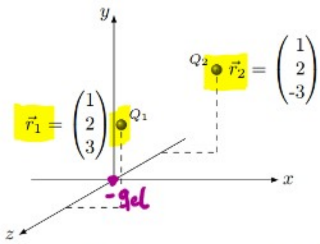
Exercise 24/02/16

Friday, 16 February 2024 05:27

Exam 2019

Q1 (5 marks)

Consider two dicrete charges $Q_1 = +q_{\rm el}$ at position $\vec{r}_1$ and $Q_2 = +q_{\rm el}$ at position $\vec{r}_2$ in vacuum, with $q_{\rm el} = \text{const.} > 0$ (see figure below).



*a) Give the electrostatic force $\vec{F}_q$ applied on a point charge $q = -q_{\rm el}$ at the position $\vec{r}_0 = (0,0,0)$ (origin).
Deduce an expression for the electric field $\vec{E}(\vec{r}_0)$ at the position $\vec{r}_0 = (0,0,0)$ (origin) from the equation of $\vec{F}_q$.

$$|\vec{r}_2| = |\vec{r}_1| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\vec{E}(\vec{r}_0) = \frac{1}{4\pi\varepsilon_0} \cdot \left[-\frac{Q_1}{|\vec{r}_1|^3}\begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{Q_2}{|\vec{r}_2|^3}\begin{pmatrix}1\\2\\-3\end{pmatrix} \right]$$

$$\vec{F}_{el} = q\cdot\vec{E}(\vec{r}_0) = -q_{el}\cdot\vec{E}(\vec{r}_0)$$

$$= \frac{-q_{el}^2}{4\pi\varepsilon_0\sqrt{14}^3}\cdot\begin{pmatrix}2\\4\\0\end{pmatrix} \Rightarrow \vec{E}(\vec{r}_0) = \frac{q_{el}}{4\pi\varepsilon_0\sqrt{14}^3}\cdot\begin{pmatrix}2\\4\\0\end{pmatrix}$$

b) Determine the electrostatic potential $\Phi(\vec{r}_0)$ generated by the above given charge constellation at the position $\vec{r}_0 = (0,0,0)$ (origin).

$$\phi(\vec{r}_0) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i|} = \frac{2q_{el}}{4\pi\varepsilon_0\sqrt{14}}$$

Q2 (4 marks)

*a) State Gauss's law in integral form for a continuous charge distribution with space charge density $\rho(\vec{r})$.

$$\int_{\partial A}\vec{D}\,d\vec{a} = \int_V \rho\, dr^3$$

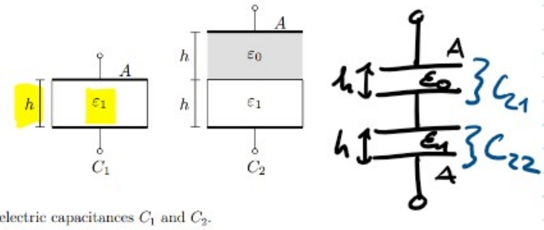
b) Deduce the differential form of Gauss's law from the integral form of subtask a). Give the units (in SI-units) of all physical quantities contained.

$$\int_{\partial A}\vec{D}\,d\vec{a} = \int_V \operatorname{div}\vec{D}\, dr^3 = \int_V \rho\, dr^3$$

$\rightarrow$ for every V: $\operatorname{div}\vec{D} = \rho$

$$[\rho] = \frac{C}{m^3} = \frac{As}{m^3} \qquad [\vec{D}] = \frac{C}{m^2} = \frac{As}{m^2}$$

Q3 (6 marks)
Consider two plate capacitors, whose interiors are partly filled with air ($\varepsilon = \varepsilon_0$) and partly filled with a dielectric medium with permittivity ε_1 according to the figure below. Both capacitors have the area A and the height of the respective dielectric layer is h.
Stray-fields are neglected.



*a) Calculate the electric capacitances C_1 and C_2.

$$C = \varepsilon \cdot \frac{A}{d} \Rightarrow C_1 = \varepsilon_0 \varepsilon_1 \frac{A}{h}$$

$$C_{21} = \varepsilon_0 \cdot \frac{A}{h} \quad \text{series} \quad C_{22} = \varepsilon_0 \varepsilon_1 \frac{A}{h}$$

$$\frac{1}{C_{tot}} = \frac{1}{C_{21}} + \frac{1}{C_{22}} = \frac{C_{21}+C_{22}}{C_{21} \cdot C_{22}} \qquad C_{tot} = \frac{C_{21} \cdot C_{22}}{C_{21}+C_{22}}$$

$$C_2 = \frac{\varepsilon_0^2 \varepsilon_1 \frac{A^2}{h^2}}{\frac{A}{h} \varepsilon_0 (1+\varepsilon_1)} = \frac{\varepsilon_0 \varepsilon_1 \frac{A}{h}}{1+\varepsilon_1} = \frac{A}{h} \frac{\varepsilon_0 \varepsilon_1}{1+\varepsilon_1}$$

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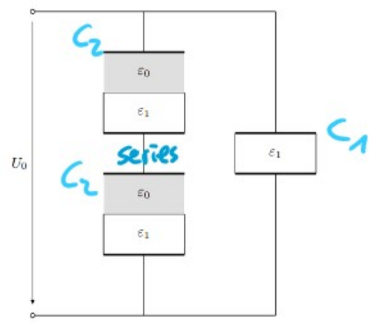
Now the capacitors are connected to form the circuit given below and the voltage U_0 is applied.



b) Calculate the total capacitance C_{tot} of the given circuit.

b) Calculate the total capacitance C_{tot} of the given circuit.

$$C_{tot} = C_1 \| (C_2 \text{ series } C_2)$$
$$= C_1 + \frac{1}{2} C_2$$
$$= \varepsilon_0 \varepsilon_1 \frac{A}{h} + \frac{1}{2} \frac{A}{h} \frac{\varepsilon_0 \varepsilon_1}{1+\varepsilon_1}$$

c) Calculate the electric field energy W_{el} stored in the capacitor aggregate as function of C_{tot} and U_0.

$$W_{el} = \frac{1}{2} C_{tot} \cdot U_0^2$$
$$= \left[\frac{\varepsilon_0 \varepsilon_1}{2} \frac{A}{h} + \frac{A}{4h} \frac{\varepsilon_0 \varepsilon_1}{1+\varepsilon_1} \right] \cdot U_0^2$$

Q4 (3 marks)

We consider a conductive material containing a certain number of mobile charge carriers. Each of the mobile carriers has the charge q. The particle density of the mobile carriers in the material is n. Under the action of an electric field $\vec{E}$ the carriers move at a mean drift velocity $\vec{v}$.

*a) Calculate the power P_{el} delivered by the electric field to one single mobile carrier.

$$P_{el} = q \vec{v} \vec{E} \quad \left(= \frac{p_{el}}{n} \right)$$

$$p_{el} = \vec{j} \cdot \vec{E} = q \cdot \underline{n} \cdot \vec{v} \vec{E}$$

*b) Calculate the power density p_{el} dissipated inside the material in terms of the given quantities.

$$p_{el} = \vec{j} \cdot \vec{E} = q \cdot n \cdot \vec{v} \cdot \vec{E}$$

a) $\quad n = \frac{N}{Vol}$

$$P_{el} = \int p_{el} \, dV = q \cdot \frac{N}{Vol} \cdot \vec{v} \cdot \vec{E} \cdot Vol$$
$$= q \cdot N \cdot \vec{v} \cdot \vec{E}$$

$N = 1$ for one carrier

$$\rightarrow P_{el} = q \vec{v} \vec{E}$$

Q5 (3 marks)

*a) State the charge balance equation in differential form.

