1. For each function below, determine the gradient vector and the maximum rate of change at point P given. Then, determine the directional derivative in the direction of vector **u** at point P.

   a) $f(x, y) = 2x - 3y, \quad P = (x, y), \quad \mathbf{u} = [-1, 2]^T$

   b) $f(x, y) = \sqrt{x^2 + y^2}, \quad P = (1, 0), \quad \mathbf{u} = [3, -1]^T$

   c) $f(x, y) = 5 - 3x \sin y, \quad P = (4, \pi/2), \quad \mathbf{u} = [4, 3]^T$

   d) $f(x, y, z) = x^2 + y^2 + z^2, \quad P = (1, 1, 1), \quad \mathbf{u} = [5, 2, 0]^T$

ANS: **a)** $\nabla f = [2, -3]^T$, $|\nabla f| = \sqrt{13}$, $D_{\mathbf{u}} f = -8/\sqrt{5}$.
b) $\nabla f(1, 0) = [1, 0]^T$, $|\nabla f| = 1$, $D_{\mathbf{u}} f(1, 0) = 3/\sqrt{10}$.
c) $\nabla f(1, \pi/2) = [-3, 0]^T$, $|\nabla f| = 3$, $D_{\mathbf{u}} f(1, \pi/2) = -12/5$.
d) $\nabla f(1, 1, 1) = [2, 2, 2]^T$, $|\nabla f| = \sqrt{12}$, $D_{\mathbf{u}} f(1, 1, 1) = 14/\sqrt{29}$.

a)

$$\vec{\nabla} f(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \qquad \vec{u} = \frac{1}{\sqrt{(-1)^2+2^2}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$|\nabla f| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$$

$$D_u f = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}(-2 + -6) = \frac{-8}{\sqrt{5}}$$

b)

$$\vec{u} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\nabla f(x,y) = \begin{pmatrix} \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x \\ \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y \end{pmatrix}$$

$$\nabla f(1,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\nabla f| = \sqrt{1} = 1$$

$$D_u \nabla f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{3}{\sqrt{10}}$$

c) $f(x,y) = 5 - 3x\sin y,\ \ P = (4, \pi/2),\ \ \mathbf{u} = [4,3]^T$

$$\nabla f(x,y) = \begin{pmatrix} -3\sin y \\ -3x\cos y \end{pmatrix} \qquad \vec{u} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\nabla f(4, \pi/2) = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \qquad D_u \nabla f = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$= \frac{-12}{5}$$

$$|\nabla f| = \sqrt{9} = 3$$

d) $f(x,y,z) = x^2 + y^2 + z^2,\ \ P = (1,1,1),\ \ \mathbf{u} = [5,2,0]^T$

$$\nabla f(1,1,1) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \qquad \vec{u} = \frac{1}{\sqrt{29}}\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$$

$$|\nabla f| = \sqrt{12} \qquad D_u \nabla f = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \frac{1}{\sqrt{29}}\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$$

$$= \frac{14}{\sqrt{29}}$$

2. Determine the directions at which each function increases and decreases most rapidly at the point P. Then, find the derivative of the functions in these directions.

a) $f(x, y) = x^2 + xy + y^2, \quad P = (-1,\ 1)$

b) $f(x, y, z) = \ln(xy) + \ln(yz) + \ln(xz), \quad P = (1, 1, 1)$

ANS: **a)** Increase most rapidly in $[-1, 1]^T/\sqrt{2}$ with $D_{\mathbf{u}}f(1, 1) = \sqrt{2}$.
Decrease most rapidly in $[1, -1]^T/\sqrt{2}$ with $D_{\mathbf{u}}f(1, 1) = -\sqrt{2}$.

b) Increase most rapidly in $[1, 1, 1]^T/\sqrt{3}$ with $D_{\mathbf{u}}f(1, 1) = \sqrt{12}$.
Decrease most rapidly in $[-1, -1, -1]^T/\sqrt{3}$ with $D_{\mathbf{u}}f(1, 1) = -\sqrt{12}$.

a) $\nabla f = \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix}$

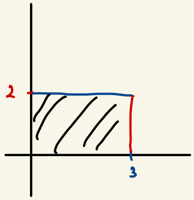
$\nabla f(-1,1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

max increase: $D_{\vec{\nabla f}} f(-1,1) = |\vec{\nabla f}(-1,1)| = \sqrt{(-1)^2+(1)^2}$

$= \sqrt{2}$

max decrease: $-\sqrt{2}$

a) $$\int_0^3 \int_0^2 4 - y^2 \, dydx$$

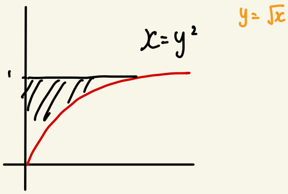


$$= \int_0^3 \left[4y - \frac{y^3}{3}\right]\Big|_0^2 \, dx$$

$$= \int_0^3 \frac{16}{3} \, dx$$

$$= \left[\frac{16x}{3}\right]\Big|_0^3$$

$$= 16$$



c) $$\int_0^1 \int_0^{y^2} 3y^3 e^{xy} \, dxdy$$

$$\int_0^1 \left[\frac{3y^3 e^{xy}}{y}\right]\Big|_0^{y^2} dy$$

$$= \int_0^1 \left[\frac{3y^2 e^{xy}}{y}\right]\Big|_0^{y^2} dy$$

$$= \int_0^1 \left[3y^2 e^{y^3} - 3y^2 \overbrace{e^0}^{1}\right] dy$$

Let $u = y^3$

$du = \frac{dy}{3y^2}$

$$= \int_0^1 3y^2 e^{y^3} dy - \int_0^1 3y^2 \, dy$$

$$= \int_0^1 e^u \, du - \left[y^3\right]\Big|_0^1$$

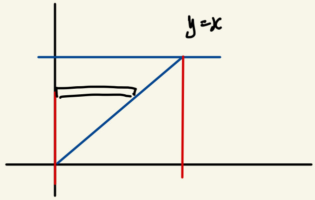
$$= e^u \Big|_0^1 - (y^3)\Big|_0^1$$

$$= e^1 - e^0 - 1^3 + 0$$

$$= \underline{\underline{e - 2}}$$

e) $\int_0^2 \int_x^2 2y^2 \sin(xy)\, dy dx$

Fubini



$$\int_0^2 \int_0^y 2y^2 \sin(xy)\, dx dy$$

$$= \int_0^2 \left[\frac{-2y^2 \cos(xy)}{y} \right]_0^y dy$$

$$= \int_0^2 \left[-2y\cos(y^2) + 2y \right] dy$$

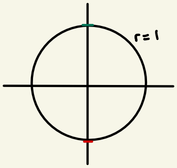
Let $u = y^2$

$dy = \frac{du}{2y}$

$$= \left[-\sin(y^2) \right]\Big|_0^2 + \left[y^2 \right]\Big|_0^2$$

$$= -\sin(4) + 4$$

b) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\sqrt{1+\underbrace{x^2+y^2}_{r^2}}}\, dydx$



$$\int_0^{2\pi}\int_0^1 \frac{2}{\sqrt{1+r^2}}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{2r}{\sqrt{1+r^2}}\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 \frac{2r}{\sqrt{1+r^2}}\, dr$$

$$= [\theta]\Big|_0^{2\pi} \cdot 2(\sqrt{2}-\sqrt{1})$$

$$= 4\pi(\sqrt{2}-1)$$

Let $1+r^2 = u$

$$dr = \frac{du}{2r}$$

$$\int_0^1 \frac{1}{\sqrt{u}}\, du \qquad \frac{(u)^{-\frac{1}{2}+1}}{\frac{1}{2}}$$

$$= [2\sqrt{u}]\Big|_1^2$$

$$= 2(\sqrt{2}-\sqrt{1})$$

a) $\int_1^e \int_1^e \int_1^e \frac{1}{xyz}\, dxdydz$

$$= \int_1^e \frac{1}{x}\, dx \cdot \int_1^e \frac{1}{y}\, dy \cdot \int_1^e \frac{1}{z}\, dz$$

$$= [\underbrace{\ln e}_{1} - 0] \cdot [\ln e - 0] \cdot [\ln e - 0]$$

$$= 1 \times 1 \times 1$$

$$= 1$$

b) $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dzdxdy$

$$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2x+y \;\; dx\, dy$$

$$= \int_0^2 \left[x^2 + xy \right] \Big|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= \int_0^2 \left\{ (4-y)^2 + y\sqrt{4-y^2} - \left[(4-y)^2 - y\sqrt{4-y^2} \right] \right\} dy$$

$$= \int_0^2 2y\sqrt{4-y^2}\; dy$$

Let $4-y^2 = u$

$$\frac{du}{dy} = -2y$$

$$dy = \frac{du}{-2y}$$

$$= \int_{4-0=4}^{4-2^2=0} -\sqrt{u}\;\; du$$

$$\frac{(u)^{\frac{1}{2}+1}}{\frac{3}{2}}$$

$$= \left[-\frac{2}{3}(u)^{3/2} \right]_4^0$$

$$= 0 - \left(-\frac{16}{3} \right) = \frac{16}{3}$$

$$\int_0^6 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\, dp\, dq\, dr$$

$$= \int_0^6 \int_0^2 \left[\frac{qp}{r+1}\right]\Bigg|_0^{\sqrt{4-q^2}} dq\, dr$$

Let $4-q^2 = u$

$du = -2q\, dq$

$$= \int_0^6 \int_0^2 \frac{q\sqrt{4-q^2}}{r+1}\, dq\, dr$$

$$= \int_0^6 \int_4^0 \frac{\cancel{q}\sqrt{u}}{r+1} \frac{du}{-2\cancel{q}}\, dr$$

$$\frac{(u)^{\frac{1}{2}+1}}{\frac{3}{2}}$$

$$= \int_0^6 \left[- \frac{\frac{2}{3}(u)^{3/2}}{2(r+1)}\right]_4^0 dr$$

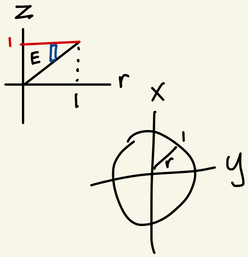
$$= \int_0^6 \left[\frac{16}{6(r+1)}\right] dr$$

$$= \frac{8}{3}\int_0^6 \frac{1}{r+1}\, dr$$

$$= \frac{8}{3}\left[\ln|r+1|\right]_0^6$$

$$= \frac{8}{3}\ln|7| + 0$$

b) The region E that is bounded by the cone $z = \underbrace{\sqrt{x^2+y^2}}_{r}$ and z = 1.



$$\int_0^{2\pi}\int_0^1\int_r^1 r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 [rz]_r^1\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r - r^2\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^3}{3}\right]\Big|_0^1\,d\theta$$

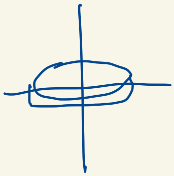
$$= \int_0^{2\pi}\left[\frac{1}{6}\right]d\theta$$

$$= \frac{2\pi}{6} - 0$$

$$= \frac{\pi}{3}$$

c) The region E that is bounded by the spherical surface $x^2+y^2+z^2=9$ and the cones $z^2 = x^2 + y^2$ and $3z^2 = x^2 + y^2$.

$r^2 \qquad \frac{r^2}{3}$



$z = \frac{r}{\sqrt{3}}$

$\rho \cos\varphi = \frac{\rho \sin\varphi}{\sqrt{3}}$

$\tan\varphi = \sqrt{3}$

$\varphi = \tan^{-1}(\sqrt{3})$

$= 60$

$z = r$

$\rho\cos\varphi = \rho\sin\varphi$

$\tan\varphi = 1$

$\varphi = 45$

$$\int_0^{2\pi}\int_{45}^{60}\int_0^3 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} 1 \, d\theta \cdot \int_{45}^{60} \sin\varphi \, d\varphi \cdot \int_0^3 \rho^2 \, d\rho$$

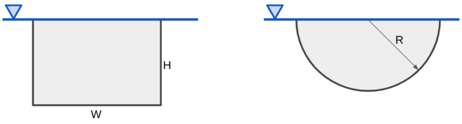
$$= 2\pi \cdot \left[-\cos\varphi\right]\Big|_{45}^{60} \cdot \left[\frac{\rho^3}{3}\right]\Big|_0^3$$

$$= 2\pi \cdot \left(-\frac{1}{2} + \frac{\sqrt{2}}{2}\right) \cdot 9$$

$$= 18\pi\left(\frac{-1+\sqrt{2}}{2}\right)$$

$$= 9\pi\left(\sqrt{2} - 1\right)$$

9. **Use of Calculus in Engineering Design**
In an underwater sea aquarium, two acrylic glass panels in the shapes shown below need to be installed for visitors to view the marine life.



In order to determine the thickness of the panels and their attachment method to the surrounding wall, the net force caused by the hydrostatic water pressure must be determined. Under a liquid of density ρ, the hydrostatic pressure is

$$P = \rho g h$$

where h is the depth from the free surface of the liquid.

a) Evaluate the net hydrostatic force on each acrylic panel as a function of other parameters.

b) What would be the hydrostatic force on the rectangular panel instead if it is submerged such that its top edge is d meters below the water surface?

ANS: **a)** $\text{Rectangle}: \ F = \frac{\rho g W H^2}{2}. \ \text{Semicircle}: \ F = \frac{2\rho g R^3}{3}.$

b) $F = \frac{\rho g W\left(H^2 + 2dH\right)}{2}.$

Semicircle : $\rho g \int_0^\pi \int_0^R (r\sin\theta)\, r\,dr\,d\theta$

$h = y$
$= r\sin\theta$

$$= \rho g \int_0^\pi \int_0^R r^2 \sin\theta \, dr\,d\theta$$

$$= \rho g \cdot \int_0^\pi \sin\theta\,d\theta \cdot \int_0^R r^2\,dr$$

$$= \rho g \cdot (-\cos\theta)\Big|_0^\pi \cdot \left[\frac{r^3}{3}\right]\Big|_0^R$$

$$= \rho g \cdot \underbrace{\left[-\cos(\pi) - (-\cos(0))\right]}_{2} \cdot \frac{R^3}{3}$$

$=$