1. Problem (17 points, marks)

a) $Q(r) = \int_V \rho(r) dV = \int_0^r \int_0^{2\pi} \int_0^{\pi} \left(\frac{3\rho_0}{R} r' - \rho_0\right) r'^2 \sin\vartheta \; d\vartheta \; d\varphi \; dr' = \frac{3\pi\rho_0 r^4}{R} - \frac{4\pi\rho_0 r^3}{3}$

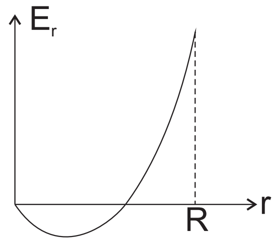
b) $Q(r) = \int_{\delta V} \vec{D} \; d\vec{a} = \int_0^{2\pi} \int_0^{\pi} D_r r^2 \sin\vartheta \; d\vartheta \; d\varphi$ and $\vec{D} = D_r \vec{e_r}$
$\rightarrow D_r = \frac{Q(r)}{4\pi r^2}$

$$D_r(r) = \begin{cases} \frac{Q(r)}{4\pi r^2} = \frac{3\rho_0 r^2}{4R} - \frac{\rho_0 r}{3} & \text{for } 0 \leq r \leq R \\ \frac{Q(R)}{4\pi r^2} = \frac{5\rho_0 R^3}{12 r^2} & \text{for } r > R \end{cases}$$

c) $\vec{D} = \epsilon \vec{E} \rightarrow E_r = \frac{1}{\epsilon} D_r$ and $\vec{E} = E_r \vec{e_r}$

$$E_r(r) = \begin{cases} \frac{1}{\epsilon_0 \epsilon_r}\left(\frac{3\rho_0 r^2}{4R} - \frac{\rho_0 r}{3}\right) & \text{for } 0 \leq r \leq R \\ \frac{1}{\epsilon_0} \frac{5\rho_0 R^3}{12 r^2} & \text{for } r > R \end{cases}$$

d)



e)

$$\begin{aligned} F_g &= F_{el} \\ mg &= Q_m E_r(h) = Q_m \frac{1}{\epsilon_0} \frac{5\rho_0 R^3}{12 h^2} \\ h &= \sqrt{\frac{5\rho_0 R^3 Q_m}{12\epsilon_0 mg}} \qquad \text{(negative solution does not make sense physically)} \end{aligned}$$

f)

$$\begin{aligned} W &= -W_{el} - W_g \\ &= -\int_h^b Q_m E_r(r) \; dr + \int_h^b mg \; dr \\ &= -Q_m \frac{1}{\epsilon_0} \frac{5\rho_0 R^3}{12} \left[-\frac{1}{r}\right]_h^b + mg(b-h) \\ &= Q_m \frac{1}{\epsilon_0} \frac{5\rho_0 R^3}{12} \left(\frac{1}{b} - \frac{1}{h}\right) - mg(h-b) \end{aligned}$$

2. Problem (14 points, marks)

a) Current density $\vec{j}(\vec{r})$ in the region $r_1 \leq r < r_2$:

$$\vec{j}(\vec{r}) = \frac{I}{A}\vec{e}_z = \frac{I}{\pi(r_2^2 - r_1^2)}\vec{e}_z.$$

b) Magnetic field:

$$\int_{\partial A} \vec{H}(\vec{r})\, \mathrm{d}\vec{r} = \int_A \vec{j}(\vec{r})\, \mathrm{d}\vec{a} = I(A).$$

Becaus of the cylindrical symmetry, it holds true: $\vec{H}(\vec{r}) = H_\varphi(r) \cdot \vec{e}_\varphi$ and hence

$$\int_{\partial A} \vec{H}(\vec{r})\, \mathrm{d}\vec{r} = \int_0^{2\pi} H_\varphi(r) \cdot \vec{e}_\varphi \cdot r\vec{e}_\varphi\, \mathrm{d}\varphi = 2\pi r H_\varphi(r).$$

Three regions: $0 \leq r < r_1$, $r_1 \leq r < r_2$ und $r_2 < r$.
1. region ($0 \leq r < r_1$): the enclosed current is zero. That means, there is no magnetic field

$$H_\varphi^{(1)}(r) = 0.$$

2. region ($r_1 \leq r < r_2$): in this region the current density calculated in a) flows. The enclosed current is

$$I(A) = \int_A \vec{j}(\vec{r})\, \mathrm{d}\vec{a} = \int_0^{2\pi} \int_{r_1}^{r} r\frac{I}{\pi(r_2^2 - r_1^2)}\, \mathrm{d}r\mathrm{d}\varphi = I\frac{r^2 - r_1^2}{r_2^2 - r_1^2}$$

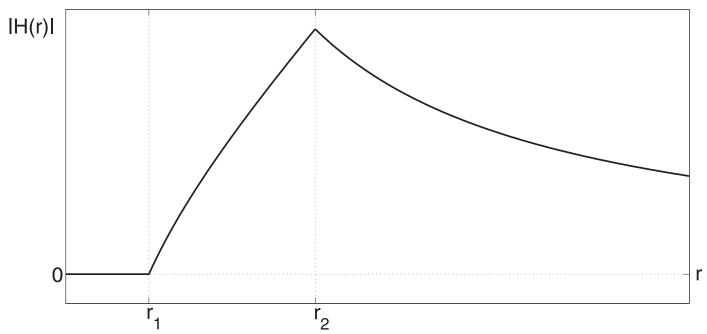
and hence

$$H_\varphi^{(2)}(r) = \frac{I}{2\pi r} \cdot \frac{r^2 - r_1^2}{r_2^2 - r_1^2}.$$

3. region ($r_2 < r$): the enclosed current in this region is I and hence

$$H_\varphi^{(3)}(r) = \frac{I}{2\pi r}.$$

c) Sketch:



d) Sketch: