CHAIR OF ELECTRONIC DESIGN AUTOMATION
TECHNICAL UNIVERSITY OF MÜNCHEN
PROFESSOR DR.-ING. U. SCHLICHTMANN

DISCRETE MATHEMATICS FOR ENGINEERS

Formulary

Lecturer	:	Prof. Dr.-Ing. U. Schlichtmann
Assistant	:	Richard Petri richard.petri@tum.de
Chair's Location	:	TUM main area, Building 9 Arcisstr. 21, 80333 München ☎ +49 89 289-23666
Internet	:	http://ce.cit.tum.de/eda

Contents of lecture:

Propositional logic: propositional forms, truth-set, laws of propositional logic, rules of inference, binary decision diagrams;
Predicate logic: predicate logic forms, laws of predicate logic, deduction scheme, induction;
Sets: notation, operation, relations between sets, boolean algebra of subsets;
Relations: binary graphs, properties, closures, order relations, equivalence relations.

Contents

1 Propositional Logic

1.1 Definitions and Terms

Proposition (two-valued)	:	Statement (assertion) which can be assigned the truth value **t**(rue) or **f**(alse) but not both.
Truth values (true-, false-symbol)	:	$\{\,\mathbf{t},\mathbf{f}\,\}$ or $\{\,\mathbf{1},\mathbf{0}\,\}$ or $\{\,\mathbf{T},\mathbf{F}\,\}$, $\{\,\mathbf{tt},\mathbf{ff}\,\}$, $\{\,\mathbf{L},\mathbf{O}\,\}$
Logical operators (Logical connectives, junctors)	:	e.g. AND; OR; NOT; IT FOLLOWS THAT; IF AND ONLY IF; EITHER, OR; $\wedge$; $\vee$; $\neg$; $\longrightarrow$; $\longleftrightarrow$; $\not\longleftrightarrow$;
Propositional variable (atomic proposition, atom, positive literal)	:	$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\ldots$,$\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ Placeholders for propositions
Propositional form (formula)	:	e.g. $\mathbf{a} \wedge (\mathbf{a} \longrightarrow \mathbf{b})$, e.g. $\mathbf{a}$
n-ary propositional form	:	$\mathbf{A}\,(\,\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\mathrm{n}\,) \;\triangleq\; \mathbf{A}(\underline{\mathbf{x}})$
Truth value of the propositional form (evaluation, interpretation)	:	$\hat{\mathbf{a}} \in \{\,\mathbf{t},\mathbf{f}\,\}$, $\hat{\mathbf{A}} \in \{\,\mathbf{t},\mathbf{f}\,\}$ $\hat{\mathbf{a}} \;\triangleq\; \mathrm{val}\,(\mathbf{a}) \;\triangleq\; \delta\,(\mathbf{a})$ $\mathbf{A}\,(\,\mathbf{t},\, \mathbf{f},\, \ldots,\, \mathbf{t}\,) \;\triangleq\; \mathbf{A}(\hat{\underline{\mathbf{x}}}) \;\triangleq\; \hat{\mathbf{A}}$
Value assignment n-tuple	:	e.g. $\hat{\underline{\mathbf{x}}} \;=\; (\,\mathbf{t},\, \mathbf{f},\, \ldots,\, \mathbf{t}\,)$
Value pattern of the propositional form	:	$\hat{\underline{\mathbf{W}}}\,[\mathbf{A}] \;=\; \big(\mathbf{A}(\hat{\underline{\mathbf{x}}}_0),\, \ldots\,,\, \mathbf{A}(\hat{\underline{\mathbf{x}}}_{2^\mathrm{n}-1})\,\big)$ e.g. $\hat{\underline{\mathbf{x}}}_5 \;=\; (\,\mathbf{f},\, \ldots\,,\, \mathbf{f},\, \mathbf{t},\, \mathbf{f},\, \mathbf{t}\,)$
On-set (truth-set, solution set; opposite: off-set, falsity set)	:	$\mathbf{E}\,[\mathbf{A}] \;=\; \{\hat{\underline{\mathbf{x}}} \mid \hat{\mathbf{A}} \Longleftrightarrow \mathbf{t}\}_{\mathbf{G}}$ $\mathbf{E}\,[\mathbf{A}] \;\subseteq\; \mathbf{G} \;=\; \{\,\mathbf{t},\, \mathbf{f}\,\}^\mathrm{n}$

1.2 Logical Operators (Connectives)

Important Logical Operators (unary, binary)

Expression	Name	Parlance
$\neg\, \mathbf{a}$	Negation, NOT	it is not true, that **a**
$\mathbf{a} \wedge \mathbf{b}$	Conjunction, AND	**a** and **b**
$\mathbf{a} \vee \mathbf{b}$	Disjunction, OR	**a** or **b**
$\mathbf{a} \not\leftrightarrow \mathbf{b}$	Alternative, XOR	either **a** or **b**
$\mathbf{a} \longrightarrow \mathbf{b}$	Conditional, material implication, subjunction	if **a**, then **b**
$\mathbf{a} \longleftrightarrow \mathbf{b}$	Biconditional, material equivalence, bijunction	**a** if and only if **b**

Truth Table

Value assignment	truth values					
a **b**	$\neg\,\mathbf{a}$	$\mathbf{a} \wedge \mathbf{b}$	$\mathbf{a} \vee \mathbf{b}$	$\mathbf{a} \not\leftrightarrow \mathbf{b}$	$\mathbf{a} \longrightarrow \mathbf{b}$	$\mathbf{a} \longleftrightarrow \mathbf{b}$
t **t**	**f**	**t**	**t**	**f**	**t**	**t**
t **f**	**f**	**f**	**t**	**t**	**f**	**f**
f **t**	**t**	**f**	**t**	**t**	**t**	**f**
f **f**	**t**	**f**	**f**	**f**	**t**	**t**

Strength of operators: $\neg \quad \wedge\ \vee \quad \not\leftrightarrow \quad \longrightarrow\ \longleftrightarrow$

Binary Operators of propositional logic

($\neg\,\mathbf{a}$: unary, $\mathbf{t}$, $\mathbf{f}$: zero-ary)

a	**t**	**t**	**f**	**f**	Expression	Name	Parlance
b	**t**	**f**	**t**	**f**			
n= 2	**t**	**t**	**t**	**t**	$\mathbf{t}$	Tautology	always true
	t	**t**	**t**	**f**	$\mathbf{a} \vee \mathbf{b}$	Disjunction	**a** or **b**
2^n $\rightarrow$	**t**	**t**	**f**	**t**	$\mathbf{b} \longrightarrow \mathbf{a}$	Conditional	if **b** , then **a**
	t	**t**	**f**	**f**	$\mathbf{a}$	Projection	**a** , **b** arbitrary
$2^{(2^n)}$	**t**	**f**	**t**	**t**	$\mathbf{a} \longrightarrow \mathbf{b}$	Conditional	if **a** , then **b**
$\downarrow$	**t**	**f**	**t**	**f**	$\mathbf{b}$	Projection	**b** , **a** arbitrary
	t	**f**	**f**	**t**	$\mathbf{a} \longleftrightarrow \mathbf{b}$	Biconditional	**a** if and only if **b**
	t	**f**	**f**	**f**	$\mathbf{a} \wedge \mathbf{b}$	Conjunction	**a** and **b**
	f	**t**	**t**	**t**	$\neg\,(\mathbf{a} \wedge \mathbf{b})$	NAND	not **a** or not **b**
	f	**t**	**t**	**f**	$\mathbf{a} \not\longleftrightarrow \mathbf{b}$	Alternative	either **a** or **b**
	f	**t**	**f**	**t**	$\neg\,\mathbf{b}$	Negation	not **b** , **a** arbitrary
	f	**t**	**f**	**f**	$\mathbf{a} \wedge \neg\,\mathbf{b}$	Inhibition	**a** and not **b**
	f	**f**	**t**	**t**	$\neg\,\mathbf{a}$	Negation	not **a** , **b** arbitrary
	f	**f**	**t**	**f**	$\neg\,\mathbf{a} \wedge \mathbf{b}$	Inhibition	not **a** and **b**
	f	**f**	**f**	**t**	$\neg\,(\mathbf{a} \vee \mathbf{b})$	NOR	neither **a** nor **b**
	f	**f**	**f**	**f**	$\mathbf{f}$	Contradiction	always false

Transformation on basis $(\{\mathbf{t}, \mathbf{f}\};\ \wedge, \vee, \neg)$

$$\left.\begin{array}{lll} \mathbf{a} \longrightarrow \mathbf{b} & \Longleftrightarrow & \neg\mathbf{a} \vee \mathbf{b} \\ \mathbf{a} \longleftrightarrow \mathbf{b} & \Longleftrightarrow & (\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{b} \longrightarrow \mathbf{a}) \\ \mathbf{a} \not\longleftrightarrow \mathbf{b} & \Longleftrightarrow & \neg\,(\mathbf{a} \longleftrightarrow \mathbf{b}) \end{array}\right.$$

1.3 Propositional forms

Discrete function (n-ary) $\mathbf{A}(\underline{\mathbf{x}}) \;:\; \{\,\mathbf{t}\,,\,\mathbf{f}\,\}^{n} \longrightarrow \{\,\mathbf{t}\,,\,\mathbf{f}\,\}$

Variable-n-Tuple : $\underline{\mathbf{x}} = (\,\mathbf{x}_1\,,\,\ldots\,,\,\mathbf{x}_n\,)$

: $\mathrm{set}(\underline{\mathbf{x}}) = \{\,\mathbf{x}_1\,,\,\ldots\,,\,\mathbf{x}_n\,\}$

Value-assignment-n-tuple : $\hat{\underline{\mathbf{x}}} = (\,\hat{\mathbf{x}}_1\,,\,\ldots\,,\,\hat{\mathbf{x}}_n\,)$

Universal set of the 2^n value-assignment n-tuples : $\mathbf{G} = \{\,\mathbf{t}\,,\,\mathbf{f}\,\}^{n}$
(Basic set) $= \{\,\hat{\underline{\mathbf{x}}}_0\,,\,\ldots\,,\,\hat{\underline{\mathbf{x}}}_{2^n-1}\,\}$

Maxterm
(n-ary disjunction of literals) : $\mathbf{D}_{\hat{\underline{\mathbf{x}}}}(\underline{\mathbf{x}}) \;:\Longleftrightarrow\; \bigvee_{\mathbf{x}\in \mathrm{set}(\underline{\mathbf{x}})} \mathbf{x} \not\longleftrightarrow \hat{\mathbf{x}}$

$\mathbf{x} \not\longleftrightarrow \mathbf{f} \Longleftrightarrow \mathbf{x}\,;\;\; \mathbf{x} \not\longleftrightarrow \mathbf{t} \Longleftrightarrow \neg\mathbf{x}$

$$\begin{aligned}\overline{\mathbf{E}}\,[\mathbf{D}_{\hat{\underline{\mathbf{x}}}}(\underline{\mathbf{x}})] &= \{\,\hat{\underline{\mathbf{x}}}\,\}\\ \mathbf{E}\,[\mathbf{D}_{\hat{\underline{\mathbf{x}}}}(\underline{\mathbf{x}})] &= \mathbf{G}\setminus\{\,\hat{\underline{\mathbf{x}}}\,\}\end{aligned}\quad ;\; \mathbf{D}_{\hat{\underline{\mathbf{x}}}}(\,\hat{\underline{\mathbf{x}}}_i) \Longleftrightarrow \begin{cases}\mathbf{f}, & \text{for } \hat{\underline{\mathbf{x}}}_i = \hat{\underline{\mathbf{x}}}\\ \mathbf{t}, & \text{otherwise}\end{cases}\qquad i = 0,\ldots,2^n-1$$

Minterm
(n-ary conjunction of literals) : $\mathbf{C}_{\hat{\underline{\mathbf{x}}}}(\underline{\mathbf{x}}) \;:\Longleftrightarrow\; \bigwedge_{\mathbf{x}\in \mathrm{set}(\underline{\mathbf{x}})} \mathbf{x} \longleftrightarrow \hat{\mathbf{x}}$

$\mathbf{x} \longleftrightarrow \mathbf{t} \Longleftrightarrow \mathbf{x}\,;\;\; \mathbf{x} \longleftrightarrow \mathbf{f} \Longleftrightarrow \neg\mathbf{x}$

$$\begin{aligned}\mathbf{E}\,[\mathbf{C}_{\hat{\underline{\mathbf{x}}}}(\underline{\mathbf{x}})] &= \{\,\hat{\underline{\mathbf{x}}}\,\}\\ \overline{\mathbf{E}}\,[\mathbf{C}_{\hat{\underline{\mathbf{x}}}}(\underline{\mathbf{x}})] &= \mathbf{G}\setminus\{\,\hat{\underline{\mathbf{x}}}\,\}\end{aligned}\quad ;\; \mathbf{C}_{\hat{\underline{\mathbf{x}}}}(\,\hat{\underline{\mathbf{x}}}_i) \Longleftrightarrow \begin{cases}\mathbf{t}, & \text{for } \hat{\underline{\mathbf{x}}}_i = \hat{\underline{\mathbf{x}}}\\ \mathbf{f}, & \text{otherwise}\end{cases}\qquad i = 0,\ldots,2^n-1$$

Canonical conjunctive normal form CCNF : conjunction of all
(Unique description of a propositional form) maxterms

$$\mathrm{CCNF}\,[\mathbf{A}] \;:\Longleftrightarrow\; \bigwedge_{\hat{\underline{\mathbf{x}}}\in\overline{\mathbf{E}}[\mathbf{A}]}\;\bigvee_{\mathbf{x}\in\mathrm{set}(\underline{\mathbf{x}})} \mathbf{x} \not\longleftrightarrow \hat{\mathbf{x}} \Longleftrightarrow \mathbf{D}_1\wedge\mathbf{D}_2\wedge\ldots\wedge\mathbf{D}_m$$

$$m = |\,\overline{\mathbf{E}}\,[\mathbf{A}]\,|$$

Canonical disjunctive normal form CDNF : disjunction of all
(Unique description of a propositional form) minterms

$$\mathrm{CDNF}\,[\mathbf{A}] \;:\Longleftrightarrow\; \bigvee_{\hat{\underline{\mathbf{x}}}\in\mathbf{E}[\mathbf{A}]}\;\bigwedge_{\mathbf{x}\in\mathrm{set}(\underline{\mathbf{x}})} \mathbf{x} \longleftrightarrow \hat{\mathbf{x}} \Longleftrightarrow \mathbf{C}_1\vee\mathbf{C}_2\vee\ldots\vee\mathbf{C}_s$$

$$s = |\,\mathbf{E}\,[\mathbf{A}]\,|\,;\;\; m+s = 2^n$$

$\mathrm{CCNF}\,[\mathbf{A}] \Longleftrightarrow \neg\,\mathrm{CDNF}\,[\neg\mathbf{A}]$; $\mathrm{CDNF}\,[\mathbf{A}] \Longleftrightarrow \neg\,\mathrm{CCNF}\,[\neg\mathbf{A}]$

Conjunctive normal form CNF	:	conjunction of disjunctive terms
disjunctive term	:	disjunction of literals
positive / negative literal	:	e.g. $\mathbf{a}$ / $\neg\mathbf{a}$
disjunctive normal form DNF	:	disjunction of conjunctive terms
conjunctive term	:	conjunction of literals
Clause $\mathbf{C_D}$ or $\mathbf{C_C}$	:	set of literals of a disjunctive or conjunctive term
Set of clauses of CNF or DNF of $\mathbf{A}$	:	$\mathbf{CC}[\mathbf{A}] = \{\mathbf{C}_{D1}, \mathbf{C}_{D2}, \ldots\}$, $\mathbf{CD}[\mathbf{A}] = \{\mathbf{C}_{C1}, \mathbf{C}_{C2}, \ldots\}$

Example:

$$\begin{aligned}
\mathbf{A}(\mathbf{a},\mathbf{b},\mathbf{c}) &\Longleftrightarrow \mathrm{CCNF}[\mathbf{A}] \Longleftrightarrow (\mathbf{a}\vee\mathbf{b}\vee\mathbf{c})\wedge(\mathbf{a}\vee\mathbf{b}\vee\neg\mathbf{c})\wedge(\neg\mathbf{a}\vee\neg\mathbf{b}\vee\mathbf{c})\\
&\Longleftrightarrow \mathrm{CNF}[\mathbf{A}] \Longleftrightarrow (\mathbf{a}\vee\mathbf{b})\wedge(\neg\mathbf{a}\vee\neg\mathbf{b}\vee\mathbf{c})\\
&\Longleftrightarrow \mathrm{DNF}[\mathbf{A}] \Longleftrightarrow (\mathbf{a}\wedge\neg\mathbf{b})\vee(\neg\mathbf{a}\wedge\mathbf{b})\vee(\mathbf{a}\wedge\mathbf{c})\vee(\mathbf{b}\wedge\mathbf{c})\\
&\Longleftrightarrow \mathrm{CDNF}[\mathbf{A}] \Longleftrightarrow (\mathbf{a}\wedge\neg\mathbf{b}\wedge\mathbf{c})\vee(\mathbf{a}\wedge\neg\mathbf{b}\wedge\neg\mathbf{c})\vee(\neg\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\vee\\
&\qquad (\neg\mathbf{a}\wedge\mathbf{b}\wedge\neg\mathbf{c})\vee(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})
\end{aligned}$$

$\mathbf{CC}[\mathbf{A}] = \{\,\{\mathbf{a},\mathbf{b}\},\{\neg\mathbf{a},\neg\mathbf{b},\mathbf{c}\}\,\}$; $\mathbf{CD}[\mathbf{A}] = \{\,\{\mathbf{a},\neg\mathbf{b}\},\{\neg\mathbf{a},\mathbf{b}\},\{\mathbf{a},\mathbf{c}\},\{\mathbf{b},\mathbf{c}\}\}$
$\mathbf{CCC}[\mathbf{A}] = \{\,\{\mathbf{a},\mathbf{b},\mathbf{c}\},\{\mathbf{a},\mathbf{b},\neg\mathbf{c}\},\{\neg\mathbf{a},\neg\mathbf{b},\mathbf{c}\}\,\}$

$\mathbf{A}$ and $\mathbf{B}$ are disjoint (pairwise)	$:\Longleftrightarrow$	$\mathbf{E}[\mathbf{A}\wedge\mathbf{B}] = \emptyset$
$\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are complete	$:\Longleftrightarrow$	$\mathbf{E}[\mathbf{A}\vee\mathbf{B}\vee\mathbf{C}] = \mathbf{G}$

Dual form of $\mathbf{A}(\underline{\mathbf{x}})$	:	$\mathrm{dual}(\mathbf{A}(\underline{\mathbf{x}}))$
$\mathrm{dual}(\mathbf{A}(\mathbf{a},\mathbf{b},\ldots;\ \mathbf{t},\mathbf{f}))$	$:\Longleftrightarrow$	$\neg\mathbf{A}(\neg\mathbf{a},\neg\mathbf{b},\ldots;\ \mathbf{t},\mathbf{f})$

Notation: $\mathbf{A}(\hat{\underline{\mathbf{x}}}) \Longleftrightarrow \hat{\mathbf{A}} \Longleftrightarrow \mathbf{t}$; $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{t}$

Agreement: $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{t} \;\hat{=}\; \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{t}$

$\mathbf{t}$: zero-ary function which continuously yields $\mathbf{t}$

$\mathbf{f}$: zero-ary function which continuously yields $\mathbf{f}$

Satisfiable propositional form	: $\mathbf{E}[\mathbf{A}] \neq \emptyset$
$\underline{\hat{\mathbf{x}}}$ verifies $\mathbf{A}(\underline{\mathbf{x}})$	: $\mathbf{A}(\underline{\hat{\mathbf{x}}}) \Longleftrightarrow \mathbf{t}$
$\underline{\hat{\mathbf{x}}}$ falsifies $\mathbf{A}(\underline{\mathbf{x}})$	: $\mathbf{A}(\underline{\hat{\mathbf{x}}}) \Longleftrightarrow \mathbf{f}$
Contingency	: $\emptyset \subset \mathbf{E}[\mathbf{A}] \subset \mathbf{G}$
	$(\mathbf{E}[\mathbf{A}] \neq \emptyset) \wedge (\mathbf{E}[\mathbf{A}] \neq \mathbf{G})$
Contradiction	: $\mathbf{E}[\mathbf{A}] = \emptyset$
(Unsatisfiable propositional form)	$\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{f}$
e.g. $\mathbf{A}(\mathbf{a}) \Longleftrightarrow \mathbf{a} \wedge \neg\mathbf{a} \Longleftrightarrow \mathbf{f}$	(denial of contradiction)
Tautology	: $\mathbf{E}[\mathbf{A}] = \mathbf{G}$
	$\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{t}$
e.g. $\mathbf{A}(\mathbf{a}) \Longleftrightarrow \mathbf{a} \vee \neg\mathbf{a} \Longleftrightarrow \mathbf{t}$	(law / principle of the excluded middle)
(Logic) Equivalence	: $\mathbf{A} \Longleftrightarrow \mathbf{B}$
	$\mathbf{A} \longleftrightarrow \mathbf{B} \Longleftrightarrow \mathbf{t}$
"**A** is (logically) equivalent to **B**"	$\mathbf{E}[\mathbf{A} \longleftrightarrow \mathbf{B}] = \mathbf{G}$
"**A** is necessary and sufficient for **B**"	$\mathbf{E}[\mathbf{A}] = \mathbf{E}[\mathbf{B}]$
"**B** is necessary and sufficient for **A**"	
"Identical value patterns of **A** and **B**"	$\underline{\hat{\mathbf{W}}}[\mathbf{A}] = \underline{\hat{\mathbf{W}}}[\mathbf{B}]$
Implication	: $\mathbf{A} \Longrightarrow \mathbf{B}$
	$\mathbf{A} \longrightarrow \mathbf{B} \Longleftrightarrow \mathbf{t}$
"From **A** follows (logically) **B**"	$\mathbf{E}[\mathbf{A} \longrightarrow \mathbf{B}] = \mathbf{G}$
"**A** implies **B**"	
"**A** is sufficient for **B**"	$\mathbf{E}[\mathbf{A}] \subseteq \mathbf{E}[\mathbf{B}]$
"**B** is necessary for **A**"	$\overline{\mathbf{E}}[\mathbf{B}] \subseteq \overline{\mathbf{E}}[\mathbf{A}]$

1.4 Laws and Rules

A) Laws (Tautologies, Equivalences)

of **Boolean propositional algebra** $(\{\mathbf{t},\ \mathbf{f}\}\ ;\wedge,\vee,\neg)$

Principle of duality:

(1)	$\mathbf{a} \wedge \mathbf{b} \Longleftrightarrow \mathbf{b} \wedge \mathbf{a}\ ; \quad \mathbf{a} \vee \mathbf{b} \Longleftrightarrow \mathbf{b} \vee \mathbf{a}$	Commutativity
(2)	$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} \Longleftrightarrow \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ $(\mathbf{a} \vee \mathbf{b}) \vee \mathbf{c} \Longleftrightarrow \mathbf{a} \vee (\mathbf{b} \vee \mathbf{c})$	Associativity
(3)	$\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c}) \Longleftrightarrow (\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c})$ $\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) \Longleftrightarrow (\mathbf{a} \vee \mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{c})$	Distributivity
(4)	$\mathbf{a} \wedge \mathbf{a} \Longleftrightarrow \mathbf{a}\ ; \quad \mathbf{a} \vee \mathbf{a} \Longleftrightarrow \mathbf{a}$	Idempotence
(5)	$\mathbf{a} \wedge (\mathbf{a} \vee \mathbf{b}) \Longleftrightarrow \mathbf{a}$ $\mathbf{a} \vee (\mathbf{a} \wedge \mathbf{b}) \Longleftrightarrow \mathbf{a}$	Absorption
(6)	$\mathbf{a} \wedge \mathbf{t} \Longleftrightarrow \mathbf{a}\ ; \quad \mathbf{a} \vee \mathbf{f} \Longleftrightarrow \mathbf{a}$	Neutral element
(7)	$\mathbf{a} \wedge \mathbf{f} \Longleftrightarrow \mathbf{f}\ ; \quad \mathbf{a} \vee \mathbf{t} \Longleftrightarrow \mathbf{t}$	Domination
(8)	$\mathbf{a} \wedge \neg\,\mathbf{a} \Longleftrightarrow \mathbf{f}\ ; \quad \mathbf{a} \vee \neg\,\mathbf{a} \Longleftrightarrow \mathbf{t}$	Complementary element
(9)	$\neg\,(\neg\,\mathbf{a}) \Longleftrightarrow \mathbf{a}$	Double negation
(10)	$\neg\,(\mathbf{a} \wedge \mathbf{b}) \Longleftrightarrow \neg\,\mathbf{a} \vee \neg\,\mathbf{b}$ $\neg\,(\mathbf{a} \vee \mathbf{b}) \Longleftrightarrow \neg\,\mathbf{a} \wedge \neg\,\mathbf{b}$	De Morgan

B) Laws (Tautologies, Equivalences) of **Alternative** ($\nleftrightarrow$, $\oplus$)

(1)	$\mathbf{a} \nleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$\mathbf{b} \nleftrightarrow \mathbf{a}$	Commutativity
(2)	$(\mathbf{a} \nleftrightarrow \mathbf{b}) \nleftrightarrow \mathbf{c}$	$\Longleftrightarrow$	$\mathbf{a} \nleftrightarrow (\mathbf{b} \nleftrightarrow \mathbf{c})$	Associativity
(3)	$\mathbf{a} \wedge (\mathbf{b} \nleftrightarrow \mathbf{c})$	$\Longleftrightarrow$	$\mathbf{a} \wedge \mathbf{b} \nleftrightarrow \mathbf{a} \wedge \mathbf{c}$	Distributivity of $\wedge$ over $\nleftrightarrow$
(4)	$\mathbf{a} \nleftrightarrow \mathbf{f}$	$\Longleftrightarrow$	$\mathbf{a}$	Neutral element
(5)	$\mathbf{a} \nleftrightarrow \mathbf{t}$	$\Longleftrightarrow$	$\neg \mathbf{a}$	Negation
	$\neg(\mathbf{a} \nleftrightarrow \mathbf{b})$	$\Longleftrightarrow$	$\mathbf{a} \nleftrightarrow \mathbf{b} \nleftrightarrow \mathbf{t} \Longleftrightarrow \neg \mathbf{a} \nleftrightarrow \mathbf{b} \Longleftrightarrow \mathbf{a} \nleftrightarrow \neg \mathbf{b}$	
(6)	$\mathbf{a} \nleftrightarrow \neg \mathbf{a}$	$\Longleftrightarrow$	$\mathbf{t}$	Complementary element
	$\mathbf{a} \nleftrightarrow \mathbf{a}$	$\Longleftrightarrow$	$\mathbf{f}$	
(7)	$\mathbf{a} \nleftrightarrow \mathbf{a} \nleftrightarrow \mathbf{a}$	$\Longleftrightarrow$	$\mathbf{a}$	Idempotence
(8)	$\mathbf{a} \nleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$\neg \mathbf{a} \nleftrightarrow \neg \mathbf{b}$	Contraposition
(9)	$\mathbf{a} \nleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$(\mathbf{a} \wedge \neg \mathbf{b}) \vee (\neg \mathbf{a} \wedge \mathbf{b})$	
(10)	$\mathbf{a} \nleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$(\mathbf{a} \vee \mathbf{b}) \wedge (\neg \mathbf{a} \vee \neg \mathbf{b})$	
(11)	$\mathbf{a} \nleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$\neg(\mathbf{a} \longleftrightarrow \mathbf{b})$	
(12)	$\mathbf{a} \nleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$(\mathbf{a} \longrightarrow \mathbf{b}) \longrightarrow \neg(\mathbf{b} \longrightarrow \mathbf{a})$	
(13)	$\mathbf{a} \vee \mathbf{b}$	$\Longleftrightarrow$	$\mathbf{a} \nleftrightarrow \mathbf{b} \nleftrightarrow (\mathbf{a} \wedge \mathbf{b})$	
(14)	$\mathbf{a} \wedge \mathbf{b}$	$\Longleftrightarrow$	$\mathbf{a} \nleftrightarrow \mathbf{b} \nleftrightarrow (\mathbf{a} \vee \mathbf{b})$	

C) Laws (Tautologies, Equivalences) of **Biconditional** $(\longleftrightarrow , \ominus)$

(1)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$\mathbf{b} \longleftrightarrow \mathbf{a}$	Commutativity
(2)	$(\mathbf{a} \longleftrightarrow \mathbf{b}) \longleftrightarrow \mathbf{c}$	$\Longleftrightarrow$	$\mathbf{a} \longleftrightarrow (\mathbf{b} \longleftrightarrow \mathbf{c})$	Associativity
(3)	$\mathbf{a} \vee (\mathbf{b} \longleftrightarrow \mathbf{c})$	$\Longleftrightarrow$	$\mathbf{a} \vee \mathbf{b} \longleftrightarrow \mathbf{a} \vee \mathbf{c}$	Distributivity of $\vee$ over $\longleftrightarrow$
(4)	$\mathbf{a} \longleftrightarrow \mathbf{t}$	$\Longleftrightarrow$	$\mathbf{a}$	Neutral element
(5)	$\mathbf{a} \longleftrightarrow \mathbf{f}$	$\Longleftrightarrow$	$\neg \mathbf{a}$	Negation
	$\neg (\mathbf{a} \longleftrightarrow \mathbf{b})$	$\Longleftrightarrow$	$\mathbf{a} \longleftrightarrow \mathbf{b} \longleftrightarrow \mathbf{f} \Longleftrightarrow \neg \mathbf{a} \longleftrightarrow \mathbf{b} \Longleftrightarrow \mathbf{a} \longleftrightarrow \neg \mathbf{b}$	
(6)	$\mathbf{a} \longleftrightarrow \neg \mathbf{a}$	$\Longleftrightarrow$	$\mathbf{f}$	Complementary element
	$\mathbf{a} \longleftrightarrow \mathbf{a}$	$\Longleftrightarrow$	$\mathbf{t}$	
(7)	$\mathbf{a} \longleftrightarrow \mathbf{a} \longleftrightarrow \mathbf{a}$	$\Longleftrightarrow$	$\mathbf{a}$	Idempotence
(8)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$\neg \mathbf{a} \longleftrightarrow \neg \mathbf{b}$	Contraposition
(9)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$(\mathbf{a} \wedge \mathbf{b}) \vee (\neg \mathbf{a} \wedge \neg \mathbf{b})$	
(10)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$(\mathbf{a} \vee \neg \mathbf{b}) \wedge (\neg \mathbf{a} \vee \mathbf{b})$	
(11)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{b} \longrightarrow \mathbf{a})$	
(12)	$\mathbf{a} \longleftrightarrow \mathbf{b}$	$\Longleftrightarrow$	$\neg (\mathbf{a} \not\longleftrightarrow \mathbf{b})$	
(13)	$\mathbf{a} \vee \mathbf{b}$	$\Longleftrightarrow$	$\mathbf{a} \longleftrightarrow \mathbf{b} \longleftrightarrow (\mathbf{a} \wedge \mathbf{b})$	
(14)	$\mathbf{a} \wedge \mathbf{b}$	$\Longleftrightarrow$	$\mathbf{a} \longleftrightarrow \mathbf{b} \longleftrightarrow (\mathbf{a} \vee \mathbf{b})$	

D) Laws (Tautologies, Equivalences) of **Conditional**
(antecedent $\longrightarrow$ consequent)

(1) $\mathbf{a} \longrightarrow \mathbf{b} \iff \neg\,\mathbf{a} \vee \mathbf{b}$

(2) $\neg\,(\mathbf{a} \longrightarrow \mathbf{b}) \iff \mathbf{a} \wedge \neg\,\mathbf{b}$ Negation

(3) $\mathbf{a} \longrightarrow \mathbf{b} \iff \neg\,\mathbf{b} \longrightarrow \neg\,\mathbf{a}$ Contraposition

(4) $\mathbf{a} \longrightarrow \mathbf{t} \iff \mathbf{t}$; $\mathbf{f} \longrightarrow \mathbf{a} \iff \mathbf{t}$

(5) $\mathbf{t} \longrightarrow \mathbf{a} \iff \mathbf{a}$; $\mathbf{a} \longrightarrow \mathbf{f} \iff \neg\,\mathbf{a}$

(6) $\mathbf{a} \longrightarrow \mathbf{a} \iff \mathbf{t}$; $\neg\,\mathbf{a} \longrightarrow \mathbf{a} \iff \mathbf{a}$; $\mathbf{a} \longrightarrow \neg\,\mathbf{a} \iff \neg\,\mathbf{a}$

(7) $\mathbf{a} \longrightarrow (\mathbf{b} \wedge \mathbf{c}) \iff (\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{a} \longrightarrow \mathbf{c})$ left-sided distributivity of $\longrightarrow$ over $\wedge$

$(\mathbf{a} \wedge \mathbf{b}) \longrightarrow \mathbf{c} \iff (\mathbf{a} \longrightarrow \mathbf{c}) \vee (\mathbf{b} \longrightarrow \mathbf{c})$

(8) $\mathbf{a} \longrightarrow (\mathbf{b} \vee \mathbf{c}) \iff (\mathbf{a} \longrightarrow \mathbf{b}) \vee (\mathbf{a} \longrightarrow \mathbf{c})$ left-sided distributivity of $\longrightarrow$ over $\vee$

$(\mathbf{a} \vee \mathbf{b}) \longrightarrow \mathbf{c} \iff (\mathbf{a} \longrightarrow \mathbf{c}) \wedge (\mathbf{b} \longrightarrow \mathbf{c})$

(9) $\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \mathbf{c}) \iff \mathbf{b} \longrightarrow (\mathbf{a} \longrightarrow \mathbf{c})$ interchange of premises

$\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \mathbf{c}) \iff (\mathbf{a} \wedge \mathbf{b}) \longrightarrow \mathbf{c}$ importation and exportation

(10) $\mathbf{a} \longrightarrow (\mathbf{a} \longrightarrow \mathbf{b}) \iff \mathbf{a} \longrightarrow \mathbf{b}$ reinforcement rule

(11) $\mathbf{a} \longrightarrow \mathbf{b} \iff \neg\,(\mathbf{a} \not\longleftrightarrow (\mathbf{a} \wedge \mathbf{b}))$

(12) $\mathbf{a} \longrightarrow \mathbf{b} \iff \mathbf{a} \longleftrightarrow (\mathbf{a} \wedge \mathbf{b}) \iff \mathbf{b} \longleftrightarrow (\mathbf{a} \vee \mathbf{b})$

(13) $\mathbf{a} \vee \mathbf{b} \iff \neg\,\mathbf{a} \longrightarrow \mathbf{b} \iff (\mathbf{a} \longrightarrow \mathbf{b}) \longrightarrow \mathbf{b}$

(14) $\mathbf{a} \wedge \mathbf{b} \iff \neg\,(\mathbf{a} \longrightarrow \neg\,\mathbf{b})$

E) Laws (Tautologies) using **Implication** $(\Longrightarrow)$

(1)	$\mathbf{f}$	$\Longrightarrow$	$\mathbf{a}$	ex falso quodlibet
(2)	$\mathbf{a}$	$\Longrightarrow$	$\mathbf{t}$	ex quodlibet verum
(3)	$\mathbf{a}$	$\Longrightarrow$	$\mathbf{a}$	identity law
(4)	$\mathbf{a} \wedge \mathbf{b}$	$\Longrightarrow$	$\mathbf{a}$	simplification
(5)	$\mathbf{a}$	$\Longrightarrow$	$\mathbf{a} \vee \mathbf{b}$	addition
(6)	$\mathbf{a} \wedge \mathbf{b}$	$\Longrightarrow$	$\mathbf{a} \vee \mathbf{b}$	conjunction implies disjunction
(7)	$\neg\, \mathbf{a}$	$\Longrightarrow$	$\mathbf{a} \longrightarrow \mathbf{b}$	denial of the antecedent
(8)	$\mathbf{b}$	$\Longrightarrow$	$\mathbf{a} \longrightarrow \mathbf{b}$	affirmation of the consequent
(9)	$\mathbf{a} \wedge (\mathbf{a} \longrightarrow \mathbf{b})$	$\Longrightarrow$	$\mathbf{b}$	modus ponens
	$\mathbf{a} \wedge (\mathbf{a} \vee \mathbf{b} \longrightarrow \mathbf{c})$	$\Longrightarrow$	$\mathbf{c}$	
(10)	$\neg\, \mathbf{b} \wedge (\mathbf{a} \longrightarrow \mathbf{b})$	$\Longrightarrow$	$\neg\, \mathbf{a}$	modus tollens
(11)	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{b} \longrightarrow \mathbf{c})$	$\Longrightarrow$	$\mathbf{a} \longrightarrow \mathbf{c}$	hypothetical syllogism (modus barbara)
	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{b} \vee \mathbf{c} \longrightarrow \mathbf{d})$	$\Longrightarrow$	$\mathbf{a} \longrightarrow \mathbf{d}$	
(12)	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\neg\mathbf{a} \longrightarrow \mathbf{b})$	$\Longrightarrow$	$\mathbf{b}$	constructive dilemma
	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\neg\mathbf{a} \longrightarrow \mathbf{b})$	$\Longleftrightarrow$	$\mathbf{b}$	
(13)	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{a} \longrightarrow \neg\mathbf{b})$	$\Longrightarrow$	$\neg\, \mathbf{a}$	destructive dilemma
	$(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{a} \longrightarrow \neg\mathbf{b})$	$\Longleftrightarrow$	$\neg\, \mathbf{a}$	(reductio ad absurdum)

Composition for conjunction:

(14) $(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{c} \longrightarrow \mathbf{d}) \Longrightarrow (\mathbf{a} \wedge \mathbf{c}) \longrightarrow (\mathbf{b} \wedge \mathbf{d})$

$(\mathbf{a} \longrightarrow \mathbf{b}) \Longrightarrow (\mathbf{a} \wedge \mathbf{c}) \longrightarrow (\mathbf{b} \wedge \mathbf{c}), \qquad \text{for } \mathbf{c} \Longleftrightarrow \mathbf{d}$

Composition for disjunction:

(15) $(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{c} \longrightarrow \mathbf{d}) \Longrightarrow (\mathbf{a} \vee \mathbf{c}) \longrightarrow (\mathbf{b} \vee \mathbf{d})$

$(\mathbf{a} \longrightarrow \mathbf{b}) \Longrightarrow (\mathbf{a} \vee \mathbf{c}) \longrightarrow (\mathbf{b} \vee \mathbf{c}), \qquad \text{for } \mathbf{c} \Longleftrightarrow \mathbf{d}$

Case differentiation:

(16) $(\mathbf{a} \vee \mathbf{b}) \wedge [\, (\mathbf{a} \longrightarrow \mathbf{c}) \wedge (\mathbf{b} \longrightarrow \mathbf{c}) \,] \Longrightarrow \mathbf{c}$

(17) $(\mathbf{a} \vee \mathbf{b}) \wedge [\, (\mathbf{a} \longrightarrow \mathbf{c}) \wedge (\mathbf{b} \longrightarrow \mathbf{d}) \,] \Longrightarrow \mathbf{c} \vee \mathbf{d}$

(18) $(\mathbf{a} \wedge \mathbf{b}) \wedge [\, (\mathbf{a} \longrightarrow \mathbf{c}) \wedge (\mathbf{b} \longrightarrow \mathbf{d}) \,] \Longrightarrow \mathbf{c} \wedge \mathbf{d}$

Frege's chain inference:

(19) $(\mathbf{a} \longrightarrow \mathbf{b}) \longrightarrow \mathbf{c} \Longrightarrow \mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \mathbf{c}) \Longleftrightarrow (\mathbf{a} \longrightarrow \mathbf{b}) \longrightarrow (\mathbf{a} \longrightarrow \mathbf{c})$

(20) $\mathbf{a} \longrightarrow \mathbf{b} \Longrightarrow (\mathbf{b} \longrightarrow \mathbf{c}) \longrightarrow (\mathbf{a} \longrightarrow \mathbf{c})$

$\mathbf{a} \longrightarrow \mathbf{b} \Longrightarrow (\mathbf{c} \longrightarrow \mathbf{a}) \longrightarrow (\mathbf{c} \longrightarrow \mathbf{b})$

Intersection of two conditionals:

(21) $(\mathbf{a} \longrightarrow \mathbf{b}) \wedge (\mathbf{c} \longrightarrow \mathbf{d}) \Longrightarrow \mathbf{a} \longrightarrow ((\mathbf{b} \longrightarrow \mathbf{c}) \longrightarrow \mathbf{d})$

RT1) Replacement theorem for tautologies

Let $\mathbf{B}(\underline{\mathbf{z}})$ be an arbitrary propositional form and $\mathbf{x}_\mathrm{i} \in \mathrm{set}(\underline{\mathbf{x}})$ of $\mathbf{A}(\underline{\mathbf{x}})$.

