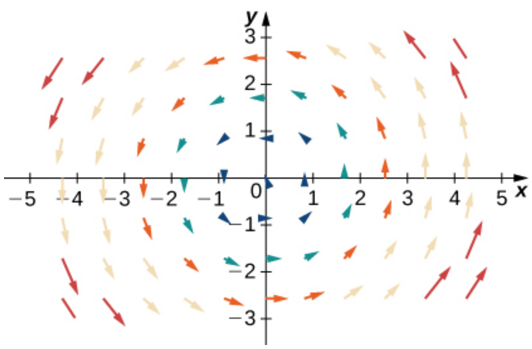
EDE1012 MATHEMATICS 2

Tutorial 5

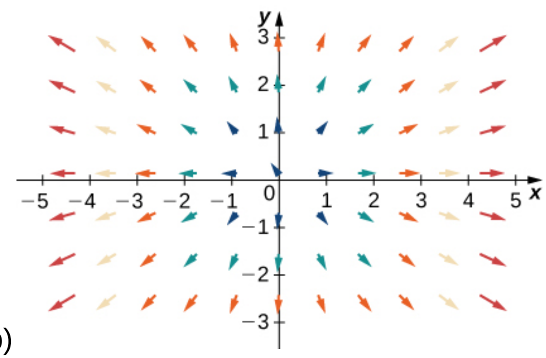
Vector Calculus I

1. (https://openstax.org/books/calculus-volume-3/pages/6-1-vector-fields)
Without using a graphing tool, match the vector fields below with their graphs. Evaluate the Jocobian and divergence of each vector field.

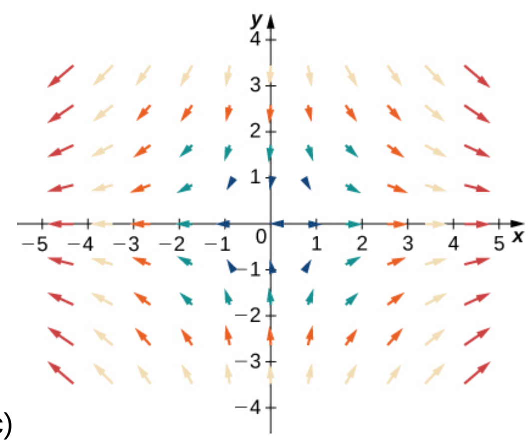
$$\mathbf{F}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{G}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}, \qquad \mathbf{H}(x,y) = \begin{bmatrix} x \\ -y \end{bmatrix}$$



a)



b)



c)

ANS: **a)** $\mathbf{G}(x,y)$. $\mathbf{J}_\mathbf{G} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $\nabla \cdot \mathbf{G} = 0$. **b)** $\mathbf{F}(x,y)$. $\mathbf{J}_\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\nabla \cdot \mathbf{F} = 2$.

c) $\mathbf{H}(x,y)$. $\mathbf{J}_\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. $\nabla \cdot \mathbf{H} = 0$.

2. Evaluate the Jacobian, divergence and curl of the velocity field below. Then, determine them at the point (1, 2, 3).

$$\mathbf{V}(x,y,z) = \left[x^2 z,\ ye^{2z},\ xyz\right]^T$$

ANS: $\mathbf{J_V} = \begin{bmatrix} 2xz & 0 & x^2 \\ 0 & e^{2z} & 2ye^{2z} \\ yz & xz & xy \end{bmatrix}$. $\nabla \cdot \mathbf{V} = 2xz + e^{2z} + xy$. $\nabla \times \mathbf{V} = \begin{bmatrix} xz - 2ye^{2z} \\ x^2 - yz \\ 0 \end{bmatrix}$.

$\text{At } (1,2,3): \mathbf{J_V} = \begin{bmatrix} 6 & 0 & 1 \\ 0 & e^6 & 4e^6 \\ 6 & 3 & 2 \end{bmatrix}$. $\nabla \cdot \mathbf{V} = 8 + e^6$. $\nabla \times \mathbf{V} = \begin{bmatrix} 3 - 4e^6 \\ -5 \\ 0 \end{bmatrix}$.

3. Using a vector field graphing tool (https://www.geogebra.org/m/QPE4PaDZ), plot the vector field below and evaluate the curl vector. Explain why the curl is the zero vector despite the vector field appearing 'rotational'.

$$\mathbf{F}(x,y,z) = \left[\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ 0\right]^T$$

ANS: $\nabla \times \mathbf{F} = \mathbf{0}$.

4. Determine if each vector field below is a gradient field (conservative). If so, evaluate the scalar potential function E such that **F** = ∇E.

a) $\mathbf{F}(x,y) = \left[2x \sin y,\ x^2 \cos y\right]^T$

b) $\mathbf{F}(x,y,z) = \left[e^x \sin y,\ e^x \cos y,\ 3z^2 + 2\right]^T$

c) $\mathbf{F}(x,y,z) = \left[e^{-yz},\ e^{xyz},\ 2xy\right]^T$

ANS: **a)** Yes. $E(x,y) = x^2 \sin y + c$. **b)** Yes. $E(x,y,z) = e^x \sin y + z^3 + 2z + c$. **c)** No.

5. A force field (in Newtons) is defined by

$$\mathbf{F}(x,y) = \left[\frac{x}{y^2} + x, \ -\frac{x^2+1}{y^3}\right]^T, \ y \neq 0$$

where coordinates x and y are in meters.

a) Show that the force field is conservative.

b) Evaluate a scalar potential E(x,y) such that **F** = ∇E.

c) Calculate the work done required to move an object subjected to the force field from point (0, 1) to point (1, 1), along the curve $y = 1 + x - x^2$.

ANS: **a)** Yes. **b)** $E(x,y) = \frac{x^2}{2y^2} + \frac{x^2}{2} + \frac{1}{2y^2} + c.$ **c)** Work done = 1 J.

6. A vector field shown below contains scalar functions f(x), g(y) and h(z) that are differentiable.

$$\mathbf{V}(x,y,z) = \begin{bmatrix} f(x) + y + z \\ g(y) + x + z \\ h(z) + x + y \end{bmatrix}$$

a) Determine if the vector field is conservative. If so, evaluate the scalar potential E such that ∇E = **V**.

b) Evaluate the line integral below, where C is any path from (x_0, y_0, z_0) to (x_1, y_1, z_1).

$$L = \int_C \mathbf{V} \cdot \mathbf{dr}$$

ANS: **a)** Yes. $E(x,y,z) = xy + xz + yz + F(x) + G(y) + H(z)$, where F(x), G(y) and H(z) are antiderivatives of f(x), g(y) and h(z).

b) $L = x_1y_1 + x_1z_1 + y_1z_1 + F(x_1) + G(y_1) + H(z_1) - x_0y_0 - x_0z_0 - y_0z_0 - F(x_0) - G(y_0) - H(z_0)$.

7. (https://openstax.org/books/calculus-volume-3/pages/6-2-line-integrals)
   Evaluate the line integral of each function defined below over the path given.

   a) $F(x,y) = xy^4$ over the curve C that is the right half of circle $x^2 + y^2 = 16$ and traversed in the clockwise direction.

   b) $F(x,y) = xe^y$ over the curve C that is the arc of curve $x = e^y$ from (1, 0) to (e, 1).

   c) $F(x,y,z) = 1/\left(x^2 + y^2 + z^2\right)$ over the curve C that is the helix $x = \cos t,\ y = \sin t,\ z = t,$ from t = 0 to t = T.

ANS: **a)** -8192/5. **b)** 7.157. c) $\sqrt{2}\tan^{-1} T$.

8. Evaluate the line integral of the vector field below over the straight line path from (1, 1) to (3, 5).

$$\mathbf{F}(x,y) = \begin{bmatrix} x^2 - y \\ x - y^2 \end{bmatrix}$$

ANS: -92/3.

9. Consider a vector field **F**(x,y) over two paths **r**(t) and **s**(t) given below.

$$\mathbf{F}(x,y) = \begin{bmatrix} x^2 + y \\ y - x \end{bmatrix}, \qquad \mathbf{r}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}, \quad 0 \le t \le 1,$$
$$\mathbf{s}(t) = \begin{bmatrix} 1 - 2t \\ 4t^2 - 4t + 1 \end{bmatrix}, \quad 0 \le t \le 1/2$$

   a) Evaluate the line integrals of **F** over each of the two paths.

   b) What are the Cartesian equations and directions of path **r** and **s**? Hence explain the values obtained in (a).

ANS: **a)** Path **r**: ½. Path **s**: -½. **b)** Path **r**: y = x^2, (0, 0) to (1, 1).
Path **s**: y = x^2, (1, 1) to (0, 0).

10. For a radial vector field **F**(x, y, z) = $[x, y, z]^T$, show that its line integral over any path that is on a sphere

$$x^2 + y^2 + z^2 = \rho^2$$

is always zero. Sketch a graph and explain why.

11. Verify Green's theorem for **F**(x, y) below over the semicircular region D given by $x^2 + y^2 \leq R^2$, $y \geq 0$.

$$\mathbf{F}(x, y) = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

ANS: 0.

12. Using Green's theorem, evaluate the line integral for **F**(x,y) below, C is the boundary of the square with vertices (0, 0), (1, 0), (0, 1) and (1, 1), oriented clockwise.

$$\mathbf{F}(x, y) = \begin{bmatrix} x - y^2 \\ x + y^2 \end{bmatrix}$$

ANS: -2.

13. Given that C is any closed path in $\mathbb{R}^2$ oriented counterclockwise, show that the line integral below is independent of the path C and only dependent on the area enclosed by C.

$$\int_C \left(x^2 y^3 - 3y\right) dx + x^3 y^2 \, dy$$

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-3/pages/6-1-vector-fields
2) https://openstax.org/books/calculus-volume-3/pages/6-2-line-integrals
3) https://openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields
4) https://openstax.org/books/calculus-volume-3/pages/6-4-greens-theorem
5) https://openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl

End of Tutorial 5

(Email to youliangzheng@gmail.com for assistance.)