-19}$ As)

Q18 Formulate the "law of charge conservation" as an integral balance equation. Write a mathematical equation which relates the electric current flux through a closed surface ∂V to the charge $Q(V)$ in the enclosed volume V.

Q19 Give a physical interpretation of Kirchhoff's current law (referring to charge conservation in a physical conductor node in the stationary case).

Q20 The resistance of a circuit with applied voltage U is stepwise extended as shown in the figure below.



Which statement is true? The power consumption of the circuit is stepwise

a) halved   b) doubled   c) increased by $\sqrt{2}$ d) unchanged

Q21 What is the differential work performed on a charge carrier in an electric field $\vec{E}$? Derive from this expression the electric power density rendered by an electric system with current density $\vec{j}$.

Q22 Charged particles move in free space under the concurrent action of an electric and a magnetic field. An observer recognizes that, under certain conditions, the motion of some particles is straight-lined with uniform velocity and independent of their masses and their amount of charges.
What are these conditions?
What could be an application of such an experimental set-up?

Q23 A current $I = 0.2\,\mathrm{A}$ flows through a coil with a winding of $n = 10000$ turns and a cross section of $A = 7.5\,\mathrm{cm}^2$. The permeability of the surrounding air is $\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{Vs/Am}$.

a) What is the magnitude of the magnetic moment $\vec{m}$ of the coil?

b) Which torque $\vec{M}$ is acting on the coil in a magnetic field $H = 1200\,\mathrm{A/m}$, when the angle between the magnetic field and the main axis of the coil is 30°?

Q24 Do magnetic monopoles exist? Justify your answer by referring to Maxwell's equations.

Q25 A voltage U_{ind} is induced by a time-variant magnetic field $\vec{B}(\vec{r}, t)$ in an almost closed time-variant conductor loop $C(t) = \partial A(t)$. Which of the following statements are true?

a) $U_{\text{ind}} = 0$, if the magnetic field does not vary with time.

b) $U_{\text{ind}} = 0$, if the magnetic field varies with time, but the conductor loop is at rest.

c) $U_{\text{ind}} = -\frac{d}{dt} \int_{A(t)} \vec{B}(\vec{r}, t) d\vec{a}$

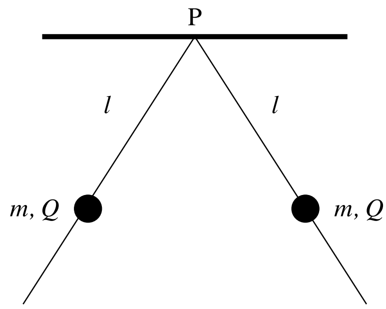
d) $U_{\text{ind}} = -\int_{A(t)} \frac{\partial \vec{B}}{\partial t}(\vec{r}, t) d\vec{a}$

e) $U_{\text{ind}} = -\int_{A(t)} \frac{\partial \vec{B}}{\partial t}(\vec{r}, t) d\vec{a} + \int_{\partial A(t)} \vec{v}(\vec{r}, t) \times \vec{B}(\vec{r}, t) d\vec{r}$

2. Problems

Problem 1 Charge Pendulum

Two small metallic balls, each with mass m, are hanging on isolating strings with length l. The strings are fixed on a common suspension point P. Each ball carries an electric charge Q. Hence, the gravitational force (gravitational acceleration $g = 9.81\,\mathrm{m/s^2}$), the electrostatic repulsion force, and the suspension force act on the balls and balance each other in mechanical equilibrium.
Calculate the angle between the strings and the distance between the balls in the case that $m = 0.5\,\mathrm{g}$, $l = 1\,\mathrm{m}$, $Q = 10^{-8}\,\mathrm{C}$.
Hint: Assume that the angle of displacement is small.



Problem 2 Twin Point Charges

We consider two different configurations of two point charges Q_1 and Q_2:

A) Two opposite charges with equal magnitude (Fig. 1):

$$Q_1 = +Q \quad \text{and} \quad Q_2 = -Q$$

(forming an electric dipole)

B) Two equal charges (Fig. 2):

$$Q_1 = Q_2 = Q$$

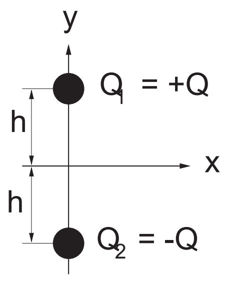


Fig. 1

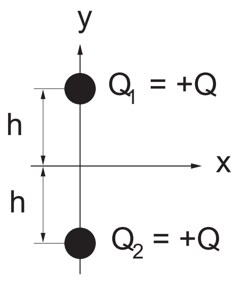


Fig. 2

The charges are located on the y-axis of a Cartesian coordinate system at $y = h$ and $y = -h$, respectively. The following subtasks shall be solved for both configurations A and B:

a) Calculate the electric field (magnitude and direction) generated by the two charges at an arbitrary point on the x-axis?

b) What are the Cartesian components E_x and E_y of the electric field at the point $H(x = h, y = h, z = 0)$?

c) Calculate the voltage U between the origin (0,0,0) and an arbitrary point $P_1(x, 0, 0)$ on the x-axis.

d) What work is necessary to move a test charge q from infinity to the origin?

