# EDE1012 MATHEMATICS 2

## Tutorial 6
### Vector Calculus II

1. A surface is defined by the vector function

[redacted]

a) Evaluate the normal vectors to the surface at (1, 0, -1).

b) Determine the Cartesian equation of the tangent plane at (1, 0, -1).

c) Determine the Cartesian equation of the surface in the form F(x, y, z) = 0.

ANS: **a)** $\mathbf{N} = \pm[1 \quad 0 \quad 2]^T$. **b)** $x + 2z = -1$. **c)** $x^2 + y^2 - z^4 = 0$.

2. Evaluate the flux of the vector field below across the triangular surface S that is the plane 2x - 2y + z = 2 cut out by the coordinate planes. The surface is orientated with an upward-pointing normal.

[redacted]

ANS: Flux = 1.

3. A surface S is the closed cylinder with its top and bottom at z = 4 and z = 0 respectively and a cylindrical surface $x^2 + y^2 = 9$. A vector field **F** is defined below.

[redacted]

a) Determine the flux of **F** across S and explain why it is zero. The surface is orientated with outward-pointing normals.

b) Verify the flux again using the divergence theorem.

ANS: **a)** Flux = 0. **b)** Flux = 0.

1. A surface is defined by the vector function

$$\mathbf{r}(s,t) = [\underbrace{s^2\cos t}_{x} \quad \underbrace{s^2\sin t}_{y} \quad \underbrace{s}_{z}]^T$$

$$x^2+y^2 = s^4(\cos^2 t + \sin^2 t) = s^4 = z^4$$

$$\Rightarrow F(x,y,z) = z^4 - x^2 - y^2 = 0 \,//$$

a) Evaluate the normal vectors to the surface at (1, 0, -1).

b) Determine the Cartesian equation of the tangent plane at (1, 0, -1).

c) Determine the Cartesian equation of the surface in the form F(x, y, z) = 0.

ANS: **a)** $\mathbf{N} = \pm[1 \quad 0 \quad 2]^T$. **b)** $x + 2z = -1$. **c)** $x^2+y^2-z^4 = 0$.

a) $$\vec{N} = \vec{r}_s \times \vec{r}_t = \begin{pmatrix} 2s\cos t \\ 2s\sin t \\ 1 \end{pmatrix} \times \begin{pmatrix} -s^2\sin t \\ s^2\cos t \\ 0 \end{pmatrix} = \begin{pmatrix} -s^2\cos t \\ -s^2\sin t \\ 2s^3\cos^2 t + 2s^3\sin^2 t \end{pmatrix} = \begin{pmatrix} -s^2\cos t \\ -s^2\sin t \\ 2s^3 \end{pmatrix}.$$

$x = 1 = s^2\cos t$.

$y = 0 = s^2\sin t \rightarrow \sin t = 0 \rightarrow t = 0$

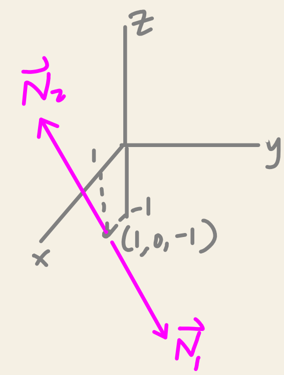
$x = 1 = s^2\cos(0)$

$s = 1, -1$

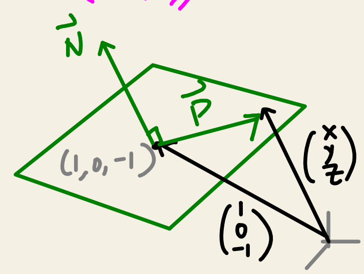
$z = -1 = s$ (na)

At $(1, 0, -1)$, $t = 0$, $s = -1$.

$$\vec{N}_1 = \begin{pmatrix} -1\cos(0) \\ -1\sin(0) \\ 2(-1)^3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} // \rightarrow \vec{N}_2 = -\vec{N}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} //$$




b) $$\vec{N} \cdot \Big[\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{p}}\Big] = 0.$$




$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \Big[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\Big] = (x-1) + 0 + 2(z+1) = 0.$$

$$x + 2z = -1 \,//$$

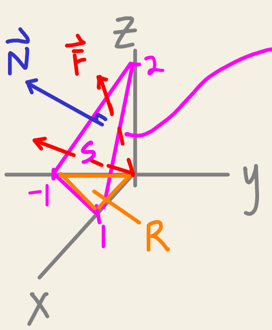
2. Evaluate the flux of the vector field below across the triangular surface S that is the plane 2x - 2y + z = 2 cut out by the coordinate planes. The surface is orientated with an upward-pointing normal.

