Topic 2
Linear Systems & Matrices

Dr Teh Yong Liang
Email: youliangzheng@gmail.com

Outline

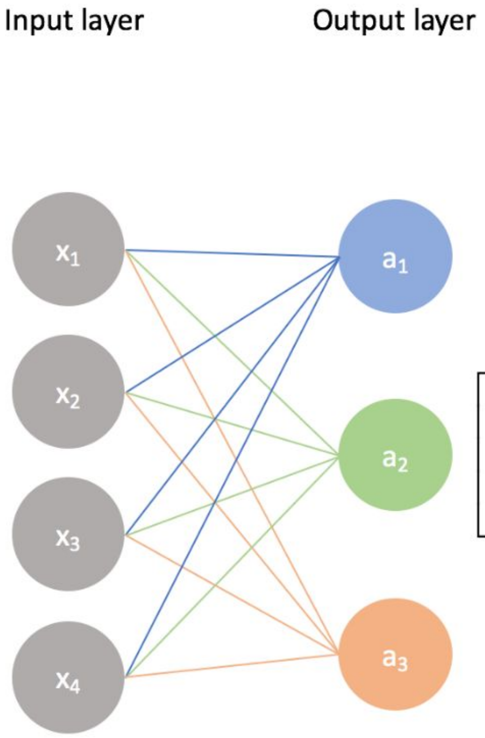
- Applications of Linear Algebra
- Systems of Linear Equations
- Basic Properties and Operations of Matrices
- Gauss-Jordan Elimination
- Types of Solution of Linear Systems

Applications of Linear Algebra

There are many applications of linear algebra in engineering and data analytics. Some examples are:

- Analysis of resistor-inductor circuits
- Analysis of multi-body heat transfer
- Analysis of harmonic oscillator systems
- Predictive modelling by linear regression - https://www.moneysmart.sg/home-loan/property-valuation-calculator
- Online recommender systems by singular value decomposition
- Data reduction by principal component analysis
- Almost all machine learning (ML) applications.

Matrices/Vectors of Weights and Biases in ML



A simple neural network

The main objective of machine learning is to 'learn' the 'best' weights (w_i) and biases (b).

$$\underbrace{\begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} = \begin{bmatrix} w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + b \\ w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + b \\ w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + b \end{bmatrix} \xrightarrow[\textit{activation}]{} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$\vec{x} + \vec{b} = \vec{a}$

A system of linear equations

Source: https://www.jeremyjordan.me/intro-to-neural-networks/

Definition of a System of Linear Equations

Generally, a system of linear equations has the form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ \ldots \ + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ \ldots \ + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ \ldots \ + a_{mn}x_n &= b_m
\end{aligned}
$$

where there are **m** equations with **n** unknown variables, x_1 to x_n. The a_{ij} are constant coefficients and the b_i are constants of the system.

System of Linear Equations (SLE)

Examples:

$$
\begin{aligned}
2x + 3y &= 1 \\
x - 7y &= -14
\end{aligned}
$$

(A system with 2 variables: x & y)

$$
\begin{aligned}
2x + y - z &= 1 \\
2x - 5y - z &= 7 \\
x + y + z &= 1
\end{aligned}
$$

(A system with 3 variables: x, y & z)

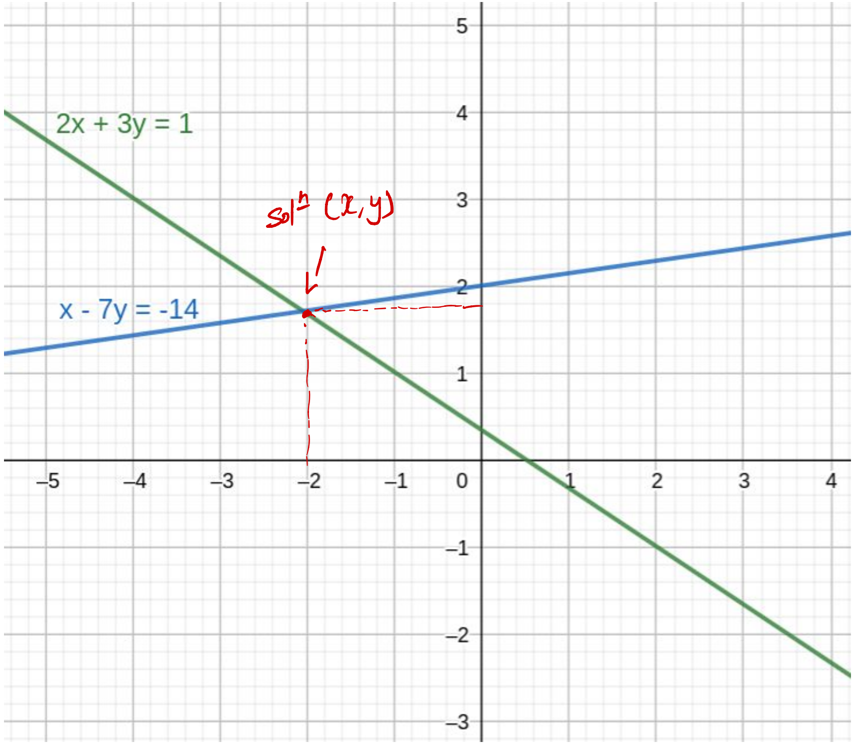
$$
\begin{aligned}
w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4 &= b_1 \\
w_{21}x_1 + w_{22}x_2 + w_{23}x_3 + w_{24}x_4 &= b_2 \\
w_{31}x_1 + w_{32}x_2 + w_{33}x_3 + w_{34}x_4 &= b_3
\end{aligned}
$$

(A system with 4 variables: x_1, x_2, x_3 & x_4)

Graphical Representations of a SLE

$$2x + 3y = 1$$
$$x - 7y = -14$$

How to solve for x & y?
What do the solutions mean?



Graphical Representations of a SLE

$$2x + y - z = 1 \quad - R_1$$
$$2x - 5y - z = 7 \quad - R_2$$
$$x + y + z = 1 \quad - R_3$$

How about for this system of 3 linear equations?

$R_2 - R_1 \rightarrow \quad 0x - 6y + 0z = -6$

$y = -1$ //

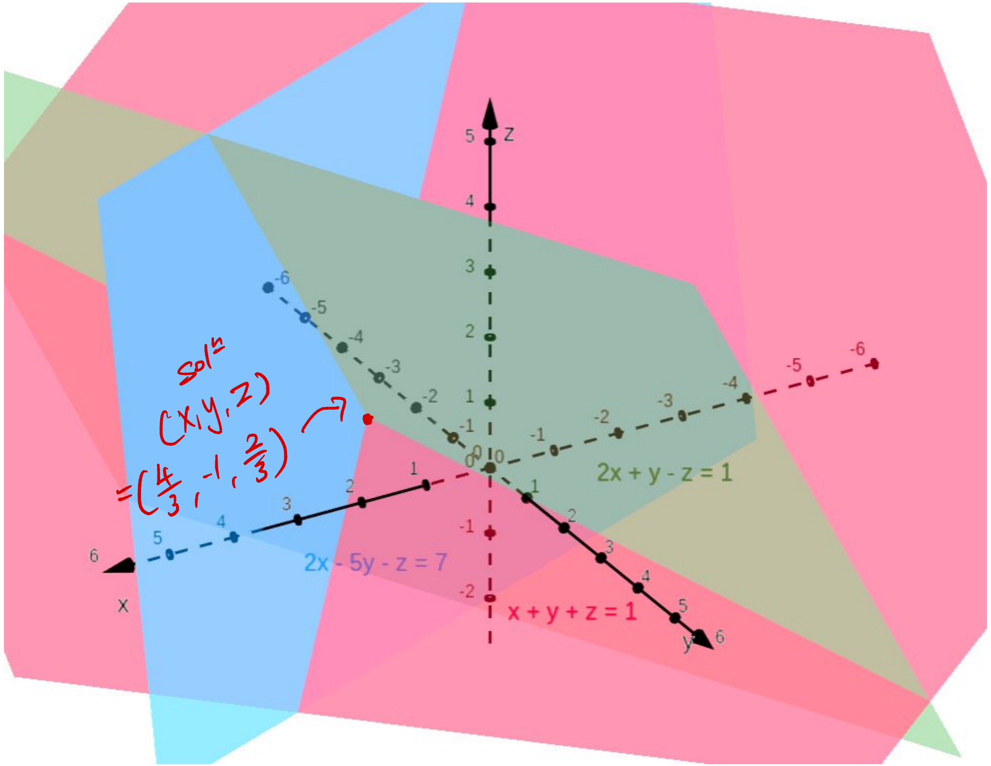
$R_1 - R_3 \rightarrow \quad x - 2z = 0 \quad - R_4$

Sub $y=-1$ into $R_3 \rightarrow \quad x + z = 2 \quad - R_5$

$R_4 - R_5 \rightarrow \quad -3z = -2$ Sub $z = \frac{2}{3}$ in R_4

$z = \frac{2}{3}$ // $x = 2(\frac{2}{3})$

$= \frac{4}{3}$ //



Real-World Representations of a SLE

Real world problems can give rise to a SLE, where its solution possibly represents the desired condition to achieve a ‘best’ outcome.

Example: A Farmer’s Problem

An animal feed is made up of corn, soybeans and cottonseed. One unit of each kind of ingredient provides units of protein, fat and fibre as shown in the table below. How many units of each ingredient is required to produce feed that contains 22 units of protein, 28 units of fat and 18 units of fibre (**to produce the ‘best’ cows**)?

	Corn	Soybeans	Cottonseed
Protein	0.25	0.4	0.2
Fat	0.4	0.2	0.3
Fibre	0.3	0.2	0.1

*The process of representing a real-world problem using math relations is known as mathematical modelling. This is a vital skill for practitioners we will be learning throughout the math modules.

Solution to a Real World SLE Problem

	x	y	z	
	Corn	**Soybeans**	**Cottonseed**	
Protein	0.25	0.4	0.2	$\leftarrow$ need $22 = 0.25x + 0.4y + 0.2z$ — R_1
Fat	0.4	0.2	0.3	$\leftarrow$ need $28 = 0.4x + 0.2y + 0.3z$ — R_2
Fibre	0.3	0.2	0.1	$\leftarrow$ need $18 = 0.3x + 0.2y + 0.1z$ — R_3

$$\left.\begin{aligned} 22 &= 0.25x + 0.4y + 0.2z \\ 28 &= 0.4x + 0.2y + 0.3z \\ 18 &= 0.3x + 0.2y + 0.1z \end{aligned}\right\} \text{SLE, Math Model}$$

$R_1 - 2R_3 \rightarrow 0.35x = -14 \rightarrow x = 40$

$R_2 - R_3 \rightarrow 0.1x + 0.2z = 10$

$0.2z = 10 - 0.1x = 10 - 4 = 6$

$z = 30$

$R_3 \rightarrow 0.2y = 18 - 0.3x - 0.1z$

$= 18 - 0.3(40) - 0.1(30) = 3$

$y = \frac{3}{0.2} = 15$

ANS: Corn = 40, Soybeans = 15, Cottonseed = 30

Solution Methods of a SLE

Substitution method

The **substitution method** solves one equation for y, then substitutes this expression into the other equation. The new equation can be solved for x.

Elimination method

The **elimination method** transforms the set of equations into another set with the same solution using the following rules:

1. Any two equations can be interchanged.
2. Any equation can be multiplied or divided by a non-zero real number.
3. Any equation can be replaced by the sum of that equation and a multiple of another equation in the system.

Using Matrix Notation for a SLE

A system of 2 linear equations can be written using matrices in the form:

$$\begin{aligned} 2x + 3y &= 1 \\ x - 7y &= -14 \end{aligned} \longrightarrow \begin{bmatrix} 2 & 3 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$$A\mathbf{v} = \mathbf{b}$$

where A is a matrix of coefficients, **v** is a vector of variables and **b** is a vector of constants.

Using Matrix Notation for a SLE

Generally, a SLE with m equations and n variables x_1 to x_n can be written in matrix form as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ \dots \ + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ \dots \ + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ \dots \ + a_{mn}x_n &= b_m \end{aligned} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A\mathbf{v} = \mathbf{b}$$

Matrices - The ‘Things’ of Linear Algebra

Generally, a matrix is just a ‘group’ (or array) of elements. A matrix with m rows and n columns is read as an m by n matrix (or $m \times n$ matrix).

$$A = \left.\overbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}^{n\ \text{col}^{n}\text{s}}\right\} m\ \text{rows}$$

where the a_{ij} ($i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$) are called the elements of the matrix.

A matrix that is $m \times 1$ is also called a column vector. A matrix that is $1 \times n$ is called a row vector.

Basic Matrix Operations

Matrix equality

Two $m \times n$ matrices A and B are equal if and only if $a_{ij} = b_{ij}$ for all i, j.

Matrices of differing dimensions cannot be equal.

Matrix addition

The sum of two $m \times n$ matrices A and B is an $m \times n$ matrix $C = A + B$ with $c_{ij} = a_{ij} + b_{ij}$ for all i, j.

Addition is not defined for matrices of differing dimensions.

Basic Matrix Operations

Scalar multiplication

The product of a scalar value k and an $m \times n$ matrix A is an $m \times n$ matrix $B = kA$ with $b_{ij} = kb_{ij}$ for all i, j.

Matrix multiplication

The product of an $m \times p$ matrix A and $p \times n$ matrix B is an $m \times n$ matrix $C = AB$ with $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$.

That is, c_{ij} is the dot product of the i'th row of A and the j'th column of B. Matrix multiplication is defined only if the number of columns in A equals the number of rows in B.

Note: matrix multiplication is **not** commutative, so $AB \neq BA$ in general.

Basic Matrix Operations

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -5 & 6 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -4 & 12 \end{bmatrix}$$

Addition / Subtraction
(Only for matrices with the same dimensions.)

$$-3\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -6 \\ 0 & -9 & 3\cancel{1} \\ -12 & 6 & -15 \end{bmatrix}$$

Scalar Multiplication
(Simply multiply each element with the scalar.)

Note that inner dimensions must be equal when multiplying matrices.

(2x2) (2x1) = (2x1)

$$\begin{bmatrix} 2 & 3 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ x - 7y \end{bmatrix}$$

(3x2) (2x2) = (3x2)

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \\ ew + fy & ex + fz \end{bmatrix}$$

Matrix Multiplication
('Rows multiply by columns')

Transpose of a Matrix

The transpose of a matrix is an operation that flips elements over the diagonal of the matrix. Formally, we have:

$$[A^T]_{ij} = A_{ji}$$

Examples:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}, \qquad B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \longrightarrow B^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Most directly, the transpose can be obtained by writing the rows as columns (or vice-versa).

Properties of the Matrix Transpose

The matrix transpose follows the following rules:

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(kA)^T = kA^T$, where k is a scalar.

The student should be able to verify each property above.

Basic Matrix Operations

Exercise: Given matrices A and B, compute matrices C, D, E and F.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \qquad C = B - A, \quad D = AB^T, \quad E = B^T A, \quad F = BA$$

$$C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -3 \end{bmatrix}$$

$$D = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} 5 & 5 \\ 11 & 11 \end{bmatrix}$$

$$E = \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \end{bmatrix}} = \begin{bmatrix} 5 & 6 & 11 \\ -3 & -2 & -5 \\ 7 & 6 & 13 \end{bmatrix}$$

$$F = \underset{2\times 3}{\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}} \times \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \end{bmatrix}}$$

$\therefore$ $(2\times 3)(2\times 3)$ cannot multiply

$\neq$

$\therefore$ F is undefined

ANS: $C = \begin{bmatrix} 1 & -2 & -2 \\ -2 & -3 & -3 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 5 \\ 11 & 11 \end{bmatrix}$, $E = \begin{bmatrix} 5 & 6 & 11 \\ -3 & -2 & -5 \\ 7 & 6 & 13 \end{bmatrix}$, F is undefined.

The Identity Matrix

The n x n identity matrix I is one with elements $a_{ij} = 1$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$:

$$I = \begin{bmatrix} 1 & 0 \ldots & 0 \\ 0 & 1 \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \ldots & 1 \end{bmatrix}$$

For any n x n matrix A, it can be verified that:

$$AI = IA = A$$

Solving a SLE in Matrix Notation

A SLE can be solved in a systematic manner by a method called Gauss-Jordan Elimination. This is especially useful for systems with a large number of equations.

The Gauss-Jordan method

The **Gauss-Jordan method** solves a linear system by representing it as an **augmented matrix** and applying the following **row transformations**:

(aka row operations)

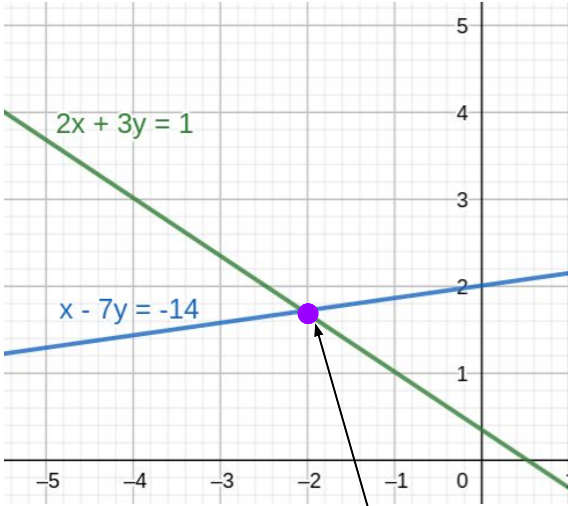
- Interchange any two rows.
- Multiply or divide the elements of any row by a non-zero real number.
- Replace any row by the sum of the elements of that row and a multiple of the elements from another row.

(These are the same rules as for solving by elimination.)

Gauss-Jordan Elimination

Example: Solve the previous example using Gauss-Jordan elimination.

$$\begin{aligned} 2x + 3y &= 1 \\ x - 7y &= -14 \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{V}} \underset{=}{=} \underset{\vec{b}}{\begin{bmatrix} 1 \\ -14 \end{bmatrix}}$$



$x = -2.059$

$y = 1.706$

Augmented matrix is:

$$[A|b] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & -7 & -14 \end{array}\right]$$

By row operations:

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ \boxed{1} & -7 & -14 \end{array}\right] \xrightarrow{2R_2 - R_1} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -17 & -29 \end{array}\right] \xrightarrow[R_2/17]{R_1/2} \underbrace{\left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{29}{17} \end{array}\right]}_{\text{Row-echelon form}}$$

make 0

make 1

$\rightarrow x + \frac{3}{2}y = \frac{1}{2}$

$\rightarrow y = \frac{29}{17}$

$x = \frac{1}{2} - \frac{3}{2}\left(\frac{29}{17}\right)$

$= -2.059$

Gauss-Jordan Elimination

General approach is to obtain the row-echelon form and then solve for the variables.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & b_{12} & \dots & b_{1n} \\ 0 & 1 & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Original Matrix

Row Echelon Form
(with leading 1s)

Some may prefer to do row operations until the coefficient matrix becomes an identity matrix and obtaining the variables directly.

Gauss-Jordan Elimination

More 1s is better

Exercise: Solve the following SLE using Gauss-Jordan elimination. What does the solution represent on a graph?

$x + 3y + 2z = 1$

$$
\left.\begin{aligned}
3x + 9y + 6z &= 3\\
2x + y - z &= 2\\
x + y + z &= 2
\end{aligned}\right\}
$$

Aug. Matrix is $[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & \boxed{1} & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \end{array}\right]$

make 0

make 0 (do not use R_1)

$R_2 - 2R_1$

$R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -2 \\ 0 & 2 & 1 & -1 \end{array}\right] \xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -5 & -5 \end{array}\right]$$

ANS: (x, y, z) = (2, -1, 1). A point.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -5 & -5 \end{array}\right] \xrightarrow[R_3/-5]{R_2/-1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

make 1s

$\rightarrow z = 1$

$y + 3z = 2$

$y = 2 - 3$

$= -1$

$x + y + z = 2$

$x = 2 - (-1) - 1$

$= 2$

$(x, y, z) = (2, -1, 1)$, a point in $\underbrace{\mathbb{R}^3}_{\text{3D space}}$

Linear Independence & Type of Solutions

Different types of solutions can occur when solving a SLE, which depends on the **linear independence** of the equations.

A unique solution occurs when all equations (rows) in the SLE are linearly independent, which means any one equation (row) cannot be expressed as a linear combination of the other equations (rows). This means:

$$k_1 R_1 + k_2 R_2 + \ \dots \ + k_m R_m = 0$$

only if all $k_i = 0$. While this condition might be difficult to check, it is easily revealed by the Gauss-Jordan Elimination - If there are no rows with all zeros in the row-echelon form of the coefficient matrix, then all equations (rows) in the SLE are linearly independent. The previous example demonstrated this.

Solution Types - 1) Unique Solution

$R_2 \to x + \frac{3}{2}y = \frac{1}{2}$

$R_1 = 2R_2$

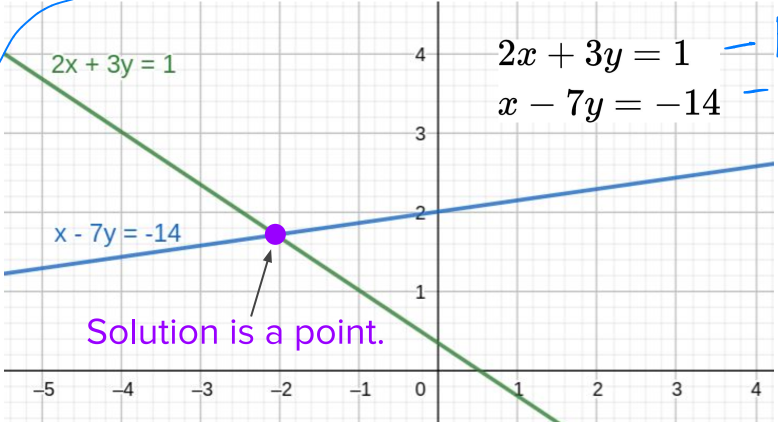
$R_1 = kR_2$ } Not L.I.

$\underbrace{R_1 \neq \overset{\text{Constant}}{C}\, R_2}_{R_1 \,\&\, R_2 \text{ are L.I.}}$

Unique Solution - Exactly one point given by the coordinates of intersection.

Occurs when all equations are linearly independent.

Such a system is known as consistent, because solution/s exist.



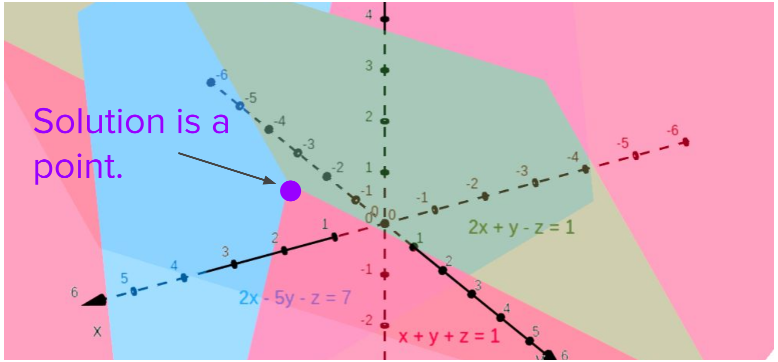
$$2x + 3y = 1 \quad - R_1$$
$$x - 7y = -14 \quad - R_2$$

$R_1 - \underbrace{C}_{k_2}R_2 \neq 0$

$k_1R_1 + k_2R_2 \neq 0$

$\Rightarrow k_1R_1 + k_2R_2 = 0$ only if $k_1 = k_2 = 0$

L.I. SLE
$$\begin{cases} 2x + y - z = 1 \\ 2x - 5y - z = 7 \\ x + y + z = 1 \end{cases}$$

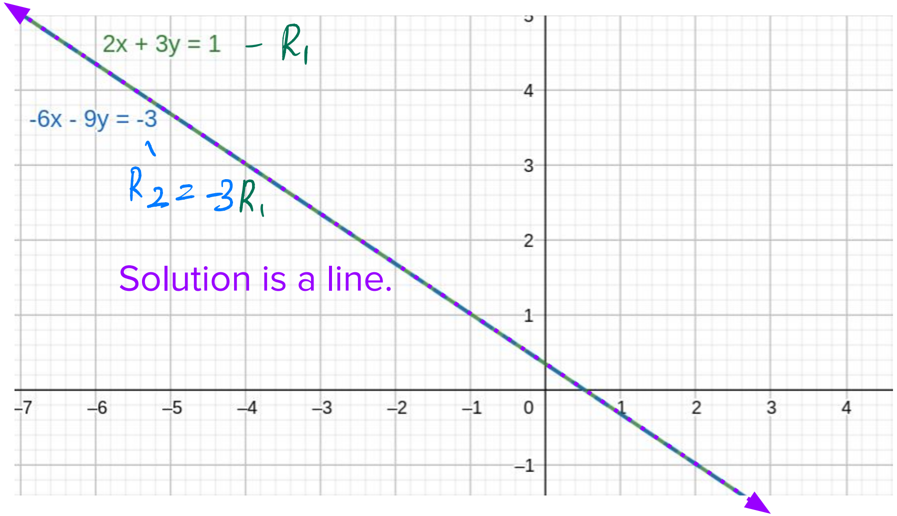


Solution Types - 2) Infinite Solutions

Infinite Solutions - Infinite points given by the intersection 'space'.

Occurs when some or all equations are not linearly independent.

Such a system is still consistent, since there are solutions.



Example:

$$\begin{aligned} 2x + 3y &= 1 \\ -6x - 9y &= -3 \end{aligned}$$

Not L. I. since $R_2 = -3R_1$

Sol$^{\underline{n}}$ is the line $2x+3y=1$

In Gauss-Jordan Elimination, such a SLE will have all zeros in its lowest row/s of the row-echelon form of the augmented matrix. Such a SLE system is still consistent, since there are solutions.

Check : $[A|b] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ -6 & 9 & -3 \end{array}\right] \xrightarrow{R_2+3R_1} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right]$

Solution Types - 2) Infinite Solutions

Infinite Solutions - Infinite points given by the intersection 'space'.