Then:

IF $\mathbf{A}(\ldots,\mathbf{x}_\mathrm{i},\ldots) \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{A}(\ldots,\mathbf{B}(\underline{\mathbf{z}}),\ldots) \Longleftrightarrow \mathbf{t}$

e.g. IF $\mathbf{a} \wedge (\mathbf{a} \longrightarrow \mathbf{b}) \Longrightarrow \mathbf{b}$, THEN $\mathbf{A}(\underline{\mathbf{x}}) \wedge (\mathbf{A}(\underline{\mathbf{x}}) \longrightarrow \mathbf{B}(\underline{\mathbf{x}})) \Longrightarrow \mathbf{B}(\underline{\mathbf{x}})$

e.g. IF $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{B}(\underline{\mathbf{x}})$, THEN $\mathrm{dual}(\mathbf{A}(\underline{\mathbf{x}})) \Longleftrightarrow \mathrm{dual}(\mathbf{B}(\underline{\mathbf{x}}))$

e.g. IF $\mathbf{A}(\underline{\mathbf{x}}) \Longrightarrow \mathbf{B}(\underline{\mathbf{x}})$, THEN $\mathrm{dual}(\mathbf{B}(\underline{\mathbf{x}})) \Longrightarrow \mathrm{dual}(\mathbf{A}(\underline{\mathbf{x}}))$

RT2) Replacement theorem for equivalent partial propositional forms

Let $\mathbf{A}(\underline{\mathbf{x}},\mathbf{B}(\underline{\mathbf{z}}))$ be an arbitrary propositional form and $\mathbf{B}$ a partial form of $\mathbf{A}$, with $\mathrm{set}(\underline{\mathbf{z}}) \subseteq \mathrm{set}(\underline{\mathbf{x}})$. Then:

IF $\mathbf{B}(\underline{\mathbf{z}}) \Longleftrightarrow \mathbf{C}(\underline{\mathbf{z}})$, THEN $\mathbf{A}(\underline{\mathbf{x}},\mathbf{B}(\underline{\mathbf{z}})) \Longleftrightarrow \mathbf{A}(\underline{\mathbf{x}},\mathbf{C}(\underline{\mathbf{z}}))$

e.g. IF $\mathbf{x} \longrightarrow \mathbf{a} \Longleftrightarrow \neg\mathbf{x} \vee \mathbf{a}$,

THEN $(\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \Longleftrightarrow (\neg\mathbf{x} \vee \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b})$

SR) Substitution rule

$\mathbf{x}_\mathrm{i} \wedge \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{x}_\mathrm{i} \wedge \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t})$; $\quad \neg\mathbf{x}_\mathrm{i} \wedge \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \neg\mathbf{x}_\mathrm{i} \wedge \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f})$

$\mathbf{x}_\mathrm{i} \vee \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{x}_\mathrm{i} \vee \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f})$; $\quad \neg\mathbf{x}_\mathrm{i} \vee \mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \neg\mathbf{x}_\mathrm{i} \vee \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t})$

ER) Expansion rules for propositional logic functions

Let $\mathbf{A}(\underline{\mathbf{x}})$ be a propositional logic function with $\mathbf{x}_\mathrm{i} \in \mathrm{set}(\underline{\mathbf{x}})$.

(1) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow [\mathbf{x}_\mathrm{i} \wedge \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t})] \vee [\neg\mathbf{x}_\mathrm{i} \wedge \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f})]$ (Shannon)

(Boole's fundamental theorem)

(2) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow [\neg\mathbf{x}_\mathrm{i} \vee \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t})] \wedge [\mathbf{x}_\mathrm{i} \vee \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f})]$

(3) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow [\mathbf{x}_\mathrm{i} \longrightarrow \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t})] \wedge [\neg\mathbf{x}_\mathrm{i} \longrightarrow \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f})]$

(4) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow [\mathbf{x}_\mathrm{i} \wedge (\mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t}) \longleftrightarrow\!\!\!\!\!\!/\;\; \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f}))] \longleftrightarrow\!\!\!\!\!\!/\;\; \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f})$ (Davio)

RS) Rule of specialization

$$\mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t}) \wedge \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f}) \Longrightarrow \underset{\mathbf{x}_\mathrm{i} \in \mathrm{set}(\underline{\mathbf{x}})}{\mathbf{A}(\underline{\mathbf{x}})} \Longrightarrow \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t}) \vee \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f})$$

RRD) Resolution rule in disjunctive form

$$(\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{b}) \Longleftrightarrow (\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{b}) \vee \underbrace{(\mathbf{a} \wedge \mathbf{b})}_{\text{resolvent}}$$

$(\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{a}) \Longleftrightarrow \mathbf{a}$ (special case)

$(\mathbf{a} \wedge \mathbf{b}) \Longrightarrow (\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{b})$ (from RRD)

RRC) Resolution rule in conjunctive form

$$(\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \Longleftrightarrow (\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \wedge \underbrace{(\mathbf{a} \vee \mathbf{b})}_{\text{resolvent}}$$

$(\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{a}) \Longleftrightarrow \mathbf{a}$ (special case)

$(\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \Longrightarrow \mathbf{a} \vee \mathbf{b}$ (from RRC)

RRP) Resolution rule in conditional clause form (premise form)

$$(\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \Longleftrightarrow (\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \wedge \underbrace{(\neg\mathbf{a} \longrightarrow \mathbf{b})}_{\text{resolvent}}$$

$(\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{a}) \Longleftrightarrow \mathbf{a}$ (special case)

$(\neg\mathbf{a} \longrightarrow \neg\mathbf{x}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \Longrightarrow (\neg\mathbf{a} \longrightarrow \mathbf{b})$ (from RRP)

FRI) Fundamental rule for implication

Let $\mathbf{A} \Longrightarrow \mathbf{B}$. Then:

IF $\mathbf{A} \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{B} \Longleftrightarrow \mathbf{t}$

FRE) Fundamental rule for equivalence

Let $\mathbf{A} \Longleftrightarrow \mathbf{B}$. Then:

$\mathbf{A} \Longleftrightarrow \mathbf{t}$ IF AND ONLY IF $\mathbf{B} \Longleftrightarrow \mathbf{t}$

FRC) Fundamental rule for tautological conjunction

$\mathbf{A} \wedge \mathbf{B} \Longleftrightarrow \mathbf{t}$ IF AND ONLY IF $\mathbf{A} \Longleftrightarrow \mathbf{t}$ AND $\mathbf{B} \Longleftrightarrow \mathbf{t}$

RI) Rules of inference (meta rules) for tautologies

Equivalence and implication

(1) $\mathbf{A} \Longleftrightarrow \mathbf{B}$ IF AND ONLY IF $\mathbf{A} \Longrightarrow \mathbf{B}$ AND $\mathbf{B} \Longrightarrow \mathbf{A}$ C11

Contraposition

(2) $\mathbf{A} \Longrightarrow \mathbf{B}$ IF AND ONLY IF $\neg\mathbf{B} \Longrightarrow \neg\mathbf{A}$ D3

Enhancement rule

(3) $\mathbf{A} \Longrightarrow \mathbf{B}$ IF AND ONLY IF $\mathbf{A} \Longleftrightarrow \mathbf{A} \wedge \mathbf{B}$ D12

$\mathbf{A} \Longrightarrow \mathbf{B}$ IF AND ONLY IF $\mathbf{B} \Longleftrightarrow \mathbf{A} \vee \mathbf{B}$ D12

Expansion rule

(4) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{t}$ IF AND ONLY IF $\mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t}) \Longleftrightarrow \mathbf{t}$ AND $\mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f}) \Longleftrightarrow \mathbf{t}$

$$\mathbf{x}_\mathrm{i} \in \mathrm{set}(\underline{\mathbf{x}})$$

Affirmation of the consequent

(5) IF $\mathbf{B} \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{A} \Longrightarrow \mathbf{B}$ E8

Modus ponens

(6) IF $\mathbf{A} \Longleftrightarrow \mathbf{t}$ AND $\mathbf{A} \Longrightarrow \mathbf{B}$, THEN $\mathbf{B} \Longleftrightarrow \mathbf{t}$ E9

Transitivity

(7) IF $\mathbf{A} \Longrightarrow \mathbf{B}$, AND $\mathbf{B} \Longrightarrow \mathbf{C}$, THEN $\mathbf{A} \Longrightarrow \mathbf{C}$ E11

IF $\mathbf{A} \Longleftrightarrow \mathbf{B}$, AND $\mathbf{B} \Longleftrightarrow \mathbf{C}$, THEN $\mathbf{A} \Longleftrightarrow \mathbf{C}$

Compatibility

(8) IF $\mathbf{A} \Longrightarrow \mathbf{B}$, THEN $\mathbf{A} \wedge \mathbf{C} \Longrightarrow \mathbf{B} \wedge \mathbf{C}$ E14

IF $\mathbf{A} \Longrightarrow \mathbf{B}$, THEN $\mathbf{A} \vee \mathbf{C} \Longrightarrow \mathbf{B} \vee \mathbf{C}$ E15

IF $\mathbf{A} \Longleftrightarrow \mathbf{B}$, THEN $\mathbf{A} \wedge \mathbf{C} \Longleftrightarrow \mathbf{B} \wedge \mathbf{C}$

IF $\mathbf{A} \Longleftrightarrow \mathbf{B}$, THEN $\mathbf{A} \vee \mathbf{C} \Longleftrightarrow \mathbf{B} \vee \mathbf{C}$

1.5 Propositional Logic Rules of Inference

Conclusions which are always true (e.g. E1 to E21):

$$\begin{array}{rcl} \mathbf{V} & \Longrightarrow & \mathbf{S} \\ \mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \wedge \ \ldots & \Longrightarrow & \mathbf{S} \\ \text{Premises} & & \text{conclusion} \\ \text{hypotheses, antecedents} & & \text{consequence} \end{array}$$

Let $\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \Longrightarrow \mathbf{S}$ be a valid conclusion. Then:
IF $\hat{\mathbf{V}}_1 \wedge \hat{\mathbf{V}}_2 \wedge \hat{\mathbf{V}}_3 \Longleftrightarrow \mathbf{t}$, THEN $\hat{\mathbf{S}} \Longleftrightarrow \mathbf{t}$

Especially:

IF $\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{S} \Longleftrightarrow \mathbf{t}$

Logically equivalent statements:

(1)	$\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \longrightarrow \mathbf{S}$	$\Longleftrightarrow$	$\mathbf{t}$,	tautology
(2)	$\neg\mathbf{V}_1 \vee \neg\mathbf{V}_2 \vee \neg\mathbf{V}_3 \vee \mathbf{S}$	$\Longleftrightarrow$	$\mathbf{t}$,	D1
(3)	$\neg\mathbf{S} \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \longrightarrow \neg\mathbf{V}_1$	$\Longleftrightarrow$	$\mathbf{t}$,	indirect conclusion
(4)	$\neg\mathbf{S} \longrightarrow \neg(\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3)$	$\Longleftrightarrow$	$\mathbf{t}$	
(5)	$\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \wedge \neg\mathbf{S}$	$\Longleftrightarrow$	$\mathbf{f}$,	contradiction
(6)	$(\mathbf{V}_1 \longrightarrow \mathbf{S}) \vee (\mathbf{V}_2 \longrightarrow \mathbf{S}) \vee (\mathbf{V}_3 \longrightarrow \mathbf{S})$	$\Longleftrightarrow$	$\mathbf{t}$	
(7)	$\neg(\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \wedge \neg\mathbf{S})$	$\Longleftrightarrow$	$\mathbf{t}$	
(8)	$\mathbf{E}[\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3]$	$\subseteq$	$\mathbf{E}[\mathbf{S}]$	
(9)	$\mathbf{E}[\mathbf{V}_1] \cap \mathbf{E}[\mathbf{V}_2] \cap \mathbf{E}[\mathbf{V}_3]$	$\subseteq$	$\mathbf{E}[\mathbf{S}]$	

Trivial conclusions:

IF $\mathbf{V} \Longleftrightarrow \mathbf{f}$, THEN $\mathbf{S}$ ($\underline{\hat{\mathbf{W}}}[\mathbf{S}]$) arbitrary
IF $\mathbf{S} \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{V}$ ($\underline{\hat{\mathbf{W}}}[\mathbf{V}]$) arbitrary
IF $\mathbf{S} \Longleftrightarrow \mathbf{f}$, THEN $\mathbf{V} \Longleftrightarrow \mathbf{f}$; IF $\mathbf{V} \Longleftrightarrow \mathbf{t}$, THEN $\mathbf{S} \Longleftrightarrow \mathbf{t}$

Equivalent statements in propositional logic and first-order predicate logic:

(1)	$\mathbf{V} \longrightarrow \mathbf{S}$ is always true	(universality, semantic term)
(2)	$\mathbf{S}$ follows from $\mathbf{V}$	(conclusion, semantic term)
(3)	$\mathbf{S}$ can be derived from $\mathbf{V}$	(derivation, syntactic term)

Possible conclusions:

$$\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \wedge \ \ldots \quad \Longrightarrow \quad \mathbf{S}_\lambda \ ; \qquad \lambda = 1, \ldots, \mathrm{l}$$

$$\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \wedge \ \ldots \quad \Longleftrightarrow \quad \underbrace{\mathbf{D}_1 \wedge \ \ldots \wedge \mathbf{D}_\mu \wedge \ \ldots \wedge \mathbf{D}_\mathrm{m}}_{\text{CCNF}}$$

$$\{\mathbf{S}_1, \ldots, \mathbf{S}_\lambda, \ldots, \mathbf{S}_\mathrm{l}\} = \{\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_1 \wedge \mathbf{D}_2, \mathbf{D}_1 \wedge \mathbf{D}_3, \ldots \ \ \ldots, \mathbf{D}_1 \wedge \mathbf{D}_2 \wedge \mathbf{D}_3, \ldots\}$$

number of possible non-trivial conclusions: $\mathrm{l} = \sum_{\mathrm{i}=1}^{\mathrm{m}} \binom{\mathrm{m}}{\mathrm{i}} = 2^\mathrm{m} - 1$

Deduction scheme:

(Proof scheme)

1) $\mathbf{V}_1$
2) $\mathbf{V}_2 \Longrightarrow \mathbf{S}$
3) $\mathbf{U}_3$
$\vdots \quad \vdots$
ν) $\mathbf{U}_\nu$
$\vdots \quad \vdots$
n) $\mathbf{S}$

$\downarrow$ Proof

(Logic analysis)

Antecedents: $\hat{\mathbf{V}}_1 \Longleftrightarrow \mathbf{t}$
$\hat{\mathbf{V}}_2 \Longleftrightarrow \mathbf{t}$
Assertion: $\hat{\mathbf{S}} \Longleftrightarrow \mathbf{t}$

Proof: $\mathbf{V}_1 \wedge \mathbf{V}_2 \longrightarrow \mathbf{S} \Longleftrightarrow \mathbf{t}$

$\mathbf{V}_2 \Longleftrightarrow \mathbf{V}_2 \wedge \mathbf{U}_\nu, \quad \text{IF} \quad \mathbf{V}_2 \Longrightarrow \mathbf{U}_\nu$ Enhancement rule
$(\mathbf{V}_2 \Longleftrightarrow \mathbf{V}_2 \wedge \mathbf{U}_\nu, \quad \text{IF} \quad \mathbf{V}_2 \Longleftrightarrow \mathbf{U}_\nu)$

$$\mathbf{E}[\mathbf{V}_2] = \mathbf{E}[\mathbf{V}_2] \cap \mathbf{E}[\mathbf{U}_\nu], \quad \text{IF} \quad \mathbf{E}[\mathbf{V}_2] \subseteq \mathbf{E}[\mathbf{U}_\nu]$$
$$(\mathbf{E}[\mathbf{V}_2] = \mathbf{E}[\mathbf{V}_2] \cap \mathbf{E}[\mathbf{U}_\nu], \quad \text{IF} \quad \mathbf{E}[\mathbf{V}_2] = \mathbf{E}[\mathbf{U}_\nu])$$

Example:

$\mathbf{a}$	:	The worker monitors the machine.
$\mathbf{b}$	:	He notices that the machine does not work properly.
$\neg\,\mathbf{c}$	:	He turns the machine off.

$\mathbf{a}$	:	Kevin loves Suzy.
$\mathbf{b}$	:	Kevin dates Suzy.
$\mathbf{c}$	:	Kevin goes dancing with Linda.

$$[\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})] \wedge \mathbf{a} \wedge \mathbf{c} \Longrightarrow \neg\mathbf{b}$$
$$[\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})] \wedge \mathbf{a} \wedge \mathbf{b} \Longrightarrow \neg\mathbf{c}$$
$$[\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})] \wedge \mathbf{b} \wedge \mathbf{c} \Longrightarrow \neg\mathbf{a}$$

Proof scheme:

				(indirect)	
1)	$\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})$		1)	$\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})$	
2)	$\mathbf{a} \wedge \mathbf{c}$	$\Longrightarrow \neg\mathbf{b}$	2)	$\mathbf{a} \wedge \mathbf{b}$	$\Longrightarrow \neg\mathbf{c}$
3)	$\neg\mathbf{a} \vee \neg\mathbf{b} \vee \neg\mathbf{c}$	1) D1	3)	$\neg\mathbf{a} \vee \neg\mathbf{b} \vee \neg\mathbf{c}$	
4)	$\mathbf{a} \wedge \mathbf{c} \longrightarrow \neg\mathbf{b}$	3) D1	4)	$\mathbf{a} \wedge \mathbf{b} \longrightarrow \neg\mathbf{c}$	
5)	$\neg\mathbf{b}$	2) 4) E9	5)	$\neg\mathbf{c}$	

Necessary and sufficient conditions for a fact (example):

Fact		Condition	Property of conditions
white horse	$\Longrightarrow$	horse	necessary
white horse	$\Longleftarrow$	white $\wedge$ ferocious $\wedge$ horse	sufficient
white horse	$\Longleftrightarrow$	white $\wedge$ horse	necessary and sufficient

deduction scheme 1

1)	$\mathbf{A} \quad \Longrightarrow \mathbf{Z}$	
$\vdots$	$\vdots$	
$\nu)$	$\wedge\, \mathbf{B}$	
$\vdots$	$\vdots$	comment
$n)$	$\wedge\, \mathbf{Z}$	

$\mathbf{A} \Longleftrightarrow \mathbf{A} \wedge \mathbf{B}$
IF AND ONLY IF
$\mathbf{A} \Longrightarrow \mathbf{B}$

special case: $\mathbf{Z} \Longleftrightarrow \mathbf{f}$, $\mathbf{A} \Longleftrightarrow \mathbf{f}$
e.g. $\mathbf{A} \Longleftrightarrow \mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 \wedge \neg\mathbf{S}$

deduction scheme 2

1)	$\mathbf{A} \quad \Longleftarrow \mathbf{Z}$	
$\vdots$	$\vdots$	
$\nu\,)$	$\vee\, \mathbf{B}$	
$\vdots$	$\vdots$	comment
$n)$	$\vee\, \mathbf{Z}$	

$\mathbf{A} \Longleftrightarrow \mathbf{A} \vee \mathbf{B}$
IF AND ONLY IF
$\mathbf{A} \Longleftarrow \mathbf{B}$

special case: $\mathbf{Z} \Longleftrightarrow \mathbf{t}$, $\mathbf{A} \Longleftrightarrow \mathbf{t}$
e.g. $\mathbf{A} \Longleftrightarrow \neg\mathbf{V}_1 \vee \neg\mathbf{V}_2 \vee \neg\mathbf{V}_3 \vee \mathbf{S}$

Resolution method (layer algorithm):

Decidability algorithm for

$\mathrm{CNF}[\mathbf{A}] \Longleftrightarrow \mathbf{f}$ or $\mathrm{DNF}[\mathbf{A}] \Longleftrightarrow \mathbf{t}$

deduction scheme 1

CNF[**A**]	layer
$\ldots\wedge(\neg\mathbf{x}\vee\mathbf{a})\wedge(\mathbf{x}\vee\mathbf{b})\wedge\ldots$	0
$\ldots \wedge (\mathbf{a} \vee \mathbf{b}) \wedge \ldots$	1
$\vdots$	$\vdots$
$\wedge\, \mathbf{f}$	n

$(\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \Longrightarrow (\mathbf{a} \vee \mathbf{b})$
RRC

deduction scheme 2

DNF[**A**]	layer
$\ldots\vee(\mathbf{x}\wedge\mathbf{a})\vee(\neg\mathbf{x}\wedge\mathbf{b})\vee\ldots$	0
$\ldots \vee (\mathbf{a} \wedge \mathbf{b}) \vee \ldots$	1
$\vdots$	$\vdots$
$\vee\, \mathbf{t}$	n

$(\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{b}) \Longleftarrow (\mathbf{a} \wedge \mathbf{b})$
RRD

Example: $[\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})] \wedge \mathbf{a} \wedge \mathbf{b} \Longrightarrow \neg\mathbf{c}$

IF AND ONLY IF	$[\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})] \wedge \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$	$\Longleftrightarrow$	$\mathbf{f}$
IF AND ONLY IF	$(\neg\mathbf{a} \vee \neg\mathbf{b} \vee \neg\mathbf{c}) \wedge \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$	$\Longleftrightarrow$	$\mathbf{f}$
IF AND ONLY IF	$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \vee \neg\mathbf{a} \vee \neg\mathbf{b} \vee \neg\mathbf{c}$	$\Longleftrightarrow$	$\mathbf{t}$

1)	$\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})$	
2)	$\mathbf{a} \wedge \mathbf{c}$	
3)	$\mathbf{b}$	$\Longrightarrow \mathbf{f}$
4)	$\neg\mathbf{a} \vee \neg\mathbf{b} \vee \neg\mathbf{c}$	1) D1
5)	$\mathbf{a} \wedge \mathbf{c} \longrightarrow \neg\mathbf{b}$	4) D1
6)	$\neg\mathbf{b}$	2) 5) E9
7)	$\mathbf{f}$	3) 6) A8

1)	$\neg[\mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \neg\mathbf{c})]$	
2)	$\neg\mathbf{a} \vee \neg\mathbf{c}$	
3)	$\neg\mathbf{b}$	$\Longleftarrow \mathbf{t}$
4)	$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$	1) D1, A10
5)	$\mathbf{a} \wedge \neg\mathbf{b} \wedge \mathbf{c}$	3) E4
6)	$\mathbf{a} \wedge \mathbf{c}$	4) 5) A3
7)	$\mathbf{t}$	2) 6) A8

$(\neg\mathbf{a}\vee\neg\mathbf{b}\vee\neg\mathbf{c})\wedge\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$	0
$(\neg\mathbf{b}\vee\neg\mathbf{c})\wedge(\neg\mathbf{a}\vee\neg\mathbf{c})\wedge(\neg\mathbf{a}\vee\neg\mathbf{b})$	1
$\neg\mathbf{c} \wedge \neg\mathbf{b} \wedge \neg\mathbf{a}$	2
$\mathbf{f}$	3

$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\vee\neg\mathbf{a}\vee\neg\mathbf{b}\vee\neg\mathbf{c}$	0
$(\mathbf{b} \wedge \mathbf{c}) \vee (\mathbf{a} \wedge \mathbf{c}) \vee (\mathbf{a} \wedge \mathbf{b})$	1
$\mathbf{c} \vee \mathbf{b} \vee \mathbf{a}$	2
$\mathbf{t}$	3

Example: $(\mathbf{a} \longrightarrow \mathbf{b}) \longrightarrow \mathbf{c} \Longrightarrow \mathbf{a} \longrightarrow (\mathbf{b} \longrightarrow \mathbf{c})$

IF AND ONLY IF	$(\neg\mathbf{a} \wedge \neg\mathbf{c}) \vee (\mathbf{b} \wedge \neg\mathbf{c}) \vee \neg\mathbf{a} \vee \neg\mathbf{b} \vee \mathbf{c}$	$\Longleftrightarrow$	$\mathbf{t}$
IF AND ONLY IF	$(\mathbf{a} \vee \mathbf{c}) \wedge (\neg\mathbf{b} \vee \mathbf{c}) \wedge \mathbf{a} \wedge \mathbf{b} \wedge \neg\mathbf{c}$	$\Longleftrightarrow$	$\mathbf{f}$

$(\mathbf{a}\vee\mathbf{c})\wedge(\neg\mathbf{b}\vee\mathbf{c})\wedge\mathbf{a}\wedge\mathbf{b}\wedge\neg\mathbf{c}$	0
$\mathbf{a} \wedge \mathbf{c} \wedge \neg\mathbf{b}$	1
$\mathbf{f}$	2

$(\neg\mathbf{a}\wedge\neg\mathbf{c})\vee(\mathbf{b}\wedge\neg\mathbf{c})\vee\neg\mathbf{a}\vee\neg\mathbf{b}\vee\mathbf{c}$	0
$\neg\mathbf{a} \vee \neg\mathbf{c} \vee \mathbf{b}$	1
$\mathbf{t}$	2

1.6 Representation of Propositional Forms

Bases :

Every propositional logic expression can be represented using the following bases (algebras).

(1)	$(\{\mathbf{t},\mathbf{f}\};\ \wedge,\vee,\neg)$	Boolean algebra
(2)	$(\{\mathbf{t},\mathbf{f}\};\ \wedge,\vee,\rightarrow,\leftrightarrow,\ \neg)$	Hilbert-Bernays-basis
(3)	$(\{\mathbf{t},\mathbf{f}\};\ \wedge,\neg)$	De Morgan-bases
(4)	$(\{\mathbf{t},\mathbf{f}\};\ \vee,\neg)$	De Morgan-bases
(5)	$(\{\mathbf{t},\mathbf{f}\};\ \wedge,\ \leftarrow\!\!\!+\!\!\!\rightarrow)$	Žegalkin algebra
(6)	$(\{\mathbf{t},\mathbf{f}\};\ \vee,\leftrightarrow)$	
(7)	$(\{\mathbf{t},\mathbf{f}\};\ \rightarrow,\neg)$	Frege-Basis
(8)	$(\{\mathbf{t},\mathbf{f}\};\ \uparrow)$	NAND-basis (Scheffer-basis)
(9)	$(\{\mathbf{t},\mathbf{f}\};\ \downarrow)$	NOR-basis (Peirce-basis)
(10)	$(\{\mathbf{t},\mathbf{f}\};\ \beta(.,.,.))$	β-$\mathbf{t}$-$\mathbf{f}$-basis

Normal forms

Example: $\mathbf{A}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \Longleftrightarrow (\mathbf{b}\rightarrow \neg\ \mathbf{c})\rightarrow(\mathbf{d}\rightarrow\mathbf{a})$

a) Disjunctive normal form in $(\{\mathbf{t},\mathbf{f}\};\ \wedge,\vee,\neg)$
e.g. $\mathrm{DNF}[\mathbf{A}] \Longleftrightarrow \mathbf{a}\vee(\mathbf{b}\wedge\mathbf{c})\vee\neg\mathbf{d}$

b) Conjunctive normal form in $(\{\mathbf{t},\mathbf{f}\};\ \wedge,\vee,\neg)$
e.g. $\mathrm{CNF}[\mathbf{A}] \Longleftrightarrow (\mathbf{a}\vee\mathbf{b}\vee\neg\mathbf{d})\wedge(\mathbf{a}\vee\mathbf{c}\vee\neg\mathbf{d})$

c) Premise normal form in $(\{\mathbf{t},\mathbf{f}\};\ \beta)$
$\mathrm{PNF}[\mathrm{A}] \Longleftrightarrow \beta(\mathbf{x},\mathbf{Y},\mathbf{Z})$, recursive expansion using ER
Premise position $\mathbf{x}$ is atomic variable; $\mathbf{Y},\mathbf{Z}\in\{\beta,\mathbf{t},\mathbf{f}\}$
e.g. $\mathbf{A} \Longleftrightarrow \beta(\mathbf{a},\mathbf{t},[\neg\mathbf{d}\vee(\mathbf{b}\wedge\mathbf{c})])$
$[\neg\mathbf{d}\vee(\mathbf{b}\wedge\mathbf{c})] \Longleftrightarrow \beta(\mathbf{d},[\mathbf{b}\wedge\mathbf{c}],\mathbf{t})$; $[\mathbf{b}\wedge\mathbf{c}] \Longleftrightarrow \beta(\mathbf{b},[\mathbf{c}],\mathbf{f})$; $[\mathbf{c}] \Longleftrightarrow \beta(\mathbf{c},\mathbf{t},\mathbf{f})$
$\mathrm{PNF}[\mathbf{A}] \Longleftrightarrow \beta(\mathbf{a},\mathbf{t},\beta(\mathbf{d},\beta(\mathbf{b},\beta(\mathbf{c},\ \mathbf{t},\mathbf{f}),\mathbf{f}),\mathbf{t}))$

d) (Canonical) Ringnormalform in $(\{\mathbf{t},\mathbf{f}\};\ \wedge,\ \leftarrow\!\!\!+\!\!\!\rightarrow)$
(Žegalkin polynomial, Muller-Reed expansion)
$\mathrm{RNF}[\mathbf{A}] :\Longleftrightarrow \mathbf{M}_1 \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{M}_2 \leftarrow\!\!\!+\!\!\!\rightarrow \dots$
$\mathbf{M} :\Longleftrightarrow \mathbf{a}\wedge\ \mathbf{b}\wedge\ \dots$ Monom; recursive expansion using ER(4)
e.g. $\mathbf{A} \Longleftrightarrow \mathbf{d}\wedge[\mathbf{a}\vee(\mathbf{b}\wedge\mathbf{c}) \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{t}] \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{t}$;
$\mathbf{a}\vee(\mathbf{b}\wedge\mathbf{c}) \Longleftrightarrow \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{b}\wedge\mathbf{c} \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{a}$
$\mathrm{RNF}[\mathbf{A}] \Longleftrightarrow \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d} \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d} \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{a}\wedge\mathbf{d} \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{d} \leftarrow\!\!\!+\!\!\!\rightarrow \mathbf{t}$

The β-Operation

$$\begin{aligned}
\mathbf{y} \Longleftrightarrow \beta(\mathbf{x},\mathbf{a},\mathbf{b}) &\Longleftrightarrow (\mathbf{x} \wedge \mathbf{a}) \vee (\neg\mathbf{x} \wedge \mathbf{b}) \\
&\Longleftrightarrow (\neg\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \\
&\Longleftrightarrow (\mathbf{x} \longrightarrow \mathbf{a}) \wedge (\neg\mathbf{x} \longrightarrow \mathbf{b}) \\
&\Longleftrightarrow [\mathbf{x} \wedge (\mathbf{a} \longleftrightarrow \mathbf{b})] \longleftrightarrow \mathbf{b}
\end{aligned}$$

x	a	b	$\beta(\mathbf{x},\mathbf{a},\mathbf{b})$
t	t	t	t
t	t	f	t
t	f	t	f
t	f	f	f
f	t	t	t
f	t	f	f
f	f	t	t
f	f	f	f

Notation options:

1) Expansion rules: $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \beta(\mathbf{x}_i, \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{t}), \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f}))$

$$\begin{aligned}
\beta(\mathbf{x}_i, \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{t}), \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f})) &\Longleftrightarrow [\mathbf{x}_i \wedge \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{t})] \vee [\neg\mathbf{x}_i \wedge \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f})] \\
&\Longleftrightarrow [\neg\mathbf{x}_i \vee \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{t})] \wedge [\mathbf{x}_i \vee \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f})] \\
&\Longleftrightarrow [\mathbf{x}_i \longrightarrow \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{t})] \wedge [\neg\mathbf{x}_i \longrightarrow \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f})] \\
&\Longleftrightarrow [\mathbf{x}_i \wedge [\mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{t}) \longleftrightarrow \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f})]] \\
&\qquad \longleftrightarrow \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f})
\end{aligned}$$

$\mathbf{x}_i \in \mathrm{set}(\underline{\mathbf{x}})$: Premise position; $\quad \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{t}), \mathbf{A}(\mathbf{x}_i \Leftrightarrow \mathbf{f})$: cofactors

2) β-basis:

e.g. $\mathbf{x} \uparrow \mathbf{y} \Longleftrightarrow \neg(\mathbf{x} \wedge \mathbf{y}) \Longleftrightarrow \beta(\mathbf{x}, \neg\mathbf{y}, \mathbf{t}) \Longleftrightarrow \beta(\mathbf{x}, \beta(\mathbf{y}, \mathbf{f}, \mathbf{t}), \mathbf{t})$

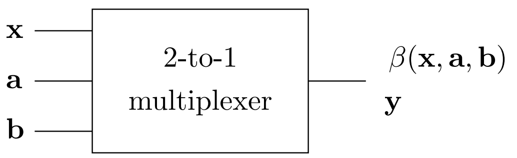
3) Resolution rules:

$\beta(\mathbf{x},\mathbf{a},\mathbf{b}) \Longleftrightarrow \beta(\mathbf{x},\mathbf{a},\mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{b}) \Longleftrightarrow \beta(\mathbf{x},\mathbf{a},\mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{b})$
$\beta(\mathbf{x},\mathbf{a},\mathbf{a}) \Longleftrightarrow a$

4) Case differentiation:

$\beta(\mathbf{x},\mathbf{a},\mathbf{b}) \Longleftrightarrow$ `if` x `then` a `else` b $\Longleftrightarrow$ `ite`$(\mathbf{x},\mathbf{a},\mathbf{b})$

5) Multiplexer:



$\beta(\mathbf{t},\mathbf{a},\mathbf{b}) \Longleftrightarrow \mathbf{a}$
$\beta(\mathbf{f},\mathbf{a},\mathbf{b}) \Longleftrightarrow \mathbf{b}$
$\mathbf{x}$: selector input
$\mathbf{a}, \mathbf{b}$: data inputs

