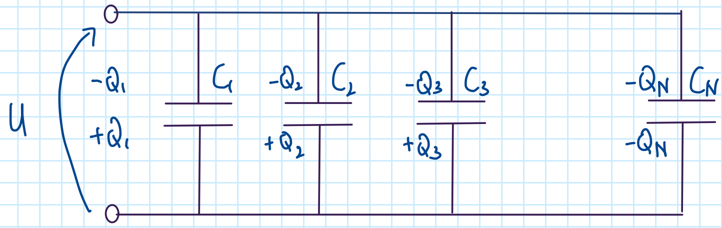
1.7.3. Capacitor Aggregations/Configurations

(i) Parallel circuit:

N capacitors in parallel $C_i \quad (i = 1 \dots N)$

(Note: they don't have to be necessarily parallel plate capacitors!)



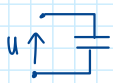
Can we find an equivalent circuit representation, which describes this configuration by a single capacitance? = total capacitance C_p

· Voltage is the same for each capacitor C_i

· For each C_i: $C_i = \frac{Q_i}{U} \Rightarrow Q_i = C_i \cdot U$

· total Q_{total}: $Q_{total} = \sum_{i=1}^{N} Q_i = \sum_{i=1}^{N} C_i \cdot U = U \sum_{i=1}^{N} C_i$

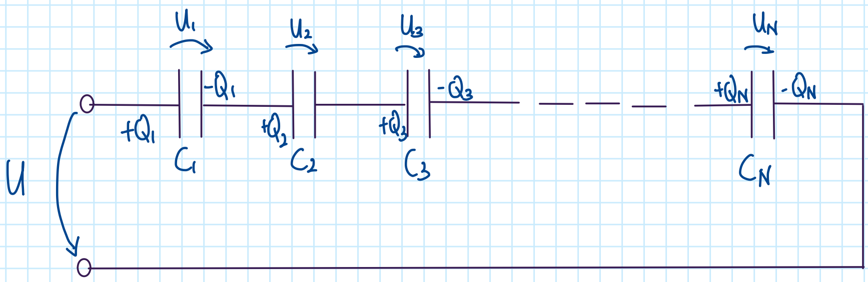
$$Q_{total} = \underset{\substack{\uparrow \\ \text{parallel}}}{C_p} \cdot U \Rightarrow \boxed{C_p = \sum_{i=1}^{N} C_i} \quad (1.48)$$

equivalent circuit representation:  $C_p = \sum_{i=1}^{N} C_i$

(ii) Serial circuit

N capacitors in serial configuration

(Note: they don't have to be necessarily parallel plate capacitors!)



in an electrostatic equilibrium, if all C_i are charge ($\hat{=}$ stable situation)

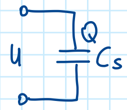
$Q_1 = Q_2 \quad Q_2 = Q_3 \quad Q_3 = Q_4 \dots Q_{N-1} = Q_N \Rightarrow Q_i$ are equal $= Q_{total}$

$Q_{total} = Q_1 = Q_2 = Q_3 = Q_N$

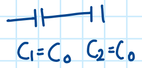
for one capacitor $C_i = \frac{Q_i}{U_i} = \frac{Q_{total}}{U_i} \Rightarrow U_i = \frac{1}{C_i} \cdot Q_{total}$

$$U = \sum_{i=1}^{N} U_i = \sum_{i=1}^{N} \frac{1}{C_i} Q_{total} = Q_{total} \cdot \underbrace{\sum_{i=1}^{N} \frac{1}{C_i}}_{\substack{C_s \\ \downarrow \\ \text{serial}}}$$

$$C_s = \frac{Q_{total}}{U} \qquad U = \frac{1}{C_s} \cdot Q$$



$$\Rightarrow \boxed{\frac{1}{C_s} = \sum_{i=1}^{N} \frac{1}{C_i}} \quad (1.46)$$

