EDE1012 MATHEMATICS 2

Tutorial 5

Vector Calculus I

1. (https://openstax.org/books/calculus-volume-3/pages/6-1-vector-fields)
   Without using a graphing tool, match the vector fields below with their graphs. Evaluate the Jocobian and divergence of each vector field.

   a ↑ correction

   (b) (a) (c)

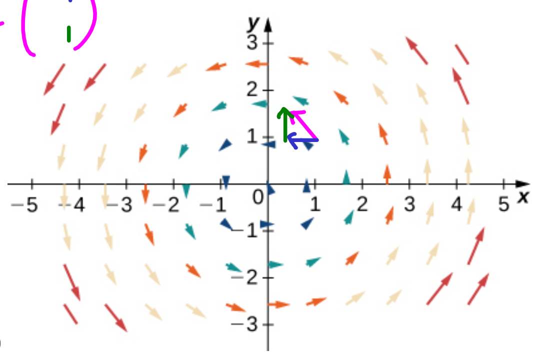
   [redacted]

$\vec{F}(1,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

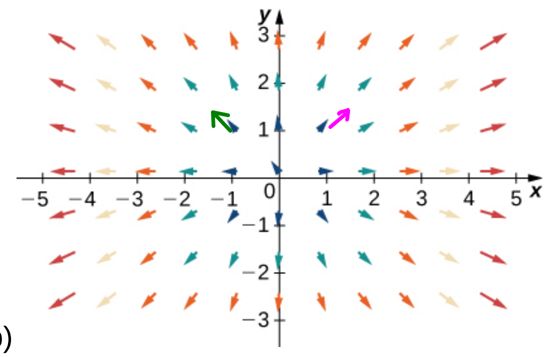
$\vec{F}(-1,1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$G(1,1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

⋮



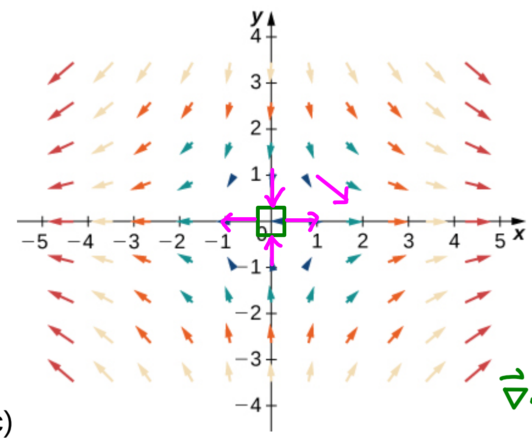
a)



b)

$H(1,1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

⋮



c)

$H(x,y) = \begin{pmatrix} x \\ -y \end{pmatrix}$

$$J_{\vec{H}}(x,y) = \begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \uparrow \ \ \ \uparrow \\ \partial_x \ \ \ \partial_y \end{array} /\!/$$

$$\vec{\nabla} \cdot \vec{H} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} x \\ -y \end{pmatrix} = 1 - 1 = 0 /\!/$$

ANS: **a)** $\mathbf{G}(x,y)$. $\mathbf{J_G} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $\nabla \cdot \mathbf{G} = 0$. **b)** $\mathbf{F}(x,y)$. $\mathbf{J_F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\nabla \cdot \mathbf{F} = 2$.

c) $\mathbf{H}(x,y)$. $\mathbf{J_H} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. $\nabla \cdot \mathbf{H} = 0$.

2. Evaluate the Jacobian, divergence and curl of the velocity field below. Then, determine them at the point (1, 2, 3).

DIY

[redacted]

ANS: $$\mathbf{J_V} = \begin{bmatrix} 2xz & 0 & x^2 \\ 0 & e^{2z} & 2ye^{2z} \\ yz & xz & xy \end{bmatrix}. \ \nabla \cdot \mathbf{V} = 2xz + e^{2z} + xy. \ \nabla \times \mathbf{V} = \begin{bmatrix} xz - 2ye^{2z} \\ x^2 - yz \\ 0 \end{bmatrix}.$$

$$\text{At } (1,2,3): \mathbf{J_V} = \begin{bmatrix} 6 & 0 & 1 \\ 0 & e^6 & 4e^6 \\ 6 & 3 & 2 \end{bmatrix}. \ \nabla \cdot \mathbf{V} = 8 + e^6. \ \nabla \times \mathbf{V} = \begin{bmatrix} 3 - 4e^6 \\ -5 \\ 0 \end{bmatrix}.$$

3. Using a vector field graphing tool (https://www.geogebra.org/m/QPE4PaDZ), plot the vector field below and evaluate the curl vector. Explain why the curl is the zero vector despite the vector field appearing 'rotational'.

[redacted]

ANS: $\nabla \times \mathbf{F} = \mathbf{0}$.

4. Determine if each vector field below is a gradient field (conservative). If so, evaluate the scalar potential function E such that **F** = ∇E.

$\nabla \times \vec{F} = \vec{\nabla} \times \vec{\nabla} E = \vec{0}.$

a) [redacted]

DIY b) [redacted]

c) [redacted]

ANS: **a)** Yes. $E(x, y) = x^2 \sin y + c$. **b)** Yes. $E(x, y, z) = e^x \sin y + z^3 + 2z + c$. **c)** No.

3. Using a vector field graphing tool (https://www.geogebra.org/m/QPE4PaDZ), plot the vector field below and evaluate the curl vector. Explain why the curl is the zero vector despite the vector field appearing 'rotational'.

$$\mathbf{F}(x,y,z) = \left[\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right]^T$$

ANS: $\nabla \times \mathbf{F} = \mathbf{0}$.

$$\vec{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rightarrow \vec{\nabla}\times\vec{F} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix} = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

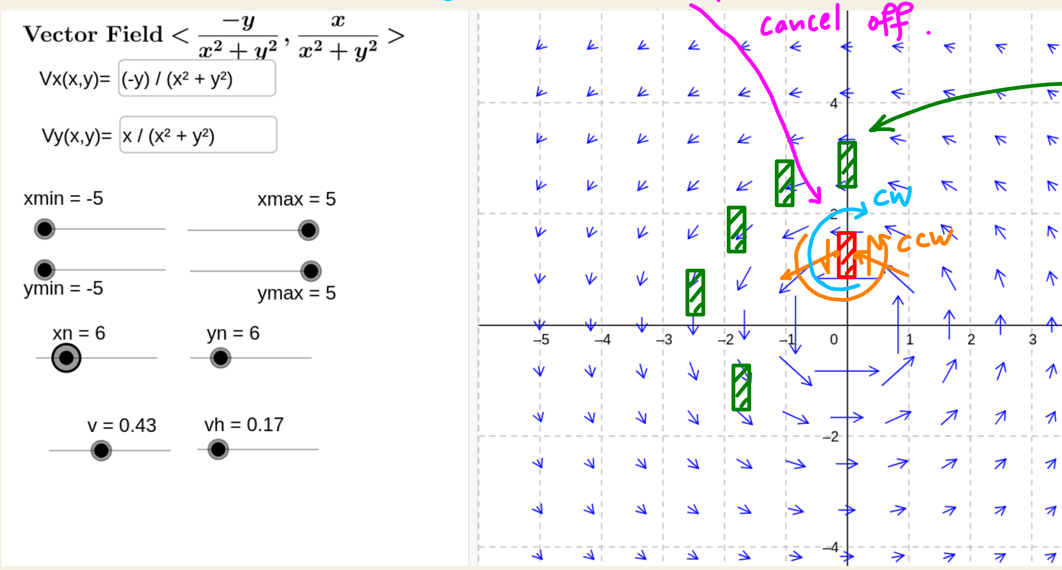
Curl, $$\vec{\nabla}\times\vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0-0 \\ -(0-0) \\ \frac{(x^2+y^2)(1)-(2x)(x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1)-(2y)(y)}{(x^2+y^2)^2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{-x^2+y^2+x^2-y^2}{(x^2+y^2)^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} //$$

$$\vec{\nabla}\times\vec{F} = \Big(\underbrace{\frac{\partial f_2}{\partial x}}_{\text{ccw rotation}} - \underbrace{\frac{\partial f_1}{\partial y}}_{\text{cw rotation.}}\Big)\hat{k} = \vec{0}$$

$\oplus$ $\oplus$

Curl components cancel off.



Orientation of object does not change (object does not rotate.)

Therefore, the curl is a point property, which means about any point, is the field rotating "an object"? If it is, then the curl is not zero.

4. Determine if each vector field below is a gradient field (conservative). If so, evaluate the scalar potential function E such that **F** = ∇E.

a) $\mathbf{F}(x,y) = [\underbrace{2x\sin y}_{f_1}, \underbrace{x^2\cos y}_{f_2}]^T$

$\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{\nabla}E = \vec{0}$.

Check: $$\vec{\nabla}\times\vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} 2x\sin y \\ x^2\cos y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x\cos y - 2x\cos y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} //$$

Insert a 0 z-comp for 2D fields.

OR $\vec{\nabla}\times\vec{F} = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$ (for 2D fields only)

$\therefore$ $\vec{F}$ is conservative $\rightarrow \vec{F} = \vec{\nabla}E$.

Evaluate $E(x,y)$ (DIY).

5. A force field (in Newtons) is defined by

[redacted]

where coordinates x and y are in meters.

DIY a) Show that the force field is conservative.

DIY b) Evaluate a scalar potential E(x,y) such that **F** = ∇E.

c) Calculate the work done required to move an object subjected to the force field from point (0, 1) to point (1, 1), along the curve [redacted].

$E(1,1) - E(0,1) = \cdots = 1J$ //

ANS: **a)** Yes. **b)** $E(x,y) = \dfrac{x^2}{2y^2} + \dfrac{x^2}{2} + \dfrac{1}{2y^2} + c.$ **c)** Work done = 1 J.

6. A vector field shown below contains scalar functions f(x), g(y) and h(z) that are differentiable.

[redacted]

a) Determine if the vector field is conservative. If so, evaluate the scalar potential E such that ∇E = **V**.

b) Evaluate the line integral below, where C is any path from (x_0, y_0, z_0) to (x_1, y_1, z_1).

[redacted]

ANS: **a)** Yes. $E(x,y,z) = xy + xz + yz + F(x) + G(y) + H(z)$, where F(x), G(y) and H(z) are antiderivatives of f(x), g(y) and h(z).

b) $$L = x_1y_1 + x_1z_1 + y_1z_1 + F(x_1) + G(y_1) + H(z_1) - x_0y_0 - x_0z_0 - y_0z_0 - F(x_0) - G(y_0) - H(z_0).$$

~~6.~~ A vector field shown below contains scalar functions f(x), g(y) and h(z) that are differentiable.

$$\mathbf{V}(x,y,z) = \begin{bmatrix} f(x)+y+z \\ g(y)+x+z \\ h(z)+x+y \end{bmatrix} \begin{matrix} \leftarrow E_x \\ \leftarrow E_y \\ \leftarrow E_z \end{matrix}$$

a) Determine if the vector field is conservative. If so, evaluate the scalar potential E such that ∇E = **V**.

b) Evaluate the line integral below, where C is any path from (x_0, y_0, z_0) to (x_1, y_1, z_1).

$$L = \int_C \mathbf{V} \cdot \mathbf{dr}$$

ANS: **a)** Yes. $E(x,y,z) = xy + xz + yz + F(x) + G(y) + H(z)$, where F(x), G(y) and H(z) are antiderivatives of f(x), g(y) and h(z).

b) $L = x_1y_1 + x_1z_1 + y_1z_1 + F(x_1) + G(y_1) + H(z_1) - x_0y_0 - x_0z_0 - y_0z_0 - F(x_0) - G(y_0) - H(z_0)$.

$L = E(x_1, y_1, z_1) - E(x_0, y_0, z_0)$

a) $\vec{\nabla} \times \vec{V} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{bmatrix} f(x)+y+z \\ g(y)+x+z \\ h(z)+x+y \end{bmatrix}$

$= \begin{pmatrix} 1-1 \\ -(1-1) \\ 1-1 \end{pmatrix} = \vec{0} \rightarrow \vec{V}$ is conservative.

$\vec{V} = \vec{\nabla} E$

$E(x,y,z) = \int f(x) + y + z \, dx$

$= F(x) + xy + xz + P(y,z)$

$E_y = x + P_y \rightarrow$ Compare with E_y,

$\rightarrow P_y = g(y) + z . \rightarrow P = \int g(y) + z \, dy$

$= G(y) + zy + q(z)$

$E(x,y,z) = F(x) + xy + xz + G(y) + zy + q(z)$

$E_z = x + y + q'(z) \rightarrow$ compare with E_z,

$\Rightarrow q'(z) = h(z) \rightarrow q(z) = \int h(z) dz = H(z) + C$.

$E(x,y,z) = F(x) + xy + xz + G(y) + zy + H(z) + C$ //

7. (https://openstax.org/books/calculus-volume-3/pages/6-2-line-integrals)
Evaluate the line integral of each function defined below over the path given.

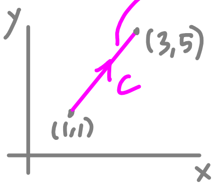
a) [redacted] over the curve C that is the right half of circle [redacted] and traversed in the clockwise direction.

b) [redacted] over the curve C that is the arc of curve [redacted] from (1, 0) to (e, 1).

c) [redacted] over the curve C that is the helix [redacted] from t = 0 to t = T.

ANS: **a)** -8192/5. **b)** 7.157. c) $\sqrt{2}\tan^{-1}T$.

8. Evaluate the line integral of the vector field below over the straight line path from (1, 1) to (3, 5).



$y-y_1=m(x-x_1)$

$m=\frac{5-1}{3-1}=\frac{4}{2}=2.$

$y-1=2(x-1)$

$y=2x-1.$

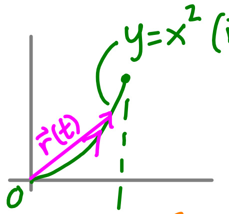
[redacted]

Parametrize : let $x=t$, $\vec{r}(t)=\begin{pmatrix} t \\ 2t-1 \end{pmatrix}$.

DIY.

ANS: -92/3.

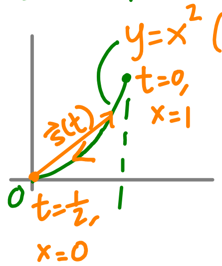
9. Consider a vector field **F**(x,y) over two paths **r**(t) and **s**(t) given below.



[redacted]

$x=1-2t$

$y=(2t-1)(2t-1)=(2t-1)^2=(-x)^2=x^2.$



a) Evaluate the line integrals of **F** over each of the two paths.

b) What are the Cartesian equations and directions of path **r** and **s**? Hence explain the values obtained in (a).

ANS: **a)** Path **r**: ½. Path **s**: -½. **b)** Path **r**: y = x^2, (0, 0) to (1, 1).
Path **s**: y = x^2, (1, 1) to (0, 0).

Since r(t) and s(t) describes the same path but with opposite directions, the path integrals are negative of each other.

$$L_s=\int_0^{1/2}\underbrace{\vec{F}(\vec{s}(t))\cdot\vec{s}\,'(t)}\,dt$$

$$\begin{bmatrix}(1-2t)^2+4t^2-4t+1 \\ 4t^2-4t+1-(1-2t)\end{bmatrix}\cdot\begin{pmatrix}-2 \\ 8t-4\end{pmatrix}=\begin{pmatrix}8t^2-8t+2 \\ 4t^2-2t\end{pmatrix}\cdot\begin{pmatrix}-2 \\ 8t-4\end{pmatrix}$$

$$=-16t^2+16t-4+32t^3-16t^2-16t^2+8t$$

$$=32t^3-48t^2+24t-4$$

$$L_s = \int_0^{1/2} 32t^3 - 48t^2 + 24t - 4 \, dt = \left[8t^4 - 16t^3 + 12t^2 - 4t \right]_0^{1/2}$$

$$= \frac{8}{16} - \frac{16}{8} + \frac{12}{4} - \frac{4}{2} = \frac{1}{2} - 2 + 3 - 2$$

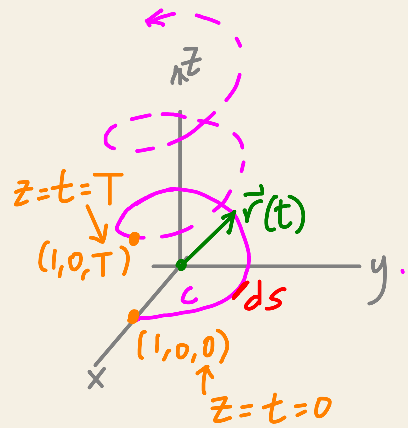
$$= -\frac{1}{2} \quad //$$

Q7c) $F(x,y,z) = 1/(x^2+y^2+z^2)$ over the curve C that is the helix $x = \cos t,\ y = \sin t,\ z = t$, from t = 0 to t = T.

ANS: **a)** -8192/5. **b)** 7.157. c) $\sqrt{2}\tan^{-1} T$.

$$\mathbf{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix} \leftarrow z$$

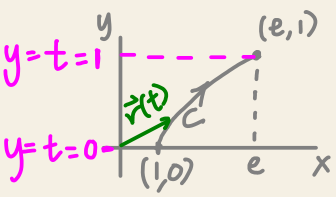
$$r'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$



$$L = \int_C F\,ds = \int_0^T \frac{1}{\underbrace{\cos^2 t}_{x^2} + \underbrace{\sin^2 t}_{y^2} + \underbrace{t^2}_{z^2}} \overbrace{\sqrt{\sin^2 t + \cos^2 t + 1}\ dt}^{|r'(t)|\,dt}$$

$$= \int_0^T \frac{1}{1+t^2}\sqrt{2}\,dt = \sqrt{2}\left(\tan^{-1} t\right)\Big|_0^T = \sqrt{2}\tan^{-1} T \ //$$

Q7b) $F(x,y) = xe^y$ over the curve C that is the arc of curve $x = e^y$ from (1, 0) to (e, 1).



Parameterize by letting $y=t$,

$$\vec{r}(t) = \begin{pmatrix} e^t \\ t \end{pmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix} \rightarrow \vec{r}'(t) = \begin{pmatrix} e^t \\ 1 \end{pmatrix} \rightarrow |\vec{r}'(t)| = \sqrt{e^{2t}+1}\,.$$

$$L = \int_0^1 F(\vec{r}(t))\,|\vec{r}'(t)|\,dt$$

$$= \int_0^1 e^t \cdot e^t \sqrt{e^{2t}+1}\,dt = \int_0^1 e^{2t}\sqrt{e^{2t}+1}\,dt$$

Let $u = e^{2t}+1 \rightarrow du = 2e^{2t}\,dt$

$$dt = \frac{1}{2e^{2t}}\,du.$$

$$= \int_{u_1}^{u_2} \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_{u_1}^{u_2}$$

$$= \frac{1}{3}(e^{2t}+1)^{3/2}\Big|_0^1$$

$$= \frac{1}{3}\left[(e^2+1)^{3/2} - 2^{3/2}\right] \ //$$

10. For a radial vector field **F**(x, y, z) = $[x, y, z]^T$, show that its line integral over any path that is on a sphere

[redacted]

is always zero. Sketch a graph and explain why.

DIY

11. Verify Green's theorem for **F**(x, y) below over the semicircular region D given by $x^2 + y^2 \le R^2$, $y \ge 0$.

[redacted]

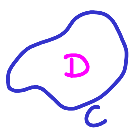
ANS: 0.

12. Using Green's theorem, evaluate the line integral for **F**(x,y) below, C is the boundary of the square with vertices (0, 0), (1, 0), (0, 1) and (1, 1), oriented clockwise.

[redacted]

ANS: -2.

13. Given that C is any closed path in $\mathbb{R}^2$ oriented counterclockwise, show that the line integral below is independent of the path C and only dependent on the area enclosed by C.



[redacted] DIY $= \cdots = 3\iint_D dA$

$= 3 \times$ Area enclosed by C. (shown).

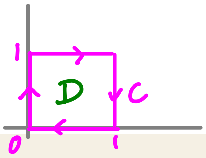
For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-3/pages/6-1-vector-fields
2) https://openstax.org/books/calculus-volume-3/pages/6-2-line-integrals
3) https://openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields
4) https://openstax.org/books/calculus-volume-3/pages/6-4-greens-theorem
5) https://openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl

End of Tutorial 5

(Email to youliangzheng@gmail.com for assistance.)

12. Using Green's theorem, evaluate the line integral for **F**(x,y) below, C is the boundary of the square with vertices (0, 0), (1, 0), (0, 1) and (1, 1), oriented clockwise.



$$\mathbf{F}(x,y) = \begin{bmatrix} x - y^2 \\ x + y^2 \end{bmatrix} \begin{array}{l} \leftarrow f_1 \\ \leftarrow f_2 \end{array}$$

ANS: -2.

Since C is CW,

Topic 4

$$\oint_C \vec{F}\cdot d\vec{r} = -\iint_D 1 + 2y \, dA = -\int_0^1\int_0^1 1+2y \, dx\, dy$$

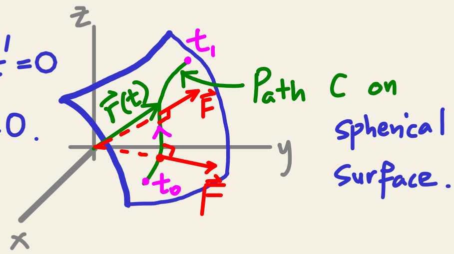
$$= -\int_0^1 dx \cdot \int_0^1 1+2y \, dy = -(1)\left[y+y^2\right]\Big|_0^1 = -2 \text{ //}$$

10. For a radial vector field **F**(x, y, z) = $[x, y, z]^T$, show that its line integral over any path that is on a sphere

$$x^2 + y^2 + z^2 = \rho^2 \rightarrow \frac{d}{dt} \rightarrow 2x\cdot x' + 2y\cdot y' + 2z\cdot z' = 0$$

$$\div 2 \quad x\cdot x' + y\cdot y' + z\cdot z' = 0.$$

is always zero. Sketch a graph and explain why.



$$L = \int_c \vec{F}\cdot d\vec{r} = \int_c \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\, dt$$

Parameterize C by

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \longrightarrow \vec{r}\,'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

$$= \int_c \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} dt$$

$$= \int_c x\cdot x' + y\cdot y' + z\cdot z'\, dt$$

$$= \int_{t_0}^{t_1} 0\, dt = 0 \;//$$

Notice that from the graph, the vector field F is perpendicular to the path at all points, so F.dr is always 0, hence leading to the line integral being 0 as well.