1.

$$|u-v|^2 = |u|^2 + |v|^2 - 2|u||v|\cos\theta$$

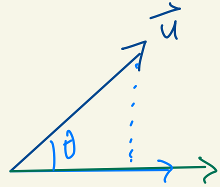
$$(u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2 = u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - 2|u||v|\cos\theta$$

$$\cancel{u_1^2} - 2u_1v_1 + \cancel{v_1^2} + \cancel{u_2^2} - 2u_2v_2 + \cancel{v_2^2} + \cancel{u_3^2} - 2u_3v_3 + \cancel{v_3^2}$$

$$= \cancel{u_1^2} + \cancel{u_2^2} + \cancel{u_3^2} + \cancel{v_1^2} + \cancel{v_2^2} + \cancel{v_3^2} - 2|u||v|\cos\theta$$

$$-\cancel{2}\sum_{i=1}^{3} u_i v_i = -\cancel{2}|u||v|\cos\theta = u.v$$ (shown)

3. a) $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad v = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$



a) $\text{proj}_v \vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|v|^2}\vec{V} = \dfrac{-2+2-3}{4+1+1}\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

$= -\dfrac{1}{2}\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

$\vec{V}$

$\text{proj}_v \vec{u} = \underbrace{|\vec{u}|\cos\theta}_{\text{length}} \; \underbrace{\dfrac{\vec{V}}{|\vec{V}|}}_{\text{direction}}$

$= \dfrac{|V||u|\cos\theta}{|V|} \cdot \dfrac{\vec{V}}{|V|}$

$= \underbrace{\dfrac{\vec{u}\cdot\vec{v}}{|v|^2}\cdot\vec{V}}_{\text{more Convenient}}$

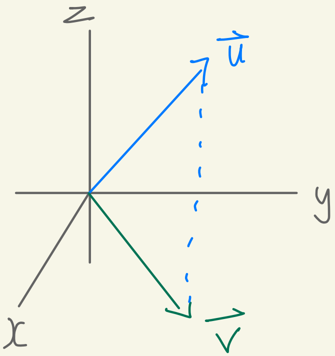
b) $u = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \quad v = \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix}$

$\text{proj}_v \vec{u} = \dfrac{-9-3+12}{9+9+36}\begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix}$

$= 0$

as $\vec{u}$ is perpendicular to $\vec{V}$

c) $u = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \quad v = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$



$\text{proj}_v \vec{u} = \vec{V}$ since the projection of $\vec{u}$ onto the xy-plane gives $\vec{V}$.

4. $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $v = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ $w = \begin{pmatrix} 5 \\ 5 \\ -5 \end{pmatrix}$

$$|u| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$|v| = \sqrt{(-2)^2+1^2+(-1)^2} = \sqrt{6}$$

a) $$u \cdot v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = -2+2-3 = -3$$

$$\theta = \cos^{-1}\left(\frac{-3}{|u||v|}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{14} \cdot \sqrt{6}}\right) = 1.9 \text{rads}$$

b)

Cross produce of $u \times v$

$$\rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2(-1) - 3(1) \\ -[1(-1) - 3(-2)] \\ 1 - 2(-2) \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ -5 \\ 5 \end{pmatrix}$$

5 a)

a. $\mathbf{u} = [-1\ \ -1\ \ 1]^T$, $\mathbf{v} = [2\ \ 1\ \ 5]^T$

b. $\mathbf{u} = [3\ \ 1\ \ -4]^T$, $\mathbf{v} = [-2\ \ 2\ \ -1]^T$

$$\vec{u} \times \vec{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -5 - 1 \\ -[-1(5) - 1(2)] \\ -1 - (-2) \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ 1 \end{pmatrix}$$

area $= \sqrt{86}$

b)

$$\vec{u} \times \vec{v} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 - (-8) \\ -[3(-1) - (-4)(-2)] \\ 6 - (-2) \end{pmatrix} = \begin{pmatrix} 7 \\ -11 \\ 8 \end{pmatrix}$$

area $= \sqrt{234}$

6)

$$\frac{x-1}{3} = y+2 = \frac{5-z}{4}$$

$$r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ mt+c \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ c \end{bmatrix} + t\begin{bmatrix} 1 \\ m \end{bmatrix}$$

Let $y = t$, $x = 3t+7$, $z = -4t-3$

$$x - 1 = 3t+6$$

$$x = 3t+7$$

$$t+2 = \frac{5-z}{4}$$

$$4t+8 = 5-z$$

$$z = -4t-3$$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 3t+7 \\ t \\ -4t-3 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

7)

8 $P_1: 2x+z=4$ $\qquad P_2: x-y+z=3$

$z=4-2x$

$(2-\frac{z}{2})-y+z=3$

$2x=4-z$

$2-y+\frac{z}{2}=3$

$x=2-\frac{z}{2}$

$\frac{z}{2}=1+y$

$z=2+2y$

$z=2y+2=4-2x$

$r(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}=\begin{bmatrix} t \\ mt+c \end{bmatrix}$

$=\begin{bmatrix} 0 \\ c \end{bmatrix}+t\begin{bmatrix} 1 \\ m \end{bmatrix}$

Let $z=2t$

$4-2x=2t$ $\qquad 2+2y=2t$

$-2x=2t-4$ $\qquad y=\frac{2t-2}{2}$

$=t-1$

$x=2-t$

$\vec{r}(t)=\begin{bmatrix} 2-t \\ t-1 \\ 2t \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}+t\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ $\begin{array}{l} \leftarrow x=2-t \\ \leftarrow y=-1+t \end{array}$

$4-2(2-t)=4-4+2t$

$=2t=z$

(Checked)

$2+2(-1+t)=2-2+2t$

$=2t=z$

(Checked)

9.

$$\vec{N} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$\vec{N} \cdot (\vec{r} - \vec{r_0}) = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x-5 \\ y-1 \\ z-3 \end{pmatrix} = 0$$

$$\Rightarrow (x-5) - 4(y-1) + 2(z-3) = 0$$

$$x - 5 - 4y + 4 + 2z - 6 = 0$$

$$x - 4y + 2z - 7 = 0$$

$$x - 4y + 2z = 7$$

10

$$r_1(s,t) = \begin{pmatrix}1\\1\\1\end{pmatrix} + s\begin{pmatrix}0\\1\\2\end{pmatrix} + t\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\vec{N} = \begin{pmatrix}0\\1\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}1(-1) - 2(1)\\ -[0(-1) - 2(2)]\\ 0(1) - 1(2)\end{pmatrix} = \begin{pmatrix}-3\\4\\-2\end{pmatrix}$$

To get cartesian eqn:

$$\vec{N} \cdot (\vec{r} - \vec{r_0}) = \begin{pmatrix}-3\\4\\-2\end{pmatrix} \cdot \begin{pmatrix}x-1\\y-1\\z-1\end{pmatrix} = 0$$

$$\Rightarrow \quad -3(x-1) + 4(y-1) - 2(z-1) = 0$$

$$-3x + 3 + 4y - 4 - 2z + 2 = 0$$

$$-3x + 4y - 2z + 1 = 0$$

$$3x - 4y + 2z = 1$$

$$\mathbf{r_2}(s,t) = \begin{bmatrix}1\\2\\3\end{bmatrix} + s\begin{bmatrix}2\\-2\\-7\end{bmatrix} + t\begin{bmatrix}6\\2\\-5\end{bmatrix} \begin{matrix} \leftarrow x = 1+2s+6t \\ \leftarrow y = 2-2s+2t \\ \leftarrow z = 3-7s-5t \end{matrix} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} \begin{matrix}\text{sub into} \\ 3x-4y+2z = 1\end{matrix}$$

Show LHS = RHS

$$3(1+2s+6t) - 4(2-2s+2t) + 2(3-7s-5t) = 1$$

$$3 + 6s + 18t - 8 + 8s - 8t + 6 - 14s - 10t = 1$$

$$1 + 0s + 0t = 1$$

(shown)

11. Show that the line given by

$$\mathbf{X}(t) = \begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}2\\1\\-1\end{bmatrix} \begin{matrix}\leftarrow x = 2t+1\\ \leftarrow y = t+1\\ \leftarrow z = -t+1\end{matrix}$$

Solve with

intersects the line given by

$$x = 5, \quad y - 4 = \frac{z-1}{2}$$

Determine the point of intersection.