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} = 0$$

*b) Show by calculation that the electric current density $\vec{j} = 2C_0yx\vec{e_x} - C_0y^2\vec{e_y}$ $(C_0 > 0)$ fulfills the stationary charge balance equation.

$$\frac{\partial \rho}{\partial t} = 0 \rightarrow \operatorname{div}\vec{j} = 0$$

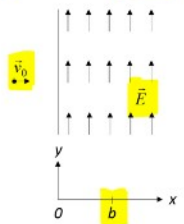
$$\operatorname{div}\vec{j} = \nabla \cdot \vec{j} = \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\right) \cdot \begin{pmatrix} 2C_0yx \\ -C_0y^2 \end{pmatrix}$$

$$= 2C_0y - 2C_0y = 0 \checkmark$$

$\Rightarrow$ fulfilled ✓

Q6 (4 marks)

A charge carrier with charge $2q$, mass m and initial velocity $\vec{v}_0 = v_0\vec{e_x}$ enters a homogenous electric field $\vec{E} = E_y\vec{e_y}$ $(E_y > 0)$ at the position $x = 0$.



Determine the velocity $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ of the charge carrier at the position $x = b$.

$v_x = v_0$

$$\vec{v} = \begin{pmatrix} v_0 \\ \frac{2q E_y}{m}\frac{b}{v_0} \end{pmatrix}$$

$F_y = m \cdot a_y = 2q \cdot E_y$

$\rightarrow a_y = \frac{2q \cdot E_y}{m}$

$v = \int a \, dt \rightarrow v_y = \int \frac{2qE_y}{m} dt$

$= \frac{2qE_y}{m} t + v_{y_0}$

$d = v \cdot t$ $\qquad v_{y_0} = 0$

$t = \frac{d}{v}$ $\quad t_b = \frac{b}{v_0}$

$= \frac{2qE_y}{m} t$

$= \frac{2qE_y}{m} \cdot \frac{b}{v_0}$

Q7 (3 marks)

*a) Determine the radius of the trajectory of a charge carrier with charge q, mass m and velocity $\vec{v} = v_0\vec{e}_x$ in a homogenous and stationary magnetic field $\vec{B} = B_0\vec{e}_z$ ($B_0 = \text{const.} > 0$).

Note: Centripedal force $F_C = m\frac{v^2}{r}$

$|\vec{F}_c| = |\vec{F}_L|$ $\quad$ $\vec{F}_L = q\,\vec{v}\times\vec{B} = q v B_0(-\vec{e}_y)$

$m\frac{v_0^2}{r} = |q| v_0 B_0$

$\rightarrow r = \frac{m v_0}{|q| B_0}$

b) How do the radiuses of the trajectories of an electron and a proton differ from each other if they move at the same velocity perpendicular to a homogenous and constant magnetic field? Justify your answer on the basis of the results of subtask a).

$r_{el} = \frac{m_e v_0}{e_0 B_0}$ with $v_0, B_0, |q|$ stay the same

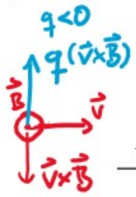
$m\uparrow \Rightarrow r\uparrow \rightarrow r_{Proton} > r_{electron}$

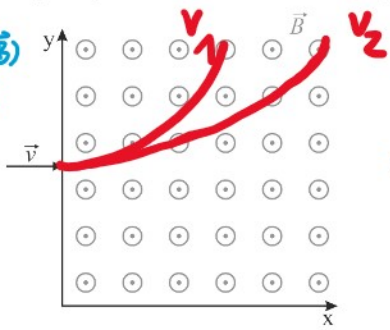
bc. $(m_{Pr} > m_{el})$

Q8 (4 marks)

Two electrons e_1 and e_2 enter a homogenous and constant magnetic field $\vec{B} = B_0\vec{e}_z$ ($B_0 = \text{const.} > 0$) at the position $x = 0$. The electrons have the initial velocities $\vec{v}_1 = v_0\vec{e}_x$ and $\vec{v}_2 = 2v_0\vec{e}_x$ respectively.

*a) Sketch qualitatively the trajectories of both electrons e_1 and e_2 in the figure below.





$r = \frac{m v}{q B}$

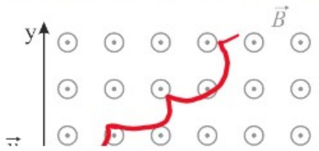
$\vec{F}_L = q\cdot\underbrace{\vec{v}\times\vec{B}}_{\text{right hand rule}}$

$\pm$

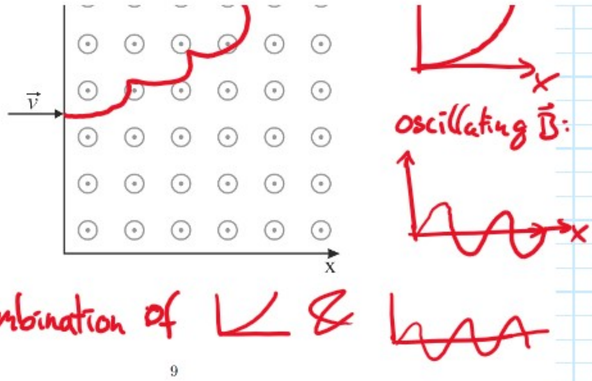
*b) The homogenous and constant magnetic field is replaced by a time-varying magnetic field $\vec{B} = B_0\left(\frac{1}{2}+\frac{1}{2}\sin(\omega t)\right)\vec{e}_z$.

Sketch the trajectory of electron e_1 which enters the magnetic field at the initial velocitiy $\vec{v}_1 = v_0\vec{e}_x$.

(Note: The electron stays in the field for several periods of oscillation.)



const. $\vec{B}$:



Q9 (2 marks)
How are magnetic fields generated?

- current density $\vec{j}$
- time variant $\dot{\vec{D}} = \frac{\partial \vec{D}}{\partial t}$

Q10 (3 marks)
State the Ampére-Maxwell circuital law in differential form and name all physical quantities contained.

$$\nabla \times \vec{H} = \text{rot}\,\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

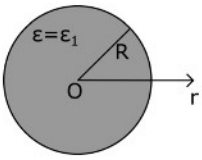
curl $\vec{H}$

$\vec{H}$: mag. field strength

$\vec{j}$: current density

$\vec{D}$: el. flux density / displacement field

1. Problem (15 marks)
Consider a sphere with radius R, electric permittivity $\varepsilon = \varepsilon_1$ and the constant space charge density $\rho(\vec{r}) = \rho_0$ for $r \in [0, R]$ ($|\vec{r}| = r$). The center of the sphere is located at the origin of the coordinate system. The sphere is placed in vacuum.
There is no charge ($\rho(\vec{r}) = 0$) outside of the sphere ($r \in [R, \infty]$).



*a) Calculate the total charge $Q(V(r))$ enclosed by the surface of a spherical control volume $\partial V(r)$ for $0 \leq r \leq \infty$. Distinguish between the different regions.

*b) Calculate the dielectric displacement field $\vec{D}(r)$ (magnitude and direction) in the region $0 \leq r \leq \infty$ in terms of the given physical quantities.

The electric charges inside the sphere generate an electrostatic potential $\Phi(\vec{r})$ outside of the sphere, which is equal to the electrostatic potential of a point charge located at the origin, which carries the same amount of charge as the sphere's total charge.

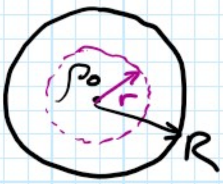
*c) Give the electrostatic potential $\Phi(\vec{r})$, which is generated by the sphere in the region $r > R$.

d) Calculate the electrostatic potential $\Phi(\vec{r})$ inside the sphere ($0 \leq r \leq R$). Choose the reference potential Φ_0 in such way, that the potential function is continuous at the surface of the sphere ($r = R$).

e) Draw a qualitative graph of the electrostatic potential $\Phi(\vec{r})$. Label the position $r = R$ on the r-axis and give the corresponding value along the Φ-axis.

f) Determine the electrostatic potential $\Phi(\vec{r} = 0)$ in the center of the sphere, if the radius R is shrinked to one half of the original value, whereby it carries still the same amount of charge.

g) How do we term the electrostatic potential, which is obtained in the limits $R \to 0$?



a) $Q(V) = \int \rho_0 \, dV$

$$= \int_0^{r}\int_0^{\pi}\int_0^{2\pi} \rho_0 \underbrace{r'^2 \sin\vartheta \, d\varphi \, d\vartheta \, dr'}_{dV}$$

$$= 2\pi \int_0^{r}\int_0^{\pi} \rho_0 \sin\vartheta \, d\vartheta \, r'^2 \, dr'$$

$$\int_0^{\pi} \sin\vartheta \, d\vartheta = \left[-\cos\vartheta\right]_0^{\pi} = \left[-(-1) - (-1)\right] = 2$$

$$= 4\pi \int_0^{r} \rho_0 \, r'^2 \, dr'$$

$$= 4\pi \rho_0 \frac{1}{3} r^3 \quad , \text{ for } r < R$$

$$= 4\pi \int_0^{r} \rho_0 \, r'^2 \, dr'$$

$$= [-(-1) - (-1)]$$
$$= 2$$

$$= 4\pi \rho_0 \frac{1}{3} r^3 \quad , \text{ for } r < R$$

$$= 4\pi \rho_0 \frac{1}{3} R^3 \quad \text{ for } r > R$$

b) Gauss: $\int_{\partial V} \vec{D} \, d\vec{a} = Q(V)$

Symmetry: $\vec{D}(\vec{r}) = D_r(r) \, \vec{e}_r$

$$\int_0^{2\pi} \int_0^{\pi} D_r(r) \, \vec{e}_r \underbrace{\vec{e}_r r^2 \sin\vartheta \, d\vartheta \, d\varphi}_{d\vec{a}} = Q(V)$$

$$r \geq R: \quad D_r(r) \cdot 4\pi r^2 = \boxed{Q(V) = \frac{4\pi}{3} R^3 \rho_0}$$

$$\rightarrow \quad D_r(r) = \frac{R^3}{3 r^2} \rho_0$$

$$r < R: \quad D_r(r) \, 4\pi r^2 = \frac{4\pi}{3} r^3 \rho_0$$

$$\rightarrow \quad \underline{D_r(r) = \frac{1}{3} r \rho_0}$$

c)
$$\phi(r) = \frac{Q}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0 r} \frac{4\pi}{3} R^3 \rho_0 = \boxed{\frac{R^3 \rho_0}{3\epsilon_0 \boxed{r}}}$$

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\underbrace{|\vec{r} - \vec{r}_i|}_{r}}$$

origin

d)
$$-\nabla\phi = \vec{E} \quad \Rightarrow \quad \frac{1}{\varepsilon} \vec{D}(\vec{r}) = -\nabla\phi$$

$$\frac{1}{\epsilon_1} \frac{1}{3} \cdot \rho_0 r = -\frac{\partial}{\partial r} \phi(r) \quad \Big| \int dr$$

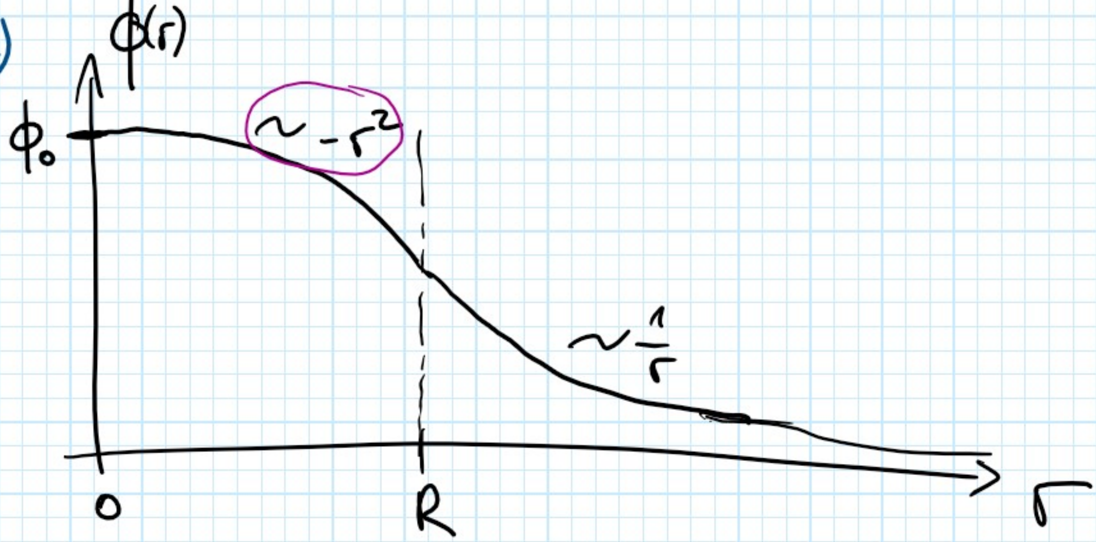
$$\hookrightarrow \phi(r) = - \frac{1}{3\epsilon_1} \rho_0 \cdot \frac{1}{2} r^2 + \phi_0$$

$$\Rightarrow \phi(r) = - \frac{1}{3\varepsilon_1}\rho_0 \cdot \frac{1}{2} r^2 + \underbrace{\phi_0}_{\text{Integration const.} \,\hat{=}\, \text{Ref. potential}}$$

for continuity: $\phi\big|_{r=R} = \frac{R^2 \rho_0}{3\varepsilon_0} = -\frac{1}{6\varepsilon_1}\rho_0 R^2 + \phi_0$

$$\Rightarrow \phi_0 = \frac{R^2}{3}\rho_0\left(\frac{1}{\varepsilon_0} + \frac{1}{2\varepsilon_1}\right)$$

e)



f) $\phi(0) = \phi_0 = \frac{R^2}{3}\rho_0\left(\frac{1}{\varepsilon_0} + \frac{1}{2\varepsilon_1}\right)$

1: $\frac{4}{3}\pi R^3 \rho_0 = Q$

2: $\frac{4}{3}\pi \left(\frac{R}{2}\right)^3 \tilde{\rho}_0 = Q$

$\rightarrow \quad R^3 \rho_0 = \left(\frac{R}{2}\right)^3 \tilde{\rho}_0$

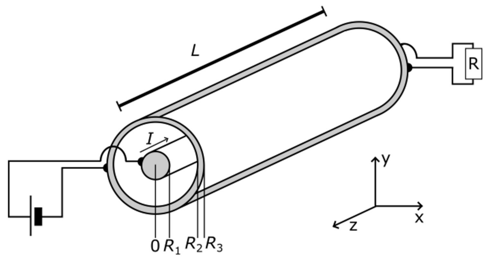
$R^2 \rho_0 = \frac{R^2}{2^3}\tilde{\rho}_0 \quad \rightarrow \quad \tilde{\rho}_0 = 2^3 \rho_0$

$$\Rightarrow \phi(0) = \frac{2^3 R^2}{3}\rho_0\left(\frac{1}{\varepsilon_0} + \frac{1}{2\varepsilon_1}\right)$$

g) Coulomb potential

2. Problem (15 marks)

A coaxial cable is used for the power supply of a load (represented by the Ohmic resistance R). The constant and homogenous electric current I is flowing through the inner conductor to the load and back through the outer conductor. Both cables are assumed to be ideal conductors. The inner conductor has the radius R_1. The ring-shaped outer conductor is extended over the region $R_2 \leq r \leq R_3$. Both conductors are separated by air and exhibit the length L ($L >> R_3$).



*a) Calculate the electric current density in the inner conductor $\vec{j}_i$ and in the outer conductor $\vec{j}_a$ under the assumption that the electric current I is homogenous and stationary.

b) Calculate the magnetostatic field $\vec{H}(r)$, which is generated by the electric current I in the region $0 < r < \infty$. Choose an appropriate coordinate system and distinguish between the different regions.

c) Draw a qualitative graph of the magnitude of the magnetostatic field $\vec{H}(r)$ depending on the radial distance r for $0 < r < \infty$. Label the positions R_1, R_2 and R_3 on the r-axis.

*d) Calculate the power loss P_{el} at the load resistance R. What is causing this loss?

*e) Which is the characteristic electrical property of coaxial cabels, that is benefical for the transmission of signals?

a) $\vec{j} = \frac{I}{A}$

Inner conductor: $I = \int_A \vec{j}\, d\vec{a}$

$$= \int_0^{2\pi}\int_0^{R_1} j_i \; r\, dr\, d\varphi$$

$|j_i| = \frac{I}{\pi R_1^2}$

$I = j_i \, \pi R_1^2$

$$I = j_i \pi R_1^2$$

outer conductor:

$$I = \int_0^{2\pi}\int_{R_2}^{R_3} j_o \, r \, dr \, d\varphi$$

$$= j_o \pi \left(R_3^2 - R_2^2\right)$$

$$|j_o| = \frac{I}{\pi\left(R_3^2 - R_2^2\right)}$$

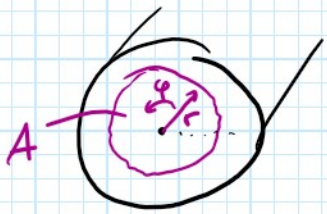
b)

$$\int_{\partial A} \vec{H}(\vec{r})\, d\vec{r} = \int_A \vec{j}\, d\vec{a}$$

Symm.: $\vec{H} = H_\varphi(r)\, \vec{e}_\varphi$

$$\Rightarrow \int_0^{2\pi} H_\varphi(r)\, \vec{e}_\varphi \vec{e}_\varphi r \, d\varphi = 2\pi r\, H_\varphi(r)$$

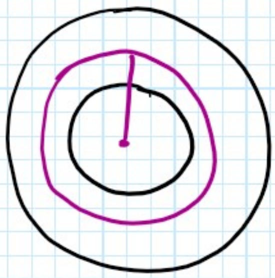
$$\rightarrow \boxed{H_\varphi(r) = \frac{1}{2\pi r}\int_A \vec{j}\, d\vec{a}}$$



i) $r \in [0, R_1]$

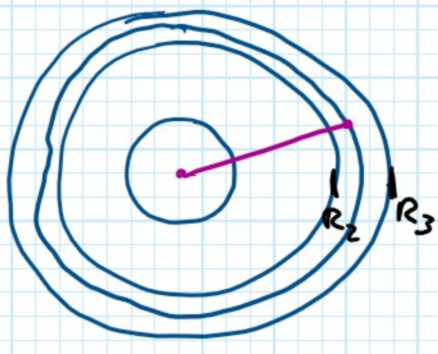
$$H_\varphi(r) = \frac{1}{2\pi r}\int_0^{2\pi}\int_0^{r} \frac{I}{\pi R_1^2} \overbrace{\,}^{(-\vec{e}_z)(-\vec{e}_z)} r'\, dr'\, d\varphi = \frac{1}{2\pi r}\cdot\frac{I}{\pi R_1^2}\cdot\frac{1}{2}r^2\cdot 2\pi$$

$$= \frac{I r}{2\pi R_1^2}$$



ii) $r \in [R_1, R_2]$ $\quad H_\varphi(r) = \frac{1}{2\pi r} I(A)$

$$= \frac{I}{2\pi r}$$



iii) $r \in [R_2, R_3]$

$$H_\varphi(r) = \frac{1}{2\pi r}\int \vec{j}\, d\vec{a}$$

$$= \frac{I}{\dots} + \frac{1}{\dots}\int_0^{2\pi}\int^{r}\left(\frac{I}{\dots}\right)\vec{e}_z(-\vec{e}_z)\, r'\, dr\, d\varphi$$

$$= \frac{I}{2\pi r} + \frac{1}{2\pi r}\int_0^{2\pi}\int_{R_2}^{r}\left(\frac{I}{\pi(R_3^2-R_2^2)}\right)\vec{e}_z(-\vec{e}_z)r'\,dr\,d\varphi$$

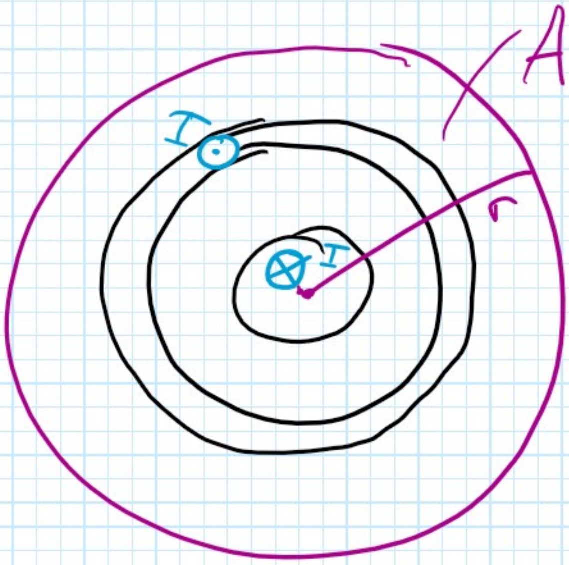
$$= \frac{I}{2\pi r} + \frac{1}{2\pi r}\,2\pi\,\frac{I}{2\pi^2 r\,(R_3^2-R_2^2)}(-1)\left[\frac{1}{2}r'^2\right]_{R_2}^{r}$$

$$= \frac{I}{2\pi r} + \frac{I\,(R_2^2-r^2)}{2\pi r\,(R_3^2-R_2^2)}$$

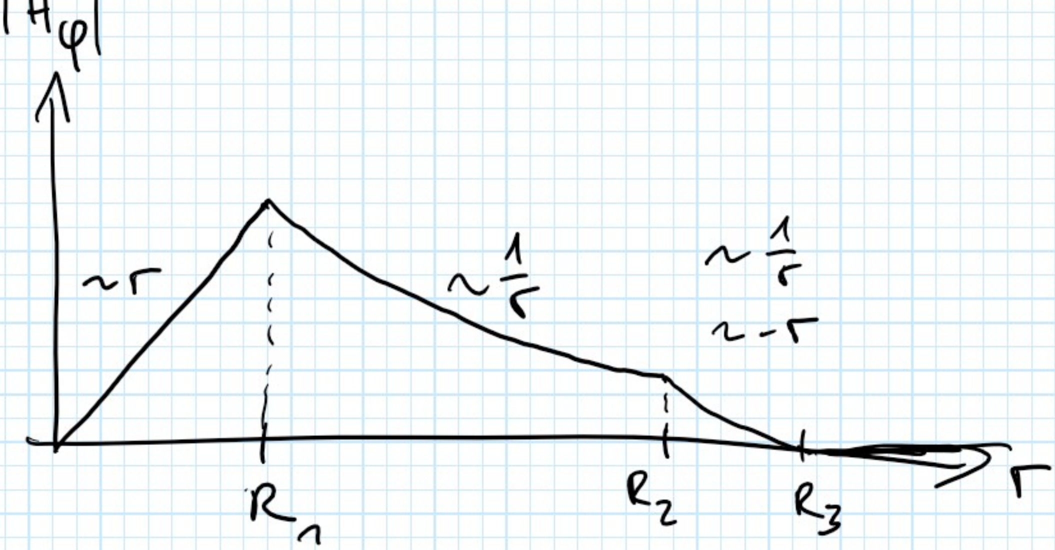
$$H_\varphi \sim \frac{1}{r}\,,\ -r$$

iv) $r \in [R_3, \infty]$

$$H_\varphi = \frac{1}{2\pi r}\cdot I(A)$$
$$= 0$$



c)

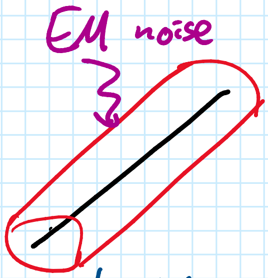


d) $P = U\cdot I = R\cdot I^2$

$U = RI$

thermal loss due to interaction of electrons thermal loss due to interaction of electrons with conductor

e) Shielding!



Outer conductor shields inner conductor from electromagnetic noise.