Problem 3 Electrostatic Field

The region $\Omega = \{(x, y, z) \in \mathbb{R}^3 \,|\, x \geq 0 \text{ and } y \geq 0\}$ in a Cartesian coordinate system is confined by two thin conductive plates, which cover the two half-planes

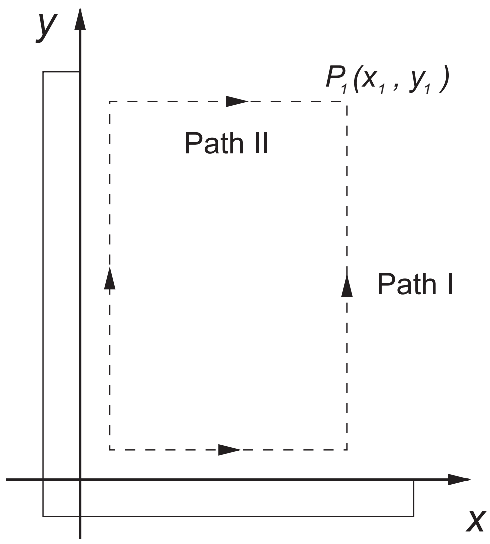
$$H_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, y = 0 \text{ and } x \geq 0\}$$

and

$$H_2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x = 0 \text{ and } y \geq 0\}$$

(see figure).
On the region Ω the following electric field is defined:

$$\vec{E}(\vec{r}) = -Ay\vec{e}_x - Ax\vec{e}_y \quad \text{with } A = \text{constant} \neq 0$$



a) Calculate the voltage U_{01} between the origin and an arbitrary position $P_1(x_1, y_1, 0)$ as a function of A, x_1 and y_1. To this end, calculate the path integral

$$\int_0^{P_1} \vec{E}\, d\vec{r}$$

along the path I and then, alternatively, also along path II (see figure).

b) What is the value of U_{01}, if P_1 is located on one of the plates H_1 or H_2?

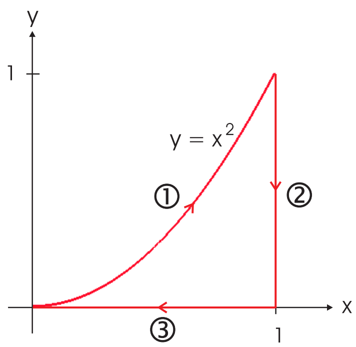
c) Is the field $\vec{E}(\vec{r})$ conservative?

d) Sketch the intersection of the equipotential surfaces with the x-y-plane (= equipotential lines) and the field lines of the electric field.

Problem 4 Force Field

The following two-dimensional force field is given in Cartesian coordinates:

$$\vec{F}(x, y) \quad = \quad c \cdot (2xy\vec{e}_x + x^2\vec{e}_y); \qquad c = \text{const.} \neq 0$$



a) Is the force field conservative?

b) Calculate the work which is performed during a motion along the sketched path.

c) Find, if possible, a potential function of the force field $\vec{F}(x, y)$.

Problem 5 Gradient

Calculate the gradient of the following functions in Cartesian coordinates:

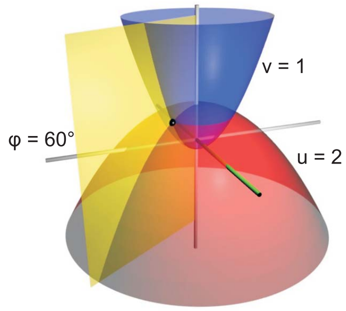
a) $f(x, y) = x$ b) $g(x, y, z) = \cos(3y) + xyz$ c) $h(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

Problem 6 Parabolic Coordinates

Parabolic coordinates are defined by

$$\vec{r}(u, v, \varphi) = uv\cos\varphi \cdot \vec{e}_x + uv\sin\varphi \cdot \vec{e}_y + \frac{1}{2}(u^2 - v^2) \cdot \vec{e}_z$$

$$(u \geq 0;\ v \geq 0;\ 0 \leq \varphi < 2\pi)$$



a) Calculate the moving basis $(\vec{b}_u, \vec{b}_v, \vec{b}_\varphi)$.

b) Are the coordinates u, v, φ orthogonal? Calculate the scale factors h_u, h_v, h_φ of the parabolic coordinates.

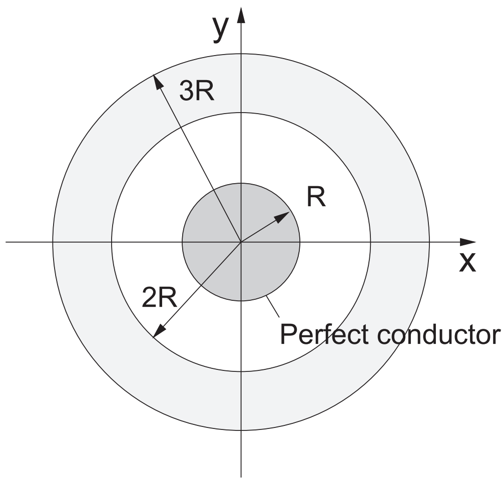
c) Express the gradient of a scalar function in parabolic coordinates.

Problem 7 Charge Distribution

Consider a perfectly conducting sphere with radius R centered at the origin. A charge $Q > 0$ is stored on the sphere (see figure). All space around the sphere is filled with a homogeneous dielectric with $\varepsilon = \varepsilon_0\varepsilon_\mathrm{r}$. In addition, there exists a space charge density:

$$\rho(r) = \begin{cases} \dfrac{\alpha}{4\pi r R} & \text{for} \quad 2R < r < 3R \\ 0 & \text{otherwise} \end{cases}$$

Choose spherical coordinates to solve the following subtasks.



