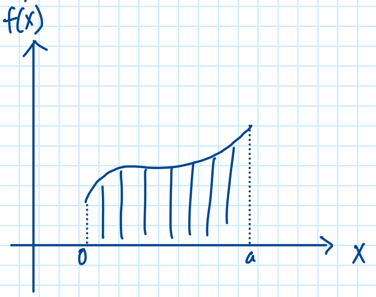
Math~~3~~4: integration in 3D space: integral over scalar fields and vector fields (= flux integral)

Integretion in one-dimensional space :



$$\int_0^a f(x)\,dx$$

slope of curve, $\Delta x \to$ small

$$df = \frac{\partial f}{\partial x} \cdot dx$$

tangent to curve, tell how much f varies w.r.t x

1. Volume integral over scalar field (analogous to integration in one dimesion)

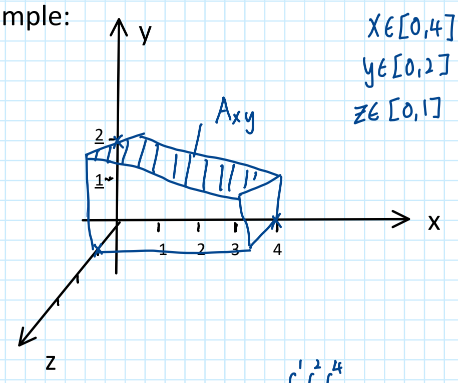
2D surface integral : $A = \int_0^2\int_0^4 1\,dx\,dy$

$x \in [0,4]$
$y \in [0,2]$
$z \in [0,1]$

$= \int_0^2 [x]_0^4\,dy = \int_0^2 4 \cdot dy$

$= [4y]_0^2 = 8$

example:



Volume integral : $\int_0^1\int_0^2\int_0^4 f(x,y,z)\,\underbrace{dx\,dy\,dz}_{dV\,=\,\text{differential volume element}} = \int_0^1\int_0^2 4\,dy\,dz = \int_0^1 8\,dz = [8z]_0^1 = 8$

If eg $f(x,y,z) = \rho_m = \rho \frac{kg}{m^3}$ mass density $\quad \int_V \rho_m\,dV = \text{mass}$

differential volume element in Cartesian coordinates : $dV = dx\,dy\,dz$

! for other coordinate systems : take into account prefactors coming from coordinate transformation

2) Surface integrals:

- We differentiate between surface integrals in scalar fields and surface integral in vector fields

(i) For scalar fields: basically like volume integral, but in 2 dimensions = surface integral over a function f(x,y)

replace differential volume element dV → differential surface element da Scalar

$$\int_A f(\vec{r}) \cdot da \quad \text{with } f(\vec{r}) = \text{scalar field}$$

in Cartesian Coordinates: $\iint_A f(x,y)\,dx\,dy \qquad dA = dx\,dy$

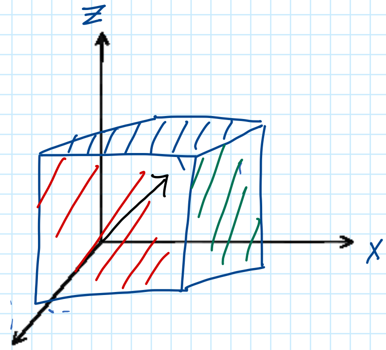
To calculate surface area (scalar, content of area; cf. volume integral, but in 2D)

For cartesian coordinates:

Top surface (parallel to xy-plane): $da = dx\,dy$

Side surface (parallel to xz-plane): $da = dx\,dz$

Side surface (parallel to yz-plane): $da = dy \cdot dz$

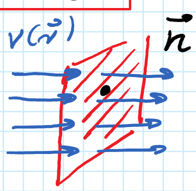


For other coordinates systems:

Take into account prefactors from Coordinate transformation.
↳ Recap on CS

Often will use integral over (Closed) surfaces in a Vector field ⇒ Flux integral
e.g., Chapter 2 → Stationary Current ; Chapter 1 → Closed surfaces Gauss's Law

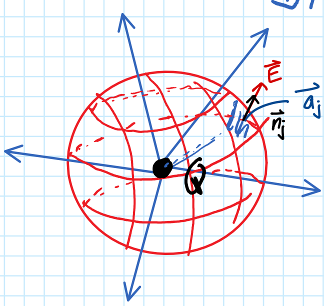
(ii) For vector fields: flux integral



project the field vector onto the normal vector of the corresponding area

$\vec{v} \cdot A \cdot \vec{n}$

$\vec{D} / \vec{E}$ electric displacement flux



$\vec{a}_j = a_j \cdot \vec{n}_j$

$\vec{E}_j \cdot \vec{a}_j = \vec{E}_j \cdot a_j \cdot \vec{n}_j$

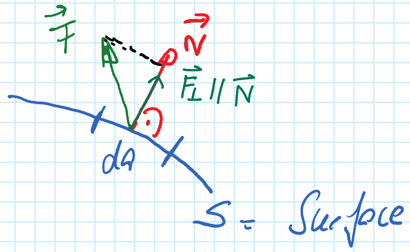
ΔA_j → very small dA

$a_j \to da$

$\underbrace{\vec{n}_j \cdot da \Rightarrow d\vec{a}}$

in contrast to scalar field → vectorial surface element $d\vec{a}$

surface integral of a vector field = flux integral



vector field $\vec{F}(\vec{r})$

$$\int_S \underbrace{\vec{F}(\vec{r}) \cdot \vec{N}}_{} \cdot da = \text{total flux of } \vec{F} \text{ through } S$$

projection of the $\vec{F}(\vec{r})$ onto the normal vector at each position is surface S

$\vec{N} \cdot da = d\vec{a}$