Topic 4
The Gradient Vector & Multiple Integration

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Outline

- Gradient Vector
- Directional Derivatives
- Volume Under a Surface
- Iterated Integrals & Fubini's Theorem
- Double Integral in Polar Coordinates
- Triple Integrals in Cartesian, Cylindrical & Spherical Coordinates

Change of a Multivariable Function

Recall that the linear approximation of a function f(x, y) is

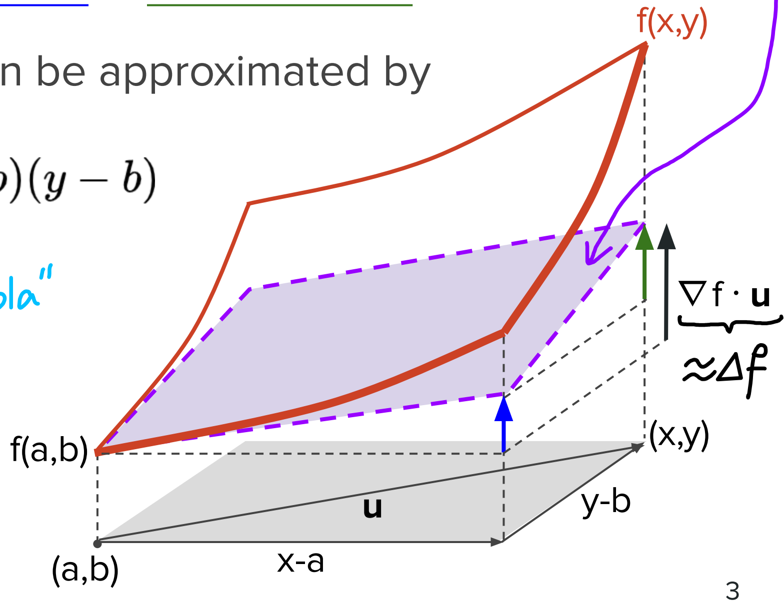
$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Rearranging, the change of the function from (a, b) can be approximated by

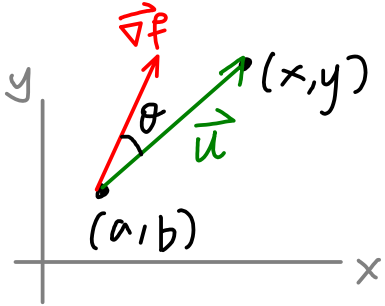
$$\begin{aligned}\Delta f = f(x,y) - f(a,b) &\approx f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\approx \begin{bmatrix} f_x(a,b) \\ f_y(a,b) \end{bmatrix} \cdot \begin{bmatrix} x-a \\ y-b \end{bmatrix} = \nabla f(a,b) \cdot \mathbf{u}\end{aligned}$$

"nabla"

which is a dot product between the gradient vector ∇f and the displacement vector **u** from (a, b). This **approximation is accurate if (x, y) is close to (a, b).**



Gradient Vector

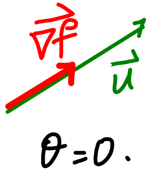


From the definition of the dot product, we can write

$$\Delta f \approx \underline{\nabla f} \cdot \underline{\mathbf{u}} = |\nabla f||\mathbf{u}| \cos\theta$$

where θ is the angle between ∇f and **u**. If we want to **maximize** the change of f(x,y) from (a, b), the displacement vector **u** must be in the **same direction** as the gradient vector ∇f such that the **dot product is at a maximum**, i.e.

$$\text{Max}\,\Delta f \approx \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos(0) = |\nabla f||\mathbf{u}|$$



Hence, we know that the gradient vector ∇f must be **pointing in the direction of maximum increase** of the function. Another observation is that the gradient vector is always perpendicular to a level curve, as illustrated in the next slide.

Gradient Vector

From the level curve of a function, say f(x, y) = c, we can express y as an implicit function of x, giving

$$f(x, y(x)) = c$$

tangent vector along level curve.

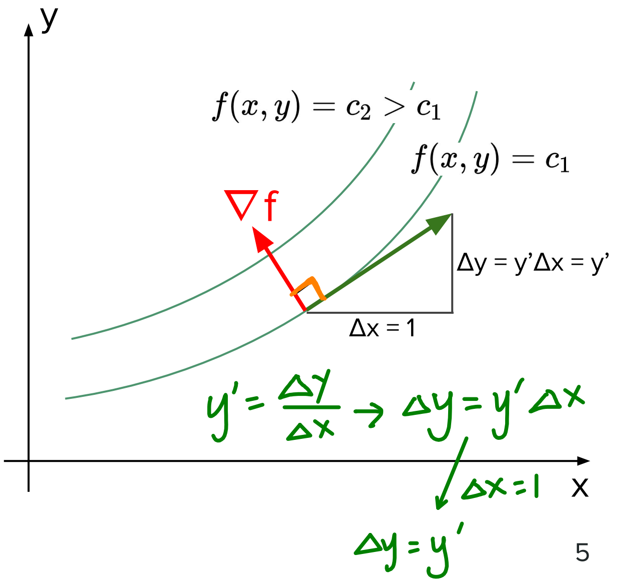
$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$

Differentiating the above w.r.t. x yields

$\frac{d}{dx}$

$$f_x + f_y\, y' = 0 \rightarrow \begin{bmatrix} f_x \\ f_y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ y' \end{bmatrix} \overbrace{=}^{\perp} 0 \rightarrow \nabla f \cdot \begin{bmatrix} 1 \\ y' \end{bmatrix} = 0$$

Since [1, y'] is a vector tangent to the level curve, the above zero dot product implies that the gradient vector is **perpendicular** to the level curve. This agrees with ∇f pointing in the **direction of max increase** because that is the **'shortest path' to the next higher level curve**.

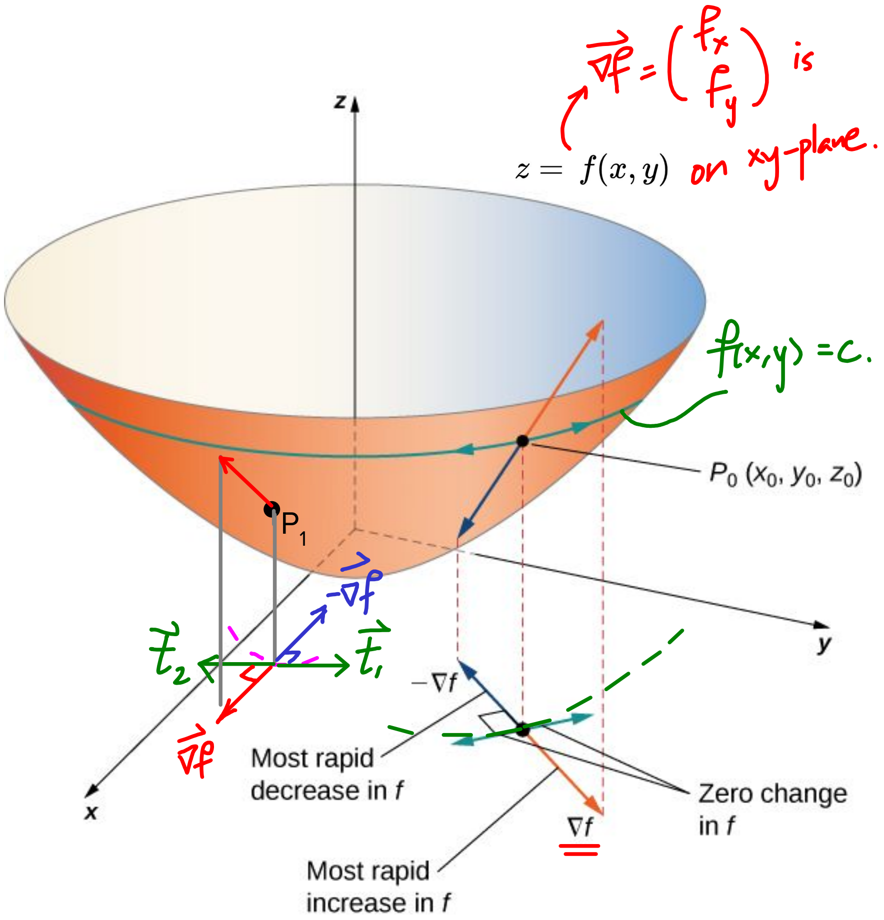


Gradient Vector

Naturally, the direction of max decrease would be that **opposite to the gradient vector**, which is $-\nabla f$. From the figure, we can observe the various vectors on the xy-plane in relation to the surface z = f(x, y) at a point P_0.

The rate of change of the function in the direction of ∇f is simply its magnitude, $|\nabla f|$.

Exercise: Sketch ∇f, $-\nabla f$ and the $\underbrace{\text{tangent vectors}}_{\vec{t}_1,\ \vec{t}_2}$ at point P_1.



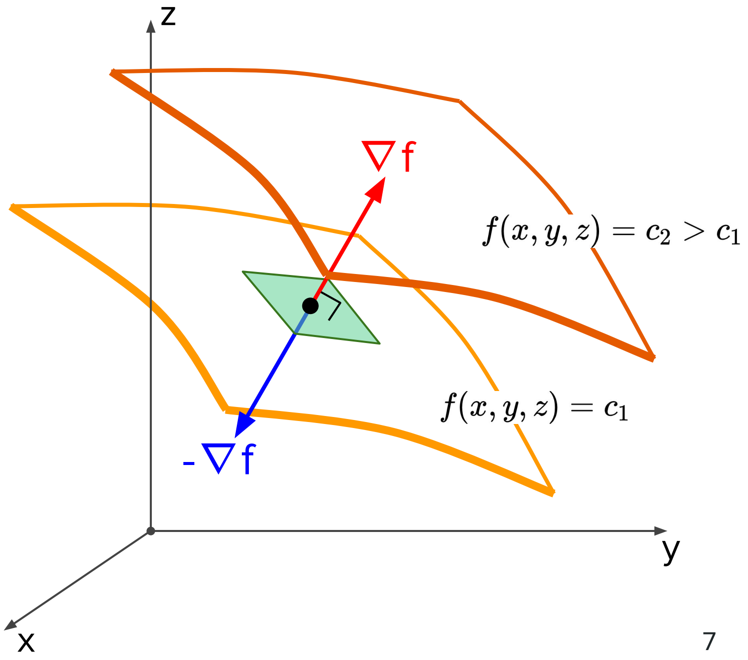
https://openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient

Gradient Vector

For a function of three variables and above, the gradient vector is similarly defined, i.e.

$$\nabla f(x_1, \ldots, x_n) = \begin{bmatrix} f_{x_1}(x_1, \ldots, x_n) \\ \vdots \\ f_{x_n}(x_1, \ldots, x_n) \end{bmatrix}$$

A function f(x, y, z) with level surfaces as illustrated will have 3D gradient vectors, where each gradient vector is **perpendicular** to the tangent plane and similarly pointing in the **direction of max increase** of the function.



Gradient Vector

$\vec{\nabla} f(0,0) = \vec{0}$. Logically, the gradient vector is the zero vector since (0,0) is at the max of f(x,y), so there is no direction to move to increase the function.

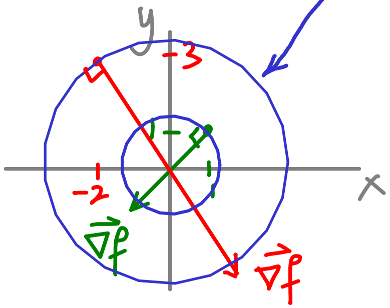
Example: Determine the gradient vector of the function below and sketch to show that it is pointing in the direction of maximum ascent and perpendicular to the level curves at a few points. What is the gradient vector at (0, 0)? Explain.

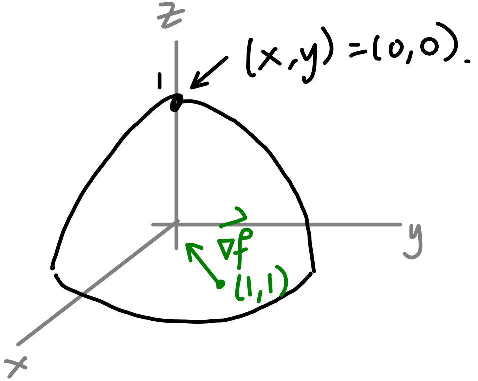
$$f(x, y) = 1 - x^2 - y^2 \longrightarrow f(x,y) = C \rightarrow x^2 + y^2 = \underbrace{1-c}_{R^2}$$

$$\vec{\nabla} f(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}.$$

Eg) $\vec{\nabla} f(1,1) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$

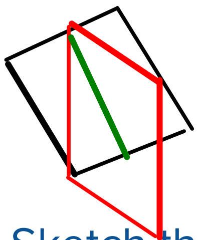
$\vec{\nabla} f(-2,3) = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$





ANS: $\nabla f = [-2x, -2y]$.

Gradient Vector



Exercise: Determine the gradient vector of the function below. Sketch the surface, its level sets and the gradient vectors on a 3D Cartesian coordinate system. What is the slope of the function in the direction of max ascent?

$$\underbrace{f(x, y) = 5 - 3x - 4y}_{=c.} = z \rightarrow$$

At $z=0, y=0$,
$5-3x=0 \rightarrow x=\frac{5}{3}$.

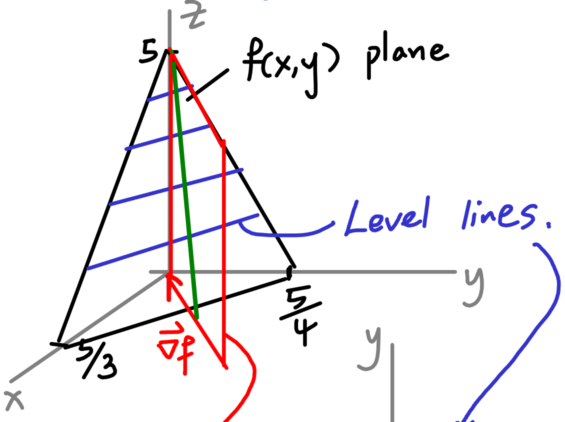
At $z=0, x=0$,
$5-4y=0 \rightarrow y=\frac{5}{4}$.

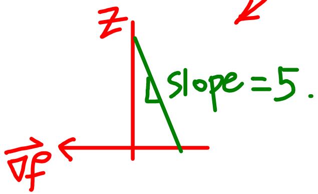
At $x=0, y=0$,
$z=5$.

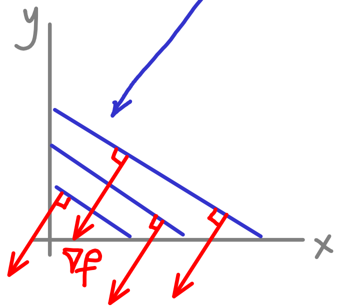
$3x+4y=5-c$.

$\vec{\nabla f} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$\text{Slope} = |\vec{\nabla f}| = \sqrt{(-3)^2+(-4)^2}$
$= \sqrt{9+16} = \sqrt{25} = 5$







ANS: ∇f = [-3, -4]. Slope of f in direction of max ascent = 5.

Directional Derivative

The directional derivative represents the rate of change of a multivariable function **in a prescribed direction**, i.e.

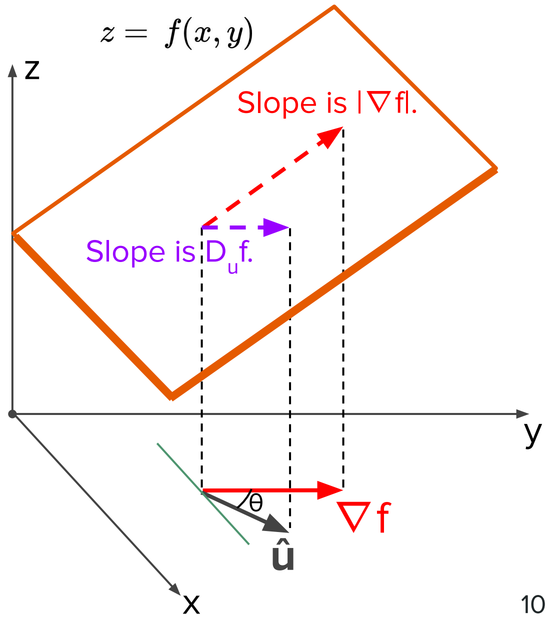
MUST Be unit vector

$$D_{\mathbf{u}} f = \nabla f \cdot \hat{\mathbf{u}} = |\nabla f| \cos \theta$$

where **û** is a **unit vector pointing in the prescribed direction**.

For example, given that f(x, y) is the plane shown, the directional derivative describes the slope in the direction indicated.

If **û** is in the direction that is 90° to ∇f, then clearly the directional derivative is zero, since the direction is along the level curve.



Directional Derivative

$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \to \hat{u} = \frac{1}{|\vec{u}|}\vec{u} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \end{pmatrix}.$

Example: For the function describing the plane, determine the directional derivatives in the directions indicated by each vector below. Sketch the vectors and indicate their directions on the surface to verify with the values obtained.

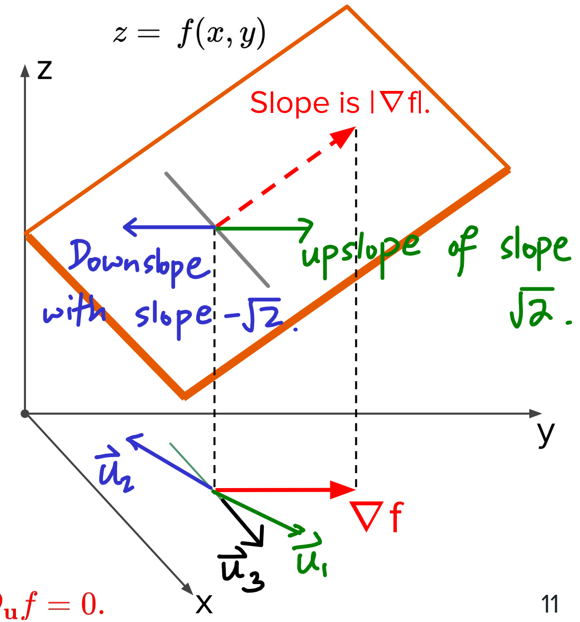
$$f(x,y) = 3 + 2y,\ \mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{u_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ \hat{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\vec{\nabla} f = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$|\vec{\nabla} f| = 2.$

$\hat{u}_1 = \frac{1}{\sqrt{1^2+1^2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$D_{u_1} f = \vec{\nabla} f \cdot \hat{u}_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 + \frac{2}{\sqrt{2}} = \sqrt{2}\ \overbrace{\left(< |\vec{\nabla} f| = 2 \right)}^{\text{logical.}}$



ANS: $\mathbf{u_1} : D_{\mathbf{u}} f = \sqrt{2},\ \mathbf{u_2} : D_{\mathbf{u}} f = -\sqrt{2},\ \mathbf{u_3} : D_{\mathbf{u}} f = 0.$

$$D_{u_2} f = \vec{\nabla} f \cdot \hat{u}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cdot \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\sqrt{2}.$$

$$D_{u_3} f = \vec{\nabla} f \cdot \hat{u}_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$ (Not going upslope or downslope, since along level curve.)

Directional Derivative

Exercise: Determine the directional derivative of the function below at the point (1, 1, 1) in the prescribed direction. Why is it negative?

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \longrightarrow \hat{u} = \frac{1}{\sqrt{1^2+2^2}}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

$$f_x = (-1)(x^2+y^2+z^2)^{-2}(2x) = \frac{-2x}{(x^2+y^2+z^2)^2} = \frac{-2x}{A}$$

At $(1,1,1)$, $A = (1^2+1^2+1^2)^2 = 9$.

$$\vec{\nabla} f(x,y,z) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} -2x/A \\ -2y/A \\ -2z/A \end{pmatrix} = \frac{-2}{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \longrightarrow \vec{\nabla} f(1,1,1) = -\frac{2}{9}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

ANS: $D_u f(1,1,1) = \dfrac{-2}{3\sqrt{5}}$.

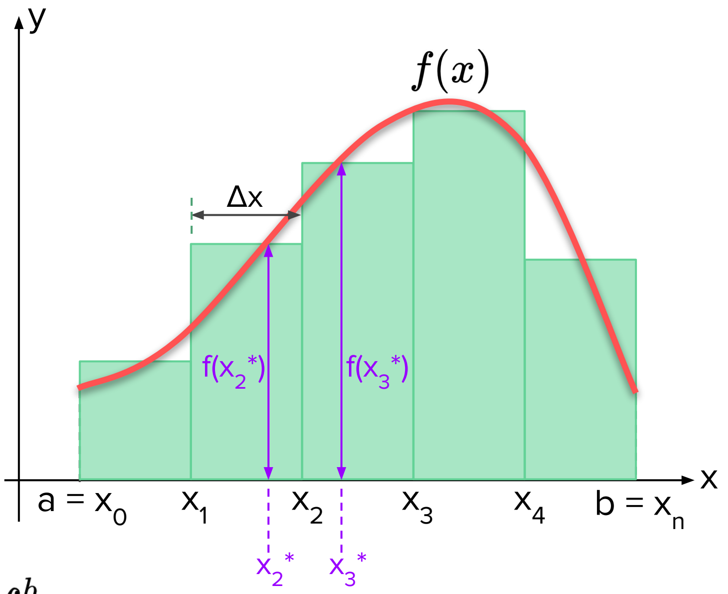
$$D_u f(1,1,1) = \vec{\nabla} f(1,1,1) \cdot \hat{u} = -\frac{2}{9}\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix}$$

$$= -\frac{2}{9}\left(\frac{1}{\sqrt{5}}\right)(0+1+2) = -\frac{2}{3\sqrt{5}} \quad //$$

Since f(x,y,z) is decreasing as (x,y,z) gets further away from the origin, the directional derivative is negative if the vector u is in a general direction pointing away from the origin. Since u = (0, 1, 2) is pointing away from the origin, hence the directional derivative is negative.

Recap: Riemann Sum & Riemann Integral

Besides using the left or right endpoint for computing each rectangle's height, one can also approximate the area by using the height at any point x_i^* in $[x_{i-1}, x_i]$, i.e.



$$Area \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

which is called the Riemann sum. Again, by having the **number of partitions approach infinity, the exact area** is given by

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$

A function f(x) is called integrable in [a, b] if the limit exists.

which is called the Riemann integral. Hence, integration is about finding the exact area bounded by a function (the integrand).

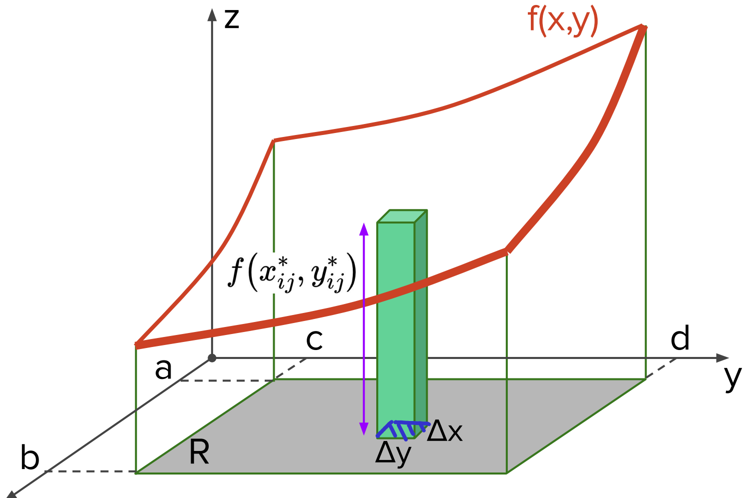
Volume Under a Surface

Analogous to finding area under a curve that results in integration, finding the volume under a surface results in a double integral. Consider a function z = f(x, y) where a portion of its surface is over a rectangular region R = [a, b] x [c, d] as shown.

Firstly, it would be intuitive to approximate the volume under the surface by using rectangular blocks of similar base area ΔxΔy , i.e.

$$\text{Volume}, V \approx \sum_{j=1}^{n}\sum_{i=1}^{m} \underbrace{\overbrace{f\left(x_{ij}^{*}, y_{ij}^{*}\right)}^{\text{height}} \overbrace{\Delta x \Delta y}^{\text{base area}}}_{\text{Volume of one block}}$$

where the height of each block $f(x_{ij}^*, y_{ij}^*)$ is evaluated any point x_{ij}^* in $[x_{i-1j}, x_{ij}]$, y_{ij}^* in $[y_{ij-1}, y_{ij}]$.



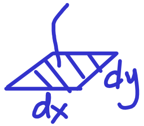
$$\Delta x = \frac{b-a}{m}, \Delta y = \frac{d-c}{n}$$

Volume Under a Surface

Similar to approximating the area under a curve, the volume approximation gets better as there are **more & thinner blocks** under the surface. Clearly, if we let $\Delta x \to 0$ and $\Delta y \to 0$, then we have $m \to \infty$ and $n \to \infty$, which means we have an **infinite number of blocks**. The exact volume is then given by taking a limit on the double Riemann sum, i.e.

dA = dxdy

$$\text{Volume}, V = \lim_{n\to\infty,\, m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta x \Delta y = \int_{c}^{d} \int_{a}^{b} f(x,y)\, \underline{dxdy}$$



which gives a double Riemann integral. Denoting the infinitesimal base area of each block as dA = dxdy, we can also state the volume as

$$\text{Volume}, V = \iint_{R} f(x,y)\, \underline{dA}$$

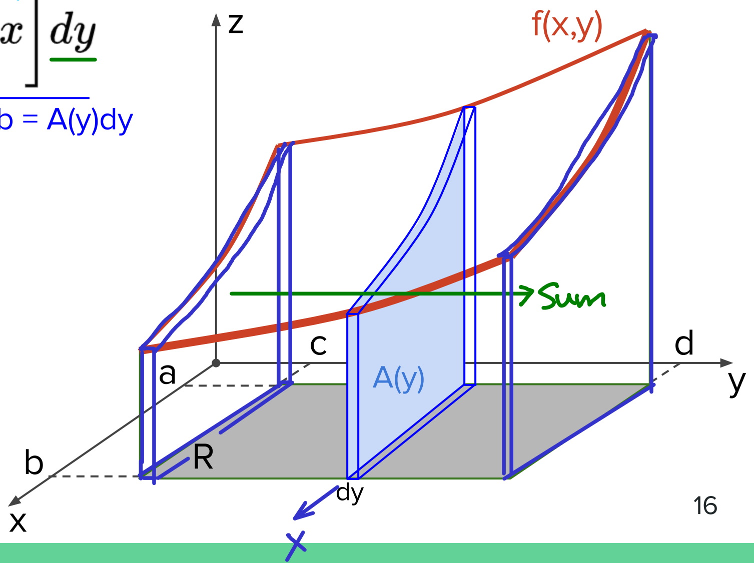
This gives the general expression for volume under surface f(x,y) over a region R.

Iterated Integrals

We can evaluate the double integral below by evaluating the inner integral w.r.t. x first while treating y as a constant, followed by evaluating the outer integral w.r.t y. Since the integration process is **repeated**, these integrals are also called **iterated** integrals.

$$\text{Volume, } V = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \underbrace{\left[\overbrace{\int_a^b f(x,y)\,dx}^{\text{Area}}\right] dy}_{\text{Volume of one slab = A(y)dy}}$$

Graphically, the inner integral gives the area of the plane at some y value, A(y). Multiply this area by dy gives the volume of the blue slab. **Integrating (summing) the volumes of all slabs** from y = c to y = d then gives the total volume over region R.

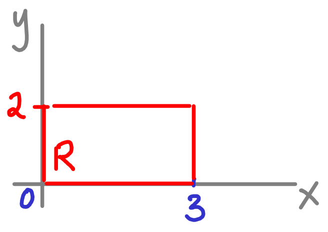


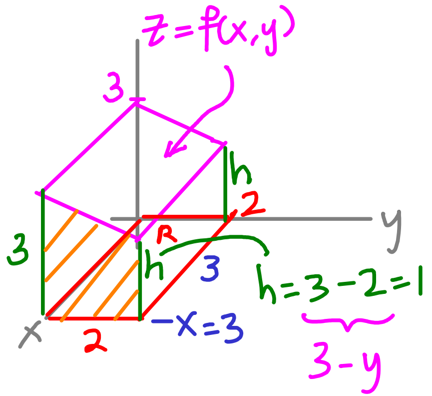
* Must be able to sketch.

Volume Under a Surface

Example: Evaluate the volume under the surface f(x, y) over the rectangular region R = [0, 3] x [0, 2]. Notice that the inner integral gives a function of y, A(y). Sketch the volume and verify the volume using a formula.

$$f(x, y) = 3 - y$$





$$Vol = \int_{y=0}^{y=2}\int_{x=0}^{x=3} 3-y \, dx \, dy = \int_{y=0}^{y=2} (3-y)x \Big|_0^3 dy$$

$$= \int_{y=0}^{y=2} (3-y)(3-0) \, dy = \int_{y=0}^{y=2} 9-3y \, dy = \left(9y - \frac{3y^2}{2}\right)\Big|_0^2$$

$$= 18-3(2) = 12 \text{ units}^3$$

Same ✓

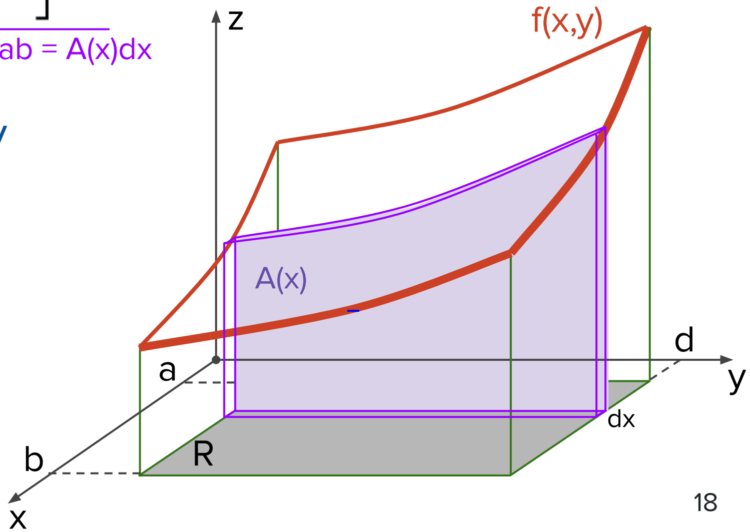
$$Vol = Area \times 3$$
$$= \frac{1}{2}(3+1)2 \times 3$$
$$= 12$$

ANS: Volume = 12 units^3.

Volume Under a Surface

One might observe that we can also sum up the slabs in the other direction. Hence, we can **switch the order of integration to get the same volume**, i.e.

$$\text{Volume}, V = \int_a^b \int_c^d f(x,y)\,dy dx = \int_a^b \underbrace{\left[\int_c^d f(x,y)\,dy\right] dx}_{\text{Volume of one slab = A(x)dx}}$$



Exercise: Repeat the last example by integrating w.r.t. y first and verify that the volume obtained is the same.

$$\text{Vol} = \int_{x=0}^{x=3} \int_{y=0}^{y=2} 3 - y \,dy\,dx = \int_{x=0}^{x=3} \left(3y - \frac{y^2}{2}\right)\Big|_0^2 dx$$

$$= \int_{x=0}^{x=3} 3(2) - \frac{2^2}{2}\, dx = \int_{x=0}^{x=3} 4\, dx = 4x \Big|_0^3 = 4(3) = 12 \;//$$

When limits are constants and the $\underbrace{\text{integrand function is separable}}_{f(x,y) = g(x)h(y)}$, you can integrated concurrently.

$h(y)$ is const. in x.

$$\int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d h(y)\, dy \int_a^b g(x)\, dx$$

Eg)

$h(y)$, $g(x) = 1$.

$$\text{Vol} = \int_{x=0}^{x=3} \int_{y=0}^{y=2} \overbrace{3-y}^{h(y)}\, dy\, dx = \int_0^2 3-y\, dy \cdot \int_0^3 1\, dx = \underbrace{\left(3y - \frac{y^2}{2}\right)\Big|_0^2}_{4} \underbrace{(x)\Big|_0^3}_{3}$$

$$= 4 \quad 3 = 12 \;//$$

Fubini's Theorem

The observation that the **order of the iterated integrals can be switched** is called Fubini's theorem. The double integral over a region R = [a, b] x [c, d] is

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dxdy = \int_a^b \int_c^d f(x,y)\,dydx$$

as long as the double integral is finite. In fact, the **region R need not be a rectangle**, as we will verify later.

In some cases, one order of integration is preferred or necessary over the other, as illustrated in the next example.

Fubini's Theorem

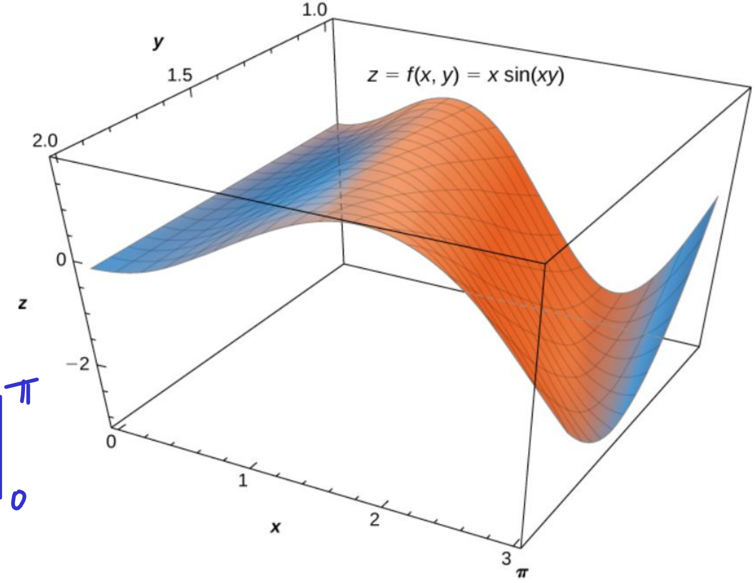
Example: Evaluate the double integral below over the rectangular region [0, π] x [1, 2]. Which order of integration is preferred?

$$\int_1^2 \int_0^\pi x \sin(xy)\, dx\, dy \quad \leftarrow \text{need int. by parts.}$$

$$\int_0^\pi \int_1^2 x\sin(xy)\, dy\, dx = \int_0^\pi \cancel{x}\left[\frac{-1}{\cancel{x}}\cos(xy)\right]_1^2 dx$$

$$= -\int_0^\pi \cos(2x) - \cos x \, dx = -\left[\frac{1}{2}\sin(2x) - \sin x\right]_0^\pi$$

$$= -[0-0] = 0$$



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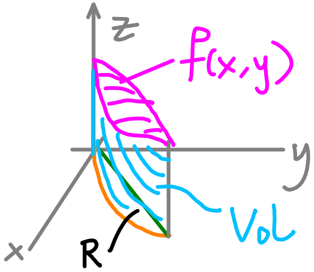
ANS: 0.

Double Integral over General Regions



When the region of integration is not a rectangle, the **same principles as discussed apply**, but some work might be required in setting up the double integral. For example, to find the integral of f(x,y) over the region R bounded by $y = x^2$ and $y = x$ as shown, we firstly need to solve for the intersection points between the curves, i.e.

$$x^2 = x \rightarrow x(x-1) = 0$$
$$\rightarrow x = 0,\ 1$$

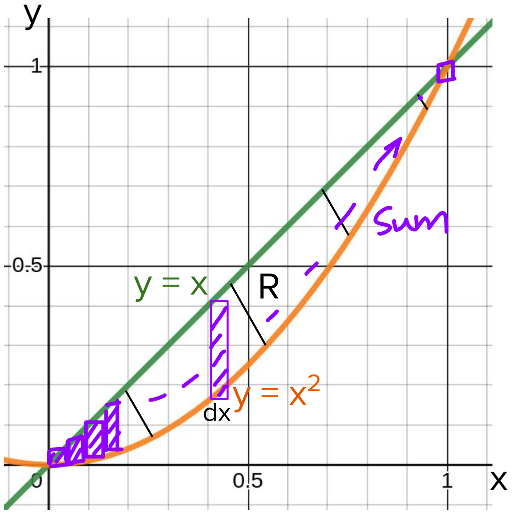
So the intersection points are (0, 0) and (1, 1). Then the double integral is

Upper bound of slab

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} f(x,y)\,dydx$$

Lower bound of slab

Volume of one slab (top view shown in graph)



Fubini’s Theorem

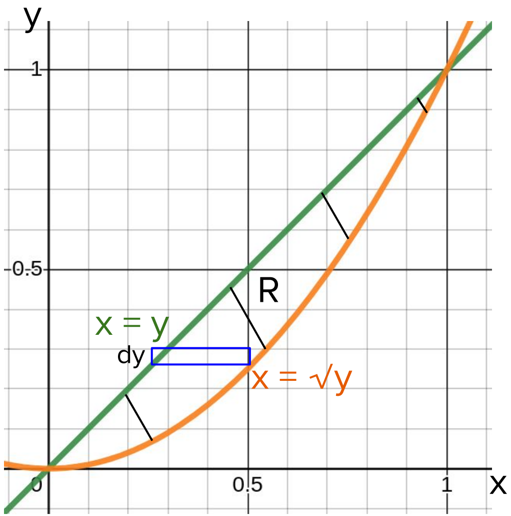
From Fubini’s theorem, we can also set up the integral in another way, i.e.

Right bound of slab

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} f(x,y)\,dx dy$$

Left bound of slab

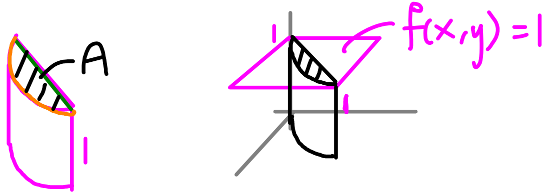
Volume of one slab (top view shown in graph)



in which the limits of integration are redefined as well.

One can observe that as long as the region of integration is the same, the double integral of f(x, y) over that region is the same, regardless of the order of integration. This is **logical because it is the same volume**.

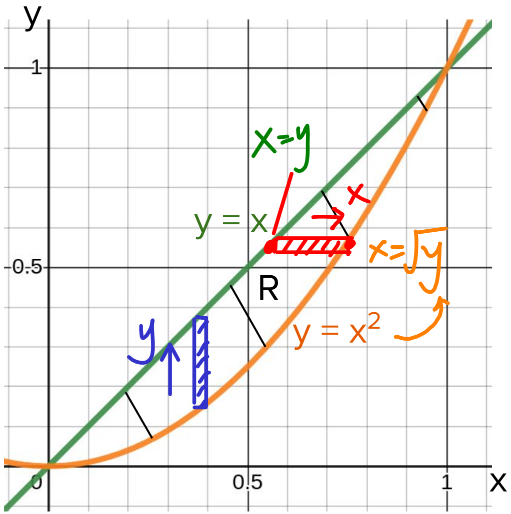
Fubini's Theorem



Example: The area bounded by the region R can be found by

$$A = \iint_R \overbrace{1\, dA}^{f(x,y)}$$

since the value of volume V = A(1) = A. Determine the area bounded by R using both orders of integration and show that they give the same area (& volume).



$$A = \int_0^1 \int_{x^2}^{x} 1\, dy\, dx = \int_0^1 y\Big|_{x^2}^{x} dx = \int_0^1 x - x^2\, dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6} \,//$$

ANS: 1/6.

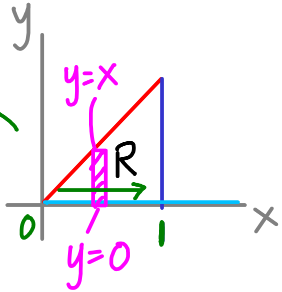
$$A = \int_0^1 \int_y^{\sqrt{y}} 1 \, dx \, dy \overset{\text{DIY}}{=} \cdots = \frac{1}{6} //$$

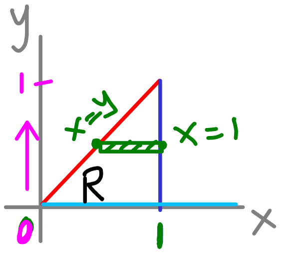
Double Integral over General Regions

Sketch R first.

Exercise: A prism has a triangular base bounded by lines y = 0, y = x, x = 1 and a top surface given by the function below. Evaluate its volume by both orders of integration and sketch the region of integration. R

$$f(x,y) = 3 - x - y$$





$$\text{Vol} = \int_0^1 \int_0^x 3-x-y \, dy \, dx = \int_0^1 \left(3y - xy - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=x} dx$$

$$= \int_0^1 3x - x^2 - \frac{x^2}{2} \, dx = \int_0^1 3x - \frac{3}{2}x^2 \, dx = \left(\frac{3x^2}{2} - \frac{1}{2}x^3\right)\Bigg|_0^1 = \frac{3}{2} - \frac{1}{2} = 1 \text{ unit}^3 //$$

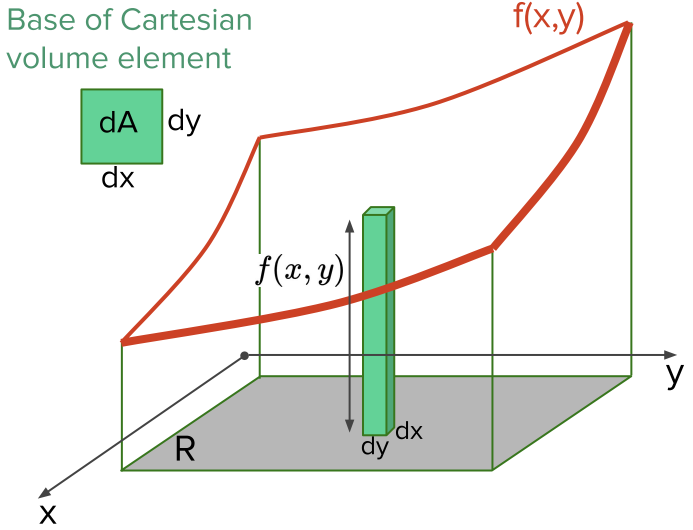
$$\text{Vol} = \int_0^1 \int_y^1 3-x-y \, dx \, dy = \cdots = 1 //$$

ANS: Volume = 1 unit^3.

Choice of Coordinate System in Double Integral

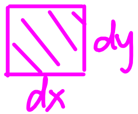
Besides evaluating a double integral in Cartesian coordinates, we **can also use other coordinates**, like polar coordinates. Firstly, notice that in the integration over a Cartesian area, the 'volume element' we are summing is f(x, y) dxdy. Hence the double integral over an area region R gives the total volume

$$V = \iint_R \underbrace{f(x,y)\,dA}_{\text{Volume element in any coordinate system}} = \iint_R \underbrace{f(x,y)\,dxdy}_{\text{Volume element in Cartesian}}$$



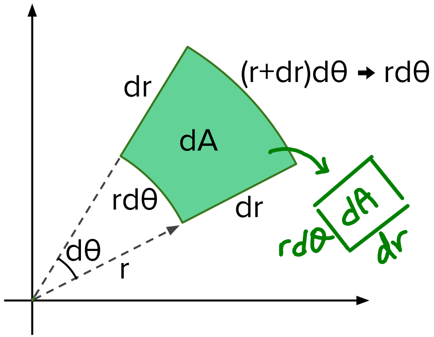
Besides using a Cartesian volume element, we **can also use other volume elements**, such as a polar volume element.

Double Integral in Polar Coordinates



Recall that a polar coordinate is defined as (r, θ). The height of the volume element is then f(r, θ). As illustrated, noting that as dr → 0 and dθ → 0, the **area dA approaches a rectangle** and is therefore

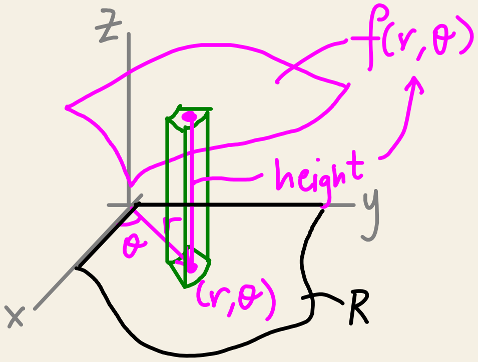
$$dA = r d\theta \, dr = r dr d\theta$$



Hence, the double integral in polar coordinates is

$$V = \iint_R f(r, \theta) \, dA = \iint_R \underbrace{f(r, \theta) \, r dr d\theta}_{\text{Polar volume element}}$$

Exercise: Sketch the polar volume element under a surface f(r, θ) over some general region R.



z
f(r,θ)
height
y
θ
r
(r,θ)
x
R

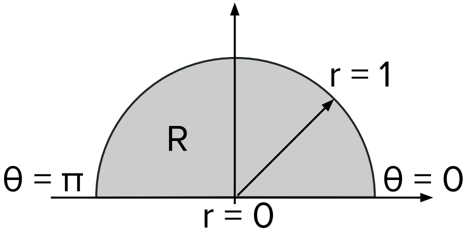
$(x,y) \xrightarrow{x=r\cos\theta,\ y=r\sin\theta} (r,\theta)$

Double Integral in Polar Coordinates

For a double integral, the choice of the coordinate system **depends on the ease of defining the region of integration and evaluating the integral**. For example, for the function $f(x, y) = x^2 + y^2$ and the region R shown, it would be easier to define and evaluate the integral in polar coordinates, i.e.

$$V = \iint_R \overbrace{x^2 + y^2}^{r^2}\, dA = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} \underline{r^2}\, r dr d\theta$$

$$= \int_0^{\pi} d\theta \int_0^1 r^3\, dr = \theta\Big|_0^{\pi} \cdot \frac{r^4}{4}\Big|_0^1 = \frac{\pi}{4}$$



However, it is **not as efficient if Cartesian coordinates** is being used, as demonstrated in the next slide.