*a) How is the charge Q distributed over the perfectly conducting sphere? Give a reason for your answer. Calculate the surface charge density on the sphere.

*b) What direction has the electric field $\vec{E}(r, \vartheta, \varphi)$? Calculate the electric field $\vec{E}(r, \vartheta, \varphi)$ in each of the four regions $0 \leq r < R$, $R < r < 2R$, $2R < r < 3R$, and $3R < r$.

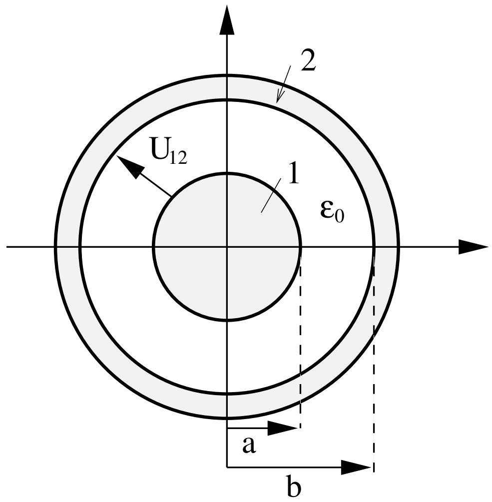
*c) Determine the factor α so that the electric field vanishes in the outer region $r > 3R$.

Use the value of α determined in c) for the following subtasks.

d) Plot the radial component of the electric field $\vec{E}$ as a function of r over all four regions. Don't forget to label the coordinate axes sufficiently.

e) A test charge q is moved from infinity onto the surface of the conducting sphere. Calculate the mechanical work.

Problem 8 Coaxial Tube



Consider two coaxial metallic cylinders, which extend to infinity in both directions perpendicular to the plane of projection (see figure). The inner conductor 1 has the radius a and carries the charge $q_1 = +q$ per unit length. The outer conductor 2 has the (inner) radius b and carries the opposite charge $q_2 = -q$. The space in between $a < r < b$ is filled with air, i.e., $\varepsilon = \varepsilon_0$.

a) Determine the radial components $E_r(r)$ and $D_r(r)$ of the electric field and the dielectric displacement, respectively, as a function of r for $a \leq r \leq b$.

b) Find the surface charge densities σ_1 and σ_2 on the inner and the outer conductor.

c) Determine the voltage U_{12} between the inner and the outer conductor as a function of q, a and b.

d) Calculate the quantity $c := q/U_{12}$ (= capacitance of the tube per unit length). How does this quantity change when, instead of air, a dielectric with relative permittivity $\varepsilon_\mathrm{r} > 1$ is filled into the space between the conductors?

Now, the following numerical parameters are given:

$U_{12} = 1000\,\mathrm{V}$, $a = 2\,\mathrm{mm}$, $b = 5.44\,\mathrm{mm}$, $\varepsilon_\mathrm{r} = 1$ and $\varepsilon_\mathrm{r} = 2.5$, respectively, $\varepsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{As/Vm}$

e) Find the numerical value of $c := q/U_{12}$.

f) Find the radial component of the electric field $E_r(a)$ on the surface of the inner conductor.

Problem 9 Hollow Sphere

Consider the following spherically symmetric charge distribution:

$$\rho(\vec{r}) = \rho(r) = \begin{cases} 0 & \text{for} \quad 0 \leq r \leq R \\ \rho_0 & \text{for} \quad R < r \leq R' \\ 0 & \text{otherwise} \end{cases} \quad (\rho_0 = \text{constant})$$

Use spherical coordinates for your calculations.

a) Calculate the total charge Q enclosed in the region $0 \leq r \leq R'$.

b) Show that the electrostatic potential can be decomposed in two contributions according to

$$\phi = \phi_1 - \phi_2 \,, \quad \text{where } \Delta\phi_\alpha = -\frac{\rho_\alpha}{\varepsilon_0} \,, \ (\alpha = 1, 2) \,, \quad \text{and } \begin{cases} \rho_1 = \rho_0 \,, & 0 \leq r \leq R' \\ \rho_2 = \rho_0 \,, & 0 \leq r \leq R \end{cases}$$

c) Calculate the electrostatic potential $\phi(r)$ using subtask b).

d) Plot the electrostatic potential $\phi(r)$.

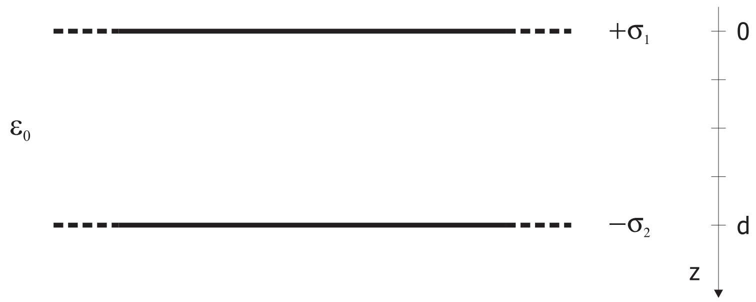
e) Calculate $\phi(r)$ in the limit $R' - R \to 0$ with $Q = \text{const.}$, and determine the discontinuity of the electric field. Does this result conform with Gauss's law?

f) Determine $\Phi(r)$ in the limit $R \to 0$. Then, calculate the force on a test charge q at a position

1. $r < R'$
2. $r > R'$

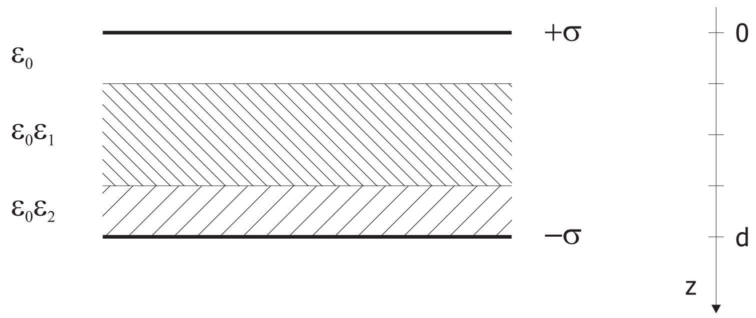
Problem 10 Plate Capacitor

Consider the following set-up of perfectly conducting, infinitely large plates. The plate at $z = 0$ carries the surface charge density $\sigma_1 > 0$, whereas the plate at $z = d$ carries the surface charge $-\sigma_2 < 0$.



a) Calculate the magnitude and the direction of the electric field in all regions outside and inside the capacitor under the assumpation that $\sigma_1 > \sigma_2$ holds.

In the following, we assume that $\sigma := \sigma_1 = \sigma_2$ holds. Both plates now have the same area A. The distance d between the plates is so small that stray fields are negligible. The space between the plates is filled with two different dielectrics ($\varepsilon_1 > 1$ and $\varepsilon_2 > 1$), as can be seen in the figure below.



We have

$$\varepsilon = \varepsilon_0 \varepsilon_{\mathrm{r}} = \begin{cases} \varepsilon_0 & \text{for} \quad 0 < z \leq \dfrac{d}{4} \\ \varepsilon_0 \varepsilon_1 & \text{for} \quad \dfrac{d}{4} < z \leq \dfrac{3d}{4} \\ \varepsilon_0 \varepsilon_2 & \text{for} \quad \dfrac{3d}{4} < z < d \end{cases}$$

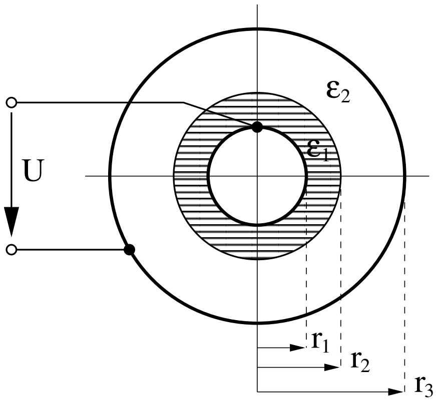
b) Calculate the electric displacement field $\vec{D}$ in all regions (magnitude and direction).

c) Calculate the electric field $\vec{E}$ in all regions (magnitude and direction).

Problem 11 Cylindrical Capacitor

The sketched cylindrical capacitor contains two different dielectric jackets in its interior.

The following data is given:



- Radius of the conducting core cylinder: $r_1 = 1\,\mathrm{cm}$
- Outer radius of the inner dielectric jacket: $r_2 = 2.72\,\mathrm{cm}$
- Inner radius of the outer conducting cylinder: $r_3 = 7.4\,\mathrm{cm}$
- Permittivities:
  - Inner medium: $\varepsilon_1 = 5\,\varepsilon_0$
  - Outer medium: $\varepsilon_2 = \varepsilon_0$

A constant voltage $U = 10\,\mathrm{kV}$ is applied to the electrodes, and the charge $+/-q$ per unit length resides on the inner and outer electrode, respectively.

Because of the cylindrical symmetry, the electric field and the displacement field have the form $\vec{D}(\vec{r}) = D(r)\,\vec{e}_r$ and $\vec{E}(\vec{r}) = E(r)\,\vec{e}_r$, respectively, where r is the distance from the cylinder axis, and $\vec{e}_r$ is the unit vector in radial direction.

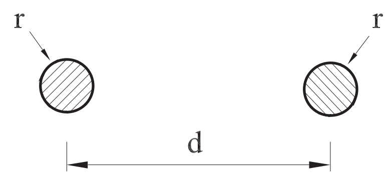
a) Calculate the electric displacement field $D(r)$ and the electric field $E(r)$ as a function of the radius r. Calculate the numerical value of the electric displacement field and the electric field for the radii $r = r_1$, $r = r_2$ and $r = r_3$ as one-sided limit in the respective dielectric.

b) Plot the graph of $D(r)$ and $E(r)$ as a function of the distance r. Where has the field its maximum?

c) Calculate the capacitance per unit length.

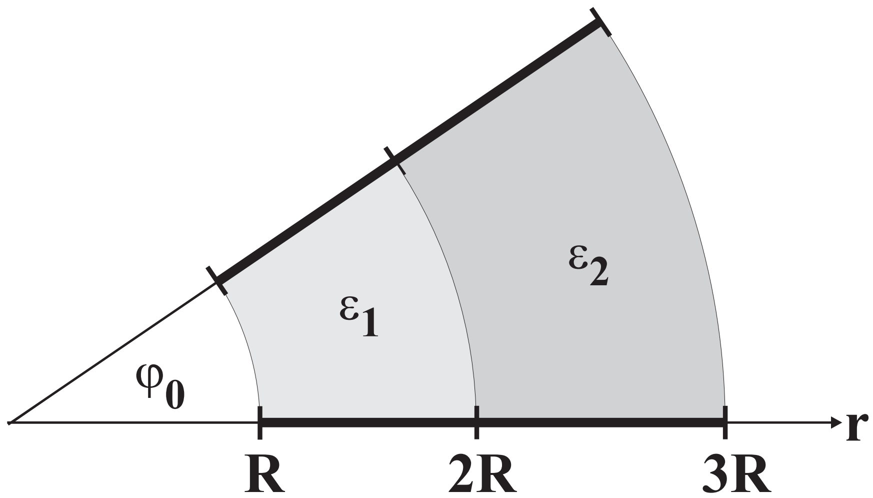
Problem 12 Two-Wire Line

Calculate the capacitance per unit length of an infinitely long two-wire line consisting of two thin wires with radius r and parallel distance d. Assume that $r << d$.