BB) The $\beta-\mathbf{t}-\mathbf{f}-$basis

$$\begin{array}{llll}
(1) \quad \mathbf{a} & \Longleftrightarrow \beta(\mathbf{a},\mathbf{t},\mathbf{f}); & \neg\mathbf{a} & \Longleftrightarrow \beta(\mathbf{a},\mathbf{f},\mathbf{t}) \\
 & \Longleftrightarrow \beta(\mathbf{a},\mathbf{a},\mathbf{f}); & & \Longleftrightarrow \beta(\mathbf{a},\neg\mathbf{a},\mathbf{t}) \\
 & \Longleftrightarrow \beta(\mathbf{a},\mathbf{t},\mathbf{a}); & & \Longleftrightarrow \beta(\mathbf{a},\mathbf{f},\neg\mathbf{a}) \\
 & \Longleftrightarrow \beta(\mathbf{t},\mathbf{a},\mathbf{b}); & & \Longleftrightarrow \beta(\mathbf{t},\neg\mathbf{a},\neg\mathbf{b}) \\
 & \Longleftrightarrow \beta(\mathbf{f},\mathbf{b},\mathbf{a}); & & \Longleftrightarrow \beta(\mathbf{f},\neg\mathbf{b},\neg\mathbf{a}) \\
 & \Longleftrightarrow \beta(\mathbf{x},\mathbf{a},\mathbf{a}); & & \Longleftrightarrow \beta(\mathbf{x},\neg\mathbf{a},\neg\mathbf{a})
\end{array}$$

$$\begin{array}{llll}
(2) \quad \mathbf{t} & \Longleftrightarrow \beta(\mathbf{a},\mathbf{t},\mathbf{t}); & \mathbf{f} & \Longleftrightarrow \beta(\mathbf{a},\mathbf{f},\mathbf{f}) \\
 & \Longleftrightarrow \beta(\mathbf{a},\mathbf{a},\mathbf{t}); & & \Longleftrightarrow \beta(\mathbf{a},\neg\mathbf{a},\mathbf{f}) \\
 & \Longleftrightarrow \beta(\mathbf{a},\mathbf{a},\neg\mathbf{a}); & & \Longleftrightarrow \beta(\mathbf{a},\neg\mathbf{a},\mathbf{a}) \\
 & \Longleftrightarrow \beta(\mathbf{a},\mathbf{t},\neg\mathbf{a}); & & \Longleftrightarrow \beta(\mathbf{a},\mathbf{f},\mathbf{a})
\end{array}$$

$$\begin{array}{llll}
(3) \quad \mathbf{a}\wedge\mathbf{b} & \Longleftrightarrow \beta(\mathbf{a},\mathbf{b},\mathbf{f}); & \mathbf{a}\vee\mathbf{b} & \Longleftrightarrow \beta(\neg\mathbf{a},\mathbf{b},\mathbf{t}) \\
 & \Longleftrightarrow \beta(\mathbf{a},\mathbf{b},\mathbf{a}); & & \Longleftrightarrow \beta(\neg\mathbf{a},\mathbf{b},\mathbf{a}) \\
 & \Longleftrightarrow \beta(\neg\mathbf{a},\mathbf{f},\mathbf{b}); & & \Longleftrightarrow \beta(\mathbf{a},\mathbf{t},\mathbf{b}) \\
 & \Longleftrightarrow \beta(\neg\mathbf{a},\mathbf{a},\mathbf{b}); & & \Longleftrightarrow \beta(\mathbf{a},\mathbf{a},\mathbf{b})
\end{array}$$

$$\begin{array}{lll}
(4) \quad \mathbf{a}\longrightarrow\mathbf{b} & \Longleftrightarrow \beta(\mathbf{a},\mathbf{b},\mathbf{t}) & \Longleftrightarrow \beta(\mathbf{b},\mathbf{t},\neg\mathbf{a}) \\
 & \Longleftrightarrow \beta(\neg\mathbf{a},\mathbf{t},\mathbf{b}) & \Longleftrightarrow \beta(\mathbf{a},\mathbf{b},\neg\mathbf{a})
\end{array}$$

$$\begin{array}{ll}
(5) \quad \mathbf{a}\longleftrightarrow\mathbf{b} & \Longleftrightarrow \beta(\mathbf{a},\mathbf{b},\neg\mathbf{b}) \\
\quad\;\; \mathbf{a}\not\longleftrightarrow\mathbf{b} & \Longleftrightarrow \beta(\mathbf{a},\neg\mathbf{b},\mathbf{b})
\end{array}$$

BO) Laws using the β-operation

(1) $\beta(\mathbf{x},\mathbf{t},\mathbf{t}) \iff \mathbf{t}$; $\quad \beta(\mathbf{x},\mathbf{f},\mathbf{f}) \iff \mathbf{f}$

(2) $\beta(\mathbf{x},\mathbf{t},\mathbf{f}) \iff \mathbf{x}$; $\quad \beta(\mathbf{x},\mathbf{f},\mathbf{t}) \iff \neg\mathbf{x}$

(3) $\beta(\mathbf{t},\mathbf{a},\mathbf{b}) \iff \mathbf{a}$; $\quad \beta(\mathbf{f},\mathbf{a},\mathbf{b}) \iff \mathbf{b}$

$$\left.\begin{array}{lrcl}
(4) & \neg\beta(\mathbf{x},\mathbf{a},\mathbf{b}) & \iff & \beta(\mathbf{x},\neg\mathbf{a},\neg\mathbf{b}) \\
(5) & \beta(\neg\mathbf{x},\mathbf{a},\mathbf{b}) & \iff & \beta(\mathbf{x},\mathbf{b},\mathbf{a}) \\
(6) & \text{dual } \beta(\mathbf{x},\mathbf{a},\mathbf{b}) & \iff & \beta(\mathbf{x},\mathbf{b},\mathbf{a})
\end{array}\right\}\text{laws of duality}$$

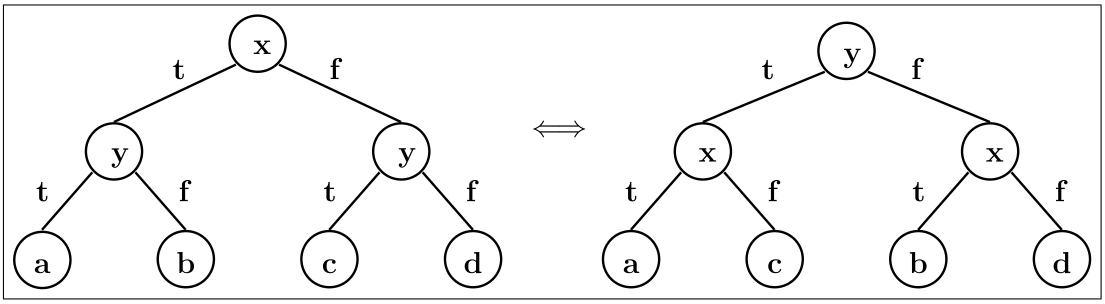
$$\left.\begin{array}{lrcl}
(7) & \beta(\mathbf{x},\mathbf{a},\mathbf{b}) & \iff & \beta(\mathbf{x},\mathbf{a},\mathbf{b}) \vee (\mathbf{a}\wedge\mathbf{b}) \\
(8) & \beta(\mathbf{x},\mathbf{a},\mathbf{b}) & \iff & \beta(\mathbf{x},\mathbf{a},\mathbf{b}) \wedge (\mathbf{a}\vee\mathbf{b}) \\
(9) & \beta(\mathbf{x},\mathbf{a},\mathbf{a}) & \iff & \mathbf{a}
\end{array}\right\}\text{resolution rules}$$

(10) $\beta(\mathbf{x},\mathbf{A},\mathbf{B}) \iff [\mathbf{x}\wedge(\mathbf{A}\wedge\neg\mathbf{B})] \vee [\neg\mathbf{x}\wedge(\neg\mathbf{A}\wedge\mathbf{B})] \vee (\mathbf{A}\wedge\mathbf{B})$
$\mathbf{A}\wedge\neg\mathbf{B},\ \neg\mathbf{A}\wedge\mathbf{B},\ \mathbf{A}\wedge\mathbf{B}$ are pairwise disjoint

(11) $\beta(\mathbf{x},\mathbf{A},\mathbf{B}) \iff [\neg\mathbf{x}\vee(\mathbf{A}\vee\neg\mathbf{B})] \wedge [\mathbf{x}\vee(\neg\mathbf{A}\vee\mathbf{B})] \wedge (\mathbf{A}\vee\mathbf{B})$

(12) $\mathbf{f} \Longrightarrow \mathbf{a}\wedge\mathbf{b} \Longrightarrow \beta(\mathbf{x},\mathbf{a},\mathbf{b}) \Longrightarrow \mathbf{a}\vee\mathbf{b} \Longrightarrow \mathbf{t}$ enclosure law using $\beta(\mathbf{x},\mathbf{a},\mathbf{b})$

(13) $\beta(\mathbf{x},\beta(\mathbf{y},\mathbf{a},\mathbf{b}),\beta(\mathbf{y},\mathbf{c},\mathbf{d})) \iff \beta(\mathbf{y},\beta(\mathbf{x},\mathbf{a},\mathbf{c}),\beta(\mathbf{x},\mathbf{b},\mathbf{d}))$



(14) $\beta(\mathbf{x}_\mathrm{i}, \mathbf{A}(\underline{\mathbf{x}}), \mathbf{B}(\underline{\mathbf{x}})) \Longleftrightarrow \beta(\mathbf{x}_\mathrm{i}, \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t}), \mathbf{B}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f}))$ — $\mathbf{x}_\mathrm{i} \in \mathrm{set}(\underline{\mathbf{x}})$
substitution rule

(15) $\beta(\mathbf{x}, \beta(\mathbf{x}, \mathbf{a}, \mathbf{b}), \mathbf{c}) \Longleftrightarrow \beta(\mathbf{x}, \mathbf{a}, \mathbf{c})$

(16) $\beta(\mathbf{x}, \mathbf{a}, \beta(\mathbf{x}, \mathbf{b}, \mathbf{c})) \Longleftrightarrow \beta(\mathbf{x}, \mathbf{a}, \mathbf{c})$

(15), (16): special cases of substitution rules

(17) $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \beta(\mathbf{x}_\mathrm{i}, \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t}), \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f}))$ — $\mathbf{x}_\mathrm{i} \in \mathrm{set}(\underline{\mathbf{x}})$;
From BO14 with $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{B}(\underline{\mathbf{x}})$
Expansion rules for propositional logic functions

(18) $\beta(\mathbf{A}(\underline{\mathbf{x}}), \mathbf{B}(\underline{\mathbf{x}}), \mathbf{C}(\underline{\mathbf{x}})) \Longleftrightarrow \mathbf{A}(\mathbf{t} := \mathbf{B}, \mathbf{f} := \mathbf{C})$ — composition rule
(if **A** in PNF)

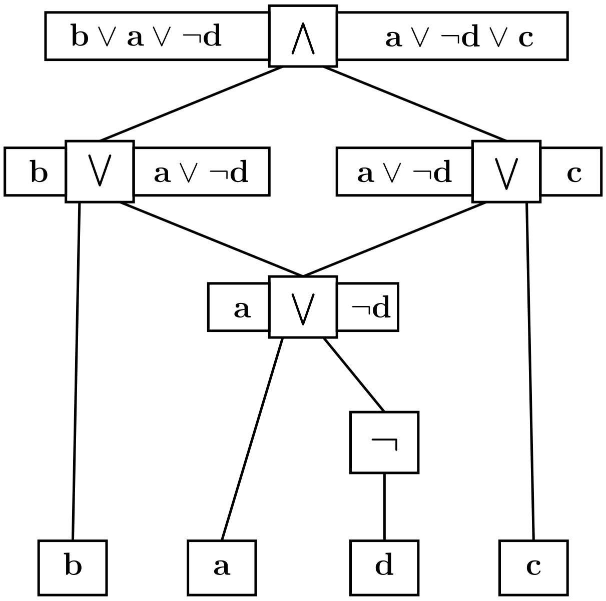
(19) $\beta(\beta(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{a}, \mathbf{b}) \Longleftrightarrow \beta(\mathbf{x}, \beta(\mathbf{y}, \mathbf{a}, \mathbf{b}), \beta(\mathbf{z}, \mathbf{a}, \mathbf{b}))$ — special case
of composition rule

$$
\begin{array}{lrcccl}
(20) & \mathbf{A} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{t}, \mathbf{f}) & & \\
(21) & \neg\mathbf{A} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{f}, \mathbf{t}) & \Longleftrightarrow & \mathbf{A}(\mathbf{t} := \mathbf{f}, \mathbf{f} := \mathbf{t}) \\
(22) & \mathbf{A} \wedge \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{B}, \mathbf{f}) & \Longleftrightarrow & \mathbf{A}(\mathbf{t} := \mathbf{B}) \\
(23) & \mathbf{A} \vee \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{t}, \mathbf{B}) & \Longleftrightarrow & \mathbf{A}(\mathbf{f} := \mathbf{B}) \\
(24) & \mathbf{A} \longrightarrow \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{B}, \mathbf{t}) & \Longleftrightarrow & \mathbf{A}(\mathbf{t} := \mathbf{B}, \mathbf{f} := \mathbf{t}) \\
(25) & \mathbf{A} \longleftrightarrow \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \mathbf{B}, \neg\mathbf{B}) & \Longleftrightarrow & \mathbf{A}(\mathbf{t} := \mathbf{B}, \mathbf{f} := \neg\mathbf{B}) \\
(26) & \mathbf{A} \not\longleftrightarrow \mathbf{B} & \Longleftrightarrow & \beta(\mathbf{A}, \neg\mathbf{B}, \mathbf{B}) & \Longleftrightarrow & \mathbf{A}(\mathbf{t} := \neg\mathbf{B}, \mathbf{f} := \mathbf{B})
\end{array}
$$

Decomposition of Boolean Functions

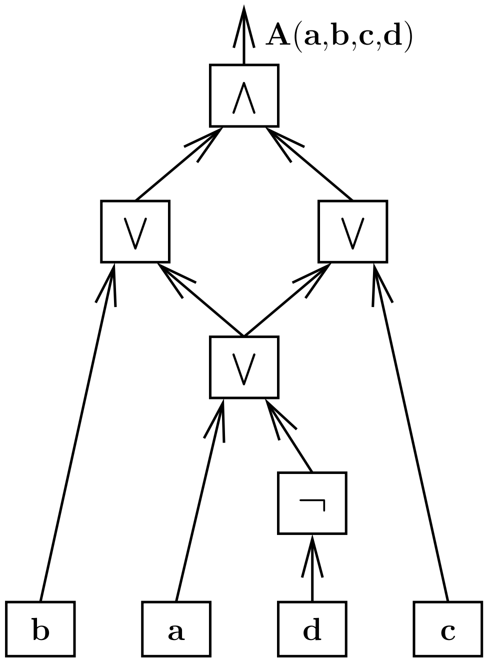
(Concatenation of subfunctions)

z.B.



Simplification:

Boolean network BN

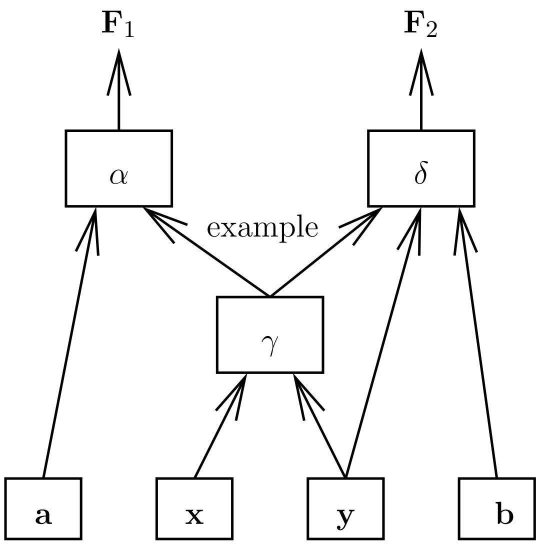


$$\mathbf{A}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \iff (\mathbf{a} \lor \neg\mathbf{d} \lor \mathbf{b}) \land (\mathbf{a} \lor \neg\mathbf{d} \lor \mathbf{c})$$

Corresponding terms

Boolean function	Boolean network	Circuit netlist
multi-output function	set of terminal vertices	circuit outputs
(atomic) variable	set of root vertices	circuit inputs
subfunction	internal vertices	subcircuit
hierarchical depth	levels	circuit levels
subfunction variable	predecessor vertex	subcircuit inputs
multiply used subfunction	fanout vertex	fanout

Boolean network BN = (V,S)



of the Boolean function

$\mathbf{F}(\underline{\mathbf{z}})=(\mathbf{F}_1\mathbf{F}_2)$

set($\underline{\mathbf{z}}$)=sup($\mathbf{F}$)={**a**,**b**,**x**,**y**}

sup($\mathbf{F}$): support of $\mathbf{F}$

$\mathbf{V}=\{\alpha, \gamma, \delta, \mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}\}$

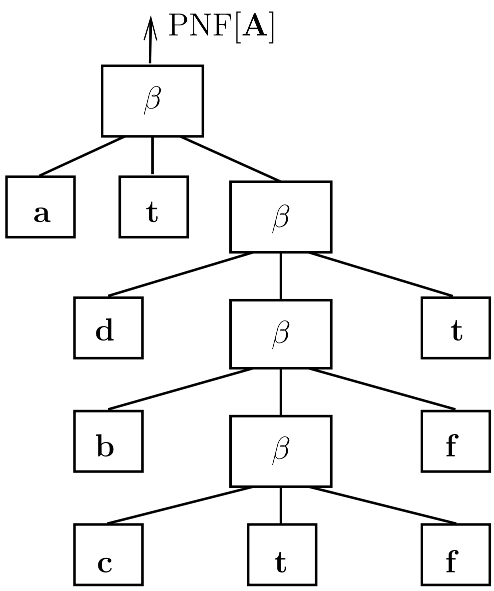
e.g. $(\gamma, \alpha) \in \mathbf{S} \Longleftrightarrow \gamma \in \sup(\alpha)$

BN is directed, acyclic graph (dag)

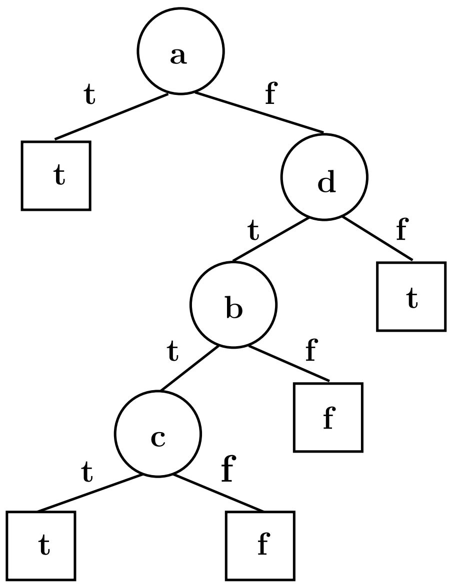
$$\alpha \Longleftrightarrow \mathbf{a} \vee \gamma \qquad \delta \Longleftrightarrow (\mathbf{b} \wedge \gamma) \vee (\neg \mathbf{b} \wedge \neg \mathbf{y}) \qquad \gamma \Longleftrightarrow \mathbf{x} \wedge \mathbf{y}$$

$$\sup(\alpha) = \{\mathbf{a},\gamma\} \qquad \sup(\delta) = \{\mathbf{b},\gamma,\mathbf{y}\} \qquad \sup(\gamma) = \{\mathbf{x},\mathbf{y}\}$$

Example:



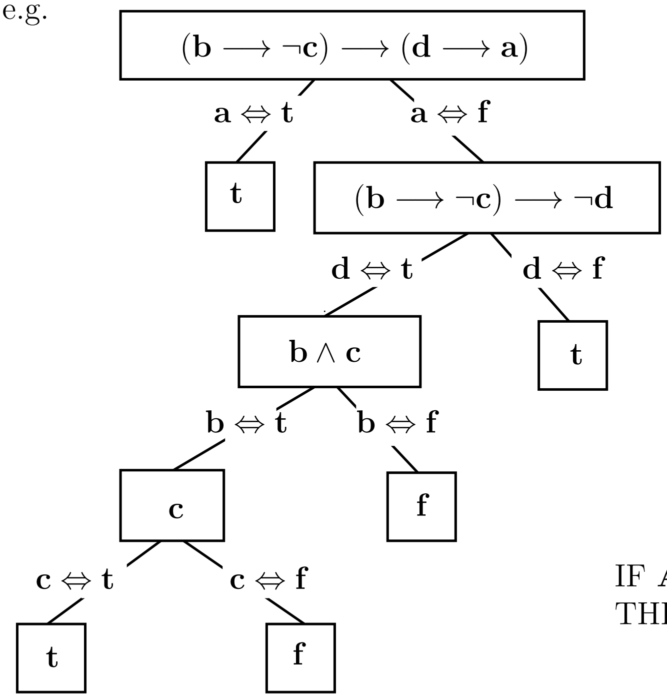
BN of PNF[**A**]



BDD of PNF[**A**]
Binary Decision Diagram

Decision trees, diagrams

e.g.

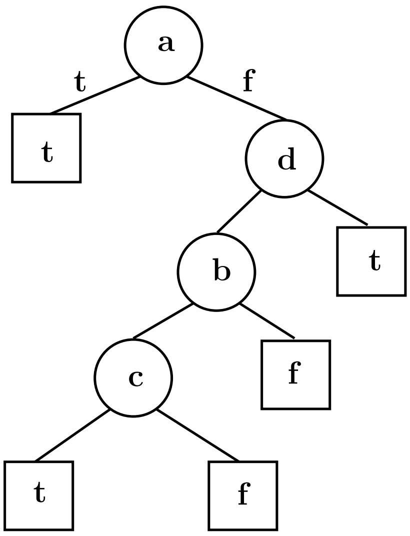


Expansion of Boolean functions using expansion rules ER (1), (2), (3)

Note regarding proof of tautology:

IF $\mathbf{A}(\underline{\mathbf{x}}) \Longleftrightarrow \mathbf{t}$,
THEN all terminal vertices are $\mathbf{t}$

Simplification: **BDT** (Binary Decision Tree), **BDD** (Binary Decision Diagram)



$$\begin{aligned}\text{“PNF}[\mathbf{A}]\text{”} \Longleftrightarrow\ & (\mathbf{a} \longrightarrow \mathbf{t}) \wedge (\neg\mathbf{a} \wedge \neg\mathbf{d} \longrightarrow \mathbf{t}) \wedge \\ & \wedge (\neg\mathbf{a} \wedge \mathbf{d} \wedge \mathbf{b} \wedge \mathbf{c} \longrightarrow \mathbf{t}) \wedge \\ & \wedge (\neg\mathbf{a} \wedge \mathbf{d} \wedge \neg\mathbf{b} \longrightarrow \mathbf{f}) \wedge \\ & \wedge (\neg\mathbf{a} \wedge \mathbf{d} \wedge \mathbf{b} \wedge \neg\mathbf{c} \longrightarrow \mathbf{f})\end{aligned}$$

$$\begin{aligned}\text{DNF}[\mathbf{A}] \Longleftrightarrow\ & \mathbf{a} \vee (\neg\mathbf{a} \wedge \neg\mathbf{d}) \vee \\ & \vee (\neg\mathbf{a} \wedge \mathbf{d} \wedge \mathbf{b} \wedge \mathbf{c})\end{aligned}$$

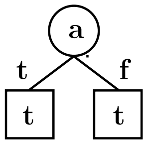
$$\begin{aligned}\text{CNF}[\mathbf{A}] \Longleftrightarrow\ & (\mathbf{a} \vee \neg\mathbf{d} \vee \mathbf{b}) \wedge \\ & \wedge (\mathbf{a} \vee \neg\mathbf{d} \vee \neg\mathbf{b} \vee \mathbf{c})\end{aligned}$$

$$\begin{array}{lcl}\text{CNF}[\mathbf{A}] & \Longleftrightarrow & \text{PNF}[\mathbf{A}] \\ \text{DNF}[\mathbf{A}] & \Longleftrightarrow & \neg\text{PNF}[\neg\mathbf{A}]\end{array}$$

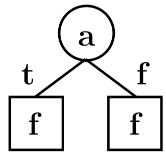
Elementary Premise Normal Forms (PNF)
– Basic structures of decision trees (BDT)

$\mathbf{t} \iff \beta(\mathbf{a},\mathbf{t},\mathbf{t})$; $\mathbf{f} \iff \beta(\mathbf{a},\mathbf{f},\mathbf{f})$

$\boxed{\mathbf{t}}$

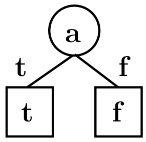


$\boxed{\mathbf{f}}$

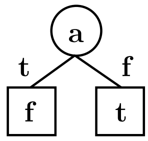


$\mathbf{a} \iff \beta(\mathbf{a},\mathbf{t},\mathbf{f})$; $\neg\mathbf{a} \iff \beta(\mathbf{a},\mathbf{f},\mathbf{t}) \iff \mathbf{a}(\mathbf{t}:=\mathbf{f},\mathbf{f}:=\mathbf{t})$

$\boxed{\mathbf{a}}$

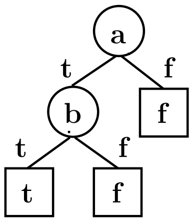


$\boxed{\neg\mathbf{a}}$

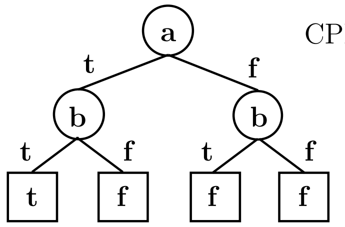


$\mathbf{a} \wedge \mathbf{b} \iff \beta(\mathbf{a},\mathbf{b},\mathbf{f}) \iff \beta(\mathbf{a},\beta(\mathbf{b},\mathbf{t},\mathbf{f}),\mathbf{f}) \iff \mathbf{a}(\mathbf{t}:=\mathbf{b})$

$\boxed{\mathbf{a} \wedge \mathbf{b}}$



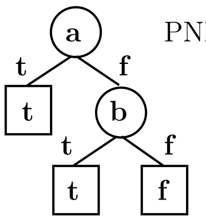
PNF[$\mathbf{a} \wedge \mathbf{b}$]



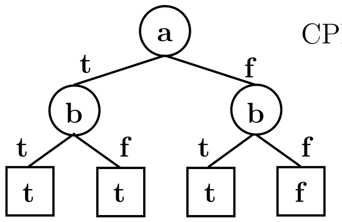
CPNF[$\mathbf{a} \wedge \mathbf{b}$]

$\mathbf{a} \vee \mathbf{b} \iff \beta(\mathbf{a},\mathbf{t},\mathbf{b}) \iff \beta(\mathbf{a},\mathbf{t},\beta(\mathbf{b},\mathbf{t},\mathbf{f})) \iff \mathbf{a}(\mathbf{f}:=\mathbf{b})$

$\boxed{\mathbf{a} \vee \mathbf{b}}$



PNF[$\mathbf{a} \vee \mathbf{b}$]

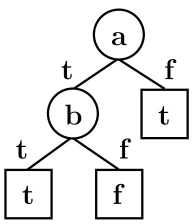


CPNF[$\mathbf{a} \vee \mathbf{b}$]

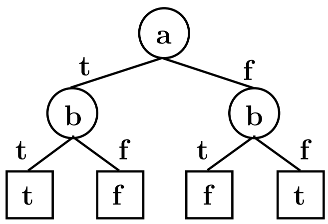
$\mathbf{a} \rightarrow \mathbf{b} \iff \beta(\mathbf{a},\mathbf{b},\mathbf{t})$; $\iff \mathbf{a}(\mathbf{t}:=\mathbf{b},\mathbf{f}:=\mathbf{t})$

$\mathbf{a} \leftrightarrow \mathbf{b} \iff \beta(\mathbf{a},\mathbf{b},\neg\mathbf{b}) \iff \mathbf{a}(\mathbf{t}:=\mathbf{b},\mathbf{f}:=\neg\mathbf{b})$

$\boxed{\mathbf{a} \rightarrow \mathbf{b}}$



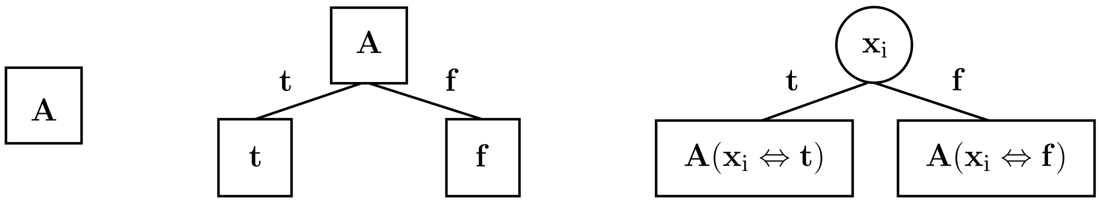
$\boxed{\mathbf{a} \leftrightarrow \mathbf{b}}$



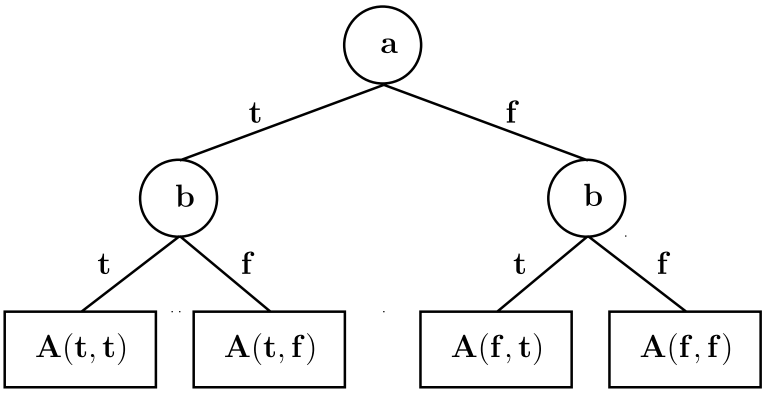
PNF[$\mathbf{a} \rightarrow \mathbf{b}$] $\iff \beta(\mathbf{a},\beta(\mathbf{b},\mathbf{t},\mathbf{f}),\mathbf{t})$; CPNF[$\mathbf{a} \leftrightarrow \mathbf{b}$] $\iff \beta(\mathbf{a},\beta(\mathbf{b},\mathbf{t},\mathbf{f}),\beta(\mathbf{b},\mathbf{f},\mathbf{t}))$

Expansion rule:

$$\mathbf{A}(\underline{\mathbf{x}}) \iff \beta(\mathbf{A}(\underline{\mathbf{x}}), \mathbf{t}, \mathbf{f}) \iff \beta(\mathbf{x}_\mathrm{i}, \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t}), \mathbf{A}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f}))$$

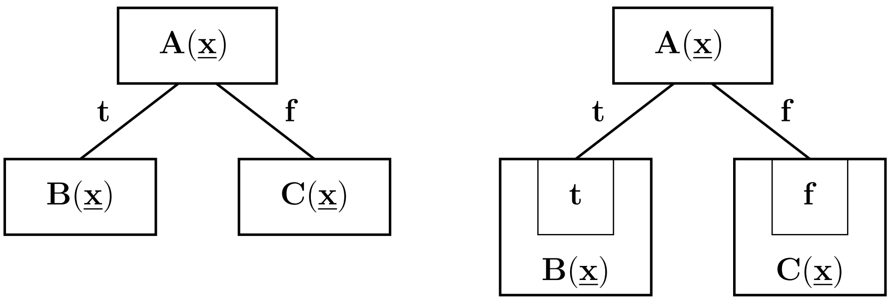


e.g. $\mathbf{A}(\mathbf{a}, \mathbf{b}) \iff \beta(\mathbf{a}, \mathbf{A}(\mathbf{t}, \mathbf{b}), \mathbf{A}(\mathbf{f}, \mathbf{b}))$
$\iff \beta(\mathbf{a}, \beta(\mathbf{b}, \mathbf{A}(\mathbf{t}, \mathbf{t}), \mathbf{A}(\mathbf{t}, \mathbf{f})), \beta(\mathbf{b}, \mathbf{A}(\mathbf{f}, \mathbf{t}), \mathbf{A}(\mathbf{f}, \mathbf{f})))$



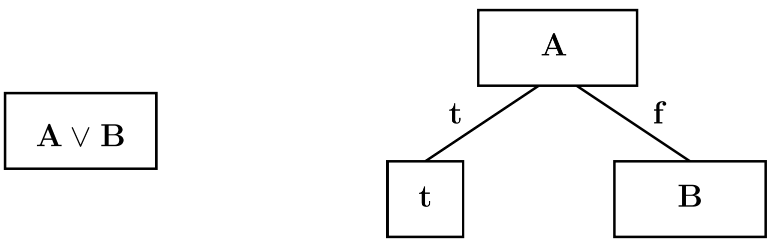
Composition rule:

$$\beta(\mathbf{A}(\underline{\mathbf{x}}), \mathbf{B}(\underline{\mathbf{x}}), \mathbf{C}(\underline{\mathbf{x}})) \iff \mathbf{A}(\mathbf{t} := \mathbf{B}, \mathbf{f} := \mathbf{C})$$



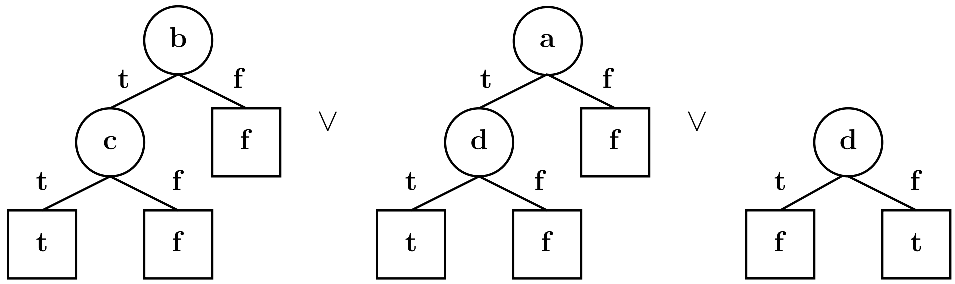
Note:
Transformation from general β-representation into PNF

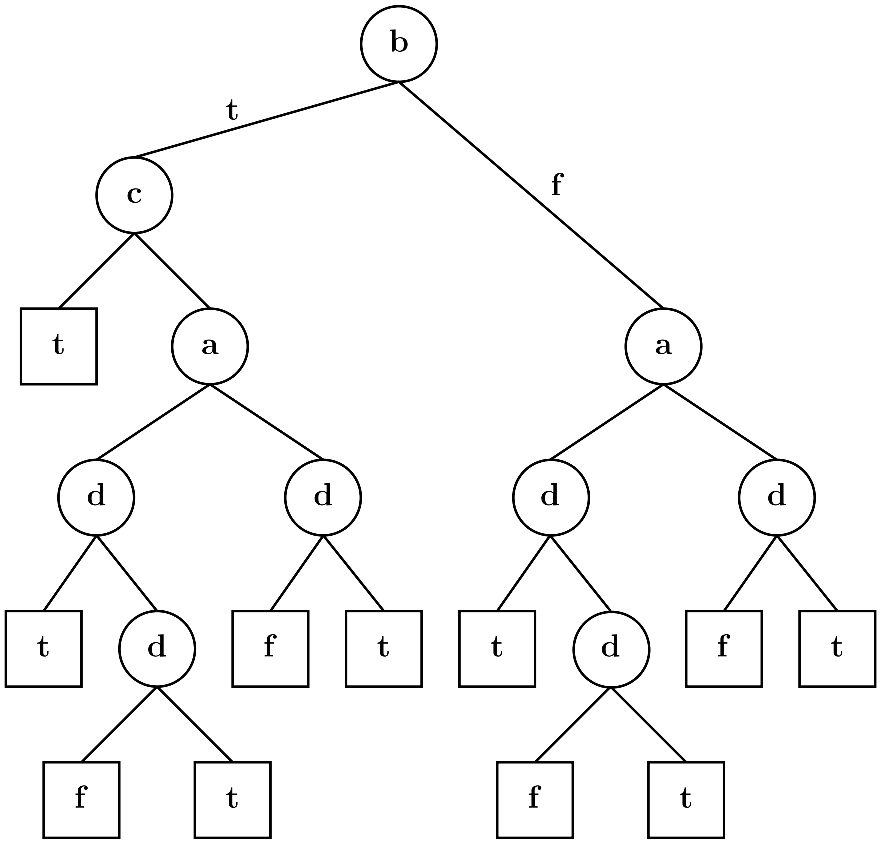
e.g. $\mathbf{A} \vee \mathbf{B} \iff \beta(\mathbf{A}, \mathbf{t}, \mathbf{B}) \iff \mathbf{A}(\mathbf{f} := \mathbf{B})$



Example:

$\mathbf{A}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \iff (\mathbf{b}\wedge\mathbf{c}) \quad \vee \quad (\mathbf{a}\wedge\mathbf{d}) \quad \vee \quad \neg\mathbf{d}$

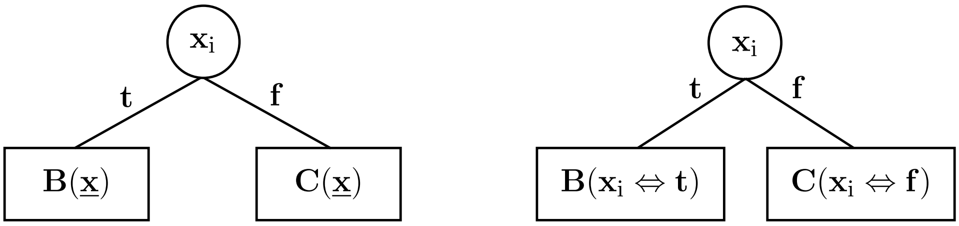




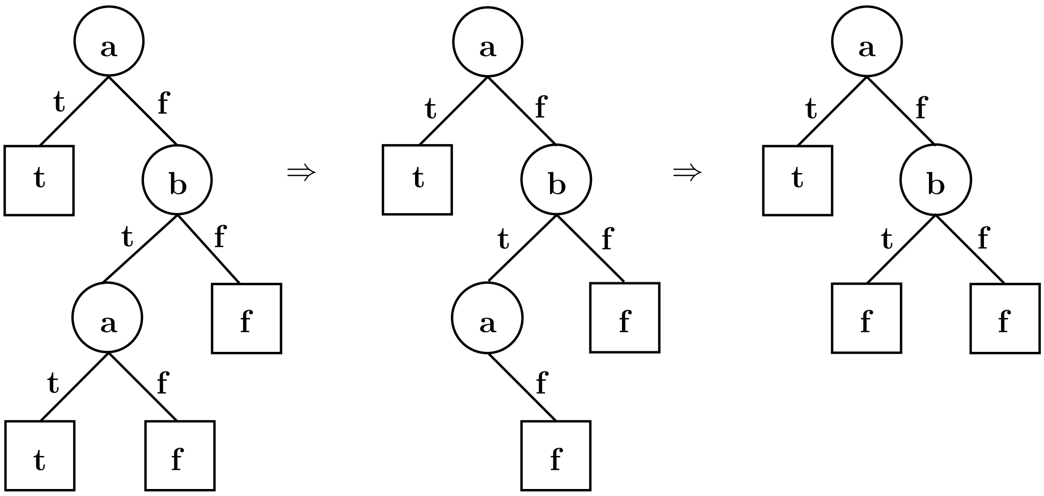
BDT[**A**]

Substitution rule (simplification rule, SPR1) :

$$\beta(\mathbf{x}_\mathrm{i}, \mathbf{B}(\underline{\mathbf{x}}), \mathbf{C}(\underline{\mathbf{x}})) \quad \Longleftrightarrow \quad \beta(\mathbf{x}_\mathrm{i}, \mathbf{B}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{t}), \mathbf{C}(\mathbf{x}_\mathrm{i} \Leftrightarrow \mathbf{f}))$$



e.g. $\mathbf{a} \vee (\mathbf{a} \wedge \mathbf{b}) \Longleftrightarrow \beta(\mathbf{a}, \mathbf{t}, \mathbf{b} \wedge \mathbf{a}) \Longleftrightarrow \beta(\mathbf{a}, \mathbf{t}, \beta(\mathbf{b}, \mathbf{a}, \mathbf{f})) \Longleftrightarrow \beta(\mathbf{a}, \mathbf{t}, \beta(\mathbf{b}, \mathbf{f}, \mathbf{f}))$

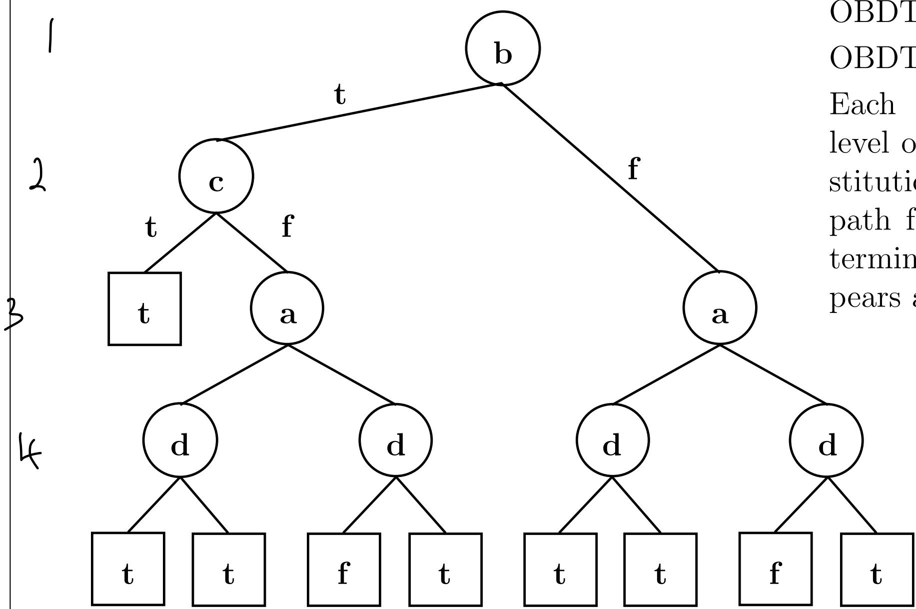


e.g. $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \Longleftrightarrow (\mathbf{b} \wedge \mathbf{c}) \vee (\mathbf{a} \wedge \mathbf{d}) \vee \neg \mathbf{d}$

OBDT: Ordered BDT

OBDT definition:

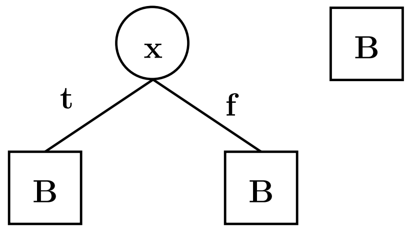
Each variable appears on one level of the BDT only (full substitution of variables: in each path from the root vertex to a terminal vertex each variable appears at most once .)



