DISCRETE MATHEMATICS FOR ENGINEERS

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Exercises

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1. Propositional Logic

$\boxed{\textbf{1.1}}$ You take a programming hands-on training. In 10 minutes is the deadline of the homework. The wheather was good during the last weeks. Thus you are doing the homework at the last moment. There are only few lines of code missing. But now the 'a' of your keyboard is broken. The next line is supposed to be '*If x* ***and*** *not(y) then ...*'. What will you do?

$X \wedge \neg Y$

$\boxed{\textbf{1.2}}$ Determine the truth tables for the following propositional forms:

1.2.a $A \iff a \wedge (b \vee c)$

1.2.b $B \iff (a \wedge b) \vee (a \wedge c)$

1.2.c $C \iff ((a \vee b) \wedge (b \vee c)) \wedge (c \vee a)$

1.2.d $D \iff ((a \wedge b) \vee (b \wedge c)) \vee (c \wedge a)$

1.2.e $E \iff \neg a \longrightarrow (b \vee c)$

1.2.f $F \iff \neg a \longleftrightarrow (b \vee c)$

1.2.g $G \iff (a \longleftrightarrow b) \longleftrightarrow c$

1.2.h $H \iff a \longleftrightarrow (b \longleftrightarrow c)$

$\boxed{\textbf{1.3}}$ Consider the following three propositional forms:

I : $(\neg a \wedge a) \vee (a \not\longleftrightarrow (b \wedge c))$
II : $[(a \longleftrightarrow b) \longrightarrow b] \wedge (a \not\longleftrightarrow a)$
III : $\neg[(c \vee \neg a) \wedge (a \vee \neg b \vee \neg c)]$

1.3.a Are these propositional forms equivalent? Determine their truth tables to check this.

1.3.b Determine the on-sets of propositional forms I to III.

$\boxed{\textbf{1.4}}$ Mr. Miller is going to have a party and wants to invite Anne, Betty, Charlotte or Dana (no "exclusive OR"!). However, some difficulties occur:

- Anne and Dana can't stand each other, so in no case both of them will appear at the party.
- Charlotte will only come to the party if Dana joins her.
- if Betty comes to the party she will in any case bring along Anne.

What possibilities does Mr. Miller have to avoid every possible conflict?
Choose appropriate logical variables and formalise the given preconditions.

Solve this problem once by setting up the truth table, and once by calculating the CDNF of the resulting propositional form.

$\boxed{\textbf{1.5}}$ The citizens of an imaginary city can be divided into two disjoint sets: Those of one set always tell the truth, the others lie all the time.

Logic variable

	a, b, c :	citizen A, B, C always tells the truth.
e.g.,	$\neg$ a :	citizen A lies all the time.

Interview 1:

A says : B always tells the truth.
B says : If A always tells the truth, then C always tells the truth, too.

Interview 2:

C says : I always tell the truth
and
if B always tells the truth, then A lies all the time.

1.5.a Formalize interview 1.

1.5.b Determine the truth table of interview 1 and its on-set.

1.5.c Formalize interview 2.

1.5.d Determine the truth table of interview 2 and its on-set. Interpret the result (conclusion) of interview 2 in your own words.

1.5.e What do you think can be concluded from the inconsistency of the two interviews?

1.6 There has been a murder. Five suspects A, B, C, D and E are being taken into custody. From the circumstances of the crime, the following premises V_1 and V_2 arise:

V_1: only these five persons are to be considered possible murderers.
V_2: the murder was committed by one, two or three persons.
(i.e. at least two suspects are not guilty)

After the first interrogation, the following assertions were made:

B_1: A says "B and C did both not do it."
B_2: B says "There are exactly three committers."
B_3: C says "D is a murderer and if A was involved, then so was E."
B_4: D says "A and B are murderers".
B_5: E says "Either A is a murderer or D, B, and C are murderers".

Use the logic variables a, b, c, d and e according to $a \iff$ "A is a murderer".

Assume that a murderer <u>may</u> lie, whereas an innocent will always tell the truth.
According to this assumption, the assertion "A says ..." can be formalized as "$\neg a \rightarrow \ldots$".

1.6.a Formalize the premises V_1 and V_2 and the assertions B_1 till B_5 of the first interrogation.

1.6.b Set up the truth table of the premises V_1 and V_2 and of the assertions B_1 till B_5 for all possible values of a, b, c, d and e. Enlist the values for the propositional forms $V \Leftrightarrow V_1 \wedge V_2$ and $B \Leftrightarrow B_1 \wedge B_2 \wedge B_3 \wedge B_4 \wedge B_5$ in addition. State the on-set $E[V \wedge B]$.

In a second interrogation, E gives the following additional evidence:

B_6: E says "A is not a murderer, but D is one."

1.6.c Formalize the assertion B_6 and add it to the truth table in b) as well as $R \Leftrightarrow V \wedge B \wedge B_6$. State the on-set $E[R]$ and interpret the facts of the case shown in the on-set in your own words. So who committed the crime?

1.7 For the following three propositional forms

$$A \iff (a \longleftrightarrow b) \wedge \neg c$$
$$B \iff a \wedge (b \not\longleftrightarrow c) \wedge b$$
$$C \iff t \not\longleftrightarrow [c \vee \neg((a \wedge b) \vee \neg(a \vee b))]$$

find out which of them are equivalent. For that purpose, transform these propositional forms into disjunctive normal forms and compare them to each other.

1.8 Calculate the $CDNF$ as well as the $CCNF$ of the following ternary propositional forms $A_i(a, b, c)$:

1.8.a $A_1 \iff [a \nleftrightarrow (b \longrightarrow c)] \vee \neg(a \longrightarrow \neg b)$

1.8.b $A_2 \iff (a \nleftrightarrow b \longrightarrow c) \wedge (a \longrightarrow b \wedge \neg c)$

1.8.c $A_3 \iff [a \longrightarrow (b \longleftrightarrow c)] \wedge (\neg a \longrightarrow b)$

1.9 Prove the laws D1 to D14 from the manuscript.

1.10 Prove the laws E1 to E21 from the manuscript.

1.11 The propositional variables a, b and c

a :	Kevin loves Suzy.
b :	Kevin dates Suzy.
c :	Kevin goes dancing with Linda.

are used to formulate the following five assertions:

$$
\begin{array}{lll}
\text{I} & : & (b \longrightarrow a) \wedge (b \longrightarrow \neg c) \iff \neg b \vee (a \wedge \neg c) \\
\text{II} & : & (b \longrightarrow a) \wedge (b \longrightarrow \neg c) \iff (a \longrightarrow c) \longrightarrow \neg b \\
\text{III} & : & b \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c) \Longrightarrow a \wedge \neg c \\
\text{IV} & : & c \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c) \Longrightarrow \neg b \\
\text{V} & : & c \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c) \Longrightarrow \neg a
\end{array}
$$

1.11.a Prove the validity of assertions I and II using equivalence transformations.

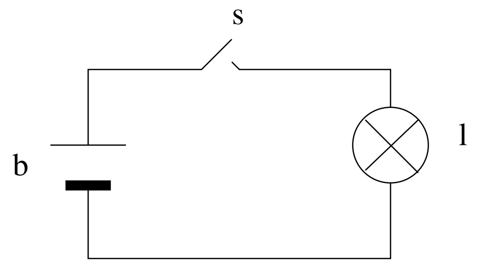
1.11.b Verify the validity of assertions III and IV using the deductive proof scheme.

1.11.c Set up the on-set $E[c \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c)]$

1.11.d How many non-trivial conclusions can be drawn from $c \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c)$?

1.11.e Verify or falsify the validity of the assertion V using the on-sets $E[c \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c)]$ and $E[\neg a]$.

1.12 The following circuit is given:



The propositional variables are:

b :	The battery is working
s :	The switch is closed
l :	The bulb is lit

The following three assertions have been formulated:

$$
\begin{array}{lll}
\text{I} & : & l \longrightarrow s \wedge b \Longrightarrow \neg s \longrightarrow \neg l \\
\text{II} & : & s \wedge b \longleftrightarrow l \Longrightarrow \neg s \longrightarrow \neg l \\
\text{III} & : & \neg l \wedge s \wedge (s \wedge b \longleftrightarrow l) \Longrightarrow \neg b
\end{array}
$$

Prove these assertions using the deductive proof scheme.