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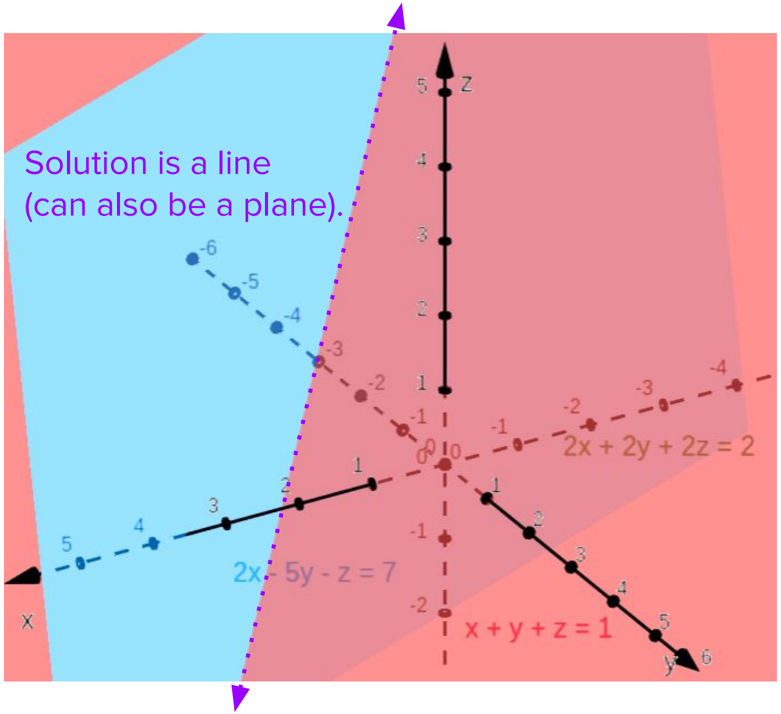


Example:

same plane

$2x + 2y + 2z = 2$ – $R_1 = 2R_3$

$2x - 5y - z = 7$ – R_2

$x + y + z = 1$

$$\underbrace{R_3 = k_1 R_1 + k_2 R_2} = 4x - 3y + z = 9$$

$$k_1 R_1 + k_2 R_2 \underbrace{-}_{k_3 = -1} R_3 = 0 \quad \leftarrow \quad \text{Not L.I.}$$

Verify

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -5 & -1 & 7 \\ 4 & -3 & 1 & 9 \end{array}\right]$$

$$\downarrow \begin{array}{l} R_3 - 4R_1 \\ R_2 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -7 & -3 & 5 \\ 0 & -7 & -3 & 5 \end{array}\right]$$

Solution Types - 3) No Solutions

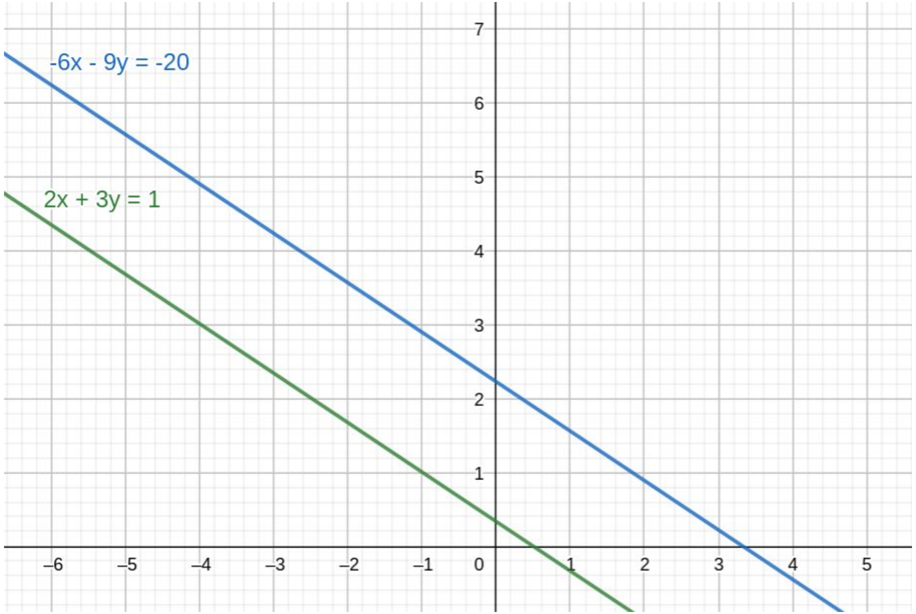
No Solution - No intersection ‘space’.

Occurs when all equations are linearly independent.

Such a system is known as inconsistent, because there are no solutions.

Example:

$$
\begin{aligned}
2x + 3y &= 1 \\
-6x - 9y &= -20
\end{aligned}
$$



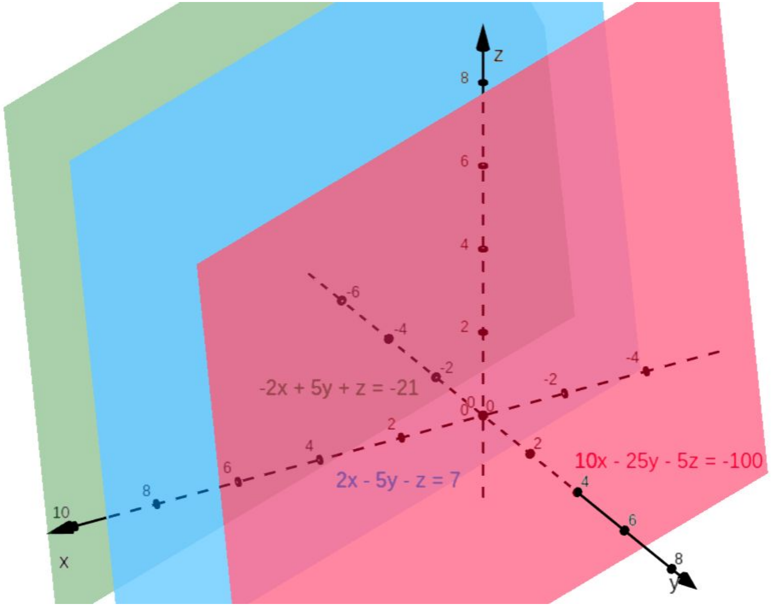
No solution occurs when all equations are linearly independent and all geometries (lines, planes, etc) are parallel to one another.

Solution Types - 3) No Solutions

No Solution - No intersection 'space'.

Occurs when all equations are linearly independent.

Such a system is known as inconsistent, because there are no solutions.



Example:

$$-2x + 5y + z = -21$$
$$2x - 5y - z = 7$$
$$10x - 25y - 5z = -100$$

In Gauss-Jordan Elimination, such a SLE will have all zeros at the lowest row/s in the row-echelon form of the coefficient matrix, but a non-zero constant at the corresponding row of the constant vector.

Aug. matrix is :

$$\left[\begin{array}{ccc|c} -2 & 5 & 1 & -21 \\ 2 & -5 & -1 & 7 \\ 10 & -25 & -5 & -100 \end{array}\right] \xrightarrow[R_3+5R_1]{R_2+R_1} \left[\begin{array}{ccc|c} -2 & 5 & 1 & -21 \\ 0 & 0 & 0 & -14 \\ 0 & 0 & 0 & -205 \end{array}\right]$$

$$\left.\begin{array}{l} 0x + 0y + 0z = -14 \\ 0x + 0y + 0z = -205 \end{array}\right\} \text{not possible}$$

inconsistent, so SLE has no sol^{n}s.

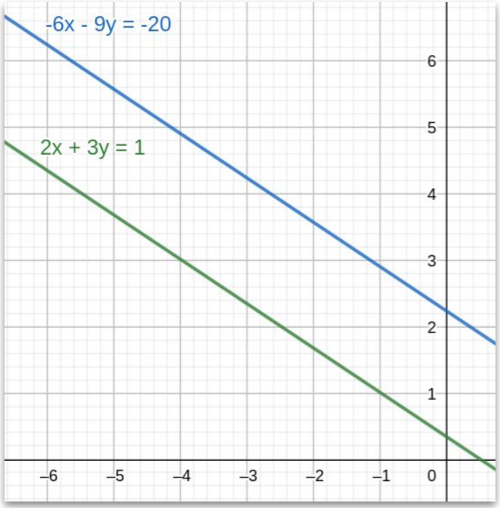
Summary of Solution Types

(Have solⁿ)

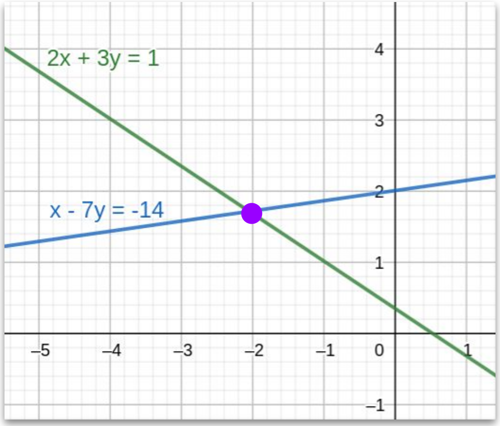
Inconsistent

Consistent

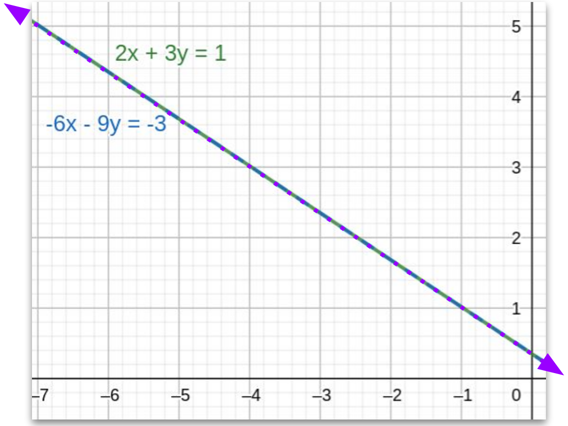
Independent SLE
No Solution



Independent SLE
Unique Solution



Dependent SLE
Infinite Solutions



Solution Types of a SLE

Exercise: Classify the following SLEs on their linear independence, consistency and type of solution.

a) $x + y = 1$
$-x + y = 1$

$R_1 \neq kR_2$, so SLE is LI, consistent with unique sol^n.

b) $\frac{x}{3} - \frac{y}{6} = 2$
$-2x + y = -12$

$R_1 = R_2 / -6$, so SLE is not LI, & it has ∞ sol^{n}s so it is consistent

c) $x + y + z = 1$
$2x - 5y - 7z = 1$
$-x - y + z = -1$

SLE is LI, consistent with unique sol^n.

d) $x + 2y + 3z = -1$
$\frac{x}{2} + y + \frac{3}{2}z = 7$
$-3x - 6y - 9z = 8$

By inspection, all rows are parallel lines, so SLE is LI & have no sol^{n}s so inconsistent

ANS: a & c) LI, consistent, unique soln. b) Not LI, consistent, infinite solns. d) LI, inconsistent, no solns.

c) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -5 & -7 & 1 \\ -1 & -1 & 1 & -1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -7 & -9 & -1 \\ 0 & 0 & 2 & 0 \end{array}\right]$

$\therefore$ c) is LI, consistent, unique soln

d) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 1 & 2 & 3 & 7 \\ -3 & -6 & -9 & 8 \end{array}\right] \xrightarrow[R_3+3R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 5 \end{array}\right]$

$\therefore$ d) is LI, inconsistent, no solutions

When rank = no. of equations, SLE is full rank & LI, so we have a unique sol^n.

Rank of a Matrix

The rank of a matrix can be defined as the number of non-zero rows in its row-echelon form, which is also the number of linearly independent equations in the corresponding SLE.

More formally, the rank of a matrix is defined as the maximum number of dimensions spanned by the column vectors (or row vectors) of the matrix. You will understand this statement better after we have covered Topics 3 & 4.

Solution Types of a SLE

Exercise: Solve the following SLE using Gauss-Jordan elimination. Is the system consistent and what is the rank of the coefficient matrix? What does the solution represent on a graph?

$$2x + 2y + 2z = 2$$
$$2x - 5y - z = 7$$
$$x + y + z = 1$$

same. Only 2 L.I. eqns, so rank 2 < Full rank of 3, so ∞ $\text{sol}^{\text{n}}\text{s}$, consistent SLE

Aug matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -5 & -1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -7 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

R_4: $-7y - 3z = 5$

$z = \dfrac{-7y-5}{3}$ //

R_5: $x + y + z = 1 \rightarrow z = 1 - x - y$

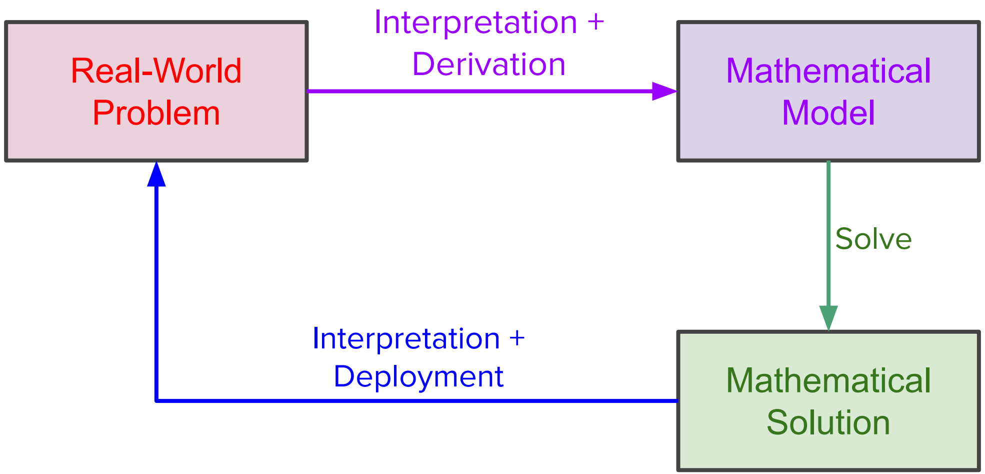
$7R_5 + R_4 \rightarrow 7x + 4z = 12$

$z = \dfrac{12-7x}{4}$ //

Ans: z = -(7y+5)/3 = -7x/4 + 3. Consistent and rank 2. A line.

Introduction to Mathematical Modelling

The main concept behind mathematical modelling is to interpret a real-world problem into a representation composed of math relations. Then, solving these relations yield results that can be interpreted back to the real world for deployment of the solution. The conceptual flowchart is shown below.



This flowchart is iterative in nature as usually it takes more than one cycle to arrive at a satisfactory solution.

Leontief Economic Input-Output Model

This is a math model that describes the interdependencies between various industries in an economy and analyses the ‘ideal’ amount of production for each industrial sector.

For example, consider a simple economy that has only two industrial sectors, namely, electricity and water production. It is known that to produce electricity, water is needed (to operate steam turbines etc) and to produce water, electricity is needed (to operate pumps etc).

In addition, to produce electricity, some electricity is also needed (to operate lights and equipment etc) and to produce water, some water is also needed (to clean equipment etc). Therefore, the production of each sector is dependent on itself and the production from other sectors. Furthermore, each commodity is also produced to satisfy demand from other consumers.

Leontief Economic Input-Output Model

Let x = electricity output, y = water output and assuming that:

To produce 1 unit of electricity, we need 0.2 units of electricity and 0.4 units of water. To produce 1 unit of water, we need 0.3 units of electricity and 0.1 units of water.

Other consumer demands for electricity and water are 20 units and 32 units respectively. Tabulating the above, we get:

	Input to x	Input to y	Other demand
Electricity output, x	0.2	0.3	20 units
Water output, y	0.4	0.1	32 units

Leontief Economic Input-Output Model

Logically, the input (demand) and output (production) of each commodity must balance, which means:

$$\begin{aligned}\text{Electricity output} = {} & \text{Electricity input for electricity output} \\ & + \text{Electricity input for water output} \\ & + \text{Other electricity demand}\end{aligned}$$

$$x = 0.2x + 0.3y + 20$$

$$\begin{aligned}\text{Water output} = {} & \text{Water input for electricity output} \\ & + \text{Water input for water output} \\ & + \text{Other water demand}\end{aligned}$$

$$y = 0.4x + 0.1y + 32$$

$$\rightarrow \begin{aligned} 8x - 3y &= 200 \\ -4x + 9y &= 320 \end{aligned}$$

This SLE is the math model that represents the (simplistic) economic problem which can be solved easily (x = 46 units, y = 56 units).

*In 1973, a Harvard professor, Wassily Leontief, was awarded the Nobel prize in economics from his work on input-output models of the economy. His model of the US economy is composed of 500 sectors, each represented by a linear equation.

End of Topic 2

*Always think about the **big picture** of why you are learning the ‘boring math’.*

I solved a system of 500 linear equations. You think you have it rough?

https://en.wikipedia.org/wiki/Input%E2%80%93output_model