Problem 13 (Exam Problem 2009I-1)

A plate capacitor has the shape of a cylindrical segment with the angle φ_0 (see figure). Two perfectly conducting electrodes A and B, which extend in r-direction from R to $3R$, confine the segment. In z-direction the capacitor has the length L, which is very large compared to R. Hence, the problem can be reduced to the two dimensions (r, φ) in cylindrical coordinates. Stray fields at the boundary of the capacitor are negligible.
There are two dielectrics placed between the electrodes. The first dielectric ($R < r < 2R$) has the constant permittivity ε_1, the second dielectric ($2R < r < 3R$) has the constant the permittivity ε_2.



The electric potential $\Phi(\varphi)$ in the interior of the capacitor ($R < r < 3R$) reads in cylindrical coordinates:

$$\Phi(\varphi) = -K \cdot \varphi\,, \quad 0 \leq \varphi \leq \varphi_0\,,$$

where K is a positive constant.

*a) Sketch the equipotential lines and the electric field lines between the two electrodes (draw two separate plots). On which electrode is the positive charge located, and on which one the negative charge?

*b) Calculate the electric field $\vec{E}$ in the capacitor.

*c) Calculate the voltage U between the two electrodes.

d) Calculate (e.g., via Gauss's law) the charge Q on the lower electrode ($\varphi = 0$). To this end, determine the electric displacement field $\vec{D}(r)$ in the interior of the capacitor. Differentiate the regions of different permittivity.

e) Calculate the capacitance C of the configuration.

f) Calculate the electric energy W_{el} stored in the capacitor.

Problem 14 Gas Discharge

In an electric gas discharge, an electron and an ion density of $n_e = n_i = 10^{15}\,{}^{1}/_{\mathrm{cm}^3}$ are measured. The ions have a positive charge (single-charged particles). In the interior of the discharge region, there is an electric field $E = 1\,{}^{\mathrm{V}}/_{\mathrm{cm}}$. The mobilities of electrons and ions are:

$$\mu_e = \frac{v_e}{E} = 10^4\,{}^{\mathrm{cm}^2}/_{\mathrm{Vs}} \qquad\qquad \mu_i = \frac{v_i}{E} = 1\,{}^{\mathrm{cm}^2}/_{\mathrm{Vs}}$$

a) Find the velocity (magnitude and direction) of electrons and ions with reference to the electric field $\vec{E}$.

b) Find the current density $\vec{j}$. In which direction is $\vec{j}$ oriented?

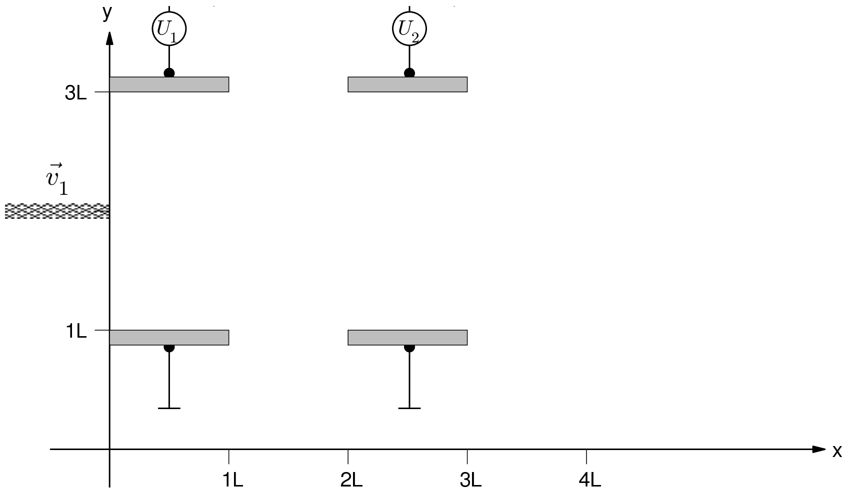
c) Calculate the electric conductivity σ.

d) Calculate the electric power density p_{el}.

Problem 15 (Exam problem 2009I-2)

In free-space, a plasma beam enters at the position $x = 0$ and $y = 2L$ a chamber with two pairs of electrodes at the positions $y = L$ and $y = 3L$. The electrically neutral beam contains electrons of mass m_e and single-charged, positive ions with mass m_i. The particles have an initial velocity $\vec{v_1} = v_1 \cdot \vec{e_x}$ at the entry. N ions are entering the chamber per second.

The electrodes have the length L in x-direction. The voltage U_1 is applied between the first pair of electrodes in such a way that the upper plate ($y = 3L$) is positively charged.



*a) At the position $x = L$, the direction of the moving ions shall include an angle of $|\alpha| = 45° = \frac{\pi}{4}$ with the x-axis. Find the magnitude of the velocity v_2 of the ions at this position.

b) What voltage U_1 must be applied to make the ions move as described in subtask a)? What happens to the electrons? (from classical physics: $\vec{F} = m \cdot \vec{a}$; $\vec{v} = \vec{a} \cdot t$ with force $\vec{F}$, mass m, acceleration $\vec{a}$, velocity $\vec{v}$ and time t)

c) Draw the path of the ions qualitatively in the sketch above.

*d) Calculate the current I carried by the ion beam. Draw the magnetic field generated by this current qualitatively in the sketch.

In the following, the magnetic field generated by the ion beam shall be neglected.

A uniform magnetic field $\vec{B}$, generated somewhere outside, shall be used to deflect the ions in the section $L < x < 2L$ in such a way that their path is parallel to the x-axis at $x = 2L$.

*e) Find the $\vec{B}$-field as a function of v_2 and other known parameters. (Hint: Centrifugal force $F = m_i \frac{v^2}{r}$, where r is the radius of gyration).

*f) What is the magnitude of the velocity v_3 with which the ions leave the magnetic field at $x = 2L$?

g) Indicate the orientation of the magnetic field and draw the path of the ions in the sketch.

In the section $2L < x < 3L$ the ions are again electrically deflected by the second pair of electrodes, but this time in the opposite direction.

h) What voltage U_2 must be applied that the ion beam includes an angle $|\alpha| = 45°$ with the x-axis at $x = 3L$?

i) Complete the sketch.

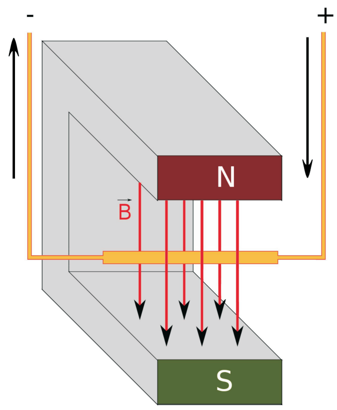
*j) The ions in the beam have all the same positive charge. Beside the magnetic field caused by the beam, another electrophysical effect has been neglected. Which other effect has been neglected?

Problem 16 Magnetic Lens

Assume a constant magnetic field with a field strength $|\vec{B}| = 0.01\,\mathrm{T}$. From a point in the magnetic field a bunch of electron beams emerge in different directions. The velocity vectors of the electrons have the same magnitude $|\vec{v}| = 3 \cdot 10^7\,{}^{\mathrm{m}}\!/\!{}_{\mathrm{s}}$ and include a fixed angle of $10°$ with the direction of the magnetic field $\vec{B}$. On what kind of trajectories do the electrons move? Is there a point where the trajectories of all electrons will meet again?

Problem 17 Wire Swing

An aluminum wire (mass density $\rho_{\mathrm{m}} = 2.7 \cdot 10^3\,{}^{\mathrm{kg}}\!/\!{}_{\mathrm{m}^3}$) is suspended between the poles of a horseshoe magnet (see figure). The wire is kept in horizontal direction by two thin flexible current supplies in such a way that it can freely swing in the magnetic field. The electric current flowing through the wire is uniformly distributed with a constant current density $j = 10^5\,{}^{\mathrm{A}}\!/\!{}_{\mathrm{m}^2}$. The magnetic field between the poles can be assumed uniform with a field strength $B = 0.08\,\mathrm{T}$. By the action of the Lorentz force, the wire is deflected by an angle α with respect to the vertical direction. Calculate this angle.



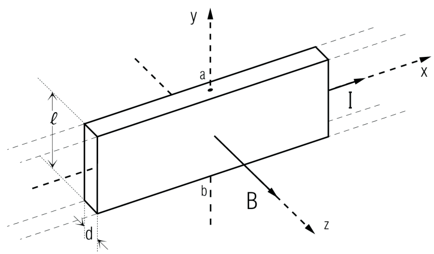
Problem 18 Hall Effect

A conductive slab with width l and thickness d carries a uniformly distributed current I (see figure). The slab is exposed to a uniform magnetic field $\vec{B}$ perpendicular to the direction of current flow. A voltage V_{ab} is measured between the two contact points a and b.

a) What is the cause of the voltage V_{ab}?

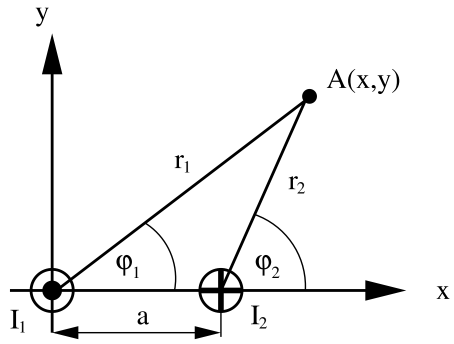
b) Calculate the drift velocity v_{drift} of the mobile charge carriers in the slab.

c) Calculate the charge density ρ of the mobile charge carriers.

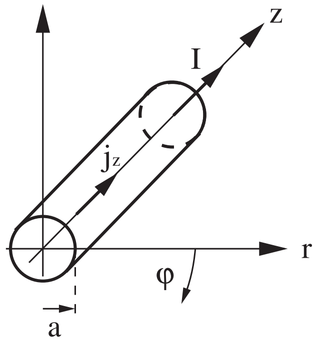


Problem 19 Parallel Straight Wires

Two infinite long, straight wires are aligned parallel to each other in a distance a. One of them coincides with the z-axis of a Cartesian coordinate system and carries the constant current $I_1 = +I$ in z-direction. The second wire is located at $x = a, y = 0$ and carries the reverse current $I_2 = -I$. Calculate the Cartesian components $H_x(x, y)$ and $H_y(x, y)$ of the total magnetic field generated by the wires at an arbitrary position $A(x, y, 0)$ in the x-y-plane.