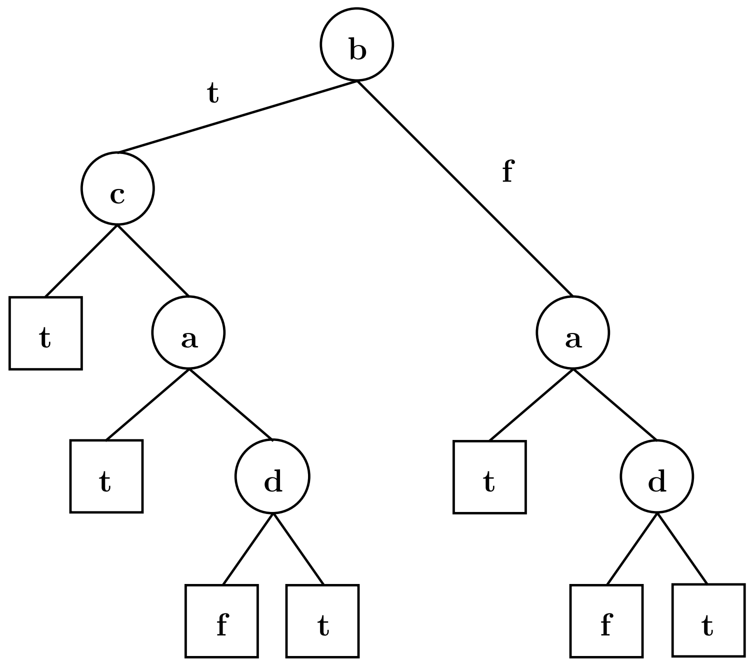
OBDT[**A**]

Resolution rule (simplification rule, SPR2) :

$\beta(\mathbf{x},\mathbf{B},\mathbf{B}) \Longleftrightarrow \qquad \mathbf{B}$



e.g. $\mathbf{A}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \Longleftrightarrow \mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) \vee \neg\mathbf{d}$

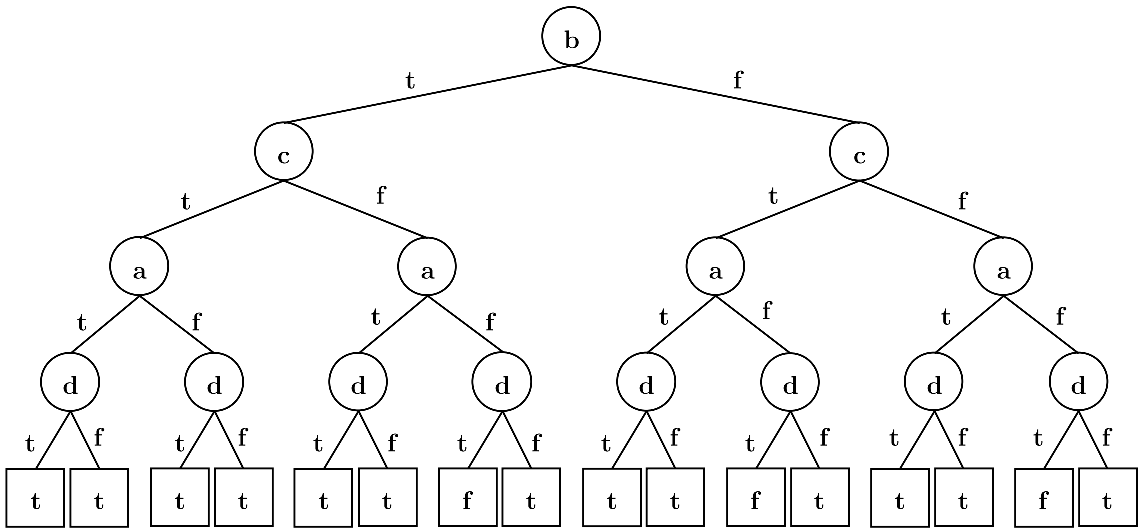


ROBDT: Reduced OBDT

ROBDT[**A**]

(complete resolution
of variables)

Canonical normal form of $\mathbf{A}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \Longleftrightarrow \mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) \vee \neg\mathbf{d}$:



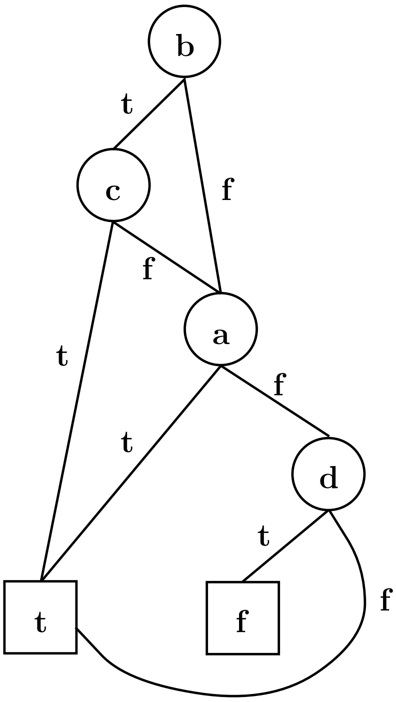
CPNF[**A**], CDNF[**A**], CCNF[**A**]

Crosslinking (simplification rule, SPR3) :

Crosslinking to equivalent sub-BDTs (merging of equivalent sub-BDTs)

BDT	$\longrightarrow$	BDD	Binary Decision Diagram
OBDT	$\longrightarrow$	OBDD	
ROBDT	$\longrightarrow$	ROBDD	

e.g. $\mathbf{A}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \Longleftrightarrow \mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) \vee \neg\mathbf{d}$



ROBDD[**A**]

(complete crosslinking)

Theorem:

An ROBDD is a canonical representation, given a defined variable ordering

2 Predicate Logic

Propositional logic : Elementary propositions

Propositional logic propositional form : $\mathbf{A}$ ($\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$)

$\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$: Propositional variable

$\mathbf{A}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$: Placeholder for propositions

$\downarrow$

Proposition : $\mathbf{A}(\mathbf{t}, \mathbf{f}, \mathbf{t}) \in \{\mathbf{t}, \mathbf{f}\}$

Predicate logic : Propositions in subject-predicate structure

2.1 Predicate (predicate logic) propositional forms

Predicate logic propositional form (Unary predicate) :	$\mathbf{P}$	$\mathbf{x}$
	Predicate (-variable)	Individual variable

$\downarrow$

Parlance : $\mathbf{x}$ has property $\mathbf{P}$

e.g. $\mathbf{Px} \Longleftrightarrow \mathbf{x}$ is a genius

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ = {Einstein, Turing, Suzy}

Proposition : $\mathbf{Px}_3 \in \{\mathbf{t}, \mathbf{f}\}$

Predicates		Parlance
$\mathbf{Px}$	unary	$\mathbf{x}$ has property $\mathbf{P}$ e.g. " $\mathbf{x}$ is whole "
$\mathbf{Pxy}$	binary	$\mathbf{x}$ is related to $\mathbf{y}$ by $\mathbf{P}$ e.g. " $\mathbf{x} < \mathbf{y}$ "
$\mathbf{Pxyz}$	ternary	$\mathbf{x}, \mathbf{y}, \mathbf{z}$ are related by $\mathbf{P}$ e.g. " $\mathbf{x} * \mathbf{y} = \mathbf{z}$ "
$\mathbf{Px}_1 \ldots \mathbf{x}_n$	n-ary (n-place)	$\mathbf{x}_1, \ldots, \mathbf{x}_n$ are related by $\mathbf{P}$

$\mathbf{Px}_1\mathbf{x}_2\mathbf{x}_3 \ldots \mathbf{x}_n$: String in prefix-notation

$\mathbf{Pxy}$: prefix-notation $(< \mathbf{x} \;\; \mathbf{y})$
$\mathbf{xPy}$: infix-notation $(\mathbf{x} < \mathbf{y})$
$\mathbf{xyP}$: postfix-notation $(\mathbf{x} \;\; \mathbf{y} <)$

Unary predicate : Unary predicates examine their associated objects for existence of a certain property.

$\mathbf{Px}$: Interpretation as set-building operator

Set-building : $\mathbf{A} = \{ \mathbf{x} \mid \underbrace{\mathbf{Px} \wedge \mathbf{x} \in \mathbf{G}}_{\text{Propositional form on basic set } \mathbf{G}.} \}$

$\mathbf{A}$ is the set of all objects $\mathbf{x}$ from $\mathbf{G}$, for which $\mathbf{Px}$ holds.

Binary relation from set $\mathbf{M}$ to $\mathbf{N}$: $\mathbf{R}_2 = \{ (\mathbf{x}, \mathbf{y}) \mid \underbrace{\mathbf{Pxy} \wedge (\mathbf{x}, \mathbf{y}) \in \mathbf{M} \times \mathbf{N}}_{\text{Propositional form on product set } \mathbf{M} \times \mathbf{N}} \}$

Cartesian product set : $\mathbf{M} \times \mathbf{N} = \{ \underbrace{(\mathbf{x}, \mathbf{y})}_{\text{ordered pair (of subjects)}} \mid \mathbf{x} \in \mathbf{M} \wedge \mathbf{y} \in \mathbf{N} \}$

Predicate logic propositional form using quantifiers:

Quantifiers	Propositional form	Parlance
Universal quantifier	$\underset{\mathbf{x}}{\forall}\ \mathbf{Px}\ ;\quad \forall \mathbf{x},\ \mathbf{Px}$	$\mathbf{Px}$ holds for all $\mathbf{x}$
Existential quantifier	$\underset{\mathbf{x}}{\exists}\ \mathbf{Px}\ ;\quad \exists \mathbf{x},\ \mathbf{Px}$	there exists (such) an $\mathbf{x}$, that $\mathbf{Px}$ holds

Propositional form : $\underset{\mathbf{x}\in\mathbf{M}}{\forall}\ \mathbf{Pxy}$

$\mathbf{x}$: bound, $\mathbf{y}$: free $\Big\}$ variables

$\downarrow$

Proposition : $\underset{\mathbf{x}\in\mathbf{M}}{\forall}\ \mathbf{Pxy}_1\ ;\qquad \underset{\mathbf{y}\in\mathbf{N}}{\exists}\ \underset{\mathbf{x}\in\mathbf{M}}{\forall}\ \mathbf{Pxy}$

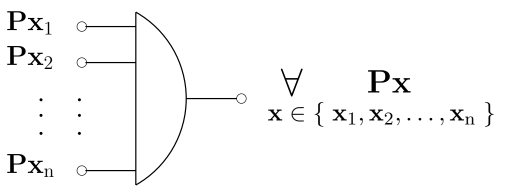
Definition and interpretation by logic circuits:

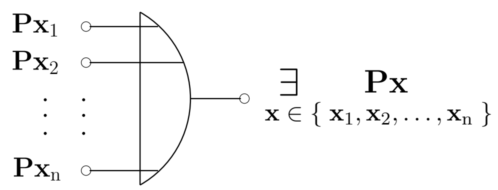
AND

$$\underset{\mathbf{x}\in\{\,\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\,\}}{\forall}\mathbf{Px} \Longleftrightarrow \mathbf{Px}_1\wedge\mathbf{Px}_2\wedge\cdots\wedge\mathbf{Px}_n\ ;$$

OR

$$\underset{\mathbf{x}\in\{\,\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\,\}}{\exists}\mathbf{Px} \Longleftrightarrow \mathbf{Px}_1\vee\mathbf{Px}_2\vee\cdots\vee\mathbf{Px}_n$$





Example: P	:	is a metal;	$\mathbf{x}\in\mathbf{M}$:
Q	:	conducts current;	$\mathbf{x}$ is an element out of the periodic system
$\mathbf{x}_1$	:	Copper	

$$\left[\underset{\mathbf{x}\in\mathbf{M}}{\forall}(\mathbf{Px}\longrightarrow\mathbf{Qx})\right]\wedge\ \mathbf{Px}_1\quad\Longrightarrow\quad\mathbf{Qx}_1$$

2.2 Laws

F) Logical Equivalence ($\Longleftrightarrow$)

(1) $$\neg \underset{\mathbf{x} \in \mathbf{M}}{\forall} \mathbf{Px} \Longleftrightarrow \underset{\mathbf{x} \in \mathbf{M}}{\exists} \neg \mathbf{Px}$$

(2) $$\neg \underset{\mathbf{x} \in \mathbf{M}}{\exists} \mathbf{Px} \Longleftrightarrow \underset{\mathbf{x} \in \mathbf{M}}{\forall} \neg \mathbf{Px}$$

(3) $$\underset{\mathbf{x} \in \mathbf{M}}{\forall} \underset{\mathbf{y} \in \mathbf{N}}{\forall} \mathbf{Pxy} \Longleftrightarrow \underset{\mathbf{y} \in \mathbf{N}}{\forall} \underset{\mathbf{x} \in \mathbf{M}}{\forall} \mathbf{Pxy}$$

(4) $$\underset{\mathbf{x} \in \mathbf{M}}{\exists} \underset{\mathbf{y} \in \mathbf{N}}{\exists} \mathbf{Pxy} \Longleftrightarrow \underset{\mathbf{y} \in \mathbf{N}}{\exists} \underset{\mathbf{x} \in \mathbf{M}}{\exists} \mathbf{Pxy}$$

(5) $$\underset{\mathbf{x} \in \mathbf{M}}{\forall} (\mathbf{Px} \wedge \mathbf{Qx}) \Longleftrightarrow (\underset{\mathbf{x} \in \mathbf{M}}{\forall} \mathbf{Px}) \wedge (\underset{\mathbf{x} \in \mathbf{M}}{\forall} \mathbf{Qx})$$

(6) $$\underset{\mathbf{x} \in \mathbf{M}}{\exists} (\mathbf{Px} \vee \mathbf{Qx}) \Longleftrightarrow (\underset{\mathbf{x} \in \mathbf{M}}{\exists} \mathbf{Px}) \vee (\underset{\mathbf{x} \in \mathbf{M}}{\exists} \mathbf{Qx})$$

(7) $$\underset{\mathbf{x} \in \mathbf{M}}{\forall} (\mathbf{S} \vee \mathbf{Qx}) \Longleftrightarrow \mathbf{S} \vee \underset{\mathbf{x} \in \mathbf{M}}{\forall} \mathbf{Qx}$$

(8) $$\underset{\mathbf{x} \in \mathbf{M}}{\exists} (\mathbf{S} \wedge \mathbf{Qx}) \Longleftrightarrow \mathbf{S} \wedge \underset{\mathbf{x} \in \mathbf{M}}{\exists} \mathbf{Qx}$$

(9) $$\underset{\mathbf{x} \in \mathbf{M}}{\forall} \underset{\mathbf{y} \in \mathbf{N}}{\exists} (\mathbf{Px} \wedge \mathbf{Qy}) \Longleftrightarrow \underset{\mathbf{y} \in \mathbf{N}}{\exists} \underset{\mathbf{x} \in \mathbf{M}}{\forall} (\mathbf{Px} \wedge \mathbf{Qy})$$

(10) $$\Longleftrightarrow (\underset{\mathbf{x} \in \mathbf{M}}{\forall} \mathbf{Px}) \wedge (\underset{\mathbf{y} \in \mathbf{N}}{\exists} \mathbf{Qy})$$

(11) $$\underset{\mathbf{x} \in \mathbf{M}}{\forall} \underset{\mathbf{y} \in \mathbf{N}}{\exists} (\mathbf{Px} \vee \mathbf{Qy}) \Longleftrightarrow \underset{\mathbf{y} \in \mathbf{N}}{\exists} \underset{\mathbf{x} \in \mathbf{M}}{\forall} (\mathbf{Px} \vee \mathbf{Qy})$$

(12) $$\Longleftrightarrow (\underset{\mathbf{x} \in \mathbf{M}}{\forall} \mathbf{Px}) \vee (\underset{\mathbf{y} \in \mathbf{N}}{\exists} \mathbf{Qy})$$

(13) $$\underset{\mathbf{x} \in \mathbf{M}}{\exists} (\mathbf{Px} \longrightarrow \mathbf{Qx}) \Longleftrightarrow (\underset{\mathbf{x} \in \mathbf{M}}{\forall} \mathbf{Px}) \longrightarrow (\underset{\mathbf{x} \in \mathbf{M}}{\exists} \mathbf{Qx})$$

$\mathbf{M}, \mathbf{N}$: Domains (of individual variables) / Universe of discourse
Prerequisite: $\mathbf{M} \neq \emptyset$, $\mathbf{N} \neq \emptyset$

G) (Logical) Implication ($\Longrightarrow$)

(1) $\forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Px} \Longrightarrow \mathbf{Px}_1 \ , \qquad \mathbf{Px}_1 \Longrightarrow \exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Px}\, , \qquad \mathbf{x}_1 \in \mathbf{M}$

(2) $\forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Px} \Longrightarrow \exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Px}$

(3) $\forall_{\mathbf{x} \in \mathbf{M}} (\mathbf{Px} \vee \mathbf{Qx}) \Longrightarrow (\forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Px}) \vee (\exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Qx})$

(4) $\exists_{\mathbf{x} \in \mathbf{M}} (\mathbf{Px} \wedge \mathbf{Qx}) \Longrightarrow (\exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Px}) \wedge (\exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Qx})$

(5) $(\forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Px}) \vee (\forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Qx}) \Longrightarrow \forall_{\mathbf{x} \in \mathbf{M}} (\mathbf{Px} \vee \mathbf{Qx})$

(6) $(\forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Px}) \wedge (\exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Qx}) \Longrightarrow \exists_{\mathbf{x} \in \mathbf{M}} (\mathbf{Px} \wedge \mathbf{Qx})$

(7) $\forall_{\mathbf{x} \in \mathbf{M}} \forall_{\mathbf{y} \in \mathbf{M}} \mathbf{Pxy} \Longrightarrow \forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Pxx}$

(8) $\exists_{\mathbf{x} \in \mathbf{M}} \forall_{\mathbf{y} \in \mathbf{N}} \mathbf{Pxy} \Longrightarrow \forall_{\mathbf{y} \in \mathbf{N}} \exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Pxy}$

(9) $\forall_{\mathbf{x} \in \mathbf{M}} (\mathbf{Px} \longrightarrow \mathbf{Qx}) \Longrightarrow \forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Px} \longrightarrow \forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Qx}$

(10) $\forall_{\mathbf{x} \in \mathbf{M}} (\mathbf{Px} \longrightarrow \mathbf{Qx}) \Longrightarrow \exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Px} \longrightarrow \exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Qx}$

(11) $(\forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Px}) \wedge \forall_{\mathbf{x} \in \mathbf{M}} (\mathbf{Px} \longrightarrow \mathbf{Qx}) \Longrightarrow \forall_{\mathbf{x} \in \mathbf{M}} \mathbf{Qx}$

(12) $(\exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Px}) \wedge \forall_{\mathbf{x} \in \mathbf{M}} (\mathbf{Px} \longrightarrow \mathbf{Qx}) \Longrightarrow \exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Qx}$

2.3 Deduction scheme

Example:

All Bavarians are Germans, and all Bavarians are Europeans,
and there exist Bavarians implies:
some Europeans are Germans.

B: is a Bavarian; D: is a German; E: is a European; M: set of all people

1) $\forall_{\mathbf{x} \in \mathbf{M}} (\mathbf{Bx} \longrightarrow \mathbf{Dx})$

2) $\forall_{\mathbf{x} \in \mathbf{M}} (\mathbf{Bx} \longrightarrow \mathbf{Ex})$

3) $\exists_{\mathbf{x} \in \mathbf{M}} \mathbf{Bx} \qquad \Longrightarrow \qquad \exists_{\mathbf{x} \in \mathbf{M}} (\mathbf{Dx} \wedge \mathbf{Ex})$

4)	$\forall_{\mathbf{x} \in \mathbf{M}} [\, (\neg \mathbf{Bx} \vee \mathbf{Dx}) \wedge (\neg \mathbf{Bx} \vee \mathbf{Ex}) \,]$	1) 2)	D1, F5
5)	$\forall_{\mathbf{x} \in \mathbf{M}} (\neg \mathbf{Bx} \vee (\mathbf{Dx} \wedge \mathbf{Ex}))$	4)	A3
6)	$\forall_{\mathbf{x} \in \mathbf{M}} \neg \mathbf{Bx} \vee \exists_{\mathbf{x} \in \mathbf{M}} (\mathbf{Dx} \wedge \mathbf{Ex})$	5)	G3!
7)	$\neg \forall_{\mathbf{x} \in \mathbf{M}} \neg \mathbf{Bx}$	3)	F2
8)	$\neg \forall_{\mathbf{x} \in \mathbf{M}} (\neg \mathbf{Bx}) \wedge \exists_{\mathbf{x} \in \mathbf{M}} (\mathbf{Dx} \wedge \mathbf{Ex})$	6) 7)	
9)	$\exists_{\mathbf{x} \in \mathbf{M}} (\mathbf{Dx} \wedge \mathbf{Ex})$	8)	E4

2.4 (Mathematical) Induction

Using the inductive rule of inference

$$\boxed{\mathbf{P0} \wedge \bigvee_{\mathbf{n}\in\mathbb{N}_0} [\mathbf{Pn} \longrightarrow \mathbf{P(n+1)}] \Longrightarrow \bigvee_{\mathbf{n}\in\mathbb{N}_0} \mathbf{Pn}}$$

we have:

FROM	$\mathbf{P0} \Longleftrightarrow \mathbf{t}$	FROM	**basis step**
AND	$\bigvee_{\mathbf{n}\in\mathbb{N}_0} [\mathbf{Pn} \longrightarrow \mathbf{P(n+1)}] \Longleftrightarrow \mathbf{t}$	AND	**induction step**
IT FOLLOWS	$\bigvee_{\mathbf{n}\in\mathbb{N}_0} \mathbf{Pn} \Longleftrightarrow \mathbf{t}$	IT FOLLOWS	**conclusion**

In the induction step we must show that
$\mathbf{Pn} \longrightarrow \mathbf{P(n+1)} \Longleftrightarrow \mathbf{t}$, i.e. $\mathbf{Pn} \Longrightarrow \mathbf{P(n+1)}$, for all $\mathbf{n} \in \mathbb{N}_0$.

Example: Show that $\sum_{\nu=0}^{\mathbf{n}} (2\nu+1) = (\mathbf{n}+1)^2$ holds for all $\mathbf{n} \in \mathbb{N}_0$.

Inductive hypothesis $\mathbf{Pn} \Longleftrightarrow \sum_{\nu=0}^{\mathbf{n}} (2\nu+1) = (\mathbf{n}+1)^2$.

Basis step:

$$\mathbf{P0} \Longleftrightarrow (2\cdot 0+1) = (0+1)^2 \Longleftrightarrow \mathbf{t}$$

Induction step:

$$\begin{aligned}\mathbf{P(n+1)} &\Longleftrightarrow \sum_{\nu=0}^{\mathbf{n+1}} (2\nu+1) = (\mathbf{n}+2)^2 \\ &\Longleftrightarrow \sum_{\nu=0}^{\mathbf{n}} (2\nu+1) + 2\mathbf{n}+3 = (\mathbf{n}+1)^2 + 2\mathbf{n}+3\end{aligned}$$

Using $\mathbf{a} = \mathbf{b} \wedge \mathbf{c} = \mathbf{d} \Longrightarrow \mathbf{a}+\mathbf{c} = \mathbf{b}+\mathbf{d}$ we have for all $\mathbf{n} \in \mathbb{N}_0$:

$$\underbrace{(\sum_{\nu=0}^{\mathbf{n}}(2\nu+1) = (\mathbf{n}+1)^2)}_{\mathbf{Pn}} \wedge \underbrace{(2\mathbf{n}+3 = 2\mathbf{n}+3)}_{\mathbf{t}} \Longrightarrow \underbrace{\sum_{\nu=0}^{\mathbf{n}}(2\nu+1) + 2\mathbf{n}+3 = (\mathbf{n}+1)^2 + 2\mathbf{n}+3}_{\mathbf{P(n+1)}}$$

Conclusion:

$$\bigvee_{\mathbf{n}\in\mathbb{N}_0} \mathbf{Pn} \Longleftrightarrow \mathbf{t}$$

Generalization in case of a basis step using an arbitrary $\mathbf{k} \in \mathbb{N}_0$:

$$\boxed{\mathbf{Pk} \wedge \bigvee_{\mathbf{n}\in\mathbb{N}_\mathbf{k}} [\mathbf{Pn} \longrightarrow \mathbf{P(n+1)}] \Longrightarrow \bigvee_{\mathbf{n}\in\mathbb{N}_\mathbf{k}} \mathbf{Pn}} \quad \text{with } \mathbb{N}_\mathbf{k} = \{\mathbf{k}, \mathbf{k}+1, \ldots\}$$

3 Sets

A set is a collection of objects that is defined precisely enough so that we can in principle determine whether any given object is or is not an object in that set.

$$\mathbf{M} = \{\underbrace{\mathbf{x}}_{\text{bound variable (by set-building operator)}} \mid \mathbf{x} \text{ has property } \mathbf{P}\}$$

Parlance: $\mathbf{M}$ is the set of all $\mathbf{x}$, for which $\mathbf{P}$ holds.

3.1 Notation

$$\mathbf{M} = \{\mathbf{x} \mid \mathbf{Px}\}$$

Set-building (describing; implicit)

$$\mathbf{M} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$$

List (enumerating; explicit)

Set notation

$\mathbb{N} = \{\mathbf{x} \mid \mathbf{x} \text{ is a positive integer (without zero)}\} \triangleq \{1, 2, 3, 4, \ldots\}$

$\mathbb{N}_0 = \{\mathbf{x} \mid \mathbf{x} \text{ is a positive integer (including zero)}\} \triangleq \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\mathbf{x} \mid \mathbf{x} \text{ is integer}\} \triangleq \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q} = \{\mathbf{x} \mid \mathbf{x} \text{ is rational}\}$
$\quad = \{\mathbf{x} \mid \mathbf{x} = \mathbf{p}/\mathbf{q}, \quad \mathbf{p} \in \mathbb{Z} \wedge \mathbf{q} \in \mathbb{N}\}$

$\mathbb{R} = \{\mathbf{x} \mid \mathbf{x} \text{ is real}\}$

$\mathbb{C} = \{\mathbf{x} \mid \mathbf{x} \text{ is complex}\}$

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

The objects of a set are called <u>elements</u> of the set.

Number of elements in $\mathbf{M}$	:	$\mid \mathbf{M} \mid$ (cardinality)	
Empty set	:	$\emptyset, \{\,\}, \quad \{\mathbf{x} \mid \mathbf{x} \neq \mathbf{x}\}$	No object differs from itself.
Universal set (universe of discourse)	:	$\mathbf{G}, \boldsymbol{\Omega}, \quad \{\mathbf{x} \mid \mathbf{x} = \mathbf{x}\}$	All objects are equal to themselves.
Power set of set $\mathbf{M}$	:	$\mathbf{P}(\mathbf{M}) = \{\mathbf{X} \mid \mathbf{X} \subseteq \mathbf{M}\}$ $\lvert\mathbf{P}(\mathbf{M})\rvert = 2^{\lvert\mathbf{M}\rvert}$	

The set $\mathbf{P}(\mathbf{M})$ of all subsets $\mathbf{X}$ of a set $\mathbf{M}$ is called power set.

Examples:

$$\mathbf{M} = \emptyset \longrightarrow \mathbf{P}(\mathbf{M}) = \{\emptyset\}$$
$$\mathbf{M} = \{\mathbf{a}\} \longrightarrow \mathbf{P}(\mathbf{M}) = \{\emptyset, \{\mathbf{a}\}\}$$
$$\mathbf{M} = \{\mathbf{a}, \mathbf{b}\} \longrightarrow \mathbf{P}(\mathbf{M}) = \{\emptyset, \{\mathbf{a}\}, \{\mathbf{b}\}, \{\mathbf{a}, \mathbf{b}\}\}$$

Notations for $\mathbf{M}$:

$$\mathbf{M} = \{\mathbf{x} \mid \mathbf{x} \in \mathbf{G} \wedge \mathbf{Px}\}$$
$$= \{\mathbf{x} \in \mathbf{G} \mid \mathbf{Px}\}$$
$$= \{\mathbf{x} \mid \mathbf{Px}\}_{\mathbf{G}}$$
$$= \{\mathbf{x} \mid \mathbf{x} \in \mathbf{M}\}_{\mathbf{G}}$$

$\mathbf{M} \subseteq \mathbf{G} \iff \mathbf{M} \in \mathbf{P}(\mathbf{G})$

$\mathbf{M}$ is the set of all objects $\mathbf{x}$ out of the universal set $\mathbf{G}$, for which $\mathbf{Px}$ holds.