ANS: (5, 3 -1).

$$x = 2t+1 = 5 \rightarrow t = 2$$

$$y - 4 = t + 1 - 4 = \frac{1}{2}(z-1) = \frac{1}{2}(-t+1-1)$$

$$t - 3 = -\frac{t}{2}$$

$$2t + t = 6$$

$$t = 2$$

Same, so

intersection at

$$x(2) = \begin{pmatrix}1+4\\1+2\\1-2\end{pmatrix}$$

$$= \begin{pmatrix}5\\3\\1\end{pmatrix}$$

12. Show that the line given by:

$$\mathbf{X}(t) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \begin{matrix} \leftarrow x = 2 - t \\ \leftarrow y = 3 + 4t \\ \leftarrow z = 1 + 2t \end{matrix}$$

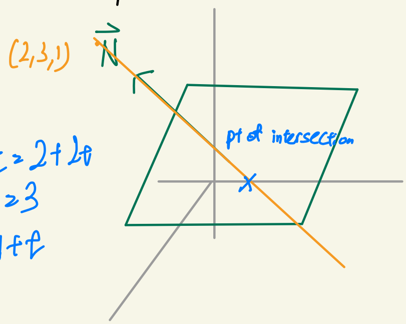
does not intersect the plane 2x + z = 9. Then, determine the equation of a line through the point (2,3,1) which is parallel to the normal vector of the plane and determine the point where it intersects the plane.

ANS: $\mathbf{r}(t) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Intersect at (18/5, 3, 9/5).

$$2x + z = 9$$

LHS : $2(2-t) + 1 + 2t = 5 \neq 9$ (RHS)

∴ Line does not intersect the plane



$$\vec{N} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \quad \vec{r}(t) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} \leftarrow x = 2 + 2t \\ \leftarrow y = 3 \\ \leftarrow 1 + t \end{matrix}$$

To find pt of intersection, solve

$\vec{r}(t)$ with $2x + z = 9$

$$2(2 + 2t) + 1 + t = 9$$

$$5t + 5 = 9 \rightarrow t = \frac{4}{5}$$

$$\Rightarrow \text{pt of intersection at } \vec{r}\left(\frac{4}{5}\right) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \frac{4}{5} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{18}{5} \\ 3 \\ \frac{9}{5} \end{pmatrix}$$

13. A linear combination of vectors, **b**, is defined by

$$\mathbf{b} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3}$$

where c_i are scalars.

a. Draw a graphical representation of this linear combination.
b. Given that the vectors $\mathbf{u}_i$ and **b** are prescribed, show that finding the unknown scalars is equivalent to solving an SLE in the form below. Define matrix A and vector **v**.

$$A\mathbf{v} = \mathbf{b}$$

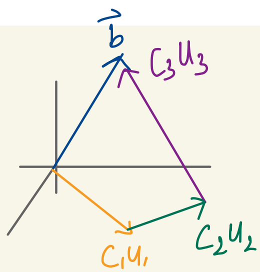
c. For the SLE in (b), what is the condition necessary of vectors $\mathbf{u}_i$ if there is to be a solution given any constant vector **b**? Explain.

ANS: **b)** $A = [\mathbf{u_1} \quad \mathbf{u_2} \quad \mathbf{u_3}],\ \mathbf{v} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. **c)** Vectors $\mathbf{u}_i$ must be linearly independent.

b) $$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = C_1\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} + C_2\begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} + C_3\begin{pmatrix} u_{31} \\ u_{32} \\ u_{33} \end{pmatrix}$$

$$\vec{b} = \underbrace{\begin{pmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}}_{\vec{v}}$$

→ Solve SLE for $\vec{v}$ to get C_1, C_2 & C_3



c) Vectors $\vec{u}_1$, $\vec{u}_2$ & $\vec{u}_3$ have to be L.I. in order to span $\mathbb{R}^3$ and hence able to denote any general vector $\vec{b}$.

Or: Vectors $\vec{u}_1$, $\vec{u}_2$ & $\vec{u}_3$ must form a basis for $\mathbb{R}^3$