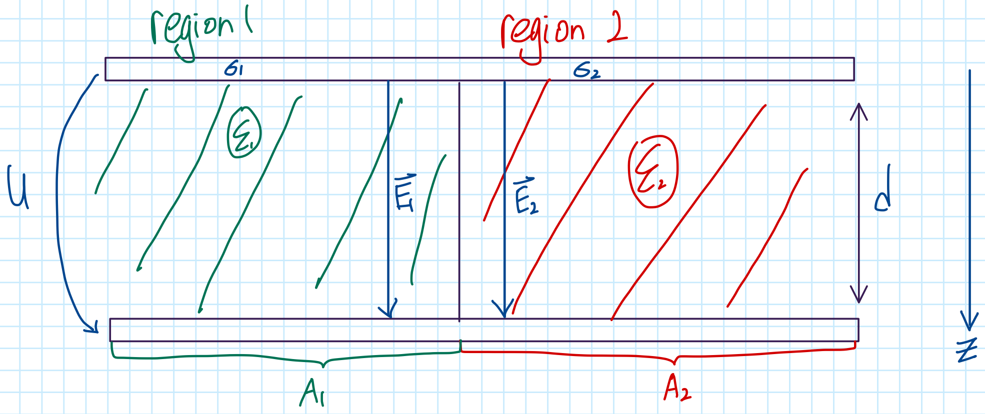


$$\frac{1}{C_{total}} = \frac{1}{C_0} + \frac{1}{C_0} = \frac{2}{C_0}$$

$$\Rightarrow C_{total} = \frac{C_0}{2}$$

(iii) Dielectric layers in parallel

Plate capacitor is filled with two dielectric layers in arranged side by side in parallel



$$\vec{E} = E_z \cdot \vec{e}_z$$

region	permittivity	area	Surface charge	E-field	D-field
1	ε_1	A_1	σ_1	$\vec{E}_1$	$\vec{D}_1$
2	ε_2	A_2	σ_2	$\vec{E}_2$	$\vec{D}_2$

Calculate total charge Q and voltage U, and hereof the total capacitance of this arrangement: $C = \frac{Q}{U}$

$$U = \int_0^d E_z \cdot \underbrace{\vec{e}_z \cdot \vec{e}_z}_{1} \cdot dz = E_z \cdot d$$

Same voltage in both regions $\Rightarrow U = E_1 d = E_2 d \Rightarrow E = \frac{U}{d} = |\vec{E}|$

$$E_1 = E_2$$

Since $\varepsilon_1 \neq \varepsilon_2 \Rightarrow |\vec{D}_1| \neq |\vec{D}_2|$ since $\vec{D} = \varepsilon \vec{E}$

$$D_1 = \varepsilon_1 E_1 = \varepsilon_1 \frac{U}{d}$$

$$D_2 = \varepsilon_2 E_2 = \varepsilon_2 \frac{U}{d}$$

$$C = \frac{Q}{U}$$

At surface of a plate: $|\vec{D}| = \sigma \Rightarrow \sigma_1 \neq \sigma_2$

$$\sigma_1 = D_1 = \varepsilon_1 \frac{U}{d}$$

$$\sigma_2 = D_2 = \varepsilon_2 \frac{U}{d}$$

Calculate total Q at the electrodes:

$$Q = \int_A \sigma \cdot da = \sigma \cdot A$$

σ = Homogeneous

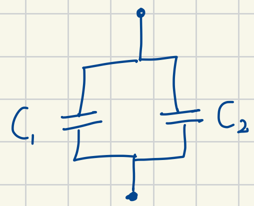
$$= (\sigma \cdot A_1 + \sigma \cdot A_2)$$

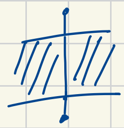
$$= \frac{\varepsilon_1 A_1 u}{d} + \frac{\varepsilon_2 A_2 u}{d}$$

$$C = \frac{Q}{u} = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = \underset{\substack{\downarrow \\ \text{region 1}}}{C_1} + \underset{\substack{\downarrow \\ \text{region 2}}}{C_2}$$

