Notes

Predicate Logic

P_x : x has property P

Predicate (P), Placeholder (x)

$x \in \{x_1, x_2, \dots\}$; $P_{x_n} \in \{t, f\}$

Quantifiers

Universal quantifiers : $\forall_{x \in \{x_1, x_2, \dots\}} P_x \Leftrightarrow P_{x_1} \wedge P_{x_2} \wedge \dots$ P_x holds for all x

Existential quantifiers : $\exists_{x \in \{x_1, x_2, \dots\}} P_x \Leftrightarrow P_{x_1} \vee P_{x_2} \vee \dots$ there exists an x such that P_x holds

Laws

1) Laws for Quantifiers (F, G)

2) All Laws from propositional Logic (A-E)

outside of Quantifiers

e.g. E6 : $\underbrace{\forall_{x \in G} P_x}_{a} \wedge \underbrace{\exists_{x \in G} Q_x}_{b} \Rightarrow \underbrace{\forall_{x \in G} P_x}_{a} \vee \underbrace{\exists_{x \in G} Q_x}_{b}$

inside of Quantifiers (inside a single Quantifier)

e.g. E7 : $\forall_{x \in G} \neg P_x \Rightarrow \forall_{x \in G} (P_x \rightarrow Q_x)$

$\neg a \Rightarrow a \rightarrow b$

Attention! Not half inside and half outside quantifiers

e.g. Law E9 : $\exists_{x \in G} P_x \wedge [\exists_{x \in G} P_x \rightarrow \exists_{x \in G} Q_x] \Rightarrow \exists_{x \in G} Q_x$ (valid)

$\exists_{x \in G} (P_x \wedge [P_x \rightarrow Q_x]) \Rightarrow \exists_{x \in G} Q_x$ (valid)

$\exists_{x \in G} P_x \wedge \exists_{x \in G} (P_x \rightarrow Q_x) \Rightarrow \exists_{x \in G} Q_x$ ✗

Deductive Proof scheme with Predicate Logic

Same scheme as with propositional logic

Hints:

- Eliminate $\neg$ in front of quantifiers and parentheses (Law F1, F2)
- Produce structures where E9, E10, E12, E13 or G11, G12 can be applied on

- Try to use universal quantifiers as much as possible

$$\forall_{x \in G} P_x \Rightarrow \exists_{x \in G} P_x \quad G2$$

2.1 The following three statements are given:

$[A_1]$: All Bavarians are German.

$[A_2]$: Every person is either a German or an Italian (multiple citizenship is not allowed).

$[A_3]$: Hans is a Bavarian.

2.1.a Formalize the given statements by means of those predicates:

$Bx \iff x$ is a Bavarian $\qquad Gx \iff x$ is a German $\qquad Ix \iff x$ is an Italian

P: set of all German and Italian people

$x_1 \iff$ "Hans"; $x_1 \in P$

$$A_1 \Leftrightarrow \forall_{x \in P} (Bx \rightarrow Gx)$$

$$A_2 \Leftrightarrow \forall_{x \in P} (Gx \not\leftrightarrow Ix)$$

$$A_3 \Leftrightarrow Bx_1$$

2.1.b Formalize the following conclusions and prove them using the given statements:

$[C_1]$: Hans is not an Italian.

$[C_2]$: There exists nobody that is Bavarian and Italian.

$C_1 \Leftrightarrow \neg Ix_1$

1) $\forall_{x \in P} (Bx \rightarrow Gx)$

2) $\forall_{x \in P} (Gx \not\leftrightarrow Ix)$

3) Bx_1 $\qquad \Rightarrow \neg Ix_1$

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4) $Bx_1 \rightarrow Gx_1$ $\qquad$ 1, G1

5) Gx_1 $\qquad$ 3, 4, E9

6) $Gx_1 \not\leftrightarrow Ix_1$ $\qquad$ 2, G1

7) $Gx_1 \leftrightarrow \neg Ix_1$ $\qquad$ 6, B11, C5

8) $Gx_1 \rightarrow \neg Ix_1$ $\qquad$ 7, C11, E4

9) $\neg Ix_1$ $\qquad$ 5, 8, E9

Idea

$Bx_1 \land (Bx_1 \rightarrow Gx_1)$

$\Rightarrow Gx_1$

$Gx_1 \land (Gx_1 \rightarrow \neg Ix_1)$

$\Rightarrow \neg Ix_1$

$C_2 \Longleftrightarrow \forall_{x \in P} \neg(B_x \wedge I_x)$

$\Longleftrightarrow \neg \exists_{x \in P} B_x \wedge I_x$

Idea:

$(B_x \to G_x) \wedge (G_x \to I_x)$

$\Rightarrow (B_x \to \neg I_x)$

1) $\forall_{x \in P} (B_x \to G_x)$

2) $\forall_{x \in P} (G_x \not\leftrightarrow I_x)$

not needed

[3) B_{x_1}]

$\Rightarrow \neg \exists_{x \in P} B_x \wedge I_x$

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4)	$\forall_{x \in P} [(G_x \to \neg I_x) \wedge (\neg I_x \to G_x)]$	2, B11, C5, C11
5)	$\forall_{x \in P} (G_x \to \neg I_x)$	4, F5, E4
6)	$\forall_{x \in P} (\neg I_x \to G_x)$	4, F5, E4
7)	$\forall_{x \in P} [(B_x \to G_x) \wedge (G_x \to \neg I_x)]$	1, 5, F5
8)	$\forall_{x \in P} (B_x \to \neg I_x)$	7, E11
9)	$\forall_{x \in P} \neg(\neg(B_x \to \neg I_x))$	8, A9
10)	$\neg \exists_{x \in P} \neg(B_x \to \neg I_x)$	9, F2
11)	$\neg \exists_{x \in P} B_x \wedge I_x$	10, D2

2.2 Formalize the following definitions. Use the predicate Kab representing $a < b$.

2.2.a A function f is continuous in the interval D, if and only if:
For every point $x_0 \in D$, there's for each $\varepsilon > 0$ a $\delta > 0$ so that:
$|f(x_0) - f(x)| < \varepsilon$ for all x with $|x_0 - x| < \delta$

Sf : function f is continuous on Interval D

$$Sf \Leftrightarrow \forall_{x_0 \in D} \; \forall_{\varepsilon \in \mathbb{R}^+} \; \exists_{\delta \in \mathbb{R}^+} \; \forall_{x \in D} \; \underbrace{K(|x_0 - x|)\delta} \to \underbrace{K(|f(x_0) - f(x)|)\varepsilon}$$

$|x_0 - x| < \delta$, then $|f(x_0) - f(x)| < \varepsilon$

2.2.b A function f, defined on the interval D, has the limit y_1 for $x \longrightarrow +\infty$ if and only if:
For every $\varepsilon \in R^+$ there is a $x_0 \in D$ with $|f(x) - y_1| < \varepsilon$ for all $x \in D$ with $x > x_0$.

Gf_x : function f has limit x

$$Gfy_1 \Leftrightarrow \forall_{\varepsilon \in \mathbb{R}^+} \; \exists_{x_0 \in D} \; \forall_{x \in D} \; Kx_0x \to K(|f(x) - y_1|)\varepsilon$$

if $x > x_0$, then $|f(x) - y_1| < \varepsilon$

2.4.d $\neg \underset{x \in \mathcal{G}}{\exists} (Px \longrightarrow \neg Qx) \wedge \left(\neg \underset{x \in \mathcal{G}}{\exists} Qx \vee \underset{x \in \mathcal{G}}{\forall} Sx\right) \quad \Longrightarrow \underset{x \in \mathcal{G}}{\exists} Sx$

1) $\neg \underset{x \in G}{\exists} (Px \to \neg Qx)$

2) $\neg \underset{x \in G}{\exists} Qx \vee \underset{x \in G}{\forall} Sx \quad \Rightarrow \underset{x \in G}{\exists} Sx$

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3) $\underset{x \in G}{\forall} \neg(Px \to \neg Qx)$ — 1, F2

4) $\underset{x \in G}{\forall} \neg(\neg Px \vee \neg Qx)$ — 3, D1

5) $\underset{x \in G}{\forall} (Px \wedge Qx)$ — 4, A10

6) $\underset{x \in G}{\forall} Px \wedge \underset{x \in G}{\forall} Qx$ — 5, F5

7) $\underset{x \in G}{\forall} Qx \wedge \underset{x \in G}{\forall} Px$ — 6, A1

8) $\underset{x \in G}{\forall} Qx$ — 7, E4

9) $\underset{x \in G}{\forall} \neg Qx \vee \underset{x \in G}{\forall} Sx$ — 2, F2

10) $\underset{x \in G}{\forall} (\neg Qx \vee Sx)$ — 9, F5

11) $\underset{x \in G}{\forall} (Qx \to Sx)$ — 10, D1

12) $\underset{x \in G}{\forall} Sx$ — 8, 11, G11

13) $\underset{x \in G}{\exists} Sx$ — 12, G2

Idea:

1) $Qx \to Sx$

2) $Qx \wedge (Qx \to Sx)$

$\Rightarrow Sx$

1) $\neg \underset{x \in G}{\exists} (Px \to \neg Qx)$

2) $\neg \underset{x \in G}{\exists} Qx \vee \underset{x \in G}{\forall} Sx \quad \Rightarrow \underset{x \in G}{\exists} Sx$

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3) $\underset{x \in G}{\forall} \neg Qx \vee \underset{x \in G}{\forall} Sx$ — 2, F2

4) $\underset{x \in G}{\forall} \neg Qx \vee Sx$ — 3, G5

5) $\underset{x \in G}{\forall} Qx \to Sx$ — 4, D1

6) $\underset{x \in G}{\forall} \neg(Px \to \neg Qx)$ — 1, F2

7) $\underset{x \in G}{\forall} Px \wedge Qx$ — 6, D2, A9

8) $\underset{x \in G}{\forall} (Px \wedge \underbrace{Qx \wedge (Qx \to Sx)}_{Sx})$ — 5, 7, F5

9) $\underset{x \in G}{\forall} (Px \wedge Sx)$ — 8, F5

E4

$a \wedge b \Rightarrow a$

10) $\underset{x \in G}{\forall} Px \wedge \underset{x \in G}{\forall} Sx$ — 9, F5

G2

$\underset{x \in G}{\forall} Px \Rightarrow \underset{x \in G}{\exists} Px$

11) $\underset{x \in G}{\forall} Sx$ — 10, E4

12) $\underset{x \in G}{\exists} Sx$ — 11, G2

2.4.e $\neg \exists_{x \in \mathcal{G}} Qx \wedge \forall_{x \in \mathcal{G}} (Tx \longrightarrow \neg(Qx \longrightarrow Px)) \quad \Longrightarrow \forall_{x \in \mathcal{G}} \neg Tx$

1) $\neg \exists_{x \in G} Qx$

2) $\forall_{x \in G} (Tx \to \neg(Qx \to Px)) \quad \Rightarrow \forall_{x \in G} \neg Tx$

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3) $\forall_{x \in G} \neg Qx$ 1, F2

4) $\forall_{x \in G} (Qx \to Px)$ 3, E7

5)

idea:

1) $Qx \Rightarrow Tx$

2) $Qx \wedge (Qx \to Tx)$

$\Rightarrow Tx$

1) $\neg \exists_{x \in G} Qx$

2) $\forall_{x \in G} (Tx \to \neg(Qx \to Px)) \quad \Rightarrow \forall_{x \in G} \neg Tx$

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3) $\forall_{x \in G} [(Qx \to Px) \to \neg Tx]$ 2, D3

4) $\forall_{x \in G} (\neg Qx \vee Px) \to \neg Tx$ 3, D1

5) $\forall_{x \in G} \neg Qx$ 1, F2

6) $\forall_{x \in G} [(\neg Qx \to \neg Tx) \wedge (Px \to \neg Tx)]$ 4, D8

7) $\forall_{x \in G} \neg Qx \to \neg Tx$ 6, F5, E4

8) $\forall_{x \in G} \neg Qx \wedge (\neg Qx \to \neg Tx)$ 5, 7, F5

9) $\forall_{x \in G} \neg Tx$ 8, E9

2.5 Check if the following are valid predicate logic implications.

Using 3rd approach

2.5.a $\underbrace{\exists_{x \in \mathcal{G}} Px}_{A(\underline{x})} \wedge \left(\underbrace{\exists_{x \in \mathcal{G}} Px}_{A(\underline{x})} \longrightarrow \underbrace{\exists_{x \in \mathcal{G}} Qx}_{B(\underline{x})} \right) \Longrightarrow \underbrace{\exists_{x \in \mathcal{G}} Qx}_{B(\underline{x})}$

$A(\underline{x}) \wedge (A(\underline{x}) \rightarrow B(\underline{x})) \Rightarrow B(\underline{x})$ Law E9

Check if $A \Rightarrow B$ is valid

Approaches

- Use $A \Rightarrow B$ IFF $A \rightarrow B \Leftrightarrow t$
- find a Counter example
- Use RTI to reduce to Known Laws

2.5.b $\exists_{x \in \mathcal{G}} Px \wedge \exists_{x \in \mathcal{G}} (Px \rightarrow Qx) \Longrightarrow \exists_{x \in \mathcal{G}} Qx$

False. 2nd approach will be used

$$\left[\exists_{x \in G} P_x \wedge \exists_{x \in G} (P_x \rightarrow Q_x)\right] \rightarrow \exists_{x \in G} Q_x \Leftrightarrow t$$

$$\overset{DI}{\Leftrightarrow} \left[\exists_{x \in G} P_x \wedge \exists_{x \in G} (\neg P_x \vee Q_x)\right] \rightarrow \exists_{x \in G} Q_x$$

$$\Leftrightarrow \neg\left[\exists_{x \in G} P_x \wedge \left(\exists_{x \in G} \neg P_x \vee \exists_{x \in G} Q_x\right)\right] \vee \exists_{x \in G} Q_x$$

$$\Leftrightarrow \neg\exists_{x \in G} P_x \vee \left(\neg\exists_{x \in G} \neg P_x \wedge \neg\exists_{x \in G} Q_x\right) \vee \exists_{x \in G} Q_x$$

$$\Leftrightarrow \left(\neg\exists_{x \in G} P_x \vee \neg\exists_{x \in G} \neg P_x \vee \exists_{x \in G} Q_x\right) \wedge \underbrace{\left(\neg\exists_{x \in G} P_x \vee \underbrace{\neg\exists_{x \in G} Q_x \vee \exists_{x \in G} Q_x}_{t}\right)}_{t}$$ Tava

$$\overset{FI}{\Leftrightarrow} \underset{f}{\neg\exists_{x \in G} P_x} \vee \underset{f}{\forall_{x \in G} P_x} \vee \underset{f}{\exists_{x \in G} Q_x} \qquad f \vee f \vee f \Leftrightarrow f$$

Counter example : Assume the following holds : $\neg\exists_{x \in G} Q_x \Leftrightarrow t$ and $\exists_{x \in G} P_x \Leftrightarrow t$ but $\neg\forall_{x \in G} P_x \Leftrightarrow t$, then the proposition form will be false

$\Rightarrow$ no tautology $\Rightarrow$ Implication not valid

2.6 Prove the general validity of the following propositions using the inductive proof scheme without using the expression to be proven as premise.

Inductive hypothesis

2.6.a $P(n) \iff \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}, \quad n \in \mathbb{N}$

Basis step: $P(1) \iff \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1+1}$ 1 is the smallest element

$\iff \frac{1}{2} = \frac{1}{2}$

$\iff$ true

inductive step: $P(n+1) \iff \sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{(n+1)}{(n+1)+1}$ show that $P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$

$\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$

$\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n+1}{n+2} - \frac{1}{(n+1)(n+2)}$

$\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{\overbrace{(n+1)^2 - 1}^{n^2+2n+1-1}}{(n+1)(n+2)}$

$\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{\overbrace{n^2+2n}^{n(\cancel{n+2})}}{(n+1)\cancel{(n+2)}}$

$\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

$\iff P(n)$

FROM $P(n) \iff P(n+1)$ IT FOLLOWS $P(n) \iff P(n+1)$

Conclusion

From $P(1) \iff t$ AND $\forall_{n \in \mathbb{N}} (P(n) \to P(n+1)) \iff t$

it follows $\forall_{n \in \mathbb{N}} P(n) \iff t$

2.6.b $P(n) \iff \left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} (i)^3, \quad n \in \mathbb{N}$

Hint: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Basis step: $P(1) \iff \underbrace{\left(\sum_{i=1}^{n} i\right)^2}_{1} = \underbrace{\sum_{i=1}^{n} (i)^3}_{1} \iff t$

Induction step: $P(n+1) \iff \left(\sum_{i=1}^{n+1} i\right)^2 = \sum_{i=1}^{n+1} (i)^3$

$$\iff \left(\sum_{i=1}^{n} i + (n+1)\right)^2 = \sum_{i=1}^{n} (i)^3 + (n+1)^3$$

$$\iff \left(\sum_{i=1}^{n} i\right)^2 + 2(n+1)\sum_{i=1}^{n} i + (n+1)^2 = \sum_{i=1}^{n} (i)^3 + (n+1)^3$$

$$\iff \left(\sum_{i=1}^{n} i\right)^2 + \cancel{2}(n+1)\left(\frac{n(n+1)}{\cancel{2}}\right) + (n+1)^2 = \sum_{i=1}^{n} (i)^3 + (n+1)^3$$

$$\iff \left(\sum_{i=1}^{n} i\right)^2 + \overbrace{n(n+1)^2 + (n+1)^2}^{(n+1)(n+1)^2} = \sum_{i=1}^{n} (i)^3 + (n+1)^3$$

$$\iff \left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} (i)^3$$

$$\iff P(n)$$

2.6.c $P(n) \iff \sum_{k=0}^{n} a^{n-k} \cdot b^k = \frac{a^{n+1}-b^{n+1}}{a-b}, \quad n \in \mathbb{N}_0$

Basis step : $P(0) \iff \underbrace{\sum_{k=0}^{0} a^{0-k} \cdot b^0}_{1} = \underbrace{\frac{a-b}{a-b}}_{1} \iff t$

Induction step : $P(n+1) \iff \sum_{k=0}^{n+1} a^{n+1-k} \cdot b^k = \frac{a^{n+2}-b^{n+2}}{a-b}$

$$\iff \sum_{k=0}^{n} a^{n+1-k} \cdot b^k + \underbrace{(a^{\overbrace{n+1-(n+1)}^{0}} \cdot b^{n+1})}_{b^{n+1}} = \frac{a^{n+2}-b^{n+2}}{a-b}$$

$$\iff \sum_{k=0}^{n} a \cdot a^{n-k} \cdot b^k + b^{n+1} = \frac{a^{n+2}-b^{n+2}}{a-b}$$

$\div a$

$$\iff \sum_{k=0}^{n} a^{n-k} \cdot b^k + \frac{b^{n+1}}{a} = \frac{a^{n+2}-b^{n+2}}{a(a-b)}$$

$$\iff \sum_{k=0}^{n} a^{n-k} \cdot b^k = \frac{a^{n+2}-b^{n+2}}{a(a-b)} - \frac{b^{n+1}}{a}$$

$$\iff \quad " \quad = \frac{a^{n+2} \cancel{-b^{n+2}} - (ab^{n+1} - \cancel{b^{n+2}})}{a(a-b)}$$

$$\iff \quad " \quad = \frac{\cancel{a}(a^{n+1} - b^{n+1})}{\cancel{a}(a-b)}$$

$$\iff \quad " \quad = \frac{a^{n+1} - b^{n+1}}{(a-b)}$$

$$\iff P(n)$$

2.6.d $P(n) \quad \iff \quad 1 + 2n \leq 3^n, \quad n \in \mathbb{N}$

Hint: The implication $a \leq c \wedge b \leq d \Longrightarrow a + b \leq c + d$ holds.

1) Basis step: $P(1) \iff 1+2 \leq 3 \iff 3 \leq 3 \iff t$

2) Induction step:

$$P(n+1) \iff 1 + 2(n+1) \leq 3^{(n+1)}$$

$$\iff 1 + 2n + 2 \leq 3^n \cdot 3$$

$$\iff (1+2n) + 2 \leq 3^n \cdot (1+2)$$

$a \iff a \wedge t$

$$\iff \underbrace{(1+2n)}_{a} + \underbrace{2}_{b} \leq \underbrace{3^n}_{c} + \underbrace{2 \cdot 3^n}_{d}$$

We want to show: $P(n) \wedge t \Rightarrow P(n+1)$

$$\overbrace{1 + 2n \leq 3^n} \wedge (2 \leq 2 \cdot 3^n) \Rightarrow (1+2n) + 2 \leq 3^n + 2 \cdot 3^n$$

$$\iff P(n+1)$$

$$P(n) \Rightarrow P(n+1)$$

3) Conclusion

From $P(1) \iff t$ AND $\forall_{n \in \mathbb{N}} P(n) \rightarrow P(n+1) \iff t$ IT FOLLOWS $\forall_{n \in \mathbb{N}} P(n) \iff t$