$z = 2 - 2x + 2y$

radial

$$\mathbf{F}(x,y,z) = [x \quad y \quad z]^T$$

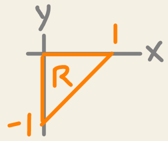
ANS: Flux = 1.




Parameterize by $\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ 2-2x+2y \end{pmatrix}$.

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$ +z, so upward.

$$\vec{F} \cdot \vec{N} = \begin{pmatrix} x \\ y \\ 2-2x+2y \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \cancel{2x} - \cancel{2y} + 2 - \cancel{2x} + \cancel{2y} = 2.$$




$$\text{Flux} = \iint_R 2\,dx\,dy = 2\,\text{Area of } R = 2\left(\tfrac{1}{2}(1)(1)\right) = 1 \,//$$

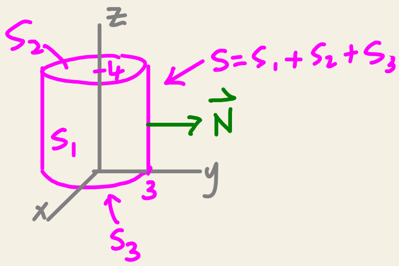
3. A surface S is the closed cylinder with its top and bottom at z = 4 and z = 0 respectively and a cylindrical surface $x^2 + y^2 = 9$. A vector field **F** is defined below.

$$\mathbf{F}(x,y,z) = [-y \quad x \quad 0]^T$$

a) Determine the flux of **F** across S and explain why it is zero. The surface is orientated with outward-pointing normals.

b) Verify the $\underbrace{\text{flux again}}_{=0.}$ using the $\underbrace{\text{divergence}}_{\vec{\nabla}\cdot\vec{F}=0.}$ theorem.

ANS: **a)** Flux = 0. **b)** Flux = 0.



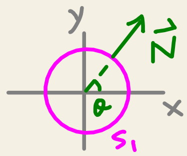

Since F has no z-comp, so it is tangent to surfaces S2 and S3. Hence the fluxes across S2 and S3 are both zero.

$$S_1: \quad \vec{r}(\theta, z) = \begin{pmatrix} 3\cos\theta \\ 3\sin\theta \\ z \end{pmatrix} \rightarrow \vec{N}_0 = \vec{r}_\theta \times \vec{r}_z = \begin{pmatrix} -3\sin\theta \\ 3\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\cos\theta \\ 3\sin\theta \\ 0 \end{pmatrix}$$

(outward)

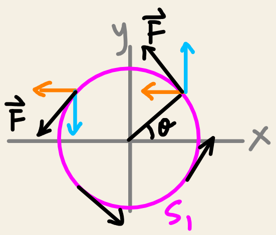
$$\vec{F}\cdot\vec{N} = \begin{pmatrix} -3\sin\theta \\ 3\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3\cos\theta \\ 3\sin\theta \\ 0 \end{pmatrix} = -9\sin\theta\cos\theta + 9\sin\theta\cos\theta$$

$$= 0.$$




$$\Rightarrow \text{Flux}_{S_1} = 0.$$

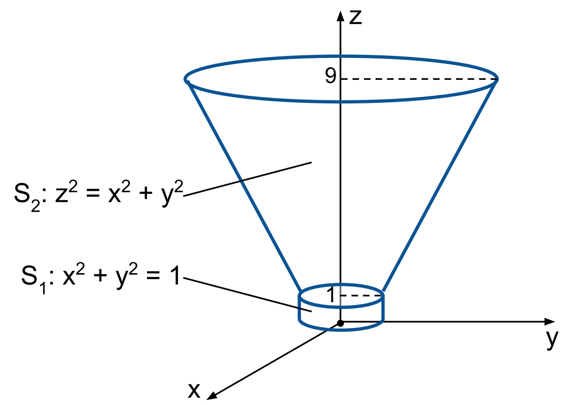
$$\therefore \text{Flux}_S = \text{Flux}_{S_1} + \text{Flux}_{S_2} + \text{Flux}_{S_3} = 0 + 0 + 0 = 0 \,//$$




$$\vec{F} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \begin{matrix} \leftarrow x\text{-dir} \\ \leftarrow y\text{-dir} \\ \phantom{0} \end{matrix} = \begin{pmatrix} -3\sin\theta \\ 3\cos\theta \\ 0 \end{pmatrix}$$

Since F is also tangent to the cylindrical surface at all points, the flux across S1 is also zero. Hence the total flux across S will be zero.

~~4.~~ DIY A surface $S = S_1 + S_2$ that looks like a funnel is shown below.




a) Determine the outward-pointing normals of surfaces $S_1$ and $S_2$.

b) Evaluate the flux of **F** below through S, which is orientated by outward-pointing normals.

[redacted]

ANS: **a)** $S_1 : \mathbf{N} = [x \quad y \quad 0]^T$. $S_2 : \mathbf{N} = [x \quad y \quad -z]^T$. **b)** -1456π/3.

DIY

5. Use the divergence theorem to evaluate the flux of the vector field below through surface S of the unit cube in the domain [0, 1] × [0, 1] × [0, 1],

[redacted]

$\int_0^1\int_0^1\int_0^1 \vec{\nabla}\cdot\vec{v}\; dx\,dy\,dz \overset{(DIY)}{=} \cdots =$ ANS: Flux = e/2 + 2.

~~6.~~ A vector field **F** and surface S is defined by the functions

[redacted]

Given that surface S is oriented by upward normal vectors, use Gauss's theorem to calculate the flux of F across S.

ANS: Flux = 0.

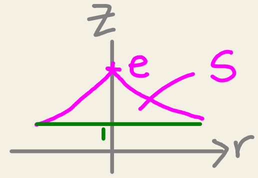
6. A vector field **F** and surface S is defined by the functions

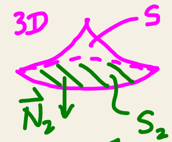
$$\mathbf{F}(x,y,z) = [x \quad y \quad 2-2z]^T, \qquad S: z = e^{1-x^2-y^2}, z \geq 1$$

$= e^{1-r^2}$

Given that surface S is oriented by upward normal vectors, use Gauss's theorem to calculate the flux of F across S.

ANS: Flux = 0.




close the surface with $S_2$

$$\text{Flux}_{S+S_2} = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV$$

$$= \iiint_V 1+1-2\, dV = 0.$$

Eg.

To use Gauss, Stokes and Green's theorem, you can close the geometry if it is not closed. Then minus off the integrals over the 'extra' geometries to get the result.

$\text{Flux}_S = \text{Flux}_{S+S_2} - \text{Flux}_{S_2}$ where

$$\text{Flux}_{S_2} = \iint_{S_2} \begin{pmatrix} x \\ y \\ 2-2(1) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dx\,dy = \iint_{S_2} 0\, dx\,dy = 0.$$

$z=1$ on $S_2$

$$\text{Flux}_S = 0 - 0 = 0$$

7. (https://openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem)
Evaluate the flux of **F** below across the surface S consisting of all faces of the tetrahedron bounded by plane x + y + z = 1 and the coordinate planes, with outward normal vectors

[redacted]

ANS: 1/8.

8. Consider a cylinder of height H with a base of radius R on the xy-plane.

a) Using a surface integral, show that the area of the cylinder mantle is $2\pi RH$.

b) Evaluate the flux of the vector field defined below through the cylinder mantle using Gauss's theorem. Orientate the cylinder with outward normals.

[redacted]

ANS: **b)** Flux = $\pi R^2 H^2$.

9. Verify Stokes' theorem for a $\overbrace{\text{conservative vector field } \mathbf{F}(x, y, z)}^{\vec{F} = \vec{\nabla} E}$ over a closed curve C that is the boundary of surface S.

$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla} E = \vec{0} \rightarrow \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{N}\, dA = 0.$

10. (https://openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem)
Use Stokes' theorem to evaluate the line integral below, where C is the curve given by x = cos t, y = sin t, z = sin t, $0 \le t \le 2\pi$, traversed in the direction of increasing t.

For curve C, $\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ \sin t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z = y \end{pmatrix}$

$\vec{r}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{r}(2\pi) \rightarrow$ C is closed.

O [redacted] $= \iint_S \underbrace{(\vec{\nabla} \times \vec{F})}_{\vec{0}} \cdot \vec{N}\, dA.$

compute this! (DIY)

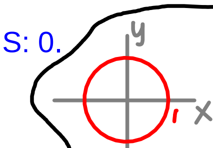
$\begin{pmatrix} 2xy^2z \\ 2x^2yz \\ x^2y^2 - 2z \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \vec{F} \cdot d\vec{r}.$

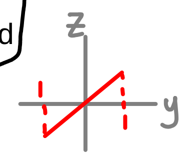
ANS: 0.

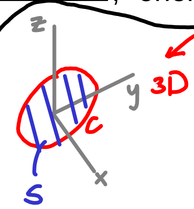
11. Use Stokes' theorem to evaluate the line integral below, where C is the intersection curve between the plane [redacted] and the cylinder [redacted], oriented counterclockwise.

DIY

[redacted]




ANS: $81\pi/2$.

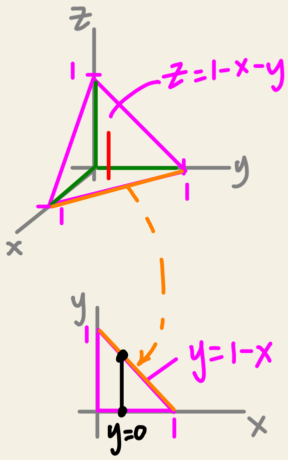
No need, just for practice.

$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ y \end{pmatrix}, \quad \vec{r}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r\sin\theta \end{pmatrix}.$

7. (https://openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem)
Evaluate the flux of **F** below across the surface S consisting of all faces of the tetrahedron bounded by plane x + y + z = 1 and the coordinate planes, with outward normal vectors

$$\mathbf{F}(x,y,z) = [x^2 \quad xy \quad x^3y^3]^T$$

ANS: 1/8.

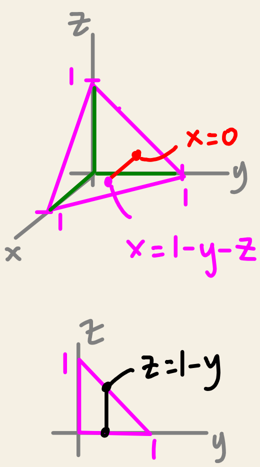



More efficient to use Gauss theorem.

$$\iint_S \vec{F}\cdot\vec{N}\,dA = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iiint_V 2x+x+0\,dV$$

$$= \iiint_V 3x\,dV = \int_0^1 \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 3x \quad dz\,dy\,dx$$

$$= \cdots = \frac{1}{8}$$ //




OR: $$= \int_0^1 \int_{z=0}^{z=1-y} \int_{x=0}^{x=1-y-z} 3x \; dx\,dz\,dy$$

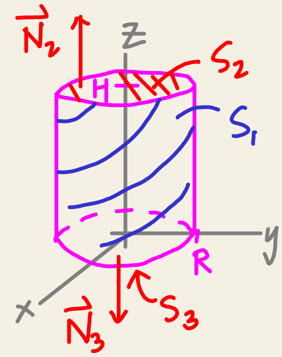
8. Consider a cylinder of height H with a base of radius R on the xy-plane.

a) Using a surface integral, show that the area of the cylinder mantle is 2πRH.

b) Evaluate the flux of the vector field defined below through the cylinder mantle using Gauss's theorem. Orientate the cylinder with outward normals.

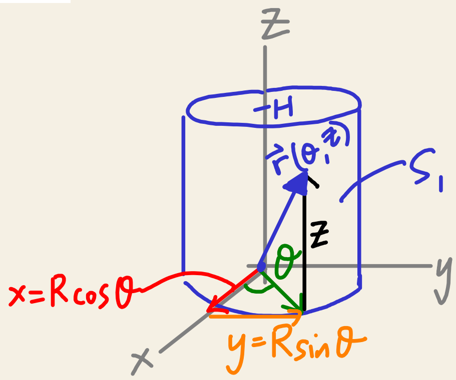
$$\mathbf{F}(x,y,z) = [xz+y \quad yz-x \quad z]^T$$

ANS: **b)** Flux = $\pi R^2 H^2$.




a) Parameterize $S_1$ by:

$$\vec{r}(\theta, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ z \end{pmatrix}$$



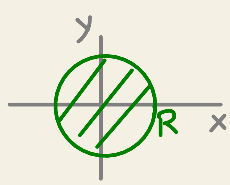

$$\vec{N} = \vec{r}_\theta \times \vec{r}_z = \ldots$$

$$\text{Area, } A = \iint_{S_1} |\vec{N}| \, d\theta \, dz = \int_0^H \int_0^{2\pi} \cdots \, d\theta \, dz$$

b) Using Gauss theorem,

$$\text{Flux}_{S_1} = \underbrace{\text{Flux}_{S_1+S_2+S_3}}_{\iiint_V \vec{\nabla}\cdot\vec{F}\, dV} - \text{Flux}_{S_2} - \text{Flux}_{S_3}$$

$$\iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \iiint_V z+z+1 \, dV$$




$$= \iiint_V 2z+1 \, dz\,dx\,dy = \int_0^{2\pi} \int_0^R \int_0^H 2z+1 \, dz \, (r\,dr\,d\theta)$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^R r\,dr \cdot \int_0^H 2z+1\,dz = 2\pi \cdot \frac{R^2}{2} \cdot (z^2+z)\Big|_0^H = \pi R^2 H(H+1)$$

$$\text{Flux}_{S_2} = \iint_{S_2} \vec{F}\cdot\vec{N}_2 \, dA = \iint \begin{pmatrix} \cdot\cdot \\ \cdot\cdot \\ H \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA = \iint_{S_2} H\,dA = H \times \text{Area of } S_2 = \pi R^2 H.$$

z=H on $S_2$

$$\text{Flux}_{S_3} = \iint_{S_3} \vec{F}\cdot\vec{N}_3 \, dA = \iint \begin{pmatrix} \cdot\cdot \\ \cdot\cdot \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dA = 0.$$

z=0 on $S_3$

$$\mathbf{F}(x,y,z) = [xz+y \quad yz-x \quad \underline{z}]^T$$

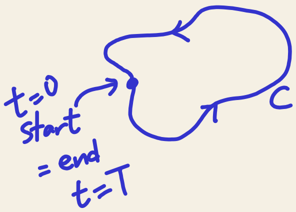
$$\text{Flux}_{S_1} = \pi R^2 H(H+1) - \pi R^2 H - 0$$
$$= \pi R^2 H^2$$

$$\overbrace{\text{conservative}}^{\vec{F} = \vec{\nabla} E}$$

9. Verify Stokes' theorem for a conservative vector field **F**(x, y, z) over a closed curve C that is the boundary of surface S.

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla} E = \vec{0} \rightarrow \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{N}\, dA = 0.$$

$$\oint_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,' dt = E(\text{end-pt}) - E(\text{start-pt}) = E(x_T, y_T, z_T) - E(x_0, y_0, z_0) = 0,$$




Since $(x_T, y_T, z_T) = (x_0, y_0, z_0)$ for any closed curve C

and so $E(x_T, y_T, z_T) = E(x_0, y_0, z_0)$.

$$\therefore \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{N}\, dA = \oint_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,' dt = 0.$$

(Stokes theorem verified.)

~~12~~. Using Stokes' theorem, evaluate the circulation of F over surface S defined below.

[redacted]

ANS: $2\pi$.

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals
2) https://openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem
3) https://openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem

*End of Tutorial 6*

*(Email to youliangzheng@gmail.com for assistance.)*

12. Using Stokes' theorem, evaluate the circulation of F over surface S defined below.

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{N}\,dA$$

$$\mathbf{F}(x,y,z) = \begin{bmatrix} e^{y+z}-2y \\ xe^{y+z}+y \\ e^{x+y} \end{bmatrix}, \quad S: \left\{(x,y,z)\,\middle|\, z = e^{-(x^2+y^2)}, z \geq 1/e\right\}$$

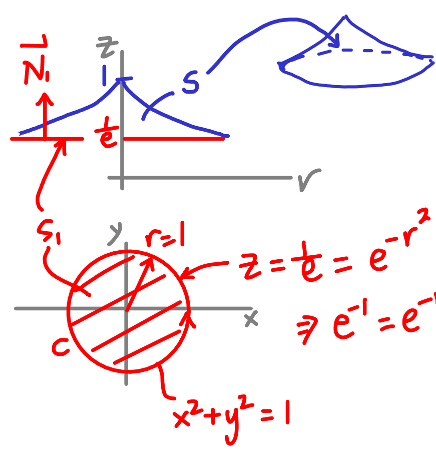
compute this! (but very tedious!)

ANS: $2\pi$.

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{N}\,dA = \oint_C \vec{F}\cdot\vec{dr}$$

relate to a simpler surface.

$$= \iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\vec{N}_1\,dA$$




$z = \frac{1}{e} = e^{-r^2}$

$\Rightarrow e^{-1} = e^{-r^2} \rightarrow r^2 = 1$

$r = 1.$

$$= \iint_{S_1} \begin{pmatrix} \vdots \\ \vdots \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA$$

$$= \iint_{S_1} e^{y+z} - (e^{y+z} - 2)\,dA$$

$$= \iint_{S_1} 2\,dA = 2\times \text{Area of } S_1 = 2\pi(1)^2 = 2\pi$$