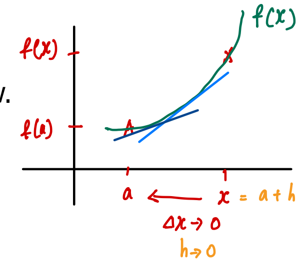
EDE1011 ENGINEERING MATHEMATICS 1

Tutorial 6

Derivatives

1. With the help of a graph, explain the meaning of the limit below.



$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right] \to \frac{\Delta f}{\Delta x}$$

2. Derive from first principles (using the limit operator) the derivative of each function below.

a) $f(x) = c$

b) $f(x) = ax^2 + bx + c$

c) $f(x) = 1/x$

d) $f(x) = \sqrt{ax-b}$

e) $f(x) = \cos x$

ANS: **a)** 0. **b)** 2ax + b. **c)** -1/x^2. **d)** $\dfrac{a}{2\sqrt{ax-b}}$. **e)** -sin x.

3. Differentiate the following functions.

a) $f(x) = 3x^4 - \dfrac{1}{\sqrt{1-2x}}$

b) $g(t) = t^3 \ln\left(t^2 - t\right) + \tan(\pi t)$

c) $h(\theta) = \dfrac{e^{\pi\theta}}{\sin(2\theta)}$

d) $z(y) = 7^y \log_7\left(y^{1/3}\right)$

e) $y(x) = \dfrac{2^{\pi x}}{\left(x^2+1\right)^3}$

f) $w(x) = xe^x \cos x$

ANS: **a)** $f'(x) = 12x^3 - \dfrac{1}{(1-2x)^{3/2}}$. **b)** $g'(t) = 3t^2 \ln\left(t^2-t\right) + \dfrac{t^2(2t-1)}{t-1} + \pi\sec^2(\pi t)$. **c)** $h'(\theta) = \dfrac{e^{\pi\theta}[\pi\sin(2\theta) - 2\cos(2\theta)]}{\sin^2(2\theta)}$. **d)** $z'(y) = \dfrac{7^y}{3\ln 7}\left[\dfrac{1}{y} + \ln 7 \cdot \ln y\right]$. **e)** $y'(x) = \dfrac{2^{\pi x}\left[\pi \ln 2\left(x^2+1\right) - 6x\right]}{\left(x^2+1\right)^4}$. **f)** $w'(x) = e^x(\cos x + x\cos x - x\sin x)$

4. Using implicit differentiation, prove the following derivatives.

a) $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

b) $\frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$

c) $\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

d) $\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x = \frac{-1}{1+x^2}$

5. Given that x is an implicit function of y in the implicit equation below, determine dx/dy.

$$x^2y + \ln x - \sin y = 0$$

ANS: $x' = \frac{\cos y - x^2}{2xy + \frac{1}{x}}$

6. Determine the equation of the tangent and normal lines on the curves defined below at the given point (0, 1). Verify your answers in Desmos.

a) $y^3 + y^2 - 5y - x^2 = -4.\ (2, 0)$

b) $xy^2 + \frac{x}{y} + y^3 = 1.\ (0, 1)$

ANS: **a)** $y = -\frac{4}{5}x + \frac{8}{5}$ (Tangent). $y = \frac{5}{4}x - \frac{5}{2}$ (Normal).

b) $y = -\frac{2}{3}x + 1$ (Tangent). $y = \frac{3}{2}x + 1$ (Normal).

7. Determine the first and second derivatives for the implicit equations below, where y is an implicit function of x.

a) $y^2 + y - x = 0$

b) $xye^y = 1$

ANS: **a)** $y' = \frac{1}{2y+1}$, $y'' = \frac{-2}{(2y+1)^3}$. **b)** $y' = \frac{-y}{x(y+1)}$, $y'' = \frac{y}{x^2}\left[\frac{1}{1+y} + \frac{1}{(1+y)^3}\right]$

8. Using logarithmic differentiation, evaluate the derivatives of the functions below.

a) $y(x) = x^{\cot x}$

b) $f(x) = \left(\dfrac{x}{1+x}\right)^x$

c) $g(x) = f(x^2)^{\sqrt{x}}$

d) $h(x) = \dfrac{x^2 e^{3x} \tan(4x)}{\sqrt{x+1}}$

ANS: **a)** $f'(x) = x^{\cot x}\left(\dfrac{\cot x}{x} - \csc^2 x \ln x\right)$. **b)** $f'(x) = \left(\dfrac{x}{1+x}\right)^x \left[\ln\left(\dfrac{x}{1+x}\right) + \dfrac{1}{1+x}\right]$.

c) $g'(x) = f(x^2)^{\sqrt{x}} \left[\dfrac{\ln f(x^2)}{2\sqrt{x}} + \dfrac{2x^{3/2} f'(x^2)}{f(x^2)}\right]$.

d) $h'(x) = \dfrac{x^2 e^{3x} \tan(4x)}{\sqrt{x+1}} \left[\dfrac{2}{x} + 3 + \dfrac{8}{\sin(8x)} - \dfrac{1}{2(x+1)}\right]$.

9. Given the following information, determine the derivatives below.

$$f(-4) = 3,\ \ f(1) = 0,\ \ f(2) = 1,\ \ f(3) = 2,$$
$$f'(-4) = 1,\ \ f'(1) = 0,\ \ f'(2) = 3,\ \ f'(3) = -1$$

$$g(-4) = 9,\ \ g(1) = 3,\ \ g(2) = -2,\ \ g(3) = 0,$$
$$g'(-4) = -3,\ \ g'(1) = 1/2,\ \ g'(2) = 6,\ \ g'(3) = -4$$

a) $p'(3)$ where $p(x) = 3f(x) - 2g(x)$

b) $q'(2)$ where $q(x) = f(x)/g(x)$

c) $r'(2)$ where $r(x) = g(3f(x))$

ANS: **a)** 5. **b)** -3. **c)** -36.

10. Determine if each function below is differentiable in $\mathbb{R}$. If not, specify where it is not differentiable.

a) $f(x) = x|x|$

b) $f(x) = e^{|2x+1|}$

c) $f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x^3 \sin x + 1, & x \geq 0 \end{cases}$

ANS: **a)** Yes. **b)** No. At x = -½. **c)** Yes.

11. Determine the values of parameters of a, b & c in order for the function below to be differentiable in $\mathbb{R}$.

$$f(x) = \begin{cases} ae^{-x} + b, & x < 1 \\ 2, & x = 1 \\ c/x & x > 1 \end{cases}$$

ANS: a = 2e, b = 0, c = 2.

12. **Mathematical Modelling**: Marginal Cost of Production

A cloth manufacturer, Muthu, produces bolts of a fabric with a fixed width. From his experience, he models the cost ($) of producing x yards of this fabric to be

$$C(x) = \begin{cases} 2x + 50, & 0 \le x \le 100 \\ a\sqrt{x} + b, & x > 100 \end{cases}$$

a) From an economic perspective, explain why possibly the cost function is linear when x is smaller and transits to a root function when x is bigger.

b) Muthu thinks that his cost model should be continuous and smooth. Determine parameters a & b to fulfil this criteria.

c) Determine $\dfrac{C(101) - C(81)}{101 - 81}$ and state its unit. What does it represent in layman?

d) State the unit of C'(x) and explain its meaning. What is C'(100) and what does it represent in layman?

e) Without calculation, is C'(100) or C'(1000) bigger and why? Do you think this trend is always true? Explain.

ANS: **b)** a = 40, b = -150. **c)** $2.1 per yard. Average increase in cost per yard when production increases from 81 to 101 yards of fabric. **d)** $ per yard. Additional cost of producing the next yard when production is at x yards. C'(100) = $2 per yard.

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-1/pages/3-2-the-derivative-as-a-function
2) https://openstax.org/books/calculus-volume-1/pages/3-3-differentiation-rules
3) https://openstax.org/books/calculus-volume-1/pages/3-6-the-chain-rule
4) https://openstax.org/books/calculus-volume-1/pages/3-8-implicit-differentiation
5) https://openstax.org/books/calculus-volume-1/pages/3-review-exercises

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End of Tutorial 6

(Email to youliangzheng@gmail.com for assistance.)