Problem 20 Gas Discharge Tube



The interior of a cylindrically symmetric gas discharge tube (radius $a = 1\,\text{cm}$) is filled with a conductive plasma of charge carries, which flow parallel to the cylinder axis (= z-axis). The electric current density has only a z-component which depends on the radial coordinate r as follows:

$$j_z(r) = \begin{cases} j_0 \left(1 - \dfrac{r^2}{a^2}\right) & \text{for} \quad r \leq a \\ 0 & \text{for} \quad r > a \end{cases} \qquad \text{with} \quad j_0 = 20\,\frac{\text{A}}{\text{cm}^2}$$

a) What is the total current I flowing through the discharge tube?

b) Calculate the magnetic field strength $H_\varphi(r)$ inside the discharge tube, i.e., for $0 < r \leq a$, and outside, i.e., for $r > a$.

c) At which distance r_m from the cylindrical axis does $H_\varphi(r)$ attain its maximum value?

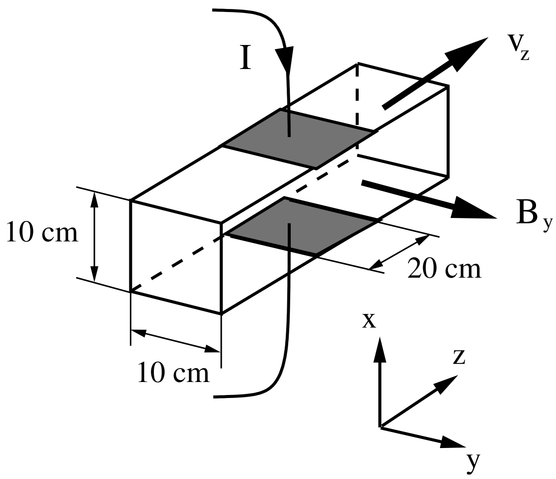
d) Find the value of $H_\varphi(a)$ and $H_\varphi(r_m)$.

e) Plot $H_\varphi(r)$ as a function of r.

Problem 21 MHD Generator

The configuration of a MHD generator (magnetohydrodynamic generator) is as follows:

An electrically conductive fluid with conductivity $\sigma = 1\,\text{S/cm}$ flows with a uniform velocity $v_z = 500\,\text{m/s}$ through a channel with a quadratic cross section of $10 \cdot 10\,\text{cm}^2$. In a channel region of length $L = 20\,\text{cm}$ a uniform magnetic field $B_y = 2\,\text{T}$ ($1\,\text{T} = 1\,\text{Vs/m}^2$) acts on the fluid perpendicular to the flow direction. On the top and on the bottom side of this region a pair of electrodes is mounted, which delivers the electric current I driven by the electromotive force in the generator. Each electrode has an area of $10 \cdot 20\,\text{cm}^2$. The additional magnetic field generated by the current I may be neglected.



a) Find the electric field $\vec{E}_{\text{ind}}$ induced in the fluid by the magnetic field.

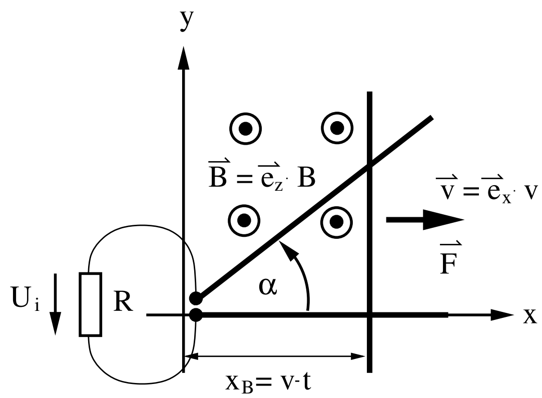
b) Find the open circuit voltage of the generator.

c) Calculate the internal resistance of the generator under the assumption that the current I is uniformly distributed over the area $10 \cdot 20\,\text{cm}^2$?

d) What is the maximal current I_k which can be taken from the MHD generator (short circuit current)?

Problem 22 Slide-Wire

Two straight wires are placed in the x-y-plane as shown in the figure. One of them is aligned along the x-axis, the other one is inclined by an angle α, so that both wires enclose an angular segment in the x-y-plane. At the origin they nearly meet and are connected to an external resistor with ohmic resistance R. A third straight wire slides on them; it is aligned parallel to the y-axis and is moved with constant velocity v in the x-direction. Its momentary position is $x_B(t) = v \cdot t$. The x-y-plane is penetrated by a uniform, constant magnetic field $\vec{B} = B \cdot \vec{e}_z$. All wires are perfect conductors with no resistance.



The magnetic field generated by the current through the wire loop is negligible. Determine as a function of time:

a) the voltage U_i on the resistor,

b) the current I flowing through wire loop and resistor,

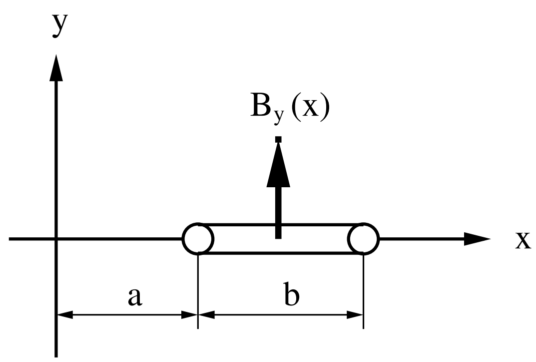
c) the magnetic force $\vec{F}_M$ acting on the moving slide-wire.