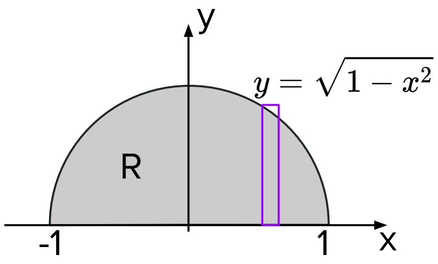
Choice of Coordinate System in Double Integral

In Cartesian coordinates, the double integral over the same region R is

$$V=\iint_R f(x,y)\,dxdy=\int_{x=-1}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} x^2+y^2\,dydx$$

$$=\int_{-1}^{1}\left(x^2y+\frac{y^3}{3}\right)\Bigg|_0^{\sqrt{1-x^2}}dx=\int_{-1}^{1}x^2\sqrt{1-x^2}+\frac{\left(1-x^2\right)^{3/2}}{3}\,dx=\ \dots\ =\frac{\pi}{4}$$

Although the last integral w.r.t. x can be evaluated by substituting trigonometric relations, the process the much more tedious. Therefore, **polar coordinates is clearly the preferred choice** here, partly due to **region R being part of a circle** and giving constant limits in the integrals.



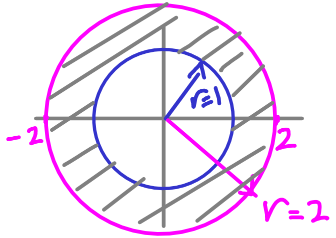
$(x,y) \xrightarrow{x=r\cos\theta,\ y=r\sin\theta} (r,\theta)$

Double Integral in Polar Coordinates

Exercise: Set up the double integrals in both Cartesian and polar coordinates for the function below in the annular region bounded by circles of radiuses 1 and 2. Evaluate the easier one. Try sketching the volume represented by the double integrals.

$f(x, y) = 5 - 2x \longrightarrow f(r,\theta) = 5 - 2r\cos\theta.$

In polar coords,



$$\text{Vol} = \int_0^{2\pi}\int_1^2 (5r - 2r^2\cos\theta)\,\cancel{r}\,dr\,d\theta = \int_0^{2\pi}\left(\frac{5r^2}{2} - \frac{2r^3}{3}\cos\theta\right)\Big|_1^2 d\theta$$

$$= \int_0^{2\pi}\left(10 - \frac{16}{3}\cos\theta\right) - \left(\frac{5}{2} - \frac{2}{3}\cos\theta\right) d\theta$$

ANS: Polar: $\int_0^{2\pi}\int_1^2 (5 - 2r\cos\theta)\, r\,dr\,d\theta = 15\pi$. Cartesian : $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 5 - 2x\,dy\,dx - \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 5 - 2x\,dy\,dx.$

$$= \int_0^{2\pi} \frac{15}{2} - \frac{14}{3}\cos\theta \, d\theta = \left(\frac{15}{2}\theta - \frac{14}{3}\sin\theta\right)\Big|_0^{2\pi}$$

$$= \frac{15}{2}(2\pi - 0) = 15\pi \quad //$$

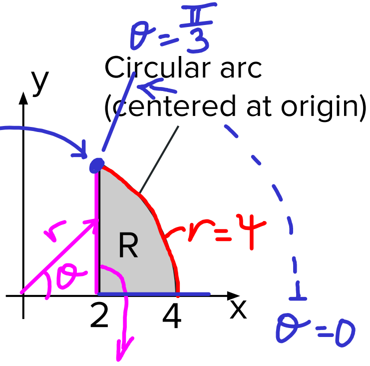
Double Integral in Polar Coordinates

Exercise: Using polar coordinates, determine the area in the region R defined as shown.

$$\text{Area} = \iint_R 1\, dA = \int_0^{\pi/3}\int_{2\sec\theta}^{4} 1\; r\,dr\,d\theta$$

$$= \int_0^{\pi/3} \frac{r^2}{2}\Bigg|_{2\sec\theta}^{4} d\theta = \frac{1}{2}\int_0^{\pi/3} 16 - 4\sec^2\theta \; d\theta$$

$$= \frac{1}{2}\left[16\theta - 4\tan\theta\right]\Bigg|_0^{\pi/3} = \frac{1}{2}\left[\frac{16}{3}\pi - 4\sqrt{3}\right] = \frac{8\pi}{3} - 2\sqrt{3}$$



$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}.$

$r\cos\theta = 2.$

$r = \frac{2}{\cos\theta} = 2\sec\theta$

ANS: $Area = \int_0^{\pi/3}\int_{2\sec\theta}^{4} r\,dr\,d\theta = \frac{8\pi}{3} - 2\sqrt{3}.$

Meaning of Integrals

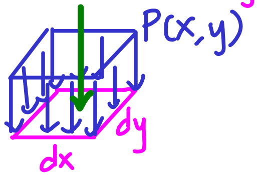
By now, it should be clear that an integral is a **sum** of the elements over a region (R) or an interval (I) of integration. What the integral represents depends on what the elements are, i.e.

$$\text{Area} = \int_I \underbrace{\text{Height}(x)\,dx}_{\text{Area element}} = \iint_R \underbrace{dx dy}_{\text{Area element}}$$

$$\text{Volume} = \int_I \underbrace{\text{Area}(x)\,dx}_{\text{Volume element}} = \iint_R \underbrace{\text{Height}(x,y)\,dx dy}_{\text{Volume element}}$$

$$\text{Net Force} = \iint_R \underbrace{\text{Pressure}(x,y)\,dx dy}_{\text{Force element}}$$

$$F = P\overbrace{(dx\,dy)}^{\text{Area } dA}$$



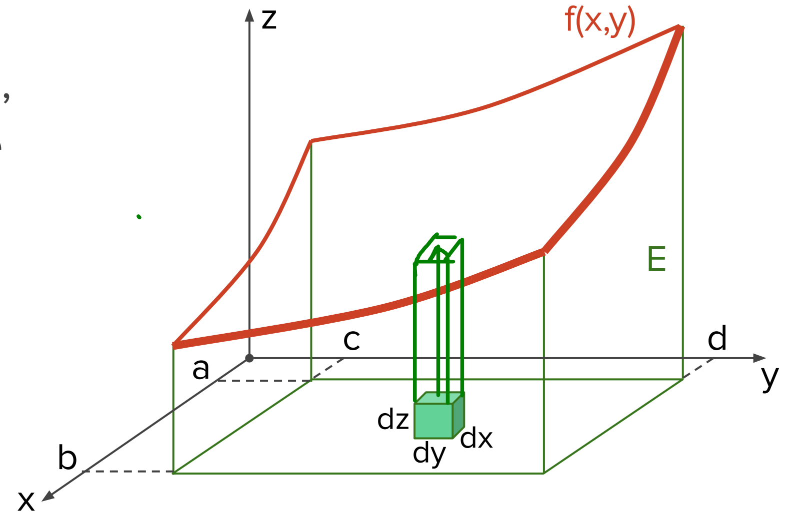
Notice that the **meaning of the integral does not depend on the number of integrals**. An area or volume can be represented by both a single and a double integral, in which the integrand can mean different entities.

Triple Integrals

Hence, a volume can also be represented by a triple integral, i.e.

$$\text{Volume} = \iiint_E dV = \iiint_E \underbrace{dxdydz}_{\text{Volume element}}$$

where the region of integration E is a volume region, such as the one shown in the figure. In this case, we integrate along z first from z = 0 to z = f(x, y), i.e.



$$\text{Volume} = \int_c^d \int_a^b \underline{\int_0^{f(x,y)} dzdxdy}$$

$$= \int_c^d \int_a^b z\Big|_0^{f(x,y)} dxdy = \int_c^d \int_a^b \underline{f(x,y)\,dxdy}$$

which results in the double integral earlier, as expected because it is the same volume.

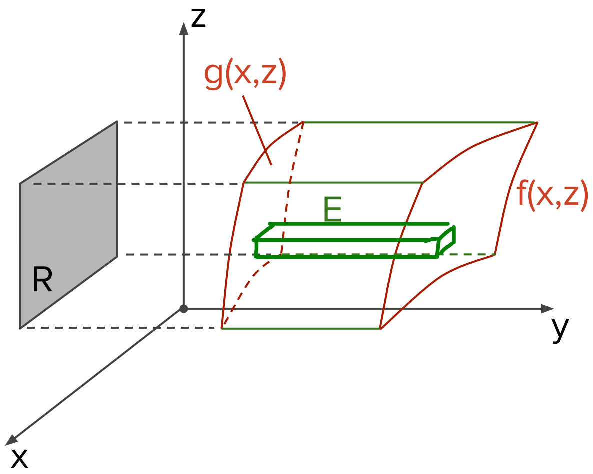
Triple Integrals

Clearly, Fubini's theorem applies for triple iterated integrals as well, **as long as the region of integration remains the same**. Depending on the orientation of region E, there can be a preferred order of integration.

Example: Set up a triple integral to evaluate the volume of region E shown and rewrite it as a double integral.

$$\text{Vol} = \iint_R \int_{g(x,z)}^{f(x,z)} dy \; dx\,dz = \iint_R f(x,z) - g(x,z) \, dx\,dz$$

$$= \iint_R f(x,z) - g(x,z) \, dz\,dx$$



Triple Integrals

Exercise: Given that the density in the cuboid shown varies according to the function below, evaluate the mass of the cuboid. Evaluate using two different orders of integration to verify Fubini's theorem. Dimensions are in meters.

