Institute for Physics of Electrotechnology
Munich University of Technology

Lecture Notes

Electricity and Magnetism

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Contents

0. Preliminary Remarks on Units and Physical Quantities

(i) A **physical quantity** (e.g., the velocity v or the length L) is described by a numerical value in conjunction with a unit of measurement.

$$\text{Physical quantity} \quad = \quad \text{numerical value} \quad \times \quad \text{measurement unit}$$

Example:

$$v = 20 \quad \frac{\mathrm{km}}{\mathrm{h}}$$
$$L = 5 \quad \text{inch}$$

(ii) There are usually several different **measurement units** for one physical quantity. To define physical quantities and physical relationships consistently, a coherent system of measurement units was created in 1960: the **SI-units** (système international d'unités). In this system, seven base units, which are independent of each other, are defined. The measurement unit of any other physical quantity can be derived algebraically from these base units.

The seven base units are the following:

Quantity	Unit	Symbol
distance	meter	m
time	second	s
mass	kilogram	kg
electric current	Ampère	A
temperature	Kelvin	K
luminous intensity	candela	cd
amount of substance	mole	mol

Derived measurement units are obtained by the defining equation of the respective physical quantity. So they are part of the physical concept formation, oftentimes in connection with the statement of a physical law. Examples are:

Quantity			Unit
velocity	=	distance/time	$\frac{\mathrm{m}}{\mathrm{s}}$
force	=	mass × acceleration	$1\,\mathrm{N}$ (Newton) $= 1\mathrm{kg} \times 1\,\frac{\mathrm{m}}{\mathrm{s}^2} = 1\frac{\mathrm{kg\,m}}{\mathrm{s}^2}$
work	=	force × distance	$1\,\mathrm{J}$ (Joule) $= 1\,\mathrm{N} \times 1\,\mathrm{m} = 1\,\mathrm{Nm}$
power	=	$\frac{\text{work}}{\text{time}}$	$1\,\mathrm{W}$ (Watt) $= 1\,\mathrm{J}/1\,\mathrm{s} = 1\frac{\mathrm{J}}{\mathrm{s}}$
charge	=	current × time	$1\,\mathrm{C}$ (Coulomb) $= 1\mathrm{As}$
electric voltage	=	$\frac{\text{work}}{\text{charge}}$	$1\,\mathrm{V}$ (Volt) $= 1\,\mathrm{J}/1\,\mathrm{C} = 1\frac{\mathrm{kg\,m}^2}{\mathrm{s}^2\,\mathrm{A\,s}} = 1\frac{\mathrm{kg\,m}^2}{\mathrm{A\,s}^3}$

(iii) **Quantity calculus** is the formal method for describing relations between physical quantities. They are expressed by mathematical equations, which are independent of the base unit system. The equality of physical quantities includes that, if they are expressed in the same measurement unit, their numerical values must be equal. Quantity calculus allows to convert different measurement units of the same physical quantity into one another:

Example 1:

The velocity v, which is given as the ratio of the distance travelled to the time elapsed, is defined by the equation

$$v = \frac{L}{t}\,.$$

Using quantity calculus, a non-SI-unit can be converted in a SI-unit as follows:

$$v = \frac{1\,\text{nautical mile}}{1\,\text{hour}} = \frac{1{,}852\,\mathrm{km}}{1\,\mathrm{h}} = \underbrace{1{,}852\frac{\mathrm{km}}{\mathrm{h}}}_{1\,\text{knot}} = 1{,}852\frac{1000\,\mathrm{m}}{3600\,\mathrm{s}} = 0{,}514\frac{\mathrm{m}}{\mathrm{s}}\,.$$

Example 2:

The kinetic energy of a twofold charged ion with charge $Q = 2q_{\mathrm{el}}$, which is accelerated in an ion accelerator with voltage $U = 20\,\mathrm{kV}$, is given by the product of charge Q and voltage U:

$$W_{\mathrm{kin}} = Q \cdot U = 2q_{\mathrm{el}} \cdot 20\,\mathrm{kV}$$

with

$$\begin{aligned} q_{\mathrm{el}} &= |e| = 1{,}602 \times 10^{-19}\,\mathrm{C} = \text{(elementary charge)} \\ \Longrightarrow W_{\mathrm{kin}} &= 6{,}408 \times 10^{-15}\,\mathrm{C}\cdot\mathrm{V} = 6{,}408 \times 10^{-15}\,\mathrm{J}\ \ \text{in SI-units} \end{aligned}$$

A particle physicist or an electrical engineer prefer the representation

$$W_{\mathrm{kin}} = \frac{Q}{\mathrm{e}} \cdot \frac{U}{\mathrm{V}} \cdot \mathrm{eV} = \frac{Q}{\mathrm{e}} \cdot \frac{U}{\mathrm{kV}} \cdot \mathrm{keV} = 2 \cdot 20\,\mathrm{keV} = 40\,\mathrm{keV}$$

The unit keV is not a SI-unit, but has a very intuitive interpretation in practice.

(iv) **Multiples of approved SI-units**

By placing the following prefixes in front of a SI-unit, multiples of a unit can be expressed easier. This allows to introduce units which are more convenient for the practical use.

10^1	deca	da
10^2	hecto	h
10^3	kilo	k
10^6	mega	M
10^9	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E
10^{21}	zetta	Z

Table 1: 10^n, $n > 0$

10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a
10^{-21}	zepto	z

Table 2: 10^n, $n < 0$

1. Electrostatics

1.1. Electric Charge

The following experimental facts about electric charges are known today:

(i) **Charge** is a **fundamental property of all elementary particles** (like mass, spin, charm, flavor, color). It is the source of the electric (to be more precise: electromagnetic) interaction, one of the four fundamental interactions in physics (in addition to strong and weak interaction as well as gravitation).

(ii) There are two classes of charges: positive and negative. Like charges repel each other, whereas unlike charges attract each other.

(iii) The total electric charge in a closed system is conserved. This means that positive and negative charges can be created and annihilated pairwise only,
e.g., matter $\leftrightarrow$ antimatter, ("real" particle)
or electron $\leftrightarrow$ hole = defect electron ("quasiparticle").

(iv) Charge is quantized: the quantum is the elementary charge (= magnitude of the charge of an electron): $|e| = q_{\mathrm{el}} = 1.602 \cdot 10^{-19}\,\mathrm{C}$, where 1 Coulomb = $1\,\mathrm{C} = 1\,\mathrm{As}$. All (separable) elementary particles have an electric charge that is an integer multiple of q_{el}:

$$q_{\mathrm{E}} = \pm N_{\mathrm{E}} \cdot q_{\mathrm{el}} \qquad \text{with } N_{\mathrm{E}} \in \mathbb{N}\,.$$

Hadrons (e.g., protons and neutrons) are composed of quarks with charge

$$q_{\mathrm{Q}} = \pm N_Q \cdot \frac{e}{3} \qquad \text{with } N_Q = 1 \text{ or } 2\,,$$

but these quarks exist only in bound states and are not separable.

1.2. Forces Between Electric Point Charges

1.2.1. Coulomb's Law

Two discrete charges q_1 at position $\vec{r}_1$ and q_2 at position $\vec{r}_2$ exert a force on each other. Let $\vec{F}_{1\leftarrow 2}$ be the force which charge q_1 experiences by the presence of charge q_2, and $\vec{F}_{2\leftarrow 1}$ the force experienced by q_2 through q_1. If the charges are at rest (electrostatics), then the following experimental facts hold:

(i) According to Newton's principle "actio = reactio", we find

$$\vec{F}_{2\leftarrow 1} = -\vec{F}_{1\leftarrow 2}\,.$$

The direction of both forces is parallel to the distance vector $\vec{r}_2 - \vec{r}_1$.

(ii) The strength of the forces is

$$|\vec{F}_{2\leftarrow 1}| = |\vec{F}_{1\leftarrow 2}| = \gamma_e \frac{|q_1 \cdot q_2|}{|\vec{r}_2 - \vec{r}_1|^2}\,,$$

with the electrostatic force constant

$$\gamma_e = \frac{1}{4\pi \cdot \varepsilon_0}$$

where $\varepsilon_0 = 8.854 \cdot 10^{-12} \dfrac{\text{As}}{\text{Vm}}$

ε_0 is called "dielectric constant", or "permittivity of free space"

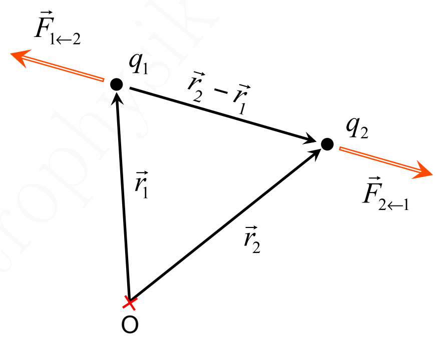


Figure 1: Force between two point charges.

(iii) Whether the charges q_1 and q_2 repel or attract each other depends on the algebraic sign of the charges. Like charges repel and unlike charges attract each other:

$$\begin{aligned} \operatorname{sgn}(q_1) &= \operatorname{sgn}(q_2) \quad \Leftrightarrow \quad \text{repulsion} \\ \operatorname{sgn}(q_1) &= -\operatorname{sgn}(q_2) \quad \Leftrightarrow \quad \text{attraction} \end{aligned}$$

The statements (i) - (iii) can be compactly summarized as vector equation:

$$\vec{F}_{2\leftarrow 1} = -\vec{F}_{1\leftarrow 2} = \frac{1}{4\pi \cdot \varepsilon_0} \cdot \frac{q_1 \cdot q_2}{|\vec{r}_2 - \vec{r}_1|^3} \cdot (\vec{r}_2 - \vec{r}_1)\,. \tag{1.1}$$

This is **Coulomb's law** in vector form.

Note that $(\vec{r}_2 - \vec{r}_1)/|\vec{r}_2 - \vec{r}_1|$ is the unit vector which points from position $\vec{r}_1$ to position $\vec{r}_2$.

1.2.2. Superposition Principle

An arrangement of N charges q_i $(i = 1, ..., N)$ at positions $\vec{r}_i$ $(i = 1, ..., N)$ exerts an electric force $\vec{F}_q(\vec{r})$ on an additional charge q at position $\vec{r}$. This force is given by vector addition of the Coulomb forces which the charges q_i exert on charge q. We obtain:

$$\vec{F}_q(\vec{r}) = \sum_{i=1}^{N} \frac{1}{4\pi \cdot \varepsilon_0} \cdot \frac{q \cdot q_i}{\mid \vec{r} - \vec{r}_i \mid^3} \cdot (\vec{r} - \vec{r}_i)$$

or rather

$$\vec{F}_q(\vec{r}) = \frac{q}{4\pi \cdot \varepsilon_0} \cdot \underbrace{\sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|^3} \cdot (\vec{r} - \vec{r}_i)}_{\text{sources of the force field}} \tag{1.2}$$

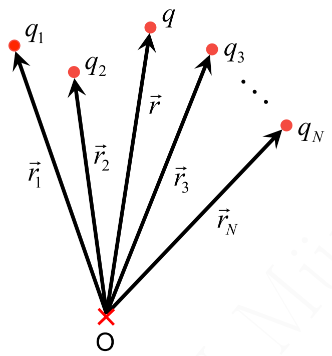


Figure 2: Discrete distribution of point charges

The forces on charge q add up as vectors in such a way that the electric forces on the charge q caused by each of the charges q_i are superposed without any mutual interference.

1.3. Electric Field

1.3.1. Definition of the Electric Field

Equation (1.2) can be interpreted as follows: A given charge distribution $(q_i, \vec{r}_i)_{i=1...N}$ generates a "force field" $\vec{E}(\vec{r})$ at every position $\vec{r}$ even without the presence of the charge q. If a "test charge" q is placed at the position $\vec{r}$, we get

$$\vec{F}_q(\vec{r}) = q \cdot \vec{E}(\vec{r}) \,.$$

This leads to the following definition of $\vec{E}(\vec{r})$:

$$\vec{E}(\vec{r}) := \frac{1}{q} \vec{F}_q(\vec{r}) \,.$$

Therefore, the electric field generated by $(q_i, \vec{r}_i)_{i=1...N}$ is explicitly given as:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi \cdot \varepsilon_0} \cdot \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|^3} \cdot (\vec{r} - \vec{r}_i) \tag{1.3}$$

The physical unit of the electric field is

$$\dim(|\vec{E}|) = \frac{\text{N}}{\text{As}} = \frac{\text{kg m}}{\text{s}^2} \cdot \frac{1}{\text{As}} = \frac{\text{V}}{\text{m}}$$

with the definition 1 Volt $= 1\,\text{V} = \dfrac{\text{kg m}^2}{\text{As}^3}$

1.3.2. Special Cases

(i) **Monopole field**: $N = 1$, one point charge q_0 at position $\vec{r}_0$ as source:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi \cdot \varepsilon_0} \cdot \frac{q_0}{|\vec{r} - \vec{r}_0|^3} \cdot (\vec{r} - \vec{r}_0) \tag{1.4}$$

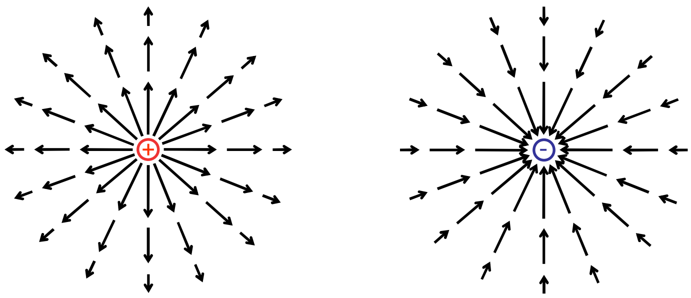


Figure 3: Arrow diagram of the electric field of a point charge q_0, with $q_0>0$ (left) and $q_0<0$ (right), respectively.

(ii) **Dipole field:** $N = 2$, charges $(Q, \vec{r}_1)$ and $(-Q, \vec{r}_2)$ as sources:

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi \cdot \varepsilon_0} \cdot \left[\frac{1}{|\vec{r} - \vec{r}_1|^3} \cdot (\vec{r} - \vec{r}_1) - \frac{1}{|\vec{r} - \vec{r}_2|^3} \cdot (\vec{r} - \vec{r}_2) \right] \tag{1.5}$$

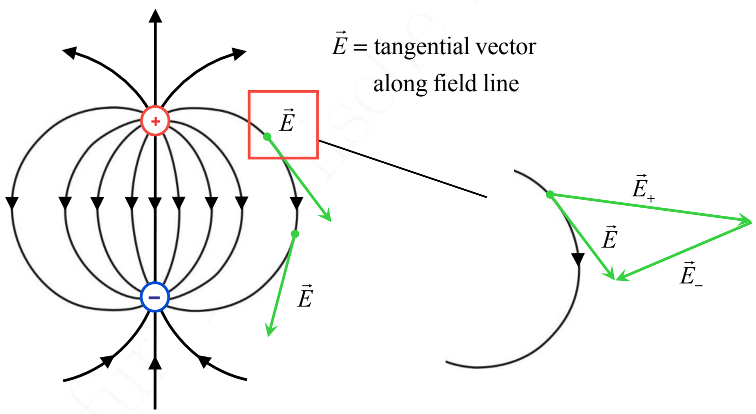


Figure 4: Electric field lines of two unlike charges with equal magnitude (dipole field)

Note: Field lines start at a positive charge and end at a negative charge.

1.3.3. Remark on the Graphical Representation of Vector Fields

Vector fields like the electric field $\vec{E}(\vec{r})$ can be represented as an "arrow plot", as shown in figure 3, by plotting the vector $\vec{E}(\vec{r})$ as an arrow at each position $\vec{r}$. However, this kind of a plot can become confusing. Alternatively, it is more practical to plot a family of curves called "field lines".

These field lines are defined in such a way that the tangential vector at each point along the field line represents the vector field (see figure 4). To calculate the parametric representation $\lambda \mapsto \vec{r}(\lambda)$ of a field line containing a given point $\vec{r}_0$, one has to solve the differential equation

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda} = \vec{E}(\vec{r}(\lambda))\,; \quad \vec{r}(\lambda_0) = \vec{r}_0$$

(= differential equation for $\vec{r}(\lambda)$).

1.4. Electric Work, Voltage and Potential

1.4.1. Electric Work

(i) **Definition of mechanical work**:
A point particle is moved by the action of a force field $\vec{F}(\vec{r})$ along a path $C\,(P_1, P_2)$ in E_3 from P_1 to P_2 (figure 5). The mechanical work performed is given by the integral over the force component tangential to the path $C\,(P_1, P_2)$. To calculate the work, we use the parametric representation of $C\,(P_1, P_2)$ with the arc length as curve parameter:

$$(0, l) \ni s \mapsto \vec{r}(s)$$

with $\vec{r}(0) = \vec{r}_1$ and $\vec{r}(l) = \vec{r}_2$.

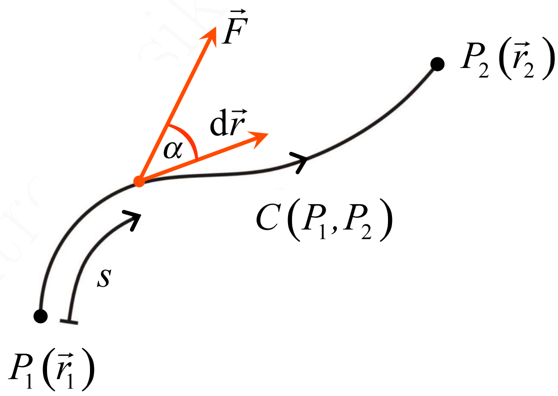


Figure 5: Path integral.

The tangential unit vector of the curve $C\,(P_1, P_2)$ is

$$\vec{t}(s) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}; \quad \left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\right| = 1\,;$$

thus, the vectorial line element is

$$\mathrm{d}\vec{r} = \vec{t} \cdot \mathrm{d}s.$$

The differential mechanical work, which is done along a line element, is (see figure 5)

$$\mathrm{d}W = |\vec{F}(\vec{r}(s))| \cos\alpha(s)\mathrm{d}s = \vec{F}(\vec{r}(s)) \cdot \vec{t}(s)\mathrm{d}s\,.$$

Hence, the total mechanical work is given by the integral

$$W_{12} = \int\limits_0^l \vec{F}(\vec{r}(s)) \cdot \underbrace{\vec{t}(s)}_{=\frac{\mathrm{d}\vec{r}}{\mathrm{d}s}} \mathrm{d}s = \int\limits_0^l \vec{F}(\vec{r}(s)) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\mathrm{d}s = \int\limits_{C(P_1,P_2)} \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r}. \tag{1.6}$$

(ii) **Electric work**:
When a point charge q is moved from P_1 to P_2 along $C\,(P_1, P_2)$ in an electric field $\vec{E}(\vec{r})$, the electric work performed by the charge is

$$W_{12} = q \int\limits_{C(P_1,P_2)} \vec{E}(\vec{r}) \cdot \mathrm{d}\vec{r}\,, \tag{1.7}$$

because $\vec{F}_q(\vec{r}) = q \cdot \vec{E}(\vec{r})$.

1.4.2. Electric Voltage

(i) **Definition of electric voltage**:
According to equation (1.7), the electrical work W_{12} is proportional to the test charge q on which the work is performed. Dividing W_{12} by q yields a quantity that depends on the electric field $\vec{E}(\vec{r})$ only. This quantity is called the **electric voltage** between P_1 and P_2:

$$U_{12} = \frac{W_{12}}{q} = \int\limits_{C(P_1,P_2)} \vec{E} \cdot \mathrm{d}\vec{r} \tag{1.8}$$

Physical unit (cf. section 1.3.1): $\dim(U_{12}) = 1\,{}^{\mathrm{J}}/_{\mathrm{As}} = 1\mathrm{V(olt)}$

(ii) **Fundamental law of electrostatics**:

Electrostatic fields are conservative!

This means that the voltage U_{12} depends on P_1 and P_2 only, but not on the choice of the path $C(P_1, P_2)$ connecting them.

This is expressed by the following notation:

$$U_{12} = \int_{P_1}^{P_2} \vec{E} \cdot \mathrm{d}\vec{r} \tag{1.9}$$

This law may be verified by showing that, in Cartesian coordinates, the “integrability conditions”

$$\frac{\partial E_j}{\partial x_i} = \frac{\partial E_i}{\partial x_j}\,, \quad (i, j = 1, 2, 3)$$

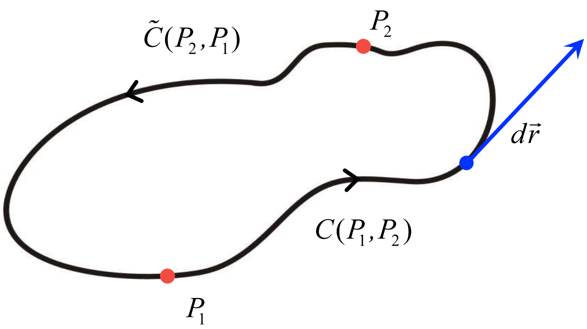
are satisfied by the Coulomb field given in equation (1.3).

(iii) **Conclusion**:
If the integration path $C(P_1, P_2)$ is inverted, i.e., directed from P_2 to P_1, the direction of the tangential vector along the path integral is inverted. Hence, we have:

$$U_{12} = \int\limits_{C(P_1,P_2)} \vec{E} \cdot \mathrm{d}\vec{r} = -\int\limits_{C(P_2,P_1)} \vec{E} \cdot \mathrm{d}\vec{r} = -U_{21} \tag{1.10}$$

(iv) **Conclusion**:
For any closed path C, the electrostatic $\vec{E}$-field satisfies the condition

$$\int_C \vec{E} \cdot \mathrm{d}\vec{r} = 0 \tag{1.11}$$



To proof this, we select two points P_1 and P_2 on the curve C, and decompose C into two segments: $C = C(P_1, P_2) + \widetilde{C}(P_2, P_1)$.
Then we have:

$$\int\limits_C \vec{E} \cdot \mathrm{d}\vec{r} = \int\limits_{C(P_1,P_2)} \vec{E} \cdot \mathrm{d}\vec{r} + \int\limits_{\tilde{C}(P_2,P_1)} \vec{E} \cdot \mathrm{d}\vec{r} = U_{12} + U_{21} = U_{12} - U_{12} = 0\,.$$

1.4.3. Electric Potential

(i) **Definition of the electric potential**
From the fundamental law of electrostatics, it follows that the electrostatic $\vec{E}$-field is a gradient field. This means that there exists a potential function $\Phi(\vec{r})$ with the property:

$$\vec{E}(\vec{r}) = -\mathrm{grad}\Phi(\vec{r}) \tag{1.12}$$

The electric potential $\Phi(\vec{r})$ can be calculated from the electric field $\vec{E}(\vec{r})$:

$$\Phi(\vec{r}) = \Phi(\vec{r}_0) - \int_{P_0}^{P} \vec{E} \cdot \mathrm{d}\vec{r} \tag{1.13}$$

where $P_0 = O + \vec{r}_0$ is a fixed reference point and $P = O + \vec{r}$ is an arbitrary point. The value of the potential $\Phi(\vec{r}_0)$ at the reference point is a constant, which can be arbitrarily chosen (reference potential), and which is often set to zero (ground, neutral conductor etc.).

(ii) **Relation to the electric voltage**
The potential difference

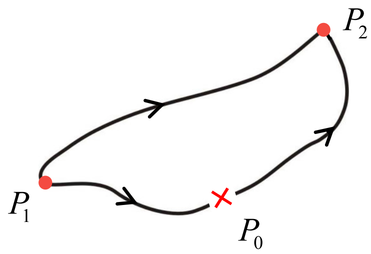
$$\Phi(\vec{r}) - \Phi(\vec{r}_0) = -\int_{P_0}^{P} \vec{E} \cdot \mathrm{d}\vec{r} = \int_{P}^{P_0} \vec{E} \cdot \mathrm{d}\vec{r} = U_{PP_0} \tag{1.14}$$

is evidently the electric voltage U_{PP_0} between the test point P and the reference point P_0. More general, the voltage U_{12} between any two points $P_1 = O + \vec{r}_1$ and $P_2 = O + \vec{r}_2$ can be calculated as

$$U_{12} = \Phi(\vec{r}_1) - \Phi(\vec{r}_2) \tag{1.15}$$

Proof: We connect P_1 with P_2 with a path $C(P_1, P_2)$ which includes the reference point P_0:

$$C(P_1, P_2) = C(P_1, P_0) + C(P_0, P_2)\,.$$



We find:

$$U_{12} = \int_{P_1}^{P_2} \vec{E}\cdot \mathrm{d}\vec{r} = \underbrace{\int_{P_1}^{P_0} \vec{E}\cdot \mathrm{d}\vec{r}}_{\Phi(\vec{r}_1)-\Phi(\vec{r}_0)} + \underbrace{\int_{P_0}^{P_2} \vec{E}\cdot \mathrm{d}\vec{r}}_{\Phi(\vec{r}_0)-\Phi(\vec{r}_2)} = \Phi(\vec{r}_1) - \Phi(\vec{r}_2)\,.$$

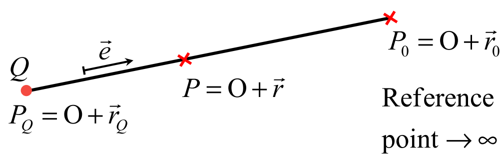
(iii) **Equipotential surface**:
For a given constant potential value Φ_0, the set $\mathcal{F}[\Phi_0] = \{P = O + \vec{r} \mid \Phi(\vec{r}) = \Phi_0\}$ is a two-dimensional surface in E_3 called the equipotential surface to Φ_0. As discussed in appendix A.5, the vector $\operatorname{grad}\Phi$ and, therefore, the electric field $\vec{E} = -\operatorname{grad}\Phi$ are perpendicular to the tangential planes of $\mathcal{F}[\Phi_0]$; they are collinear with the surface normal. Varying Φ_0 results in a family of equipotential surfaces which are all perpendicularly intersected by the electric field lines.

1.4.4. Coulomb Potential of a Point Charge

(i) Let us calculate the electric potential of a point charge Q located at a position $P_Q = \mathrm{O} + \vec{r}_Q$ in free space. The point charge generates the electric field (cf. equation (1.4))

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{(\vec{r}-\vec{r}_Q)}{|\vec{r}-\vec{r}_Q|^3}\,.$$

For an arbitrary point $P = O + \vec{r}$, we take the straight line passing through the points P_Q and P. We will use this line to calculate the path integral from P to the reference point P_0. P_0 is also placed on this line; eventually, it will be shifted to infinity.



So we have to calculate the path integral

$$\Phi(\vec{r}) = \Phi(\vec{r}_0) + \int_P^{P_0} \vec{E}\cdot \mathrm{d}\vec{r} = \Phi(\vec{r}_0) + \int_P^{P_0} \frac{Q}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}_Q)}{|\vec{r}-\vec{r}_Q|^3}\cdot \mathrm{d}\vec{r}\,.$$

Parametric representation of the straight path C from P to P_0 :

$$C : \vec{r}(\lambda) = \vec{r}_Q + \lambda\vec{e}; \;\; \lambda_1 \leq \lambda \leq \lambda_0$$

$$\text{with } \vec{e} = \frac{\vec{r} - \vec{r}_Q}{|\vec{r} - \vec{r}_Q|}; \;\; \lambda_1 = |\vec{r} - \vec{r}_Q|; \;\; \lambda_0 = |\vec{r}_0 - \vec{r}_Q| \, .$$

Tangential vector:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda} = \vec{e} \, .$$

Electric field in parametric representation:

$$\vec{E}(\vec{r}(\lambda)) = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{r}(\lambda) - \vec{r}_Q}{|\vec{r}(\lambda) - \vec{r}_Q|^3} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\lambda\vec{e}}{\lambda^3} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{e}}{\lambda^2} \, .$$

Path integral:

$$\int_P^{P_0} \vec{E} \cdot \mathrm{d}\vec{r} = \int_{\lambda_1}^{\lambda_0} \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{e}}{\lambda^2} \cdot \vec{e}\mathrm{d}\lambda = \int_{\lambda_1}^{\lambda_0} \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{\lambda^2}\mathrm{d}\lambda = \frac{Q}{4\pi\varepsilon_0} \cdot \left(-\frac{1}{\lambda_0} + \frac{1}{\lambda_1} \right) \, .$$

Hence, we obtain:

$$\Phi(\vec{r}) = \Phi(\vec{r}_0) + \frac{Q}{4\pi\varepsilon_0} \cdot \left(\frac{1}{|\vec{r} - \vec{r}_Q|} - \frac{1}{|\vec{r}_0 - \vec{r}_Q|} \right) \tag{1.16}$$

It is convenient to shift the reference point P_0 to infinity, $|\vec{r}_0| \to \infty$, and to set $\Phi(\vec{r}_0) = 0$; the result is:

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{|\vec{r} - \vec{r}_Q|} \, . \tag{1.17}$$

The equipotential surfaces are surfaces of concentric spheres with common center $\vec{r}_Q$:

$$\Phi(\vec{r}) = \text{const.} = \Phi_0 \;\Leftrightarrow\; |\vec{r} - \vec{r}_Q| = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{\Phi_0}$$

(ii) **Coulomb potential of a discrete charge distribution**

Using the principle of linear superposition of fields (1.3) and equation (1.17), we obtain the electrostatic potential of a discrete distribution of point charges $(q_i, \vec{r}_i)_{i=1,\ldots,N}$:

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|} \tag{1.18}$$

1.5. Electric Fields in Electrically Polarizable Materials

1.5.1. Electric Polarization

An electrically polarizable material, also called “dielectric”, has the property that an external primary electric field induces electric dipoles in the material on an atomic length scale (or forces already existing dipoles to point to the same direction). These generate a secondary “polarization field”, which partially screens the primary field. Hence, a test charge experiences a reduced electric force (compared to the vacuum case). If the electric polarization is proportional to the electric field, we call this a linear material law. In this case, we find:

$$\vec{F}_q(\vec{r}) = \frac{1}{\varepsilon_r} \cdot \vec{F}_{q,\mathrm{vac}}(\vec{r}) = \frac{q}{4\pi \underbrace{\varepsilon_0 \varepsilon_r}_{=\varepsilon}} \cdot \sum_{i=1}^{N} \frac{q_i}{|\vec{r}-\vec{r}_i|^3} (\vec{r}-\vec{r}_i)\,. \tag{1.19}$$

As the strength of the force is reduced, we have $\varepsilon_r \geq 1$.

The product $\varepsilon := \varepsilon_0 \varepsilon_r$ is called **(absolute) permittivity** (or dielectric constant); ε_r is called **relative permittivity**, and ε_0 is denoted as **vacuum permittivity** (or dielectric constant of vacuum).

Typical values of ε_r are:

gases(air)	:	$\varepsilon_r = 1,0005 \ldots 1,0010$
organic material, oil	:	$\varepsilon_r = 1,5 \ldots 10$
water	:	$\varepsilon_r = 81$
special ceramic materials (high-k-materials)	:	$\varepsilon_r = 10^3 \ldots 10^4$

1.5.2. Electric Displacement Field

(i) The force law (1.19) suggests to factorize the right hand side as $\vec{F}_q(\vec{r}) = q \cdot \frac{1}{\varepsilon} \cdot \vec{D}(\vec{r})$. The term $\vec{D}(\vec{r})$ is solely determined by the presence of the charge distribution $(q_i, \vec{r}_i)_{i=1,\ldots,N}$ generating the primary field, whereas the polarizability of the dielectric material is taken into account by the factor $\frac{1}{\varepsilon}$. Thus, we define

$$\vec{D}(\vec{r}) = \varepsilon \vec{E}(\vec{r}) = \varepsilon_0 \varepsilon_r \vec{E}(\vec{r})\,. \tag{1.20}$$

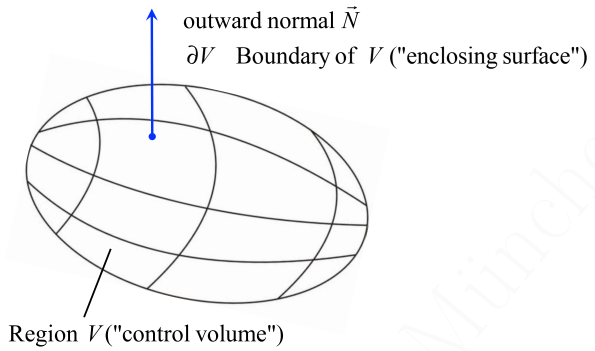
as **electric displacement field**. From equation (1.19) it is evident:

$$\vec{D}(\vec{r}) = \frac{1}{4\pi} \cdot \sum_{i=1}^{N} \frac{q_i}{|\vec{r}-\vec{r}_i|^3} (\vec{r}-\vec{r}_i)\,. \tag{1.21}$$

Indeed, this expression depends on the field-generating charge distribution $(q_i, \vec{r}_i)_{i=1,\ldots,N}$ solely, but is independent of the surrounding material.

(ii) Imagine a three-dimensional control volume V enclosed by its boundary surface ∂V. The outward unit normal on ∂V is denoted by $\vec{N}$. We define the "**displacement flux**" through ∂V as

$$\int_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = \int_{\partial V} \left(\vec{D} \cdot \vec{N}\right) \mathrm{d}a\,. \qquad (1.22)$$



1.5.3. Gauss's Law (Flux theorem)

(i) Using the displacement field allows us to formulate one of the fundamental laws of electromagnetism, namely, "Gauss's flux theorem" or shortly "Gauss's law". We demonstrate this law by considering a very simple situation, where the displacement flux flows through the surface of a sphere, while the electric field is generated by a point charge Q located in its center. We may choose the origin as center of the sphere; then the displacement flux is given by

$$\vec{D}(\vec{r}) = \frac{1}{4\pi} \cdot \frac{Q}{r^3} \cdot \vec{r} \quad \text{with } r = |\vec{r}|\,.$$

As control volume, we take the sphere $K(\mathrm{O}, R)$ around the origin with radius R. The enclosing surface

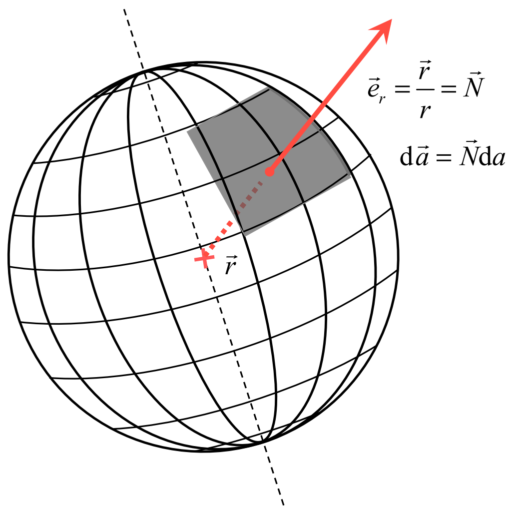
$$\partial K(\mathrm{O}, R) = \{\mathrm{O} + \vec{r} \in E_3 \big| |\vec{r}| = R\}$$

has the outward normal vector

$$\vec{N} = \vec{e}_r = \frac{\vec{r}}{r}$$

and the vectorial surface element

$$\mathrm{d}\vec{a} = \vec{N}\mathrm{d}a = \frac{\vec{r}}{r}\mathrm{d}a\,.$$



Hence, the displacement flux through the enclosing surface $\partial K(\mathrm{O}, R)$ is

$$\int\limits_{\partial K(\mathrm{O},R)} \vec{D} \cdot \mathrm{d}\vec{a} = \frac{Q}{4\pi} \cdot \int\limits_{\partial K(\mathrm{O},R)} \underbrace{\frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{r}}_{\frac{1}{r^2} = \frac{1}{R^2}} \mathrm{d}a = \frac{Q}{4\pi R^2} \int\limits_{\partial K(\mathrm{O},R)} \mathrm{d}a = Q\,\frac{4\pi R^2}{4\pi R^2} = Q\,. \qquad (1.23)$$

(ii) **Generalization** (without proof):
Let now the point charge Q be placed at an arbitrary position $P_0 = \mathrm{O} + \vec{r}_0$ inside or outside of an arbitrarily shaped control volume V with enclosing boundary surface ∂V. Q generates

the displacement field

$$\vec{D}(\vec{r}) = \frac{1}{4\pi} \cdot \frac{Q}{|\vec{r} - \vec{r}_0|^3} (\vec{r} - \vec{r}_0) \,. \tag{1.24}$$

Then the following is true:

$$\int\limits_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = \begin{cases} Q & \text{for} \quad P_0 \in V \setminus \partial V \\ 0 & \text{for} \quad P_0 \notin V \setminus \partial V \end{cases} \tag{1.25}$$

(iii) **Gauss's law** (flux theorem for point charges)
We consider an arbitrary, discrete charge distribution $(q_i, \vec{r}_i)_{i=1\ldots N}$ and a control volume V of general shape with enclosing surface ∂V. We denote by

$$Q(V) = \sum_{\vec{r}_i \in V} q_i \tag{1.26}$$

the total amount of charge in the interior of V, which is enclosed by the surface ∂V (i.e., we sum over that part of the charge distribution which is enclosed by ∂V).
Then Gauss's flux theorem states:

$$\boxed{\int\limits_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = Q(V) = \sum_{\vec{r}_i \in V} q_i \,.} \,. \tag{1.27}$$

This law is a consequence of the superposition principle (1.21) and relation (1.25)! It is mathematically equivalent to Coulomb's field law expressed by equation (1.3).

1.6. Continuous Charge Distribution

1.6.1. Space Charge Density

Discrete point charge distributions make sense only on an atomistic length scale. On technically relevant length scales, we have to deal with a giant number of charge carriers per unit volume (e.g. 10^{22} electrons per cm^3 in a conductor). In this situation, it is reasonable to assume that the charge carriers are distributed in space with a continuous volume density called "space charge density". Its naive definition is as follows: We enclose a given position $\vec{r}$ by a small volume $\Delta V(\vec{r})$ and define the space charge density $\varrho(\vec{r})$ as

$$\rho(\vec{r}) = \frac{\text{amount of charge (net) in } \Delta V(\vec{r})}{|\Delta V(\vec{r})|} \text{ , for } |\Delta V(\vec{r})| \to 0\,.$$

This means that $\rho(\vec{r})\mathrm{d}^3r$ $(= \rho(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ in Cartesian coordinates) is the differential charge $\mathrm{d}Q$ enclosed in the volume element d^3r, so that for any arbitrary control volume V holds:

$$Q(V) = \int\limits_V \rho(\vec{r})\mathrm{d}^3r \tag{1.28}$$

is the charge enclosed in the volume V. The mathematical precise definition of $\rho(\vec{r})$ is therefore:

$$\rho(\vec{r}) = \lim_{|\Delta V(\vec{r})|\to 0} \frac{Q(\Delta V(\vec{r}))}{|\Delta V(\vec{r})|}\,. \tag{1.29}$$

1.6.2. Surface Charge Density

Under certain conditions the electric charges are continuously distributed in a very thin layer along the surface of a body or along the interface between two adjacent materials. In this case, we count the number of charge carriers per unit surface area and introduce a surface charge density $\sigma(\vec{r})$ on a two-dimensional surface S as follows: Enclose a given position $\vec{r} \in S$ by a small surface area $\Delta A(\vec{r})$ and define

$$\sigma(\vec{r}) = \frac{\text{amount of charge (net) in } \Delta A(\vec{r})}{|\Delta A(\vec{r})|} \text{ for } |\Delta A(\vec{r})| \to 0\,.$$

Let $\mathrm{d}a$ denote the scalar surface element along the surface S. Then $\sigma\mathrm{d}a$ is the differential charge $\mathrm{d}Q$ contained in the surface area $\mathrm{d}a$, so that for any arbitrary control area $A \subset S$ holds:

$$Q(A) = \int\limits_A \sigma(\vec{r})\mathrm{d}a \tag{1.30}$$

is the charge contained in the control area A. The mathematical precise definition is therefore:

$$\sigma(\vec{r}) = \lim_{|\Delta A(\vec{r})|\to 0} \frac{Q(\Delta A(\vec{r}))}{|\Delta A(\vec{r})|}\,. \tag{1.31}$$

1.6.3. Gauss's Law for Charge Distributions (in Integral Form)

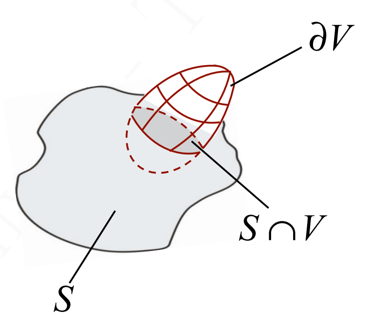
(i) Gauss's flux theorem, as stated in equation (1.27) for point charges, stays true when the enclosed charge $Q(V)$ is expressed by a continuous space charge density $\rho(\vec{r})$ using (1.28). It can therefore be re-stated as follows:

A space charge distribution $\rho(\vec{r})$ generates a displacement field $\vec{D}(\vec{r})$ in such a way that for any arbitrary control volume V with enclosing surface ∂V holds:

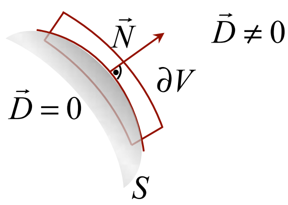
$$\int_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = Q(V) = \int_{V} \rho(\vec{r}) \mathrm{d}^3 r \,. \tag{1.32}$$

(ii) In the case that the charges generating the $\vec{D}$-field are concentrated on a surface S with surface charge density $\sigma(\vec{r})$, then, for any control volume V that intersects the surface S, Gauss's law states:

$$\int_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = Q(S \cap V) = \int_{S \cap V} \sigma(\vec{r}) \mathrm{d}a \,. \tag{1.33}$$



(iii) A special case is the surface S of a perfectly conducting body which has a surface charge density $\sigma(\vec{r})$ along S. The interior of the body, due to electrostatic induction (cf. section 1.7.1), is field-free (i.e., $\vec{D} = 0$).
Therefore, the surface S of the body is an equipotential surface, and the one-sided limit of the $\vec{D}$-field outside the conducting body must be perpendicular to S (cf. section 1.4.3), i.e., it is directed parallel to the outward unit normal $\vec{N}$.
As control volume V in Gauss's flux theorem (1.33), we choose a small thin plate whose top surface lies outside the body and is aligned parallel to S, while its bottom surface is parallel-shifted into the interior of the body.



When the thickness of the plate approaches zero, we get as limit of equation (1.33)

$$\int_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = \int_{A} \vec{D} \cdot \vec{N} \mathrm{d}a = \int_{A} \sigma \mathrm{d}a \,,$$

where $\vec{D} \cdot \vec{N}$ is the one-sided limit from the exterior of the body and A denotes a small portion of the surface S (covered by the topside of the control volume V). Because A can be chosen arbitrarily, it follows:

$$\boxed{\vec{D} \cdot \vec{N} = \sigma \text{ in the limit from outside}} \,. \tag{1.34}$$

1.6.4. Gauss's Law in Differential Form and Poisson's Equation

(i) According to equation (1.32) we have the following relation between the space charge density $\rho(\vec{r})$ and the displacement field $\vec{D}(\vec{r})$ caused by the space charge:

$$\int\limits_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = \int\limits_{V} \rho \mathrm{d}^3 r$$

for any control volume V. On the other hand, the displacement flux through ∂V can be converted into a volume integral over V by applying Gauss's integral theorem:

$$\int\limits_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = \int\limits_{V} \operatorname{div} \vec{D} \mathrm{d}^3 r \,.$$

Hence, we obtain:

$$\int\limits_{V} \operatorname{div} \vec{D} \, \mathrm{d}^3 r = \int\limits_{V} \rho \mathrm{d}^3 r \,.$$

As this equality holds for any arbitrary control volume V, the integrands must be equal, and we get **Gauss's law in the differential formulation**:

$$\boxed{\operatorname{div} \vec{D} = \rho} \,. \tag{1.35}$$

This is one of the fundamental laws of electromagnetism. It is generally valid, also in the case of a time-variant space charge density $\rho(\vec{r}, t)$.

(ii) In electrostatics, the $\vec{E}$-field is a gradient field: $\vec{E} = -\operatorname{grad} \Phi$. Furthermore, $\vec{D}$ and $\vec{E}$ are related by the constitutive law $\vec{D} = \varepsilon \vec{E}$. Inserting this in equation (1.35) yields **Poisson's equation**:

$$\boxed{\operatorname{div}(\varepsilon \operatorname{grad} \Phi) = -\rho} \,. \tag{1.36}$$

(iii) If, moreover, the permittivity ε is not position-dependent, we get:

$$\operatorname{div}(\operatorname{grad} \Phi) = -\frac{\rho}{\varepsilon} \,.$$

Using the **Laplace operator**

$$\Delta \Phi := \operatorname{div}(\operatorname{grad} \Phi)$$

leads to the simplified form of Poisson's equation

$$\boxed{\Delta \Phi = -\frac{\rho}{\varepsilon}} \,. \tag{1.37}$$

(Note that in Cartesian coordinates $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$)
This equation is the starting point for the calculation of electrostatic fields in practice-oriented applications. Typically, it has to be solved in a finite domain $\Omega \subset E_3$ for a given charge distribution $\rho(\vec{r})$. The "world outside Ω" is modelled by appropriate boundary conditions on $\partial\Omega$ (e.g., potential boundary values on contacts). However, systematic solution methods can not be treated in the context of this lecture.

1.6.5. Coulomb Potential

(i) The problem of finding the electric potential $\Phi(\vec{r})$ for a given discrete charge distribution $(q_i, \vec{r}_i)_{1\dots N}$, has already been solved in section 1.4.4 (equation (1.18)):

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon} \cdot \sum_{i=1}^{n} \frac{q_i}{|\vec{r} - \vec{r}_i|} \,. \tag{1.38}$$

This potential is a solution of Poisson's equation (1.37) in $E_3 \setminus \{P_1 \dots P_N\}$, where $P_i = \mathrm{O} + \vec{r}_i$ are the source points from which the electric field originates. Moreover, it satisfies the "boundary condition" $\Phi(\vec{r}) \to 0$ for $|\vec{r}| \to \infty$.

Unfortunately, this solution is not very useful for real engineering problems as it requires the knowledge of all field-creating charges in E_3. In practice, we have usually a different situation: The charges may be, for example, situated in a conductive body, where they can move freely around and arrange themselves self-consistently in space so that the interior of the body is field-free (i.e., it is an equipotential region, see section 1.7.1).

Yet, equation (1.38) shall be extended to the case of a continuous charge distribution $\rho(\vec{r})$, because this "Coulomb-Potential" is essential for electromagnetic field theory.

(ii) Derivation of Coulomb integral from equation (1.38):

We have to find the electric potential $\Phi(\vec{r})$ of a continuous charge distribution $\rho(\vec{r})$ defined everywhere in E_3. Instead of solving Poisson's equation (1.37) to this end, we start from equation (1.38). We imagine that $\rho(\vec{r})$ is the result of a quasi-continuous, discrete charge carrier distribution $(q_i, \vec{r}_i)_{1\dots N}$ with $N \to \infty$. Each charge q_i is smeared over a small cell d^3r around $\vec{r}_i$ so that the differential charge $\mathrm{d}Q(\vec{r}_i)$ contained in the volume d^3r equals q_i:

$$q_i = \mathrm{d}Q(\vec{r}_i) = \rho(\vec{r}_i)\mathrm{d}^3r \,.$$

In the limit $N \to \infty$, we replace the discrete sum over $\vec{r}_i$ by a continuous integral over $\vec{r}$:

$$\sum_i q_i \mapsto \int \mathrm{d}Q(\vec{r}) = \int \rho(\vec{r})\mathrm{d}^3r$$

More general, for expressions of the form $func(\vec{r}_i;\ \text{parameter})$ we use the following substitution rule:

$$\sum_i func(\vec{r}_i;\ \text{parameter}) q_i \to \int_{E_3} func(\vec{r};\ \text{parameter})\rho(\vec{r})\mathrm{d}^3r \,. \tag{1.39}$$

Applying this rule to equation (1.38) yields

$$\boxed{\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon} \int_{E_3} \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\mathrm{d}^3r' \qquad \text{(Coulomb potential)}} \,. \tag{1.40}$$

The **Coulomb potential** solves the equation $\Delta\Phi = -\frac{\rho}{\varepsilon}$ everywhere in space and satisfies the boundary condition $\Phi(\vec{r}) \to 0$ for $|\vec{r}| \to \infty$.

(iii) **Coulomb field**

The electric field $\vec{E}(\vec{r})$ generated by the space charge density $\rho(\vec{r})$ could be calculated from equation (1.40) by taking the gradient of Φ: $\vec{E} = -\operatorname{grad}\Phi$. However, it is easier to start from the field of a quasi-continuous point charge distribution given in equation (1.3) and to apply the substitution rule (1.39). The result is the **Coulomb field**

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon} \int\limits_{E_3} \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \rho(\vec{r}\,')\, \mathrm{d}^3 r'\,. \tag{1.41}$$

1.7. Electric Fields Between Conducting Media

1.7.1. Electrostatic Induction

(i) An **electric conductor** is a material containing a huge number of mobile charge carriers ($\approx 10^{21}$ - 10^{23} elementary charges per cm^3). In electrostatic equilibrium, they organize themselves in such a way that no internal electric field is generated and their charges are exactly compensated by the background charge of the substrate material. Hence, there is no space charge (local charge neutrality). Even a small disturbance of the field-free state will immediately be counterbalanced by the effect of the dielectric screening. As a consequence of the vanishing electric field in the interior of the conductor, the conductor together with its surface is an equipotential region:

$$\vec{E} = -\nabla\Phi = 0 \quad \Leftrightarrow \quad \Phi = \text{constans}\,.$$

(ii) When a neutral conductor is exposed to an external $\vec{E}$-field, the electrostatic equilibrium condition $\vec{E} = 0$ is maintained in its interior. To this end, electric charge must be shifted so that a surface charge density $\sigma(\vec{r})$ is induced on the boundary of the conducting body, while the interior stays neutral. This effect is called **electrostatic induction**.

The situation can be characterized by the following conditions:

a) $\vec{E} = 0$ in the interior of a conducting body.

b) $\vec{E} \perp$ conductor surface in the limit from outside.

c) The surface charge induced on the surface of the conductor satisfies the condition $\sigma = \vec{D} \cdot \vec{N}$ (in the limit from outside, see equation (1.34)).

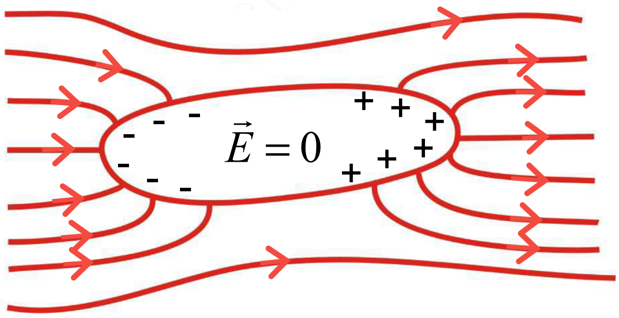


Figure 6: Electrostatic induction

1.7.2. Electric Capacitance

(i) **Definition of the capacitance of a two-conductor system**
We consider two conducting bodies L_1 and L_2 embedded in a dielectric material ("two-electrode capacitor"). The two conductors L_1 and L_2 are loaded with opposite charges Q and $-Q$, respectively, and have different electric potentials Φ_1 and Φ_2. The electric voltage

between the two conductors is obtained by integrating the electric field along a field line starting at conductor 1 and ending at conductor 2:

$$U_{12} = \Phi_1 - \Phi_2 = \int_{L_1}^{L_2} \vec{E} \cdot \mathrm{d}\vec{r} \, .$$

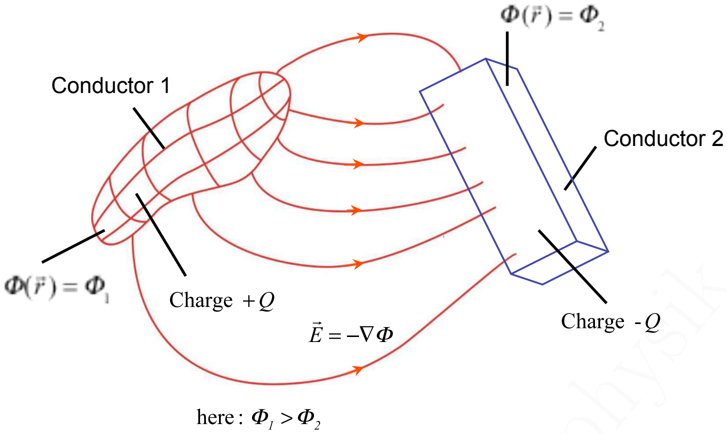


Figure 7: Schematic view of a two-electrode capacitor

As the charge Q located on conductor L_1 is equal in magnitude, but opposite in sign to the charge $-Q$ on conductor L_2, the capacitor as a whole has zero charge. We determine the charge Q on conductor L_1 by enveloping it by a closed surface S and calculate the displacement flux:

$$Q = \int_{S \text{ around } L_1} \vec{D} \cdot \mathrm{d}\vec{a} = \int_{S \text{ around } L_1} \varepsilon \vec{E} \cdot \mathrm{d}\vec{a} \, .$$

Here, ε is the permittivity of the material around both conductors. The **capacitance** is defined as

$$C = \frac{Q}{U_{12}} \tag{1.42}$$

The capacitance only depends on the geometry of the conductor system and the permittivity of the dielectric, but it does **not** depend on the applied voltage U_{12}. This can be immediately recognized from the relation

$$C = \frac{\int_{H_1} \varepsilon \vec{E} \cdot \mathrm{d}\vec{a}}{\int_{L_1}^{L_2} \vec{E} \cdot \mathrm{d}\vec{r}} \, . \tag{1.43}$$

As the electric field scales linearly with the voltage applied between the conductors, the magnitude of $\vec{E}$ changes by the same factor in numerator and denominator of equation (1.43) if the voltage is changed, so that C remains unchanged.

(ii) Example 1: **Plate capacitor**:

A plate capacitor consists of two parallel conducting plates with area A and plate distance d. The space in between is filled with a dielectric with constant permittivity ε. The plates are charged with opposite charges $\pm Q$. The resulting electric field $\vec{E}$ between the plates is constant and is perpendicular to the plates (stray fields at the boundary of the plates are neglected).
The electric voltage between the plates is

$$U_{12} = \int_{L_1}^{L_2} \vec{E} \cdot \mathrm{d}\vec{r} = E_z \cdot d$$

and the charge Q is

$$Q = \int_{S \text{ around } L_1} \vec{D} \cdot \mathrm{d}\vec{a} = \int_{L_1} \vec{D} \cdot \vec{e}_z \mathrm{d}x \mathrm{d}y = D_z \cdot A = \varepsilon \cdot E_z \cdot A$$

Hence, we get the capacitance:

$$C = \frac{Q}{U_{12}} = \varepsilon \cdot \frac{A}{d}\,. \tag{1.44}$$

Evidently, this expression depends on ε and the geometry of the capacitor only.

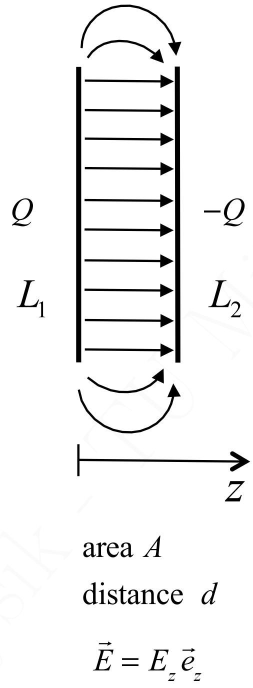


Figure 8: Plate capacitor

(iii) Example 2: **Spherical capacitor**:

A spherical capacitor consists of an inner, spherical conductor with radius a, which is concentrically placed in a spherical cavity with radius $b > a$. The material around the cavity is a perfect conductor. The cavity $a \leq r \leq b$ is filled with a dielectric with constant permittivity ε.
The capacitor is loaded with a charge $+Q$ on the surface of the inner sphere $|\vec{r}| = a$ and a counter charge $-Q$ on the spherical surface of the cavity $|\vec{r}| = b$. For symmetry reasons, the respective surface charge distributions are uniform and, hence, generate a spherically symmetric $\vec{E}$-field $\vec{E}(\vec{r}) = E(r)\vec{e}_r$ in the dielectric $a < r < b$. Applying Gauss's flux theorem for a concentric spherical surface within the cavity yields:

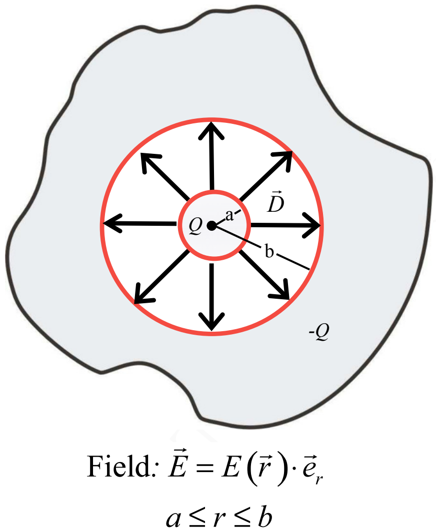


Figure 9: Spherical capacitor

$$Q = \int\limits_{|\vec{r}|=r} \vec{D} \cdot \mathrm{d}\vec{a} = \varepsilon \cdot E(r) 4\pi r^2\,, \text{ for } a \leq r \leq b\,.$$

Hence, we find

$$E(r) = \frac{Q}{4\pi \cdot \varepsilon} \cdot \frac{1}{r^2}$$

Integrating this electric field along a radial line between $a \leq r \leq b$, we obtain:

$$\begin{aligned} U_{ab} &= \int\limits_a^b E(r)\mathrm{d}r = \frac{Q}{4\pi\varepsilon} \cdot \int\limits_a^b \frac{1}{r^2}\mathrm{d}r\,, \\ &= \frac{Q}{4\pi\varepsilon} \cdot \left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon} \cdot \frac{b-a}{ab}\,. \end{aligned}$$

Thus, the capacitance of a spherical capacitor is

$$\boxed{C = \frac{Q}{U_{ab}} = 4\pi\varepsilon \frac{a \cdot b}{b-a}}\,. \tag{1.45}$$

In the limit $b \to \infty$, we obtain the "capacitance of a sphere w.r.t. infinity"

$$C_\infty = 4\pi\varepsilon a\,.$$

1.7.3. Capacitor Aggregates

(i) **Parallel circuit**:
N capacitors with capacitances $(C_i)_{i=1\ldots N}$ are connected in parallel. They are charged with opposite charges $\pm Q_i$ when a common voltage U is applied to each of the capacitors. We have

$$Q_i = C_i \cdot U \;\Rightarrow\; Q_{\text{total}} = \sum_{i=1}^{N} Q_i = U \cdot \sum_{i=1}^{N} C_i \,.$$

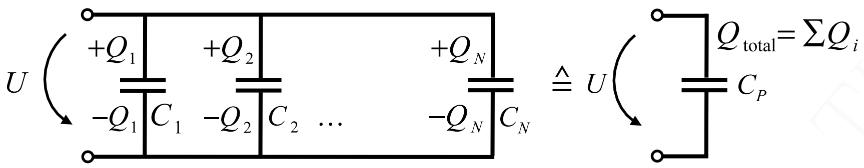


Figure 10: Parallel circuit of capacitors

From this we find the equivalent capacitance C_p of the parallel circuit:

$$\boxed{C_p = \frac{Q_{\text{total}}}{U} = \sum_{i=1}^{N} C_i} \,. \tag{1.46}$$

(ii) **Series circuit**:
N capacitors with capacitances $(C_i)_{i=1\ldots N}$ are connected in series. They are charged with opposite charges $\pm Q_i$ when a voltage U is applied at the terminals. Due to charge conservation on the connections between adjacent capacitors all charges Q_i have the same value $Q_1 = Q_2 = \cdots = Q_N =: Q$. The total voltage U is the sum of the partial voltages U_i applied to the individual capacitors. Hence, we conclude:

$$U_i = \frac{Q}{C_i} \;\Rightarrow\; U = \sum_{i=1}^{N} U_i = \sum_{i=1}^{N} \frac{Q}{C_i} = Q \sum_{i=1}^{N} \frac{1}{C_i} \,.$$

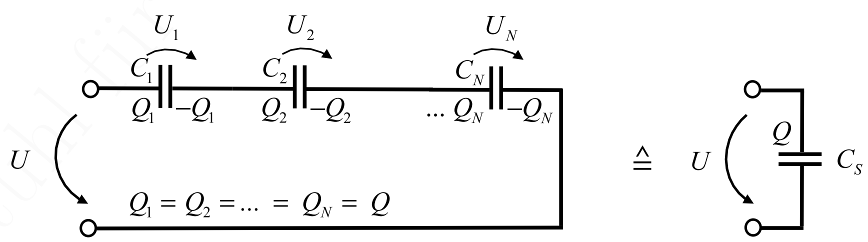


Figure 11: Series circuit of capacitors

From this we find the reciprocal value of the equivalent capacitance C_s for the series circuit:

$$\boxed{\frac{1}{C_s} = \sum_{i=1}^{N} \frac{1}{C_i}} \,. \tag{1.47}$$

(iii) **Dielectric layers in parallel**:
We consider a plate capacitor with plate distance d, where the space between the plates is divided into two parallel regions R_1 and R_2 with area A_1 and A_2, which are filled with two different dielectrics with permittivities ε_1 and ε_2. Applying a voltage U between the plates generates piecewise uniform electric fields $\vec{E}_1$ and $\vec{E}_2$ and displacement fields $\vec{D}_1$ and $\vec{D}_2$ in R_1 and R_2, respectively. The charge on the plates is piecewise uniformly distributed with constant surface charge densities σ_1 and σ_2. Integrating the electric field along a straight line between the plates in region R_1 and R_2 yields

$$|\vec{E}_1| \cdot d = U = |\vec{E}_2| \cdot d \quad \Rightarrow \quad |\vec{E}_1| = |\vec{E}_2| \,.$$

Using relation (1.34) allows us to determine the surface charge densities σ_1 and σ_2 as functions of U:

$$\sigma_1 = |\vec{D}_1| = \varepsilon_1 |\vec{E}_1| = \varepsilon_1 \cdot \frac{U}{d} \quad \text{and} \quad \sigma_2 = |\vec{D}_2| = \varepsilon_2 |\vec{E}_2| = \varepsilon_2 \cdot \frac{U}{d} \,.$$

Thus, we obtain as total charge

$$Q = Q_1 + Q_2 = \sigma_1 A_1 + \sigma_2 A_2 = \frac{U}{d}(\varepsilon_1 A_1 + \varepsilon_2 A_2) \,.$$

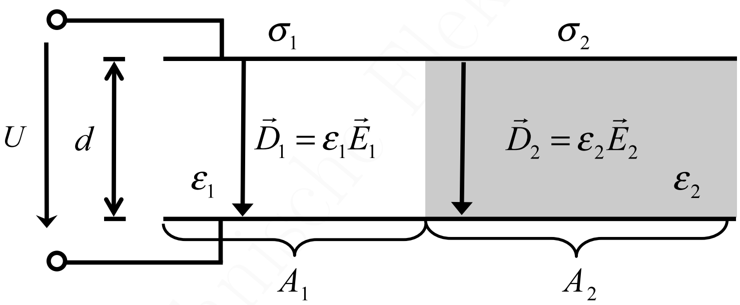


Figure 12: Dielectric layers in parallel

The capacitance of the whole structure is therefore:

$$C = \frac{Q}{U} = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} \,. \tag{1.48}$$

This result may be interpreted as parallel circuit of two plate capacitors with capacitances

$$C_1 = \frac{\varepsilon_1 A_1}{d} \quad \text{and} \quad C_2 = \frac{\varepsilon_2 A_2}{d} \,.$$

(iv) **Dielectric layers in series**:
We consider a plate capacitor, where the space between the layers is divided into two subsequent horizontal dielectric layers R_1 and R_2 with thickness d_1 and d_2 and different permittivities ε_1 and ε_2. The plate area is A. Applying a voltage U between the plates generates piecewise uniform electric fields $\vec{E}_1$ and $\vec{E}_2$ and displacement fields $\vec{D}_1$ and $\vec{D}_2$ in R_1 and R_2,

respectively. The charges $\pm Q$ on the plates are uniformly distributed with constant surface charge density σ_{top} and σ_{bottom}, respectively. With relation (1.34) follows:

$$|\vec{D}_1| = |\sigma_{\text{top}}| = \frac{Q}{A} = |\sigma_{\text{bottom}}| = |\vec{D}_2|\,.$$

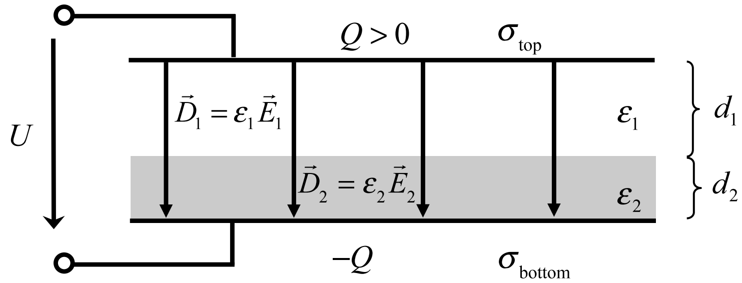


Figure 13: Dielectric layers in series

On the other hand, integrating the electric field along a straight line between the plates through region R_1 and R_2 yields

$$U = U_1 + U_2 = |\vec{E}_1| d_1 + |\vec{E}_2| d_2 = \frac{|\vec{D}_1|}{\varepsilon_1} d_1 + \frac{|\vec{D}_2|}{\varepsilon_2} d_2\,.$$

Combining the two relations above leads to

$$U = \frac{Q}{A}\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right)$$

with the final result:

$$C = \frac{Q}{U} = \frac{A}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}}\,. \tag{1.49}$$

This result may be interpreted as a serial circuit with two plate capacitors with capacitance

$$C_1 = \frac{\varepsilon_1 A}{d_1} \quad \text{and} \quad C_2 = \frac{\varepsilon_2 A}{d_2}\,.$$

1.7.4. Electric Field Energy

(i) **Energy stored in a charged capacitor**:

Imagine a capacitor composed of two conducting bodies L_1 and L_2, with a dielectric medium in between. Let L_1 be charged with a positive charge Q and L_2 with $-Q$. The voltage between L_1 and L_2 amounts to $U(Q) = Q/C$. In order to continue charging the capacitor, Q must be increased by $\Delta Q > 0$. This is done by transporting the charge ΔQ against the electrostatic force $\vec{F}_{\text{el}} = \Delta Q \cdot \vec{E}$ from L_2 to L_1. The mechanic work ΔW_{mech} (= energy taken from the mechanical transport system < 0) results in an increase ΔW_{el} of the electric energy stored in the capacitor. The energy balance reads:

$$\Delta W_{\mathrm{el}} = -\Delta W_{\mathrm{mech}} = -\int_{L_2}^{L_1} \Delta Q \cdot \vec{E} \cdot \mathrm{d}\vec{r} = \Delta Q \cdot \int_{L_1}^{L_2} \vec{E} \cdot \mathrm{d}\vec{r} = \Delta Q \cdot U(Q)\,.$$

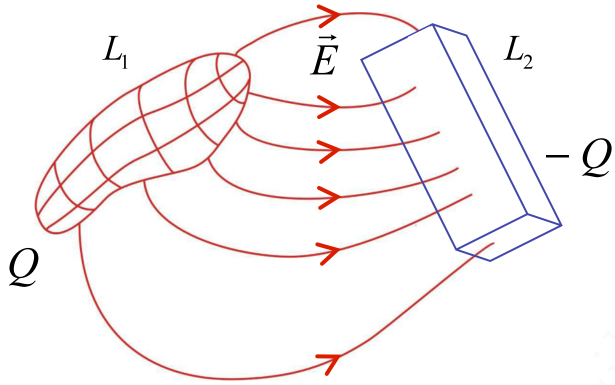


Figure 14: Schematic view of a two-electrode capacitor

So the differential increase of the electric energy $\mathrm{d}W_{\mathrm{el}}$ by charging the capacitor from Q to $Q + \mathrm{d}Q$ is

$$\mathrm{d}W_{\mathrm{el}} = U(Q)\mathrm{d}Q\,.$$

When we charge the capacitor from the empty state $Q = 0$ up to the final charge Q, the total energy stored in the capacitor is

$$W_{\mathrm{el}} = \int_0^Q U(Q')\mathrm{d}Q'\,. \tag{1.50}$$

In the case of an ideal linear capacitor with capacitance C, we have $U(Q) = Q/C$. So we get as final result

$$W_{\mathrm{el}} = \int_0^Q \frac{Q'}{C}\mathrm{d}Q' = \frac{1}{2} \cdot \frac{Q^2}{C}\,.$$

Alternative representations are:

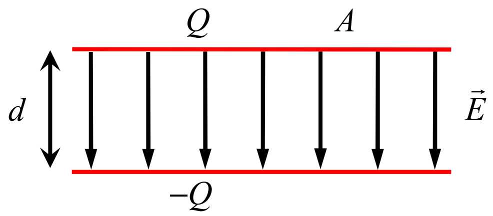
$$\boxed{W_{\mathrm{el}} = \frac{1}{2} \cdot \frac{Q^2}{C} = \frac{1}{2} \cdot U \cdot Q = \frac{1}{2} \cdot C \cdot U^2}\,. \tag{1.51}$$

(ii) **Energy density of the electric field**:

It is an interesting question in which part of a capacitor the stored energy is physically deposited. Our intuition suggests that the energy is stored as polarization energy in the dielectric. Following this idea, the energy must be spatially distributed in the dielectric with an energy density w_{el}. In the special case of a plate capacitor with plate area A and plate distance d, the energy density can be easily calculated. The electric energy stored in the volume $V = A \cdot d$ is

$$W_{\text{el}} = \frac{1}{2} \cdot U \cdot Q = \frac{1}{2} \cdot |\vec{E}| \cdot d \cdot |\vec{D}| \cdot A = \frac{1}{2} \cdot |\vec{E}| \cdot |\vec{D}| \cdot V$$

because $U = |\vec{E}| \cdot d$ and $Q = \sigma \cdot A = |\vec{D}| \cdot A$.



Hence, the **electric energy density** is

$$w_{\text{el}} = \frac{W_{\text{el}}}{V} = \frac{1}{2} \cdot |\vec{E}| \cdot |\vec{D}| = \frac{\varepsilon}{2} \cdot |\vec{E}|^2 = \frac{1}{2\varepsilon} \cdot |\vec{D}|^2 \,. \tag{1.52}$$

This conforms with the result derived from general electromagnetic field theory:

$$\boxed{w_{\text{el}} = \frac{1}{2} \cdot \vec{E} \cdot \vec{D}} \,. \tag{1.53}$$

One should note that this relation is also true in empty space without any dielectric material. This means that the electric field itself is the carrier of the electric energy!

2. Stationary Currents

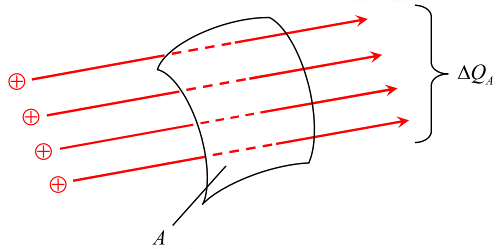
2.1. Electric Current and Current Density

We consider electrically conductive materials, which contain a huge number ($10^{15} - 10^{23}$ per cm^3) of mobile charge carriers. Under the action of an electric field or other driving forces (like particle diffusion or thermo-diffusion) the carriers move collectively like a fluid in a continuous current flow. This is described by the following concepts of continuum theory:

(i) **Definition of the electric current**
We place a control surface A in the flow of charges and determine the amount of charges $\mathrm{d}Q(A)$ passing through the area A per time $\mathrm{d}t$:

$$\text{Current } I(A) := \frac{\mathrm{d}Q(A)}{\mathrm{d}t}\,. \tag{2.1}$$

$$\text{Phys. unit: } \dim(I) = 1\frac{\mathrm{C}}{\mathrm{s}} = 1\ \mathrm{A(mpère)}\,.$$



(ii) **Electric Current Density**
The flow field of electric charges is described by a vector field $\vec{j}(\vec{r},t)$ defined by the following defining properties:

a) The direction of $\vec{j}$ is tangential to the flow lines (= trajectories of charge carriers).

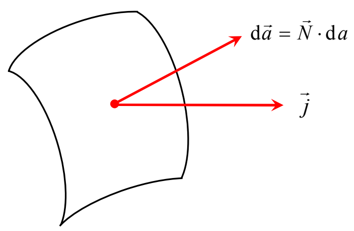
b) The absolute value of $\vec{j}$ is defined by the the following limit: A small control surface $\triangle A$ is placed **orthogonal to the flow direction** of the charges. The current density is defined as the current $I(\triangle A)$ per area $|\triangle A|$ that flows through the surface $\triangle A$ in the limit $|\triangle A| \to 0$:

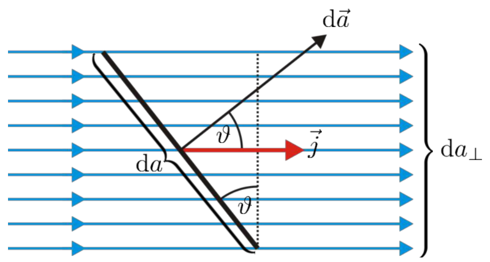
$$|\vec{j}| := \lim_{|\triangle A|\to 0} \frac{I(\triangle A)}{|\triangle A|}\,.$$

$$\text{Phys. unit: } \dim(|\vec{j}|) = 1\frac{\mathrm{A}}{\mathrm{m}^2}\ \left(\frac{\mathrm{A}}{\mathrm{cm}^2}\right)\,.$$

Using the current density $\vec{j}$ allows to calculate the current $I(S)$ flowing through a surface S of arbitrary shape. Let $\mathrm{d}\vec{a}$ be the vectorial surface element on S; then

$$\mathrm{d}Q = (\vec{j}\cdot\vec{N})\mathrm{d}a\mathrm{d}t = \vec{j}\cdot\mathrm{d}\vec{a}\mathrm{d}t$$





$\mathrm{d}a_\perp = \cos\vartheta \mathrm{d}a$
$\mathrm{d}I = |\vec{j}|\mathrm{d}a_\perp = |\vec{j}|\cos\vartheta\mathrm{d}a = \vec{j}\cdot\vec{N}\mathrm{d}a = \vec{j}\cdot\mathrm{d}\vec{a}$

is the differential amount of charge which passes through the surface da in the time dt. The inner product $\vec{j}\cdot\vec{N}$ takes into account that the current density is defined for a control surface orthogonal to the flow direction, whereas $\mathrm{d}\vec{a}$ may point in a direction oblique to the flow lines. The differential current flowing through the surface element $\mathrm{d}\vec{a}$ is then given as

$$\mathrm{d}I(\mathrm{d}\vec{a}) = \vec{j}\cdot\mathrm{d}\vec{a}\,.$$

Therefore, the total current flowing through the surface S is:

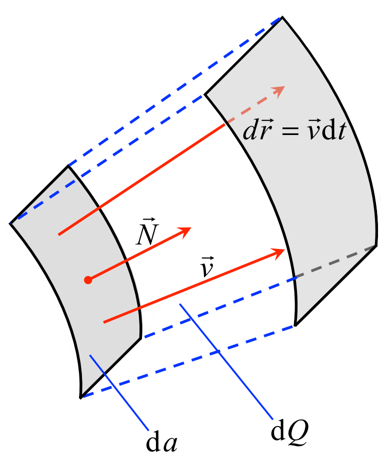
$$\boxed{I(S) = \int\limits_S \vec{j}\cdot\mathrm{d}\vec{a}\,.} \tag{2.2}$$

(iii) **Relation to the space charge density**

In section 1.6 we introduced the notion of the space charge density ρ to describe a continuous distribution of charge carriers. If we assume that these carriers are mobile, then there must be an interrelation between $\vec{j}$ and ρ.

Let $n(\vec{r})$ be the particle density of mobile charge carriers (= number of charge carriers per volume) and q the specific charge (= charge per carrier). Then we have

$$\rho(\vec{r}) = q\cdot n(\vec{r})\,. \tag{2.3}$$



Furthermore, let $\vec{v}(\vec{r})$ be the mean drift velocity (average velocity) of the charge carriers in the flow field. We imagine a small frame enclosing a control surface $\mathrm{d}\vec{a}$ and mark all carriers which are located on the control surface at the time $t = t_0$. During the time period $\mathrm{d}t$, they travel a distance $\mathrm{d}\vec{r} = \vec{v}\mathrm{d}t$ and are then located on a surface parallel-shifted from $\mathrm{d}\vec{a}$ by the distance $\mathrm{d}\vec{r}$. The charge carriers contained in the parallelepiped between $\mathrm{d}\vec{a}$ and the surface shifted by $\mathrm{d}\vec{r}$ are exactly those which have passed the left surface $\mathrm{d}\vec{a}$ in the time period between t_0 and $t_0 + \mathrm{d}t$. Therefore, the charge in the parallelepiped

$$\mathrm{d}Q = \rho \cdot \mathrm{d}V = qn\mathrm{d}V = qn \cdot \mathrm{d}\vec{a} \cdot \mathrm{d}\vec{r} = qn\vec{v} \cdot \mathrm{d}\vec{a}\mathrm{d}t$$

equals the charge which passed through $\mathrm{d}\vec{a}$:

$$\mathrm{d}Q = \vec{j} \cdot \mathrm{d}\vec{a}\mathrm{d}t\,.$$

Comparing both expressions leads to the conclusion

$$\vec{j} = q \cdot n \cdot \vec{v} = \rho \cdot \vec{v}\,. \tag{2.4}$$

In the case that the electric current density is composed of K different species of charge carriers, we obtain the general relation

$$\vec{j} = \sum_{\alpha=1}^{K} q_\alpha \cdot n_\alpha \cdot \vec{v}_\alpha\,, \tag{2.5}$$

where q_α, n_α and $\vec{v}_\alpha$ are the specific charge, carrier concentration and drift velocity of the carrier species α, respectively.

2.2. Transport of Charge Carriers in the Electric Field

2.2.1. Transport in Free Space without Scattering Processes

Mobile charge carriers are accelerated by the action of an electric field. If they are not deflected by collisions with other charge carriers or other scattering events, we can easily determine their motion: A charge carrier with mass m and charge q satisfies Newton's law of motion

$$m \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{F}_{el} = q \cdot \vec{E}(\vec{r})\,.$$

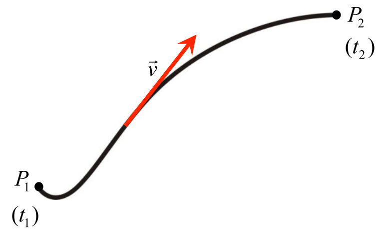
Taking the inner product of this equation with the velocity $\vec{v}(t)$ and integrating over the time period $[t_1, t_2]$, we obtain

$$\int_{t_1}^{t_2} m\vec{v} \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\mathrm{d}t = \int_{t_1}^{t_2} q \cdot \vec{E} \cdot \vec{v} \cdot \mathrm{d}t = \int_{t_1}^{t_2} q \cdot E \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot \mathrm{d}t\,.$$

The left- and right-hand side can be simplified to

$$\int_{t_1}^{t_2} \frac{1}{2} \cdot m \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{v}^2\right)\mathrm{d}t = q\int_{P_1}^{P_2} \vec{E} \cdot \mathrm{d}\vec{r}\,,$$

where P_1 and P_2 are the positions of the charge at the times t_1 and t_2, respectively.



Evaluating the two integrals yields

$$\frac{1}{2} \cdot m \cdot (v(t_2)^2 - v(t_1)^2) = q \cdot U_{12}\,, \tag{2.6}$$

where $v(t_i) = |\vec{v}(t_i)|$ is the velocity at time t_i and U_{12} denotes the electric voltage which the charge carrier has traversed, when it "runs down" the potential slope from P_1 to P_2. Equation (2.6) expresses the conservation of total energy: the gain in kinetic energy is balanced by the loss of electrostatic energy along the trajectory. If $v_1 = 0$ is set as initial velocity, and $v(U)$ denotes the velocity after traversing the voltage U, we find:

$$v(U) = \sqrt{\frac{2q}{m}} \cdot \sqrt{U}, \quad \text{if } v(t_1) = 0\,. \tag{2.7}$$

Note that the carrier velocity and, therefore, the current is a non-linear function of the voltage U.

2.2.2. Transport with Scattering Processes (Drift Model)

(i) **Carrier mobility**

The density of mobile charge carriers in a conductive material (typically a metal or a semiconductor) is extremely high. The charge carriers moving by the action of an electric field experience a high rate of collisions: collisions with other charge carriers and with scattering centres in the host material.

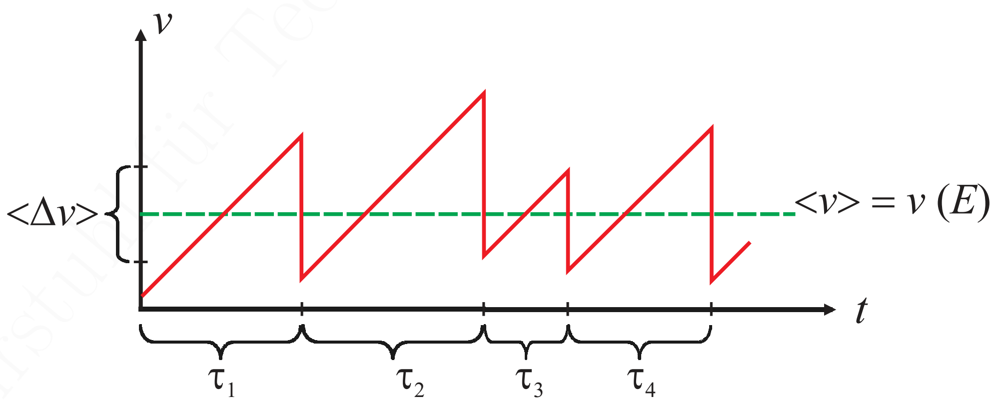


Figure 15: Particle velocity in a statistical sequence of collision events

The collective motion of particles in an ensemble must be treated by statistical methods to obtain the average over many subsequent collision events with collision times τ_1, τ_2, τ_3, ...

where the charge carriers are deflected and decelerated. As a result, we obtain a mean drift velocity $\vec{v}\left(\vec{E}\right)$ which describes the averaged collective motion of the charge carriers in an effective one-particle model. A simple approach is Drude's drift model. The model parameters are an effective mass m^* of the charge carriers and a mean collision time $\tau = \langle \tau_i \rangle$ between subsequent collisions. A statistical analysis leads to the conclusion that the variance $\langle \Delta\vec{v} \rangle$ of the velocity distribution has about the same value as the mean drift velocity:

$$\langle \Delta\vec{v} \rangle \approx \vec{v}(\vec{E})$$

Then the time-averaged equation of motion for a statistically representative charge carrier reads

$$q\vec{E} = m^* \cdot \left\langle \frac{\Delta\vec{v}}{\Delta t} \right\rangle = m^* \frac{\langle \Delta\vec{v} \rangle}{\tau} = m^* \frac{\vec{v}}{\tau} \,,$$

from which we derive a mean drift velocity proportional to the driving $\vec{E}$-field:

$$\vec{v} = \frac{q \cdot \tau}{m^*} \cdot \vec{E} = \mathrm{sgn}(q) \cdot \mu \cdot \vec{E} \,, \tag{2.8}$$

with the carrier mobility defined as

$$\mu := \frac{|q|\tau}{m^*} > 0 \tag{2.9}$$

This quantity is central for characterizing the drift motion. Inserting the linear relation (2.8) in equation (2.4) results in

$$\vec{j} = q \cdot n \cdot \vec{v} = q \cdot n \cdot \mathrm{sgn}(q) \cdot \mu \cdot \vec{E} \,,$$

i.e., we obtain a linear relation between current density and electric field.

$$\vec{j} = |q| \cdot n \cdot \mu \cdot \vec{E}. \tag{2.10}$$

If the current $\vec{j}$ is composed of K different species of carriers, we find the general relation

$$\vec{j} = \sum_{\alpha=1}^{K} |q_\alpha| \cdot n_\alpha \cdot \mu_\alpha \cdot \vec{E} \,, \tag{2.11}$$

where μ_α denotes the mobility of species α.

(ii) **Ohm's law (local formulation)**

Regardless of the number of carrier species contributing to the current flow, we find a linear relation between the current density $\vec{j}$ and the electric field$\vec{E}$:

$$\boxed{\vec{j} = \sigma \cdot \vec{E} \quad \text{(Ohm's law)}} \tag{2.12}$$

with the positive transport coefficient

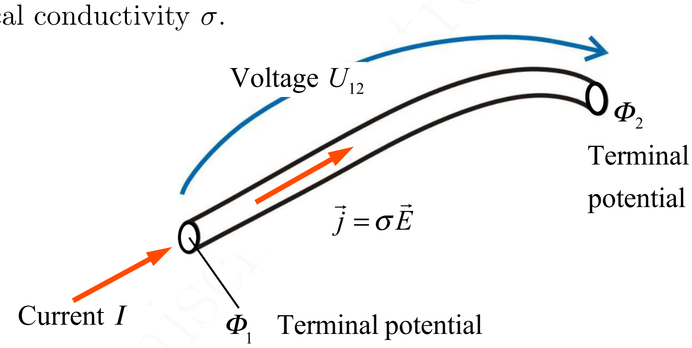
$$\sigma = \sum_{\alpha=1}^{K} |q_\alpha| n_\alpha \mu_\alpha \,. \tag{2.13}$$

This quantity is called "**electric conductivity**". Note that $\sigma > 0$ implies that the drift current and the electric field point in the same direction. If $\vec{E}$ is a quasi-static gradient field, $\vec{E} = -\nabla\Phi$, then the electric current flows from the higher potential in the direction of the lower potential.

$$\text{Phys. unit:}\, \dim(\sigma) = 1\frac{\text{A}}{\text{Vm}} = 1\frac{1}{\Omega\text{m}} \,.$$

(iii) **Ohm's law (integral formulation)**

We consider a wire with length l and uniform cross section A, which is made of a material with uniform electrical conductivity σ.



Applying different terminal potentials Φ_1 and Φ_2 at the contacts on the two ends of the wire causes a voltage drop $U = \Phi_1 - \Phi_2 > 0$ along the wire and, inside the wire, an electric field $\vec{E}$ and a current density $\vec{j}$ pointing in tangential direction of the wire with constant magnitude

$$|\vec{E}| = \frac{U}{l} \quad \text{and} \quad |\vec{j}| = \sigma|\vec{E}| \,.$$

The total current I flowing through the wire is

$$I = \int_A \vec{j} \cdot \mathrm{d}\vec{a} = \int_A \sigma\vec{E} \cdot \mathrm{d}\vec{a} = \sigma \cdot |\vec{E}| \cdot A = \sigma \frac{A}{l} U \,.$$

With the definitions

$$\text{electric conductance } G := \sigma \cdot \frac{A}{l} \,, \tag{2.14}$$

$$\text{electric resistance } R := \frac{1}{G} = \frac{1}{\sigma} \cdot \frac{l}{A} \tag{2.15}$$

the above equation can be re-written as **Ohm's law in integral form**:

$$I = G \cdot U \,, \tag{2.16}$$

$$\boxed{U = R \cdot I \,.} \tag{2.17}$$

It is common to define the

$$\text{electric resistivity } \rho := \frac{1}{\sigma} \tag{2.18}$$

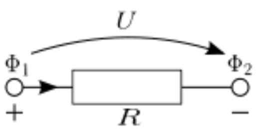
so that the electric resistance can be expressed as

$$R = \rho \cdot \frac{l}{A} \,. \tag{2.19}$$

Physical unit:

$$\dim(R) = 1\frac{V}{A} = 1\Omega(\text{Ohm}); \;\; \dim(\rho) = 1\Omega\text{m, often also } 1\Omega\frac{\text{mm}^2}{\text{m}}$$

(iv) Ohm's law (2.17) is applicable to any conductive body of arbitrary shape, if it has two terminals with applied voltage U and $\vec{j}(\vec{r}) = \sigma\vec{E}(\vec{r})$ holds everywhere inside the body. However, the explicit calculation of R may be sophisticated. In a circuit, regardless of the geometric shape of a resistor, it is represented by the following block symbol:



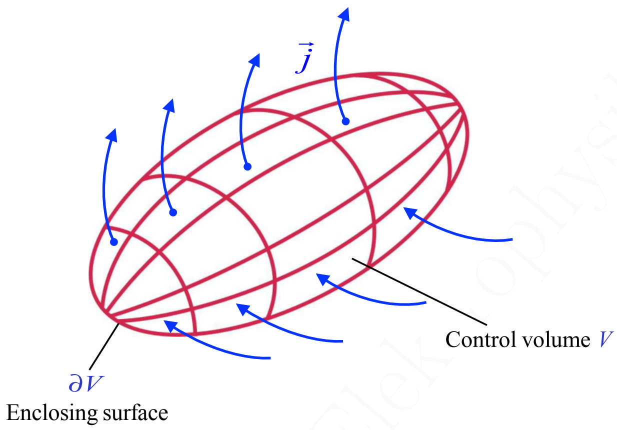
$$U = R \cdot I, \;\; \Phi_1 > \Phi_2$$

2.3. Charge Conservation and Kirchhoff's Current Law

2.3.1. Charge Balance in Integral Representation

Electric charges can neither be generated nor annihilated. The amount of charge $Q(V)$ contained in a control volume V can only change in time by a flow of charge carriers into V or out of V. The net outflow of charge is given by the flux integral of the current density $\vec{j}$ along the closed surface ∂V. It is counterbalanced by the decrease (= negative increase) of $Q(V)$ with time, leading to the **charge balance equation**:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(V) = -\int\limits_{\partial V} \vec{j} \cdot \mathrm{d}\vec{a} \,. \tag{2.20}$$



In the case of stationary current flow, the charge $Q(V)$ must not change with time. As a consequence, the condition

$$\int\limits_{\partial V} \vec{j} \cdot \mathrm{d}\vec{a} = 0 \tag{2.21}$$

must be satisfied for any closed surface ∂V.

2.3.2. Kirchhoff's Current Law

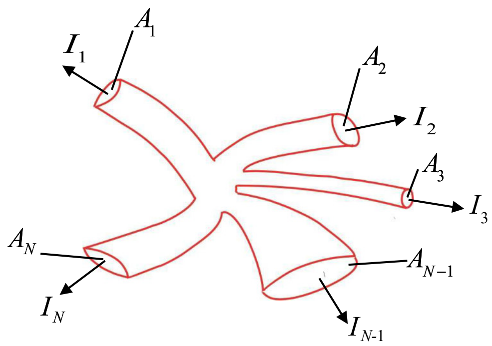
We consider a conductive body with N contacts A_1, A_2,...A_N. The outward-directed terminal current through the contact A_k is

$$I_k = \int\limits_{A_k} \vec{j} \cdot \mathrm{d}\vec{a}$$

so that $I_k > 0$ indicates an outgoing current. We enclose the body with a closed surface ∂V which coincides piecewise with each of the contacts A_k.

We assume stationary current flow through the body and evaluate the balance equation (2.21). Evidently, only the contact areas A_k contribute to the flux integral, and we obtain:

$$0 = \int\limits_{\partial V} \vec{j} \cdot \mathrm{d}\vec{a} = \sum_{k=1}^{N} \int\limits_{A_k} \vec{j} \cdot \mathrm{d}\vec{a} = \sum_{k=1}^{N} I_k \,.$$



This is **"Kirchhoff's current law"**, which says that in an electrical network the algebraic sum of all branch currents I_k meeting at a node must be zero:

$$\sum_{k=1}^{N} I_k = 0 \tag{2.22}$$

2.3.3. Charge Conservation in Differential Form

The charge enclosed in a control volume V can be expressed by the space charge density ρ as

$$Q(V) = \int\limits_{V} \rho(\vec{r}, t) \mathrm{d}^3 r \,.$$

We assume that the domain V does not vary with time. Then we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(V) = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{V} \rho(\vec{r}, t) \mathrm{d}^3 r = \int\limits_{V} \frac{\partial \rho}{\partial t}(\vec{r}, t) \mathrm{d}^3 r \,. \tag{2.23}$$

The flux integral over the current density $\vec{j}$ can be converted into a volume integral by virtue of Gauss's integral theorem:

$$\int\limits_{\partial V} \vec{j} \cdot \mathrm{d}\vec{a} = \int\limits_{V} \operatorname{div} \vec{j} \, \mathrm{d}^3 r \,. \tag{2.24}$$

Inserting these results in the charge balance equation (2.20) yields

$$\int\limits_{V} \operatorname{div} \vec{j} \, \mathrm{d}^3 r + \int\limits_{V} \frac{\partial \rho}{\partial t} \, \mathrm{d}^3 r = 0 \,,$$

or, equivalently,

$$\int\limits_V \left(\operatorname{div} \vec{j} + \frac{\partial \rho}{\partial t} \right) \mathrm{d}^3 r = 0 \,.$$

As this equation holds for any arbitrarily shaped control volume V, the integrand must vanish, and we obtain the **charge balance in differential form**

$$\boxed{\operatorname{div} \vec{j} + \frac{\partial \rho}{\partial t} = 0} \,. \tag{2.25}$$

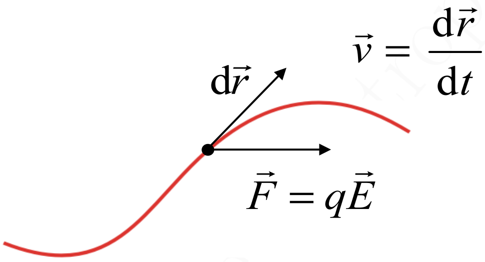
This relation is also called **(charge) continuity equation** or **charge conservation equation**.

2.4. Electric Power and Energy Transmission

2.4.1. Electric Power Performed by a Point Charge

When a point charge q moves in an electric field $\vec{E}$, the electric field performs the differential electric work

$$\mathrm{d}W_{el} = \vec{F} \cdot \mathrm{d}\vec{r} = q \cdot \vec{E} \cdot \mathrm{d}\vec{r} \,.$$



Hence, when the charged particle moves with the velocity $\vec{v}$, it performs the electric power

$$P_{\mathrm{el}} = \frac{\mathrm{d}W_{\mathrm{el}}}{\mathrm{d}t} = q \cdot \vec{E} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = q \cdot \vec{E} \cdot \vec{v} \,. \tag{2.26}$$

Note that this power is rendered by the charge (i.e. from the electrical energy reservoir), if P_{el} is positive.

2.4.2. Electric Power Performed by an Electric Flow Field

We assume that K different species of charge carriers with specific charge q_α, carrier concentration n_α and drift velocity $\vec{v}_\alpha$ contribute to the flow of current. Then, according to equation (2.5), the total current density is given by

$$\vec{j} = \sum_{\alpha=1}^{K} q_\alpha \cdot n_\alpha \cdot \vec{v}_\alpha \,.$$

The electric power performed by one carrier of species α amounts to

$$P_{\mathrm{el}}^{(\alpha)} = q_\alpha \cdot \vec{v}_\alpha \cdot \vec{E}$$

(see equation (2.26)). It is practical to introduce the concept of power density. This is the electrical power rendered by the flow field per volume element. The total power density is the sum of the contributions from the individual charge species:

$$p_{\text{el}} = \sum_{\alpha=1}^{K} n_\alpha \cdot P_{\text{el}}^{(\alpha)} = \sum_{\alpha=1}^{K} (n_\alpha \cdot q_\alpha \cdot \vec{v}_\alpha) \cdot \vec{E} = \left(\sum_{\alpha=1}^{K} n_\alpha \cdot q_\alpha \cdot \vec{v}_\alpha \right) \cdot \vec{E} = \vec{j} \cdot \vec{E}\,.$$

This is an interesting and surprising result: Regardless of the composition of the flow field, the electric power density is, in the case of pure drift current, given by the simple relation

$$\boxed{p_{\text{el}} = \vec{j} \cdot \vec{E}}\,. \tag{2.27}$$

2.4.3. Electric Power Loss in Ohmic Resistors

The basic law for ohmic drift motion is

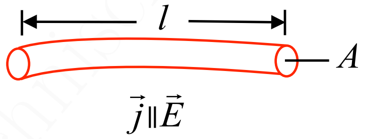
$$\vec{j} = \underbrace{\sigma}_{>0} \cdot \vec{E}\,,$$

where the electric conductivity σ is always positive. Therefore, the electric power density rendered by the electric system is

$$p_{el} = \vec{j} \cdot \vec{E} = \sigma |\vec{E}|^2 = \frac{1}{\sigma} |\vec{j}|^2 \geq 0\,. \tag{2.28}$$

Since $\sigma > 0$, this quantity is positive, i.e., the electric system always dissipates energy. Therefore, p_{el} is also called **power loss density**. The dissipated energy is converted into heat.

The total power loss in a wire-shaped ohmic resistor can be calculated as follows: Consider a wire with length l and uniform cross section A.



As discussed in section 2.2.2, the magnitudes of the electric field $|\vec{E}|$ and the current density $|\vec{j}|$ are constant in the interior of the wire. Therefore, the power loss density $p_{\text{el}} = |\vec{j}||\vec{E}|$ is constant. Integrating p_{el} over the wire yields the total power loss:

$$P_{\text{el}} = \int_{\text{Wire}} p_{\text{el}}(\vec{r})\, \mathrm{d}^3 r = p_{\text{el}} \cdot l \cdot A = |\vec{j}| \cdot A \cdot |\vec{E}| \cdot l\,.$$

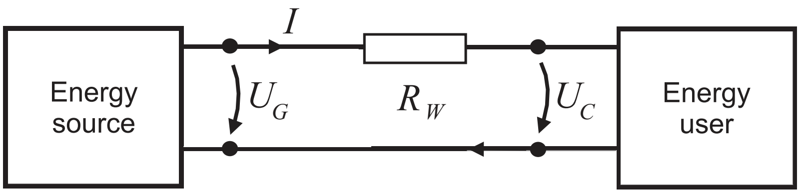
Since the voltage drop along the wire is $U = |\vec{E}| \cdot l$ and the current through the wire is $I = |\vec{j}| \cdot A$, we conclude (using $U = I \cdot R$):

$$\boxed{P_{\text{el}} = U \cdot I = \frac{U^2}{R} = R \cdot I^2}\,, \tag{2.29}$$

where R denotes the ohmic resistance of the wire. It can be shown that this result is generally valid, regardless of the shape of the resistor.
Unit of the electric power: $\dim(P_{\text{el}}) = 1\,\text{VA} = 1\,\text{W(att)}$.

2.4.4. Electric Power Transmission Line

Electric power has to be transported from a power generator (e.g., power plant) to the energy consumer. To this end, a transmission line is used, which consists of an outgoing wire and a return wire.



The wires have a distributed ohmic resistance, which we can imagine as an equivalent lumped resistance R_W. At the generator side a voltage U_G is fed into the transmission line, but at the consumer side a smaller voltage U_C arrives because of the voltage drop along the wires. For a current I flowing through the outgoing wire and the return wire, we find:

$$U_C = U_G - R_W \cdot I \,. \tag{2.30}$$

The power delivered by the power generator is

$$P_G = U_G \cdot I. \tag{2.31}$$

The power received by the consumer is

$$P_C = U_C \cdot I \,. \tag{2.32}$$

For assessing the quality of the transmission line, we introduce the transmission efficiency

$$\eta := \frac{P_C}{P_G} \tag{2.33}$$

as a figure of merit. Using equations (2.30) – (2.32), we obtain

$$\eta = \frac{P_C}{P_G} = \frac{U_C}{U_G} = \frac{U_G - R_W \cdot I}{U_G} = 1 - \frac{R_W \cdot I \cdot U_G}{U_G^2} \,,$$

and thus:

$$\boxed{\eta = 1 - \frac{R_W \cdot P_G}{U_G^2} \,.} \tag{2.34}$$

Evidently, η is always smaller than 100%. In order to optimize the efficiency for a given generator power P_G, the transmission voltage U_G should be chosen as high as possible, because $\eta \to 1$ for $U_G \to \infty$. Therefore, voltages around some 100 kV are used for overhead power lines.

3. Magnetostatics

3.1. Forces on Charges Moving in Magnetic Fields

3.1.1. Lorentz Force and Magnetic Field

(i) The Lorentz force:

The fundamental property of the magnetic field consists in that it exerts a force on **moving** point charges — the so-called **Lorentz force**. The magnetic field can be represented as a vectorial field quantity $\vec{B}(\vec{r},t)$. When a point charge q moves along a trajectory $\vec{r}(t)$ with the velocity $\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$, it experiences a force $\vec{F}_L$ which is directed perpendicular to $\vec{v}$ and $\vec{B}$. This means that the Lorentz force is always acting in a direction orthogonal to the trajectory.

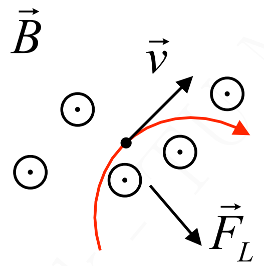


Figure 16: Action of the Lorentz force. The magnetic field is orthogonal to the plane of projection.

The basic vectorial relation is

$$\boxed{\vec{F}_{\mathrm{L}} = q(\vec{v} \times \vec{B})} \tag{3.1}$$

The field quantity $\vec{B}$ is called **magnetic induction** or **magnetic flux density**; very often it is just referred to as "$\vec{B}$-field".

Equation (3.1) also defines the physical unit of the $\vec{B}$-field:

$$\dim(|\vec{B}|) = \frac{\mathrm{Vs}}{\mathrm{m}^2} = 1\mathrm{T(esla)}\,.$$

(ii) **Electromagnetic force**: If an electric field $\vec{E}$ and a magnetic field $\vec{B}$-field concurrently act on a moving charged particle, the superposition principle can be applied and the electric and magnetic force vectors are added to constitute the **electromagnetic force**

$$\vec{F}_{\mathrm{em}} = q \cdot (\vec{E} + \vec{v} \times \vec{B})\,. \tag{3.2}$$

(iii) By the Lorentz force, no energy can be fed in or detracted from a moving charge q, because the force is always perpendicular to the direction of motion $\vec{v}(t)$. Since

$$\mathrm{d}W_{\mathrm{mag}} = \vec{F}_L \cdot \mathrm{d}\vec{r} = q\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{B}\right) \cdot \mathrm{d}\vec{r}$$

is the differential work performed by the magnetic field, we obtain for the power performed on the charge

$$P_{\mathrm{mag}} = \frac{\mathrm{d}W_{\mathrm{mag}}}{\mathrm{d}t} = q\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{B}\right) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = 0\,.$$

If the Lorentz force solely acts on a free charge, the kinetic energy and, therefore, the absolute value of the velocity $|\vec{v}(t)|$ stays constant in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{kin}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\cdot m\cdot \vec{v}^2\right) = m\vec{v}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{v}\cdot\vec{F}_L \overset{\vec{v}\perp\vec{F}_L}{=} 0 \quad \Rightarrow |\vec{v}(t)| = \text{const.}$$

3.1.2. Motion of a Charged Particle in a Constant Magnetic Field

We consider the case that a particle with charge q and mass m is moving under the action of a constant $\vec{B}$-field, without any other forces involved. Then Newton's law of motion says

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = q\left(\vec{v}\times\vec{B}\right). \tag{3.3}$$

This a linear differential equation of first order for the vector function $\vec{v}(t)$. Once this equation is solved, the trajectory $\vec{r}(t)$ can be calculated by integration of $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v}(t)$. For a unique solution $\vec{r}(t)$, the initial values

$$\vec{r}(t=t_0) = \vec{r}_0 \quad \text{and} \quad \vec{v}(t=t_0) = \vec{v}_0$$

must be specified. Without loss of generality, the z-axis of a Cartesian coordinate system (O, $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$) can be aligned with the direction of the constant $\vec{B}$-field:

$$\vec{B} = B\cdot\vec{e}_z; \quad B \geq 0\,.$$

With the vector product

$$\vec{v}\times\vec{B} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = B\cdot(v_y\vec{e}_x - v_x\vec{e}_y)$$

the three components of equation (3.3) read:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = q\cdot B\cdot v_y \tag{3.4}$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -q\cdot B\cdot v_x \tag{3.5}$$

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = 0 \tag{3.6}$$

The third equation has the solution

$$v_z(t) = \text{constant} = v_{0z}\,.$$

Hence, we focus on the projection of $\vec{v}(t)$ on the v_x-v_y-plane

$$\vec{v}_\perp(t) := v_x(t)\cdot\vec{e}_x + v_y(t)\cdot\vec{e}_y$$

We know already that the kinetic energy and, therefore, $|\vec{v}(t)| = |\vec{v}_0| = v_0$ is a constant of motion. Therefore

$$|\vec{v}_\perp(t)|^2 = v_x(t)^2 + v_y(t)^2 = |\vec{v}(t)|^2 - v_z^2(t) = v_0^2 - v_{0z}^2 = \text{constant} =: v_\perp^2$$

is a constant of motion, too.

This means that the vector $\vec{v}_\perp(t)$ moves on a circle with radius $v_\perp$ around the origin in the v_x-v_y-plane, leading us to the following ansatz:

$$\begin{aligned} v_x(t) &= v_\perp \cdot \sin\left(\Omega(t-t_0)\right) \ , \\ v_y(t) &= v_\perp \cdot \cos\left(\Omega(t-t_0)\right) \ . \end{aligned}$$

(uniform circular motion with angular velocity Ω)

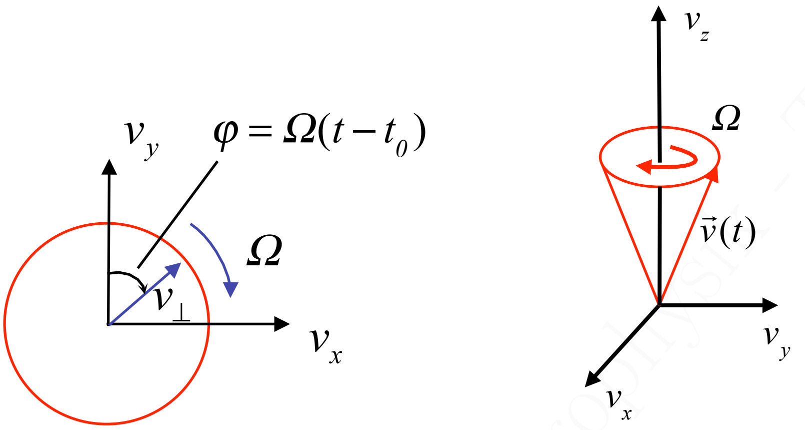


Figure 17: Motion of a point charge in a uniform magnetic field

Inserting this ansatz in the equations of motion (3.4) and (3.5) shows that they are solved, if and only if the **gyrofrequency** Ω takes the value

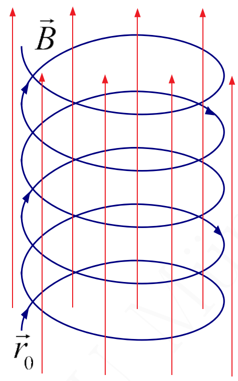
$$\boxed{\Omega = \frac{q \cdot B}{m}} . \tag{3.7}$$

Finally, we calculate the trajectory $\vec{r}(t)$ by integrating $\vec{v}(t)$:

$$\begin{aligned} \vec{r}(t) &= \vec{r}_0 + \int\limits_{t_0}^{t} \vec{v}(\tau)\mathrm{d}\tau = \\ &= \vec{r}_0 + v_\perp \left[\int\limits_{t_0}^{t} \sin\left(\Omega(\tau - t_0)\right) \mathrm{d}\tau \cdot \vec{e}_x + \int\limits_{t_0}^{t} \cos\left(\Omega(\tau - t_0)\right) \mathrm{d}\tau \cdot \vec{e}_y \right] + v_{0z} \cdot (t - t_0) \cdot \vec{e}_z \, . \end{aligned}$$

The result is

$$\vec{r}(t) = \vec{r}_0 - v_\perp \cdot \frac{1}{\Omega} \cdot \Big[(\cos(\varphi(t)) - 1) \cdot \vec{e}_x - \sin(\varphi(t)) \cdot \vec{e}_y\Big] + v_\parallel (t - t_0) \cdot \vec{e}_z$$



with

$$\varphi(t) = \Omega(t - t_0)\,.$$

The trajectory is a helix in z-direction with initial point $\vec{r}_0$ and radius

$$R = \frac{v_\perp}{\Omega} = \frac{v_\perp \cdot m}{q \cdot B}\,. \tag{3.8}$$

The charged particle is guided along the $\vec{B}$-field lines in a helical motion. With increasing $\vec{B}$-field the radius R of the helix shrinks, while the gyrofrequency Ω increases.

3.1.3. Lorentz Force Acting on a Current Distribution

We assume that a current distribution is composed of K different carriers species (cf. equation (2.5))

$$\vec{j} = \sum_{\alpha=1}^{K} q_\alpha n_\alpha \vec{v}_\alpha\,.$$

On the statistical average (drift model), a particle of species α experiences the Lorentz force

$$\vec{F}_{L,\alpha} = q_\alpha (\vec{v}_\alpha \times \vec{B})\,.$$

If there are n_α carriers per volume element, they experience the force density $n_\alpha \vec{F}_{L,\alpha}$. Hence, the **Lorentz force density** acting on a volume element is

$$\vec{f}_L = \sum_{\alpha=1}^{K} \Big[n_\alpha q_\alpha (\vec{v}_\alpha \times \vec{B})\Big] = \left(\sum_{\alpha=1}^{K} q_\alpha n_\alpha \vec{v}_\alpha\right) \times \vec{B} = \vec{j} \times \vec{B}\,.$$

The final result

$$\boxed{\vec{f}_\mathrm{L} = \vec{j} \times \vec{B}} \tag{3.9}$$

is independent of the composition of the current flow.

If an electric field $\vec{E}$ is additionally acting on the charge carriers, we have to extend equation (3.9) by the electric force:

$$\vec{f}_\mathrm{em} = \sum_{\alpha=1}^{K} n_\alpha \Big[q_\alpha \vec{E} + q_\alpha \Big(\vec{v}_\alpha \times \vec{B}\Big)\Big] = \rho \vec{E} + \vec{j} \times \vec{B} \tag{3.10}$$

with the space charge density

$$\rho = \sum_{\alpha=1}^{K} q_\alpha n_\alpha\,.$$

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3.2. Lorentz Force and Torque on Current-Carrying Conductors

3.2.1. Force on a Conducting Body with Arbitrary Shape

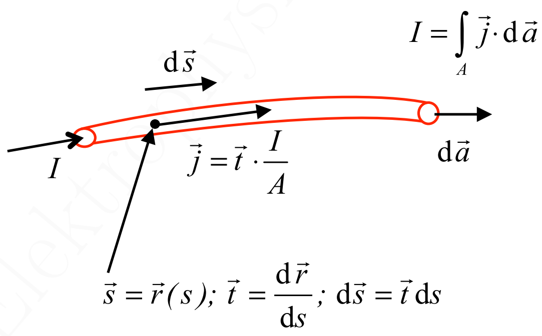
Imagine an electric current flowing through a solid body. Under the action of a magnetic field, the charged particles moving in the host substrate experience the Lorentz force. We assume that, by collisions of the mobile particles with the substrate, the Lorentz force density acting on a volume element is completely transferred to the substrate itself. As a result, the Lorentz force density (3.9) can be thought to act on the solid body itself. Therefore, the total force acting on a current-carrying solid body is given by the integral

$$\vec{F}_{\text{body}} = \int_{\text{body}} \vec{j}(\vec{r}) \times \vec{B}(\vec{r}) \mathrm{d}^3 r \,. \tag{3.11}$$

3.2.2. Force on a Current-Carrying Wire

In the case of a wire, equation (3.11) can be simplified as follows: The wire is represented by a curve C with parametrization $s \mapsto \vec{r}(s)$, where s is the arc length. The tangential vector $\vec{t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}$ has unit length. The cross sections $A(s)$ of the wire are orthogonal to $\vec{t}$ and have a constant area A along the wire. We assume that the current I through the wire is uniformly distributed over the cross section, resulting in the current density

$$\vec{j}(\vec{r}(s)) = \frac{I}{A}\,\vec{t}(s) \,.$$



The volume integral in equation (3.11) is calculated in two steps: For a fixed position $\vec{r}(s)$ on the wire, we integrate over the cross section $A(s)$; then we integrate along the curve:

$$\vec{j}(\vec{r})\,\mathrm{d}^3 r = \vec{j}(\vec{r})\,\mathrm{d}a\,\mathrm{d}s = \frac{I}{A}\,\mathrm{d}a\,\vec{t}\,\mathrm{d}s \,.$$

Since $\vec{t}\,\mathrm{d}s = \mathrm{d}\vec{s}$ is the vectorial line element along the curve C, we obtain:

$$\vec{F}_{\text{wire}} = \int_V \vec{j}(\vec{r}) \times \vec{B}(\vec{r})\ \mathrm{d}^3 r = -\int_C \int_{A(\vec{s})} \vec{B}(\vec{r}(s)) \times \left(\frac{I}{A}\right)\ \mathrm{d}a\,\mathrm{d}\vec{s}$$

The integral over the cross section has the constant value A, leading to the result:

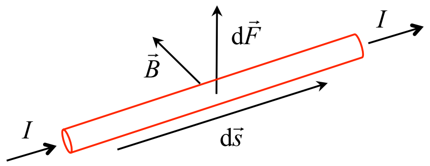
$$\boxed{\vec{F}_{\text{wire}} = -I \int_C \vec{B}(\vec{s}) \times \mathrm{d}\vec{s}} \,. \tag{3.12}$$

The force $\vec{F}_{\text{wire}}$ may be interpreted as the integral of infinitesimal contributions $\mathrm{d}\vec{F}_{\mathrm{L}}$, where $\mathrm{d}\vec{F}_{\mathrm{L}}$ is the differential Lorentz force on a current-carrying wire element $\mathrm{d}\vec{s}$:

$$\vec{F}_{\text{wire}} = \int_C \mathrm{d}\vec{F}_L \tag{3.13}$$

with

$$\mathrm{d}\vec{F}_L = I \cdot \mathrm{d}\vec{s} \times \vec{B} \tag{3.14}$$



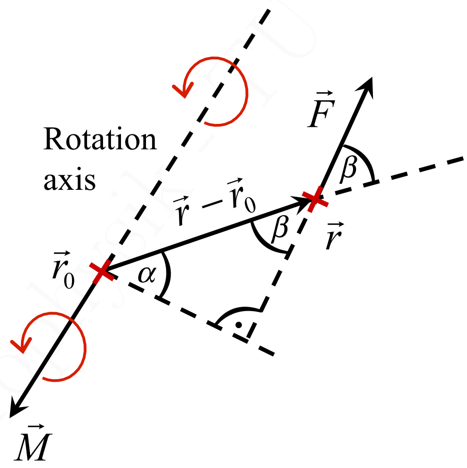
3.2.3. Torque on a Current-Carrying Loop

(i) **Torque on a fixed axis of rotation**

We consider a lever arm mounted perpendicular to a fixed axis of rotation. The mounting point of the arm is $\vec{r}_0$. If a force $\vec{F}$ acts on a point $\vec{r}$ on the arm, it causes a vectorial torque

$$\vec{M} = (\vec{r} - \vec{r}_0) \times \vec{F}\,. \tag{3.15}$$

The projection of $\vec{M}$ onto the axis of rotation conforms with the scalar definition of a torque as "lever arm length × force". The direction of $\vec{M}$ indicates, according to the "right-hand rule", in which rotational direction it turns a body mounted on the axis. Note that several torques add up as vectors.



$$|\vec{M}| = |\vec{r} - \vec{r}_0| \cdot |\vec{F}| \cdot \sin\beta$$

(ii) **Lorentz force and torque acting on a rotatable current-carrying loop**

A rigid conductor loop C is mounted on a fixed axis of rotation (see figure 18).

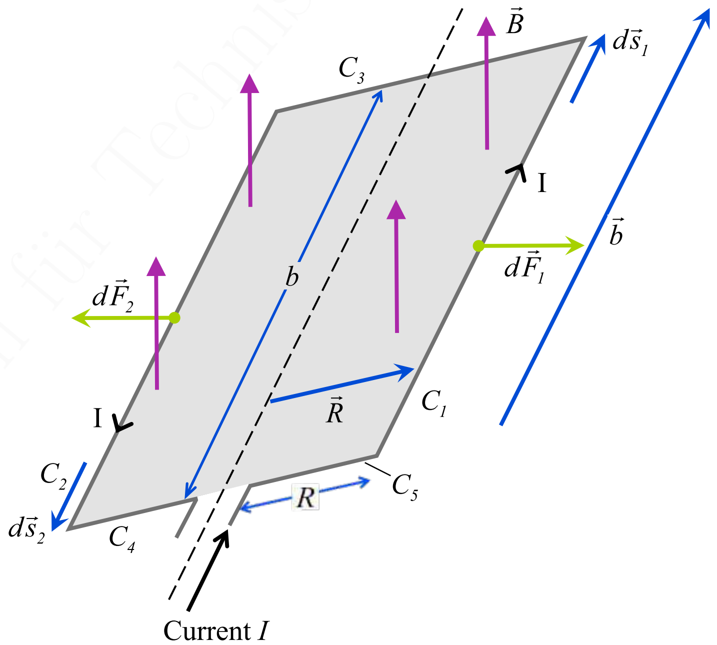


Figure 18: Rectangular conductor loop hinged to a fixed rotation axis.

For simplicity we assume a rectangular conductor loop with side length b along and side length $2R$ perpendicular to the axis of rotation. The conductor loop is symmetrically mounted on the axis along its centerline and carries a current I. In the space around the loop, there is a constant magnetic field $\vec{B}$, which is directed orthogonal to the axis of rotation.

a) **Total force acting on the conductor loop**

We decompose the conductor loop C in 5 segments C_i $(i = 1, \dots, 5)$ as shown in figure 18. Using equation (3.13), the total force acting on the conductor loop can be split in 5 contributions from the segments C_i:

$$\vec{F}_{\text{wire}} = \int\limits_C \mathrm{d}\vec{F} = \sum_{i=1}^{5} \int\limits_{C_i} \mathrm{d}\vec{F}_i \,.$$

On each segment C_i, the differential Lorentz force is given as $\mathrm{d}\vec{F}_i = I\mathrm{d}\vec{s}_i \times \vec{B}$, where $\mathrm{d}\vec{s}_i$ denotes the vectorial line element along C_i, while I and $\vec{B}$ are constant. Evidently, $\mathrm{d}\vec{s}_1 = -\mathrm{d}\vec{s}_2$ and $\mathrm{d}\vec{s}_3 = -\mathrm{d}\vec{s}_4 = -\mathrm{d}\vec{s}_5$, so that the contributions from "opposite" segments exactly compensate each other: $\mathrm{d}\vec{F}_1 = -\mathrm{d}\vec{F}_2$ and $\mathrm{d}\vec{F}_3 = -\mathrm{d}\vec{F}_4$ or $\mathrm{d}\vec{F}_3 = -\mathrm{d}\vec{F}_5$, respectively. Hence, we find as result:

$$\vec{F}_{\text{wire}} = \vec{0}\,.$$

b) **Total torque acting on a conductor loop**

Each vectorial line element $\mathrm{d}\vec{s}_i$ on C_i contributes a differential torque $\mathrm{d}\vec{M}_i = (\vec{s}_i - \vec{r}_{0i}) \times \mathrm{d}\vec{F}_i$ to the total torque obtained by integration over C_i and summation over i. Let $\vec{R} := \vec{s}_1 - \vec{r}_{01}$ be the lever arm along C_1, then $-\vec{R}$ is the lever arm along C_2. The total torque is

$$\vec{M} = \int\limits_C \mathrm{d}\vec{M} = \int\limits_{C_1} \vec{R} \times \mathrm{d}\vec{F}_1 + \int\limits_{C_2} (-R) \times \mathrm{d}\vec{F}_2 + \int\limits_{C_3} \mathrm{d}\vec{M}_3 + \int\limits_{C_4} \mathrm{d}\vec{M}_4 + \int\limits_{C_5} \mathrm{d}\vec{M}_5 \,.$$

The contributions from C_3, C_4 and C_5 are all orthogonal to the axis of rotation and, hence, have no effect. The contributions from C_1 and C_2 are parallel to the axis of rotation and, moreover, are equal:

$$\mathrm{d}\vec{M}_2 = -\vec{R} \times \mathrm{d}\vec{F}_2 = \vec{R} \times \mathrm{d}\vec{F}_1 = \mathrm{d}\vec{M}_1 \,.$$

Thus we obtain

$$\begin{aligned}\vec{M} &= 2 \cdot \int\limits_{C_1} \vec{R} \times \mathrm{d}\vec{F}_1 = 2\vec{R} \times \int\limits_{C_1} \mathrm{d}F_1 = 2\vec{R} \times (I \cdot \underbrace{\int\limits_{C_1} \mathrm{d}\vec{s}_1}_{=\vec{b}} \times \vec{B}) \\ &= 2I\,\vec{R} \times \left(\vec{b} \times \vec{B}\right) = 2I\,[\vec{b}(\vec{R} \cdot \vec{B}) - \vec{B}(\underbrace{\vec{R} \cdot \vec{b}}_{=0})]\,.\end{aligned}$$

On the other hand, with the vectorial surface of the conductor loop $\vec{A} = 2\vec{R} \times \vec{b}$ we find the relation

$$\vec{A} \times \vec{B} = 2\left(\vec{R} \times \vec{b}\right) \times \vec{B} = 2\vec{b}(\vec{R} \cdot \vec{B}) - 2\vec{R}(\underbrace{\vec{B} \cdot \vec{b}}_{=0}) = 2\vec{b}(\vec{R} \cdot \vec{B})$$

leading us to the very compact result:

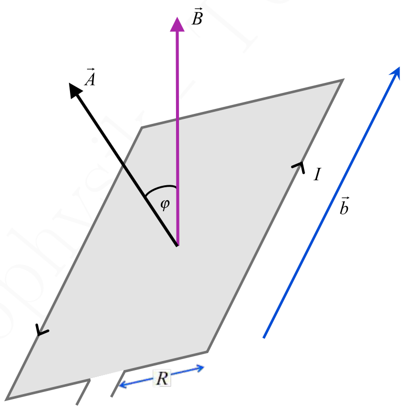
$$\boxed{\vec{M} = I\vec{A} \times \vec{B}}\,. \tag{3.16}$$

It can be shown that this simple expression is valid for any conductor loop regardless of its shape. Therefore it makes sense to define the quantity $I\vec{A}$ as a characteristic property of a current loop:

$$\vec{m} := I\vec{A} \tag{3.17}$$

is called the "**magnetic moment of a conductor loop**". The torque on a conductor loop can now be simply expressed as

$$\boxed{\vec{M} = \vec{m} \times \vec{B}}\,. \tag{3.18}$$



3.3. Permanent Magnets

A permanent magnet is made of a material where, on an atomistic length scale, a huge number ($\sim 10^{23}$ per cm^3) of atomic current loops form domains of parallel-oriented magnetic moments $\vec{m}_0$. The resulting density of magnetic moments (i.e., total magnetic moment per volume) is called "magnetization" and denoted by $\vec{\mathcal{M}}$. If n is the density of atomic current loops, then

$$\vec{\mathcal{M}} = n \left\langle \vec{m}_0 \right\rangle ,$$

where $\langle \vec{m}_0 \rangle$ denotes the statistical average of the atomic magnetic moments.

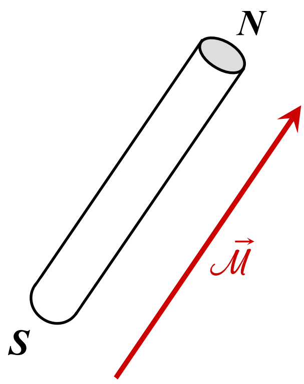
A macroscopic permanent magnet with volume V has a total magnetic moment

$$\vec{m} = V\vec{\mathcal{M}}\,.$$

When it is exposed to a magnetic field $\vec{B}$, it experiences a torque

$$\vec{M} = \vec{m} \times \vec{B}\,.$$

This conforms exactly with equation (3.18) and leads to the conclusion that macroscopic current loops and permanent magnets show equal behaviour when they are exposed to a magnetic field.



3.4. Solenoidality of the $\vec{B}$-Field

(i) The electric field is generated by electric charges, from which the electric field lines emerge or in which they disappear. In the vicinity of an electric point charge, the electric field has the form of a monopole field. In contrast, the field lines of the $\vec{B}$-field are always closed, which demonstrates that magnetic point charges (magnetic monopoles) do not exist.

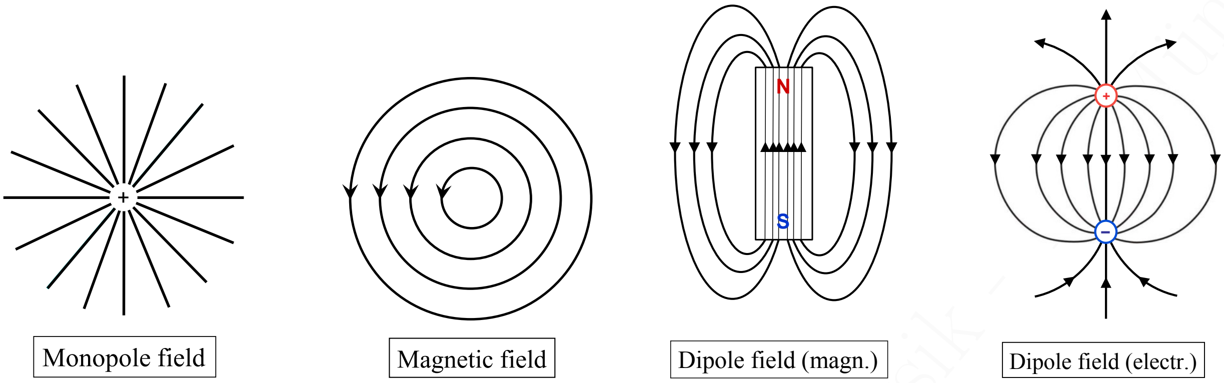


Figure 19: Qualitative difference between electric and magnetic fields

This statement is expressed in a mathematical formulation by referring to Gauss's flux theorem:

$$\int\limits_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = Q(V) = \sum_{\vec{r}_\alpha \in V} q_\alpha \,.$$

By a proper choice of the control volume V the flux integral indicates that we have localized an electric charge q_α. As magnetic charges do not exist, the reverse conclusion is:

$$\boxed{\int\limits_{\partial V} \vec{B} \cdot \mathrm{d}\vec{a} = 0 \quad \text{for any control volume } V} \,. \tag{3.19}$$

(ii) The integral formulation of the solenoidality of the $\vec{B}$-field can be converted in a more compact differential form by using Gauss's integral theorem. We obtain:

$$0 = \int_{\partial V} \vec{B} \mathrm{d}\vec{a} = \int_V \operatorname{div} \vec{B} \mathrm{d}^3 r \qquad \text{for any control volume } V \,,$$

from which we conclude

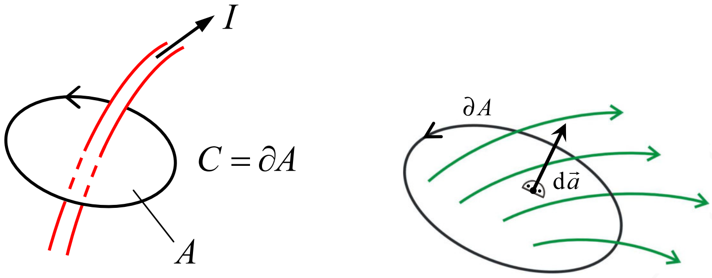
$$\boxed{\begin{array}{c} \operatorname{div} \vec{B} = 0 \\ \text{“}\vec{B}\ \textbf{is solenoidal}\text{”} \end{array}} \tag{3.20}$$

The comparison with Gauss's law $\operatorname{div} \vec{D} = \rho$ makes evident that $\operatorname{div} \vec{B} = 0$ expresses the non-existence of magnetic charges.

3.5. Generation of Magnetic Fields

3.5.1. Ampère's Circuital Law (Quasi-Stationary Form)

(i) Static magnetic fields are not generated by magnetic charges, but by moving electric charges, i.e., by an electric current density $\vec{j}$. The relationship between the generating current and the generated magnetic field is expressed by **Ampère's Circuital Law**.



We consider an orientable control surface A with right-hand oriented boundary curve ∂A. The electric current enclosed by ∂A is

$$I(A) = \int_A \vec{j} \cdot \mathrm{d}\vec{a}\,.$$

In free space, the magnetic field $\vec{B}$ generated by $\vec{j}$ satisfies **Ampère's law**

$$\int_{\partial A} \vec{B} \cdot \mathrm{d}\vec{r} = \mu_0 I(A) = \mu_0 \int_A \vec{j} \cdot \mathrm{d}\vec{a}\,. \tag{3.21}$$

μ_0 is called "**magnetic field constant**" or "**vacuum permeability**". Its value is

$$\mu_0 = 4\pi \cdot 10^{-7}\,\frac{\Omega\mathrm{s}}{\mathrm{m}}\,.$$

Note that in equation (3.21) the orientation of the boundary curve ∂A and the orientation of the vectorial surface element $\mathrm{d}\vec{a} = \vec{N}\mathrm{d}a$ are related by the "right-hand rule".

(ii) In a magnetizable material, an external primary magnetic field generated by a current density $\vec{j}$ induces magnetic dipoles ("magnetization") on an atomistic length scale. The magnetization field contributes to the total magnetic field in such a way that Ampère's law (3.21) stays valid, but with a modified field constant:

$$\int_{\partial A} \vec{B} \cdot \mathrm{d}\vec{r} = \mu_0 \mu_r I(A) = \mu I(A)\,. \tag{3.22}$$

The quantity μ_r is called **relative permeability**, the product

$$\mu = \mu_r \mu_0 \tag{3.23}$$

is called **absolute permeability**. μ_r is a measure of the magnetizability of the material.

3.5.2. Magnetic Field Strength ($\vec{H}$-Field)

(i) It is convenient to define a magnetic field quantity which depends on the generating current distribution solely, but not on the magnetizability of the surrounding material.

To this end, we define the **magnetic field strength** as

$$\vec{H} := \frac{1}{\mu} \cdot \vec{B} \quad (\text{i.e.} \quad \vec{B} = \mu \cdot \vec{H})\,. \tag{3.24}$$

This field is also termed "magnetic field intensity" or simply $\vec{H}$-field (in order to avoid any confusion with the $\vec{B}$-field). Now Ampère's circuital law can be re-written as

$$\int\limits_{\partial A} \vec{H} \cdot \mathrm{d}\vec{r} = I(A) = \int\limits_{A} \vec{j} \cdot \mathrm{d}\vec{a}\,. \tag{3.25}$$

Evidently, $\vec{H}$ depends on the generating current distribution only, but not on the magnetization of the surrounding material. Note that this formulation of Ampère's law holds only for quasi-static conditions. It must be extended in the case that there exists also a time-variant electric field (see section 3.7)!

(ii) **Analogy between electrostatics and magnetostatics**:
Between the pairs of fields ($\vec{E}$, $\vec{D}$) and ($\vec{B}$, $\vec{H}$) exists an analogy, which can be put in the following scheme:

Material-dependent fields:

$$\text{force on} \left\{ \begin{array}{c} \text{motionless} \\ \text{moving} \end{array} \right\} \text{test charge} \quad \begin{array}{c} \text{(electric force)} \\ \Longrightarrow \\ \text{(Lorentz force)} \end{array} \quad \left\{ \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right\}$$

Fields depending on sources only:

$$\text{generated by} \left\{ \begin{array}{c} \text{charge distribution } \rho \\ \text{current distribution } \vec{j} \end{array} \right\} \quad \begin{array}{c} \text{(Gauss's law)} \\ \Longrightarrow \\ \text{(Ampère's law)} \end{array} \quad \left\{ \begin{array}{c} \vec{D} \\ \vec{H} \end{array} \right\}$$

The relationship between the electric field quantities $\vec{E}$ and $\vec{D}$ and the magnetic field quantities $\vec{B}$ and $\vec{H}$, respectively, is expressed by the **constitutive equations**:

$$\vec{D} = \epsilon \vec{E} \quad \text{und} \quad \vec{H} = \frac{1}{\mu} \vec{B}\,.$$

3.5.3. Permeability and Magnetic Susceptibility

(i) In magnetic materials, we often find a linear relation between the magnetization $\vec{\mathcal{M}}$ and the external $\vec{H}$-field ("linear response"):

$$\vec{\mathcal{M}} = \chi_m \vec{H} \tag{3.26}$$

χ_m is called **magnetic susceptibility**.

The total magnetic field $\vec{B}$ is obtained by adding the internal magnetization to the external $\vec{H}$-field:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{\mathcal{M}} = \underbrace{\mu_0 \cdot \vec{H}}_{\substack{\text{generated by external} \\ \text{current distribution } \vec{j}}} + \underbrace{\mu_0 \cdot \chi_m \cdot \vec{H}}_{\substack{\text{generated by induced} \\ \text{parallel-oriented current loops} \\ \text{in the material}}}$$

By comparison with the constitutive law $\vec{B} = \mu \vec{H}$, we find

$$\mu = \mu_0 (1 + \chi_m)\,. \tag{3.27}$$

(ii) The magnetic susceptibility and, hence, the permeability are determined by atomistic mechanisms which cause magnetism. Therefore they can be used to set up the following classification of magnetic materials:

- **Diamagnetism**:
  The external $\vec{H}$-field alters the orbital velocity of counterrotating electrons and, thus, their magnetic moments. This results in a (very small) uncompensated magnetic moment, which opposes the $\vec{H}$-field (according to Lenz's law). Therefore, $\vec{\mathcal{M}}$ is anti-parallel to $\vec{H}$:

  $$\vec{\mathcal{M}} = \chi_m \vec{H} \quad \text{with} \quad \chi_m < 0, \quad |\chi_m| \ll 1, \quad \text{hence} \quad \mu_r < 1\,.$$

  Examples: Au, Ag, Cu, H_2O .

- **Paramagnetism**:
  Under the action of an external $\vec{H}$-field existing permanent magnetic dipoles are forced to be parallel-oriented in the direction of $\vec{H}$, i.e., $\vec{\mathcal{M}}$ is parallel to $\vec{H}$:

  $$\vec{\mathcal{M}} = \chi_m \vec{H} \quad \text{with} \quad \chi_m > 0, \quad |\chi_m| \ll 1, \quad \text{hence} \quad \mu_r > 1\,.$$

  Examples: Pt, Al.

- **Ferromagnetism**:
  At sufficiently low temperature $T < T_\mathrm{C}$ (Curie temperature), existing permanent magnetic dipoles form domains with spontaneous self-orientation along one common direction (Weiss domains). Under the action of an external $\vec{H}$-field, domains with $\vec{\mathcal{M}} \upuparrows \vec{H}$ grow, while the other domains shrink. A small "guiding field" $\vec{H}$ is sufficient for a collective re-orientation of a huge number of magnetic moments in the preferential direction parallel to $\vec{H}$. A strong mutual magnetic interaction in a domain leads to hysteresis.

  $$\vec{\mathcal{M}} = \chi_m \vec{H} \quad \text{with} \quad \chi_m \gg 1 \quad (\approx 10^4 ... 10^5)\,.$$

Since $\mu_r = 1 + \chi_m \approx \chi_m$, we have in this case:

$$\vec{B} = \mu_0 \cdot \vec{H} + \mu_0 \chi_m \vec{H} \approx \mu_0 \cdot \vec{\mathcal{M}} \,,$$

i.e., the $\vec{B}$-field is determined mostly by the magnetization $\vec{\mathcal{M}}$.

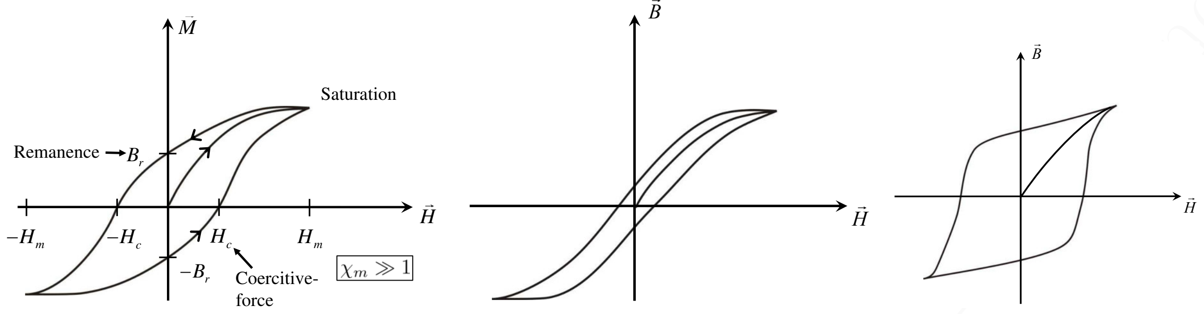


Figure 20: Ferromagnetic hysteresis curve (left); magnetic "soft" (middle) and magnetic "hard" materials (right)

For $T > T_\mathrm{C}$, the ferromagnetic state turns into the paramagnetic state (phase transition) with susceptibility

$$\chi_m = \frac{\chi_0}{T - T_C} \qquad \text{for } T > T_C \,.$$

Examples: Ni: $T_\mathrm{C} = 360°C$; Fe: $T_\mathrm{C} = 770°\mathrm{C}$; Co: $T_\mathrm{C} = 1075°\mathrm{C}$.

3.6. Calculation of Magnetostatic Fields and Forces

In problems with high spatial symmetry, the $\vec{H}$-field can be calculated analytically from a given current distribution $\vec{j}(\vec{r})$ by exploiting Ampère's circuital law

$$\int_{\partial A} \vec{H} \cdot \mathrm{d}\vec{r} = I(A) = \int_{A} \vec{j} \cdot \mathrm{d}\vec{a} \,.$$

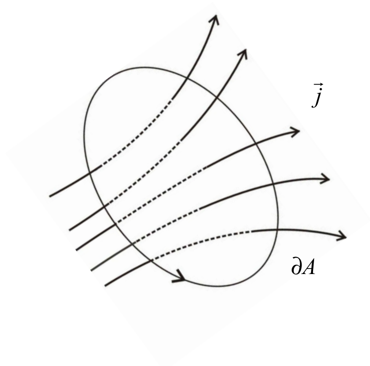


Figure 21: Current through a surface with boundary ∂A

In the following, we consider selected examples.

3.6.1. Magnetic Field Around an Infinitely Long Straight Wire

As a basic example, we calculate the magnetic field generated by an infinitely long, straight wire that carries a current I.

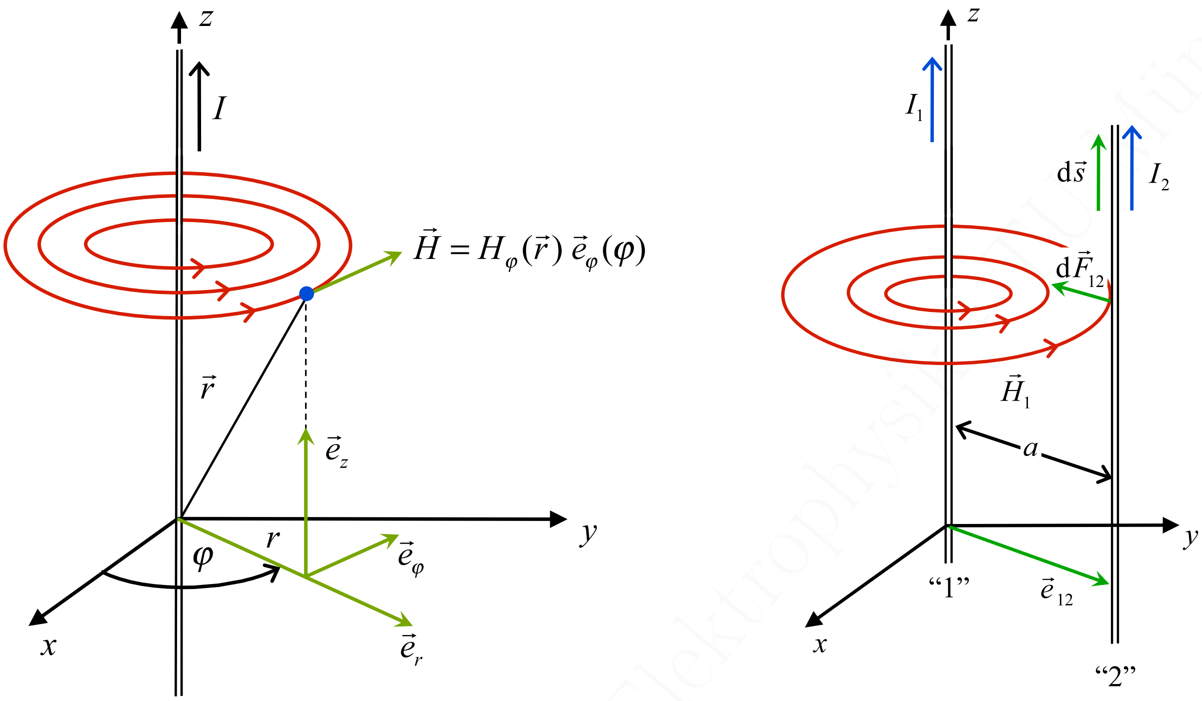


Figure 22: Magnetic field of an infinitely long, straight wire carrying a current I (left) and force between two parallel straight wires (right).

The cylindrical symmetry of the problem suggests to use cylindrical coordinates (r, φ, z), where the z-axis is aligned along the wire in the direction of current flow. As the magnetic field lines must be closed and cylindrically symmetric, we expect that they are concentric circular lines around the wire with radius r at a height z (denoted by $\partial K(r, z)$). Hence, in the local cylindrical basis $(\vec{e}_r, \vec{e}_\varphi, \vec{e}_z)$ the $\vec{H}$-field has the representation

$$\vec{H}(\vec{r}) = H_\varphi(r)\,\vec{e}_\varphi(\varphi)\,.$$

Using the circle $\vec{K}(r, z)$ enclosed by $\partial K(r, z)$ as control surface in Ampère's law, we find

$$\int\limits_{\partial K(r,z)} \vec{H} \cdot \mathrm{d}\vec{r} = \int\limits_0^{2\pi} H_\varphi(r)\,\underbrace{\vec{e}_\varphi \cdot \vec{e}_\varphi}_{=1}\, r\mathrm{d}\varphi = H_\varphi(r) \cdot r \cdot 2\pi = I\,.$$

Solving for $H_\varphi(r)$ yields

$$H_\varphi(r) = \frac{I}{2\pi r}\,.$$

Hence, the result is

$$\boxed{\vec{H}(\vec{r}) = \frac{I}{2\pi r}\,\vec{e}_\varphi(\varphi)}\,. \tag{3.28}$$

3.6.2. Force Between Two Parallel Current-Carrying Wires

Now we consider the situation that in the $\vec{H}$-field of a wire L1 along the z-axis carrying the current I_1, there is a second straight wire L2 parallel to L1 in a distance a, which carries a current I_2 (see figure 22, right). A vectorial line element $\mathrm{d}\vec{s}_2$ of L2 experiences a differential Lorentz force $\mathrm{d}\vec{F}_{12}$ as stated in equation (3.14):

$$\mathrm{d}\vec{F}_{12} = I_2\,\mathrm{d}\vec{s}_2 \times \vec{B}_1 = \mu I_2 \left(\vec{e}_z \times \vec{H}_1\right)\mathrm{d}z\,,$$

where μ is the permeability of the surrounding material and $\vec{H}_1$ is the magnetic field generated by L1 in the distance $r = a$:

$$\vec{H}_1(r = a) = \frac{I_1}{2\pi a}\,\vec{e}_\varphi\,.$$

The resulting force acting per line element $\mathrm{d}z$ on wire L2 is then

$$\mathrm{d}\vec{F}_{12} = \frac{\mu I_1 I_2}{2\pi a}\,\underbrace{(\vec{e}_z \times \vec{e}_\varphi)}_{=-\vec{e}_r=-\vec{e}_{12}}\,\mathrm{d}z = -\frac{\mu I_1 I_2}{2\pi a}\,\vec{e}_{12}\,\mathrm{d}z\,,$$

where $\vec{e}_{12}$ is the unit vector pointing form L1 to L2 in the direction orthogonal to the two wires. Hence, the differential force per line element between two parallel straight wires is

$$\frac{\mathrm{d}\vec{F}_{12}}{\mathrm{d}z} = -\mu\,\frac{I_1 I_2}{2\pi a}\,\vec{e}_{12}\,. \tag{3.29}$$

As it can be expected from the symmetry of the problem, this expression is symmetric with respect to an interchange of L1 and L2. Furthermore, we recognize that parallel wires carrying parallel-directed currents attract each other, while they repel each other with anti-parallel current flow.

3.6.3. $\vec{H}$-Field Generated by a General Cylindrically Symmetric Current Distribution

As a further example, we calculate the magnetic field generated by a cylindrically symmetric, continuously distributed current density $\vec{j}(\vec{r}) = j(r)\vec{e}_z$ flowing parallel to the z-axis. Following the same argument as in the case of the one-dimensional wire (section 3.6.1), we assume again

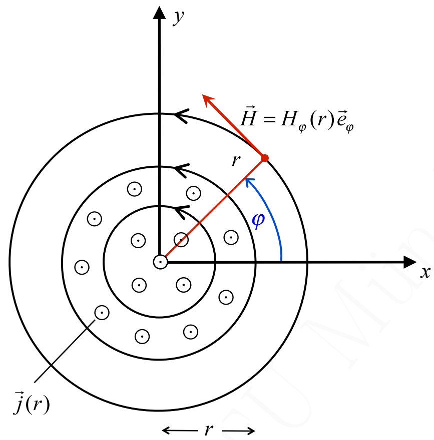
$$\vec{H}(\vec{r}) = H_\varphi(r)\vec{e}_\varphi(\varphi)\,.$$

But in contrast to the one-dimensional wire, the current $I(A)$ in Ampère's circuital law has now to be determined by integrating over the circular surface $K(r,z)$:

$$2\pi r H_\varphi(r) = \int\limits_{\partial K(r,z)} \vec{H}\cdot\mathrm{d}\vec{r} = \int\limits_{K(r,z)} \vec{j}\cdot\mathrm{d}\vec{a} = \int_0^{2\pi}\int_0^{r} j(r')r'\mathrm{d}r'\mathrm{d}\varphi = 2\pi\int_0^{r} j(r')r'\mathrm{d}r'\,.$$

Solving for $\vec{H}_\varphi(r)$ yields:

$$\boxed{H_\varphi(r) = \frac{1}{r}\int\limits_0^r j(r')r'\mathrm{d}r'}\,, \tag{3.30}$$



which is the general solution of the problem.

Example: Infinitely long, straight wire with radius a and uniform current distribution:

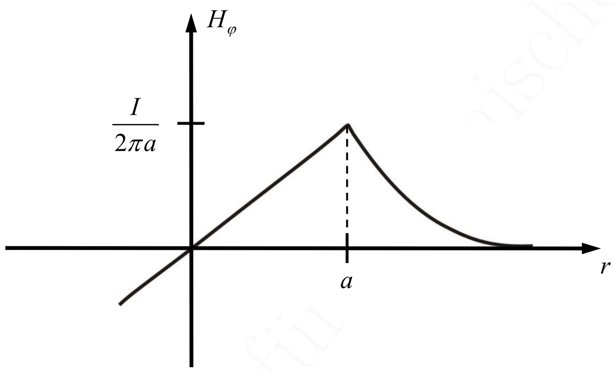
$$j(r) = \begin{cases} \frac{I}{a^2\pi} & \text{for } 0 \le r \le a\,, \\ 0 & \text{for } r > a\,. \end{cases}$$

Evaluation of the integral (3.30) gives

$$\int\limits_0^r j(r')r'\,\mathrm{d}r' = \begin{cases} \frac{I}{a^2\pi}\cdot\frac{1}{2}r^2 & \text{for } 0 \le r \le a\,, \\ \frac{I}{2\pi} & \text{for } r > a\,. \end{cases}$$

Hence:

$$H_\varphi(r) = \begin{cases} \frac{I}{a^2\pi}\cdot\frac{1}{2}r & \text{for } 0 \le r \le a\,, \\ \frac{I}{2\pi}\cdot\frac{1}{r} & \text{for } r > a\,. \end{cases}$$



Outside the wire, i.e., for $r > a$, the magnetic field $H_\varphi(r)$ behaves like the field of a perfect line-shaped conductor concentrated on the z-axis.

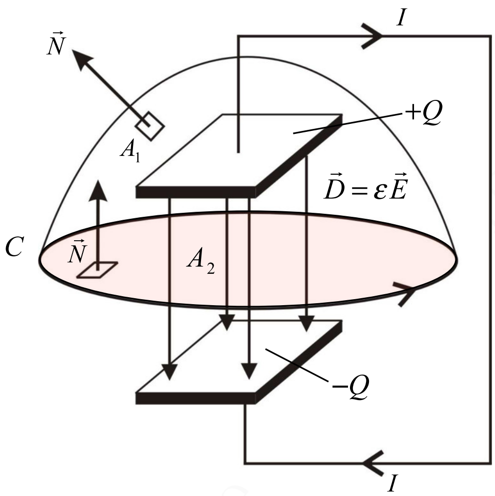
3.7. Extension of Ampère's Circuital Law to Fast Time-Variant Phenomena

3.7.1. Incompleteness of Ampère's Law and its Correction

So far, the right-hand side $I(A)$ in Ampère's law (3.25) was assumed to be a material current of charge carriers moving in a flow field $\vec{j}(\vec{r})$:

$$\int\limits_{\partial A} \vec{H} \cdot \mathrm{d}\vec{r} = I(A) = \int\limits_{A} \vec{j} \cdot \mathrm{d}\vec{a}\,.$$

The following thought experiment demonstrates that this formulation needs an extension. Consider the electrodes of a plate capacitor loaded with charges $+Q$ and $-Q$, respectively.



The charges generate a displacement field $\vec{D}$ in the dielectric between the two capacitor plates, but there is no electric current flow ($\vec{j} = 0$), because the dielectric is an insulator.

We surround the dielectric with a closed loop C, which is the common boundary curve of two right-hand oriented surfaces A_1 and A_2 with respective surface unit normal $\vec{N}$. The surface A_1 is chosen such that it covers the upper capacitor plate like a dome. It is intersected by a wire which connects the two capacitor plates and carries a discharge current I; hence $I(A_1)$ equals I. The second surface A_2 is chosen in such a way that it is located between the capacitor plates, where no current is flowing; therefore $I(A_2)$ is zero. Thus, the closed-loop integral $\int_C \vec{H} \cdot \mathrm{d}\vec{r}$ takes the value

$$\int\limits_{C} \vec{H} \cdot \mathrm{d}\vec{r} = \left\{ \begin{array}{ll} I(A_1) = I & \text{, if } C = \partial A_1 \\ \\ 0 & \text{, if } C = \partial A_2 \end{array} \right\} \Rightarrow \text{contradiction!}$$

This contradiction can be resolved as follows: Let Q be the charge on the upper electrode; then the discharge current I is related to Q by $I = -\frac{\mathrm{d}Q}{\mathrm{d}t}$. On the other hand, we may connect the surfaces A_1 and A_2 along their common boundary $C = \partial A_1 = \partial A_2$ so as to build a closed surface ∂V around the upper electrode. The control volume V confined by A_1 and A_2 contains the charge Q; hence, Gauss's flux theorem yields

$$Q = \int\limits_{A_1 \cup A_2} \vec{D} \cdot \mathrm{d}\vec{a} = -\int\limits_{A_2} \vec{D} \cdot \vec{N}\,\mathrm{d}a = -\int_{A_2} \vec{D} \cdot \mathrm{d}\vec{a}\,.$$

(Note: $\vec{N}$ = right-hand oriented unit normal on A_2 w.r.t. the orientation of C = − outward unit normal w.r.t. V)

Since A_2 does not depend on time, we get

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int\limits_{A_2} \frac{\partial \vec{D}}{\partial t} \cdot \mathrm{d}\vec{a}\,.$$

So we conclude:

$$\int\limits_{A_1} \vec{j} \cdot \mathrm{d}\vec{a} = I(A_1) = -\frac{\mathrm{d}Q}{\mathrm{d}t} = \int\limits_{A_2} \frac{\partial \vec{D}}{\partial t} \cdot \mathrm{d}\vec{a}$$

With this result, the above-stated contradiction can be resolved by the following hypothesis:

$$\int\limits_{C} \vec{H} \cdot \mathrm{d}\vec{r} = \begin{cases} \int_{A_1} \vec{j} \cdot \mathrm{d}\vec{a} & \text{, if } C = \partial A_1 \\ \int_{A_2} \frac{\partial \vec{D}}{\partial t} \cdot \mathrm{d}\vec{a} & \text{, if } C = \partial A_2 \end{cases}$$

The quantity $\frac{\partial \vec{D}}{\partial t}$ is called "**displacement current density**".

Extending Ampère's circuital law by this term leads to the unified formulation

$$\boxed{\int\limits_{\partial A} \vec{H} \cdot \mathrm{d}\vec{r} = \int\limits_{A} \left(\vec{j} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \mathrm{d}\vec{a} \quad \text{for any orientable control surface } A}\,. \tag{3.31}$$

(Maxwell's extension of Ampère's Circuital Law)

3.7.2. Ampère-Maxwell's Circuital Law in Differential Form

The integral form of Ampère-Maxwell's circuital law (3.31) can be rewritten by applying Stokes' integral theorem:

$$\int\limits_{A} \operatorname{curl} \vec{H} \cdot \mathrm{d}\vec{a} = \int\limits_{\partial A} \vec{H} \cdot \mathrm{d}\vec{r} = \int\limits_{A} \left(\vec{j} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \mathrm{d}\vec{a}\,.$$

As this statement is true for any arbitrary control surface A, the integrands in the two surface integrals must be equal, and we obtain:

$$\boxed{\operatorname{curl} \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}}\,. \tag{3.32}$$

(general Ampère's circuital law)

4. Electromagnetic Induction

The generation of electric fields and voltages by the motion of a conductor through a magnetic field ("electromotive force", EMF) is termed "motional induction". In contrast, "motionless induction" is the generation of an electric field in a motionless conductor loop by the action of a time-variant magnetic field ("transformer EMF").

4.1. Motional Induction

4.1.1. Electromotive Force in a Moving Conductive Body

Imagine that an electrically conductive body moves through a magnetic field $\vec{B}$ with velocity $\vec{V}$. A test charge q resting in the body experiences the Lorentz force $\vec{F}_L = q(\vec{V} \times \vec{B})$. Inside the conductor, we measure an "electromotive force" $\vec{F}_L/q$ and interpret it as an electric field

$$\vec{E}_{\mathrm{ind,m}} = \vec{V} \times \vec{B} \tag{4.1}$$

induced by the motion of the body.

4.1.2. Induced EMF in a Time-Variant Conductor Loop

Imagine the following thought experiment:

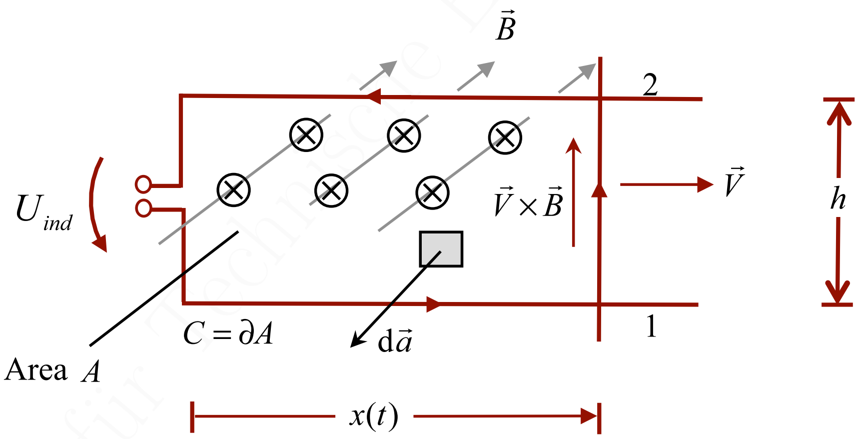


Figure 23: Time-variant conductor loop

Two parallel wire bows fixed in a distance h are electrically connected by a mobile wire bridge, which moves with velocity $\vec{V} = V\vec{e}_x$ in the direction of the wire bows. The momentary position of the wire bridge is $x(t)$, hence $V = \mathrm{d}x/\mathrm{d}t$. The wire bows and the bridge form a (nearly) closed loop with two terminals at the left, between which a voltage U_{ind} is measured. The loop encloses a rectangle with area $|A(t)| = h \cdot x(t)$, which is penetrated by a constant magnetic field $\vec{B}$ directed orthogonal to the loop into the plane of projection. The orientation of the wire loop is counterclockwise, so that the vectorial surface element $\mathrm{d}\vec{a}$ of the loop area points in the direction antiparallel to $\vec{B}$. The magnetic flux through the loop area amounts to

$$\Phi(A) = \int\limits_{A(t)} \vec{B}\cdot\mathrm{d}\vec{a} = -B \cdot |A(t)| = -B \cdot h \cdot x(t)\,. \tag{4.2}$$

Along the moving wire bridge the electric field $\vec{E}_{\text{ind,m}} = \vec{V} \times \vec{B}$ is induced, while in the other parts of the conductor loop there is no electric field. Hence, the induced voltage U_{ind} can be calculated as the line integral along the wire bridge between the two bows 1 and 2:

$$U_{\text{ind}} = \int\limits_1^2 \vec{E}_{\text{ind,m}} \cdot \mathrm{d}\vec{r} = \int\limits_1^2 \left(\vec{V} \times \vec{B}\right) \cdot \mathrm{d}\vec{r} = V \cdot B \cdot h = \frac{\mathrm{d}x}{\mathrm{d}t} \cdot B \cdot h\,.$$

Using equation (4.2), we find:

$$U_{\text{ind}} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(A)\,. \tag{4.3}$$

This result is generally true for any time-variant conductor loop with arbitrary shape, and it is irrelevant whether the change of shape is caused by a continuous deformation of the loop or a rigid motion such as a rotation. The mathematical reason is the following: Imagine a time-variant closed loop $C(t) = \partial A(t)$ in a position-dependent, but not time-dependent magnetic field $\vec{B}(\vec{r})$, which causes a magnetic flux

$$\Phi(A) = \int\limits_{A(t)} \vec{B}(\vec{r}) \cdot \mathrm{d}a$$

through the area $A(t)$ surrounded by $C(t)$. Let $\vec{V}(\vec{r}, t)$ denote the local velocity of a point $P(\vec{r})$ which sits on a fixed position on the moving loop. Then the voltage induced in the loop is given by the relation

$$U_{\text{ind}} = \int\limits_{\partial A(t)} \left(\vec{V}(\vec{r}, t) \times \vec{B}(\vec{r})\right) \cdot \mathrm{d}\vec{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\int\limits_{A(t)} \vec{B}(\vec{r}) \cdot \mathrm{d}\vec{a}\right]\,. \tag{4.4}$$

The mathematical proof is quite sophisticated and not given here.

4.1.3. Homopolar Generator

In a homopolar generator a motional EMF is induced without using a wire loop. Instead, the EMF is generated in a conductive solid body, which moves in a uniform static magnetic field. A typical example is **Barlow's wheel**:

A conductive disk with radius a rotates around a fixed axis (z-axis) with angular velocity Ω in a uniform magnetic field $\vec{B}$, which is directed parallel to the rotational axis: $\vec{B} = B\vec{e}_z$. Two sliding contacts are placed on the rim of the disk and on the rotating axis to pick up the induced voltage U_{ind}. A volume element of the disk in a distance r form the axis has the local tangential velocity $\vec{V} = \Omega \cdot r \cdot \vec{e}_\phi$ (in cylindrical coordinates with local basis $(\vec{e}_r, \vec{e}_\phi, \vec{e}_z)$). Therefore the induced EMF is

$$\vec{E}_{\text{ind}} = \vec{V} \times \vec{B} = \Omega B r \left(\vec{e}_\varphi \times \vec{e}_z\right) = \Omega B r \vec{e}_r\,.$$

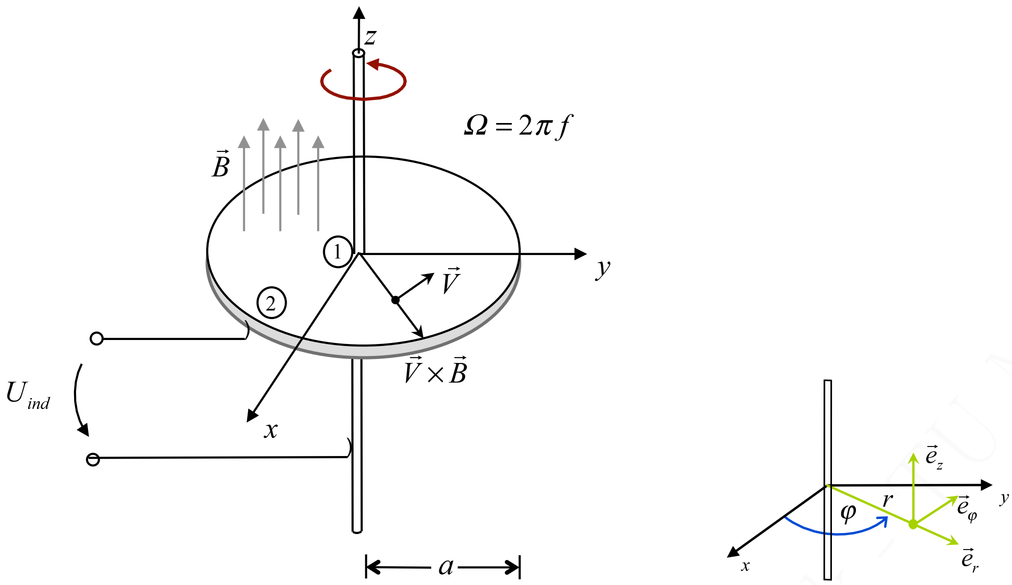


Figure 24: Barlow's wheel

By integrating the induced $\vec{E}$-field along a radial path from the axis (point 1) to the slider at the rim (point 2), we obtain the induced voltage U_{ind} measured between the sliding contacts. With the parametric representation of this path

$$r \mapsto r \cdot \vec{e}_r\,, \quad 0 \le r \le a \quad \Rightarrow \quad \mathrm{d}\vec{r} = \vec{e}_r \cdot \mathrm{d}r\,,$$

we calculate the induced voltage as

$$U_{\text{ind}} = \int\limits_1^2 \left(\vec{V} \times \vec{B}\right) \cdot \mathrm{d}\vec{r} = \Omega B \int\limits_0^a r \cdot \mathrm{d}r = \frac{1}{2}\Omega B a^2\,. \tag{4.5}$$

Note that this result cannot be represented as time-derivative of the magnetic flux according to equation (4.4), because there is no moving conductor loop to refer to.

4.2. Motionless Induction

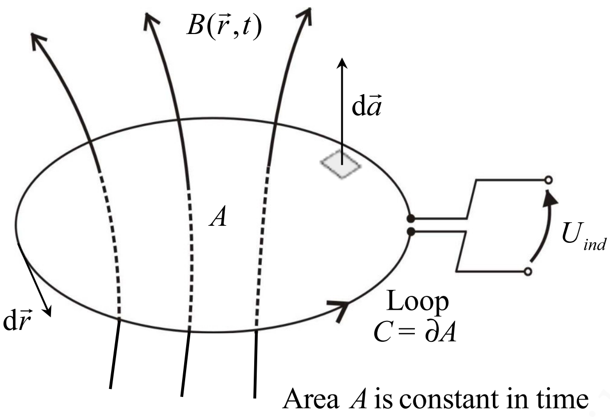
There is experimental evidence that, in the case of conductor loops, the general form of Faraday's magnetic induction law reads

$$U_{\text{ind}} = -\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{A(t)} \vec{B}(\vec{r}, t) \cdot \mathrm{d}\vec{a}\,. \tag{4.6}$$

This means: The time-dependence of the magnetic flux may be caused by a variation of the conductor loop with time, $A(t)$, as well as by a variation of the magnetic field with time, $\vec{B}(\vec{r}, t)$. Therefore, even in a motionless conductor loop (i.e., without the presence of the Lorentz force), an electric field $\vec{E}_{\text{ind,ml}}$ is induced by the explicit time-dependence of the magnetic field $\vec{B}(\vec{r}, t)$. This effect is called "motionless induction" (or "transformer EMF").

4.2.1. Induced EMF in a Motionless Conductor Loop

A change of the magnetic field with time $\partial\vec{B}/\partial t$ causes the generation of an electric field $\vec{E}_{\text{ind,ml}}$ along a motionless conductor loop as follows:



As the integration area is constant in time, the magnetic flux

$$\Phi(A) = \int_A \vec{B}(\vec{r},t)\cdot \mathrm{d}\vec{a}$$

depends on time solely through the integrand $\vec{B}(\vec{r},t)$. Hence, the time-derivative is

$$\begin{aligned}
\frac{\mathrm{d}\Phi(A)}{\mathrm{d}t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_A \vec{B}(\vec{r},t+\Delta t)\cdot\mathrm{d}\vec{a} - \int_A \vec{B}(\vec{r},t)\cdot\mathrm{d}\vec{a}\right) = \\
&= \int_A \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\vec{B}(\vec{r},t+\Delta t) - \vec{B}(\vec{r},t)\right)\cdot\mathrm{d}\vec{a} = \\
&= \int_A \frac{\partial\vec{B}}{\partial t}(\vec{r},t)\cdot\mathrm{d}\vec{a}\,.
\end{aligned}$$

So we arrive at the result that the voltage generated in the conductor loop by motionless induction is given by

$$U_{\text{ind}} = -\int_A \frac{\partial\vec{B}}{\partial t}\cdot\mathrm{d}\vec{a}\,. \tag{4.7}$$

On the other hand, the induced voltage can also be calculated by integrating the induced electric field $\vec{E}_{\text{ind,ml}}(\vec{r},t)$ along the conductor loop, and the result must be equal to that in equation (4.7). This leads us to the following integral relation between $\vec{E}_{\text{ind,ml}}$ and $\partial\vec{B}/\partial t$:

$$\boxed{\int_{\partial A} \vec{E}_{\text{ind,ml}}(\vec{r},t)\cdot\mathrm{d}\vec{r} = -\int_A \frac{\partial\vec{B}}{\partial t}(\vec{r},t)\cdot\mathrm{d}\vec{a}}\,. \tag{4.8}$$

4.2.2. Maxwell's Generalization: Differential Form of Faraday's Law of Induction

Maxwell recognized that the integral relation (4.8) between the induced EMF and the time-derivative of the $\vec{B}$-field constitutes a natural law which holds for any mathematical control surface A, regardless of the physical realization of its boundary curve ∂A as a conductor loop. He set up the hypothesis that equation (4.8) is generally valid for any surface A with right-hand oriented boundary curve ∂A, which implies that even in free space a time-variant magnetic field generates an electric field!

This "law of motionless induction" is the key for extending the fundamental law of electrostatics $\int_{\partial A} \vec{E} \cdot \mathrm{d}\vec{r} = 0$ to the general situation encountered in the world of electrodynamics. Here the total electric field is composed of two contributions:

$$\vec{E} = \vec{E}_{\text{pot}} + \vec{E}_{\text{ind,ml}}\,,$$

where $\vec{E}_{pot}$ is a gradient field generated by an electric space charge:

$$\vec{E}_{\text{pot}} = -\nabla\Phi \quad \text{with } \operatorname{div}(\varepsilon\nabla\Phi) = -\rho\,,$$

while $\vec{E}_{\text{ind,ml}}$ is a source-free solenoidal field generated by $\partial\vec{B}/\partial t$. Thus an electrodynamic $\vec{E}$-field is the linear combination of two qualitatively different field types:

$$\vec{E} = \underbrace{-\nabla\Phi}_{\substack{\text{irrotational,}\\ \text{source } \rho}} + \underbrace{\vec{E}_{\text{ind,ml}}}_{\substack{\text{solenoidal,}\\ \text{source-free}}} \tag{4.9}$$

Integrating $\vec{E}$ along the closed boundary curve ∂A of a control surface A yields

$$\int\limits_{\partial A} \vec{E} \cdot \mathrm{d}\vec{r} = \int\limits_{\partial A} -\nabla\Phi \cdot \mathrm{d}\vec{r} + \int\limits_{\partial A} \vec{E}_{\text{ind,ml}} \cdot \mathrm{d}\vec{r}.$$

As the contour integral of a gradient field over a closed curve vanishes, we conclude from equation 4.8:

$$\int\limits_{\partial A} \vec{E} \cdot \mathrm{d}\vec{r} = -\int\limits_{A} \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{a} \quad \text{for any control surface } A\,.$$

Using Stokes' integral theorem, we obtain

$$\int\limits_{A} \operatorname{curl}\vec{E} \cdot \mathrm{d}\vec{a} = \int\limits_{\partial A} \vec{E} \cdot \mathrm{d}\vec{r} = -\int\limits_{A} \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{a}\,,$$

and since this statement is true for any control surface A, we finally arrive at the differential form of Faraday's law of induction

$$\boxed{\operatorname{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}}\,. \tag{4.10}$$

4.3. General Integral Form of the Induction Law for Conductor Loops

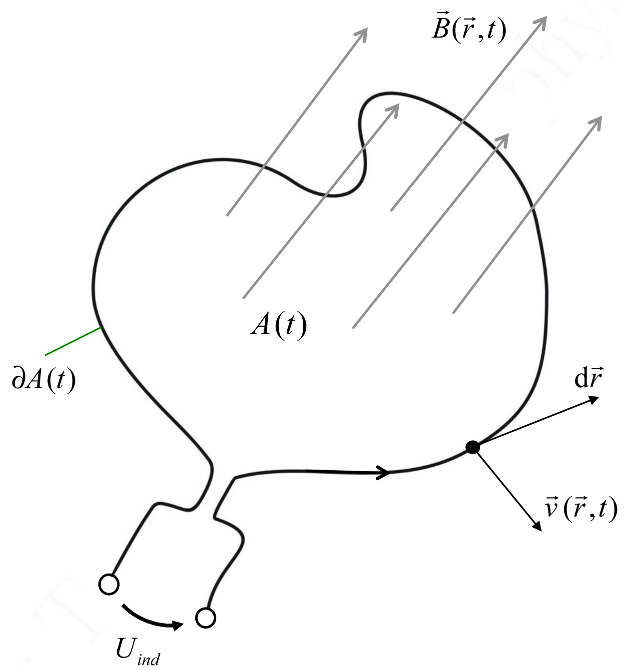
The results for motional and motionless induction in a conductor loop can be summarized in a unified description as follows: We assume that both the conductor loop $\partial A(t)$ and the magnetic field $\vec{B}(\vec{r},t)$ are time-dependent. Then the induced voltage is the sum of a motional and a motionless contribution:

$$U_{\text{ind}} = \int\limits_{\partial A(t)} [\underbrace{\vec{E}_{\text{ind,ml}}(\vec{r},t)}_{\text{motionless EMF}} + \underbrace{\vec{V}(\vec{r},t) \times \vec{B}(\vec{r},t)}_{\text{motional EMF}}] \cdot \mathrm{d}\vec{r}. \tag{4.11}$$

Here, as in equation (4.4), $\vec{V}(\vec{r},t)$ denotes the local velocity of a point $P(\vec{r})$ which sits on a fixed position on the moving wire loop.

Substituting equation (4.8) for the motionless EMF leads to

$$U_{\text{ind}} = -\int\limits_{A(t)} \frac{\partial \vec{B}}{\partial t}(\vec{r},t)\mathrm{d}\vec{a} + \int\limits_{\partial A(t)} \left[\vec{V}(\vec{r},t) \times \vec{B}(\vec{r},t)\right] \mathrm{d}\vec{r}. \tag{4.12}$$



Without mathematical proof we note that this expression can also be obtained from equation (4.6) by an explicit calculation of the time-derivative of the magnetic flux. The time-derivative with respect to $\vec{B}(\vec{r},t)$ leads to the first term of equation (4.12), while the time-derivative w.r.t. the control surface $A(t)$ results in the second term. Hence, the combined effect of both motional and motionless induction is described in a compact and elegant way by

$$\boxed{U_{\text{ind}} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{A(t)} \vec{B}(\vec{r},t) \cdot \mathrm{d}\vec{a} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(A(t)\right) .} \tag{4.13}$$

4.4. Summary of the Electromagnetic Field Equations (Maxwell's Equations)

We are now able to express the fundamental laws of electromagnetism in a consistent system of coupled partial differential equations. These equations are referred to as "Maxwell's equations":

$$\operatorname{div} \vec{D} = \rho \qquad \text{cf. (1.35)} \tag{4.14}$$

$$\operatorname{div} \vec{B} = 0 \qquad \text{cf. (3.20)} \tag{4.15}$$

$$\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{cf. (4.10)} \tag{4.16}$$

$$\operatorname{curl} \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad \text{cf. (3.32)} \tag{4.17}$$

This set of equations describes **natural laws**. Their physical interpretation is as follows:

- An electric field is generated
  - by an electric charge distribution ρ
    (quasi-static, equation (4.14))
  - or by a rapidly varying magnetic field $\partial B/\partial t$
    (magnetic induction, equation (4.16)).
- A magnetic field is generated
  - by an electric current $\vec{j}$
    (quasi-static, equation (4.17))
  - or by a rapidly varying electric field $\partial \vec{D}/\partial t$
    (displacement current = "electrodynamic induction", equation (4.17)).
- Evidently there is a strong dynamic coupling between the electric and the magnetic field, $\vec{E}$ and $\vec{H}$, as expressed by Faraday's law of induction (4.16) and Ampère's circuital law (4.17). Therefore, $\vec{E}$ and $\vec{H}$ have to be conceived as the two components of one single physical field quantity $(\vec{E}, \vec{H})$ called "**electromagnetic field**". Only under static conditions, $\partial \vec{B}/\partial t = 0$ and $\partial \vec{D}/\partial t = 0$, the "electric world" and the "magnetic world" are decoupled, and $\vec{E}$ and $\vec{H}$ may be considered as independent physical quantities.

In order to form a closed system for the electromagnetic field $(\vec{E}, \vec{H})$, Maxwell's equations (4.14)–(4.17) must be completed by **constitutive laws**, for example

$$\vec{B} = \mu \vec{H} \tag{4.18}$$

$$\vec{D} = \epsilon \vec{E} \tag{4.19}$$

$$\vec{j} = \sigma \vec{E} + \vec{j}_0 \tag{4.20}$$

These relations are not fundamental, but rather **phenomenological model equations** with a confined range of validity.

A. Gradient Fields and Potential Function

A.1. Definition and Uniqueness of Potential Functions

(i) **Definition:** Vector field $V_3 \supset \Omega \to V_3$, $\vec{r} \mapsto \vec{E}(\vec{r})$ is "gradient field" if there exists a scalar "potential function" $\Phi(\vec{r})$, defined on Ω, with

$$\boxed{\vec{E}(\vec{r}) = -\operatorname{grad}\Phi(\vec{r})}\,. \tag{A.1}$$

Alternative notations: $\operatorname{grad}\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial\vec{r}}$

(ii) Uniqueness of potential function:

For a given $\vec{E}$, a potential Φ satisfying $\vec{E} = -\nabla\Phi$ can be determined only up to an additive constant. This is because $\Phi(\vec{r})$ and $\tilde{\Phi}(\vec{r}) := \Phi(\vec{r}) + c$ with $c \in \mathbb{R}$ give the same vector field $\vec{E}$:

$$\nabla\tilde{\Phi} = \nabla\Phi + \underbrace{\nabla c}_{=0} = -\vec{E}$$

A.2. Existence of a Potential Function

(i) Necessary condition for a gradient field:

Let $\vec{E}(\vec{r})$ be a gradient field (i.e., $\vec{E} = -\nabla\Phi$ with appropriate Φ). In Cartesian coordinates, we have

$$\nabla\Phi = \sum_{j=1}^{3} \frac{\partial\Phi}{\partial x_j}\vec{e}_j \quad \text{and} \quad \vec{E} = \sum_{j=1}^{3} E_j\vec{e}_j\,,$$

hence

$$E_j = -\frac{\partial\Phi}{\partial x_j} \quad (j = 1,2,3)\,.$$

Taking the second partial derivatives yields:

$$\frac{\partial E_j}{\partial x_k} = -\frac{\partial^2\Phi}{\partial x_k \partial x_j} = -\frac{\partial^2\Phi}{\partial x_j \partial x_k} = \frac{\partial E_k}{\partial x_j} \quad (j,k = 1,2,3)$$

(provided Φ is in function class $\mathcal{C}^2(\Omega)$).

Result: The "integrability conditions"

$$\boxed{\frac{\partial E_j}{\partial x_k} = \frac{\partial E_k}{\partial x_j} \quad (j,k = 1,2,3)} \tag{A.2}$$

are a necessary criterion for the property "$\vec{E}$ is gradient field".

(ii) Since in Cartesian coordinates

$$\operatorname{curl}\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\vec{e}_z\,,$$

equation (A.2) is equivalent to

$$\boxed{\operatorname{curl}\vec{E} = 0}\,. \tag{A.3}$$

This criterion is valid in any coordinate system.

(iii) Important is the reverse statement (without proof):

Let $\vec{E} : \Omega \to V_3$ be a vector field, defined on a simply connected domain Ω, which satisfies the criterion (A.3) (or (A.2)). Then $\vec{E}$ is a gradient field, $\vec{E} = -\nabla\Phi$, and its potential function is uniquely defined by $\vec{E}$ up to an additive constant.

A.3. Calculation of a Potential Function

Given: Vector field $\vec{E}(\vec{r})$ defined on a simply connected domain $\Omega \subset V_3$, which satisfies the condition $\operatorname{curl}\vec{E} = 0$.

Find: Potential function $\Phi(\vec{r})$ defined on Ω, with $\vec{E} = -\nabla\Phi$.

Method of explicit calculation of Φ:

Choose fixed reference point $P(\vec{r}_0) = P_0$ and connect it with an arbitrary point $P(\vec{r}) = P$ by a (piecewise) smooth curve $C(P_0, P)$ (see figure 25). Parametric representation of $C(P_0, P)$:

$$[\lambda_0, \lambda_1] \ni \lambda \mapsto \vec{r}(\lambda) \in V_3\,;\ \vec{r}(\lambda_0) = \vec{r}_0 \text{ and } \vec{r}(\lambda_1) = \vec{r}\,.$$

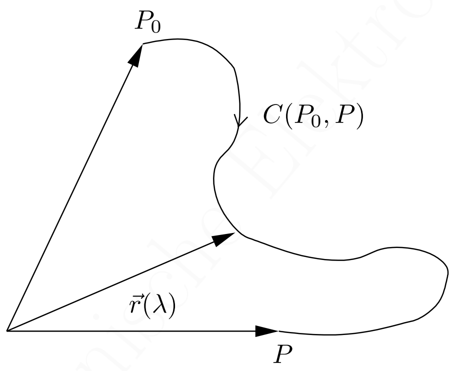


Figure 25: Smooth curve $C(P_0, P)$ connecting P_0 with P

We evaluate the (existing) potential along the curve $\vec{r}(\lambda)$ and calculate its derivative along $C(P_0, P)$ using the "chain rule":

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi(\vec{r}(\lambda)) = \nabla\Phi(\vec{r}(\lambda)) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}(\lambda)\,.$$

Hence, we obtain

$$\int\limits_{C(P_0,P)} \vec{E} \cdot \mathrm{d}\vec{r} = \int\limits_{\lambda_0}^{\lambda_1} \vec{E}(\vec{r}(\lambda)) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}(\lambda)\mathrm{d}\lambda = -\int\limits_{\lambda_0}^{\lambda_1} \nabla\Phi(\vec{r}(\lambda)) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}(\lambda)\mathrm{d}\lambda$$

$$= -\int\limits_{\lambda_0}^{\lambda_1} \frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi(\vec{r}(\lambda))\mathrm{d}\lambda = \Phi(\vec{r}(\lambda_0)) - \Phi(\vec{r}(\lambda_1)) = \Phi(\vec{r}_0) - \Phi(\vec{r})\,,$$

and find as result:

$$\boxed{\Phi(\vec{r}) = \Phi(\vec{r}_0) - \int\limits_{C(P_0,P)} \vec{E} \cdot \mathrm{d}\vec{r} = \Phi(\vec{r}_0) - \int\limits_{P(\vec{r}_0)}^{P(\vec{r})} \vec{E} \cdot \mathrm{d}\vec{r}}\,. \tag{A.4}$$

Remarks:

- The result does evidently not depend on the special choice of the path connecting P_0 with P.
- $\Phi(\vec{r}_0)$ is the above-mentioned additive constant, which can be arbitrarily chosen.

A.4. Equivalent Characterizations of Gradient Fields

For a differentiable vector field $\vec{E} : \Omega \to V_3$ defined on a simply connected domain $\Omega \subset V_3$, the following statements are equivalent:

a) $\vec{E}$ satisfies the "integrability conditions"

$$\frac{\partial E_j}{\partial x_k} = \frac{\partial E_k}{\partial x_j} \quad (j, k = 1, 2, 3)$$

in Cartesian coordinates.

b) $\mathrm{curl}\,\vec{E} = 0$.

c) $\vec{E}$ is a gradient field.

d) The path integral $\int_{C(P_0,P_1)} \vec{E} \cdot \mathrm{d}\vec{r}$ does not depend on the curve connecting P_0 with P_1 (i.e., $\vec{E}$ is conservative).

e) $\int_C \vec{E} \cdot \mathrm{d}\vec{r} = 0$ for any closed curve $C \subset \Omega$.

A.5. Geometric Interpretation of Potential Function and Gradient Field

(i) Let $\vec{E}(\vec{r})$ be a gradient field with potential $\Phi(\vec{r})$, i.e., $\vec{E} = -\nabla\Phi$. For a given fixed value of the potential $\Phi_0 \in \mathbb{R}$, we consider the "equipotential set"

$$\mathcal{F}[\Phi_0] = \{\vec{r} \in \Omega \mid \Phi(\vec{r}) = \Phi_0\}\,.$$

Assume that $\Phi(\vec{r}) = \Phi_0$ has solutions (i.e., $\mathcal{F}[\Phi_0] \neq \emptyset$) and that $\nabla\Phi(\vec{r}) \neq 0$ on $\mathcal{F}[\Phi_0]$. Then $\mathcal{F}[\Phi_0]$ is two-dimensional surface, called "**equipotential surface**".

(ii) For a point $P(\vec{r}) \in \mathcal{F}[\Phi_0]$, we construct the tangential plane $T_{P(\vec{r})}$, which touches $\mathcal{F}[\Phi_0]$ at $P(\vec{r})$. Then we find: $-\nabla\Phi(\vec{r}) = \vec{E}(\vec{r})$ is perpendicular to $T_{P(\vec{r})}$, i.e., $\nabla\Phi$ and $\vec{E}$ always point in the direction of the surface normal vector along an equipotential surface.

Proof: For an arbitrary tangential vector $\vec{t}_P \in T_P$, we can find a curve C that is completely contained in $\mathcal{F}[\Phi_0]$ and passes the point P with tangent vector $\vec{t}_P$:
$C : \lambda \mapsto \vec{r}(\lambda) \in \mathcal{F}[\Phi_0]$; $\vec{r}(\lambda_0) = \vec{r}_P$; $\frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}(\lambda_0) = \vec{t}_P$. Since the potential $\Phi(\vec{r})$ does not change its value along the curve, the function $\lambda \mapsto \Phi(\vec{r}(\lambda))$ is constant. Hence:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi(\vec{r}(\lambda))\bigg|_{\lambda_0} = \nabla\Phi(\vec{r}_P) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}\lambda}(\lambda_0) = \nabla\Phi(\vec{r}_P) \cdot \vec{t}_P\,,$$

i.e., $\nabla\Phi(\vec{r}_P) \perp \vec{t}_P$ for all $\vec{t}_P \in T_P$.