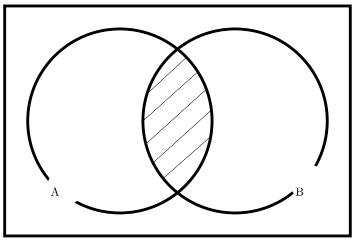
$\mathbf{x} \in \mathbf{M} \underset{\mathbf{G}}{\iff} \mathbf{Px}$;
($\mathbf{x}$ is element of $\mathbf{M}$)

$\mathbf{x} \notin \mathbf{M} \underset{\mathbf{G}}{\iff} \neg (\mathbf{x} \in \mathbf{M}) \underset{\mathbf{G}}{\iff} \neg \mathbf{Px}$
($\mathbf{x}$ is not element of $\mathbf{M}$)

Additional notation: card($\mathbf{M}$) for $\lvert\mathbf{M}\rvert$

3.2 Operations and Definitions

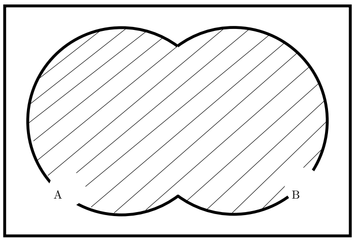
$\mathbf{A},\mathbf{B}\in\mathbf{P}(\mathbf{G})$



Intersection

$$\mathbf{A}\cap\mathbf{B} \;=\; \{\mathbf{x}\in\mathbf{G}\,|\,\mathbf{x}\in\mathbf{A} \;\wedge\; \mathbf{x}\in\mathbf{B}\}$$

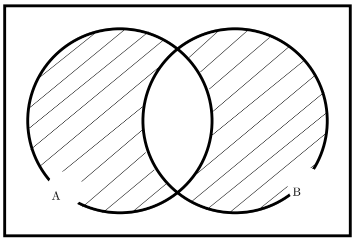
$$\mathbf{x}\in\; \mathbf{A}\cap\mathbf{B} \Longleftrightarrow \mathbf{x}\in\mathbf{A} \;\wedge\; \mathbf{x}\in\mathbf{B}$$



Union

$$\mathbf{A}\cup\mathbf{B} \;=\; \{\mathbf{x}\in\mathbf{G}\,|\,\mathbf{x}\in\mathbf{A} \;\vee\; \mathbf{x}\in\mathbf{B}\}$$

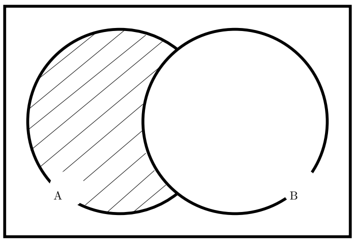
$$\mathbf{x}\in\; \mathbf{A}\cup\mathbf{B} \Longleftrightarrow \mathbf{x}\in\mathbf{A} \;\vee\; \mathbf{x}\in\mathbf{B}$$



Symmetric difference (of two sets)

$$\mathbf{A} \bigtriangleup \mathbf{B} \;=\; \{\mathbf{x}\in\mathbf{G}\,|\,\mathbf{x}\in\mathbf{A} \;\not\leftrightarrow\; \mathbf{x}\in\mathbf{B}\}$$

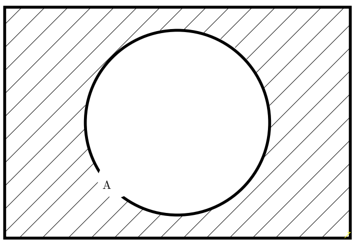
$$\mathbf{x}\in\; \mathbf{A} \bigtriangleup \mathbf{B} \Longleftrightarrow \mathbf{x}\in\mathbf{A} \;\not\leftrightarrow\; \mathbf{x}\in\mathbf{B}$$



Difference of sets (relative complement of **B** w.r.t. **A**)
(parlance: **A** without **B**)

$$\mathbf{A}\backslash\mathbf{B} \;=\; \{\mathbf{x}\in\mathbf{G}\,|\,\mathbf{x}\in\mathbf{A} \;\wedge\; \mathbf{x}\notin\mathbf{B}\} \quad\triangleq\quad \mathbf{A}-\mathbf{B}$$

$$\mathbf{x}\in\; \mathbf{A}\backslash\mathbf{B} \Longleftrightarrow \mathbf{x}\in\mathbf{A} \;\wedge\; \mathbf{x}\notin\mathbf{B}$$



Complement

$$\overline{\mathbf{A}} = \;\{\mathbf{x}\in\mathbf{G}\,|\,\mathbf{x} \notin\mathbf{A}\} \quad=\quad \mathbf{G}\setminus\mathbf{A}$$

$$\mathbf{x}\in\overline{\mathbf{A}} \Longleftrightarrow \mathbf{x} \notin\mathbf{A} \Longleftrightarrow \neg\,(\mathbf{x}\in\mathbf{A})$$

Strength of operators: $\overline{}$ $\quad\cap\quad \cup \quad \bigtriangleup \quad \backslash$

Aggregation of indexed sets:

$$\mathbf{C} = \{\mathbf{S}_1, \ldots, \mathbf{S}_\nu, \ldots, \mathbf{S}_\mathrm{n}\} = \{\mathbf{S} \mid \mathbf{S} \in \mathbf{C}\}_\Omega \,; \qquad \Omega = \mathbf{P}(\mathbf{G})$$

Index set:

$$\mathbf{N} = \{1, \ldots, \nu, \ldots, \mathrm{n}\} \,; \qquad \mathbf{C} = \{\mathbf{S}_\nu \mid \nu \in \mathbf{N}\}_\Omega$$

Union of sets:

$$\bigcup_{\mathbf{S} \in \mathbf{C}} \mathbf{S} = \{\mathbf{x} \mid \underset{\mathbf{S} \in \mathbf{C}}{\exists} \mathbf{x} \in \mathbf{S}\}_\mathbf{G}$$

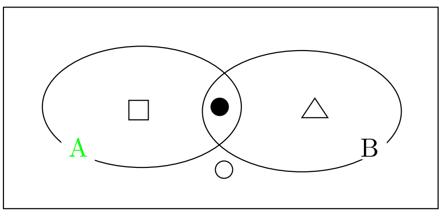
$$\bigcup_{\nu \in \mathbf{N}} \mathbf{S}_\nu = \bigcup_{\nu=1}^{\mathrm{n}} \mathbf{S}_\nu = \bigcup_{1 \le \nu \le \mathrm{n}} \mathbf{S}_\nu = \{\mathbf{x} \mid \underset{\nu \in \mathbf{N}}{\exists} \mathbf{x} \in \mathbf{S}_\nu\}_\mathbf{G}$$

Intersection of sets:

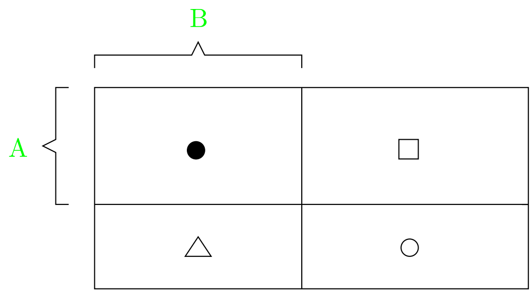
$$\bigcap_{\mathbf{S} \in \mathbf{C}} \mathbf{S} = \{\mathbf{x} \mid \underset{\mathbf{S} \in \mathbf{C}}{\forall} \mathbf{x} \in \mathbf{S}\}_\mathbf{G}$$

$$\bigcap_{\nu \in \mathbf{N}} \mathbf{S}_\nu = \bigcap_{\nu=1}^{\mathrm{n}} \mathbf{S}_\nu = \bigcap_{1 \le \nu \le \mathrm{n}} \mathbf{S}_\nu = \{\mathbf{x} \mid \underset{\nu \in \mathbf{N}}{\forall} \mathbf{x} \in \mathbf{S}_\nu\}_\mathbf{G}$$

Representation of sets:



Venn diagram



Karnaugh diagram

Conjunctions of propositional forms and on-sets:

$$\mathbf{E}[\mathbf{A}] = \{\hat{\underline{\mathbf{x}}} \mid \mathbf{A}(\hat{\underline{\mathbf{x}}})\}_{\mathbf{G}}, \quad \mathbf{G} = \{\mathbf{t},\mathbf{f}\}^{\mathrm{n}}, \quad \mathbf{E}[\mathbf{A}], \mathbf{E}[\mathbf{B}] \in \mathbf{P}(\mathbf{G}), \quad \mathbf{A} : \{\mathbf{t},\mathbf{f}\}^{\mathrm{n}} \longrightarrow \{\mathbf{t},\mathbf{f}\}$$

$$\begin{aligned}
\mathbf{E}[\mathbf{A} \wedge \mathbf{B}] &= \{\hat{\underline{\mathbf{x}}} \mid \mathbf{A} \wedge \mathbf{B}\}_{\mathbf{G}} &&= \mathbf{E}[\mathbf{A}] \cap \mathbf{E}[\mathbf{B}] \\
\mathbf{E}[\mathbf{A} \vee \mathbf{B}] &= \{\hat{\underline{\mathbf{x}}} \mid \mathbf{A} \vee \mathbf{B}\}_{\mathbf{G}} &&= \mathbf{E}[\mathbf{A}] \cup \mathbf{E}[\mathbf{B}] \\
\mathbf{E}[\neg \mathbf{A}] &= \{\hat{\underline{\mathbf{x}}} \mid \neg \mathbf{A}\}_{\mathbf{G}} &&= \overline{\mathbf{E}}[\mathbf{A}] \\
\mathbf{E}[\mathbf{A} \longrightarrow \mathbf{B}] &= \{\hat{\underline{\mathbf{x}}} \mid \mathbf{A} \longrightarrow \mathbf{B}\}_{\mathbf{G}} &&= \overline{\mathbf{E}}[\mathbf{A}] \cup \mathbf{E}[\mathbf{B}] \\
\mathbf{E}[\mathbf{A} \not\longleftrightarrow \mathbf{B}] &= \{\hat{\underline{\mathbf{x}}} \mid \mathbf{A} \not\longleftrightarrow \mathbf{B}\}_{\mathbf{G}} &&= \mathbf{E}[\mathbf{A}] \triangle \mathbf{E}[\mathbf{B}] \\
\mathbf{E}[\mathbf{A} \longleftrightarrow \mathbf{B}] &= \{\hat{\underline{\mathbf{x}}} \mid \mathbf{A} \longleftrightarrow \mathbf{B}\}_{\mathbf{G}} &&= \mathbf{E}[\mathbf{A} \longrightarrow \mathbf{B}] \cap \mathbf{E}[\mathbf{B} \longrightarrow \mathbf{A}]
\end{aligned}$$

$\mathbf{A}$ coding $\mathbf{z} := \hat{\underline{\mathbf{x}}}$ can be used to represent a set $\mathbf{M}$ as the on-set $\mathbf{E}[\mathbf{A}]$ of a propositional form $\mathbf{A}(\underline{\mathbf{x}})$.

$$\mathbf{M} = \{\mathbf{z} \in \mathbf{G} \mid \mathbf{Pz}\} := \mathbf{E}[\mathbf{A}]; \qquad |\mathbf{G}| = 2^{\mathrm{n}}$$

$\mathbf{A}(\underline{\mathbf{x}}) : \{\mathbf{t},\mathbf{f}\}^{\mathrm{n}} \longrightarrow \{\mathbf{t},\mathbf{f}\}$

$\hat{\underline{\mathbf{x}}} \in \mathbf{M} \iff \mathbf{P}\hat{\underline{\mathbf{x}}} \iff \hat{\underline{\mathbf{x}}} \in \mathbf{E}[\mathbf{A}] \iff \mathbf{A}(\hat{\underline{\mathbf{x}}})$

$\mathbf{A}(\underline{\mathbf{x}})$: characteristic function of $\mathbf{M}$

Note: $\underline{\hat{\mathbf{W}}}[\mathbf{A}]$ is a vector representation of $\mathbf{E}[\mathbf{A}]$ or $\mathbf{M}$

Definitions: $\mathbf{A} \subseteq \mathbf{G}$

$$\forall_{\mathbf{x} \in \mathbf{G}} \; \mathbf{x} \in \mathbf{A} \iff \mathbf{A} = \mathbf{G}; \qquad \exists_{\mathbf{x} \in \mathbf{G}} \; \mathbf{x} \in \mathbf{A} \iff \mathbf{A} \neq \emptyset$$

$$\forall_{\mathbf{x} \in \mathbf{G}} \; \mathbf{x} \notin \mathbf{A} \iff \mathbf{A} = \emptyset; \qquad \exists_{\mathbf{x} \in \mathbf{G}} \; \mathbf{x} \notin \mathbf{A} \iff \mathbf{A} \neq \mathbf{G}$$

$$\mathbf{A} = \mathbf{G} \iff \overline{\mathbf{A}} = \emptyset; \qquad \mathbf{A} \neq \emptyset \iff \overline{\mathbf{A}} \neq \mathbf{G}$$

$$\mathbf{A} = \emptyset \iff \overline{\mathbf{A}} = \mathbf{G}; \qquad \mathbf{A} \neq \mathbf{G} \iff \overline{\mathbf{A}} \neq \emptyset$$

3.3 Relations between Sets

$\mathbf{A} = \mathbf{B}$ (Equality of sets)

$$\mathbf{A} = \mathbf{B} \iff \underset{\mathbf{x} \in \mathbf{G}}{\forall} (\mathbf{x} \in \mathbf{A} \longleftrightarrow \mathbf{x} \in \mathbf{B}) \iff$$
$$\underset{\mathbf{x} \in \mathbf{G}}{\forall} \neg (\mathbf{x} \in \mathbf{A} \not\longleftrightarrow \mathbf{x} \in \mathbf{B}) \iff$$
$$\underset{\mathbf{x} \in \mathbf{G}}{\forall} \mathbf{x} \notin \mathbf{A} \triangle \mathbf{B} \iff \mathbf{A} \triangle \mathbf{B} = \emptyset$$

$$\mathbf{A} \neq \mathbf{B} \iff \underset{\mathbf{x} \in \mathbf{G}}{\exists} (\mathbf{x} \in \mathbf{A} \not\longleftrightarrow \mathbf{x} \in \mathbf{B}) \iff$$
$$\underset{\mathbf{x} \in \mathbf{G}}{\exists} \mathbf{x} \in \mathbf{A} \triangle \mathbf{B} \iff \mathbf{A} \triangle \mathbf{B} \neq \emptyset$$

$\mathbf{A} \subseteq \mathbf{B}$ — Subset, Superset (Inclusion)

$$\mathbf{A} \subseteq \mathbf{B} \iff \underset{\mathbf{x} \in \mathbf{G}}{\forall} (\mathbf{x} \in \mathbf{A} \longrightarrow \mathbf{x} \in \mathbf{B}) \iff$$
$$\underset{\mathbf{x} \in \mathbf{G}}{\forall} \neg (\mathbf{x} \in \mathbf{A} \wedge \mathbf{x} \notin \mathbf{B}) \iff$$
$$\underset{\mathbf{x} \in \mathbf{G}}{\forall} \mathbf{x} \notin \mathbf{A} \setminus \mathbf{B} \iff \mathbf{A} \setminus \mathbf{B} = \emptyset$$

$$\mathbf{A} \nsubseteq \mathbf{B} \iff \underset{\mathbf{x} \in \mathbf{G}}{\exists} (\mathbf{x} \in \mathbf{A} \wedge \mathbf{x} \notin \mathbf{B}) \iff$$
$$\underset{\mathbf{x} \in \mathbf{G}}{\exists} \mathbf{x} \in \mathbf{A} \setminus \mathbf{B} \iff \mathbf{A} \setminus \mathbf{B} \neq \emptyset$$

$$\mathbf{A} \subset \mathbf{B} \iff (\mathbf{A} \subseteq \mathbf{B}) \wedge (\mathbf{A} \neq \mathbf{B})$$

proper subset

$$\mathbf{A} \not\subset \mathbf{B} \iff (\mathbf{A} \nsubseteq \mathbf{B}) \vee (\mathbf{A} = \mathbf{B})$$

3.4 Laws $\quad \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{P}(\mathbf{G})$

H) Set algebra $\quad (\mathbf{P}(\mathbf{G});\ \cap,\ \cup,\ \overline{}\ ;\ \mathbf{G}\ ,\ \emptyset)$

Principle of duality:

(1) $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A} \quad ; \quad \mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A}$ — Commutativity

(2) $(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C} = \mathbf{A} \cap (\mathbf{B} \cap \mathbf{C})$ — Associativity
$(\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C} = \mathbf{A} \cup (\mathbf{B} \cup \mathbf{C})$

(3) $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$ — Distributivity
$\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$

(4) $\mathbf{A} \cap \mathbf{A} = \mathbf{A} \quad ; \quad \mathbf{A} \cup \mathbf{A} = \mathbf{A}$ — Idempotence

(5) $\mathbf{A} \cap (\mathbf{A} \cup \mathbf{B}) = \mathbf{A}$ — Absorption
$\mathbf{A} \cup (\mathbf{A} \cap \mathbf{B}) = \mathbf{A}$

(6) $\mathbf{A} \cap \mathbf{G} = \mathbf{A} \quad ; \quad \mathbf{A} \cup \emptyset = \mathbf{A}$ — Neutral element

(7) $\mathbf{A} \cap \emptyset = \emptyset \quad ; \quad \mathbf{A} \cup \mathbf{G} = \mathbf{G}$

(8) $\mathbf{A} \cap \overline{\mathbf{A}} = \emptyset \quad ; \quad \mathbf{A} \cup \overline{\mathbf{A}} = \mathbf{G}$ — Complementary element

(9) $\overline{(\overline{\mathbf{A}})} = \mathbf{A}$ — Double negation

(10) $\overline{\mathbf{A} \cap \mathbf{B}} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$ — De Morgan
$\overline{\mathbf{A} \cup \mathbf{B}} = \overline{\mathbf{A}} \cap \overline{\mathbf{B}}$

J) Set and Subset Relations

(1) $\mathbf{A} = \mathbf{A}$

(2) $\mathbf{A} \cap \mathbf{B} = \emptyset$ disjoint sets (no common elements)

(3) $\mathbf{A} \setminus \mathbf{B} = \mathbf{A} \cap \overline{\mathbf{B}}$

(4) $\mathbf{A} \setminus \emptyset = \mathbf{A}$; $\mathbf{A} \setminus \mathbf{A} = \emptyset$; $\mathbf{G} \setminus \mathbf{A} = \overline{\mathbf{A}}$

(5) $\mathbf{A} \setminus (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \setminus \mathbf{B}) \cap (\mathbf{A} \setminus \mathbf{C})$

(6) $\mathbf{A} \setminus (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \setminus \mathbf{B}) \cup (\mathbf{A} \setminus \mathbf{C})$

(7) $\mathbf{A} \cap (\mathbf{B} \setminus \mathbf{A}) = \emptyset$

(8) $\mathbf{A} \triangle \mathbf{B} = (\mathbf{A} \cup \mathbf{B}) \setminus (\mathbf{A} \cap \mathbf{B})$

(9) $= (\mathbf{A} \cup \mathbf{B}) \cap (\overline{\mathbf{A} \cap \mathbf{B}})$

(10) $= (\mathbf{A} \setminus \mathbf{B}) \cup (\mathbf{B} \setminus \mathbf{A})$

(11) $= (\mathbf{A} \cap \overline{\mathbf{B}}) \cup (\overline{\mathbf{A}} \cap \mathbf{B})$

(12) $\mathbf{A} \triangle \mathbf{B} = \mathbf{B} \triangle \mathbf{A}$

(13) $(\mathbf{A} \triangle \mathbf{B}) \triangle \mathbf{C} = \mathbf{A} \triangle (\mathbf{B} \triangle \mathbf{C})$

(14) $\mathbf{A} \cap (\mathbf{B} \triangle \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \triangle (\mathbf{A} \cap \mathbf{C})$

(15) $\mathbf{A} \triangle \mathbf{A} = \emptyset$; $\mathbf{A} \triangle \emptyset = \mathbf{A}$; $\mathbf{G} \triangle \mathbf{A} = \overline{\mathbf{A}}$

(16) $\mathbf{A} \triangle \overline{\mathbf{A}} = \mathbf{G}$; $\mathbf{A} \triangle \mathbf{A} \triangle \mathbf{A} = \mathbf{A}$

(20) $\mathbf{A} \subseteq \mathbf{A}\,; \qquad \emptyset \subseteq \mathbf{A}\,; \qquad \mathbf{A} \subseteq \mathbf{G}$

(21) $\mathbf{A} \subseteq \mathbf{A} \cup \mathbf{B}\,; \qquad \mathbf{A} \cap \mathbf{B} \subseteq \mathbf{A}$

(22) $\mathbf{A} \cap \mathbf{B} \subseteq \mathbf{A} \cup \mathbf{B}$

(23) $\mathbf{A} \cap (\overline{\mathbf{A}} \cup \mathbf{B}) \subseteq \mathbf{B}$ modus ponens

(24) $\mathbf{A} = \mathbf{B} \iff \mathbf{B} = \mathbf{A}$

(25) $\mathbf{A} = \mathbf{B} \iff (\overline{\mathbf{A}} \cup \mathbf{B}) \cap (\mathbf{A} \cup \overline{\mathbf{B}}) = \mathbf{G}$

(26) $\iff (\mathbf{A} \cap \overline{\mathbf{B}}) \cup (\overline{\mathbf{A}} \cap \mathbf{B}) = \emptyset$

(27) $\iff \mathbf{A} \bigtriangleup \mathbf{B} = \emptyset$

(28) $\mathbf{A} = \mathbf{B} \iff (\mathbf{A} \subseteq \mathbf{B}) \wedge (\mathbf{B} \subseteq \mathbf{A})$

(29) $\mathbf{A} \subseteq \mathbf{B} \iff \overline{\mathbf{A}} \cup \mathbf{B} = \mathbf{G} \iff \mathbf{A} \cap \overline{\mathbf{B}} = \emptyset$

(30) $\iff \mathbf{A} \cap \mathbf{B} = \mathbf{A} \iff \mathbf{A} \cup \mathbf{B} = \mathbf{B}$

(31) $\mathbf{A} \subseteq \mathbf{B} \iff \overline{\mathbf{B}} \subseteq \overline{\mathbf{A}}\,; \qquad \mathbf{A} = \mathbf{B} \iff \overline{\mathbf{A}} = \overline{\mathbf{B}}$

(32) $\mathbf{A} \subseteq \mathbf{B} \cup \mathbf{C} \iff \mathbf{A} \cap \overline{\mathbf{B}} \subseteq \mathbf{C}$

(33) $(\mathbf{A} = \mathbf{B}) \wedge (\mathbf{B} = \mathbf{C}) \implies \mathbf{A} = \mathbf{C}$

(34) $(\mathbf{A} \subseteq \mathbf{B}) \wedge (\mathbf{B} \subseteq \mathbf{C}) \implies \mathbf{A} \subseteq \mathbf{C}$

(35) $(\mathbf{A} \subseteq \mathbf{B}) \wedge (\mathbf{C} \subseteq \mathbf{D}) \implies (\mathbf{A} \cup \mathbf{C}) \subseteq (\mathbf{B} \cup \mathbf{D})$

(36) $(\mathbf{A} \subseteq \mathbf{B}) \wedge (\mathbf{C} \subseteq \mathbf{D}) \implies (\mathbf{A} \cap \mathbf{C}) \subseteq (\mathbf{B} \cap \mathbf{D})$

(37) $\mathbf{A} \subset \mathbf{B} \implies \mathbf{B} \setminus \mathbf{A} \neq \emptyset$

Comparison of notation using unary predicates vs. sets:

$$
\begin{array}{ccccc}
(\forall_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Px}) \wedge (\forall_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Qx}) & \overset{F5}{\Longleftrightarrow} & \forall_{\mathbf{x}\in\mathbf{M}}\ (\mathbf{Px} \wedge \mathbf{Qx}) & & \\
\Downarrow & & & & \\
(\forall_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Px}) \wedge (\exists_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Qx}) & \overset{G6}{\Longrightarrow} & \exists_{\mathbf{x}\in\mathbf{M}}\ (\mathbf{Px} \wedge \mathbf{Qx}) & \overset{G4}{\Longrightarrow} & (\exists_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Px}) \wedge (\exists_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Qx}) \\
\Downarrow & & & & \Downarrow \\
(\forall_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Px}) \vee (\forall_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Qx}) & \overset{G5}{\Longrightarrow} & \forall_{\mathbf{x}\in\mathbf{M}}\ (\mathbf{Px} \vee \mathbf{Qx}) & \overset{G3}{\Longrightarrow} & (\forall_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Px}) \vee (\exists_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Qx}) \\
 & & & & \Downarrow \\
 & & \exists_{\mathbf{x}\in\mathbf{M}}\ (\mathbf{Px} \vee \mathbf{Qx}) & \overset{F6}{\Longleftrightarrow} & (\exists_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Px}) \vee (\exists_{\mathbf{x}\in\mathbf{M}}\ \mathbf{Qx})
\end{array}
$$

$$\mathbf{Px} \Longleftrightarrow \mathbf{x} \in \mathbf{A}; \qquad \mathbf{Qx} \Longleftrightarrow \mathbf{x} \in \mathbf{B}$$

$$\mathbf{A} \subseteq \mathbf{M}; \qquad \mathbf{B} \subseteq \mathbf{M}; \qquad \mathbf{M} \neq \emptyset$$

$$
\begin{array}{ccccc}
(\mathbf{A} = \mathbf{M}) \wedge (\mathbf{B} = \mathbf{M}) & \overset{F5}{\Longleftrightarrow} & \mathbf{A} \cap \mathbf{B} = \mathbf{M} & & \\
\Downarrow & & & & \\
(\mathbf{A} = \mathbf{M}) \wedge (\mathbf{B} \neq \emptyset) & \overset{G6}{\Longrightarrow} & \mathbf{A} \cap \mathbf{B} \neq \emptyset & \overset{G4}{\Longrightarrow} & (\mathbf{A} \neq \emptyset) \wedge (\mathbf{B} \neq \emptyset) \\
\Downarrow & & & & \Downarrow \\
(\mathbf{A} = \mathbf{M}) \vee (\mathbf{B} = \mathbf{M}) & \overset{G5}{\Longrightarrow} & \mathbf{A} \cup \mathbf{B} = \mathbf{M} & \overset{G3}{\Longrightarrow} & (\mathbf{A} = \mathbf{M}) \vee (\mathbf{B} \neq \emptyset) \\
 & & & & \Downarrow \\
 & & \mathbf{A} \cup \mathbf{B} \neq \emptyset & \overset{F6}{\Longleftrightarrow} & (\mathbf{A} \neq \emptyset) \vee (\mathbf{B} \neq \emptyset)
\end{array}
$$

4 Relations

4.1 Definitions

Ordered pair (of objects) : $(\mathbf{x}, \mathbf{y})$; $\mathbf{x}$: first component, $\mathbf{y}$: second component

$$\mathbf{x} \neq \mathbf{y} \Longrightarrow (\mathbf{x}, \mathbf{y}) \neq (\mathbf{y}, \mathbf{x})$$

Ordered triple : $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \triangleq ((\mathbf{x}, \mathbf{y}), \mathbf{z})$

Ordered quadruple : $(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}) \triangleq ((\mathbf{w}, \mathbf{x}, \mathbf{y}), \mathbf{z})$

Ordered n-tuple : $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\mathrm{n}) \triangleq \underline{\mathbf{x}}_{<\mathrm{n}>};\quad \underline{\mathbf{x}}$

Cartesian product (Cross product) : $\mathbf{A} \times \mathbf{B} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathbf{A} \wedge \mathbf{y} \in \mathbf{B}\}$

$$(\mathbf{x}, \mathbf{y}) \in \mathbf{A} \times \mathbf{B} \Longleftrightarrow \mathbf{x} \in \mathbf{A} \wedge \mathbf{y} \in \mathbf{B}$$

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| \cdot |\mathbf{B}|;$$

Notation : $\mathbf{A} \times \mathbf{A} = \mathbf{A}^2$

Example: $\mathbf{A} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

$\mathbf{B} = \{1, 2, \mathbf{a}\}$

Table :

A \ B	1	2	a
a	(**a**,1)	(**a**,2)	(**a**,**a**)
b	(**b**,1)	(**b**,2)	(**b**,**a**)
c	(**c**, 1)	(**c**, 2)	(**c**, **a**)

$\mathbf{A} \times \mathbf{B}$

$$\mathbf{N} = \{1, \ldots, \nu, \ldots, \mathrm{n}\}$$

$$\mathop{\times}_{\nu \in \mathbf{N}} \mathbf{A}_\nu = \mathop{\times}_{\nu=1}^{\mathrm{n}} \mathbf{A}_\nu = \mathbf{A}_1 \times \mathbf{A}_2 \times \ldots \times \mathbf{A}_\nu \times \ldots \times \mathbf{A}_\mathrm{n}$$

$$= \{(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\mathrm{n}) \mid \forall_{\nu \in \mathbf{N}}\ \mathbf{x}_\nu \in \mathbf{A}_\nu\}$$

$$\left|\mathop{\times}_{\nu=1}^{\mathrm{n}} \mathbf{A}_\nu\right| = \prod_{\nu=1}^{\mathrm{n}} |\mathbf{A}_\nu|; \qquad \forall_{\nu \in \mathbf{N}} (\mathbf{A}_\nu = \mathbf{A}) \Longrightarrow \mathop{\times}_{\nu=1}^{\mathrm{n}} \mathbf{A}_\nu = \mathbf{A}^\mathrm{n}$$

$\mathbf{R}$ is a binary relation from set $\mathbf{A}$ to set $\mathbf{B}$ (over $\mathbf{A} \times \mathbf{B}$) $:\Longleftrightarrow$ $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$
Heterogeneous relation
Inhomogeneous relation

$\mathbf{R}$ is a binary relation within the set $\mathbf{A}$ (over $\mathbf{A}^2$); $\mathbf{A} = \mathbf{B}$ $:\Longleftrightarrow$ $\mathbf{R} \subseteq \mathbf{A}^2$
Homogeneous relation

$\mathbf{A}$: domain

$\mathbf{B}$: codomain

$$\mathbf{R} = \{(\mathbf{x}, \mathbf{y}) \mid (\mathbf{x}, \mathbf{y}) \in \mathbf{R}\}_{\mathbf{A}\times\mathbf{B}} \quad \triangleq \quad \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{xRy}\}_{\mathbf{A}\times\mathbf{B}}$$

$(\mathbf{x}, \mathbf{y}) \in \mathbf{R} \Longleftrightarrow \mathbf{xRy} \Longleftrightarrow \mathbf{Rxy}$ (Parlance: $\mathbf{x}$ is related to $\mathbf{y}$)

Empty relation: $\emptyset$
Universal relation: $\mathbf{A} \times \mathbf{B}$ or $\mathbf{A}^2$
Identity relation: $\mathbf{I} = \{(\mathbf{x}, \mathbf{x}) \mid \mathbf{x} \in \mathbf{A}\}_{\mathbf{A}^2}$;

$$\mathbf{x} = \mathbf{y} \Longleftrightarrow (\mathbf{x}, \mathbf{y}) \in \mathbf{I} \Longleftrightarrow \mathbf{xIy}$$
$$\mathbf{x} \neq \mathbf{y} \Longleftrightarrow (\mathbf{x}, \mathbf{y}) \notin \mathbf{I} \Longleftrightarrow \mathbf{x\bar{I}y}$$

$\mathbf{R}$ is an n-ary relation between the sets $\mathbf{A}_1, \ldots, \mathbf{A}_n$ (over $\bigtimes_{\nu\in\mathbf{N}} \mathbf{A}_\nu$) $:\Longleftrightarrow$ $\mathbf{R} \subseteq \bigtimes_{\nu\in\mathbf{N}} \mathbf{A}_\nu$

$\mathbf{R}$ is an n-ary relation within the set $\mathbf{A}$ (over $\mathbf{A}^n$) $:\Longleftrightarrow$ $\mathbf{R} \subseteq \mathbf{A}^n$

$$\mathbf{R} = \{(\mathbf{x}_1, \ldots, \mathbf{x}_n) \mid \mathbf{Rx}_1 \ldots \mathbf{x}_n\}_{\bigtimes_{\nu\in\mathbf{N}} \mathbf{A}_\nu} \quad \triangleq \quad \{\underline{\mathbf{x}}_{<n>} \mid \underline{\mathbf{x}}_{<n>} \in \mathbf{R}\}_{\bigtimes_{\nu\in\mathbf{N}} \mathbf{A}_\nu}$$

$\mathbf{R} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{xRy}\}_{\mathbf{A}\times\mathbf{B}}$; $\quad \mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$; $\quad \mathbf{R} \in \mathbf{P}(\mathbf{A} \times \mathbf{B})$
$\mathbf{S} = \{(\mathbf{y},\mathbf{z}) \mid \mathbf{ySz}\}_{\mathbf{B}\times\mathbf{C}}$; $\quad \mathbf{S} \subseteq \mathbf{B} \times \mathbf{C}$; $\quad \mathbf{S} \in \mathbf{P}(\mathbf{B} \times \mathbf{C})$

Complementary relation $\overline{\mathbf{R}}$ of $\mathbf{R}$:

$$\overline{\mathbf{R}} = \{(\mathbf{x},\mathbf{y}) \mid (\mathbf{x},\mathbf{y}) \notin \mathbf{R}\}_{\mathbf{A}\times\mathbf{B}} = (\mathbf{A} \times \mathbf{B}) \setminus \mathbf{R}$$

Converse (transposed, reciprocal) relation $\mathbf{R}^{-1}$ of $\mathbf{R}$:

$$\mathbf{R}^{-1} = \{(\mathbf{y},\mathbf{x}) \mid (\mathbf{x},\mathbf{y}) \in \mathbf{R}\}_{\mathbf{B}\times\mathbf{A}}; \qquad \mathbf{R}^{-1} \subseteq \mathbf{B} \times \mathbf{A}$$

Composition of relations $\mathbf{R}$ and $\mathbf{S}$:

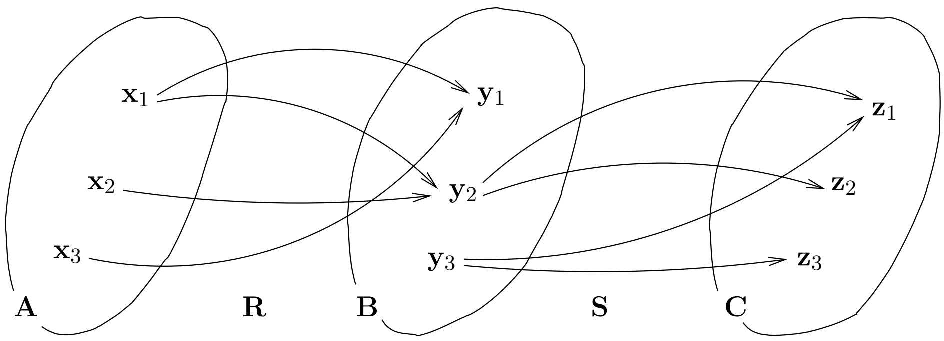
$$\mathbf{R} \circ \mathbf{S} = \mathbf{RS} = \{(\mathbf{x},\mathbf{z}) \mid \underset{\mathbf{y}\in\mathbf{B}}{\exists} [(\mathbf{x},\mathbf{y}) \in \mathbf{R} \ \wedge \ (\mathbf{y},\mathbf{z}) \in \mathbf{S}]\}_{\mathbf{A}\times\mathbf{C}}$$

$$\mathbf{xRSz} \iff \underset{\mathbf{y}\in\mathbf{B}}{\exists} [\mathbf{xRy} \wedge \mathbf{ySz}] \ ; \qquad \mathbf{RS} \subseteq \mathbf{A} \times \mathbf{C}$$

Strength of operators: $\overline{}$, $^{-1}$, $\circ$, $\cap$, $\cup$, ...

Example:

arrow
diagram



$\mathbf{R} = \{(\mathbf{x}_1,\mathbf{y}_1), (\mathbf{x}_1,\mathbf{y}_2), (\mathbf{x}_2,\mathbf{y}_2), (\mathbf{x}_3,\mathbf{y}_1)\}$; $\quad \mathbf{S} = \{(\mathbf{y}_2,\mathbf{z}_1), (\mathbf{y}_2,\mathbf{z}_2), (\mathbf{y}_3,\mathbf{z}_1), (\mathbf{y}_3,\mathbf{z}_3)\}$
$\mathbf{RS} = \{(\mathbf{x}_1,\mathbf{z}_1), (\mathbf{x}_1,\mathbf{z}_2), (\mathbf{x}_2,\mathbf{z}_1), (\mathbf{x}_2,\mathbf{z}_2)\}$

$\mathbf{x}$ is child of $\mathbf{y}$ $\iff$ $\mathbf{xRy}$
$\mathbf{y}$ has a brother $\mathbf{z}$ $\iff$ $\mathbf{ySz}$
$\mathbf{x}$ has an uncle $\mathbf{z}$ $\iff$ $\mathbf{xRSz}$

4.2 Binary graphs

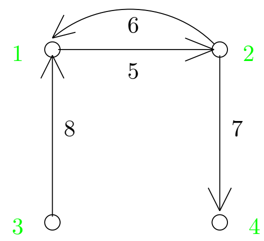
Directed (binary) graph	:	$\mathbf{G} = (\mathbf{A}, \mathbf{R})$
(DAG, Digraph), graph of $\mathbf{R}$	or	$\mathbf{G} = (\mathbf{A} \cup \mathbf{B}, \mathbf{R})$
Set of nodes (points, vertices) of $\mathbf{G}$	:	$\mathbf{A} = \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}\}$
Adjacency relation of $\mathbf{G}$	:	$\mathbf{R} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{xRy}\}_{\mathbf{A}^2}$
	or	$\mathbf{R} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{xRy}\}_{\mathbf{A} \times \mathbf{B}}$
Order of a graph	:	$\mid \mathbf{A} \mid$
Complement of $\mathbf{G}$	:	$\overline{\mathbf{G}} = (\mathbf{A}, \overline{\mathbf{R}})$
$\mathbf{G}$ is complete	$:\Longleftrightarrow$	$\mathbf{R} = \mathbf{A} \times \mathbf{A} = \mathbf{A}^2$
$\mathbf{G}$ is empty	$:\Longleftrightarrow$	$\mathbf{R} = \emptyset$

$$\mathbf{x} \circ\!\!\longrightarrow\!\!\circ \ \mathbf{y}$$

Directed edge (arc, arrow)	:	$(\mathbf{x}, \mathbf{y})$ incident with $\mathbf{x}$ and $\mathbf{y}$
Initial, terminal node	:	$\mathbf{x}, \mathbf{y}$
Loop	:	$(\mathbf{x}, \mathbf{x})$; $\mathbf{x}$
plain graph	:	no multiple edges
$\mathbf{x}$ is predecessor of $\mathbf{y}$ ($\mathbf{y}$ is successor of $\mathbf{x}$)	$:\Longleftrightarrow$	$(\mathbf{x}, \mathbf{y}) \in \mathbf{R} \Longleftrightarrow \mathbf{xRy} \Longleftrightarrow \mathbf{yR}^{-1}\mathbf{x}$
Set of successors of $\mathbf{x}$	:	$\mathbf{\Gamma}^{+}(\mathbf{x}) = \mathrm{suc}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) = \{\mathbf{y} \mid \mathbf{xRy}\}_{\mathbf{A}}$
Set of predecessors of $\mathbf{x}$	:	$\mathbf{\Gamma}^{-}(\mathbf{x}) = \mathrm{pre}(\mathbf{x}) = \mathbf{R}^{-1}(\mathbf{x}) = \{\mathbf{y} \mid \mathbf{yRx}\}_{\mathbf{A}}$
Set of proper successors or predecessors of $\mathbf{x}$	:	$\mathbf{\Gamma}^{+}(\mathbf{x}) \setminus \{\mathbf{x}\}$ or $\mathbf{\Gamma}^{-}(\mathbf{x}) \setminus \{\mathbf{x}\}$

Degree of node $\mathbf{x}$	:	$\mathbf{d}^+(\mathbf{x}) = \mid \mathbf{\Gamma}^+(\mathbf{x}) \mid$; $\mathbf{d}^-(\mathbf{x}) = \mid \mathbf{\Gamma}^-(\mathbf{x}) \mid$ outdegree (successor degree); indegree (predecessor degree)
$(\mathbf{x},\mathbf{y})\in\mathbf{R}$	$\Longleftrightarrow$	$\mathbf{y}\in\mathbf{\Gamma}^+(\mathbf{x})$
$(\mathbf{y},\mathbf{x})\in\mathbf{R} \Longleftrightarrow (\mathbf{x},\mathbf{y})\in\mathbf{R}^{-1}$	$\Longleftrightarrow$	$\mathbf{y}\in\mathbf{\Gamma}^-(\mathbf{x})$
$\mathbf{x}$ has a loop	$:\Longleftrightarrow$	$(\mathbf{x},\mathbf{x})\in\mathbf{R} \Longleftrightarrow \mathbf{xRx}$
$\mathbf{G}$ has no loops	$:\Longleftrightarrow$	$\mathbf{R} \subseteq \bar{\mathbf{I}}$
$\mathbf{x}$ is initial (source)	$:\Longleftrightarrow$	$\mathbf{\Gamma}^-(\mathbf{x}) = \emptyset \Longleftrightarrow \mathbf{d}^-(\mathbf{x}) = 0$
$\mathbf{x}$ is terminal (sink)	$:\Longleftrightarrow$	$\mathbf{\Gamma}^+(\mathbf{x}) = \emptyset \Longleftrightarrow \mathbf{d}^+(\mathbf{x}) = 0$
$\mathbf{x}$ is functional	$:\Longleftrightarrow$	$\mathbf{d}^+(\mathbf{x}) \leq 1$
$\mathbf{x}$ is cofunctional	$:\Longleftrightarrow$	$\mathbf{d}^-(\mathbf{x}) \leq 1$
$\mathbf{x}$ is branch point	$:\Longleftrightarrow$	$\mathbf{d}^+(\mathbf{x}) \geq 2$
$\mathbf{x}$ is reconvergence point	$:\Longleftrightarrow$	$\mathbf{d}^-(\mathbf{x}) \geq 2$
set of successors and predecessors of $\mathbf{A}' \subseteq \mathbf{A}$	:	$\bigcup_{\mathbf{x}\in\mathbf{A}'} \mathbf{\Gamma}^+(\mathbf{x})$ or $\bigcup_{\mathbf{x}\in\mathbf{A}'} \mathbf{\Gamma}^-(\mathbf{x})$
set of initial and terminal vertices in $\mathbf{G}$	:	$\overline{\bigcup_{\mathbf{x}\in\mathbf{A}} \mathbf{\Gamma}^+(\mathbf{x})}$ or $\overline{\bigcup_{\mathbf{x}\in\mathbf{A}} \mathbf{\Gamma}^-(\mathbf{x})}$
partial subgraph $\mathbf{G}'$ of $\mathbf{G}$	:	$\mathbf{A}' \subseteq \mathbf{A}$; $\mathbf{R}' \subseteq \mathbf{R} \cap (\mathbf{A}' \times \mathbf{A}')$
(induced) subgraph $\mathbf{G}''$ of $\mathbf{G}$	:	$\mathbf{A}'' \subseteq \mathbf{A}$; $\mathbf{R}'' = \mathbf{R} \cap (\mathbf{A}'' \times \mathbf{A}'')$
spanning subgraph $\mathbf{G}'''$ of $\mathbf{G}$	:	$\mathbf{A}''' = \mathbf{A}$; $\mathbf{R}''' \subseteq \mathbf{R}$

Representations of $\mathbf{G} = (\mathbf{A}, \mathbf{R})$
Example: $\mathbf{A} = \{\, \mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4 \}$; $\mathbf{R} = \{\, (\mathbf{x}_1, \mathbf{x}_2)\ ,\ (\mathbf{x}_2, \mathbf{x}_1)\ ,\ (\mathbf{x}_2, \mathbf{x}_4)\ ,\ (\mathbf{x}_3, \mathbf{x}_1) \,\}$



Arrow-diagram

(1, 2)
(2, 1)
(2, 4)
(3, 1)

List of ordered pairs

$\mathbf{x}$	$\boldsymbol{\Gamma}^{+}(\mathbf{x})$
1	$\{2\}$
2	$\{1, 4\}$
3	$\{1\}$
4	$\{\ \}$

Successor table/list

$\mathbf{x}$	$\boldsymbol{\Gamma}^{-}(\mathbf{x})$
1	$\{2, 3\}$
2	$\{1\}$
3	$\{\ \}$
4	$\{2\}$

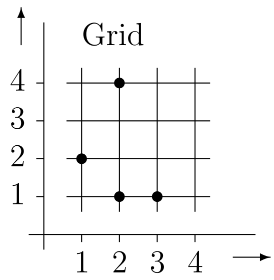
Predecessor table/list

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

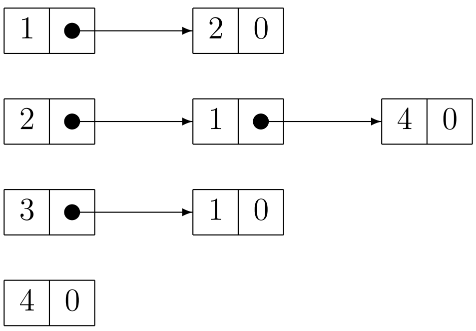
Adjacency matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Incidence-matrix



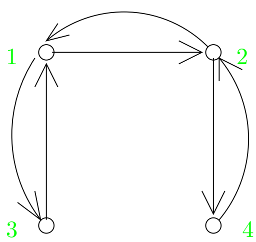
Cartesian coordinate system



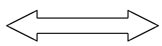
Chained successor list

i	1	2	3	4	5	6	7	8
NODE [i]					2	1	4	1
NEXT [i]	5	6	8	0	0	7	0	0

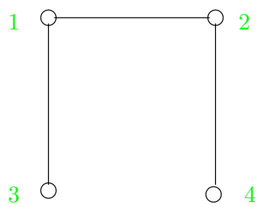
Successor table



Symmetric directed graph



Undirected graph



$\mathbf{P}_\mathrm{n}(\mathbf{z}_0,\ \mathbf{z}_1,\ \ldots,\mathbf{z}_\mathrm{n})$ is a (directed) path of length n from node $\mathbf{z}_0$ to $\mathbf{z}_\mathrm{n}$ in $\mathbf{G} = (\mathbf{A}, \mathbf{R})$	$:\Longleftrightarrow$	$\underset{1 \le \nu \le \mathrm{n}}{\forall}\ \mathbf{z}_{\nu-1}\mathbf{R}\mathbf{z}_\nu$
$\mathbf{V}_\mathrm{n}(\mathbf{z}_0,\ \mathbf{z}_1,\ \ldots,\mathbf{z}_\mathrm{n})$ is an undirected path of length n from node $\mathbf{z}_0$ to $\mathbf{z}_\mathrm{n}$ in $\mathbf{G} = (\mathbf{A}, \mathbf{R})$	$:\Longleftrightarrow$	$\underset{1 \le \nu \le \mathrm{n}}{\forall}\ [\,\mathbf{z}_{\nu-1}\mathbf{R}\mathbf{z}_\nu \vee \mathbf{z}_\nu\mathbf{R}\mathbf{z}_{\nu-1}\,]$
Open path	:	$\mathbf{z}_0 \ \neq\ \mathbf{z}_\mathrm{n}$
Cycle (closed path)	:	$\mathbf{z}_0 \ =\ \mathbf{z}_\mathrm{n}$
Trivial cycle	:	$\mathbf{P}_1(\mathbf{z}, \mathbf{z}),\ \mathbf{P}_2(\mathbf{z}, \mathbf{z}, \mathbf{z}),\ \ldots$
Proper Cycle	:	trivial cycles excluded
Elementary (simple) path	:	$\mathbf{z}_0,\ \ldots,\ \mathbf{z}_{\mathrm{n}-1}$ and $\mathbf{z}_1,\ \ldots,\ \mathbf{z}_\mathrm{n}$ differ from each other, respectively
$\mathbf{y}$ is accessible from $\mathbf{x}$ in $\mathbf{G}$ ($\mathbf{y}$ is descendent of $\mathbf{x}$ in $\mathbf{G}$, $\mathbf{x}$ is ancestor of $\mathbf{y}$ in $\mathbf{G}$)	$:\Longleftrightarrow$	$\mathbf{P}(\mathbf{x},\ \ldots,\mathbf{y})$ exists in $\mathbf{G}$ (empty path $\mathbf{P}_0(\mathbf{x})$ included)
$\mathbf{x}$ and $\mathbf{y}$ are mutually accessible in $\mathbf{G}$ (cycle through $\mathbf{x}$ and $\mathbf{y}$)	$:\Longleftrightarrow$	$\mathbf{P}(\mathbf{x},\ \ldots,\mathbf{y})$ AND $\mathbf{P}(\mathbf{y},\ \ldots,\mathbf{x})$ exist in $\mathbf{G}$
$\mathbf{x}$ and $\mathbf{y}$ can be connected in $\mathbf{G}$	$:\Longleftrightarrow$	$\mathbf{V}(\mathbf{x},\ \ldots,\mathbf{y})$ exists in $\mathbf{G}$ ($\mathbf{V}_0(\mathbf{x})$ included)
$\mathbf{G} = (\mathbf{A}, \mathbf{R})$ is connected	$:\Longleftrightarrow$	$\underset{\mathbf{x},\mathbf{y} \in \mathbf{A}}{\forall}\ [\,\mathbf{V}(\mathbf{x},\ \ldots,\mathbf{y})$ exists in $\mathbf{G}\,]$
$\mathbf{G} = (\mathbf{A}, \mathbf{R})$ is strongly connected	$:\Longleftrightarrow$	$\underset{\mathbf{x},\mathbf{y} \in \mathbf{A}}{\forall}\ [\,\mathbf{P}(\mathbf{x},\ \ldots,\mathbf{y})$ exists in $\mathbf{G}\,]$
$\mathbf{C} = (\mathbf{A}_\mathrm{C}, \mathbf{R}_\mathrm{C})$ is a complete subgraph (clique) of $\mathbf{G} = (\mathbf{A}, \mathbf{R})$	$:\Longleftrightarrow$	$(\mathbf{A}_\mathrm{C} \subseteq \mathbf{A}) \wedge (\mathbf{A}_\mathrm{C}^2 \subseteq \mathbf{R})$ $(\mathbf{R}_\mathrm{C} \ =\ \mathbf{A}_\mathrm{C}^2)$

(Note: Literature also uses “walk” instead of “path”, and “path” instead of “elementary path”.)

4.3 Properties of relations

$\mathbf{R} = \{ (\mathbf{x},\mathbf{y}) \mid \mathbf{xRy} \}_{\mathbf{A}^2}$; $\mathbf{R} \subseteq \mathbf{A}^2$;

$\mathbf{R}$ is reflexive $:\Longleftrightarrow \forall_{\mathbf{x}\in\mathbf{A}} \mathbf{xRx} \Longleftrightarrow \mathbf{I} \subseteq \mathbf{R}$

$\mathbf{R}$ is irreflexive $:\Longleftrightarrow \forall_{\mathbf{x}\in\mathbf{A}} \neg\mathbf{xRx} \Longleftrightarrow \mathbf{I} \subseteq \overline{\mathbf{R}} \Longleftrightarrow \mathbf{R} \subseteq \bar{\mathbf{I}}$

(anti-reflexive)

$\mathbf{R}$ is symmetric $:\Longleftrightarrow \forall_{\mathbf{x},\mathbf{y}\in\mathbf{A}} [\mathbf{xRy} \longrightarrow \mathbf{yRx}] \Longleftrightarrow \mathbf{R} = \mathbf{R}^{-1}$

$\mathbf{R}$ is antisymmetric $:\Longleftrightarrow \forall_{\mathbf{x},\mathbf{y}\in\mathbf{A}} [\mathbf{xRy} \wedge \mathbf{yRx} \longrightarrow \mathbf{x}=\mathbf{y}] \Longleftrightarrow \mathbf{R} \cap \mathbf{R}^{-1} \subseteq \mathbf{I}$

$\Longleftrightarrow \mathbf{R}^{-1} \subseteq \overline{\mathbf{R}} \cup \mathbf{I}$

$\mathbf{R}$ is asymmetric $:\Longleftrightarrow \forall_{\mathbf{x},\mathbf{y}\in\mathbf{A}} [\mathbf{xRy} \longrightarrow \neg\mathbf{yRx}]$

$\Longleftrightarrow \mathbf{R} \cap \mathbf{R}^{-1} = \emptyset \Longleftrightarrow \mathbf{R} \subseteq (\overline{\mathbf{R}})^{-1} \Longleftrightarrow \mathbf{R}^{-1} \subseteq \overline{\mathbf{R}}$

$\mathbf{R}$ is transitive $:\Longleftrightarrow \forall_{\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{A}} [\mathbf{xRy} \wedge \mathbf{yRz} \longrightarrow \mathbf{xRz}] \Longleftrightarrow \mathbf{R}^2 \subseteq \mathbf{R}$

$\mathbf{R}$ is intransitive $:\Longleftrightarrow \forall_{\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{A}} [\mathbf{xRy} \wedge \mathbf{yRz} \longrightarrow \neg\mathbf{xRz}] \Longleftrightarrow \mathbf{R}^2 \subseteq \overline{\mathbf{R}}$

$\mathbf{R}$ is connex $:\Longleftrightarrow \forall_{\mathbf{x},\mathbf{y}\in\mathbf{A}} [\mathbf{xRy} \vee \mathbf{yRx}] \Longleftrightarrow \overline{\mathbf{R}}$ is asymmetric

(all elements of A are comparable) $\Longleftrightarrow \overline{\mathbf{R}} \cap (\overline{\mathbf{R}})^{-1} = \emptyset \Longleftrightarrow \mathbf{R} \cup \mathbf{R}^{-1} = \mathbf{A}^2$

$\mathbf{R}$ is semiconnex $:\Longleftrightarrow \forall_{\mathbf{x},\mathbf{y}\in\mathbf{A}} [(\mathbf{x}\neq\mathbf{y}) \longrightarrow \mathbf{xRy} \vee \mathbf{yRx}] \Longleftrightarrow \overline{\mathbf{R}}$ is antisymmetric

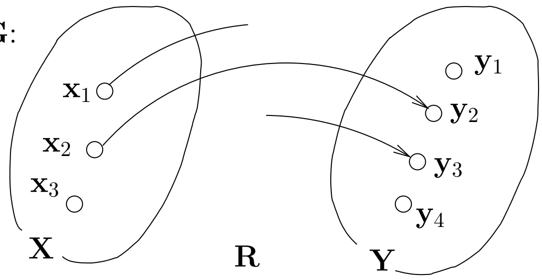
(all elements of A are comparable, but not necessarily to themselves) $\Longleftrightarrow \overline{\mathbf{R}} \cap (\overline{\mathbf{R}})^{-1} \subseteq \mathbf{I} \Longleftrightarrow \bar{\mathbf{I}} \subseteq \mathbf{R} \cup \mathbf{R}^{-1} \Longleftrightarrow \overline{\mathbf{R}} \subseteq \mathbf{R}^{-1} \cup \mathbf{I}$

$\mathbf{R} \subseteq \mathbf{X} \times \mathbf{Y}$

$\mathbf{R} = \{ (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y} \mid \mathbf{xRy} \}$

$\mathbf{G} = (\mathbf{X} \cup \mathbf{Y}, \mathbf{R})$

G:



$\mathbf{X}$: domain; $\mathbf{D}_\mathrm{R} \subseteq \mathbf{X}$

$\mathbf{Y}$: codomain; $\mathbf{W}_\mathrm{R} \subseteq \mathbf{Y}$ ($\mathbf{W}_\mathrm{R}$: range)

$$\mathbf{D}_\mathrm{R} = \{\mathbf{x} \in \mathbf{X} \mid \mathbf{d}^+(\mathbf{x}) \geq 1\}; \quad \mathbf{W}_\mathrm{R} = \{\mathbf{y} \in \mathbf{Y} \mid \mathbf{d}^-(\mathbf{y}) \geq 1\};$$

$$\mathbf{x} \in \mathbf{D}_\mathrm{R} \Longleftrightarrow \mathbf{d}^+(\mathbf{x}) \geq 1; \qquad \mathbf{y} \in \mathbf{W}_\mathrm{R} \Longleftrightarrow \mathbf{d}^-(\mathbf{y}) \geq 1;$$

"left One": $\mathbf{I}_\mathrm{X} \subseteq \mathbf{X}^2$; $\mathbf{I}_\mathrm{X} \cup \bar{\mathbf{I}}_\mathrm{X} = \mathbf{X}^2$; "right One": $\mathbf{I}_\mathrm{Y} \subseteq \mathbf{Y}^2$; $\mathbf{I}_\mathrm{Y} \cup \bar{\mathbf{I}}_\mathrm{Y} = \mathbf{Y}^2$;

$$\mathbf{I}_\mathrm{X} \circ \mathbf{R} = \mathbf{R} \circ \mathbf{I}_\mathrm{Y} = \mathbf{R}\ ; \qquad \mathbf{R}^{-1} \circ \mathbf{I}_\mathrm{X} = \mathbf{I}_\mathrm{Y} \circ \mathbf{R}^{-1} = \mathbf{R}^{-1};$$

R is total $:\Longleftrightarrow \forall_{\mathbf{x}\in\mathbf{X}} \mathbf{d}^+(\mathbf{x}) \geq 1 \Longleftrightarrow \mathbf{D}_\mathrm{R} = \mathbf{X} \Longleftrightarrow \forall_{\mathbf{x}\in\mathbf{X}} \exists_{\mathbf{y}\in\mathbf{Y}} \mathbf{xRy}$
$\Longleftrightarrow \mathbf{I}_\mathrm{X} \subseteq \mathbf{RR}^{-1} \Longleftrightarrow \overline{\mathbf{R}} \subseteq \mathbf{R}\bar{\mathbf{I}}_\mathrm{Y} \Longleftrightarrow \mathbf{X} \times \mathbf{Y} = \mathbf{R} \cup \mathbf{R}\bar{\mathbf{I}}_\mathrm{Y}$

R is surjective $:\Longleftrightarrow \forall_{\mathbf{y}\in\mathbf{Y}} \mathbf{d}^-(\mathbf{y}) \geq 1 \Longleftrightarrow \mathbf{W}_\mathrm{R} = \mathbf{Y} \Longleftrightarrow$ $\mathbf{R}^{-1}$ is total
$\Longleftrightarrow \mathbf{I}_\mathrm{Y} \subseteq \mathbf{R}^{-1}\mathbf{R} \Longleftrightarrow \overline{\mathbf{R}} \subseteq \bar{\mathbf{I}}_\mathrm{X}\mathbf{R} \Longleftrightarrow \mathbf{X} \times \mathbf{Y} = \mathbf{R} \cup \bar{\mathbf{I}}_\mathrm{X}\mathbf{R}$

R is functional $:\Longleftrightarrow \forall_{\mathbf{x}\in\mathbf{X}} \mathbf{d}^+(\mathbf{x}) \leq 1 \Longleftrightarrow \forall_{\mathbf{x}\in\mathbf{X}} \forall_{\mathbf{y}_1,\mathbf{y}_2\in\mathbf{Y}} [\mathbf{xRy}_1 \wedge \mathbf{xRy}_2 \longrightarrow \mathbf{y}_1 = \mathbf{y}_2]$
$\Longleftrightarrow \mathbf{R}^{-1}\mathbf{R} \subseteq \mathbf{I}_\mathrm{Y} \Longleftrightarrow \mathbf{R}\bar{\mathbf{I}}_\mathrm{Y} \subseteq \overline{\mathbf{R}} \Longleftrightarrow \emptyset = \mathbf{R} \cap \mathbf{R}\bar{\mathbf{I}}_\mathrm{Y}$

R is injective $:\Longleftrightarrow \forall_{\mathbf{y}\in\mathbf{Y}} \mathbf{d}^-(\mathbf{y}) \leq 1 \Longleftrightarrow$ $\mathbf{R}^{-1}$ is functional
$\Longleftrightarrow \mathbf{RR}^{-1} \subseteq \mathbf{I}_\mathrm{X} \Longleftrightarrow \bar{\mathbf{I}}_\mathrm{X}\mathbf{R} \subseteq \overline{\mathbf{R}} \Longleftrightarrow \emptyset = \mathbf{R} \cap \bar{\mathbf{I}}_\mathrm{X}\mathbf{R}$

R is functional $:\Longleftrightarrow$ **G** has no branch point

R is injective $:\Longleftrightarrow$ **G** has no reconvergence point

R is a total function $:\Longleftrightarrow$ **R** is total AND functional
(mapping) $\Longleftrightarrow \forall_{\mathbf{x}\in\mathbf{X}} \mathbf{d}^+(\mathbf{x}) = 1 \Longleftrightarrow \mathbf{R}\bar{\mathbf{I}}_\mathrm{Y} = \overline{\mathbf{R}}$

R is bijective $:\Longleftrightarrow$ **R** is surjective AND injective
($\mathbf{R}^{-1}$ is a mapping) $\Longleftrightarrow \forall_{\mathbf{y}\in\mathbf{Y}} \mathbf{d}^-(\mathbf{y}) = 1 \Longleftrightarrow \bar{\mathbf{I}}_\mathrm{X}\mathbf{R} = \overline{\mathbf{R}}$

$\mathbf{R}$ is total	$:\Longleftrightarrow$	$\mathbf{D}_{\mathrm{R}} = \mathbf{X}$	;	$\mathbf{R}$ is surjective	$:\Longleftrightarrow$	$\mathbf{W}_{\mathrm{R}} = \mathbf{Y}$
$\mathbf{R}$ is functional	$\Longrightarrow$	$\lvert\mathbf{W}_{\mathrm{R}}\rvert \leq \lvert\mathbf{D}_{\mathrm{R}}\rvert$	;	$\mathbf{R}$ is injective	$\Longrightarrow$	$\lvert\mathbf{D}_{\mathrm{R}}\rvert \leq \lvert\mathbf{W}_{\mathrm{R}}\rvert$
$\mathbf{R}$ is a mapping	$\Longrightarrow$	$\lvert\mathbf{W}_{\mathrm{R}}\rvert \leq \lvert\mathbf{X}\rvert$	;	$\mathbf{R}$ is bijective	$\Longrightarrow$	$\lvert\mathbf{D}_{\mathrm{R}}\rvert \leq \lvert\mathbf{Y}\rvert$

$\mathbf{R}$ is total AND injective $\Longrightarrow$ $|\mathbf{X}| \leq |\mathbf{Y}|$

$\mathbf{R}$ is surjective AND functional $\Longrightarrow$ $|\mathbf{Y}| \leq |\mathbf{X}|$

$\mathbf{R}$ is a mapping AND bijective $\Longrightarrow$ $|\mathbf{X}| = |\mathbf{Y}|$
(bijection or permutation)

$$\left.\begin{array}{l}\mathbf{f} \text{ is a (total)}\\ \text{function (mapping)}\\ \mathbf{f}:\mathbf{X}\longrightarrow\mathbf{Y}\end{array}\right\} :\Longleftrightarrow \left\{\begin{array}{l}\text{for } \underline{\text{every}}\ \mathbf{x}\in\mathbf{X} \text{ there exists}\\ \text{exactly } \underline{\text{one}}\ \mathbf{y}\in\mathbf{Y},\\ \text{such that } \mathbf{xfy}\end{array}\right.$$

$$\mathbf{f} = \{\,(\mathbf{x},\mathbf{y})\ \mid\ \mathbf{f}(\mathbf{x}) = \mathbf{y}\}_{\mathbf{X}\times\mathbf{Y}}\ ; \quad \mathbf{f}(\mathbf{x}) = \mathbf{y} \Longleftrightarrow (\mathbf{x},\mathbf{y}) \in \mathbf{f} \Longleftrightarrow \mathbf{xfy} \Longleftrightarrow \mathbf{x} \mapsto \mathbf{y}$$

$\mathbf{X} = \mathbf{D}_{\mathrm{f}}$: domain of $\mathbf{f}$

$\mathbf{f}(\mathbf{X}) = \mathbf{W}_{\mathrm{f}}$: codomain of $\mathbf{f}$
(image of function $\mathbf{f}$)

$\mathbf{f}(\mathbf{X}_0) = \mathbf{Y}_0$: image of $\mathbf{X}_0$ given by $\mathbf{f}$; $\mathbf{X}_0 \subseteq \mathbf{X}$, $\mathbf{Y}_0 \subseteq \mathbf{Y}$

$\mathbf{x}$: argument of function $\mathbf{f}$

$\mathbf{y}$: function value, image value of $\mathbf{f}$

$\mathbf{f}$ is surjective function (surjection) $:\Longleftrightarrow$ $\underset{\mathbf{x}\in\mathbf{X}}{\forall}\ \underset{\mathbf{y}\in\mathbf{Y}}{\forall}\ [\mathbf{d}^+(\mathbf{x}) = 1 \wedge \mathbf{d}^-(\mathbf{y}) \geq 1]$

$\mathbf{f}$ is injective function (injection) $:\Longleftrightarrow$ $\underset{\mathbf{x}\in\mathbf{X}}{\forall}\ \underset{\mathbf{y}\in\mathbf{Y}}{\forall}\ [\mathbf{d}^+(\mathbf{x}) = 1 \wedge \mathbf{d}^-(\mathbf{y}) \leq 1]$

$\mathbf{f}$ is bijective function (bijection) $:\Longleftrightarrow$ $\underset{\mathbf{x}\in\mathbf{X}}{\forall}\ \underset{\mathbf{y}\in\mathbf{Y}}{\forall}\ [\mathbf{d}^+(\mathbf{x}) = 1 \wedge \mathbf{d}^-(\mathbf{y}) = 1]$

$\mathbf{p}$ is bijection (permutation) $:\Longleftrightarrow$ $\left(\mathbf{p}\circ\mathbf{p}^{-1} = \mathbf{I}_{\mathrm{X}}\right) \wedge \left(\mathbf{p}^{-1}\circ\mathbf{p} = \mathbf{I}_{\mathrm{Y}}\right)$

$\mathbf{p}:\mathbf{X}\longrightarrow\mathbf{Y}$; $|\mathbf{X}| = |\mathbf{Y}|$ $\qquad$ $\mathbf{p} = \{(\mathbf{x}_1,\mathbf{p}(\mathbf{x}_1)),(\mathbf{x}_2,\mathbf{p}(\mathbf{x}_2)),\ldots\}$

K) Laws for cartesian product

$(\mathbf{A} \neq \mathbf{B}) \wedge (\mathbf{A},\mathbf{B} \neq \emptyset) \Longrightarrow \mathbf{A}\times\mathbf{B} \neq \mathbf{B}\times\mathbf{A}$ (~~Commutativity~~)

(1) $(\mathbf{A}\times\mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B}\times\mathbf{C})$ Associativity

(2) $\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A}\times\mathbf{B}) \cup (\mathbf{A}\times\mathbf{C})$ Distributivity of $\times$ over $\cup, \cap$

(3) $\mathbf{A} \times (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A}\times\mathbf{B}) \cap (\mathbf{A}\times\mathbf{C})$

(4) $(\mathbf{A} \cup \mathbf{B}) \times \mathbf{C} = (\mathbf{A}\times\mathbf{C}) \cup (\mathbf{B}\times\mathbf{C})$

(5) $(\mathbf{A} \cap \mathbf{B}) \times \mathbf{C} = (\mathbf{A}\times\mathbf{C}) \cap (\mathbf{B}\times\mathbf{C})$

(6) $(\mathbf{A} = \emptyset) \vee (\mathbf{B} = \emptyset) \Longrightarrow \mathbf{A}\times\mathbf{B} = \emptyset$

(7) $(\mathbf{A}\times\mathbf{B}) \circ (\mathbf{B}\times\mathbf{C}) = \mathbf{A}\times\mathbf{C}, \quad \text{IF } \mathbf{B} \neq \emptyset$

L) Laws for converse relations

$\mathbf{Q},\mathbf{R} \subseteq \mathbf{A}\times\mathbf{B}$

(1) $(\mathbf{A}\times\mathbf{B})^{-1} = \mathbf{B}\times\mathbf{A}\,; \quad \emptyset^{-1} = \emptyset\,; \quad \mathbf{I}^{-1} = \mathbf{I}$

(2) $(\overline{\mathbf{R}})^{-1} = \overline{(\mathbf{R}^{-1})}\,; \quad (\mathbf{R}^{-1})^{-1} = \mathbf{R}$

(3) $(\mathbf{R} \cup \mathbf{Q})^{-1} = \mathbf{R}^{-1} \cup \mathbf{Q}^{-1}$

(4) $(\mathbf{R} \cap \mathbf{Q})^{-1} = \mathbf{R}^{-1} \cap \mathbf{Q}^{-1}$

(5) $(\mathbf{R} \setminus \mathbf{Q})^{-1} = \mathbf{R}^{-1} \setminus \mathbf{Q}^{-1}$

(6) $\mathbf{R} \subseteq \mathbf{Q} \Longleftrightarrow \mathbf{R}^{-1} \subseteq \mathbf{Q}^{-1}$

M) Laws for composition of relations

$\mathbf{Q},\mathbf{R} \subseteq \mathbf{A}\times\mathbf{B}\,; \quad \mathbf{S},\mathbf{T} \subseteq \mathbf{B}\times\mathbf{C}\,; \quad \mathbf{V} \subseteq \mathbf{A}\times\mathbf{C}\,; \quad \mathbf{W} \subseteq \mathbf{C}\times\mathbf{D}$

$(\mathbf{R} \neq \mathbf{S}) \wedge (\mathbf{R}, \mathbf{S} \neq \mathbf{I}, \emptyset) \Longrightarrow \mathbf{R} \circ \mathbf{S} \neq \mathbf{S} \circ \mathbf{R}$ (~~Commutativity~~)

(1) $(\mathbf{R} \circ \mathbf{S}) \circ \mathbf{W} = \mathbf{R} \circ (\mathbf{S} \circ \mathbf{W})$ Associativity

(2) $\mathbf{R} \circ (\mathbf{S} \cup \mathbf{T}) = \mathbf{R} \circ \mathbf{S} \cup \mathbf{R} \circ \mathbf{T}$ Distributivity of $\circ$ over $\cup$

(3) $(\mathbf{S} \cup \mathbf{T}) \circ \mathbf{W} = \mathbf{S} \circ \mathbf{W} \cup \mathbf{T} \circ \mathbf{W}$

(4) $\mathbf{R} \circ (\mathbf{S} \cap \mathbf{T}) \subseteq \mathbf{R} \circ \mathbf{S} \cap \mathbf{R} \circ \mathbf{T}$

(5) $(\mathbf{S} \cap \mathbf{T}) \circ \mathbf{W} \subseteq \mathbf{S} \circ \mathbf{W} \cap \mathbf{T} \circ \mathbf{W}$

(6) $(\mathbf{R} \circ \mathbf{S})^{-1} = \mathbf{S}^{-1} \circ \mathbf{R}^{-1}$

(7) $(\mathbf{Q} \subseteq \mathbf{R}) \wedge (\mathbf{S} \subseteq \mathbf{T}) \implies \mathbf{Q} \circ \mathbf{S} \subseteq \mathbf{R} \circ \mathbf{T}$ Monotony

(8) $\mathbf{R} \circ \mathbf{S} \subseteq \mathbf{V} \iff \mathbf{R}^{-1} \circ \overline{\mathbf{V}} \subseteq \overline{\mathbf{S}} \iff \overline{\mathbf{V}} \circ \mathbf{S}^{-1} \subseteq \overline{\mathbf{R}}$ Schröder rule

$\mathbf{R} \circ \mathbf{S} \subseteq \mathbf{R} \circ \mathbf{S} \iff \mathbf{R}^{-1} \circ \overline{\mathbf{R} \circ \mathbf{S}} \subseteq \overline{\mathbf{S}}$

(9) $\mathbf{R} \circ \mathbf{S} \cap \mathbf{V} \subseteq (\mathbf{R} \cap \mathbf{V} \circ \mathbf{S}^{-1}) \circ (\mathbf{S} \cap \mathbf{R}^{-1} \circ \mathbf{V})$ Dedekind formula

$\mathbf{R} \subseteq \mathbf{A}^2$

(10) $\mathbf{I} \circ \mathbf{R} = \mathbf{R} \circ \mathbf{I} = \mathbf{R}\ ; \quad \mathbf{R} \circ \emptyset = \emptyset \circ \mathbf{R} = \emptyset$

(11) $\mathbf{R} \neq \emptyset \implies \mathbf{A}^2 \circ \mathbf{R} \circ \mathbf{A}^2 = \mathbf{A}^2$

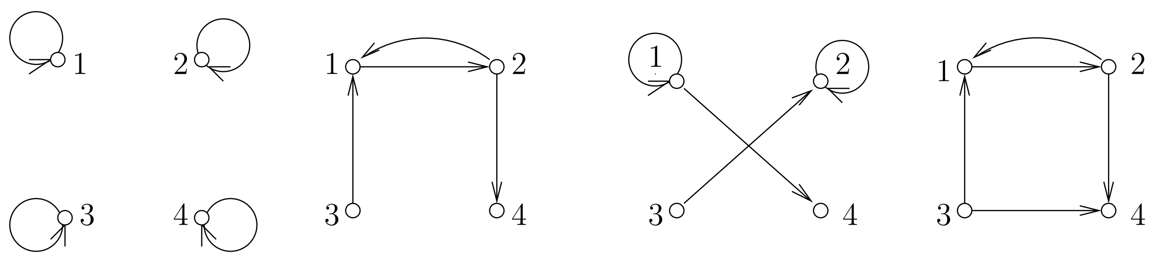
$$\left.\begin{array}{lll} (12) & \mathbf{R}^{\mathrm{m}} \circ \mathbf{R}^{\mathrm{n}} = & \mathbf{R}^{\mathrm{m+n}} \\ (13) & (\mathbf{R}^{\mathrm{m}})^{\mathrm{n}} = & \mathbf{R}^{\mathrm{m \cdot n}} \end{array}\right\} \text{ if } \left\{\begin{array}{l} \mathbf{R}^{\mathrm{n}} = \mathbf{R} \circ \mathbf{R} \circ \ldots \circ \mathbf{R} \\ \mathrm{m}, \mathrm{n} \in \mathbb{N}_0 \end{array}\right.$$

$\mathbf{R}^0 = \mathbf{I} = \{(\mathbf{x}, \mathbf{x}) \mid \mathbf{x} \in \mathbf{A}\}_{\mathbf{A}^2}$ Identity relation

$\mathrm{s} \in \mathbb{N}_0\ ;\ \mathrm{p} \in \mathbb{N}\ ; \quad \mathbf{R}^{\mathrm{s+p}} = \mathbf{R}^{\mathrm{s}} \implies \forall_{\mathrm{n} \in \mathbb{N}_0} [\mathbf{R}^{\mathrm{n}} \in \{\mathbf{R}^0, \mathbf{R}, \mathbf{R}^2, \ldots \mathbf{R}^{\mathrm{s+p-1}}\}]$

$\mathrm{n} \in \mathbb{N}\ ; \quad (\mathbf{x}, \mathbf{y}) \in \mathbf{R}^{\mathrm{n}} \iff \mathbf{P}_{\mathrm{n}}(\mathbf{x}, \ldots \mathbf{y})$ exists in $\mathbf{G}$

Example: $\mathbf{R} \subseteq \mathbf{A}^2$, $\mathbf{G} = (\mathbf{A}, \mathbf{R})$



$\mathbf{R}^0$ $\mathbf{R}$ $\mathbf{R}^2, \mathbf{R}^4, \ldots$ $\mathbf{R}^3, \mathbf{R}^5, \ldots$

4.4 Closure operations on relations

$$\left.\begin{array}{l}\mathbf{t}(\mathbf{R}) \text{ is transitive} \\ \text{closure of } \mathbf{R} \\ \mathbf{R} \subseteq \mathbf{A}^2;\ \mathbf{t}(\mathbf{R}) \subseteq \mathbf{A}^2\end{array}\right\} :\Longleftrightarrow \left\{\begin{array}{l}\mathbf{t}(\mathbf{R}) \text{ is the smallest transitive relation} \\ \text{which contains } \mathbf{R} \\ (\mathbf{R} \subseteq \mathbf{t}(\mathbf{R}) \subseteq \mathbf{A}^2)\end{array}\right.$$

$\mathbf{r}(\mathbf{R})$: reflexive closure of $\mathbf{R}$
$\mathbf{s}(\mathbf{R})$: symmetric closure of $\mathbf{R}$

$$\begin{array}{lclcl}
\mathbf{R} \text{ is reflexive} & \Longleftrightarrow & \mathbf{R} = \mathbf{r}(\mathbf{R}) & \Longleftrightarrow & \mathbf{I} \subseteq \mathbf{R} \\
\mathbf{R} \text{ is symmetric} & \Longleftrightarrow & \mathbf{R} = \mathbf{s}(\mathbf{R}) & \Longleftrightarrow & \mathbf{R} = \mathbf{R}^{-1} \\
\mathbf{R} \text{ is transitive} & \Longleftrightarrow & \mathbf{R} = \mathbf{t}(\mathbf{R}) & \Longleftrightarrow & \mathbf{R}^2 \subseteq \mathbf{R}
\end{array}$$

$$\mathbf{r}(\mathbf{R}) = \mathbf{R} \cup \mathbf{I} \ ; \qquad \mathbf{s}(\mathbf{R}) = \mathbf{R} \cup \mathbf{R}^{-1} ; \qquad \mathbf{t}(\mathbf{R}) = \bigcup_{\nu=1}^{\infty} \mathbf{R}^{\nu}$$

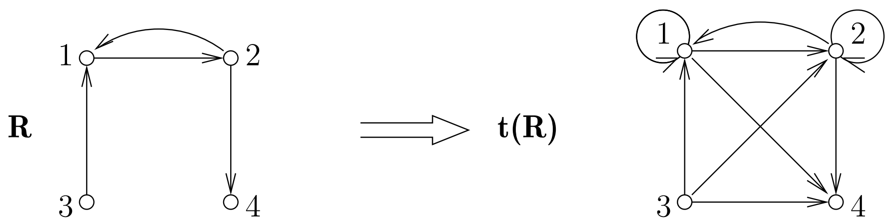
$$\begin{array}{lcl}
(\mathbf{x}, \mathbf{y}) \in \mathbf{t}(\mathbf{R}) & \Longleftrightarrow & \mathbf{P}(\mathbf{x}, \ldots, \mathbf{y}) \text{ exists in } \mathbf{G} \\
& \Longleftrightarrow & \mathbf{y} \text{ is descendent of } \mathbf{x} \\
& \Longleftrightarrow & \mathbf{x} \text{ is ancestor of } \mathbf{y}
\end{array}$$

$$(\mathbf{R} \subseteq \mathbf{A}^2) \ \wedge \ (\ |\mathbf{A}| \ < \infty \ ) \ \Longrightarrow \mathbf{t}(\mathbf{R}) = \bigcup_{\nu=1}^{|\mathbf{A}|} \mathbf{R}^{\nu}$$

$$\mathrm{n} \in \mathbb{N} ; \qquad \mathbf{R}^{\mathrm{n}} \subseteq \bigcup_{\nu=1}^{|\mathbf{A}|} \mathbf{R}^{\nu}$$

$$(\mathbf{x}, \mathbf{y}) \in \mathbf{t}(\mathbf{R}) \ \Longleftrightarrow \ \underset{1 \leq \nu \leq |\mathbf{A}|}{\exists} \ [\mathbf{P}_{\nu}(\mathbf{x}, \ \ldots, \mathbf{y}) \text{ exists in } \mathbf{G}] \ ; \qquad \nu \in \mathbb{N}$$

Example : $\mathbf{R} \subseteq \mathbf{A}^2$; $\mathbf{G}$=($\mathbf{A}$,$\mathbf{R}$)



N) Laws for closures on relations

(1)	$\mathbf{R} \subseteq \mathbf{r}(\mathbf{R})$;	$\mathbf{R} \subseteq \mathbf{s}(\mathbf{R})$;	$\mathbf{R} \subseteq \mathbf{t}(\mathbf{R})$	Extensity
(2)	$\mathbf{r}(\mathbf{R}) = \mathbf{rr}(\mathbf{R})$;	$\mathbf{s}(\mathbf{R}) = \mathbf{ss}(\mathbf{R})$;	$\mathbf{t}(\mathbf{R}) = \mathbf{tt}(\mathbf{R})$	Idempotence

(3) $\mathbf{R} = \mathbf{r}(\mathbf{R}) \implies \mathbf{s}(\mathbf{R}) = \mathbf{rs}(\mathbf{R})$ or $\mathbf{t}(\mathbf{R}) = \mathbf{rt}(\mathbf{R})$