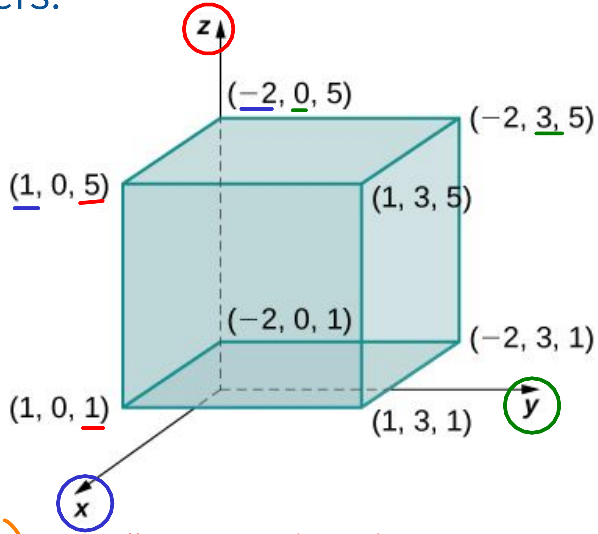
$$\rho(x, y, z) = x^2yz \quad (\mathrm{kg/m^3})$$

$$\text{mass} = \int_1^5 \int_0^3 \int_{-2}^1 \underbrace{x^2yz}_{\text{separable}}\, dx\, dy\, dz = \int_{-2}^1 x^2\, dx \int_0^3 y\, dy \int_1^5 z\, dz$$

$$= \left(\frac{x^3}{3}\right)\Big|_{-2}^1 \left(\frac{y^2}{2}\right)\Big|_0^3 \left(\frac{z^2}{2}\right)\Big|_1^5 = 162\ \mathrm{kg}$$

$$\int_0^3 \int_{-2}^1 \int_1^5 x^2yz\, dz\, dx\, dy \quad \text{or} \quad \ldots$$

(6 combinations of different orders)



https://openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals

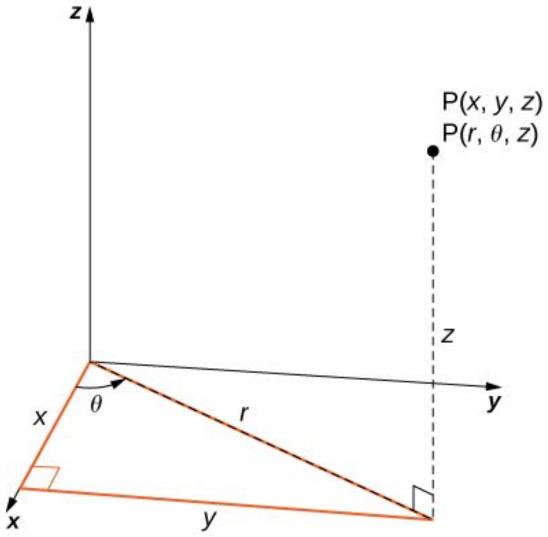
ANS: Mass = 162 kg.

Triple Integrals in Cylindrical Coordinates

Similar to how double integrals can be set up in polar coordinates, triple integrals can be set up in cylindrical coordinates. As illustrated, a **cylindrical coordinate system** is simply a **z-axis added to a polar coordinate system**.

A point P at coordinate (x, y, z) in Cartesian can also be located by (r, θ, z) in cylindrical coordinates.

Exercise: Sketch the volume element in cylindrical coordinates and define its volume. Hence, state the triple integral of f(r, θ, z) in cylindrical coordinates.

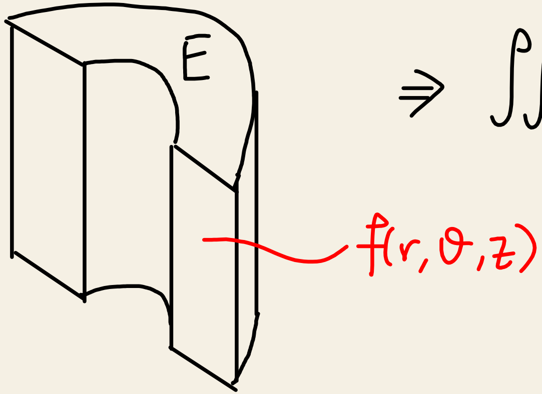


https://openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates

cylindrical element.

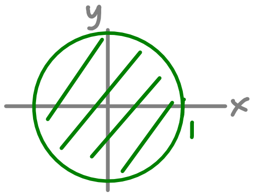
Vol, $dV = r\,d\theta\,dr\,dz$

Eg)



$$\Rightarrow \iiint_E f(r,\theta,z)\, r\,d\theta\,dr\,dz$$

Triple Integrals in Cylindrical Coordinates



Exercise: Set up a triple integral to evaluate the mass of the solid below which has a circular base. The solid has a uniform density of ρ = 2000 kg/m^3. Compare your answer with that given by the formula $\underbrace{\rho \times \pi R^2 H/2}_{\text{mass} = \rho \text{Vol}}$.

* Since ρ = const.

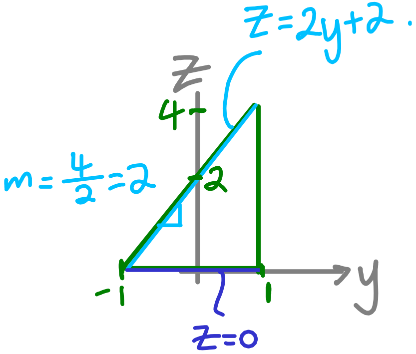
$$\text{mass} = \rho \,\text{Vol}$$

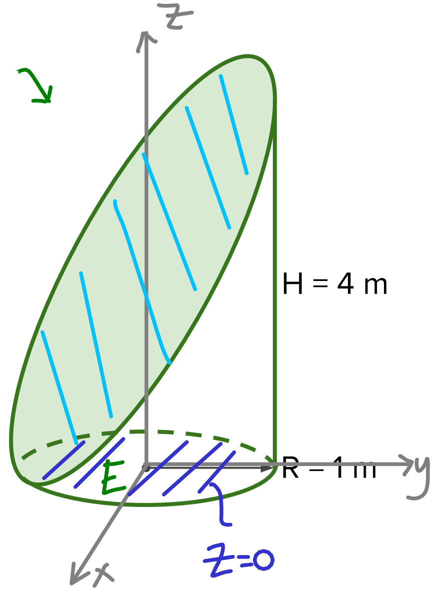
$$= 2000 \iiint_E 1\, dV = 2000 \int_0^{2\pi} \int_0^{1} \int_{z=0}^{z=2y+2} 1\, dz . r\,dr\,d\theta$$

$$= 2000 \int_0^{2\pi} \int_0^{1} (2y+2)\, r\,dr\,d\theta$$

$$= \overset{4000}{\cancel{2000}} \int_0^{2\pi} \int_0^{1} (\cancel{2} r^{2} \sin\theta + \cancel{2r})\cancel{r}\,dr\,d\theta$$

cylinder cut in half





ANS: Mass = 4000π kg.

$$= 4000\int_0^{2\pi}\left(\frac{r^3}{3}\sin\theta + \frac{r^2}{2}\right)\Bigg|_0^1 d\theta = 4000\int_0^{2\pi}\frac{1}{3}\sin\theta + \frac{1}{2}\,d\theta$$

$$= 4000\left(-\frac{1}{3}\cos\theta + \frac{1}{2}\theta\right)\Bigg|_0^{2\pi}$$

$$= 4000\left(\frac{1}{2}(2\pi)\right) = 4000\pi\text{ kg}\,//$$

same ✓

$$\text{mass} = \rho\text{Vol} = 2000\left(\frac{\pi(1^2)(4)}{2}\right) = 4000\pi\text{ kg}$$

Spherical Coordinate System

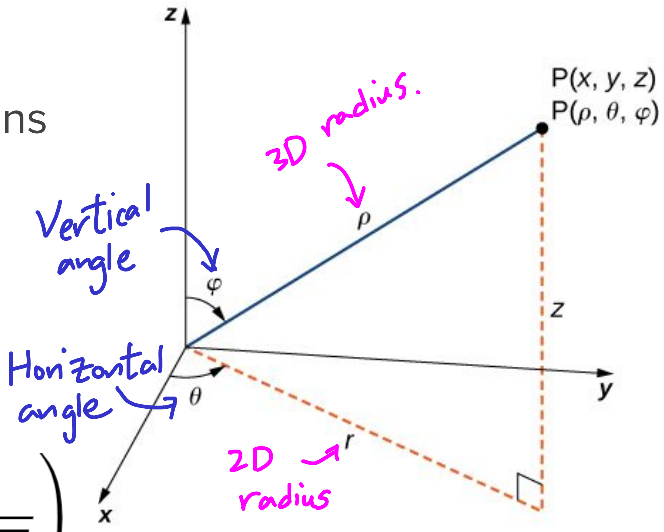
Another coordinate system commonly used is the spherical coordinate system as shown, where a point P at (x, y, z) in Cartesian can also be located using spherical coordinates (ρ, θ, φ).

From the figure, we can derive the transformation relations from spherical to Cartesian coordinates as

$$x = \rho \sin \varphi \cos \theta,\ y = \rho \sin \varphi \sin \theta,\ z = \rho \cos \varphi$$

And, from Cartesian to spherical coordinates, we have

$$\rho = \sqrt{x^2 + y^2 + z^2},\ \theta = \tan^{-1} \frac{y}{x},\ \varphi = \cos^{-1} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$



https://openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates

Triple Integrals in Spherical Coordinates

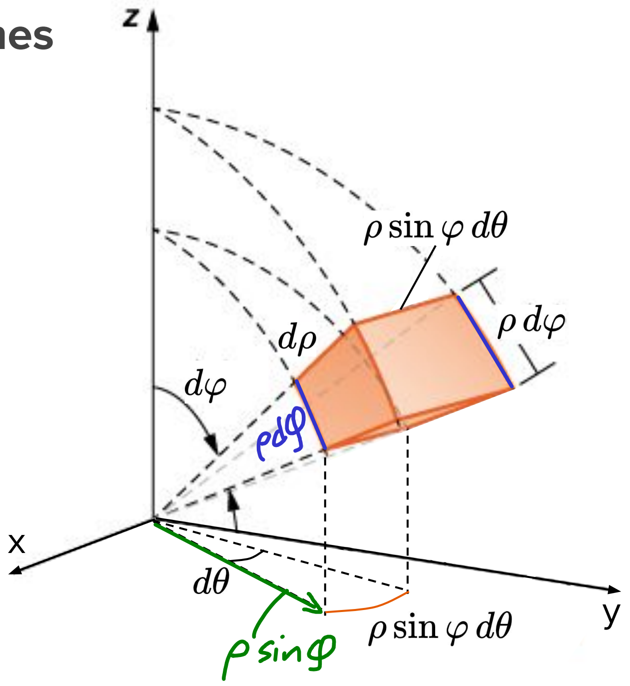
From the spherical volume element shown, we can observe that as dρ → 0, dθ → 0 and dφ → 0, the **element approaches a cuboid** that has the volume

$$dV = \rho \sin\varphi\, d\theta \cdot d\rho \cdot \rho\, d\varphi = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Hence, the triple integral of a function f(ρ, θ, φ) over a volume region E in spherical coordinates is

$$\iiint_E f(\rho, \theta, \varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

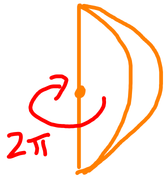
Generally, the use of spherical coordinates is suitable for regions of integration that is part of a sphere.

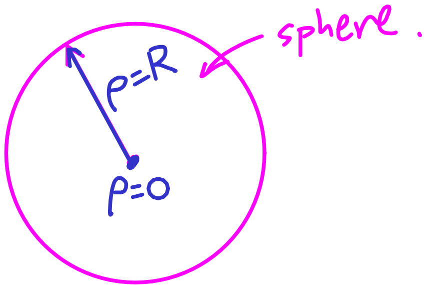


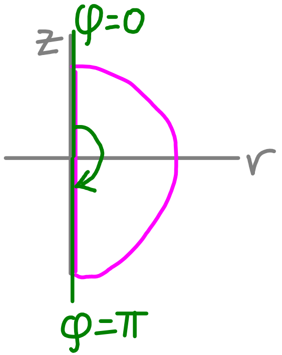
Triple Integrals in Spherical Coordinates

Example: Using a triple integral in spherical coordinates, show that the volume of a sphere of radius R is $4\pi R^3/3$.

$$\text{Vol} = \iiint_E 1\,dV = \underbrace{\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \sin\varphi \, d\rho \, d\varphi}_{} \, d\theta$$







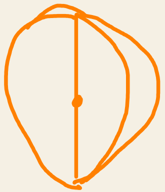
$$= \int_0^{2\pi} 1\,d\theta \int_0^{\pi} \sin\varphi\,d\varphi \int_0^{R} \rho^2\,d\rho = 2\pi\left(-\cos\varphi\right)\Big|_0^{\pi} \cdot \frac{\rho^3}{3}\Big|_0^{R}$$

$$= 2\pi\left(1-(-1)\right)\frac{R^3}{3} = 4\pi R^3/3 \quad \text{(shown)} \,/\!/$$

DIY

OR:

$$\text{Vol} = \iiint_E 1\,dV = \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=R} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

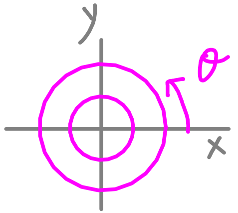


2 slices of "watermelon"

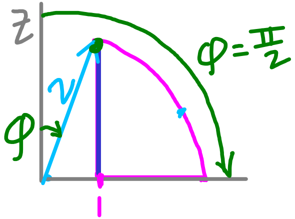
Triple Integrals in Spherical Coordinates

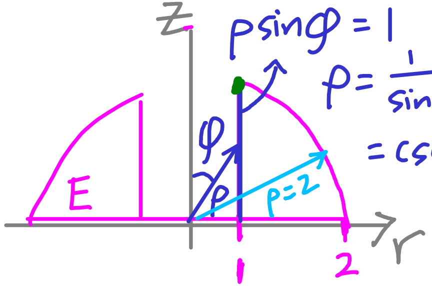
Exercise: Set up a triple integral in spherical coordinates to find the volume of the region E that is bounded by the sphere $x^2 + y^2 + z^2 = 4$, the cylinder $x^2 + y^2 = 1$ and the xy-plane from below as shown. Do not evaluate the integral.

$$Vol = \iiint_E dV = \int_0^{2\pi} \int_{\varphi=\frac{\pi}{6}}^{\varphi=\frac{\pi}{2}} \int_{\rho=\csc\varphi}^{\rho=2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \quad //$$



θ from 0 to 2π.

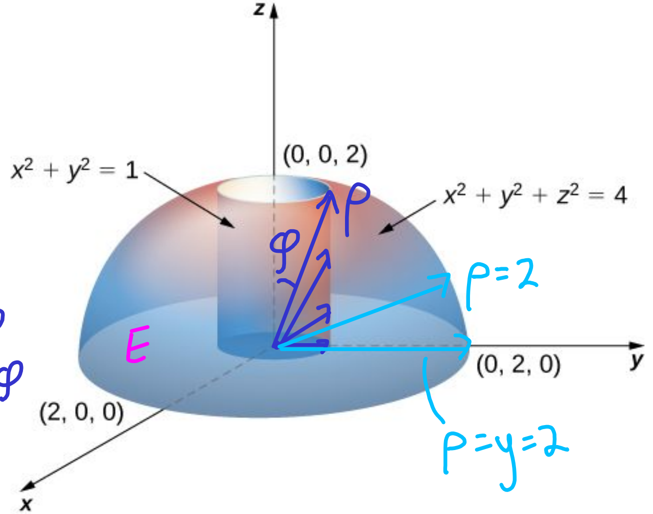




$\rho \sin\varphi = 1$

$\rho = \frac{1}{\sin\varphi}$

$= \csc\varphi$



$\sin\varphi = \frac{1}{2}$

$\varphi = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$

ANS: $\text{Volume} = \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_{\csc\varphi}^{2} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

https://openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates

Double & Triple Integrals

In summary, the double & triple integrals of a function in the various coordinate systems are

$$\text{Cartesian: } \iint_R f(x,y)\,dx\,dy$$

$$\text{Polar: } \iint_R f(r,\theta)\,rdr\,d\theta$$

$$\text{Cartesian: } \iiint_E f(x,y,z)\,dx\,dy\,dz$$

$$\text{Cylindrical: } \iiint_E f(r,\theta,z)\,rdr\,d\theta\,dz$$

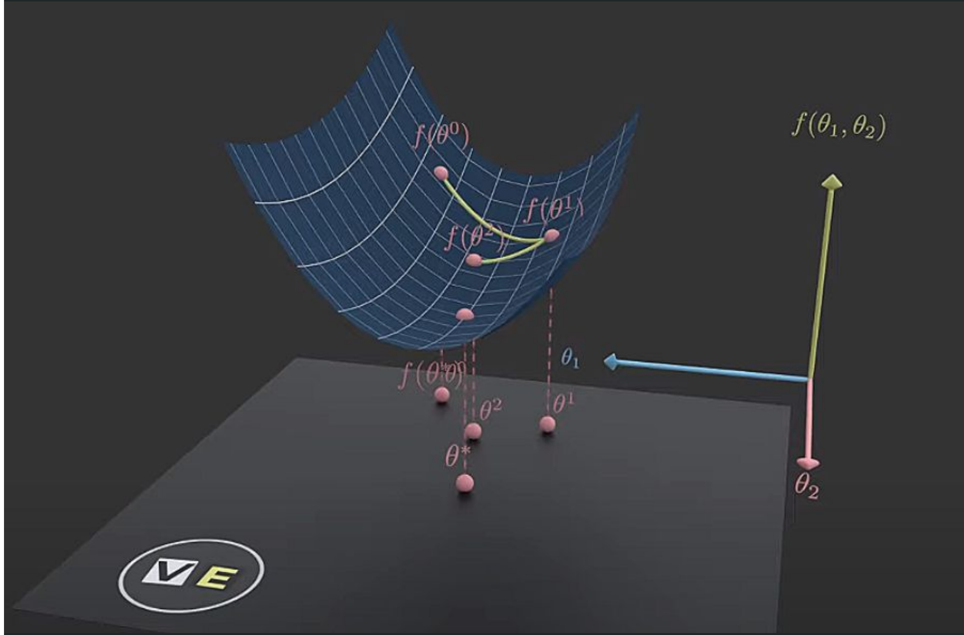
$$\text{Spherical: } \iiint_E f(\rho,\theta,\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

As long as the **integrand and the region of integration are the same**, Fubini's theorem applies and one can **convert between coordinate systems** when evaluating the integral.

End of Topic 4

*By taking steps in the **direction opposite to the gradient vector**, one can find the **local minimum** of a function. That's **machine learning**.*

A machine learns by minimizing an error function.



Gradient Descent in 3 minutes (Visually Explained)
https://youtu.be/qg4PchTECck