Problem 23 Induction

A non-uniform static magnetic field has only a y-component given as $B_y(x) = B_0 e^{-kx}$, where B_0 and k are positive constants. A rectangular wire loop is placed in the plane $y = 0$.

The geometric data are:
b = side length in x-direction,
l = side length in z-direction,
a = distance from origin.

The loop is open and, therefore, carries no current.



a) Calculate the magnetic flux Φ through the wire loop.

Now assume that both the magnetic field and the position of the loop are time-dependent as follows:

$$B_0(t) = C \cdot \sin(\omega t)$$
$$a(t) = a_0 + a_1 \cdot \sin(\omega t)$$

where C, a_0 and $a_1 < a_0$ are constants.

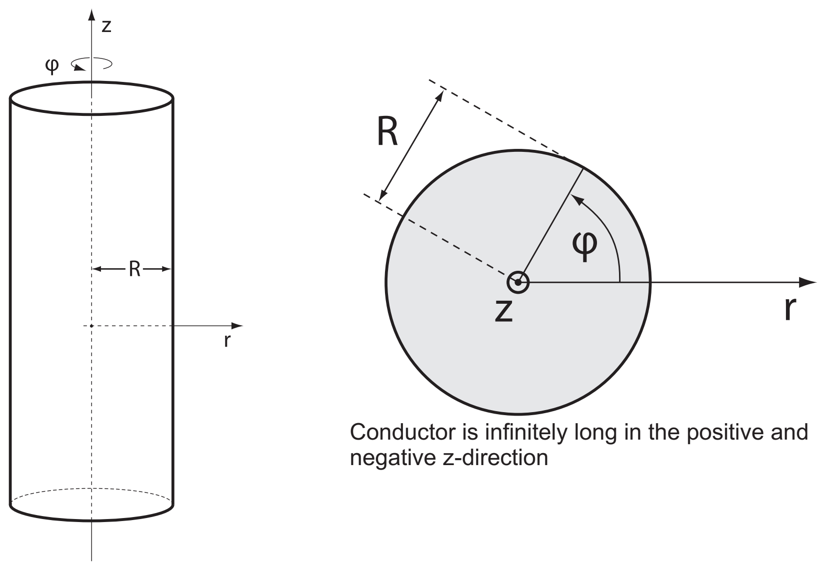
b) Determine the time-dependent voltage $U_{\text{ind}}(t)$ which is induced in the loop.

Problem 24 (Exam Problem 2009I-3)

A cylindrical conductor with radius R has an infinite long extension along the $\pm z$-direction (see figure). The conductor carries a time-variant current in $+\vec{e}_z$-direction. The current density $\vec{j}(r,t)$ in the conductor is given in cylindrical coordinates:

$$\vec{j}(r,t) = \begin{cases} j_0 \cdot \left(\frac{r}{R}\right)^2 \cdot \sin(2\pi t/T) \cdot \vec{e}_z & \text{for } 0 \leq r < R \\ 0 & \text{for } r \geq R \end{cases}$$

Assume for all calculations that the conductor is placed in free space (i.e., $\mu = \mu_0 =$ vacuum permeability).

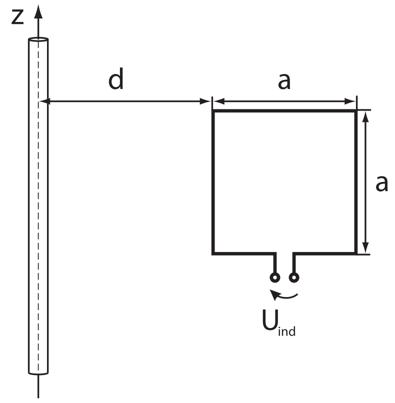


*a) Sketch the magnetic field lines in a plane perpendicular to the z-direction. Also sketch the direction of the current. Differentiate between the regions inside and outside of the conductor.

*b) Calculate the total current I flowing through the conductor by integration over its cross section.

c) Determine the magnitude and the direction of the magnetic field strength $\vec{H}(r,\varphi,t)$ by using Ampère's circuital law.

d) Plot the magnitude of the magnetic field strength $\vec{H}(r,\varphi,t)$ as a function of r at the time $t_1 = T/4$ and $t_2 = T/8$.



In a distance $d \gg R$, there is a quadratic conductor loop with edge length a (see figure). Assume for the following calculations that the magnetic field is given as

$$\vec{H}(r, \varphi, t) = \frac{H_0 R}{r} \cdot \sin(2\pi t/T) \cdot \vec{e}_\varphi(\varphi) \qquad \text{for } r \geq R$$

*e) Calculate the voltage $U_{\text{ind}}(t)$ induced in the motionless conductor loop in the period $0 < t < T$ and draw the result in an appropriate sketch.

*f) Does the induced voltage $U_{\text{ind}}(t)$ change when the conductor loop is rotating around the z-axis? (Note that the normal vector of the area surrounded by the conductor loop is always directed in φ-direction.) How do you include this motion in your calculation?

*g) Does the induced voltage change, when the conductor loop is parallel-shifted in radial direction with constant velocity $\vec{V}$? How do you account for this effect in the calculation of the magnetic flux?