(4) $\mathbf{R} = \mathbf{s}(\mathbf{R}) \implies \mathbf{r}(\mathbf{R}) = \mathbf{sr}(\mathbf{R})$ or $\mathbf{t}(\mathbf{R}) = \mathbf{st}(\mathbf{R})$

(5) $\mathbf{R} = \mathbf{t}(\mathbf{R}) \implies \mathbf{r}(\mathbf{R}) = \mathbf{tr}(\mathbf{R})$

(6) $\mathbf{rs}(\mathbf{R}) = \mathbf{sr}(\mathbf{R})$; $\mathbf{rt}(\mathbf{R}) = \mathbf{tr}(\mathbf{R})$; $\mathbf{st}(\mathbf{R}) \subseteq \mathbf{ts}(\mathbf{R})$

(7) $\mathbf{ts}(\mathbf{R}) = \mathbf{sts}(\mathbf{R})$; $\mathbf{tr}(\mathbf{R}) = \mathbf{rtr}(\mathbf{R})$

(8) $\mathbf{R} = \mathbf{tr}(\mathbf{R}) \iff \mathbf{R} = \mathbf{r}(\mathbf{R}) \wedge \mathbf{R} = \mathbf{t}(\mathbf{R})$

(9) $\mathbf{R} = \mathbf{sr}(\mathbf{R}) \iff \mathbf{R} = \mathbf{r}(\mathbf{R}) \wedge \mathbf{R} = \mathbf{s}(\mathbf{R})$

(10) $\mathbf{R} = \mathbf{ts}(\mathbf{R}) \iff \mathbf{R} = \mathbf{s}(\mathbf{R}) \wedge \mathbf{R} = \mathbf{t}(\mathbf{R})$

(11) $\mathbf{R} = \mathbf{s}(\mathbf{R}) \wedge \mathbf{R} = \mathbf{t}(\mathbf{R}) \implies \mathbf{R} = \mathbf{st}(\mathbf{R})$

$$\left.\begin{array}{lll} (12) & \mathbf{R}_2 \subseteq \mathbf{R}_1 \implies & \mathbf{r}(\mathbf{R}_2) \subseteq \mathbf{r}(\mathbf{R}_1) \\ (13) & \mathbf{R}_2 \subseteq \mathbf{R}_1 \implies & \mathbf{s}(\mathbf{R}_2) \subseteq \mathbf{s}(\mathbf{R}_1) \\ (14) & \mathbf{R}_2 \subseteq \mathbf{R}_1 \implies & \mathbf{t}(\mathbf{R}_2) \subseteq \mathbf{t}(\mathbf{R}_1) \end{array}\right\} \text{Monotony}$$

(15) $\mathbf{r}(\mathbf{R}_1) \cup \mathbf{r}(\mathbf{R}_2) = \mathbf{r}(\mathbf{R}_1 \cup \mathbf{R}_2)$

(16) $\mathbf{s}(\mathbf{R}_1) \cup \mathbf{s}(\mathbf{R}_2) = \mathbf{s}(\mathbf{R}_1 \cup \mathbf{R}_2)$

(17) $\mathbf{t}(\mathbf{R}_1) \cup \mathbf{t}(\mathbf{R}_2) \subseteq \mathbf{t}(\mathbf{R}_1 \cup \mathbf{R}_2)$

(18) $\mathbf{r}(\mathbf{R}) = \mathbf{R} \cup \mathbf{I}$

(19) $\mathbf{s}(\mathbf{R}) = \mathbf{R} \cup \mathbf{R}^{-1}$

(20) $\mathbf{t}(\mathbf{R}) = \bigcup_{\nu=1}^{\infty} \mathbf{R}^{\nu}$

$$
\begin{array}{llcll}
(21) & \mathbf{R}^{+} & = & \mathbf{t}(\mathbf{R}) \ ; & \mathbf{R}^{*} = \mathbf{tr}(\mathbf{R}) \\
(22) & \mathbf{tsr}(\mathbf{R}) & = & \mathbf{trs}(\mathbf{R}) & = (\mathbf{R} \cup \mathbf{R}^{-1})^{*} \\
(23) & (\mathbf{R}^{+})^{+} & = & \mathbf{R}^{+} \ ; & (\mathbf{R}^{*})^{*} = \mathbf{R}^{*} \\
(24) & (\mathbf{R}^{-1})^{+} & = & (\mathbf{R}^{+})^{-1} \ ; & (\mathbf{R}^{-1})^{*} = (\mathbf{R}^{*})^{-1} \\
(25) & \mathbf{R}^{+} & = & \mathbf{R}\mathbf{R}^{*} & = \mathbf{R}^{*}\,\mathbf{R} \\
(26) & \mathbf{R}^{*} & = & \mathbf{R}^{+} \cup \mathbf{I} & \\
(27) & (\mathbf{R} \cup \mathbf{S})^{*} & = & (\mathbf{R}^{*}\,\mathbf{S})^{*}\,\mathbf{R}^{*} & \\
(28) & \mathbf{R}^{*}\,\mathbf{S}^{*} & \subseteq & (\mathbf{R} \cup \mathbf{S})^{*} & \\
(29) & (\mathbf{R} \cup \mathbf{S})^{+} & = & \mathbf{R}^{+} \cup (\mathbf{R}^{*}\,\mathbf{S})^{+}\,\mathbf{R}^{*} &
\end{array}
$$

$$(30) \quad \mathbf{R}^{+} = \inf\{\mathbf{H} \in \mathbf{P}(\mathbf{A}^2) \mid (\mathbf{R} \subseteq \mathbf{H}) \wedge (\mathbf{H}^2 \subseteq \mathbf{H})\} = \inf \mathbf{M} = \bigcap_{\mathbf{H} \in \mathbf{M}} \mathbf{H}$$

$$(31) \quad \mathbf{R}^{+} = \sup\{\mathbf{R}^{\nu} \in \mathbf{P}(\mathbf{A}^2) \mid \nu \geq 1\} = \bigcup_{\nu=1}^{\infty} \mathbf{R}^{\nu}$$

$$(32) \quad \mathbf{R}^{*} = \inf\{\mathbf{H} \in \mathbf{P}(\mathbf{A}^2) \mid (\mathbf{R} \cup \mathbf{I} \subseteq \mathbf{H}) \wedge (\mathbf{H}^2 \subseteq \mathbf{H})\} = \inf \mathbf{M} = \bigcap_{\mathbf{H} \in \mathbf{M}} \mathbf{H}$$

$$(33) \quad \mathbf{R}^{*} = \sup\{\mathbf{R}^{\nu} \in \mathbf{P}(\mathbf{A}^2) \mid \nu \geq 0\} = \bigcup_{\nu=0}^{\infty} \mathbf{R}^{\nu}$$

Warshall Algorithm to compute $\mathbf{R}^{+}$ in $\mathbf{G} = (\mathbf{A}, \mathbf{R})$:

$\mathbf{G} = (\mathbf{A}, \mathbf{R})$; $\quad \mathbf{R} \subseteq \mathbf{A}^2$; $\quad \mathbf{x}_{\mathrm{i}} \in \mathbf{A}$, $\mathrm{i} = 1, \ldots, \mathrm{n}$; $\mathrm{n} = |\mathbf{A}|$

$$
\begin{array}{lcl}
\mathbf{R}_0 & := & \mathbf{R} \\
\mathbf{\Gamma}^{-}(\mathbf{x}_{\mathrm{i}}) & := & (\mathbf{R}_{\mathrm{i}-1})^{-1}(\mathbf{x}_{\mathrm{i}}) \\
\mathbf{\Gamma}^{+}(\mathbf{x}_{\mathrm{i}}) & := & \mathbf{R}_{\mathrm{i}-1}(\mathbf{x}_{\mathrm{i}}) \\
\mathbf{R}_{\mathrm{i}} & := & \mathbf{R}_{\mathrm{i}-1} \cup \mathbf{\Gamma}^{-}(\mathbf{x}_{\mathrm{i}}) \times \mathbf{\Gamma}^{+}(\mathbf{x}_{\mathrm{i}}) \\
\mathbf{R}^{+} & := & \mathbf{R}_{\mathrm{n}}
\end{array}
$$

4.5 Accessibility in binary graphs

$$\mathbf{R} \subseteq \mathbf{A}^2 \quad ; \quad \mathbf{G} = (\mathbf{A},\mathbf{R})$$

$\mathbf{x}$ is ancestor of $\mathbf{y}$ ($\mathbf{y}$ is descendent of $\mathbf{x}$) $\quad :\Longleftrightarrow \quad (\mathbf{x},\mathbf{y}) \in \mathbf{R}^+ \Longleftrightarrow \mathbf{x}\mathbf{R}^+\mathbf{y} \Longleftrightarrow \mathbf{y}(\mathbf{R}^+)^{-1}\mathbf{x}$

Set of descendents of $\mathbf{x}$ $\quad : \quad \mathrm{des}(\mathbf{x}) = \{\mathbf{y} \mid \mathbf{x}\mathbf{R}^+\mathbf{y}\}_{\mathbf{A}}$

Set of ancestors of $\mathbf{x}$ $\quad : \quad \mathrm{anc}(\mathbf{x}) = \{\mathbf{y} \mid \mathbf{y}\mathbf{R}^+\mathbf{x}\}_{\mathbf{A}}$

$$(\mathbf{x},\mathbf{y}) \in \mathbf{R}^+ \Longleftrightarrow \mathbf{y} \in \mathrm{des}(\mathbf{x}) \Longleftrightarrow \mathbf{x} \in \mathrm{anc}(\mathbf{y})$$

A cycle through $\mathbf{x}$ exists $\quad :\Longleftrightarrow \quad (\mathbf{x},\mathbf{x}) \in \mathbf{R}^+ \Longleftrightarrow \mathbf{x}\mathbf{R}^+\mathbf{x}$

$\mathbf{G}$ has no cycles $\quad :\Longleftrightarrow \quad \mathbf{R}^+ \subseteq \bar{\mathbf{I}}$

A proper cycle exists $\mathbf{G}$ $\quad :\Longleftrightarrow \quad \exists_{(\mathbf{x},\mathbf{y}) \in \mathbf{A}^2} (\mathbf{x}\bar{\mathbf{I}}\mathbf{y}) \wedge (\mathbf{x}\mathbf{R}^+\mathbf{y}) \wedge (\mathbf{y}\mathbf{R}^+\mathbf{x})$

$\mathbf{G}$ has no proper cycles $\quad :\Longleftrightarrow \quad \mathbf{R}^+ \cap (\mathbf{R}^+)^{-1} \subseteq \mathbf{I}$

$\mathbf{y}$ is accessible from $\mathbf{x}$ in $\mathbf{G}$ $\quad :\Longleftrightarrow \quad (\mathbf{x},\mathbf{y}) \in \mathbf{R}^* \Longleftrightarrow \mathbf{x}\mathbf{R}^*\mathbf{y}$

$\mathbf{G}$ is strongly connected (every node is accessible from every node) $\quad :\Longleftrightarrow \quad \mathbf{R}^* = \mathbf{A}^2$

$\mathbf{x}$ and $\mathbf{y}$ share (at least) one common predecessor ($\mathbf{z}$) $\quad :\Longleftrightarrow \quad \exists_{\mathbf{z} \in \mathbf{A}} \mathbf{z}\mathbf{R}\mathbf{x} \wedge \mathbf{z}\mathbf{R}\mathbf{y}$

$\mathbf{x}$ and $\mathbf{y}$ share a common
- successor $\quad :\Longleftrightarrow \quad (\mathbf{x},\mathbf{y}) \in \mathbf{R} \circ \mathbf{R}^{-1}$
- predecessor $\quad :\Longleftrightarrow \quad (\mathbf{x},\mathbf{y}) \in \mathbf{R}^{-1} \circ \mathbf{R}$

$\mathbf{x}$ and $\mathbf{y}$ share a common
- descendent $\quad :\Longleftrightarrow \quad (\mathbf{x},\mathbf{y}) \in \mathbf{R}^+ \circ (\mathbf{R}^+)^{-1}$
- ancestor $\quad :\Longleftrightarrow \quad (\mathbf{x},\mathbf{y}) \in (\mathbf{R}^+)^{-1} \circ \mathbf{R}^+$

$\mathbf{r}$ is root of $\mathbf{G}$ (Every node is accessible from $\mathbf{r}$) $\quad : \quad \forall_{\mathbf{x} \in \mathbf{A}} (\mathbf{r},\mathbf{x}) \in \mathbf{R}^*$

Set of all roots in $\mathbf{G}$ $\quad : \quad \mathbf{W}(\mathbf{G}) = \{\mathbf{r} \mid \forall_{\mathbf{x} \in \mathbf{A}} \mathbf{r}\mathbf{R}^*\mathbf{x}\}_{\mathbf{A}}$

$\mathbf{G}$ has a root $\quad :\Longleftrightarrow \quad \mathbf{W}(\mathbf{G}) \neq \emptyset \Longleftrightarrow (\mathbf{R}^*)^{-1} \circ \mathbf{R}^* = \mathbf{A}^2$

Directed distance (length of shortest path) from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbf{G}$: $\mathbf{d}(\mathbf{x},\mathbf{y}) = \min\{ n \mid (\mathbf{x},\mathbf{y}) \in \mathbf{R}^n\}_{\mathbb{N}_0}$

$\mathbf{d}(\mathbf{x},\mathbf{x}) = 0$; $(\mathbf{x},\mathbf{y}) \notin \mathbf{R}^* \Longrightarrow \mathbf{d}(\mathbf{x},\mathbf{y}) = \infty$

$\mathbf{d}(\mathbf{x},\mathbf{y}) \le \mathbf{d}(\mathbf{x},\mathbf{z}) + \mathbf{d}(\mathbf{z},\mathbf{y})$, Triangle inequality

Degrees of accessibility:

$\mathbf{xR^*y} \wedge \mathbf{yR^*x}$	$\Longrightarrow$	$\mathbf{xR^*y}$	$\Longrightarrow$	$\mathbf{xR^*y} \vee \mathbf{yR^*x}$	$\Longrightarrow$	$\mathbf{x}(\mathbf{R}^*)^{-1} \circ \mathbf{R^*y}$	$\Longrightarrow$	$\mathbf{x}(\mathbf{R} \cup \mathbf{R}^{-1})^*\mathbf{y}$
$\mathbf{x}$ and $\mathbf{y}$ are mutually accessible (cycle)		$\mathbf{y}$ is accessible from $\mathbf{x}$		$\mathbf{x}$ and $\mathbf{y}$ are at least one-sided accessible		$\mathbf{x}$ and $\mathbf{y}$ are indirectly accessible		$\mathbf{x}$ and $\mathbf{y}$ can be connected

$$\mathbf{R}^* \cap (\mathbf{R}^*)^{-1} \subseteq \mathbf{R}^* \subseteq \mathbf{R}^* \cup (\mathbf{R}^*)^{-1} \subseteq (\mathbf{R}^*)^{-1} \circ \mathbf{R}^* \subseteq (\mathbf{R} \cup \mathbf{R}^{-1})^*$$

$$\mathbf{R}^* \cap (\mathbf{R}^*)^{-1} = \mathbf{A}^2 \Longleftrightarrow \mathbf{R}^* = \mathbf{A}^2 \Longrightarrow \mathbf{R}^* \cup (\mathbf{R}^*)^{-1} = \mathbf{A}^2 \Longrightarrow (\mathbf{R}^*)^{-1} \circ \mathbf{R}^* = \mathbf{A}^2 \Longrightarrow (\mathbf{R} \cup \mathbf{R}^{-1})^* = \mathbf{A}^2$$

$\mathbf{R}^* \cap (\mathbf{R}^*)^{-1} = \mathbf{A}^2 \Longleftrightarrow \mathbf{R}^* = \mathbf{A}^2$	$\mathbf{R}^* \cup (\mathbf{R}^*)^{-1} = \mathbf{A}^2$	$(\mathbf{R}^*)^{-1} \circ \mathbf{R}^* = \mathbf{A}^2$	$(\mathbf{R} \cup \mathbf{R}^{-1})^* = \mathbf{A}^2$
$\mathbf{G}$ is strongly connected	$\mathbf{G}$ is one-sided strongly connected	$\mathbf{G}$ is quasi strongly connected	$\mathbf{G}$ is connected

$(\mathbf{R} \cup \mathbf{R}^{-1})^* = \mathbf{tsr}(\mathbf{R}) = \mathrm{ECL}(\mathbf{R})$ Connected components in $\mathbf{G}$
Equivalence closure of $\mathbf{R}$ (equivalence relation)

$\mathbf{R}^* \cup (\mathbf{R}^*)^{-1} = \mathbf{str}(\mathbf{R})$

$\mathbf{R}^* = \mathbf{tr}(\mathbf{R})$

$\mathbf{R}^* \cap (\mathbf{R}^*)^{-1} = \mathrm{ECO}(\mathbf{R}^*)$ strongly connected components in $\mathbf{G}$
Equivalence core of $\mathbf{R}^*$ (equivalence relation)

every node has at most one predecessor — no cycles

$\mathbf{T} = (\mathbf{A}, \mathbf{R})$ is a tree $:\iff \mathbf{W}(\mathbf{T}) \neq \emptyset \,\wedge\, \mathbf{R}\mathbf{R}^{-1} \subseteq \mathbf{I} \,\wedge\, \mathbf{R}^+ \subseteq \bar{\mathbf{I}}$

$\mathbf{W}(\mathbf{T}) \neq \emptyset \,\wedge\, \mathbf{R}^+ \subseteq \bar{\mathbf{I}} \implies |\mathbf{W}(\mathbf{T})| = 1$

root

$$\mathbf{T} = (\mathbf{A}, \mathbf{R}) \text{ is a tree} \quad :\iff \quad \begin{cases} \mathbf{d}^-(\mathbf{r}) = 0\,, \quad \mathbf{r} \in \mathbf{W}(\mathbf{T}) \\ \forall_{\mathbf{x} \in \mathbf{A} \setminus \{\mathbf{r}\}} \ \mathbf{d}^-(\mathbf{x}) = 1 \\ \forall_{\mathbf{x} \in \mathbf{A} \setminus \{\mathbf{r}\}} [\mathbf{P}(\mathbf{r}, \ldots, \mathbf{x}) \text{ exists in } \mathbf{T}] \end{cases}$$

$\mathbf{r}$ is root of $\mathbf{T}$ $:\iff \mathbf{d}^-(\mathbf{r}) = 0$

$\mathbf{a}$ is leaf of $\mathbf{T}$ $:\iff \mathbf{d}^+(\mathbf{a}) = 0$

$\mathbf{x}$ is interior node of $\mathbf{T}$ $:\iff \mathbf{d}^-(\mathbf{x}) = 1 \,\wedge\, \mathbf{d}^+(\mathbf{x}) \geq 1$

$\mathbf{x}$ is father of $\mathbf{y}$ ($\mathbf{y}$ is son of $\mathbf{x}$) $:\iff (\mathbf{x}, \mathbf{y}) \in \mathbf{T} \iff \mathbf{x}\mathbf{T}\mathbf{y}$

$$\left.\begin{array}{l} \text{subtree} \\ \mathbf{T_s} = (\mathbf{A_s}, \mathbf{R_s}) \text{ of } \mathbf{T} = (\mathbf{A}, \mathbf{R}) \end{array}\right\} : \quad \begin{cases} \mathbf{A_s} = \{\, \mathbf{x} \mid \mathbf{s}\mathbf{R}^*\mathbf{x} \}_{\mathbf{A}}\,; \quad \mathbf{A_s} \subseteq \mathbf{A} \\ \mathbf{R_s} = \mathbf{R} \cap \mathbf{A_s}^2\,; \qquad \mathbf{s} \in \mathbf{W}(\mathbf{T_s}) \end{cases}$$

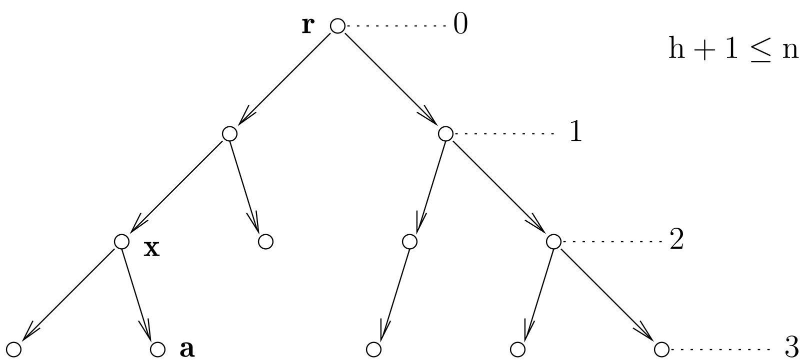
n-ary tree : $\mathbf{d}^+(\mathbf{x}) \leq \mathrm{n}\,; \quad (\mathbf{x} \in \mathbf{A})$

Complete n-ary tree : $\mathbf{d}^+(\mathbf{x}) = \mathrm{n}$ or $\mathbf{d}^+(\mathbf{x}) = 0$

Ordered tree : $\mathbf{\Gamma}^+(\mathbf{x}) = \{\, \mathbf{x}_1, \mathbf{x}_2, \ldots \}$

Height h of $\mathbf{T}$: $\mathrm{h} = \max\{\mathbf{d}(\mathbf{r}, \mathbf{x}) \mid \mathbf{x} \in \mathbf{A}\}_{\mathbb{N}_0}$

For a binary tree with $|\mathbf{A}| = \mathrm{n}$ it holds:



$$\mathrm{h} + 1 \leq \mathrm{n} \leq 2^0 + 2^1 + \ldots + 2^{\mathrm{h}-1} + 2^{\mathrm{h}}$$

$$\mathrm{h} + 1 \leq \mathrm{n} \leq 2^{\mathrm{h}+1} - 1$$

$$\lfloor \log_2 \mathrm{n} \rfloor \leq \mathrm{h} \leq \mathrm{n} - 1$$

4.6 Order relations

$\mathbf{R}$ is a partial order (relation) $:\Longleftrightarrow$ $\mathbf{R} \subseteq \mathbf{A}^2$, $\mathbf{R}$ is reflexive, antisymmetric and transitive; $\mathbf{I} \subseteq \mathbf{R}$, $\mathbf{R} \cap \mathbf{R}^{-1} \subseteq \mathbf{I}$, $\mathbf{R}^2 \subseteq \mathbf{R}$

($\mathbf{A}$,$\mathbf{R}$) or ($\mathbf{A}$, $\preceq$) is a <u>p</u>artially <u>o</u>rdered <u>set</u> (poset) $:\Longleftrightarrow$ $\mathbf{G} = (\mathbf{A}, \mathbf{R})$ is a (binary) digraph with $\mathbf{R}$ as a partial order

$\mathbf{R}$ is a strict (quasi) order (relation) $:\Longleftrightarrow$ $\mathbf{R} \subseteq \mathbf{A}^2$, $\mathbf{R}$ is asymmetric and transitive $\mathbf{R}^{-1} \subseteq \overline{\mathbf{R}}$, $\mathbf{R}^2 \subseteq \mathbf{R}$

($\mathbf{A}$,$\mathbf{R}$) or ($\mathbf{A}$, $\prec$) is a strictly ordered set $:\Longleftrightarrow$ $\mathbf{G} = (\mathbf{A}, \mathbf{R})$ is a (binary) digraph with $\mathbf{R}$ as a strict order

$\mathbf{R}$ is a total (linear, simple) order (relation) $:\Longleftrightarrow$ $\mathbf{R}$ is an order and $\mathbf{R}$ is connex ($\mathbf{R} \cup \mathbf{R}^{-1} = \mathbf{A}^2$) (All elements of set $\mathbf{A}$ are pairwise comparable)

($\mathbf{A}$, $\preceq$) is a total ordered set (chain) $:\Longleftrightarrow$ $\mathbf{G} = (\mathbf{A}, \preceq)$ is a (binary) digraph with $\preceq$ as total order

$$\mathbf{R} \text{ is a strict order} \Longrightarrow \mathbf{r}(\mathbf{R}) = \mathbf{R} \cup \mathbf{I} \text{ is a (not strict) order}$$

$$\mathbf{R} \text{ is a (not strict) order} \Longrightarrow \mathbf{R} \backslash \mathbf{I} \text{ is a strict order}$$

$$\mathbf{R} \text{ is a } \left\{ \begin{array}{c} \text{strict} \\ \text{not strict} \\ \text{total} \end{array} \right\} \text{ order} \Longrightarrow \mathbf{R}^{-1} \text{ is a } \left\{ \begin{array}{c} \text{strict} \\ \text{not strict} \\ \text{total} \end{array} \right\} \text{ order}$$

The order $\preceq$ in $\mathbf{A}$ is a well order (or ($\mathbf{A}$, $\preceq$) is a well ordered set $:\Longleftrightarrow$ $\underset{\mathbf{B} \in \mathbf{P}(\mathbf{A}) \backslash \{\emptyset\}}{\forall} \operatorname{lst}(\mathbf{B}) \neq \emptyset$ (Every non-empty subset of $\mathbf{A}$ contains a least element)

$$\preceq \text{ is well order in } \mathbf{A} \Longrightarrow \preceq \text{ is total order in } \mathbf{A}$$

Bounds and Extrema

Let $(\mathbf{A}, \preceq)$ be a well ordered set and $\mathbf{B} \subseteq \mathbf{A}$

bounds of set B

set of upper bounds:

$$\mathrm{ub}(\mathbf{B}) = \{\mathbf{a}\in\mathbf{A} \mid \forall_{\mathbf{x}\in\mathbf{A}}\ \mathbf{x}\in\mathbf{B} \longrightarrow \mathbf{x}\preceq\mathbf{a}\}$$

$$\mathrm{ub}(\mathbf{B}) = \bigcap_{\mathbf{x}\in\mathbf{B}} \mathbf{\Gamma}^{+}(\mathbf{x})$$

$\mathbf{a}\in\mathbf{A}$ is upper bound of set $\mathbf{B}$
IF AND ONLY IF
$\mathbf{a} \in \mathrm{ub}(\mathbf{B})$ or $\mathbf{B} \subseteq \mathbf{\Gamma}^{-}(\mathbf{a})$.

set of lower bounds:

$$\mathrm{lb}(\mathbf{B}) = \{\mathbf{a}\in\mathbf{A} \mid \forall_{\mathbf{x}\in\mathbf{A}}\ \mathbf{x}\in\mathbf{B} \longrightarrow \mathbf{a}\preceq\mathbf{x}\}$$

$$\mathrm{lb}(\mathbf{B}) = \bigcap_{\mathbf{x}\in\mathbf{B}} \mathbf{\Gamma}^{-}(\mathbf{x})$$

$\mathbf{a}\in\mathbf{A}$ is lower bound of set $\mathbf{B}$
IF AND ONLY IF
$\mathbf{a} \in \mathrm{lb}(\mathbf{B})$ or $\mathbf{B} \subseteq \mathbf{\Gamma}^{+}(\mathbf{a})$.

greatest and least elements of set B

set of greatest elements of set $\mathbf{B}$:

$$\mathrm{grt}(\mathbf{B}) = \{\mathbf{b}^*\in\mathbf{B} \mid \forall_{\mathbf{x}\in\mathbf{A}}\ \mathbf{x}\in\mathbf{B} \longrightarrow \mathbf{x}\preceq\mathbf{b}^*\}$$

$$\mathrm{grt}(\mathbf{B}) = \mathbf{B} \cap \mathrm{ub}(\mathbf{B})$$

$$\mathrm{grt}(\mathbf{B}) \neq \emptyset \implies |\,\mathrm{grt}(\mathbf{B})\,| = 1$$

$\mathbf{b}^*$ is greatest element of $\mathbf{B}$
IF AND ONLY IF
$\mathbf{b}^*\in \mathbf{B}$ AND $\mathbf{b}^*\in \mathrm{ub}(\mathbf{B})$ or $\mathbf{B} \subseteq \mathbf{\Gamma}^{-}(\mathbf{b}^*)$.

set of least elements of set $\mathbf{B}$:

$$\mathrm{lst}(\mathbf{B}) = \{\mathbf{b}^*\in\mathbf{B} \mid \forall_{\mathbf{x}\in\mathbf{A}}\ \mathbf{x}\in\mathbf{B} \longrightarrow \mathbf{b}^*\preceq\mathbf{x}\}$$

$$\mathrm{lst}(\mathbf{B}) = \mathbf{B} \cap \mathrm{lb}(\mathbf{B})$$

$$\mathrm{lst}(\mathbf{B}) \neq \emptyset \implies |\,\mathrm{lst}(\mathbf{B})\,| = 1$$

$\mathbf{b}^*$ is least element of $\mathbf{B}$
IF AND ONLY IF
$\mathbf{b}^*\in \mathbf{B}$ AND $\mathbf{b}^*\in \mathrm{lb}(\mathbf{B})$ or $\mathbf{B} \subseteq \mathbf{\Gamma}^{+}(\mathbf{b}^*)$.

maximal and minimal elements of B

set of maximal elements of $\mathbf{B}$:

$$\max(\mathbf{B}) = \{\mathbf{b}\in\mathbf{B} \mid \forall_{\mathbf{x}\in\mathbf{A}}\ \mathbf{x}\in\mathbf{B} \rightarrow (\mathbf{b}\preceq\mathbf{x} \rightarrow \mathbf{x}=\mathbf{b})\}$$

$$\max(\mathbf{B}) = \mathbf{B} \cap \Big[\bigcap_{\mathbf{x}\in\mathbf{B}} \overline{\mathbf{\Gamma}^{-}(\mathbf{x})\setminus\{\mathbf{x}\}}\Big]$$

$\mathbf{b}$ is maximal element of $\mathbf{B}$
IF AND ONLY IF
$\mathbf{b}\in\mathbf{B}$ AND $\mathbf{b}\in \bigcap_{\mathbf{x}\in\mathbf{B}} \overline{\mathbf{\Gamma}^{-}(\mathbf{x})\setminus\{\mathbf{x}\}}$
or $\mathbf{B} \subseteq \overline{\mathbf{\Gamma}^{+}(\mathbf{b})\setminus\{\mathbf{b}\}}$ or $\mathbf{b}\notin \bigcup_{\mathbf{x}\in\mathbf{B}} \mathbf{\Gamma}^{-}(\mathbf{x})\setminus\{\mathbf{x}\}$.

set of minimal elements of $\mathbf{B}$:

$$\min(\mathbf{B}) = \{\mathbf{b}\in\mathbf{B} \mid \forall_{\mathbf{x}\in\mathbf{A}}\ \mathbf{x}\in\mathbf{B} \rightarrow (\mathbf{x}\preceq\mathbf{b} \rightarrow \mathbf{x}=\mathbf{b})\}$$

$$\min(\mathbf{B}) = \mathbf{B} \cap \Big[\bigcap_{\mathbf{x}\in\mathbf{B}} \overline{\mathbf{\Gamma}^{+}(\mathbf{x})\setminus\{\mathbf{x}\}}\Big]$$

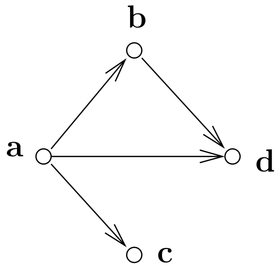
$\mathbf{b}$ is minimal element of $\mathbf{B}$
IF AND ONLY IF
$\mathbf{b}\in\mathbf{B}$ AND $\mathbf{b}\in \bigcap_{\mathbf{x}\in\mathbf{B}} \overline{\mathbf{\Gamma}^{+}(\mathbf{x})\setminus\{\mathbf{x}\}}$
or $\mathbf{B} \subseteq \overline{\mathbf{\Gamma}^{-}(\mathbf{b})\setminus\{\mathbf{b}\}}$ or $\mathbf{b}\notin \bigcup_{\mathbf{x}\in\mathbf{B}} \mathbf{\Gamma}^{+}(\mathbf{x})\setminus\{\mathbf{x}\}$.

least upper / greatest lower bounds of set $\mathbf{B}$

Set of least upper bounds (suprema) of $\mathbf{B}$:	Set of greatest lower bounds (infima) of $\mathbf{B}$:
$\mathrm{lub}(\mathbf{B}) = \{\mathbf{a}^*\in\mathrm{ub}(\mathbf{B}) \mid \forall_{\mathbf{x}\in\mathbf{A}}\ \mathbf{x}\in\mathrm{ub}(\mathbf{B})\rightarrow\mathbf{a}^*\preceq\mathbf{x}\}$	$\mathrm{glb}(\mathbf{B}) = \{\mathbf{a}^*\in\mathrm{lb}(\mathbf{B}) \mid \forall_{\mathbf{x}\in\mathbf{A}}\ \mathbf{x}\in\mathrm{lb}(\mathbf{B})\rightarrow\mathbf{x}\preceq\mathbf{a}^*\}$
$\mathrm{lub}(\mathbf{B}) = \mathrm{lst}(\mathrm{ub}(\mathbf{B}))$	$\mathrm{glb}(\mathbf{B}) = \mathrm{grt}(\mathrm{lb}(\mathbf{B}))$
$\mathrm{lub}(\mathbf{B}) = \mathrm{ub}(\mathbf{B}) \cap \mathrm{lb}(\mathrm{ub}(\mathbf{B}))$	$\mathrm{glb}(\mathbf{B}) = \mathrm{lb}(\mathbf{B}) \cap \mathrm{ub}(\mathrm{lb}(\mathbf{B}))$
$\mathrm{lub}(\mathbf{B}) \neq \emptyset \implies \mid \mathrm{lub}(\mathbf{B}) \mid = 1$	$\mathrm{glb}(\mathbf{B}) \neq \emptyset \implies \mid \mathrm{glb}(\mathbf{B}) \mid = 1$
$\mathbf{a}^*\in \mathrm{ub}(\mathbf{B})$ is the least upper bound (supremum) of $\mathbf{B}$ IF AND ONLY IF $\mathbf{a}^*\in \mathrm{lb}(\mathrm{ub}(\mathbf{B}))$ or $\mathrm{ub}(\mathbf{B}) \subseteq \mathbf{\Gamma}^+(\mathbf{a}^*)$.	$\mathbf{a}^*\in \mathrm{lb}(\mathbf{B})$ is the greatest lower bound (infimum) of $\mathbf{B}$ IF AND ONLY IF $\mathbf{a}^*\in \mathrm{ub}(\mathrm{lb}(\mathbf{B}))$ or $\mathrm{lb}(\mathbf{B}) \subseteq \mathbf{\Gamma}^-(\mathbf{a}^*)$.

Representation of ordered sets (example):

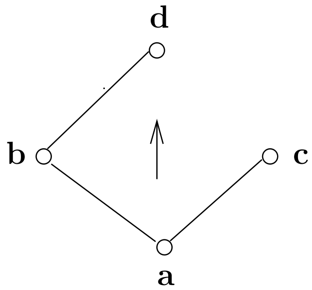
strict order



$\mathbf{H} = \mathbf{R} \cap \overline{\mathbf{R}^2}$

$\Longleftrightarrow$

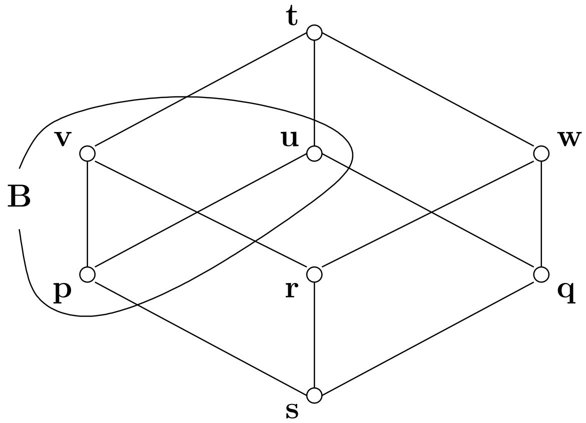
$\mathbf{R} = \mathbf{H}^+$



Arrow diagram $\mathbf{G} = (\mathbf{A}, \mathbf{R})$

Hasse diagram $\mathbf{G}_\mathrm{H} = (\mathbf{A}, \mathbf{H})$

Example regarding bounds and extrema:

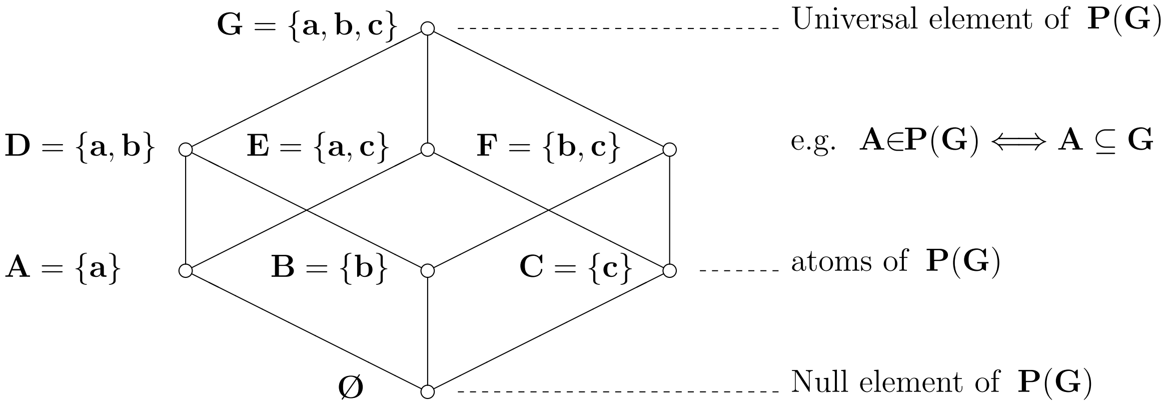


Hasse diagram $\mathbf{G}_\mathrm{H} = (\mathbf{A}, \mathbf{H})$

$\mathbf{B} = \{\mathbf{p}, \mathbf{u}, \mathbf{v}\}$

$\mathrm{ub}(\mathbf{B}) = \{\mathbf{t}\}$	$\mathrm{lb}(\mathbf{B}) = \{\mathbf{s}, \mathbf{p}\}$
$\mathrm{grt}(\mathbf{B}) = \{\}$	$\mathrm{lst}(\mathbf{B}) = \{\mathbf{p}\}$
$\max(\mathbf{B}) = \{\mathbf{u}, \mathbf{v}\}$	$\min(\mathbf{B}) = \{\mathbf{p}\}$
$\mathrm{lub}(\mathbf{B}) = \{\mathbf{t}\}$	$\mathrm{glb}(\mathbf{B}) = \{\mathbf{p}\}$

Example — containment order of a power set
$(\mathbf{P}(\mathbf{G}), \subseteq)$ is an ordered set



Hasse diagram of $(\mathbf{P}(\mathbf{G}), \subseteq)$; e.g. with $\mathbf{G} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

Supremum of $\mathbf{M}$: $\quad \sup \mathbf{M} = \bigcup_{\mathbf{X} \in \mathbf{M}} \mathbf{X}$

$\mathbf{M} \subseteq \mathbf{P}(\mathbf{G})$
(least upper bound) e.g. $\sup\{\mathbf{A}, \mathbf{D}, \mathbf{E}\} = \mathbf{A} \cup \mathbf{D} \cup \mathbf{E} = \mathbf{G}$

Infimum of $\mathbf{M}$: $\quad \inf \mathbf{M} = \bigcap_{\mathbf{X} \in \mathbf{M}} \mathbf{X}$

$\mathbf{M} \subseteq \mathbf{P}(\mathbf{G})$
(greatest lower bound) e.g. $\inf\{\mathbf{A}, \mathbf{D}, \mathbf{E}\} = \mathbf{A} \cap \mathbf{D} \cap \mathbf{E} = \mathbf{A}$

$(\mathbf{P}(\mathbf{G});\ \cup,\ \cap,\ \overline{};\ \emptyset,\ \mathbf{G})$ is a $2^{|\mathbf{G}|}$-valued Boolean algebra

$$(\mathbf{P}(\mathbf{G}), \subseteq) \quad \underset{\text{extension}}{\Longrightarrow} \quad (\mathbf{P}(\mathbf{M} \times \mathbf{N}), \subseteq)$$

$$\text{set algebra} \quad \Longrightarrow \quad \text{relational algebra}$$

4.7 Equivalence relations

$$\left.\begin{array}{l}\mathbf{R} \text{ is an} \\ \text{equivalence relation (on } \mathbf{A})\end{array}\right\} :\Longleftrightarrow \left\{\begin{array}{l}\mathbf{R} \subseteq \mathbf{A}^2,\ \ \mathbf{R} \text{ is reflexive,} \\ \text{symmetric and transitive} \\ \mathbf{I} \subseteq \mathbf{R},\ \mathbf{R} = \mathbf{R}^{-1},\ \mathbf{R}^2 \subseteq \mathbf{R}\end{array}\right.$$

$$\begin{array}{rcl}\mathbf{R} \text{ is an equivalence relation} & \Longleftrightarrow & \mathbf{R} = \mathbf{tsr}(\mathbf{R}) \\ \mathbf{R} = \mathbf{tsr}(\mathbf{R}) & \Longleftrightarrow & \mathbf{R} = \mathbf{r}(\mathbf{R}) \wedge \mathbf{R} = \mathbf{s}(\mathbf{R}) \wedge \mathbf{R} = \mathbf{t}(\mathbf{R})\end{array}$$

$$\left.\begin{array}{l}\mathrm{ECL}(\mathbf{Q}) \text{ is the} \\ \text{equivalence closure of } \mathbf{Q} \\ \text{(the equivalence relation} \\ \text{induced by } \mathbf{Q}) \\ \mathbf{Q} \subseteq \mathbf{A}^2,\ \mathrm{ECL}(\mathbf{Q}) \subseteq \mathbf{A}^2\end{array}\right\} :\Longleftrightarrow \left\{\begin{array}{l}\mathrm{ECL}(\mathbf{Q}) \text{ is the smallest equivalence} \\ \text{relation which contains } \mathbf{Q} \\ (\mathbf{Q} \subseteq \mathrm{ECL}(\mathbf{Q}) \subseteq \mathbf{A}^2)\end{array}\right.$$

$$\mathrm{ECL}(\mathbf{Q}) = \mathbf{tsr}(\mathbf{Q}) = (\mathbf{Q} \cup \mathbf{Q}^{-1})^*$$

$$\left.\begin{array}{l}\mathrm{ECO}(\mathbf{Q}^*) \text{ is the} \\ \text{equivalence core of } \mathbf{Q}^* \\ \mathbf{Q} \subseteq \mathbf{A}^2,\ \mathbf{Q}^* = \mathbf{rt}(\mathbf{Q}) \\ \mathrm{ECO}(\mathbf{Q}^*) \subseteq \mathbf{A}^2\end{array}\right\} :\Longleftrightarrow \left\{\begin{array}{l}\mathrm{ECO}(\mathbf{Q}^*) \text{ is the largest equivalence} \\ \text{relation which is contained in } \mathbf{Q}^* \\ (\mathrm{ECO}(\mathbf{Q}^*) \subseteq \mathbf{Q}^*)\end{array}\right.$$

$$\mathrm{ECO}(\mathbf{Q}^*) = \mathbf{Q}^* \cap (\mathbf{Q}^*)^{-1}$$

Example: $\mathbf{R} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{y} \bmod \mathbf{k} = \mathbf{x} \bmod \mathbf{k}\}_{\mathbf{A}^2}$; $\mathbf{k} \in \mathbb{N},\ \mathbf{x},\mathbf{y} \in \mathbb{Z}$

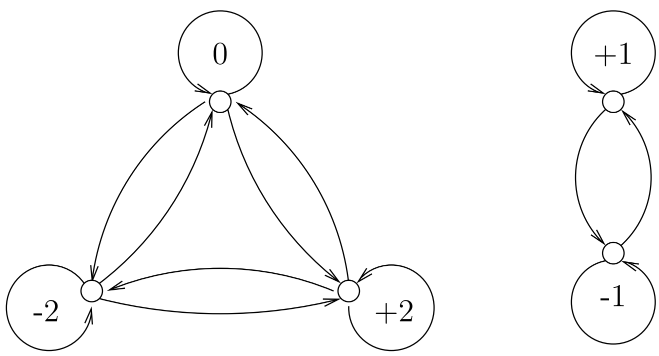
$$\mathbf{R} = \mathbf{tsr}(\mathbf{R})\,;\ \mathbf{y} \bmod \mathbf{k} = \mathbf{x} \bmod \mathbf{k} \Longleftrightarrow \underset{\mathbf{n}\in\mathbb{Z}}{\exists} (\mathbf{y} = \mathbf{x} + \mathbf{n} \cdot \mathbf{k})$$

$\mathbf{k} = 2$

$\mathbf{A} = \{\mathbf{x} \mid -2 \leq \mathbf{x} \leq +2\}_{\mathbb{Z}}$

$\mathbf{A} = \{-2, -1, 0, +1, +2\}$

$\mathbf{G} = (\mathbf{A},\mathbf{R})$:



$$\left.\begin{array}{l}[\mathbf{a}]_{\mathrm{R}} \text{ is} \\ \text{equivalence class of } \mathbf{a} \\ \text{with respect to } \mathbf{R}\end{array}\right\} :\Longleftrightarrow \begin{cases}\mathbf{R} \subseteq \mathbf{A}^2, \quad \mathbf{R} = \mathbf{tsr}(\mathbf{R}) \\ [\mathbf{a}]_{\mathrm{R}} = \{\mathbf{x} \mid \mathbf{xRa}\}_{\mathbf{A}} \\ \mathbf{a} \text{ is a representative of } [\mathbf{a}]_{\mathrm{R}}\end{cases}$$

$$[\mathbf{x}]_{\mathrm{R}} = \{\mathbf{y} \mid \mathbf{yRx} \wedge \mathbf{R} = \mathbf{tsr}(\mathbf{R})\}_{\mathbf{A}}$$

$$= \mathbf{\Gamma}^-(\mathbf{x}) = \mathbf{\Gamma}^+(\mathbf{x}) = \mathbf{\Gamma}(\mathbf{x})$$

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{yRx} \iff (\mathbf{x}, \mathbf{y}) \in \mathbf{R}$$

$$\iff \mathbf{x} \in [\mathbf{y}] \iff \mathbf{y} \in [\mathbf{x}] \iff [\mathbf{x}] = [\mathbf{y}]$$

$$[\mathbf{x}] \neq [\mathbf{y}] \iff [\mathbf{x}] \cap [\mathbf{y}] = \emptyset$$

quotient set: $\mathbf{A}/\mathbf{R} = \{\, [\mathbf{x}]_{\mathrm{R}} \mid \mathbf{x} \in \mathbf{A}\}_{\mathbf{P}(\mathbf{A})}$

$$\left.\begin{array}{l}\Pi \text{ is partition of } \mathbf{A} \neq \emptyset; \\ \Pi \subseteq \mathbf{P}(\mathbf{A}) \\ \Pi = \{\, \mathbf{K} \in \mathbf{P}(\mathbf{A}) \mid \mathbf{K} \in \Pi\} \\ (\mathbf{K} \text{ is a block} \\ \text{of partition } \Pi)\end{array}\right\} :\Longleftrightarrow \begin{cases}\forall_{\mathbf{K} \in \Pi}\ \mathbf{K} \neq \emptyset \\ \forall_{\mathbf{K}, \mathbf{L} \in \Pi} [\mathbf{K} \neq \mathbf{L} \longrightarrow \mathbf{K} \cap \mathbf{L} = \emptyset] \\ (\bigcup_{\mathbf{K} \in \Pi} \mathbf{K}) = \mathbf{A}\end{cases}$$

$$\begin{aligned}\Pi = \{\, \mathbf{K} \in \mathbf{P}(\mathbf{A}) \mid \mathbf{K} \in \Pi\} \quad &= \quad \{\, [\mathbf{x}]_{\mathrm{R}} \in \mathbf{P}(\mathbf{A}) \mid \mathbf{x} \in \mathbf{A}\} = \mathbf{A}/\mathbf{R} \\ \mathbf{R} \quad &= \quad \{\, (\mathbf{x}, \mathbf{y}) \mid \exists_{\mathbf{K} \in \Pi} [\mathbf{x} \in \mathbf{K} \wedge \mathbf{y} \in \mathbf{K}]\}_{\mathbf{A}^2} \\ \mathbf{x} \sim \mathbf{y} \quad &\iff \quad \exists_{\mathbf{K} \in \Pi} [\mathbf{x} \in \mathbf{K} \wedge \mathbf{y} \in \mathbf{K}] \iff [\mathbf{x}]_{\mathrm{R}} = [\mathbf{y}]_{\mathrm{R}}\end{aligned}$$

$$\mathbf{R}_1 = \mathbf{R}_2 \iff \mathbf{A}/\mathbf{R}_1 = \mathbf{A}/\mathbf{R}_2 \iff \Pi_1 = \Pi_2$$

$$\mathrm{rank}(\mathbf{R}) = \mathrm{rank}(\mathbf{A}/\mathbf{R}) = \mathrm{rank}(\Pi) = |\Pi|$$

Partition $\Pi' = \mathbf{A}/\mathbf{R}'$ is a refinement of partition $\Pi = \mathbf{A}/\mathbf{R}$; $|\,\Pi'\,| \geq |\,\Pi\,|$

$$:\Longleftrightarrow \underset{\mathbf{K}'\in\Pi'}{\forall} \;\; \underset{\mathbf{K}\in\Pi}{\exists} \; \mathbf{K}' \subseteq \mathbf{K}\,,$$

$$:\Longleftrightarrow \underset{\mathbf{a}\in\mathbf{A}}{\forall} \; ([\mathbf{a}]_{\mathrm{R}'} \subseteq [\mathbf{a}]_{\mathrm{R}})$$

$$:\Longleftrightarrow \mathbf{R}' \subseteq \mathbf{R}; \quad (|\,\mathbf{R}'\,| \leq |\,\mathbf{R}\,|)$$

coarsest partition of $\mathbf{A}$:

$\mathbf{R}_{\max} = \mathbf{A}^2; \quad |\,\mathbf{R}_{\max}\,| = |\mathbf{A}|^2$

$\Pi_{\min} = \{\mathbf{A}\}; \quad |\,\Pi_{\min}\,| = 1$

finest partition of $\mathbf{A}$:

$\mathbf{R}_{\min} = \mathbf{I}\,; \quad |\,\mathbf{R}_{\min}\,| = |\mathbf{A}|$

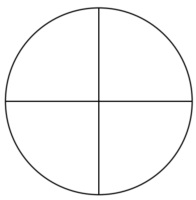
$\Pi_{\max} = \{\,\{\mathbf{a}\} \mid \mathbf{a} \in \mathbf{A}\,\}; \; |\,\Pi_{\max}\,| = |\mathbf{A}|$

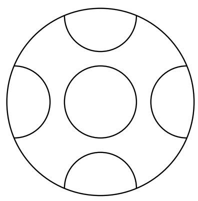
$\hat{\Pi} = \mathbf{A}/\hat{\mathbf{R}}$ is the coarsest partition, which is a refinement of $\Pi_1 = \mathbf{A}/\mathbf{R}_1$ as well as of $\Pi_2 = \mathbf{A}/\mathbf{R}_2$
$\hat{\Pi} = \Pi_1 \cdot \Pi_2$ is a product partition
$(|\hat{\Pi}| \geq |\Pi_1|; \; |\hat{\Pi}| \geq |\Pi_2|; \; |\hat{\Pi}| \overset{!}{=} \min)$

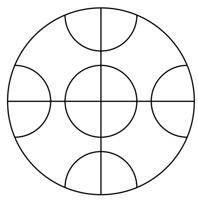
$$:\Longleftrightarrow \underset{\mathbf{a}\in\mathbf{A}}{\forall} \; ([\mathbf{a}]_{\hat{\mathrm{R}}} \subseteq [\mathbf{a}]_{\mathrm{R}_1} \cap [\mathbf{a}]_{\mathrm{R}_2}),$$

$$\underset{\mathbf{a}\in\mathbf{A}}{\forall} \; ([\mathbf{a}]_{\hat{\mathrm{R}}} \overset{!}{=} \max)$$

$$:\Longleftrightarrow \hat{\mathbf{R}} = \mathbf{R}_1 \cap \mathbf{R}_2$$

Π_1: 

Π_2: 

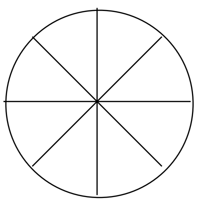
$\hat{\Pi}$: 

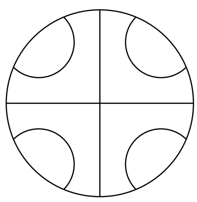
$\check{\Pi} = \mathbf{A}/\check{\mathbf{R}}$ is the finest partition, which is a coarsening of $\Pi_1 = \mathbf{A}/\mathbf{R}_1$ as well as of $\Pi_2 = \mathbf{A}/\mathbf{R}_2$
$\check{\Pi} = \Pi_1 + \Pi_2$ is a sum partition
$(|\check{\Pi}| \leq |\Pi_1|; \; |\check{\Pi}| \leq |\Pi_2|; \; |\check{\Pi}| \overset{!}{=} \max)$

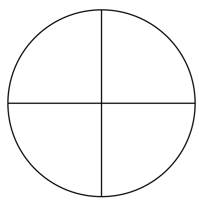
$$:\Longleftrightarrow \underset{\mathbf{a}\in\mathbf{A}}{\forall} \; ([\mathbf{a}]_{\mathrm{R}_1} \cup [\mathbf{a}]_{\mathrm{R}_2} \subseteq [\mathbf{a}]_{\check{\mathrm{R}}}),$$

$$\underset{\mathbf{a}\in\mathbf{A}}{\forall} \; ([\mathbf{a}]_{\check{\mathrm{R}}} \overset{!}{=} \min)$$

$$:\Longleftrightarrow \check{\mathbf{R}} = (\mathbf{R}_1 \cup \mathbf{R}_2)^+$$

Π_1: 

Π_2: 

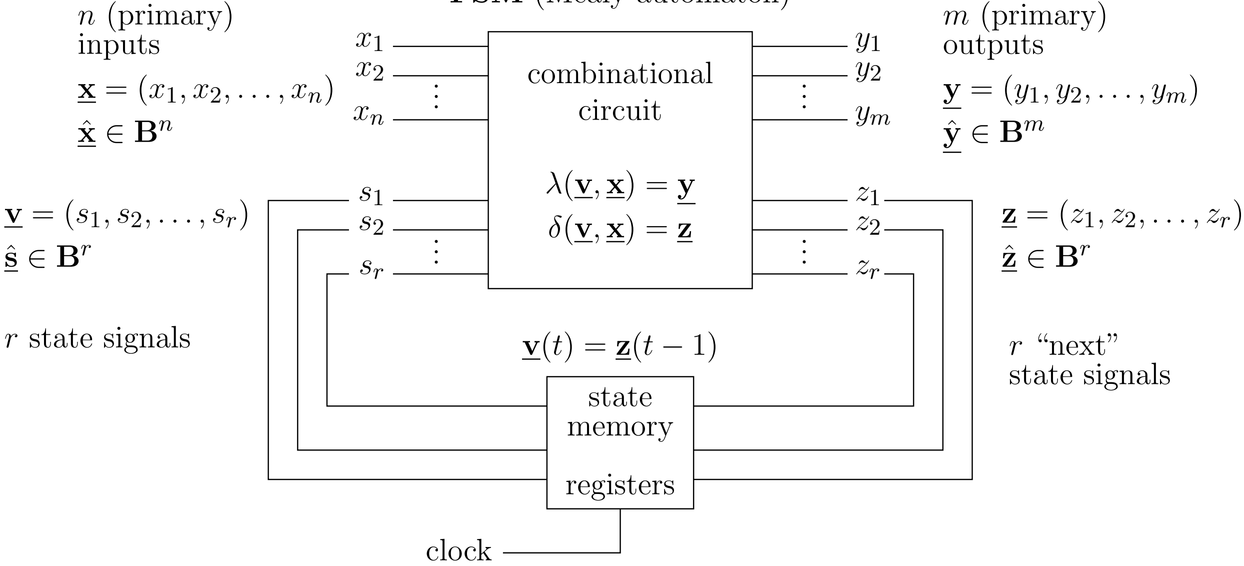
$\check{\Pi}$: 

5 Finite State Machines

5.1 Finite State Machine FSM

(Automaton with output)

FSM (Mealy automaton)



$$\mathbf{FSM} = (S, I, O, \delta, \lambda, S^0) \qquad \text{6-tuple}$$

1) $S = \{\underline{\hat{\mathbf{s}}}_0, \underline{\hat{\mathbf{s}}}_1, \ldots, \underline{\hat{\mathbf{s}}}_{2^r-1}\} = \mathbf{B}^r$: finite set of states, $0 < |S| < \infty$
$\quad = \{S_0, S_1, \ldots, S_{2^r-1}\}$

2) $I = \{\underline{\hat{\mathbf{x}}}_0, \underline{\hat{\mathbf{x}}}_1, \ldots, \underline{\hat{\mathbf{x}}}_{2^n-1}\} = \mathbf{B}^n$: set of input patterns
$\quad = \{X_0, X_1, \ldots, X_{2^n-1}\}$ input alphabet

3) $O = \{\underline{\hat{\mathbf{y}}}_0, \underline{\hat{\mathbf{y}}}_1, \ldots, \underline{\hat{\mathbf{y}}}_{2^m-1}\} = \mathbf{B}^m$: set of output patterns
$\quad = \{Y_0, Y_1, \ldots, Y_{2^m-1}\}$ output alphabet

4) $\delta \;:\; S \times I \to S\;,\quad (\underline{\mathbf{s}}, \underline{\mathbf{x}}) \to \underline{\mathbf{z}}$: (state-) transition function

5) $\lambda \;:\; S \times I \to O\;,\quad (\underline{\mathbf{s}}, \underline{\mathbf{x}}) \to \underline{\mathbf{y}}$: output function

6) $S^0 \in S$: initial state

Types of machines:

Mealy automaton $\lambda : \; S \times I \to O\;, \quad \underline{\mathbf{y}} = \lambda(\underline{\mathbf{s}}, \underline{\mathbf{x}})$
Moore automaton $\lambda : \; S \to O\;, \quad \underline{\mathbf{y}} = \lambda(\underline{\mathbf{s}})$

5.2 General description of FSM

X_j : Input pattern $\quad$ Y_l : Output pattern
S_i : State $\quad$ S_k : Next state $\quad$ $i, j, k, l \in \{0, 1, 2, 3, \dots\}$

δ	$\cdots$	X_j	$\cdots$
$\vdots$		$\vdots$	
S_i	$\cdots$	S_k	$\cdots$
$\vdots$		$\vdots$	

state transition table, state table

λ	$\cdots$	X_j	$\cdots$
$\vdots$		$\vdots$	
S_j	$\cdots$	Y_l	$\cdots$
$\vdots$		$\vdots$	

output table

$$\delta \;:\; S \times I \to S$$
$$S_k = \delta(S_i, X_j)$$

state transition function, next-state function

$$\lambda \;:\; S \times I \to O$$
$$Y_l = \lambda(S_i, X_j)$$

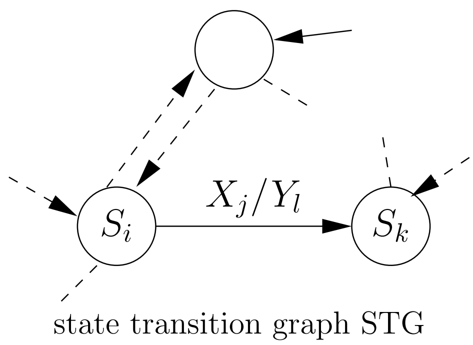
output function

$$\mu \;:\; S \times I \to S \times O$$
$$(S_k, Y_l) = \mu(S_i, X_j)$$

state output function

μ	$\cdots$	X_j	$\cdots$
$\vdots$		$\vdots$	
S_i	$\cdots$	(S_k, Y_l)	$\cdots$
$\vdots$		$\vdots$	

state output table, state table
state diagram



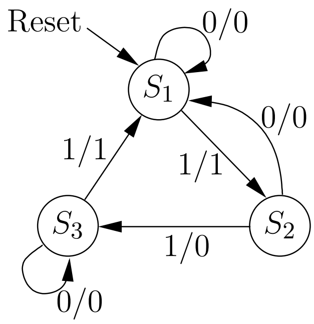
state transition graph STG

Example (10): $I = O = \{0, 1\}, \quad n = m = 1; \quad S = \{S_1, S_2, S_3\}, \quad S^0 = S_1$
$(S_j, \hat{\mathbf{y}}) = \mu(S_i, \hat{\mathbf{x}}), \quad i, j = 1, 2, 3$

μ	$x = 0$	$x = 1$
S_1	$(S_1, 0)$	$(S_2, 1)$
S_2	$(S_1, 0)$	$(S_3, 0)$
S_3	$(S_3, 0)$	$(S_1, 1)$

state output table

$\equiv$



STG

$\equiv$

$\hat{\mathbf{x}}$	state	next state	$\hat{\mathbf{y}}$
0	S_1	S_1	0
1	S_1	S_2	1
0	S_2	S_1	0
1	S_2	S_3	0
0	S_3	S_3	0
1	S_3	S_1	1

cube table

x	s_1 s_2	z_1 z_2	y
0	0 0	0 0	0
1	0 0	0 1	1
0	0 1	0 0	0
1	0 1	1 0	0
0	1 0	1 0	0
1	1 0	0 0	1
0	1 1	$(1)* \; *(0)$	$*(0)$
1	1 1	$(1)* \; *(0)$	$*(0)$

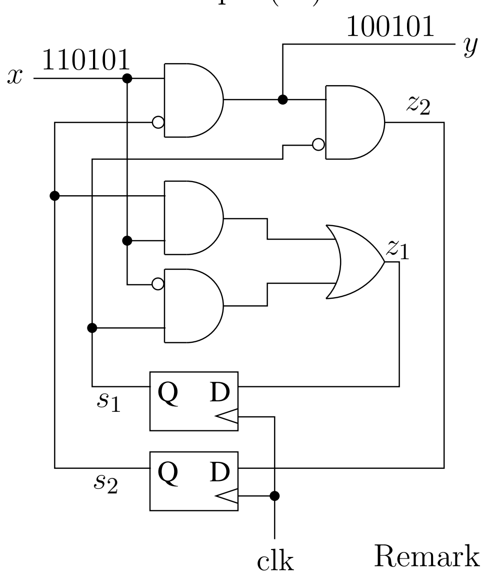
coded cube table

state coding

$S_1 \triangleq 00 \triangleq \overline{s}_1 \cdot \overline{s}_2$
$S_2 \triangleq 01 \triangleq \overline{s}_1 \cdot s_2$
$S_3 \triangleq 10 \triangleq s_1 \cdot \overline{s}_2$

$$y = x \cdot \overline{s}_1 \cdot \overline{s}_2 + x \cdot s_1 \cdot \overline{s}_2 = x \cdot \overline{s}_2$$
$$z_2 = x \cdot \overline{s}_1 \cdot \overline{s}_2$$
$$z_1 = x \cdot \overline{s}_1 \cdot s_2 + \overline{x} \cdot s_1 \cdot \overline{s}_2 \underbrace{+ \overline{x} \cdot s_1 \cdot s_2 + x \cdot s_1 \cdot s_2}_{\text{optional}}$$
$$z_1 = x \cdot s_2 + \overline{x} \cdot s_1$$

FSM for example (10):



FSM processes strings (words, concatenation of symbols). Starting with the initial state $S_1 \triangleq 00 \triangleq \overline{s}_1 \cdot \overline{s}_2$, e.g. the string

$$\text{str}(x) = x(0) \circ x(1) \circ x(2) \circ x(3) \circ x(4) \circ x(5) = 110101$$

is transformed into the string

$$\text{str}(y) = y(0) \circ y(1) \circ y(2) \circ y(3) \circ y(4) \circ y(5) = 100101$$

one symbol at a time. The sequential system cycles through a number of system states during this processing.

$$\text{str}(S) = S(0) \circ S(1) \circ S(2) \circ S(3) \circ S(4) \circ S(5) \circ S(6) = S_1\, S_2\, S_3\, S_3\, S_1\, S_1\, S_2 \qquad \text{(see STG)}$$

Remark: State sequences or strings with an initial state describe paths in the state transition graph STG.

5.3 State Minimization

The first step in state minimization is to eliminate non-reachable states.

Definition: A state S_j is reachable from an initial state S_i, if a directed path from S_i to S_j exists in the state transition graph.

The key task of state minimization is to reduce the number of states by merging equivalent states.

Definition of equivalence of two states S_i and S_j:

$S_i \sim S_j$ Equivalent (non differentiable) states	$\Longleftrightarrow$	$\lambda(S_i, \mathrm{str}(x)) = \lambda(S_j, \mathrm{str}(x))$, for all $\mathrm{str}(x)$ identical output patterns for identical input patterns, originating from S_i and S_j

The equivalence relation $\sim$ partitions the state set S into disjoint subsets (equivalence classes). S_i and S_j are elements of an equivalence class K_l of the partition Π of S. $\Pi \subseteq P(S)$

$$\Pi(S) = \{K_1, \ldots, K_l, \ldots, K_L\} \ ; \quad S_i, S_j \in K_l \ , \quad \bigcup_l K_l = S$$

Remark:	$S_i \not\sim S_j$ differentiable states	$\Longleftrightarrow$	there exists a $\mathrm{str}(x)$, such that$\lambda(S_i, \mathrm{str}(x)) \neq \lambda(S_j, \mathrm{str}(x))$ different output patterns

Definition of k-equivalence of two states S_i, S_j:

$$S_i \overset{k}{\sim} S_j \quad \Longleftrightarrow \quad \lambda(S_i, \mathrm{str}(x)|_k) = \lambda(S_j, \mathrm{str}(x)|_k) \ , \quad \text{for all } \mathrm{str}(x)|_k$$

$$\Pi^{(k)}(S) = \{K_1^{(k)}, \ldots, K_l^{(k)}, \ldots, K_L^{(k)}\} \ ; \quad S_i, S_j \in K_l^{(k)}$$

$$S_i \overset{1}{\sim} S_j \quad \Longleftrightarrow \quad \lambda(S_i, x^0) = \lambda(S_j, x^0) \ , \quad \text{for all } x^0 \in \{1, 0\}$$

$$S_i \overset{1}{\sim} S_j \quad \Longleftrightarrow \quad [\lambda(S_i, 0) = \lambda(S_j, 0)] \wedge [\lambda(S_i, 1) = \lambda(S_j, 1)]$$

The iterative computation of k-equivalent (or simply: equivalent) states starts with the computation of the 1-equivalent states, since:

$$S_i \overset{k}{\sim} S_j \Longleftrightarrow \lambda(S_i, \epsilon) = \lambda(S_j, \epsilon) \Longleftrightarrow \epsilon = \epsilon \ , \quad \text{for all } S_i, S_j \in S \ , \qquad \Pi^{(0)} = \{S\}$$

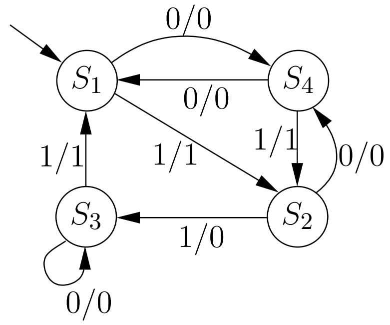
coarsest partition of S

Example (11): $I = O = \{0, 1\}, \quad n = m = 1; \quad S = \{S_1, S_2, S_3, S_4\}, \quad S^0 = S_1$
$(S_j, \hat{\mathbf{y}}) = \mu(S_i, \hat{\mathbf{x}}), \quad i, j = 1, 2, 3, 4$

μ	$x = 0$	$x = 1$
S_1	$(S_4, 0)$	$(S_2, 1)$
S_2	$(S_4, 0)$	$(S_3, 0)$
S_3	$(S_3, 0)$	$(S_1, 1)$
S_4	$(S_1, 0)$	$(S_2, 1)$

state output table

$\equiv$



STG

From the state output table we obtain the following relations:

$$S_1 \overset{1}{\sim} S_3 \iff [0 = 0] \wedge [1 = 1]\ , \quad S_1 \overset{1}{\sim} S_4 \quad (S_3 \overset{1}{\sim} S_4)$$

$$S_1 \overset{1}{\not\sim} S_2 \iff [0 = 0] \wedge \underbrace{[1 = 0]}_{\text{false}}\ , \quad S_1 \overset{1}{\not\sim} S_2 \quad (S_3 \overset{1}{\not\sim} S_2, S_4 \overset{1}{\not\sim} S_2)$$

$$\Pi^{(1)} = \Big\{\{S_1, S_3, S_4\}, \{S_2\}\Big\}$$

$$S_i \overset{2}{\sim} S_j \iff [S_i \overset{1}{\sim} S_j] \wedge [\delta(S_i, 0) \overset{1}{\sim} \delta(S_j, 0)] \wedge [\delta(S_i, 1) \overset{1}{\sim} \delta(S_j, 1)]$$

$$\text{e.g. } S_1 \overset{2}{\sim} S_4 \iff [S_1 \overset{1}{\sim} S_4] \wedge [S_4 \overset{1}{\sim} S_1] \wedge [S_2 \overset{1}{\sim} S_2]$$

$$S_1 \overset{2}{\not\sim} S_3 \iff [S_1 \overset{1}{\sim} S_4] \wedge [S_4 \overset{1}{\sim} S_3] \wedge \underbrace{[S_2 \overset{1}{\sim} S_1]}_{\text{false}}\ , \quad S_1 \overset{2}{\not\sim} S_3$$

$$\Pi^{(2)} = \Big\{\{S_1, S_4\}, \{S_3\}, \{S_2\}\Big\}$$

$$S_1 \overset{3}{\sim} S_4 \iff [S_1 \overset{2}{\sim} S_4] \wedge [\delta(S_1, 0) \overset{2}{\sim} \delta(S_4, 0)] \wedge [\delta(S_1, 1) \overset{2}{\sim} \delta(S_4, 1)]\text{rcl}$$

$$\iff [S_1 \overset{2}{\sim} S_4] \wedge [S_4 \overset{2}{\sim} S_1] \wedge [S_2 \overset{2}{\sim} S_2]$$

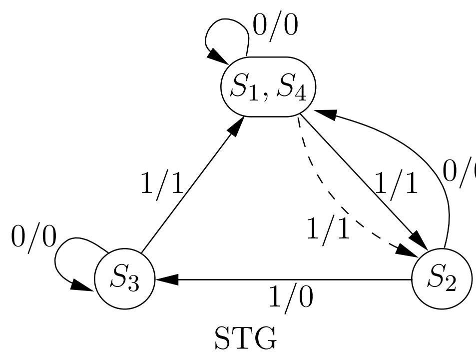
$$\Pi^{(3)} = \Pi^{(2)} = \Pi(S)$$

Result: state S_1 and S_4 are equivalent and can be merged in STG (see example (10)!).

μ	$x = 0$	$x = 1$	
S_1	$(S_4, 0)$	$(S_2, 1)$	
S_4	$(S_1, 0)$	$(S_2, 1)$	2
S_3	$(S_3, 0)$	$(S_1, 1)$	1
S_2	$(S_4, 0)$	$(S_3, 0)$	

state output table

$\equiv$



STG

Iterative checking for equivalent states in an FSM:

$$S_i \overset{k+1}{\sim} S_j \iff [S_i \overset{k}{\sim} S_j] \wedge [\delta(S_i, x^0) \overset{k}{\sim} \delta(S_j, x^0)] \;, \quad \text{for all } x^0 \in \{0,1\}$$
$$S_i \overset{k+1}{\sim} S_j \iff [S_i \overset{k}{\sim} S_j] \wedge [\delta(S_i, 0) \overset{k}{\sim} \delta(S_j, 0)] \wedge [\delta(S_i, 1) \overset{k}{\sim} \delta(S_j, 1)]$$

Comment: $S_i \overset{k+1}{\sim} S_j \iff [S_i \overset{1}{\sim} S_j] \wedge [\delta(S_i, 0) \overset{k}{\sim} \delta(S_j, 0)] \wedge [\delta(S_i, 1) \overset{k}{\sim} \delta(S_j, 1)]$

Determination of equivalent states:

$$\Pi = \Pi^{(k)} \;, \quad \text{IF} \quad \Pi^{(k+1)} = \Pi^{(k)} \;, \quad \text{with} \quad k \le |S| - 1$$

Additional remark: $\Pi^{(k)}$ is a refinement of $\Pi^{(k-1)}$; $\Pi^{(k)}, \Pi^{(k-1)}, \ldots, \Pi^{(2)}, \Pi^{(1)}$

Because $S_i \overset{1}{\not\sim} S_j \Longrightarrow S_i \overset{2}{\not\sim} S_j \Longrightarrow \ldots \Longrightarrow S_i \overset{k}{\not\sim} S_j \Longrightarrow S_i \not\sim S_j$

$S_i \sim S_j \Longrightarrow S_i \overset{k}{\sim} S_j \Longrightarrow S_i \overset{k-1}{\sim} S_j \Longrightarrow \ldots \Longrightarrow S_i \overset{1}{\sim} S_j$

Example (12): Completely specified FSM; $I = O = \{0,1\}, \quad n = m = 1;$
$S = \{A, B, C, D, E, F, G\}$

μ	$x=0$	$x=1$	
A	$(E,0)$	$(C,0)$	2
D	$(G,0)$	$(A,0)$	3
G	$(D,0)$	$(G,0)$	2
F	$(E,0)$	$(D,0)$	1
B	$(C,0)$	$(A,1)$	3
C	$(B,0)$	$(G,1)$	1
E	$(F,1)$	$(B,0)$	

state output table
(appropriately sorted)

$$\Pi^{(1)} = \{\underset{00}{\{A, D, G, F\}}, \underset{01}{\{B, C\}}, \underset{10}{\{E\}}\} \qquad \hat{\mathbf{y}}\text{−assignment}$$

$$\Pi^{(2)} = \{\{A\}, \{D, G\}, \{F\}, \{B, C\}, \{E\}\}$$

e.g. $A \overset{2}{\not\sim} D \iff [A \overset{1}{\sim} D] \wedge \underset{\text{false}}{[E \overset{1}{\sim} G]} \wedge \underset{\text{false}}{[C \overset{1}{\sim} A]}$

$D \overset{2}{\sim} G \iff [D \overset{1}{\sim} G] \wedge [G \overset{1}{\sim} D] \wedge [A \overset{1}{\sim} G]$

$$\Pi^{(3)} = \{\{A\}, \{D\}, \{G\}, \{F\}, \{B\}, \{C\}, \{E\}\}$$
finest decomposition of S

Result : All states can be pairwise differentiated by appropriate strings with $k \le 3$.

Remark : Typically, the state output table will be sorted again after each partitioning step.

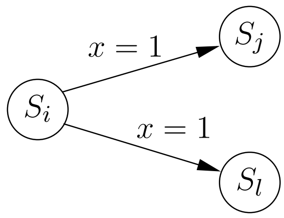
5.4 General comments on machine models

Machine types:

Mealy Automaton	$\lambda :$	$S \times I \rightarrow O$,	$Y_l = \lambda(S_i, X_j)$
Moore Automaton	$\lambda :$	$S \rightarrow O$,	$Y_l = \lambda(S_i)$
Medwedew Automaton	$\lambda :$	$I \rightarrow O$,	$Y_l = \lambda(X_j)$

Properties of machines:

completely specified – incompletely specified
simplified
minimal
deterministic – non-deterministic



$$\{S_j, S_l\} = \delta(\{S_i\}, 1)$$

In a non-deterministic machine the state transition is not necessarily unambiguous anymore.

FSM design flow: