Topic 5
Limits & Continuity

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Outline

- Concept of a Limit
- The Euler's Number, *e*
- Existence & Non-existence of a Limit
- Indeterminate Forms
- Methods of Evaluating a Limit
- Continuity
- Intermediate Value Theorem
- Numerical Method: Bisection Method

Recap: Asymptotes

Recall from Topic 1 that the vertical asymptote of

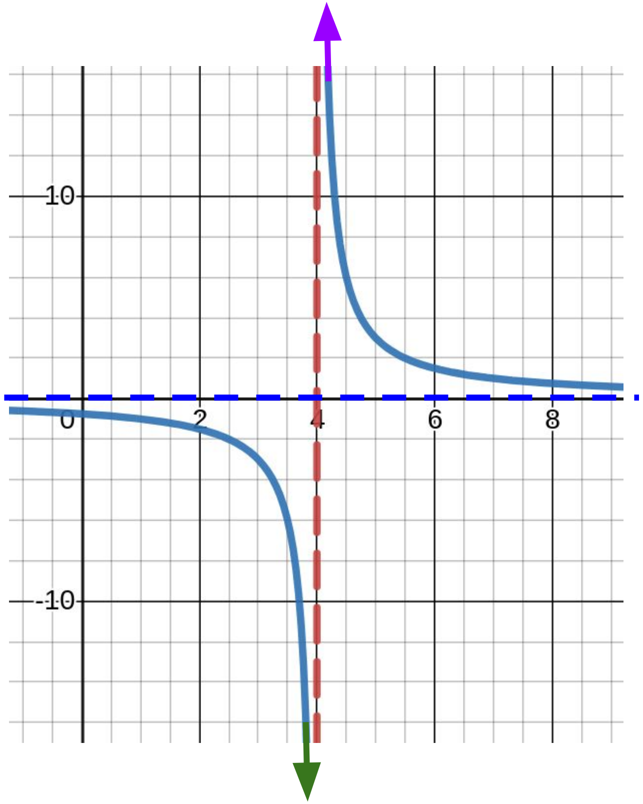
$$f(x) = \frac{3}{x-4}$$

can be evaluated to be at x = 4. When x approaches 4 from the left (3.99...), f(x) approaches -∞. When x approaches 4 from the right (4.00...1), f(x) approaches +∞. Using limit notation, we can write

$$\lim_{x \to 4^-} f(x) = -\infty, \qquad \lim_{x \to 4^+} f(x) = +\infty$$

And we can also find the horizontal asymptote at y = 0 by

$$\lim_{x \to \pm\infty} f(x) = 0$$



Concept of a Limit

So, the concept of a limit, say *L*, is simply a value that a function approaches as its input variable approaches some prescribed value, say *a*. Mathematically, this means

$$\lim_{x \to a} f(x) = L$$

If *L* is the same (finite) value regardless of how x approaches a, then we say the limit exists. Otherwise, the limit does not exist (DNE). (More on this later.)

The concept of a **limit is pivotal to both differential & integral calculus**. The derivative of a function, say df/dx, is derived from the limit of the change of a function. And the integral of a function is derived from the limit of the sum of areas under the function.

Hence, this is a concept I hope you can **truly understand**.

Euler's Number, e

Perhaps the most famous example of a limit originates in 1683 when a Swiss mathematician, Jacob Bernoulli, discovered the number **e = 2.71828…** while studying compound interest.

He asks: If I have $1 in an investment that pays 100% per year, at the end of year one, I would have $2. But, what would happen if the returns are paid and compounded more frequently?

This is a logical question, because there are investments that pay returns multiple times per year. For example, a bond typically pays 2 times per year. So, if the interest yield is 6% annually, the bond might pay 3% in June and 3% in December.

Assuming that the returns are reinvested, more payments in a year will lead to more compounding effect which makes the accumulated returns higher.

Euler's Number, *e*

For a 100% annual return, the return per payment is 1/n, where n is the number of payment (compounding) periods.

Given that the initial investment is $1, the formula to calculate the accumulated amount as a function of the number of compounding periods n can be worked out to be

$$A(n) = \left(1 + \frac{1}{n}\right)^n$$

n	\$ $A(n)$
0	1
1	$(1)\left(1+\frac{1}{n}\right)$
2	$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right) = \left(1+\frac{1}{n}\right)^2$
$\vdots$	
n	$\left(1+\frac{1}{n}\right)^n$

So Bernoulli wanted to know

$$\lim_{n\to\infty} A(n) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

and if the limit exists.

Euler's Number, e

Tabulating the limit numerically, we observe that

$$
\begin{array}{rcl}
n = 10 & \rightarrow & A(10) = 2.59374\ldots \\
n = 100 & \rightarrow & A(100) = 2.70481\ldots \\
n = 1000 & \rightarrow & A(1000) = 2.71692\ldots \\
n = 10,000 & \rightarrow & A(10,000) = 2.71815\ldots \\
n = 100,000 & \rightarrow & A(100,000) = 2.71827\ldots \\
n = 1,000,000 & \rightarrow & A(1,000,000) = 2.71828\ldots \\
n = 10,000,000 & \rightarrow & A(10,000,000) = 2.71828\ldots
\end{array}
$$

The maximum gain there can be achieved by increasing the number of compounding periods is 2.718 times the initial investement principle

which indicates that A(~~r~~ n) approaches 2.71828... asymptotically. So the limit is

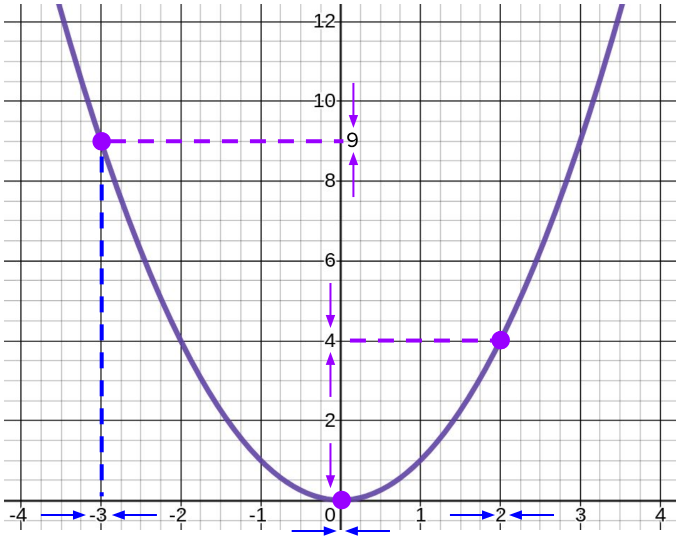
$$\lim_{n \to \infty} A(n) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828\ldots$$

which is called the **Euler's number, e**. What does this imply on the investment?

Limit VS Function Value

Most simplistically, the limit of a function **can be equal** to its function value. For example, for the function f(x) = x^2, we find that

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} x^2 = (-3)^2 = 9,$$
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0^2 = 0,$$
$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 = 2^2 = 4$$
$$\vdots$$
$$\rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} x^2 = a^2 = f(a)$$

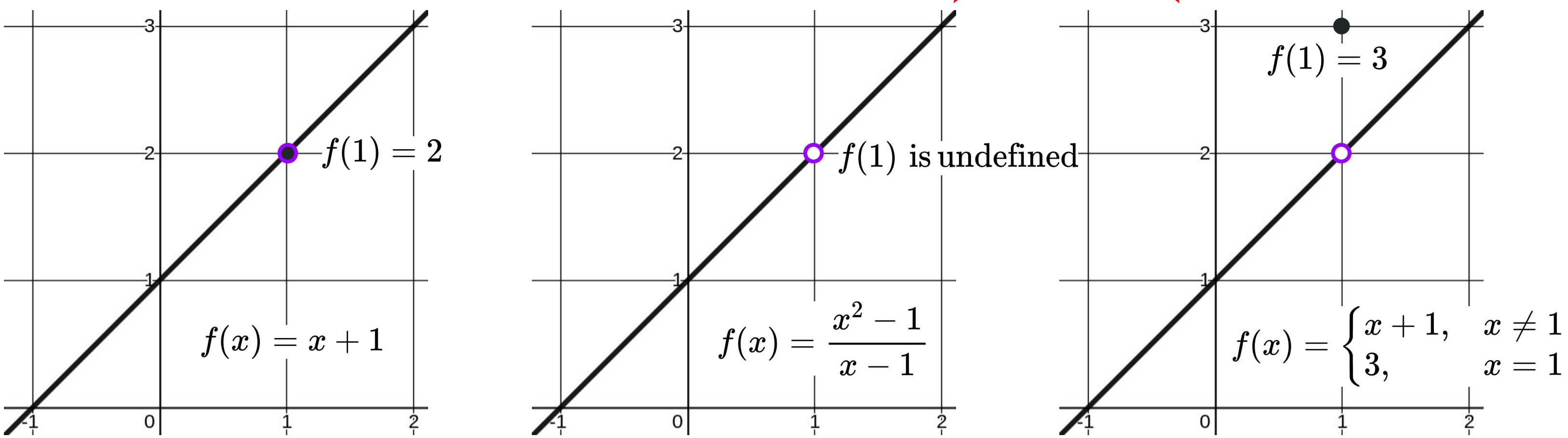


Hence, the first method of evaluating a limit is simply by direct substitution. This works if the function is continuous at x = a.

Limit VS Function Value

However, it is also possible that the limit of a function is not equal to the function value. Consider the functions below where all three satisfy

$$\lim_{x \to 1} f(x) = 2$$



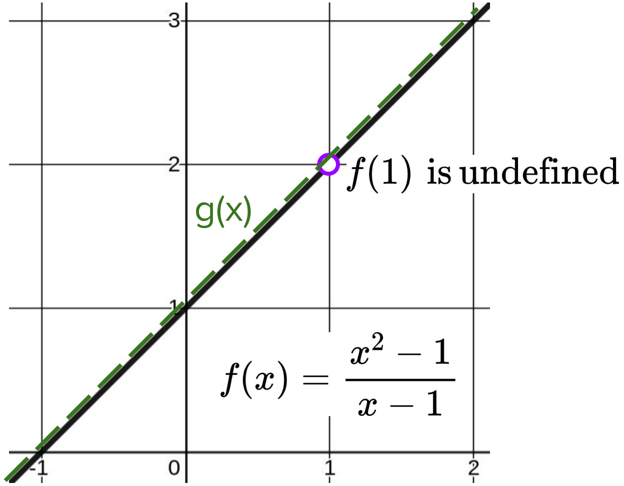
So, a limit is what a function approaches as the input approaches a prescribed value **without reaching it**.

Limit VS Function Value

Hence, a limit can still exist even if the function is undefined at x = a. A second method of evaluating a limit is by factoring & dividing off, i.e.

$\frac{(1)^2-1}{1-1} = \frac{0}{0}!$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$
$$= \lim_{x\to 1} \underbrace{x+1}_{g(x)} = 2$$



In this case, the limit of f(x) is found by making use of a 'similar' function, g(x) = x + 1, which is defined at x = 1. After some algebraic manipulation, direct substitution is used to evaluate the limit.

That begs the question: Why does direct substitution not work in the first place?

Indeterminate Forms

In many cases, an indeterminate form arises when evaluating a limit by direct substitution, such as

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{1^2 - 1}{1 - 1} = \frac{0}{0}$$

And there are other indeterminate forms as shown below.

$$\lim_{x \to a} f(x) = \frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 \times \infty, \infty - \infty, 1^{\infty}, 0^{0}, \infty^{0}$$

As shown, an indeterminate form can arise when a function is undefined at an input value. However, a limit **might** still exist even when this happens. It is **important not to confuse an indeterminate form with a limit that D.N.E., like infinities**.

Indeterminate Forms

To better understand the prior statement, consider the following limits:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{e^x} = \frac{\infty}{\infty} \qquad \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^3}{\ln x} = \frac{\infty}{\infty} \qquad \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{x^3}{x^3+1} = \frac{\infty}{\infty}$$

While all three limits result in the same indeterminate form, they do not result in the same limit. At large x, since e^x increases faster than x^n which increases faster than ln x, we have

(DNE)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{e^x} = 0 \qquad \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^3}{\ln x} = \infty \qquad \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{x^3}{x^3+1} = 1$$

The above limits can be verified easily in Desmos. Hence, an indeterminate form simply means 'cannot be determined directly', and so necessitates further evaluation. The limit might exist like in f(x) & h(x) or might not exist like in g(x).

Indeterminate Forms

In the earlier example, we have the limit for f(x) equal to 0 because the denominator gets bigger than the numerator as x approaches infinity. Understand that ∞/∞ really means a big number divided by another big number, so the limit depends on which number is bigger (or getting bigger faster) as x ➔ ∞.

Similarly, 0/0 is indeterminate because it is a small number divided by another small number, and the limit depends on which number is smaller as x ➔ a.

But why is ∞^0 indeterminate? Again, read it as a big number to the power of a small number. If the base is getting bigger faster than the power getting smaller as x ➔ a, then the limit is infinity. Conversely, if the power is getting to zero faster than the base getting bigger, then the limit is 1. It could also be somewhere in-between.

reach 0 faster

$x^{\frac{1}{\infty}}$ still reaches ∞

$$\lim_{x \to a} f(x)^{g(x)} = \infty^{0.000\ldots1} = \infty \qquad \lim_{x \to a} f(x)^{g(x)} = 999\ldots9^{0} = 1$$

Indeterminate Forms

indeterminate means cannot be determined exactly, need to evaluate f(x) & g(x) further

if limit is ∞ then it DNE and other versions

Exercise: Explain why the limit below is indeterminate. Isn't 1^∞ always equal to 1? What did you notice about the limit of A(n)?

$$\lim_{x\to a} f(x)^{g(x)} = 1^\infty$$

$$0.9999\ldots^{\infty} = 0$$

$$1.000\ldots1^{\infty} = \infty$$

$$1^{999\ldots} = 1$$

$$\lim_{n\to\infty} A(n) = \underbrace{\lim_{n\to\infty} \Big(\overbrace{1+\frac{1}{n}}^{f(n)}\Big)}^{\quad}{}^{\overset{g(n)}{n}} = 2.71828\ldots$$

$$\left(1+\frac{1}{\infty}\right)^{\infty} = (1+0)^{\infty}$$

$$= 1^{\infty}$$

since there can be multiple limits attained, 1^∞ is indeterminate

*An important reason why we have to know how to evaluate indeterminate forms is because the derivative is precisely an indeterminate form, to be studied in the next topic.

** always try substitution first

Evaluating Limits

Exercise: Evaluate the following limits.

a) $\lim_{x \to 5} \sqrt{x-1}$

$= \sqrt{5-1} = 2$

b) $\lim_{x \to 2} \frac{3}{2-x}$

$= \frac{3}{0}$ (DNE)

c) $\lim_{x \to 2} \frac{x^2-4}{2-x} = \frac{0}{0}$

$= \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{-\cancel{(x-2)}}$

$= -(2+2)$

$= -4$

d) $\lim_{x \to -1} \frac{x^2-x-2}{x+1} = \frac{1+1-2}{0} = \frac{0}{0}$

$= \lim_{x \to -1} \frac{\cancel{(x+1)}(x-2)}{\cancel{(x+1)}}$

$= -1-2 = -3$

e) $\lim_{x \to \infty} \frac{0.5^x}{x} = \frac{0}{\infty}$

$= 0$

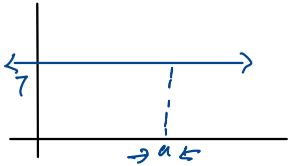
f) $\lim_{x \to \infty} \frac{x^{999}}{(1.1)^x - 999} = \frac{\infty}{\infty}$

$= 0$

since $x^{999} \to \infty$ slower than $(1.1)^x \to \infty$

g) $\lim_{x \to a} 7 = 7$

constant



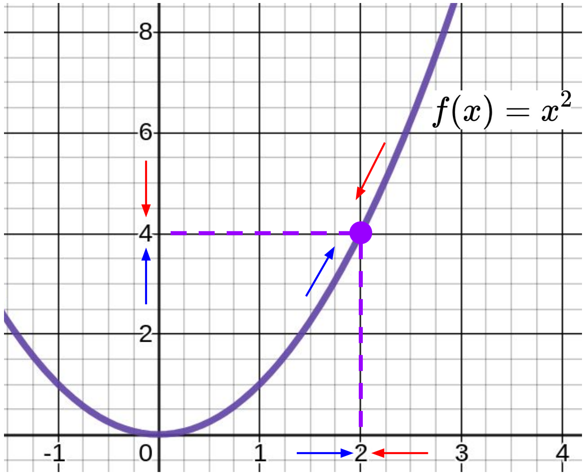
One-sided Limits

To define the existence of a limit more formally, we require

$$\underline{\lim_{x \to a^-} f(x) = L}, \quad \underline{\lim_{x \to a^+} f(x) = L}$$
$$\to \lim_{x \to a} f(x) = L$$

in which the limit at $x \to a^-$ is the limit from the left of a and that at $x \to a^+$ is the limit from the right of a.

This means limit of the function from **both sides must be equal** to the same (finite) value L in order for the limit of f(x) to exist as x approaches a.



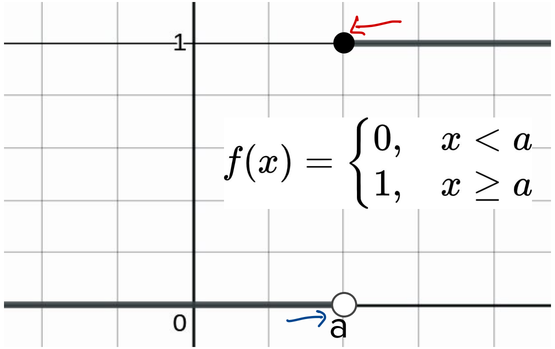
$$\underline{\lim_{x \to 2^-} f(x) = 4}, \quad \underline{\lim_{x \to 2^+} f(x) = 4}$$
$$\to \lim_{x \to 2} f(x) = 4$$

Nonexistent Limits

Now we can define the cases when a limit does not exist (DNE) as shown below.

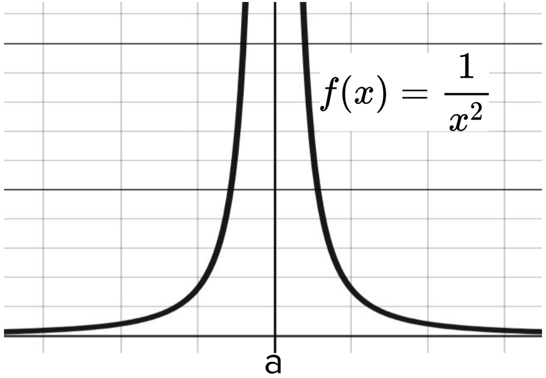
$$\lim_{x \to a^-} f(x) = 0, \quad \lim_{x \to a^+} f(x) = 1$$

$$\lim_{x \to a} f(x) \text{ D.N.E.}$$



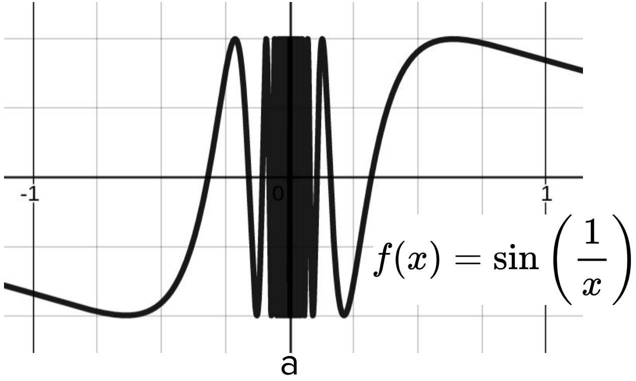
$$\lim_{x \to a} f(x) = \infty$$

$$\lim_{x \to a} f(x) \text{ D.N.E.}$$



Function oscillates ‘wildly’ near x = a.

$$\lim_{x \to a} f(x) \text{ D.N.E.}$$



Realize that in all cases above, the limit is NOT the same (finite) value as x ➔ a from different directions, so the limit DNE.

Evaluating Limits

Exercise: From the graph, state each limit below.

$\lim_{x \to -3^+} f(x) = 2$

$\lim_{x \to -2} f(x) = 1$

$\lim_{x \to -1^-} f(x) = 2$

$\lim_{x \to -1^+} f(x) = 2.5$

$\lim_{x \to -1} f(x) = \text{DNE}$

$f(-1) = 2.5$

function value is defined

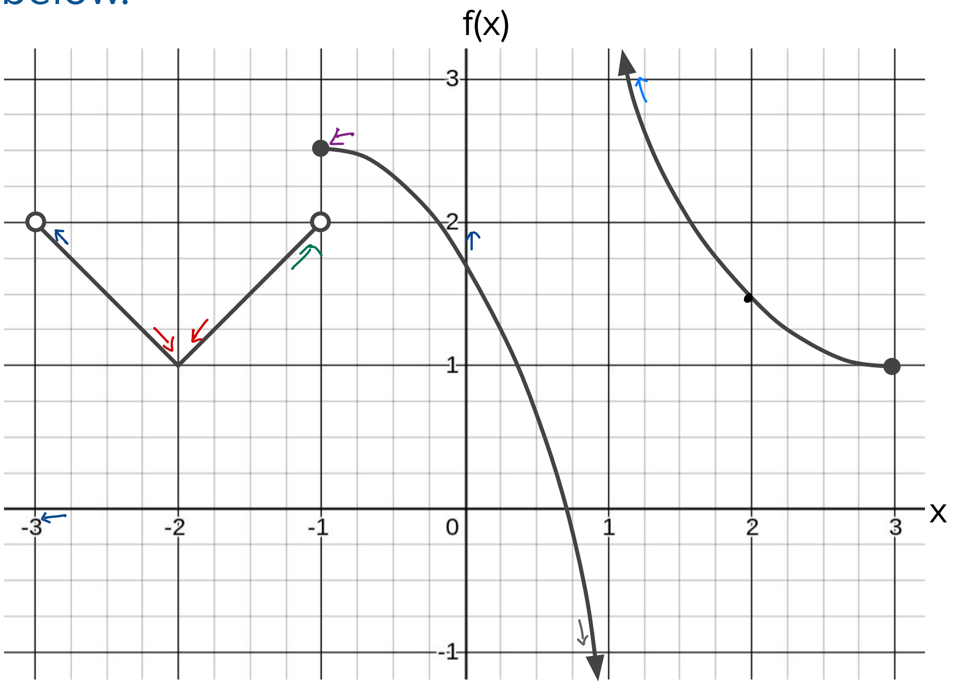
$\lim_{x \to 1^-} f(x) = -\infty$ (DNE)

$\lim_{x \to 1^+} f(x) = +\infty$ (DNE)

$\lim_{x \to 1} f(x) = \text{DNE}$

$\lim_{x \to 2} f(x) = f(2) = 1.5$

$\lim_{x \to 3^-} f(x) = f(3) = 1$



Properties of Limits

Given the limits below, the limit operator follows the following properties.

$$\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} g(x) = K$$

a) $\displaystyle \lim_{x \to a} \{f(x) \pm g(x)\} = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm K$

b) $\displaystyle \lim_{x \to a} \{f(x) \cdot g(x)\} = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = LK$

c) $\displaystyle \lim_{x \to a} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{K}$

d) $\displaystyle \lim_{x \to a} \left\{ f(x)^{g(x)} \right\} = \lim_{x \to a} f(x)^{\lim_{x \to a} g(x)} = L^K$

The limit operator 'operates' on any function that has the input x.

Dividing by x to the Largest Power

When evaluating limits at infinity, such as when finding asymptotes, one of the methods is by dividing by x to the largest power at both the numerator and denominator. For example,

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 1}{x^2 + 5} = \frac{\infty}{\infty}$$

results in an indeterminate form after direct substitution. Notice that x to the largest power is x^2, so

$$\lim_{x\to\infty} \frac{(2x^2 - 3x + 1)}{(x^2 + 5)} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{1}{x^2}}{1 + \frac{5}{x^2}} = \frac{2 - 0 + 0}{1 + 0} = 2$$

This means a horizontal asymptote of the rational function is y = 2 (verify in Desmos).

Dividing by x to the Largest Power

Exercise: Evaluate the following limits.

$\frac{-\infty}{\infty} =$

a) $$\lim_{x\to-\infty} \frac{(x^5+x^3-x+1)\cdot\frac{1}{x^5}}{\underbrace{3x(1-x^2-x^4)}_{3(x-x^3-x^5)}\cdot\frac{1}{x^5}}$$

$$\lim_{x\to-\infty} \frac{1+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^5}}{3\left(\frac{1}{x^4}-\frac{1}{x^2}-1\right)}$$

$$= \frac{1+0-0+0}{3(0-0-1)} = -\frac{1}{3}$$

b) $$\lim_{x\to\infty} \overbrace{\left\{ \underbrace{\frac{7x\sqrt{x^2+1}}{2x^2-1}} - \frac{1}{\sqrt{x}-1} \right\}}^{f(x)} = \frac{\infty}{\infty} - \underbrace{\frac{1}{\infty}}_{0}$$

$$7\lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{x^2+1}}{\sqrt{(2x^2-1)^2}} = 7\lim_{x\to\infty}\sqrt{\frac{x^2(x^2+1)}{(2x^2-1)^2}}$$

$$= 7\lim_{x\to\infty}\sqrt{\frac{x^4+x^2}{4x^4-4x^2+1}} \cdot \frac{\sqrt{\frac{1}{x^4}}}{\sqrt{\frac{1}{x^4}}}$$

$$= 7\lim_{x\to\infty}\sqrt{\frac{1+\frac{1}{x^2}}{4-\frac{4}{x^2}+\frac{1}{x^4}}} = 7\sqrt{\frac{1}{4}} = \frac{7}{2}$$

$$\lim_{x\to\infty} f(x) = \frac{7}{2} - 0$$
$$= \frac{7}{2}$$

ANS: a) -1/3. b) 7/2.

Rationalizing

Rationalizing the numerator or denominator (or both) is another technique to evaluate a limit. Consider

$$\lim_{x\to 1} \frac{\sqrt{2-x}-1}{x-1} = \frac{0}{0}$$

which gives an indeterminate form after direct substitution. And factoring does not work. Notice that the numerator is an irrational expression, which can be rationalized by multiplying by its conjugate, i.e.

$$\lim_{x\to 1} \frac{\sqrt{2-x}-1}{x-1} \cdot \frac{\sqrt{2-x}+1}{\sqrt{2-x}+1} = \lim_{x\to 1} \frac{2-x-1}{(x-1)\left(\sqrt{2-x}+1\right)}$$

$$= \lim_{x\to 1} \frac{-(x-1)}{(x-1)\left(\sqrt{2-x}+1\right)} = \lim_{x\to 1} \frac{-1}{\sqrt{2-x}+1} = -\frac{1}{2}$$

(Verify in Desmos.)

Rationalizing

OR : $\lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{2-\sqrt{x}} = \frac{-\cancel{(2-\sqrt{x})}(\sqrt{x}+2)}{\cancel{2-\sqrt{x}}} = -(\sqrt{x}+2)$

$= -(\sqrt{4}+2) = -4$

Exercise: Evaluate the limits below. (**Always try direct substitution first.**)

a) $\lim_{x\to 4} \frac{x-4}{2-\sqrt{x}} \frac{(2+\sqrt{x})}{(2+\sqrt{x})}$

$\frac{0}{0}$

$= \lim_{x\to 4} \frac{(x-4)(2+\sqrt{x})}{4-x}$

$= \lim_{x\to 4} \frac{\cancel{(x-4)}(2+\sqrt{x})}{-\cancel{(x-4)}}$

$= -(2+\sqrt{4}) = -4$

b) $\lim_{x\to 2} \frac{3-\sqrt{x+7}}{2-\sqrt{x+2}} \left(\frac{3+\sqrt{x+7}}{3+\sqrt{x+7}}\right)\left(\frac{2+\sqrt{x+2}}{2+\sqrt{x+2}}\right)$

$\frac{0}{0}$

$= \lim_{x\to 2} \frac{[9-(x+7)][2+\sqrt{x+2}]}{(4-(x+2))(3+\sqrt{x+7})}$

$= \lim_{x\to 2} \frac{\cancel{(2-x)}[2+\sqrt{x+2}]}{\cancel{(2-x)}(3+\sqrt{x+7})}$

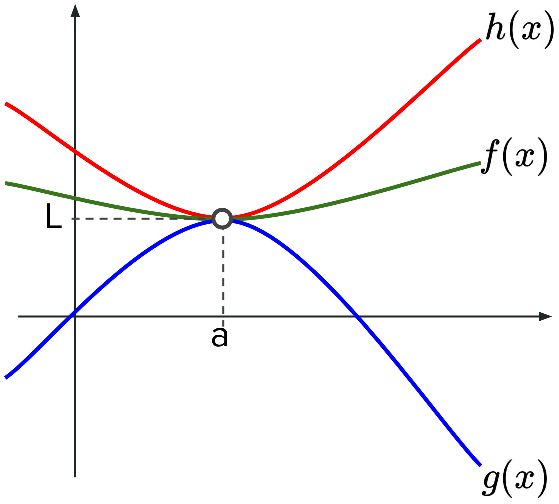
$= \frac{2+\sqrt{4}}{3+\sqrt{9}} = \frac{4}{6} = \frac{2}{3}$

ANS: a) -4. b) ⅔.

Squeeze Theorem

As illustrated on the graph, the squeeze theorem works by 'squeezing' a function f(x) in-between another two functions, h(x) & g(x), such that

$$g(x) \leq f(x) \leq h(x)$$



for all x ≠ a in an open interval containing a. If

$$\lim_{x \to a} g(x) = L, \quad \lim_{x \to a} h(x) = L$$

then the squeeze theorem states that

$$\lim_{x \to a} f(x) = L$$

which is logical. The main idea is, find h(x) and g(x) that 'squeezes' f(x) and use them to evaluate the limit for f(x).

Squeeze Theorem

For example, try to evaluate the limit

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Notice that although the limit of sin(1/x) DNE, we know that the range of the sine function is [-1, 1]. So

$$-1 \leq \sin(1/x) \leq 1$$

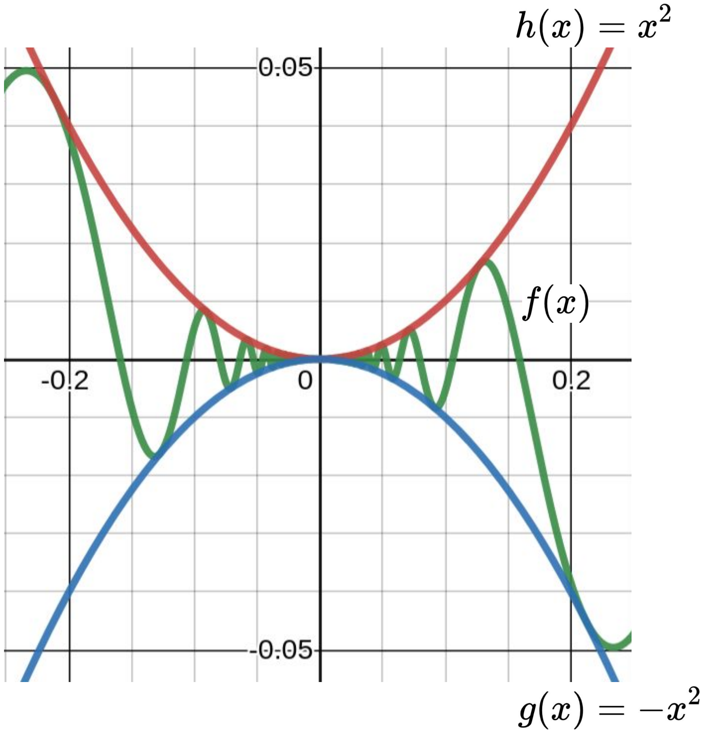
While on inequality from the function that is bounded and work towards f(x)

$$\times x^2 \to -x^2 \leq x^2 \sin(1/x) \leq x^2$$

$$\text{And } \underline{\lim_{x\to 0} -x^2 = 0}, \quad \underline{\lim_{x\to 0} x^2 = 0}$$

Hence, by squeeze theorem,

$$\underline{\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0}$$



Squeeze Theorem

Exercise: Evaluate the following limits.

a) $\lim\limits_{x\to\infty}\left\{\underbrace{\frac{\cos\sqrt{x}}{x}-9}_{f(x)}\right\}$.

$$-1 \le \cos\sqrt{x} \le 1$$

$\div x$

$$-\frac{1}{x} \le \frac{\cos\sqrt{x}}{x} \le \frac{1}{x}$$

-9

$$\underbrace{-\frac{1}{x}-9}_{g(x)} \le \underbrace{\frac{\cos\sqrt{x}}{x}-9}_{f(x)} \le \underbrace{\frac{1}{x}-9}_{h(x)}$$

$$\lim_{x\to\infty}\left(-\frac{1}{x}-9\right)=-9 \qquad \lim_{x\to\infty}\left(\frac{1}{x}-9\right)=-9$$

$\Rightarrow$ so by squeeze theorem, $\lim\limits_{x\to\infty} f(x) = -9$

b) $\lim\limits_{x\to 3}(x-3)f(x)$, where $-5 \le f(x) \le 7 \;\forall x \ne 3$

$$-5 \le f(x) \le 7$$

$\times(x-3)$

$$-5(x-3) \le f(x)(x-3) \le 7(x-3)$$

$$\lim_{x\to 3}\{-5(x-3)\} = 0 = \lim_{x\to 3}\{7(x-3)\}$$

By squeeze theorem,

$$\lim_{x\to 3}(x-3)f(x) = 0$$

ANS: a) -9. b) 0.

Squeeze Theorem

The squeeze theorem can also be used to derive the following important limits, which will be required in the next topic on derivatives.

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$\to \frac{0}{0}$

$\frac{\sin x}{\cos x} = \frac{DF}{1}$

$\Rightarrow DF = \tan x$

By similar triangles.

From the figure shown, observe that the arc length x is in-between lengths BE and FD, so

$$\sin x \le x \le \tan x$$

$1 < 2 < 3$

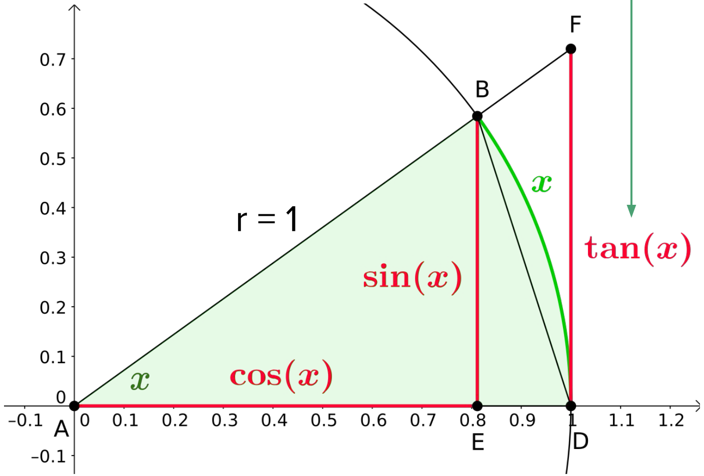
$$\div \sin x \to 1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

Inverse signs

Take reciprocal

$$\to \cos x \le \frac{\sin x}{x} \le 1$$

$1 > \frac{1}{2} > \frac{1}{3}$



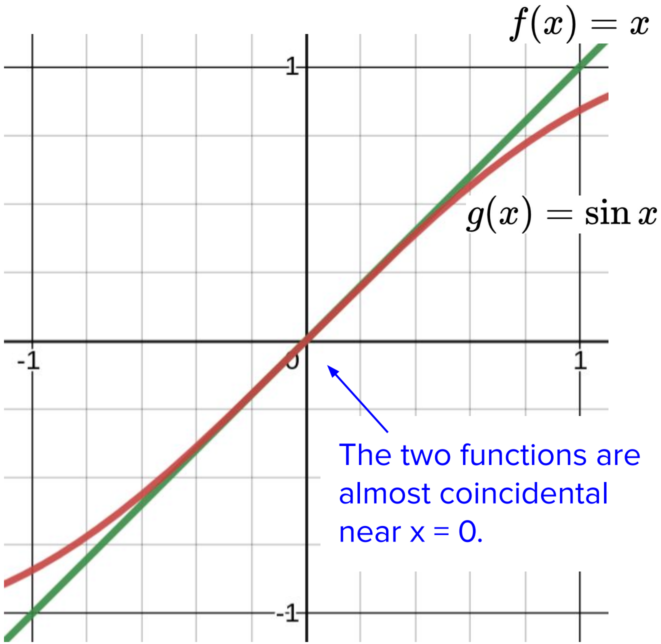
https://en.wikipedia.org/wiki/Squeeze_theorem#/media/File:Limit_sin_x_x.svg

Squeeze Theorem

Since it is clear that $\lim_{x \to 0} \cos x = 1, \ \lim_{x \to 0} 1 = 1$, by squeeze theorem we have

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

which agrees with the graph shown. This explains the small angle approximation formula, sin x ≈ x, when angle x is small.



From another geometrical perspective, notice from the previous slide that the arc length x approaches tan x (which approaches sin x) as the angle x approaches 0. So that's why for small x we have

$$\sin x \approx x, \quad \cos x \approx 1, \quad \underset{\approx \frac{x}{1}}{\tan x} \approx x \quad \text{(Small-angle approximation formulas)}$$

Squeeze Theorem + Rationalizing

Exercise: Prove the limit below by rationalizing using conjugate multiplication.

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0} \frac{1-\cos x}{x}\left(\frac{1+\cos x}{1+\cos x}\right)$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\frac{\sin x}{(1+\cos x)}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right) \lim_{x\to 0} \frac{\sin x}{(1+\cos x)}$$

$$= 1\left(\frac{0}{2}\right) = 0$$

TRY :

$$-1 \le -\cos x \le 1$$

$+1$

$$0 \le 1-\cos x \le 2$$

$\div x$

$$0 \le \frac{1-\cos x}{0} \le \frac{2}{x}$$

But $\lim_{x\to 0} \frac{2}{x} = \pm\infty$

squeeze theorem fails

Evaluating Limits

We have covered quite a few techniques of limit evaluation. The table below provides a general guide of when to use which (may not necessarily work).

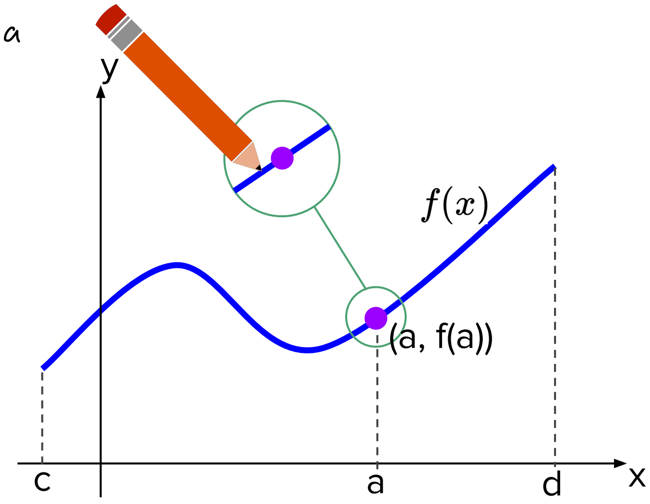
Technique	Generally works for:
Direct substitution	Continuous functions. **Always try first.**
Factoring and divide off	Rational functions giving indeterminate form of 0/0.
Compare growth rates	Ratio of elementary functions as x approaches ±∞.
Divide by x to the largest power	Rational functions giving indeterminate form of ±∞/±∞.
Rationalizing	Functions with irrational (root) expressions.
Squeeze theorem	Functions with oscillatory behaviour.
L'Hopital's Rule	Functions giving giving indeterminate form of 0/0 or ±∞/±∞. (Covered in a later topic.)

Continuity

In layman, continuity of a function at x = a implies that a line can be drawn in one stroke across the point (a, f(a)). Mathematically, this means

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Check: If $f(x)$ is continuous at $x = a$
1) Limit must exist at $x = a$
2) Limit must be equal to function value



If the above equation is satisfied at every point in an interval (c, d), then we say the function is continuous in (c, d).

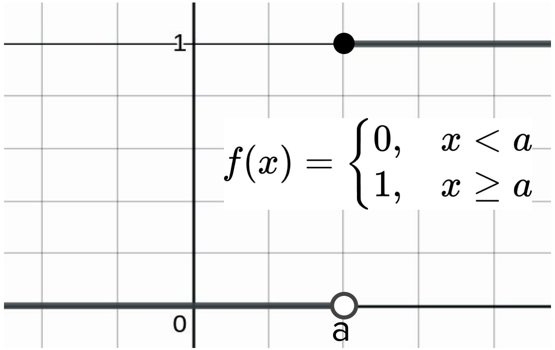
For example, any polynomial function is continuous everywhere since the above equation is satisfied for all x = a in ℝ.

Types of Discontinuity

We have seen the various types of discontinuity in earlier slides. They are described below. The removable discontinuity is called ‘removable’ because one can define f(a) = L to ‘remove’ the discontinuity. This cannot be done for other types.

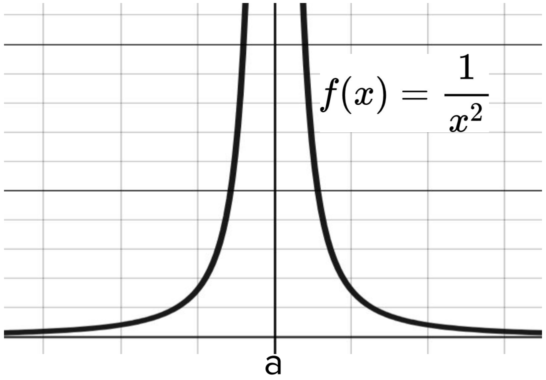
Jump discontinuity

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$
$$\lim_{x \to a} f(x) \text{ D.N.E.}$$



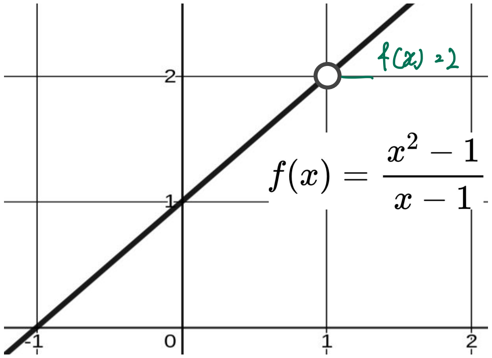
Infinite discontinuity

$$\lim_{x \to a^\pm} f(x) = \pm\infty$$
$$\lim_{x \to a} f(x) \text{ D.N.E.}$$



Removable discontinuity

$$\lim_{x \to a} f(x) = L$$
$$f(a) \neq L$$



Evaluating Intervals of Continuity

To evaluate the interval/s where a function is continuous, one can look for discontinuous point/s and exclude them (if necessary). For example, consider the piecewise function

$$f(x) = \begin{cases} 1/x, & x < -1 \\ 3x - 2, & -1 \leq x < 2 \\ \frac{x^2 - x - 6}{x - 3}, & x \geq 2 \end{cases}$$

In the first function 1/x, an infinite discontinuity exists at x = 0, but it is outside of its interval (-∞, -1), so we ignore this discontinuity. This means f(x) is continuous in (-∞, -1).

In the third function, we find a removable discontinuity at x = 3, which is inside its interval, so f(x) is discontinuous at x = 3.

After **evaluating discontinuities in each function**, we have to **check at the interval transitions**.

Evaluating Intervals of Continuity

At x = -1, we have

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 1/x = -1, \ \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 3x - 2 = 3(-1) - 2 = -5$$
$$\to \lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$$

So there is jump discontinuity at x = -1. And at x = 2, we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 3x - 2 = 4, \ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2 - x - 6}{x - 3} = \frac{4 - 2 - 6}{2 - 3} = 4$$
$$\to \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2} f(x) = f(2)$$

So f(x) is continuous at x = 2. Consolidating all results, we found two discontinuities, at x = -1 and x = 3. Hence, f(x) is continuous in (-∞, -1) U (-1, 3) U (3,∞).

Evaluating Intervals of Continuity

Exercise: Determine the intervals where f(x) is continuous.

$$f(x) = \begin{cases} \underbrace{\dfrac{x^2-9}{x^2+2x-3}}_{f_1(x)}, & x < 0 \\ \underbrace{3e^x}_{f_2(x)}, & x \geq 0 \end{cases}$$

Step 1: Look for discontinuities in each function

In $f_1(x) = \dfrac{(x-3)(x+3)}{(x-1)(x+3)}$, discontinuity at $x=1$ & $x=-3$, but ignore $x=1$ since domain is $x<0$

(removable)

And $f_2(x)$ has no discontinuity

Step 2: Look for discontinuity at interval transitions

At $x=0$, $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} f_1(x) = \dfrac{-9}{-3} = 3$, $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} 3e^x = 3 = f(0)$

ANS: (-∞, -3) U (-3, ∞)

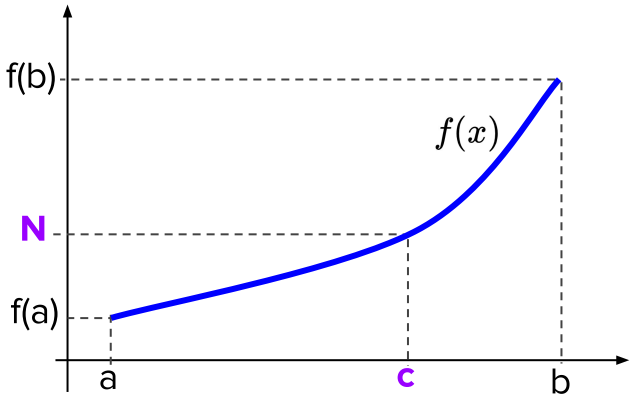
$\Rightarrow \lim_{x \to 0} f(x) = f(0)$, so $f(x)$ is continuous at $x=0$. $\therefore$ Only 1 discontinuity at $x=-3$, so $f(x)$ is continuous in $\{\mathbb{R} \mid x \neq -3\}$

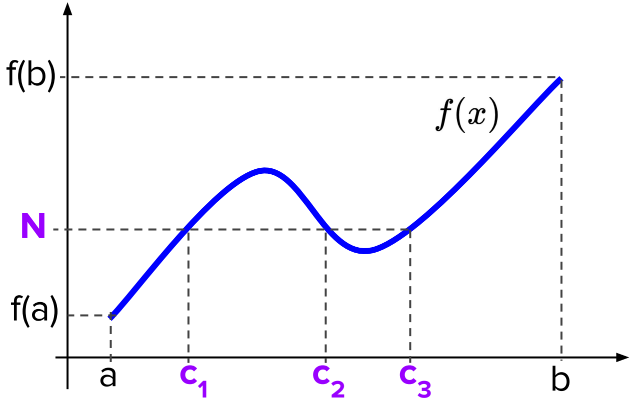
Intermediate Value Theorem

The intermediate value theorem (IVT) states a logical consequence from the continuity of a function over a closed interval.

If a function f(x) is **continuous in [a, b]** and N is any real number between f(a) and f(b), then the IVT states that there must be at **least one c in [a, b]** such that **f(c) = N**.

In a layman analogy of growing up, one's height has to reach 120 cm first before reaching 160 cm, because the growth in height is continuous. (120 cm here is the intermediate value.)





Intermediate Value Theorem

To apply the IVT, it is important to verify first the continuity of the function in the given interval. If any discontinuity exists in the interval, IVT cannot be applied.

For example, we cannot apply IVT for f(x) = 1/x in [-2, 2], because the function is discontinuous at x = 0, inside [-2, 2]. Clearly, there is no c in [-2, 2] such that f(c) = 1/c = 0.

Exercise: Which of the functions below can the IVT apply in [-5, 9]?

a) $g(x) = x^3 + 2x - 7$ ✓

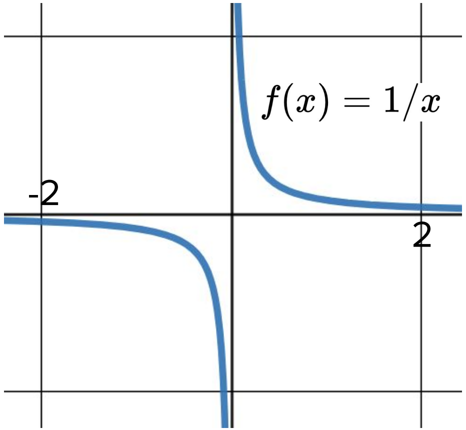
b) $h(x) = \ln x$ ✗ since $\ln x$ is undefined $\forall\ x \leq 0$

c) $p(x) = \dfrac{1}{x+6}$ ✓

since $p(x)$ is cont. in $[-5, 9]$

d) $q(x) = \dfrac{x^2 - 1}{x + 1}$ ✗

since $q(x)$ is discont. at $x = -1$, in $[-5, 9]$



Intermediate Value Theorem

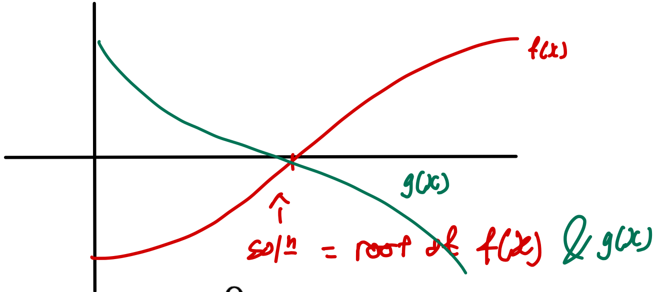
The IVT provides a logical approach to check the existence of a solution of an equation. This is extremely useful because we cannot solve many equations analytically. For example, there is no analytical solution for

$$x = \cos x$$

which looks deceptively simple to solve. Truth is, besides polynomial equations of degree less than 5, there is no general analytical method to solve most equations.

However, there is another approach, called **numerical methods, that can potentially solve any equation**. Hence, we first use the IVT to check if a solution exist. If so, numerical methods are applied.

Intermediate Value Theorem



For example, we can rewrite

$$x = \cos x \to f(x) = x - \cos x = 0$$

or $g(x) = \cos x - x = -f(x) = 0$

which is continuous everywhere. Now, we want to find a closed interval [a, b] such that f(x) = 0 (x is a root of f(x) in [a, b]). Hence, we want f(a) < 0 & f(b) > 0 or vice-versa. By trial and error,

$$f(0) = 0 - 1 = -1 < 0$$
$$f(1) = 1 - \cos(1) = 0.46 > 0$$

Hence, the IVT states that there is at least one root in [0, 1] such that f(x) = 0. In the next slide, we introduce a numerical method for finding the root in the interval.

Exercise: Explain why g(x) = cos(x) - x will give the same root as f(x).

Numerical Method: Bisection Method

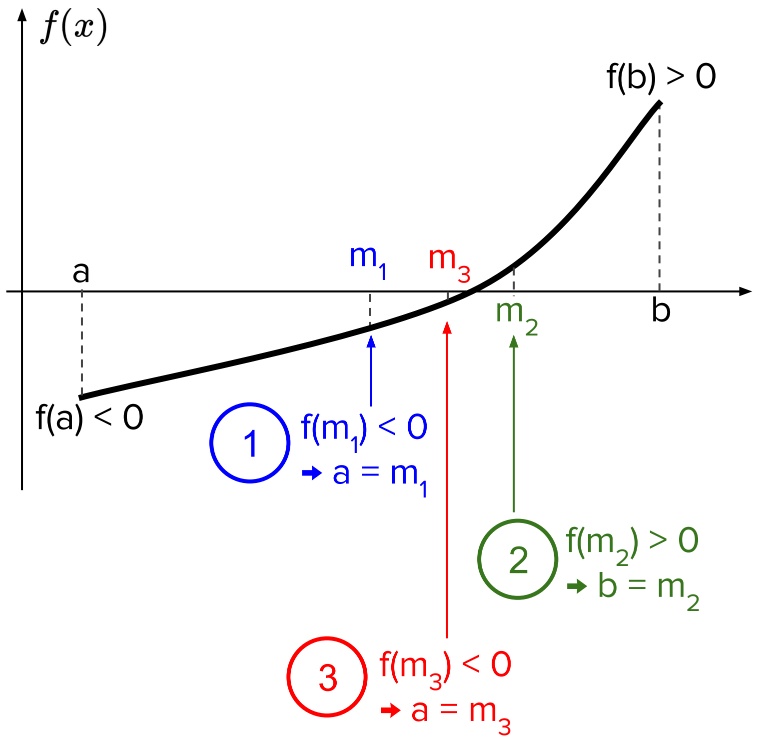
As implied by its name, this method bisects the interval containing the root repeatedly until f(x) is close enough to zero. As illustrated by the graph, the iterative procedure is:

i. Bisect: $m_i = ½(a+b)$
ii. Compute $f(m_i)$.
iii. Move a or b or stop:

   if $f(m_i) \approxeq 0$, root found & stop.
   else if $f(m_i) < 0$, then a = m.
   else if $f(m_i) > 0$, then b = m.

iv. Repeat until max number of iterations reached.

*The first 3 bisections are illustrated below. Notice that the **root is being straddled** consistently.*



Numerical Method: Bisection Method

Certainly, numerical methods are best applied by a computer program or spreadsheet, but let's apply it manually first for clearer understanding. Let's continue to solve

$$f(x) = x - \cos x = 0$$

in the interval [0, 1]. We get

$$\begin{aligned}
m_1 &= \frac{0+1}{2} = 0.5 \to f(0.5) = 0.5 - \cos(0.5) = -0.3776 < 0 \to a = 0.5 \\
m_2 &= \frac{0.5+1}{2} = 0.75 \to f(0.75) = 0.75 - \cos(0.75) = 0.0183 > 0 \to b = 0.75 \\
m_3 &= \frac{0.5+0.75}{2} = 0.625 \to f(0.625) = 0.625 - \cos(0.625) = -0.186 > 0 \to a = 0.625 \\
&\vdots
\end{aligned}$$

See python program to continue iterations to obtain the solution.

Numerical Method: Bisection Method

Exercise: Find an interval where the solution to the equation below lies and use the bisection method program to give an approximation of the solution.

$$x^2 + \ln(x+1) = 2$$

Let $f(x) = x^2 + \ln(x+1) - 2 = 0$

Trial: $f(0) = 0^2 + \ln(1) - 2 = 0 - 2 = -2 < 0$

$f(1) = 1^2 + \ln(2) - 2 = 1^2 + 0.693 - 2 < 0$

$f(2) = 4 + \ln(3) - 2 > 0$

$[1, 2]$ is where root/s of $f(x)$ lies

since $f(x)$ is continuous in $[1,2]$, by IVT, at least one root exists in $[1,2]$

ANS: [1, 2]. x ≈ 1.1179

End of Topic 5

Evaluating a limit is something like asking your crush out for a date.