$$\vec{\mathbf{V}} = V_1\vec{\mathbf{e}}_1 + V_2\vec{\mathbf{e}}_2 + V_3\vec{\mathbf{e}}_3. \tag{13.16a}$$

If we take the coordinate unit vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ as the base vectors and express the coefficients V_1, V_2, V_3 in Cartesian coordinates, then we get:

$$\vec{\mathbf{V}} = V_x(x,y,z)\vec{\mathbf{i}} + V_y(x,y,z)\vec{\mathbf{j}} + V_z(x,y,z)\vec{\mathbf{k}}. \tag{13.16b}$$

So, the vector field can be defined with the help of three scalar functions of three scalar variables.

2. Vector Field in Cylindrical and Spherical Coordinates

In cylindrical and spherical coordinates, the coordinate unit vectors

$$\vec{\mathbf{e}}_\rho,\ \vec{\mathbf{e}}_\varphi,\ \vec{\mathbf{e}}_z\ (=\vec{\mathbf{k}}), \qquad \text{and} \qquad \vec{\mathbf{e}}_r\ (=\frac{\vec{\mathbf{r}}}{r}),\ \vec{\mathbf{e}}_\vartheta,\ \vec{\mathbf{e}}_\varphi \tag{13.17a}$$

are tangents to the coordinate lines at each point **(Fig. 13.6, 13.7)**. In this order they always form a right-handed system. The coefficients are expressed as functions of the corresponding coordinates:

$$\vec{\mathbf{V}} = V_\rho(\rho,\varphi,z)\vec{\mathbf{e}}_\rho + V_\varphi(\rho,\varphi,z)\vec{\mathbf{e}}_\varphi + V_z(\rho,\varphi,z)\vec{\mathbf{e}}_z\,, \tag{13.17b}$$

$$\vec{\mathbf{V}} = V_r(r,\vartheta,\varphi)\vec{\mathbf{e}}_r + V_\vartheta(r,\vartheta,\varphi)\vec{\mathbf{e}}_\vartheta + V_\varphi(r,\vartheta,\varphi)\vec{\mathbf{e}}_\varphi. \tag{13.17c}$$

At transition from one point to the other, the coordinate unit vectors change their directions, but remain mutually perpendicular.

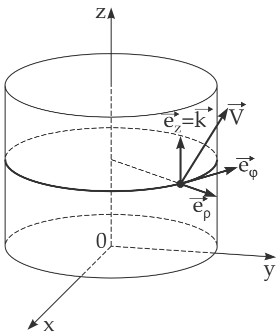

Figure 13.6

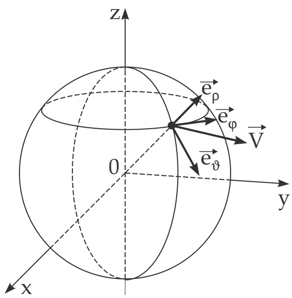

Figure 13.7

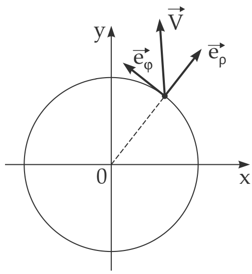

Figure 13.8

13.1.3.4 Transformation of Coordinate Systems

See also **Table 13.1**.

1. Cartesian Coordinates in Terms of Cylindrical Coordinates

$$V_x = V_\rho\cos\varphi - V_\varphi\sin\varphi, \qquad V_y = V_\rho\sin\varphi + V_\varphi\cos\varphi, \qquad V_z = V_z. \tag{13.18}$$

2. Cylindrical Coordinates in Terms of Cartesian Coordinates

$$V_\rho = V_x\cos\varphi + V_y\sin\varphi, \qquad V_\varphi = -V_x\sin\varphi + V_y\cos\varphi, \qquad V_z = V_z. \tag{13.19}$$

3. Cartesian Coordinates in Terms of Spherical Coordinates

$$\begin{aligned} V_x &= V_r\sin\vartheta\cos\varphi - V_\varphi\sin\varphi + V_\vartheta\cos\varphi\cos\vartheta, \\ V_y &= V_r\sin\vartheta\sin\varphi + V_\varphi\cos\varphi + V_\vartheta\sin\varphi\cos\vartheta, \\ V_z &= V_r\cos\vartheta - V_\vartheta\sin\vartheta. \end{aligned} \tag{13.20}$$

4. Spherical Coordinates in Terms of Cartesian Coordinates

$$\begin{aligned} V_r &= V_x\sin\vartheta\cos\varphi + V_y\sin\vartheta\sin\varphi + V_z\cos\vartheta, \\ V_\vartheta &= V_x\cos\vartheta\cos\varphi + V_y\cos\vartheta\sin\varphi - V_z\sin\vartheta, \end{aligned} \tag{13.21}$$

$$V_\varphi = -V_x \sin\varphi + V_y \cos\varphi.$$

5. Expression of a Spherical Vector Field in Cartesian Coordinates

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2+y^2+z^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}). \tag{13.22}$$

6. Expression of a Cylindrical Vector Field in Cartesian Coordinates

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2+y^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}}). \tag{13.23}$$

In the case of a spherical vector field, spherical coordinates are most convenient for investigations, i.e., the form $\vec{\mathbf{V}} = V(r)\vec{\mathbf{e}}_r$; and for investigations in cylindrical fields, cylindrical coordinates are most convenient, i.e., the form $\vec{\mathbf{V}} = V(\varphi)\vec{\mathbf{e}}_\varphi$. In the case of a plane field **(Fig. 13.8)**, we have

$$\vec{\mathbf{V}} = V_x(x,y)\vec{\mathbf{i}} + V_y(x,y)\vec{\mathbf{j}} = V_\rho(\rho,\varphi)\vec{\mathbf{e}}_\rho + V_\varphi(\rho,\varphi)\vec{\mathbf{e}}_\varphi, \tag{13.24}$$

and for a circular field

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2+y^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}}) = \varphi(\rho)\vec{\mathbf{e}}_\rho. \tag{13.25}$$

Table 13.1 Relations between the components of a vector in Cartesian, cylindrical, and spherical coordinates

Cartesian coordinates	**Cylindrical coord.**	**Spherical coordinates**
$\vec{\mathbf{V}} = V_x\vec{\mathbf{e}}_x + V_y\vec{\mathbf{e}}_y + V_z\vec{\mathbf{e}}_z$	$V_\rho\vec{\mathbf{e}}_\rho + V_\varphi\vec{\mathbf{e}}_\varphi + V_z\vec{\mathbf{e}}_z$	$V_r\vec{\mathbf{e}}_r + V_\vartheta\vec{\mathbf{e}}_\vartheta + V_\varphi\vec{\mathbf{e}}_\varphi$
V_x	$= V_\rho\cos\varphi - V_\varphi\sin\varphi$	$= V_r\sin\vartheta\cos\varphi + V_\vartheta\cos\vartheta\cos\varphi - V_\varphi\sin\varphi$
V_y	$= V_\rho\sin\varphi + V_\varphi\cos\varphi$	$= V_r\sin\vartheta\sin\varphi + V_\vartheta\cos\vartheta\sin\varphi + V_\varphi\cos\varphi$
V_z	$= V_z$	$= V_r\cos\vartheta - V_\vartheta\sin\vartheta$
$V_x\cos\varphi + V_y\sin\varphi$	$= V_\rho$	$= V_r\sin\vartheta + V_\vartheta\cos\vartheta$
$-V_x\sin\varphi + V_y\cos\varphi$	$= V_\varphi$	$= V_\varphi$
V_z	$= V_z$	$= V_r\cos\vartheta - V_\vartheta\sin\vartheta$
$V_x\sin\vartheta\cos\varphi + V_y\sin\vartheta\sin\varphi + V_z\cos\vartheta$	$= V_\rho\sin\vartheta + V_z\cos\vartheta$	$= V_r$
$V_x\cos\vartheta\cos\varphi + V_y\cos\vartheta\sin\varphi - V_z\sin\vartheta$	$= V_\rho\cos\vartheta - V_z\sin\vartheta$	$= V_\vartheta$
$-V_x\sin\varphi + V_y\cos\varphi$	$= V_\varphi$	$= V_\varphi$

13.1.3.5 Vector Lines

A curve C is called a *line of a vector* or a *vector line* of the vector field $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ **(Fig. 13.9)** if the vector $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is a tangent vector of the curve at every point P. There is a vector line passing through every point of the field. Vector lines do not intersect each other, except, maybe, at points where the function $\vec{\mathbf{V}}$ is not defined, or where it is the zero vector. The differential equations of the vector lines of a vector field $\vec{\mathbf{V}}$ given in Cartesian coordinates are

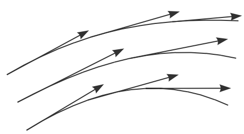

Figure 13.9

$$\operatorname{div}(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) = \operatorname{div}\vec{\mathbf{V}}_1 + \operatorname{div}\vec{\mathbf{V}}_2. \tag{13.86}$$

$$\operatorname{div}(c\vec{\mathbf{V}}) = c\operatorname{div}\vec{\mathbf{V}}. \tag{13.87}$$

$$\operatorname{div}(U\vec{\mathbf{V}}) = \vec{\mathbf{V}}\cdot\operatorname{grad}U + U\operatorname{div}\vec{\mathbf{V}}. \tag{13.88}$$

$$\operatorname{rot}(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) = \operatorname{rot}\vec{\mathbf{V}}_1 + \operatorname{rot}\vec{\mathbf{V}}_2. \tag{13.89}$$

$$\operatorname{rot}(c\vec{\mathbf{V}}) = c\operatorname{rot}\vec{\mathbf{V}}. \tag{13.90}$$

$$\operatorname{rot}(U\vec{\mathbf{V}}) = U\operatorname{rot}\vec{\mathbf{V}} - \vec{\mathbf{V}}\times\operatorname{grad}U. \tag{13.91}$$

$$\operatorname{div}\operatorname{rot}\vec{\mathbf{V}} \equiv 0. \tag{13.92}$$

$$\operatorname{rot}\operatorname{grad}U \equiv \vec{\mathbf{0}} \qquad \text{(zero vector)}. \tag{13.93}$$

$$\operatorname{div}\operatorname{grad}U = \Delta U. \tag{13.94}$$

$$\operatorname{rot}\operatorname{rot}\vec{\mathbf{V}} = \operatorname{grad}\operatorname{div}\vec{\mathbf{V}} - \Delta\vec{\mathbf{V}}. \tag{13.95}$$

$$\operatorname{div}(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2) = \vec{\mathbf{V}}_2\cdot\operatorname{rot}\vec{\mathbf{V}}_1 - \vec{\mathbf{V}}_1\cdot\operatorname{rot}\vec{\mathbf{V}}_2. \tag{13.96}$$

13.2.7.3 Expressions of Vector Analysis in Cartesian, Cylindrical, and Spherical Coordinates (see Table 13.3)

Table 13.3 Expressions of vector analysis in Cartesian, cylindrical, and spherical coordinates

	Cartesian coordinates	**Cylindrical coordinates**	**Spherical coordinates**
$d\vec{\mathbf{s}} = d\vec{\mathbf{r}}$	$\vec{\mathbf{e}}_x dx + \vec{\mathbf{e}}_y dy + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_\rho d\rho + \vec{\mathbf{e}}_\varphi \rho d\varphi + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_r dr + \vec{\mathbf{e}}_\vartheta r d\vartheta + \vec{\mathbf{e}}_\varphi r\sin\vartheta d\varphi$
grad$U$	$\vec{\mathbf{e}}_x\frac{\partial U}{\partial x} + \vec{\mathbf{e}}_y\frac{\partial U}{\partial y} + \vec{\mathbf{e}}_z\frac{\partial U}{\partial z}$	$\vec{\mathbf{e}}_\rho\frac{\partial U}{\partial \rho} + \vec{\mathbf{e}}_\varphi\frac{1}{\rho}\frac{\partial U}{\partial \varphi} + \vec{\mathbf{e}}_z\frac{\partial U}{\partial z}$	$\vec{\mathbf{e}}_r\frac{\partial U}{\partial r} + \vec{\mathbf{e}}_\vartheta\frac{1}{r}\frac{\partial U}{\partial \vartheta} + \vec{\mathbf{e}}_\varphi\frac{1}{r\sin\vartheta}\frac{\partial U}{\partial \varphi}$
div$\vec{\mathbf{V}}$	$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(V_\vartheta\sin\vartheta)$ $+\frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial\varphi}$
rot$\vec{\mathbf{V}}$	$\vec{\mathbf{e}}_x\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)$ $+\vec{\mathbf{e}}_y\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$ $+\vec{\mathbf{e}}_z\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$	$\vec{\mathbf{e}}_\rho\left(\frac{1}{\rho}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right)$ $+\vec{\mathbf{e}}_\varphi\left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)$ $+\vec{\mathbf{e}}_z\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\varphi) - \frac{1}{\rho}\frac{\partial V_\rho}{\partial\varphi}\right)$	$\vec{\mathbf{e}}_r\frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(V_\varphi\sin\vartheta) - \frac{\partial V_\vartheta}{\partial\varphi}\right]$ $+\vec{\mathbf{e}}_\vartheta\frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial}{\partial r}(rV_\varphi)\right]$ $+\vec{\mathbf{e}}_\varphi\frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\vartheta) - \frac{\partial V_r}{\partial\vartheta}\right]$
ΔU	$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\varphi^2}$ $+\frac{\partial^2 U}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)$ $+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right)$ $+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 U}{\partial\varphi^2}$

13.3 Integration in Vector Fields

Integration in vector fields is usually performed in Cartesian, cylindrical or in spherical coordinate systems. Usually we integrate along curves, surfaces, or volumes. The line, surface, and volume elements needed for these calculations are collected in **Table 13.4**.

Table 13.4 Line, surface, and volume elements in Cartesian, cylindrical, and spherical coordinates

	Cartesian coordinates	**Cylindrical coordinates**	**Spherical coordinates**
$d\vec{\mathbf{r}}$	$\vec{\mathbf{e}}_x dx + \vec{\mathbf{e}}_y dy + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_\rho d\rho + \vec{\mathbf{e}}_\varphi \rho d\varphi + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_r dr + \vec{\mathbf{e}}_\vartheta r d\vartheta + \vec{\mathbf{e}}_\varphi r \sin\vartheta d\varphi$
$d\vec{\mathbf{S}}$	$\vec{\mathbf{e}}_x dydz + \vec{\mathbf{e}}_y dxdz + \vec{\mathbf{e}}_z dxdy$	$\vec{\mathbf{e}}_\rho \rho d\varphi dz + \vec{\mathbf{e}}_\varphi d\rho dz + \vec{\mathbf{e}}_z \rho d\rho d\varphi$	$\vec{\mathbf{e}}_r r^2 \sin\vartheta d\vartheta d\varphi$ $+\vec{\mathbf{e}}_\vartheta r \sin\vartheta dr d\varphi$ $+\vec{\mathbf{e}}_\varphi r dr d\vartheta$
dv^*	$dxdydz$	$\rho d\rho d\varphi dz$	$r^2 \sin\vartheta dr d\vartheta d\varphi$
	$\vec{\mathbf{e}}_x = \vec{\mathbf{e}}_y \times \vec{\mathbf{e}}_z$ $\vec{\mathbf{e}}_y = \vec{\mathbf{e}}_z \times \vec{\mathbf{e}}_x$ $\vec{\mathbf{e}}_z = \vec{\mathbf{e}}_x \times \vec{\mathbf{e}}_y$	$\vec{\mathbf{e}}_\rho = \vec{\mathbf{e}}_\varphi \times \vec{\mathbf{e}}_z$ $\vec{\mathbf{e}}_\varphi = \vec{\mathbf{e}}_z \times \vec{\mathbf{e}}_\rho$ $\vec{\mathbf{e}}_z = \vec{\mathbf{e}}_\rho \times \vec{\mathbf{e}}_\varphi$	$\vec{\mathbf{e}}_r = \vec{\mathbf{e}}_\vartheta \times \vec{\mathbf{e}}_\varphi$ $\vec{\mathbf{e}}_\vartheta = \vec{\mathbf{e}}_\varphi \times \vec{\mathbf{e}}_r$ $\vec{\mathbf{e}}_\varphi = \vec{\mathbf{e}}_r \times \vec{\mathbf{e}}_\vartheta$
	$\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$	$\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$	$\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$
	The indices i and j take the place of x, y, z or ρ, φ, z or r, ϑ, φ.		
*	The volume is denoted here by v to avoid confusion with the absolute value of the vector function $\lvert\vec{\mathbf{V}}\rvert = V$.		

13.3.1 Line Integral and Potential in Vector Fields

13.3.1.1 Line Integral in Vector Fields

1. Definition The scalar-valued curvilinear integral or line integral of a vector function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ along a rectifiable curve $\overset{\frown}{AB}$ **(Fig. 13.13)** is the scalar value

$$P = \int\limits_{\overset{\frown}{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.97a}$$

2. Evaluation of this Integral in Five Steps

a) We divide the path $\overset{\frown}{AB}$ **(Fig. 13.13)** by division points $A_1(\vec{\mathbf{r}}_1), A_2(\vec{\mathbf{r}}_2), \ldots, A_{n-1}(\vec{\mathbf{r}}_{n-1})$ $(A \equiv A_0,$ $B \equiv A_n)$ into n small arcs which are approximated by the vectors $\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_{i-1} = \Delta\vec{\mathbf{r}}_{i-1}$.

b) We choose arbitrarily the points P_i with position vectors $\vec{\mathbf{r}}_i$ lying inside or at the boundary of each small arc.

c) We calculate the dot product of the value of the function $\vec{\mathbf{V}}(\vec{\mathbf{r}}_i)$ at these chosen points with the corresponding $\Delta\vec{\mathbf{r}}_{i-1}$.

d) We add all the n products.

e) We calculate the limit of the sums got this way $\sum\limits_{i=1}^{n} \tilde{\vec{\mathbf{V}}}(\vec{\mathbf{r}}_i) \cdot \Delta\vec{\mathbf{r}}_{i-1}$ for $\Delta\vec{\mathbf{r}}_{i-1} \to 0$, while $n \to \infty$ obviously.