EDE1012 MATHEMATICS 2

Tutorial 4

The Gradient Vector & Multiple Integration

DIY

1. For each function below, determine the gradient vector and the maximum rate of change at point P given. Then, determine the directional derivative in the direction of vector **u** at point P.

   a) [redacted]

   b) [redacted]

   c) [redacted]

   d) [redacted]

ANS: **a)** $\nabla f = [2, -3]^T,\ |\nabla f| = \sqrt{13},\ D_{\mathbf{u}} f = -8/\sqrt{5}.$

b) $\nabla f(1,0) = [1,0]^T,\ |\nabla f| = 1,\ D_{\mathbf{u}} f(1,0) = 3/\sqrt{10}.$

c) $\nabla f(1,\pi/2) = [-3,0]^T,\ |\nabla f| = 3,\ D_{\mathbf{u}} f(1,\pi/2) = -12/5.$

d) $\nabla f(1,1,1) = [2,2,2]^T,\ |\nabla f| = \sqrt{12},\ D_{\mathbf{u}} f(1,1,1) = 14/\sqrt{29}.$

2. Determine the directions at which each function increases and decreases most rapidly at the point P. Then, find the derivative of the functions in these directions.

   a) [redacted]

   b) [redacted]

ANS: **a)** Increase most rapidly in $[-1,1]^T/\sqrt{2}$ with $D_{\mathbf{u}} f(1,1) = \sqrt{2}.$
Decrease most rapidly in $[1,-1]^T/\sqrt{2}$ with $D_{\mathbf{u}} f(1,1) = -\sqrt{2}.$

b) Increase most rapidly in $[1,1,1]^T/\sqrt{3}$ with $D_{\mathbf{u}} f(1,1) = \sqrt{12}.$
Decrease most rapidly in $[-1,-1,-1]^T/\sqrt{3}$ with $D_{\mathbf{u}} f(1,1) = -\sqrt{12}.$

Q2 b) $f(x,y,z) = \ln(xy) + \ln(yz) + \ln(xz), \quad P = (1,1,1)$

$$\vec{\nabla} f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \frac{\cancel{y}}{x\cancel{y}} + \frac{\cancel{z}}{x\cancel{z}} \\ \frac{\cancel{x}}{\cancel{x}y} + \frac{\cancel{z}}{y\cancel{z}} \\ \frac{\cancel{y}}{\cancel{y}z} + \frac{\cancel{x}}{\cancel{x}z} \end{pmatrix} = \begin{pmatrix} 2/x \\ 2/y \\ 2/z \end{pmatrix} \rightarrow \vec{\nabla} f(1,1,1) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} //$$

is the direction of max increase of f.

OR

$$\underbrace{\frac{\vec{\nabla} f(1,1,1)}{|\vec{\nabla} f(1,1,1)|} = \frac{2}{\sqrt{3(2^2)}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{unit vector}} //$$

$\Rightarrow$ The direction of max decrease of f is

$$-\vec{\nabla} f(1,1,1) = -\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad \text{or} \quad \frac{-1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} //$$

$$D_{\vec{\nabla} f} f(1,1,1) = |\vec{\nabla} f(1,1,1)| = \sqrt{3(2^2)} = \sqrt{12} //$$

$$D_{-\vec{\nabla} f} f(1,1,1) = -|\vec{\nabla} f(1,1,1)| = -\sqrt{12} //$$

3. Using Fubini's theorem, show that the mixed partial derivatives for a differentiable function f(x, y) are equal (Clairaut's theorem).

4. Sketch the region of integration for each integral below and evaluate it. Change the order of integration if necessary.

a) [redacted]

b) [redacted]

c) [redacted]

d) [redacted]

e) [redacted]

f) [redacted]

ANS: **a)** 16. **b)** $2 + \pi^2/2$. **c)** e - 2. **d)** e - 1. **e)** 4 - sin(4). **f)** $\frac{1}{80\pi}$.

5. Using polar coordinates, evaluate the following integrals.

a) [redacted]

b) [redacted]

ANS: **a)** $\pi/8$. **b)** $4\pi\left(\sqrt{2}-1\right)$.

6. Evaluate the triple integral

[redacted]

where

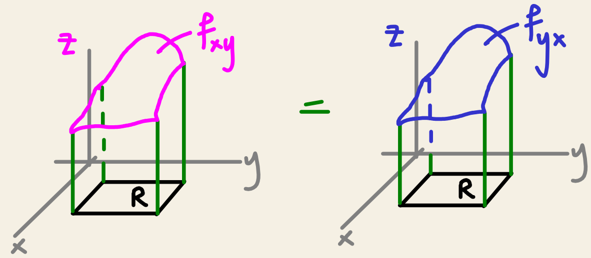
[redacted]

ANS: 63π.

$f_{xy},\ f_{yx}$

3. Using Fubini's theorem, show that the $\underbrace{\text{mixed partial derivatives}}_{}$ for a differentiable function f(x, y) are equal (Clairaut's theorem).

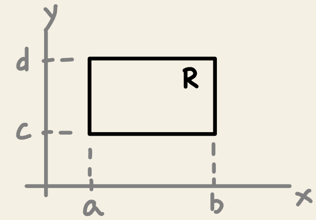
If the volumes of integration are the same for both functions f_xy and f_yx for any region R, then both functions must be equal.



$$\int_a^b \underbrace{\int_c^d f_{xy}(x,y)\,dy}_{f_x(x,y)\big|_{y=c}^{y=d}}\,dx = \int_a^b f_x(x,d) - f_x(x,c)\,dx$$

$$= f(b,d) - f(a,d) - [f(b,c) - f(a,c)]$$

Fubini.



$$\int_c^d \int_a^b f_{yx}(x,y)\,dx\,dy = \int_c^d f_y(b,y) - f_y(a,y)\,dy$$

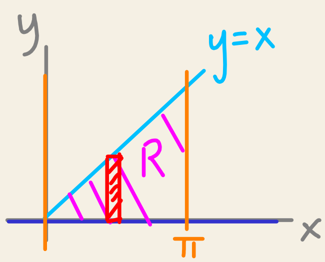
$$= f(b,d) - f(b,c) - [f(a,d) - f(a,c)]$$

= shown //

$\therefore\ f_{xy} = f_{yx}$.

Q4 b) $I = \underbrace{\int_0^{\pi} \int_0^{x}}_{} x \sin y \, dy dx$

From limits, sketch region R.



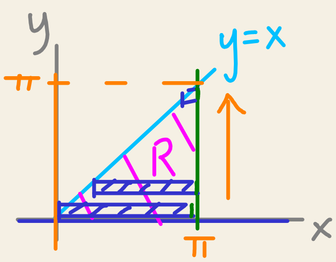
$$= \int_0^{\pi} -x\cos y \Big|_0^x \, dx = \int_0^{\pi} -x(\cos x - 1)\,dx$$

$$= \left[-x(\sin x - x) - \cos x - \frac{x^2}{2}\right]_0^{\pi}$$

	u	dv
+	$-x$	$\cos x - 1$
−	-1	$\sin x - x$
	0	$-\cos x - \frac{x^2}{2}$

$$= \left(-\pi(-\pi) + 1 - \frac{\pi^2}{2}\right) - (-1)$$

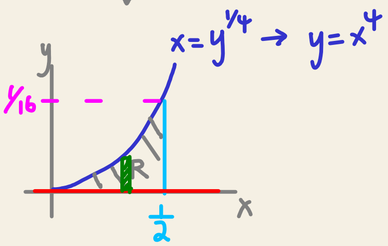
$$= \frac{\pi^2}{2} + 2 \ //$$



$$I = \int_0^{\pi} \int_y^{\pi} x \sin y \, dx dy \overset{\text{DIY}}{=} \cdots = \frac{\pi^2}{2} + 2.$$

Q4 f) $\underbrace{\int_0^{1/16}\int_{y^{1/4}}^{1/2}}\cos\left(16\pi x^5\right)dx\,dy$

Difficult to integrate, so use Fubini.



$$\int_0^{1/2}\int_0^{x^4}\cos(16\pi x^5)\,dy\,dx = \int_0^{1/2}\cos(16\pi x^5)\cdot y\Big|_0^{x^4}dx$$

$$= \int_0^{1/2} x^4\cdot\cos(16\pi x^5)\,dx \to \text{let } u=x^5,\ du=5x^4\,dx,$$

limits for u.

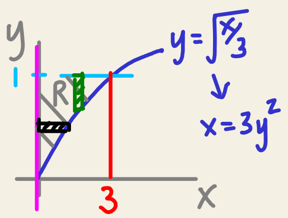
$$= \int_0^{1/32}\cancel{x^4}\cos(16\pi u)\left(\frac{1}{5\cancel{x^4}}du\right) = \frac{1}{80\pi}\sin(16\pi u)\Big|_0^{1/32}$$

$$= \frac{1}{80\pi}\left(\sin\frac{\pi}{2}-\sin 0\right)$$

$$= \frac{1}{80\pi} \,/\!/$$

Q4 d) $I = \underbrace{\int_0^3\int_{\sqrt{x/3}}^1} e^{y^3}\,dy\,dx$

Difficult to integrate, so use Fubini.



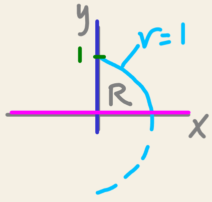
$$I = \int_0^1\int_0^{3y^2} e^{y^3}\,dx\,dy = \dots$$

DIY

Q5 a) $\underbrace{\int_0^1\int_0^{\sqrt{1-y^2}}}\underbrace{x^2+y^2}_{r^2}\,dx\,dy$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^{\cancel{2}\,3}\cdot\cancel{r}\,dr\,d\theta = \int_0^{\frac{\pi}{2}}d\theta\int_0^1 r^3\,dr = \frac{\pi}{2}\left(\frac{r^4}{4}\right)\Big|_0^1$$

$$= \frac{\pi}{8}\,/\!/$$



6. Evaluate the triple integral

$$I = \iiint_E 2+\sqrt{x^2+y^2}\,dV$$

where

$$E = \left\{(x,y,z) \,\middle|\, \underbrace{\sqrt{x^2+y^2}}_{r} \le z/2 \le 3\right\}$$

$\times 2$: $\overset{z=2r}{2r} \le z \le \overset{z=6}{6}$

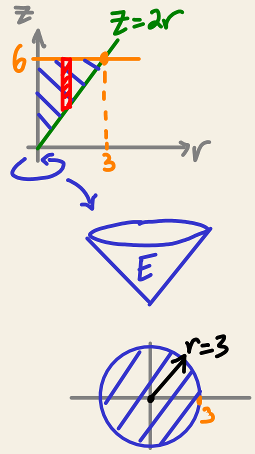
ANS: 63π.

In cylindrical coords,

$$I = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=3}\int_{z=2r}^{z=6}(2+r)\overbrace{r\,dz\,dr\,d\theta}^{dV}$$

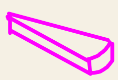
$$= \underset{\text{DIY}}{\cdots} = 63\pi.$$

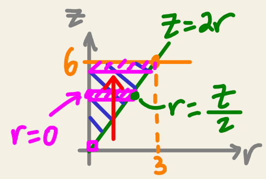
DIY.



OR:

$$I = \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=6}\underbrace{\int_{r=0}^{r=\frac{z}{2}}(2+r)r\,dr\,dz\,d\theta}$$





7. Evaluate the following triple integrals.

DIY
~~a)~~ [redacted]

DIY
~~b)~~ [redacted]

~~c)~~ [redacted]
DIY

ANS: **a)** 1. **b)** 16/3. **c)** $\frac{8}{3}\ln 7$.

8. Using triple integrals, evaluate the volume of each region described below. An appropriate coordinate system will make the integral easier to evaluate.

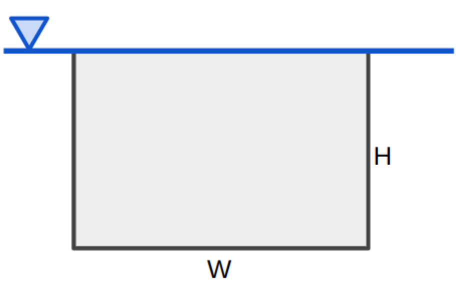
   ~~a)~~ The region E that is between the parabolic surface $z = y^2$ and the xy-plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$ and $y = 1$.

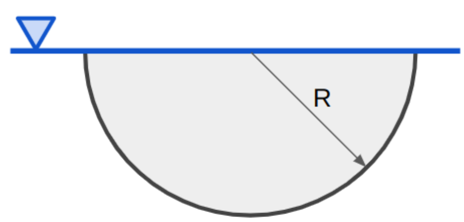
   DIY
   ~~b)~~ The region E that is bounded by the cone [redacted] and $z = 1$.

   Try
   c) The region E that is bounded by the spherical surface [redacted] and the cones [redacted] and [redacted].

ANS: **a)** ⅔. **b)** π/3. **c)** 9π(√2 - 1).

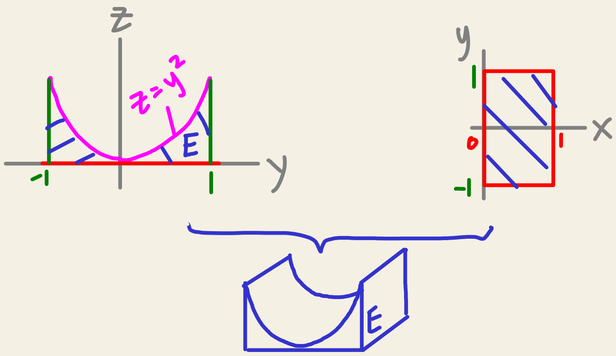
9. **Use of Calculus in Engineering Design**
   In an underwater sea aquarium, two acrylic glass panels in the shapes shown below need to be installed for visitors to view the marine life.





Q8

a) The region E that is between the parabolic surface $z = y^2$ and the xy-plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$ and $y = 1$.

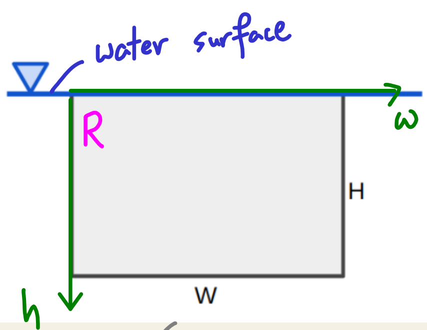


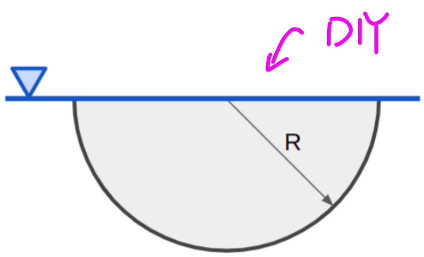
$$Vol = \int_0^1 \int_{-1}^1 \int_0^{y^2} 1 \, dz\,dy\,dx$$

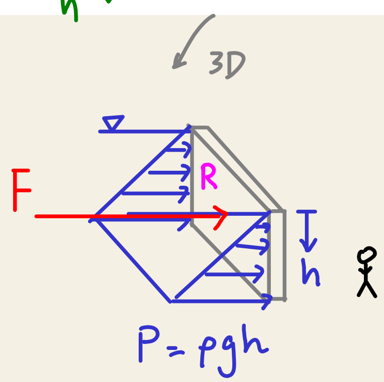
$$= \dots$$

DIY.

9. **Use of Calculus in Engineering Design**
In an underwater sea aquarium, two acrylic glass panels in the shapes shown below need to be installed for visitors to view the marine life.







$$F = \iint_R \rho g h \, dA = \rho g \int_0^W \int_0^H h \, dh\,dw$$

$$= \rho g \int_0^W dw \int_0^H h \, dh$$

$$= \rho g W \frac{h^2}{2} \Big|_0^H = \frac{\rho g W H^2}{2} \ //$$

$$= \rho g \int_0^W \int_0^H (h+d) \, dh\, dw$$

$$= \rho g \int_0^W dw \int_0^H h \, dh + \int_0^H d \, dh$$

$$= \rho g \left[W \cdot \left[\frac{H^2}{2} - dH \right] \right]$$

$$= \rho g W \cdot \frac{H^2 - 2dH}{2}$$

In order to determine the thickness of the panels and their attachment method to the surrounding wall, the net force caused by the hydrostatic water pressure must be determined. Under a liquid of density ρ, the hydrostatic pressure is

[redacted]

where h is the depth from the free surface of the liquid.

a) Evaluate the net hydrostatic force on each acrylic panel as a function of other parameters.

b) What would be the hydrostatic force on the rectangular panel instead if it is submerged such that its top edge is d meters below the water surface?

ANS: **a)** $\text{Rectangle}: \ F = \dfrac{\rho g W H^2}{2}. \ \text{Semicircle}: \ F = \dfrac{2\rho g R^3}{3}.$

b) $F = \dfrac{\rho g W \left(H^2 + 2dH\right)}{2}.$

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient
2) https://openstax.org/books/calculus-volume-3/pages/5-1-double-integrals-over-rectangular-regions
3) https://openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions
4) https://openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates
5) https://openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals
6) https://openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates

End of Tutorial 4

(Email to youliangzheng@gmail.com for assistance.)