$\triangleq$ parallel connection of capacitors

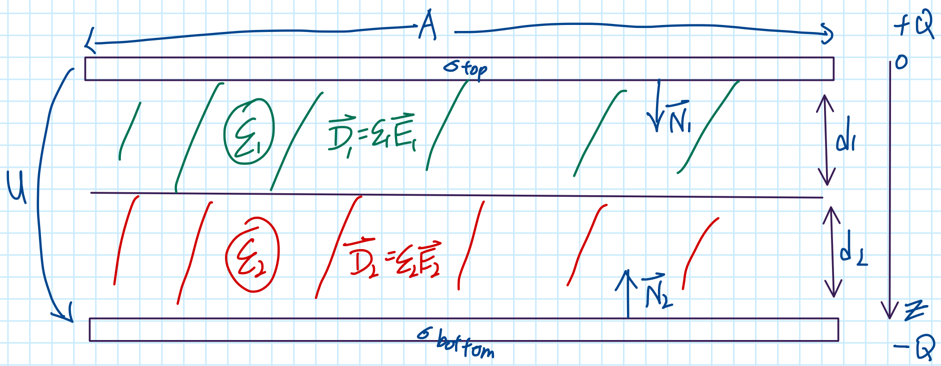
Equivalent circuit representation:





(iii) Dielectric layers in series

Plate capacitor is filled with two dielectric materials, which are arranged in series



$\Rightarrow \vec{N}_1 = -\vec{N}_2$

region	permittivity	area	Surface charge	E-field	D-field
1	ε_1	A	$\sigma_{top} \Rightarrow Q = \sigma_{top} \cdot A$	$\vec{E}_1$	$\vec{D}_1$
2	ε_2	A	$\sigma_{bot} \Rightarrow Q = \sigma_{bot} \cdot A$	$\vec{E}_2$	$\vec{D}_2$

Calculate total charge Q and voltage U, and hereof the total capacitance of this arrangement: $C_{tot} = \frac{Q}{U}$

· Calculate Q: $\sigma_{top} = +\frac{Q}{A}$

$\sigma_{bottom} = -\frac{Q}{A}$

· We know at Conductor's surface: $\vec{D}_1 \cdot \vec{N}_1 = \sigma_{top} = \frac{Q}{A}$ $\qquad \vec{N}_1 = -\vec{N}_2$

$\vec{D}_2 \cdot \vec{N}_2 = \sigma_{bot} = -\frac{Q}{A}$

$-\vec{D}_2 \cdot \vec{N}_1 = \sigma_{bot} = -\frac{Q}{A}$

$\Rightarrow \vec{D}_2 \cdot \vec{N}_1 = \frac{Q}{A}$

$\Rightarrow \vec{D}_1 = \vec{D}_2 \Rightarrow \vec{D}$ is Continuous

$|\vec{D}_1| = |\vec{D}_2| = \frac{Q}{A}$

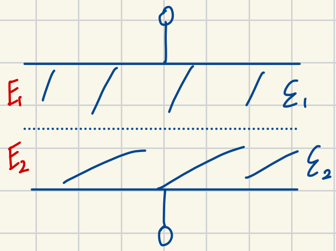
$\Rightarrow \vec{E}_1 \neq \vec{E}_2$ due to $\varepsilon_1 \neq \varepsilon_2$

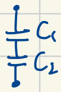
· Calculate Voltage :

$$U = \int_{top}^{bottom} \vec{E}\, d\vec{r} = \int_0^{d_1} E_1\, dz + \int_{d_1}^{d_2} E_2\, dz = d_1 \cdot E_2 + d_2 \cdot E_2$$

$$U = d_1 \frac{D_1}{\varepsilon_1} + d_2 \frac{D_2}{\varepsilon_2} = \left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right) \cdot D_1 = \left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right) \cdot \frac{Q}{A}$$

$$\boxed{\cdot\ \text{Capacitance}: C = \frac{Q}{U} = \frac{\cancel{Q} \cdot A}{\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right) \cdot \cancel{Q}} = \frac{A}{\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right)}}$$





$$C_1 = \frac{\varepsilon_1 A}{d_1} \qquad C_2 = \frac{\varepsilon_2 A}{d_2}$$

$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d_1}{\varepsilon_1 A} + \frac{d_2}{\varepsilon_2 A}$$

$$\Rightarrow C_{total} = \frac{1}{\left(\frac{d_1}{\varepsilon_1 A} + \frac{d_2}{\varepsilon_2 A}\right)} = \frac{1}{\frac{1}{A}\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right)}$$

$$= \frac{A}{\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right)}$$