1.13 Use the deductive proof scheme to prove the following assertions:

1.13.a $\neg d \wedge b \wedge [a \longrightarrow (d \longrightarrow (b \longrightarrow c))] \quad \Longrightarrow \quad d \longrightarrow c$

1.13.b $[\neg a \longrightarrow (b \vee d)] \wedge (a \longleftrightarrow c) \wedge (b \longrightarrow c) \wedge (d \longrightarrow c) \Longrightarrow \quad c$

1.13.c $[a \longrightarrow (c \longrightarrow ((d \longrightarrow b) \longrightarrow b))] \wedge (\neg c \longrightarrow d) \wedge \neg d \Longrightarrow \quad a \longrightarrow b$

1.13.d $a \longleftrightarrow c \quad \Longrightarrow \quad (b \longrightarrow a) \leftrightarrow (b \longrightarrow c)$

1.13.e $[(\neg a \longrightarrow \neg b) \longrightarrow c] \wedge [b \longrightarrow ((a \longrightarrow c) \longrightarrow \neg c)] \quad \Longrightarrow \quad b \longleftrightarrow c$

1.13.f $\neg a \wedge [b \longrightarrow (c \longrightarrow a)] \wedge [\neg(c \longrightarrow b) \longrightarrow a] \quad \Longrightarrow \quad a \longleftrightarrow c$

1.14 The state secretary of the ministry of finance receives the following analysis of the economic situation:

- [I)] Tax increase leads to price instability.
- [II)] If the taxes are not increased the national budget has to be cut.
- [III)] If keeping the national budget constant implies instable prices the taxes must be increased.

Hence, the state secretary concludes that the taxes have to be increased.

1.14.a Formalize the statements I, II and III.
Use the following propositional variables:

- [i:] Taxes are being increased
- [p:] Prices are stable
- [b:] The national budget has to be cut

1.14.b Use the deductive proof scheme to find out if the state secretary's decision was correct.

1.15 Use only the following laws in this problem, in connection with rule RT1 if necessary:

$$
\begin{array}{rcll}
a \vee (b \vee c) & \Longleftrightarrow & (a \vee b) \vee c & \text{(A2)} \\
\neg(\neg a) & \Longleftrightarrow & a & \text{(A9)} \\
a \longleftrightarrow b & \Longleftrightarrow & (a \longrightarrow b) \wedge (b \longrightarrow a) & \text{(C11)} \\
a \longrightarrow b & \Longleftrightarrow & \neg a \vee b & \text{(D1)} \\
a \longrightarrow b & \Longleftrightarrow & \neg b \longrightarrow \neg a & \text{(D3)} \\
\neg a & \Longrightarrow & a \longrightarrow b & \text{(E7)} \\
a \wedge (a \longrightarrow b) & \Longrightarrow & b & \text{(E9)} \\
(a \longrightarrow b) \wedge (a \longrightarrow \neg b) & \Longrightarrow & \neg a & \text{(E13)}
\end{array}
$$

Prove the following assertions with the deductive proof scheme by exclusively using these laws:

1.15.a $[(b \vee c) \longrightarrow a] \wedge (a \vee b \vee c) \Longrightarrow a$

1.15.b $b \wedge (\neg a \longrightarrow \neg c) \wedge (a \longrightarrow \neg b) \Longrightarrow a \longleftrightarrow c$

$\boxed{\textbf{1.16}}$ In this excercise, it is to be proven, that Sherlock Holmes drew the right conclusion about the motive of a murder from the evidence. To this, an excerpt from "A Study in Scarlet":

> And now we come to the great question as to the reason why. Robbery has not been the object of the murder, for nothing was taken. Was it politics, then, or was it a woman? That is the question which confronted me. I was inclined from the first to the latter supposition. Political assassins are only too glad to do their work and fly. This murder had, on the contrary, been done most deliberately, and the perpetrator had left his tracks all over the room, showing he had been there all the time.

From these circumstances, Sherlock Holmes drew the conclusion that the motive was a woman.

Use the following propositional variables to formalise the given excerpt:

r : It was robbery.
p : It was politics.
w : It was a woman.
s : Something was taken.
i : The assassin left immediately.
l : The assassin left tracks all over the room.

On the basis of the excerpt above, the following premises can be formulated:

P_1 : If it was a robbery, something would have been taken.
P_2 : It was robbery or it was politics or it was a woman.
P_3 : Nothing was taken.
P_4 : If it was politics, the assassin would have left immediately.
P_5 : If the assassin left tracks all over the room, he cannot have left immediately.
P_6 : The assassin left tracks all over the room.

1.16.a Formalize the premises P_1 till P_6 using the given propositional variables.

1.16.b Prove that the motive was a woman, i.e.:

$$P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge P_6 \Longrightarrow w$$

using the deductive proof scheme.

$\boxed{\textbf{1.17}}$ **1.17.a** Prove:

$$(a \vee b) \wedge (\neg a \vee c) \iff (a \vee b) \wedge (\neg a \vee c) \wedge (b \vee c)$$

1.17.b Prove:

$$(a \wedge b) \vee (\neg a \wedge c) \iff (a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c)$$

1.17.c Use the resolution method to prove that

$A \iff (a \wedge b) \vee (b \wedge \neg c) \vee \neg b \vee (\neg a \wedge c)$

is a tautology.

$\boxed{\textbf{1.18}}$ Determine if the following propositional forms are contradictions, tautologies or partly valid using the resolution method:

1.18.a $A \iff (a \vee b) \wedge (\neg b \vee \neg c) \wedge (c \vee \neg d)$

1.18.b $B \iff (\neg a \vee b) \wedge (a \vee \neg c) \wedge (c \vee \neg b) \wedge a \wedge (\neg a \vee \neg b)$

1.18.c $C \iff a \vee (\neg a \wedge b) \vee (\neg a \wedge \neg c) \vee (\neg b \wedge c)$

1.18.d $D \iff \neg a \wedge (a \vee \neg b) \wedge (a \vee c) \wedge (b \vee \neg c)$

$\boxed{\textbf{1.19}}$ Prove by only using the resolution rule in condition clause form (RRP) that it holds:

$$A(a, b, c) \iff (a \longrightarrow \neg b) \wedge (\neg a \longrightarrow \neg b) \wedge (\neg b \longrightarrow f) \iff f.$$

$\boxed{\textbf{1.20}}$ Prove the following assertions about properties of the β-operation. Name all laws you use.

1.20.a $\beta(p, a, a) \iff a$

1.20.b $\beta(p, t, f) \iff p$

1.20.c $\beta(p, f, t) \iff \neg p$

1.20.d $\beta(t, a, b) \iff a$

1.20.e $\beta(f, a, b) \iff b$

1.20.f $p \wedge \beta(p, a, b) \iff p \wedge a \iff \beta(p, a, f)$

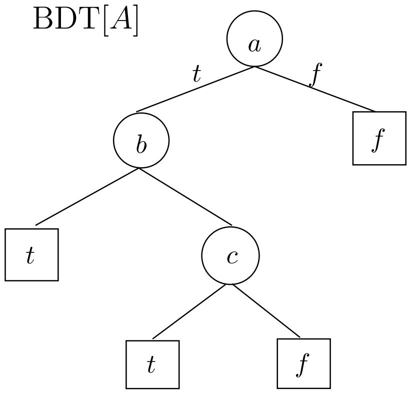
1.20.g $\neg p \wedge \beta(p, a, b) \iff \neg p \wedge b \iff \beta(p, f, b)$

1.20.h $\beta(p, \beta(p, a, b), c) \iff \beta(p, a, c)$

1.20.i $\beta(p, a, \beta(p, b, c)) \iff \beta(p, a, c)$

1.20.j Describe what effects the above assertions about the β-operation have on a BDT.

$\boxed{\textbf{1.21}}$ Consider the following BDT of a propositional form $A(a, b, c)$.



1.21.a How many t- and f-assignments does A own?
Set up the CDNF and the CCNF by reading off the BDT.

1.21.b Determine the premise normal form $PNF[A]$ by reading off the BDT.

1.21.c Calculate the CDNF from the PNF.

1.21.d How to obtain the $\mathrm{BDT}[\neg A]$?

1.21.e Set up the $\mathrm{ROBDD}[A \vee B]$ with $B \iff a \vee c$.

$\boxed{\textbf{1.22}}$ Consider the following propositional form $C(a, b, c)$:

$$C(a, b, c) \iff [a \not\longleftrightarrow (b \wedge \neg c)] \longleftrightarrow [b \longrightarrow (a \wedge c)]$$

1.22.a Calculate the $PNF[C]$.
Choose $a,\ b,\ c$ as processing order of the variables.

1.22.b Draw the BDT of C.

1.22.c Calculate the ring normal form RNF of C.

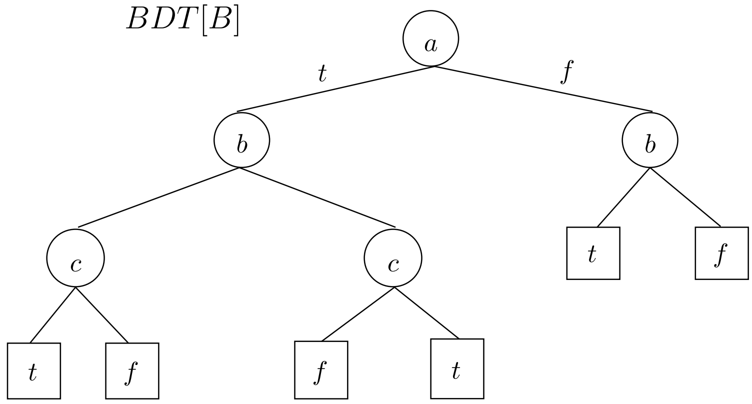
$\boxed{\textbf{1.23}}$ Consider the propositional form

$$A(a,b,c) \iff (a \nleftrightarrow b \longrightarrow \neg c) \vee (a \wedge b \longleftrightarrow c)$$

Use law $BO17$ to calculate the premise normal form (PNF) of A and draw the binary decision tree (BDT).
Choose $c,\ a,\ b$ as processing order of the variables.

$\boxed{\textbf{1.24}}$ Consider the following binary decision tree (BDT) of a propositional form $B(a,b,c)$:

$BDT[B]$



Determine $PNF[B]$ by reading off the BDT.
Calculate $CDNF[B]$ from the PNF.

$\boxed{\textbf{1.25}}$ Prove: $A \iff (a \vee \neg b \vee c) \vee (\neg a \wedge b \wedge \neg c) \iff t$

1.25.a by setting up a truth table.

1.25.b by use of transformations.

1.25.c by replacement (e.g., $E \iff a \vee \neg b \vee c$).

1.25.d by setting up a binary decision diagram (BDD).

1.25.e by using the resolution method.

$\boxed{\textbf{1.26}}$ Given is the propositional form $A(a,b,c) \iff a \longrightarrow (b \vee c)$.

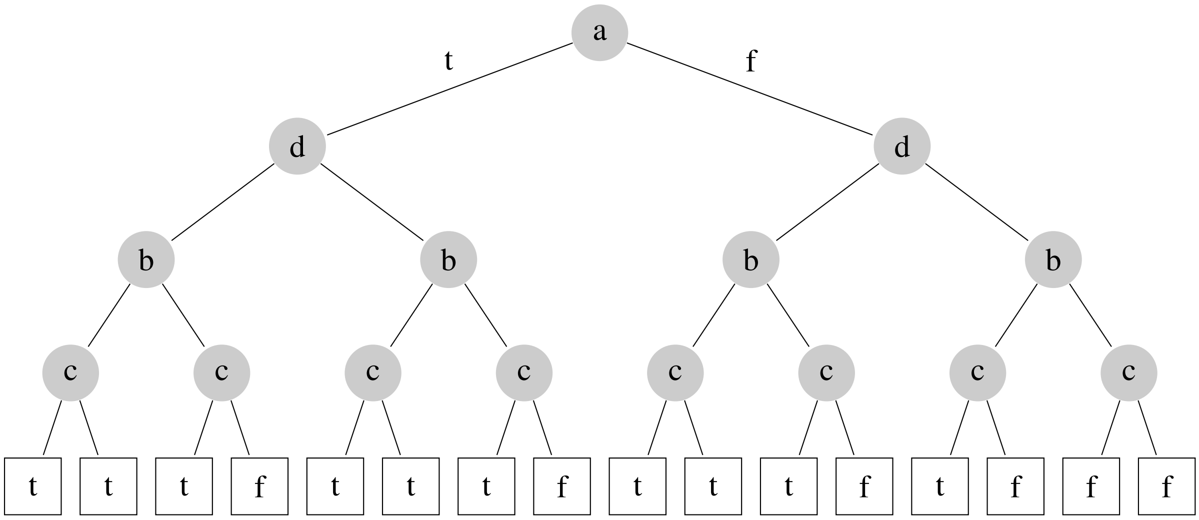
1.26.a Give the dual form dual$(A(a,b,c))$ of $A(a,b,c)$.

1.26.b Calculate the number of non-trivial conclusions that can be drawn from dual$(A(a,b,c))$.

1.26.c Name two non-trivial conclusions that can be drawn from dual$(A(a,b,c))$.

$\boxed{\textbf{1.27}}$ Let the number of non-trivial conclusions that can be drawn from an arbitrary k-ary contingency D, be denoted with n.
The number of value assignments that falsify the dual form of D equals the number of value assignments that verify D.
Calculate the number of non-trivial conclusions l that can be drawn from $dual(D)$, dependent on k und n.

$\boxed{\textbf{1.28}}$ Given is the following Ordered Binary Decision Tree (OBDT) of the propositional form A:



1.28.a Draw the corresponding ROBDT[A] and the ROBDD[A].

1.28.b Draw the ROBDD[dual(A)].

Further exercises:

1.29 By constructing a BDD, the following tautology shall be verified:
$b \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c) \Longrightarrow (a \wedge \neg c)$
For that purpose, this proposition is first being transformed into an equivalent representation:
$b \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c) \longrightarrow (a \wedge \neg c) \iff w$

1.29.a Set up the binary decision tree BDT(A) of the propositional form
$A \iff b \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c)$.
Use the following order of decision variables: $b \succ a \succ c$.

1.29.b Transform BDT(A) into BDD(A).

1.29.c Set up the binary decision tree BDT(B) of the propositional form
$B \iff (a \wedge \neg c)$.
Use the following order of decision variables: $a \succ c$.

1.29.d Transform BDT(B) into ROBDD(B).

1.29.e Restate the propositional form $A \longrightarrow B$ as PNF: $A \longrightarrow B \iff \beta(?, ?, ?)$

1.29.f Use the composition rule and the result of d) to merge ROBDD(A) and ROBDD(B). (Attach ROBDD(B) to ROBDD(A), not the other way round!)
State the PNF of the resulting BDD. Note it down exactly according to the structure of the BDD without applying any simplifications.

1.29.g Now apply the *substitution rule* to the obtained BDD.
(SPR1 from the lecture, $\beta\,(x_i, A(\underline{x}), B(\underline{x})) \iff \beta\,(x_i, A(x_i \iff t), B(x_i \iff f))$).
Give the the resulting PNF.

1.29.h Simplify the result further on by use of transformation laws of the β-form, especially the resolution law $\beta(x, A, A) \iff A$.
Name the applied law for every step as well as the modified PNF.

1.29.i Is the originally considered implication a tautology or not?

1.30 Set up OBDTs for the following propositional forms. Use the given order of the decision variables.
Transform the OBDTs to ROBDDs afterwards.
Specify for each propositional form, if it turns out to be a tautology or a contradiction. Furthermore, denote the propositional forms which are equivalent to one another.

1.30.a $A(x_1, \ldots, x_6) \iff (x_1 \wedge x_2) \;\vee\; (x_3 \wedge x_4) \;\vee\; (x_5 \wedge x_6)$
Order: $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$

1.30.b $A(x_1, \ldots, x_6) \iff (x_1 \wedge x_4) \;\vee\; (x_2 \wedge x_5) \;\vee\; (x_3 \wedge x_6)$
Order: $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$

1.30.c $A(x_1, \ldots, x_4) \iff (((x_1 \wedge x_2) \longrightarrow x_3) \longrightarrow x_4) \longrightarrow ((x_1 \longrightarrow x_3) \longrightarrow x_4)$
Order: $x_1 \succ x_2 \succ x_3 \succ x_4$

1.30.d $A(x_1, \ldots x_3) \iff (x_1 \longrightarrow (x_2 \vee x_3)) \wedge (\neg x_1 \longrightarrow (x_2 \longleftrightarrow x_3))$
Order: $x_3 \succ x_2 \succ x_1$

1.30.e $A(x_1, \ldots, x_3) \iff \neg(((x_1 \not\longleftrightarrow x_3) \wedge \neg x_2) \vee ((x_2 \not\longleftrightarrow x_3) \wedge \neg x_1))$
Order: $x_3 \succ x_2 \succ x_1$

$\boxed{\textbf{1.31}}$ Assume $\mathcal{M}_1 = \{V_1, \ldots, V_n\}$ and $\mathcal{M}_2 = \{U_1, \ldots, U_m\}$ to be premise sets. Under which prerequisites $\Big(a),\ b)\ or\ c)\Big)$ does $V_1 \wedge \ldots \wedge V_n \Rightarrow$ A
for all propositional forms A also lead to $U_1 \wedge \ldots \wedge U_m \Rightarrow$ A ?

1.31.a $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

1.31.b $V_1 \wedge \ldots \wedge V_n \Rightarrow U_1 \wedge \ldots \wedge U_m$.

1.31.c $E[U_1 \wedge \ldots \wedge U_m] \subseteq E[V_1 \wedge \ldots \wedge V_n]$.

Thus, in the first part of this excercise, it is to show if
$\{(V_1 \wedge \ldots \wedge V_n) \longrightarrow A\} \wedge \mathcal{M}_1 \subseteq \mathcal{M}_2 \overset{?}{\Longrightarrow} \{(U_1 \wedge \ldots \wedge U_m) \longrightarrow A\}$ (exact justification!)

2. Predicate Logic

2.1 The following three statements are given:

$[A_1]$: All Bavarians are German.

$[A_2]$: Every person is either a German or an Italian (multiple citizenship is not allowed).

$[A_3]$: Hans is a Bavarian.

2.1.a Formalize the given statements by means of those predicates:

$Bx \iff x$ is a Bavarian $\quad Gx \iff x$ is a German $\quad Ix \iff x$ is an Italian

P: set of all German and Italian people

$x_1 \iff$ "Hans"; $x_1 \in P$

2.1.b Formalize the following conclusions and prove them using the given statements:

$[C_1]$: Hans is not an Italian.

$[C_2]$: There exists nobody that is Bavarian and Italian.

2.2 Formalize the following definitions. Use the predicate Kab representing $a < b$.

2.2.a A function f is continuous in the interval D, if and only if:

For every point $x_0 \epsilon D$, there's for each $\varepsilon > 0$ a $\delta > 0$ so that:
$|f(x_0) - f(x)| < \varepsilon$ for all x with $|x_0 - x| < \delta$

2.2.b A function f, defined on the interval D, has the limit y_1 for $x \longrightarrow +\infty$ if and only if:

For every $\varepsilon \epsilon R^+$ there is a $x_0 \in D$ with $|f(x) - y_1| < \varepsilon$ for all $x \in D$ with $x > x_0$.

2.3 Prove the equivalences of predicate logic given in the manuscript (F1 ... F13). The first laws (F1 ... F8) can be proven by tracing back the predicative expressions to equivalent expressions of the propositional logic. For the remaining laws (F9 ... F13) try to find additional proofs which use the preceding laws and therefore avoid a reduction of the equivalences to propositional logic.

2.4 Prove the following predicative logic forms using the deductive proof scheme. Try to find several different approaches to solve the problems.

2.4.a $\underset{x\,\in\,\mathcal{G}}{\forall}(Px \longrightarrow Sx) \wedge \underset{x\,\in\,\mathcal{G}}{\forall}(Sx \longrightarrow Qx) \wedge \underset{x\,\in\,\mathcal{G}}{\exists} Px \implies \underset{x\,\in\,\mathcal{G}}{\exists} Qx$

2.4.b $\underset{x\,\in\,\mathcal{G}}{\forall}(Sx \longrightarrow Qx) \wedge \underset{x\,\in\,\mathcal{G}}{\exists}(Px \wedge Sx) \implies \underset{x\,\in\,\mathcal{G}}{\exists}(Px \wedge Qx)$

2.4.c $\underset{x\,\in\,\mathcal{G}}{\forall}(Tx \longrightarrow \neg Px) \wedge \left(\underset{x\,\in\,\mathcal{G}}{\forall} Tx \vee \neg \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right) \implies \underset{x\,\in\,\mathcal{G}}{\forall}(Qx \longrightarrow \neg Px)$

2.4.d $\neg \underset{x\,\in\,\mathcal{G}}{\exists}(Px \longrightarrow \neg Qx) \wedge \left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Qx \vee \underset{x\,\in\,\mathcal{G}}{\forall} Sx\right) \implies \underset{x\,\in\,\mathcal{G}}{\exists} Sx$

2.4.e $\neg \underset{x\,\in\,\mathcal{G}}{\exists} Qx \wedge \underset{x\,\in\,\mathcal{G}}{\forall} (Tx \longrightarrow \neg\,(Qx \longrightarrow Px)) \implies \underset{x\,\in\,\mathcal{G}}{\forall} \neg Tx$

2.4.f $\left(\underset{x\,\in\,\mathcal{G}}{\exists} Px \longleftrightarrow \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right) \wedge \underset{x\,\in\,\mathcal{G}}{\exists} Qx \implies \underset{x\,\in\,\mathcal{G}}{\exists} \neg Px$

2.4.g $\left(\underset{x\,\in\,\mathcal{G}}{\forall} \neg Px \vee \neg \underset{x\,\in\,\mathcal{G}}{\exists} Tx\right) \wedge \underset{x\,\in\,\mathcal{G}}{\forall} (\neg Px \longrightarrow Qx) \implies \underset{x\,\in\,\mathcal{G}}{\forall} (Tx \longrightarrow Qx)$

2.5 Check if the following are valid predicate logic implications.

2.5.a $\underset{x\,\in\,\mathcal{G}}{\exists} Px \quad \wedge \left(\underset{x\,\in\,\mathcal{G}}{\exists} Px \longrightarrow \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right) \implies \underset{x\,\in\,\mathcal{G}}{\exists} Qx$

2.5.b $\underset{x\,\in\,\mathcal{G}}{\exists} Px \quad \wedge \underset{x\,\in\,\mathcal{G}}{\exists}(Px \rightarrow Qx) \implies \underset{x\,\in\,\mathcal{G}}{\exists} Qx$

2.6 Prove the general validity of the following propositions using the inductive proof scheme without using the expression to be proven as premise.

2.6.a $P(n) \quad \iff \quad \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}\,, \quad n \in \mathbb{N}$

2.6.b $P(n) \quad \iff \quad \left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} (i)^3\,, \quad n \in \mathbb{N}$

Hint: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2.6.c $P(n) \iff \sum\limits_{k=0}^{n} a^{n-k} \cdot b^k = \frac{a^{n+1}-b^{n+1}}{a-b}\,, \quad n \in \mathbb{N}_0$

2.6.d $P(n) \iff 1 + 2n \leq 3^n\,, \quad n \in \mathbb{N}$

Hint: The implication $a \leq c \wedge b \leq d \Longrightarrow a + b \leq c + d$ holds.

2.6.e $P(n) \iff \prod\limits_{i=0}^{n} \left(1 + x^{(2^i)}\right) = \frac{1-x^{(2^{n+1})}}{1-x}\,, \quad n \in \mathbb{N}_0,\ x \in \mathbb{R} \setminus \{1\}$

Hints: $x^{(a+b)} = x^a \cdot x^b, \quad x^{a \cdot b} = (x^a)^b; \quad a, b \in \mathbb{Z} \setminus \{0\}$

2.6.f $P(n) \iff \sum\limits_{k=1}^{n-1} (x_k - x_{k+1}) = x_1 - x_n\,, \quad n \in \mathbb{N} \setminus \{1\}$

2.7 The formalization of the proposition "There is only one $x \in \mathcal{M}$ for which Px is valid." leads to:

$\underset{x \in \mathcal{M}}{\exists} \left[Px \wedge \underset{y \in \mathcal{M}}{\forall} (Py \longrightarrow (x = y)) \right].$

Demonstrate that this proposition implies the conclusion $\underset{y \in \mathcal{M}}{\forall} \left[Py \longrightarrow \underset{x \in \mathcal{M}}{\exists} (x = y) \right]$:

$\underset{x \in \mathcal{M}}{\exists} \left[Px \wedge \underset{y \in \mathcal{M}}{\forall} (Py \longrightarrow (x = y)) \right] \Longrightarrow \underset{y \in \mathcal{M}}{\forall} \left[Py \longrightarrow \underset{x \in \mathcal{M}}{\exists} (x = y) \right]$

Use the following auxiliary rule for the deductive proof scheme:

$\underset{x \in \mathcal{M}}{\exists} \ \underset{y \in \mathcal{N}}{\forall} Qxy \Longrightarrow \underset{y \in \mathcal{N}}{\forall} \ \underset{x \in \mathcal{M}}{\exists} Qxy$

2.8 Given is the recursion formula for the following integral:

$$I_n = \int_0^\pi \sin^n(x)dx = \frac{n-1}{n} \int_0^\pi sin^{n-2}(x)dx$$

Prove by utilizing the inductive proof scheme that for $n = 2k + 1$, $k \in \mathbb{N}$ this formula can be expressed as:

$$I_{2k+1} = 2 \prod_{i=1}^{k} \frac{2i}{2i+1}$$

3. Sets

3.1 Given are the sets A, B and the universal set G:

$$\begin{aligned} A &= \{\{\}, \{\emptyset\}, \{b\}, \{a, c\}, \{a, b, c\}\} \\ B &= \{\{c\}, \{a, b\}, \{a, c\}\} \\ \text{with: } A, B &\subseteq G \end{aligned}$$

Give the following sets in explicite notation:

$$\begin{aligned} M_1 &= G \setminus \overline{A} \\ M_2 &= \Big\{X \in A \,\Big|\, \underset{Y \in B}{\exists} X \subset Y\Big\} \\ M_3 &= \big\{(X, Y) \in A \times B \,\big|\, |X| + |Y| = 2\big\} \\ M_4 &= P(A \cap B) \end{aligned}$$

3.2 Consider the following two laws in propositional logic and the rule of compatibility with three arbitrary propositional forms A, B and C. Name the corresponding relationships between the on-sets $E[A]$, $E[B]$ and $E[C]$.

3.2.a $A \wedge B \Longrightarrow (A \wedge C) \vee (B \wedge \neg C)$

3.2.b $A \longleftrightarrow B \iff (A \vee \neg B) \wedge (\neg A \vee B)$

3.2.c IF $A \Longrightarrow B$, THEN $A \vee C \Longrightarrow B \vee C$

3.3 Which of the following propositions about the propositional forms A and B are correct? Justify your answer.

3.3.a If A is a tautology and $E[B] \neq \emptyset$, then $E[A \to B] \neq \emptyset$.

3.3.b If A is a tautology and $E[B] \neq \emptyset$, then $A \to B$ is a tautology.

3.3.c If $E[B] \neq \emptyset \iff t$, then $B \to (B \wedge \neg B) \iff t$.

Hint: Transform the propositions into relations between on-sets and prove them formally.

3.4 Decide and justify if the following propositions about the propositional forms A and B are true or false:

3.4.a If B is a contradiction and A is not a tautology, then $E[A \rightarrow B] \neq \emptyset$.

3.4.b If B is not a contradiction, then $(A \vee B) \rightarrow A$ is a tautology for any given A.
Hint: Transform the propositions into relations between on-sets and prove them formally.

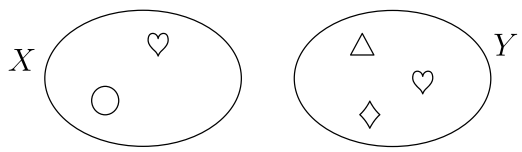
3.5 Given is the following implication (IMP) and the corresponding on-set of A, B and C:

$$(A \longrightarrow B) \longleftrightarrow C \Longrightarrow A \qquad \text{(IMP)}$$

$$\begin{aligned} \mathbf{E}[A] &= \{(0,1,0),(1,0,0),(1,1,0)\} \\ \mathbf{E}[B] &= \{(0,0,1),(0,1,0),(0,1,1),(1,1,1)\} \\ \mathbf{E}[C] &= \{(0,1,0),(1,1,0)\} \end{aligned}$$

The universal set is $G = \{0,1\}^3$. Show that the implication IMP is valid for the given sets.

3.6 Given are the sets X und Y:



To describe these sets in propositional logic, the elements are coded using the following table:

Element	$\bigcirc$	$\diamondsuit$	$\triangle$	$\heartsuit$
Coding (a,b)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$

Give for both sets X and Y a valid characteristic function A and B, respectively (see formulary p. 49).

3.7 Prove the following relations between sets:

3.7.a $(A\triangle C)\setminus D \subseteq \overline{\left[\overline{(A\setminus D)}\setminus\left(C\cap\overline{D}\right)\right]\cap B} \iff t$

3.7.b $\overline{(A\setminus B)}\cap[A\setminus(B\cap C)] \subseteq A\setminus C \iff t$

3.7.c $(A \triangle B) \subseteq A \iff B \subseteq A$

Name all laws you use.

3.8 Prove the following implications in set notation using the deductive proof scheme. Make sure you solely use sets of laws. Are all implications valid? The universal set is G.

3.8.a $(\mathcal{A} \subseteq \mathcal{C}) \wedge (\mathcal{B} \subseteq \mathcal{D}) \implies (\mathcal{A} \cup \mathcal{B}) \subseteq (\mathcal{C} \cup \mathcal{D})$

3.8.b $[(\mathcal{A} \triangle \mathcal{G}) \cup (\mathcal{B} \cap \mathcal{C}) = \mathcal{G}] \wedge [(\mathcal{A} \cup \mathcal{B}) \triangle \mathcal{G} \neq \mathcal{C}] \implies \mathcal{B} \cap \mathcal{C} \neq \emptyset$

3.8.c $(\mathcal{A} \subseteq \mathcal{B}) \wedge \left(\overline{\mathcal{B}} \cup \mathcal{A} = \mathcal{G}\right) \implies \mathcal{B} \backslash \mathcal{A} = \emptyset$

3.8.d $(\mathcal{G} \triangle \mathcal{A} = \emptyset) \wedge (\mathcal{B} \subseteq \mathcal{C}) \implies \mathcal{B} \subseteq (\mathcal{C} \cap \mathcal{A})$

3.8.e $\left(\overline{\mathcal{A}} \cup \overline{\mathcal{B}} = \mathcal{G}\right) \wedge \left(\overline{\mathcal{A}} \subseteq \overline{\mathcal{C}}\right) \implies \mathcal{C} \backslash \overline{\mathcal{B}} = \emptyset$

3.8.f $(\mathcal{A} \triangle \mathcal{B} = \emptyset) \implies \left(\overline{\mathcal{B}} \subseteq \overline{\mathcal{A}}\right) \wedge (\mathcal{B} \backslash \mathcal{A} = \emptyset)$

3.9 Transform the implications given in set form in the previous exercise into implications in predicate logic.
The predicates are defined as follows:

$$x \in \mathcal{A} \iff Px \qquad x \in \mathcal{C} \iff Sx$$
$$x \in \mathcal{B} \iff Qx \qquad x \in \mathcal{D} \iff Tx\,,$$
The universal set is $\mathcal{G}$.

3.10 Transform the implications in predicate logic given in exercise 2.4 into set notation.
The predicates are defined as follows:

$$x \in \mathcal{A} \iff Px \qquad x \in \mathcal{C} \iff Sx$$
$$x \in \mathcal{B} \iff Qx \qquad x \in \mathcal{D} \iff Tx\,,$$
The universal set is $\mathcal{G}$.

$\boxed{\textbf{3.11}}$ Consider the following proposition in predicate logic:

$$\underset{x \in G}{\forall}(Px \wedge Rx) \vee \underset{x \in G}{\forall} Qx \Longrightarrow \underset{x \in G}{\exists}(Px \vee Qx)$$

3.11.a Prove this proposition using laws of predicate logic.

3.11.b Transform the proposition into set notation. Use the sets

$$P = \{x \in G | Px\} \qquad Q = \{x \in G | Qx\} \qquad R = \{x \in G | Rx\}.$$

3.11.c Prove the propostion using laws of set algebra.

3.11.d Compare the two ways of proof stepwise. What conclusion can you draw?

$\boxed{\textbf{3.12}}$ Given is the following implication in set notation:

$$A = B \Longrightarrow A \cup C = B \cup C$$

$$x \in A \iff Px;\; x \in B \iff Qx;\; x \in C \iff Sx; x \in D \iff Tx$$
$$A, B, C, D \subseteq G$$

Transform the expressions into predicate logic.
Perform the deductive proof once in predicate logic notation and once in set notation.

$\boxed{\textbf{3.13}}$ Given is the following implication:

$$V_1 \wedge V_2 \wedge V_3 \Longrightarrow S$$

with:

$$\begin{aligned}
V_1 &\iff \neg \underset{x \in G}{\exists}(Qx \wedge \neg Sx) \\
V_2 &\iff \underset{x \in G}{\forall}(\neg Sx \longleftrightarrow Tx) \vee Tx \\
V_3 &\iff \neg \underset{x \in G}{\exists}(Px \wedge \neg Qx \wedge \neg Sx) \\
S &\iff \underset{x \in G}{\forall}(Px \vee Qx \longrightarrow Tx)
\end{aligned}$$

$$x \in A \iff Px;\; x \in B \iff Qx;\; x \in C \iff Sx; x \in D \iff Tx$$
$$A, B, C, D \subseteq G$$

Transform the implication into set notation and perform the deduction.

$\boxed{\textbf{3.14}}$ Given is the set $M = \{1, 2, 3\}$.

3.14.a Give the power set $P(M)$ and the cardinality $|P(M)|$ of M.

3.14.b Give the cardinality $|P(M)^2|$.

3.14.c Give the following set explicitly: $N = \{(x, y)|((x, y) \in P(M)^2) \wedge (x \subset y)\}$.

$\boxed{\textbf{3.15}}$ Consider the sets M and N.

Prove the equivalence in subtask a) and examine the truth value of the propositions on power sets in the remaining subtasks by proof or showing a counter-example:

3.15.a $N \subseteq M \iff P(N) \subseteq P(M)$

3.15.b $P(M) \cup P(N) \subseteq P(M \cup N)$

3.15.c $P(M) \cup P(N) \supseteq P(M \cup N)$

3.15.d $P(M) \cup P(N) = P(M \cup N)$

3.15.e $P(M) \cap P(N) = P(M \cap N)$

4. Relations

4.1 Transform the following expressions given in component free notation into an expression in predicate logic component notation.

4.1.a $R \subseteq S^{-1} \cup (T \setminus S), \qquad R, S, T \subseteq A^2$

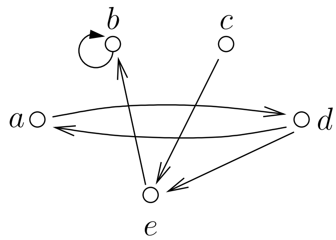
4.1.b $R \setminus (S^2 \setminus T) \subseteq R \triangle \overline{S}, \qquad R, S, T \subseteq A^2$

4.2 Transform the following expressions given in predicate logic component notation into component free notation without any predicate logic expressions.

4.2.a $\underset{x,\,y\,\in\,A}{\forall}\left[xRy \longleftrightarrow \left(\underset{z\,\in\,A}{\exists}(xSz \wedge zSy)\right) \wedge \neg xTy\right]$

4.2.b $\underset{(x,\,y)\,\in\,A^2}{\exists}\left[(xRy \vee xSy) \not\longleftrightarrow \underset{z\,\in\,A}{\exists}(xRz \wedge zSy)\right]$

4.3 Consider the arrow diagram of a homogeneous relation R:



4.3.a Which of the nodes in R are initial, which are terminal?

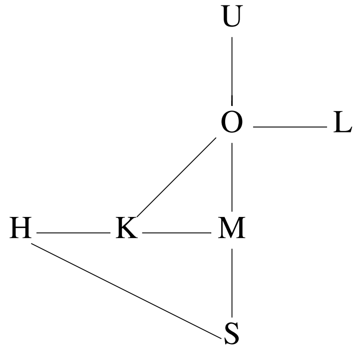
4.3.b Name all branch and reconvergence points.

4.3.c Does R have loops?

Justify your answers.

4.4 Prove the laws K3, K4, and K5 from the manuscript. Proceed like it was shown in the lecture.

4.5 Consider the following extract of Munich's public transport (MVV) map:



4.5.a Write down the adjacency relation R of G as list of ordered pairs. What properties does R have?

4.5.b Calculate $R' = R \circ R$ as list of ordered pairs. What properties does R' have?

4.5.c What direct meaning has $S = R \cup R'$ for someone who does not own a commutation ticket ("MVV Zeitkarte") ?

4.5.d Determine S graphically by inserting bypass edges into the transport map.

4.5.e Determine the adjacency matrix M_R of G and determine the adjacency matrices of $M_{R'}$ and M_S by operations on M_G.

4.6 Prove that the following formulations of properties of relations are each equivalent to one another.

4.6.a R is symmetric: $\underset{x,y\,\in\,A}{\forall}[xRy \longrightarrow yRx] \iff R = R^{-1}$

4.6.b R is antisymmetric: $\underset{x,y\,\in\,A}{\forall}[xRy \wedge yRx \longrightarrow x = y] \iff R \cap R^{-1} \subseteq I$

4.6.c R is asymmetric: $\underset{x,y\,\in\,A}{\forall}[xRy \longrightarrow \neg yRx] \iff R \cap R^{-1} = \emptyset$

4.6.d R is transitive: $\underset{x,y,z\,\in\,A}{\forall}[xRy \wedge yRz \longrightarrow xRz] \iff R^2 \subseteq R$

4.6.e R is intransitive: $\underset{x,\,y,\,z\,\in\,A}{\forall}[xRy \wedge yRz \longrightarrow \neg xRz] \iff R^2 \subseteq \overline{R}$

4.6.f R is connex: $\underset{x,\,y\,\in\,A}{\forall}[xRy \vee yRx] \iff \overline{R} \cap (\overline{R})^{-1} = \emptyset$

4.6.g R is semiconnex: $\underset{x,\,y\,\in\,A}{\forall}[(x \neq y) \longrightarrow xRy \vee yRx] \iff \overline{R} \cap (\overline{R})^{-1} \subseteq I$

4.7 In chapter 4.3 of the manuscript, nine properties of relations are listed. These are formally denoted and they are given in component notation as well as in closed notation. In the following, you'll find nine propositions on graphs of relations. Determine the properties of relations described by the propositions.

4.7.a All elements of A are comparable to each other, including each element to itself.

4.7.b Arrow and "reverse arrow" only occur as a loop.

4.7.c All elements of A have a loop.

4.7.d All consecutive arrows lack the bypass arrow.

4.7.e Every arrow has a "reverse arrow".

4.7.f All distinct elements of A are comparable.

4.7.g No element of A has a loop.

4.7.h No arrow has a "reverse arrow".

4.7.i All consecutive arrows have a bypass arrow.

4.8 Examine the following relations and find out, which of the properties discussed in the manuscript they actually exhibit. The basic set is the set of all human beings.

4.8.a $R_1 \iff$ "is brother of"

4.8.b $R_2 \iff$ "is married to" (does it make any difference if we consider a polygamous instead of a monogamous society?)

4.8.c $R_3 \iff$ "is mother of"

4.8.d $R_4 \iff$ "is a relative of"

4.8.e $R_5 \iff$ "is at least as tall as"

$\boxed{\textbf{4.9}}$ **4.9.a** Try to give a definition of the adjacency matrix M of a relation R in predicate logic.

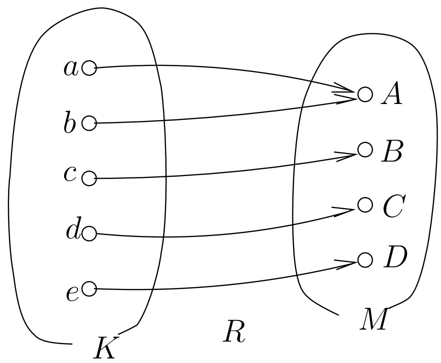
4.9.b Name the properties of M according to the properties of relations in chapter 4.3 of the manuscript.

$\boxed{\textbf{4.10}}$ Assume a set of children $K = \{a, b, c, d, e\}$, and a set of mothers $M = \{A, B, C, D\}$ and a relation $R \subseteq K \times M$ with

$$R = \{(x, y)|\ x \text{ is child of mother } y\}_{K \times M}.$$

The graph $G = (K \cup M, R)$ looks as follows:

$G = (K \cup M, R)$:



4.10.a Determine the adjacency matrices of R and R^{-1}.

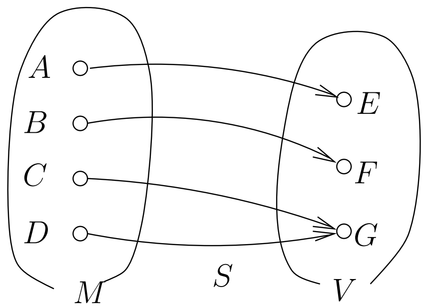
4.10.b Determine the adjacency matrices of RR^{-1} and of $R^{-1}R$.
How can these relations be interpreted?
Derive properties of R and R^{-1} from that.

In addition, a relation

$$S = \{(x, y)|\ \text{mother } x \text{ and father } y \text{ have common children }\}_{M \times V}$$

and its graph $G = (M \cup V, S)$ is given. $V = \{E, F, G\}$ is supposed to be a set of fathers.

$G = (M \cup V, S)$:



4.10.c Draw the graph of the composition of R with S and determine the adjacency matrix of RS.
Which properties has RS?

4.10.d Prove that generally, the composition of two functional relations produces a functional relation again.

4.10.e How can the composition RS be interpreted, considering that RS is functional?

4.11 The specification of a combinational circuit is usually not done as a function but as a relation. The specification S assigns one or more valid output assignments $\underline{z}$ to an input assignment $\underline{x}$. If there are several output assignments assigned to one input assignment, then degrees of freedom, so called “don’t cares” exist for the realization of the circuit. A circuit that fulfills a specification S realizes a function C, which is included in the specification S, i.e. $C \subseteq S$ must hold. The task of logic synthesis is to determine and synthesize a function C for a specification S in a way that the final circuit can be produced at minimum cost.

A circuit is often given as a netlist and the task is to resynthesize the netlist. In this case $S = C$ holds. For embedded subcircuits (blocks) degrees of freedom can be calculated and these degrees of freedom can be used during resynthesis of the block.

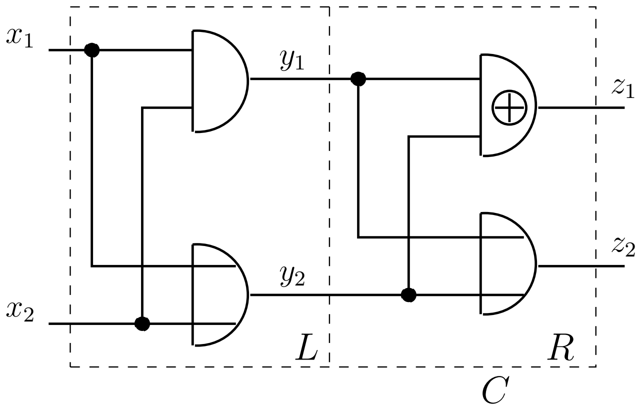
The following figure shows the truth table of a two-dimensional boolean function C as well as the adjacency matrix of the respective boolean relation.

x_1	x_2	z_1	z_2
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1

$$C: \begin{array}{c|c} {}_{x_1x_2} \diagdown {}^{z_1z_2} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

4.11.a Draw an arrow diagram of the (heterogenous) relation C. Prove that the relation C is a (total) function. How can a function be recognized considering the arrow diagram or the adjacency matrix respectively?

The following figure shows a realization of C, whereas C has been realized by series connection of blocks L and R.



4.11.b Determine the adjacency matrices of L and R and show that C emerges from the composition of L with R. Draw an arrow diagram of the composed relations L and R.

To optimize block R, the degrees of freedom of R have to be determined and utilized.

4.11.c Determine the maximum relation R_{max} for block R. Then, choose an appropriate function R_{opt} and set up the corresponding realization of the whole circuit.
Hint: Use the Schröder rule to determine relation R.

4.11.d Now, optimize block L of the original circuit C. Proceed according to subtask c).

4.12 Consider the set $A_1 = \{a, b, c\}$ and the relation $R_1 = \{(a, b), (b, c), (c, a)\}_{A_1^2}$.

4.12.a Draw the arrow diagram of $G_1 = (A_1, R_1)$.

4.12.b Determine the relations R_1^2 and R_1^3 and state the graph.
Determine $p_1 \in \mathcal{N}$, such that $R_1^{s+p_1} = R_1^s$, $s \in \mathcal{N}_0, p_1 \in \mathcal{N}$ holds.

4.12.c Compose the adjacency matrix M_1 of relation R_1.

4.12.d Determine the adjaceny matrices M_1^2 and M_1^3 of relations R_1^2 and R_1^3 using matrix multiplication.

4.12.e Now consider the set $A_2 = \{d, e, f, g, h\}$ and the relation $R_2 = \{(d, e), (e, f), (f, g), (g, h), (h, d)\}_{A_2^2}$. State its graph $G_2 = (A_2, R_2)$ as well as its adjacency matrix M_2.

4.12.f Determine $p_2 \in \mathcal{N}$, such that $R_2^{s+p_2} = R_2^s, s \in \mathcal{N}_0, p_2 \in \mathcal{N}$ holds.
For this purpose, calculate $M_2^2, M_2^3, \ldots$, using matrix multiplication.

4.12.g Now the relation $R = R_1 \cup R_2$ over the basic set $A = A_1 \cup A_2$ is to be analyzed.
Determine $p \in \mathcal{N}$ such that $R^{s+p} = R^s$, $s \in \mathcal{N}_0$ holds.
(Think before doing!)

4.12.h For a relation R over the basic set A with $|A| = n$ it's usually sufficient to calculate the union of the relations R^ν and $\nu = 1, \ldots, n$ to determine the transitive closure. Until which ν do you have to calculate the union if you want to build the transitive closure of $R = R_1 \cup R_2$? Justify your answer.

$\boxed{4.13}$ Determine the closures of the following relations depicted by their adjacency matrices.

4.13.a

$$R : \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$ Determine $r(R)$, $s(R)$ and $t(R)$

4.13.b

$$R : \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Determine $r(R)$, $s(R)$ and $t(R)$

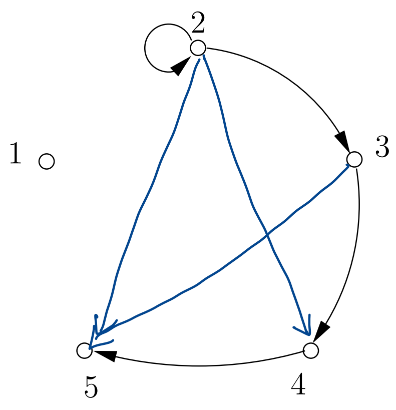
4.13.c

$$R : \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ Determine $s(R)$ and $t(R)$

$\boxed{4.14}$ Determine the transitive closure R_i^+ for each of the following relations $R_i \subseteq A^2$ with $A = \{1, 2, 3, 4, 5\}$ using the Warshall-Algorithm.

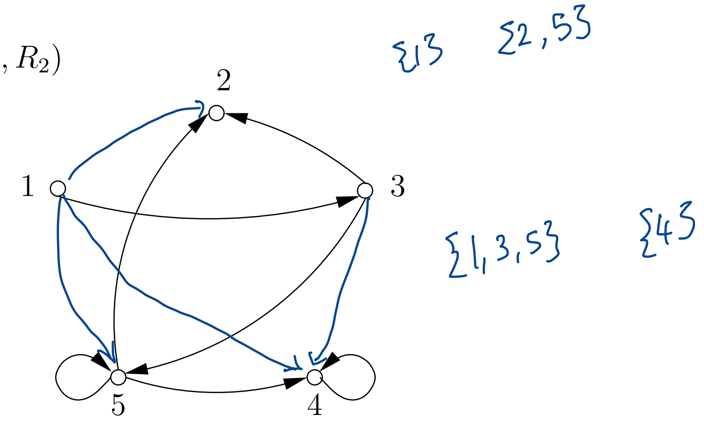
4.14.a

$G_{R_1} = (A, R_1)$ $\{2\}$ $\{3\}$



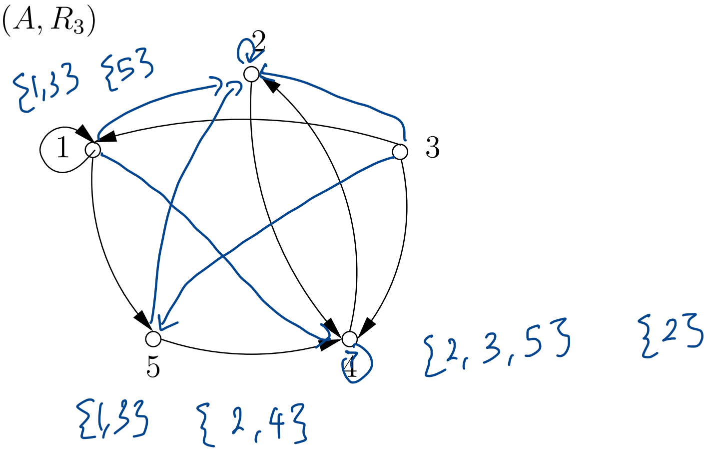
4.14.b

$G_{R_2} = (A, R_2)$



4.14.c

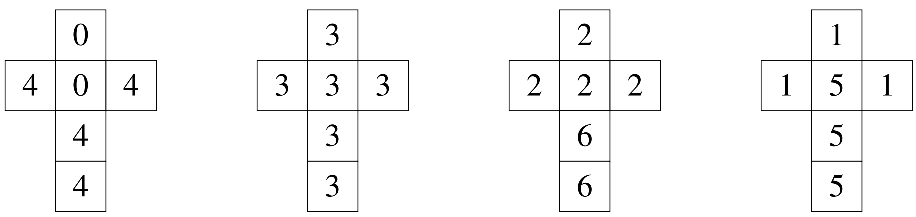
$G_{R_3} = (A, R_3)$



$\boxed{4.15}$ Relation $R \subseteq A^2$ with $A = \{a, b, c, d, e\}$ is given as list of ordered pairs:
$R = \{(a, c), (b, b), (c, a), (c, e), (d, b), (e, d)\}$

Carry out the Warshall-Algorithm step by step to build the transitive closure of R. Determine the cross product $\Gamma^-(x) \times \Gamma^+(x)$ for every node.
Write down R^+ as set of ordered pairs.

$\boxed{4.16}$ Consider four dice, denoted as A, B, C and D. In the following, the surfaces of the dice are depicted:



A B C D

After two throws (even with the same dice) the higher score wins.

4.16.a For each pair of dice x and y with $x, y \in \{A, B, C, D\}$ determine the probability $P(x, y)$, that dice x beats dice y.

4.16.b Now consider the binary relation R ("has got a higher chance of winning"), which is defined on {A,B,C,D} as follows:

$$xRy \iff P(x \text{ beats } y) > \frac{1}{2}$$

State the relation explicitly and draw its graph.

4.16.c State $t(R)$.

4.16.d Is R transitive? Justify your answer.

4.16.e What does transitivity of a relation R mean in the case of the proposed game?

$\boxed{\textbf{4.17}}$ Assume a set $P = \{P_1, P_2, ..., P_n\}$ of procedures of a computer program.
Relation $\rightsquigarrow$ on P is defined as follows:

$$P_i \rightsquigarrow P_j \iff P_i \text{ calls } P_j$$

For a given set of procedures and the relation $\rightsquigarrow$ defined on it, the task is to find out if recursive procedures exist. A procedure P_i is (directly) recursive if it calls itself or (transitively) recursive if it is being called by another procedure, which (or whose predecessor) has been called by the procedure P_i.
There is a set of five procedures $P = \{A, B, C, D, E\}$ and the following facts:
A calls B and E
B calls C
C calls E
D calls C and
E calls B .

4.17.a Write down relation $\rightsquigarrow$ in set notation as well as its graph $G = (P, \rightsquigarrow)$.

4.17.b Does P contain directly recursive procedures?

4.17.c Does P contain transitively recursive procedures? How can that be seen?

4.17.d Set up $G = (P, \rightsquigarrow^+)$. Use matrices ($\rightsquigarrow^+ = \bigcup_{\nu=1}^{|P|} \rightsquigarrow^\nu$).

4.17.e How can recursive procedures be recognized in $G = (P, \rightsquigarrow^+)$?

4.17.f Can you distinguish between directly recursive and transitively recursive procedures in this graph?

4.17.g From the adjacency matrix $M_{\rightsquigarrow}$ of the original relation $\rightsquigarrow$, determine the predecessors and successors of procedure C by an appropriate multiplication of the matrix by a vector.

4.18 Consider relation $R = \{(a, e), (b, a), (b, c), (c, d), (e, b)\}$ with $R \subseteq A^2$ and $A = \{(a, b, c, d, e)\}$.
Draw the arrow diagram of relation R.
Are the pairs $\{a, c\}$ and $\{a, e\}$

- connected?
- indirectly accessible?
- mutually accessible?

Determine the set of roots in $G = (A, R)$.
Justify your answers with the aid of the arrow diagram.

4.19 Consider relation $S \subseteq B^2$ with $B = \{a, b, c, d\}$.

$$\underset{\sim}{S} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

4.19.a Determine the adjacency matrix of S^*.
Calculate which of the nodes in S are mutually accessible and name them.

4.19.b Relation T is defined on B as follows:
$(x, y) \in T \iff x$ and y have common successors in S .
Use the formulas from chapter “Accessibility in binary graphs” to calculate the adjacency matrix of T.
Starting from T, name all different nodes x and y that have common successors in S.

4.20 Consider the adjacency matrix of relation $T \subseteq B^2$ with $B = \{a, b, c, d\}$:

$$\underset{\sim}{T} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Calculate the adjacency matrices of T^+ and T^* and determine, if T has no cycles and which elements of B in T are not mutually accessible.

4.21 In the following, eight relations with their respective basic sets are listed. Furthermore, four kinds of order relations are given. For each relation, determine which kinds of order relation it belongs to.

	partial order	strict order	total order	well order
$(N, <)$				
$(N, \leq)$				
$(Z, \leq)$				
$(R, \leq)$				
$(P(N), \subset)$				
$(P(N), \subseteq)$				
$(P(\{a\}), \subseteq)$				
$(P(\emptyset), \subseteq)$				

Instead of checking all possible 32 cases, reflect on the relations between partial order, strict order, total order and well order that are known to you. Try to simplify this exercise with that knowledge.

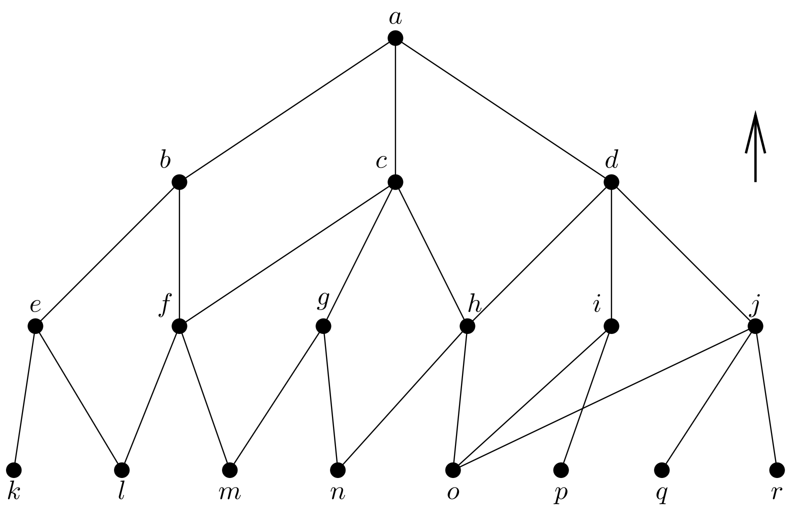
4.22 Consider the following relation $R \subseteq A^2$ with $A = \{2, 3, 4, 5, 6\}$:

$$(a, b) \in R \iff (a \text{ is factor of } b) \vee \big[(a \text{ is a prime number}) \wedge (a < b)\big]$$

4.22.a Write down relation R as set of ordered pairs and draw its Hasse diagram $G_H = (A, H)$.

4.22.b Now consider set $B \subseteq A$ with $B = \{3, 4, 5\}$.
Determine ub(B), grt(B), lub(B), lb(B), lst(B), glb(B), max(B) and min(B) by conception or by calculation.

4.23 Consider the following organigram of a company with staff members $A = \{a, b, \ldots, r\}$:



4.23.a Can this organigram be interpreted as Hasse diagram? What properties does the underlying relation "is subordinate to" have to fulfill?

4.23.b Determine ub(B_1), $B_1 = \{n, o\}$ and interpret it with regard to the company's hierarchy.

4.23.c Determine and interpret lb(B_2), $B_2 = \{b, c\}$.

4.23.d For a specific task, a "task force" has to be set up. The boss a considers the alternatives $B_3 = \{d, i, j, p, q, r, n\}$ and $B_4 = \{c, f, e, h, l, m\}$. Which of these alternatives is to be preferred in terms of a well-defined assignment of responsability? For this purpose, determine and interpret $\max(B_3)$, $\max(B_4)$, grt(B_3) and grt(B_4).

4.23.e Employees $B_5 = \{l, n, o\}$ are ordered to a winter project on the isle of Spitzbergen during the polar night. One of their supervisors has to accompany them. Which employee joins them if the boss does not want to go himself? For that purpose, determine lub(B_5).

4.24 Assume a relation T over the basic set $A = \mathcal{R} \times \mathcal{R}$ which is defined as follows:

$$(x_1, y_1)T(x_2, y_2) \iff (x_1 \leq x_2) \wedge (y_1 \leq y_2)$$

Which of the following assertions are true? Justify your answers.

4.24.a T is an order relation

4.24.b T is a total order

4.24.c T is a well order

4.24.d Every subset B of $\mathcal{R} \times \mathcal{R}$, which has a lower bound, has a greatest lower bound concerning T.

4.25 Prove the following in general for an order relation R over the basic set A with $B \subseteq A$:

4.25.a $b \in \text{grt}(B) \Longrightarrow b \in \max(B)$

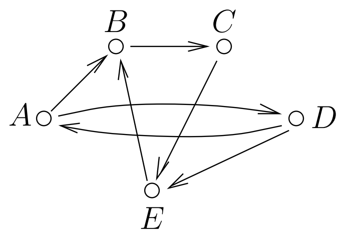
4.25.b $|\text{lub}(B)| \leq 1$

4.26 Assume an equivalence relation R over the basic set A.

4.26.a Describe the structure of $G(A, R)$ in your own words.

4.26.b Try to describe it formally.

4.27 Over the basic set $M = \{A, B, C, D, E\}$, relation R is defined by the following graph G:



4.27.a Determine the adjacency matrices R and R^*

4.27.b Compose the equivalence core $\text{EC}(R^*)$. Re-sort the resulting adjacency matrix so that the connectivities become apparent.

4.27.c Draw the graph of the equivalence core.

4.28 Assume a basic set $A = \{a, b, c, d\}$. Determine the underlying equivalence relations for the following partitions π_i.

4.28.a $\pi_1 = \{\{a, b, c\}, \{d\}\}$

4.28.b $\pi_2 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$

4.28.c $\pi_3 = \{\{a, b, c, d\}\}$

4.28.d Draw the Hasse diagram of the order relation $(\{\pi_1, \pi_2, \pi_3\}, R)$ with $\pi_a R \pi_b \iff \pi_a$ is a refinement of π_b

$\boxed{4.29}$ Over the basic set $A = \mathcal{Z}$, a number of relations R_k is defined as follows:

$$aR_kb \iff a \bmod k = b \bmod k \iff \underset{c \in \mathcal{Z}}{\exists} (a-b) = c \cdot k$$

4.29.a Draw the Hasse diagram of the order relation

$$(\{A/R_3, A/R_5, A/R_6\}, \text{ is refinement of })$$

4.29.b Give the relations $\hat{\pi}(A/R_3, A/R_6)$, $\check{\pi}(A/R_3, A/R_6)$, $\hat{\pi}(A/R_3, A/R_5)$ and $\check{\pi}(A/R_3, A/R_5)$

4.29.c Generally, what is the structure of partitions $\hat{\pi}(A/R_j, A/R_k)$ and $\check{\pi}(A/R_j, A/R_k)$?

$\boxed{4.30}$ Eddie Know-it-all wants to prove that every transitive and symmetric relation is an equivalence relation.

His proof looks as follows:

1. R is symmetric $\iff$ if xRy, then yRx
2. R is transitive $\Longrightarrow$ if $xRy \wedge yRx$, then xRx
3. $xRx \iff R$ is reflexive
4. R is an equivalence relation

Is this proof correct or faulty? Justify your answer.

$\boxed{4.31}$ Prove that the following holds for arbitrary relations U, V, W:

$$\overline{U^{-1}\overline{V}} \subseteq \overline{(WU)^{-1}\overline{WV}}$$

Name all laws you use and do not use any laws of predicate logic.
Hint: Start with $WV \subseteq WV$ and use the Schröder rule.

4.32 Consider a relation V for which $V = t(V)$ holds.
Show without using any laws of predicate logic that the following expression is valid:

$$str(V) \cap \overline{V} \cap \overline{V}^{-1} \subseteq I$$

Name all laws you use.

4.33 Prove without using any laws of predicate logic that for a transitive and reflexive relation $V \subseteq A^2$, the following is fulfilled:

$$(V = V^2) \wedge (V \cap \overline{V^2} = \emptyset)$$

Name all laws you use.

4.34 Prove the rule

$$st(R) = tst(R) \text{ for all } R$$

or refute it by a counter-example.

Further exercises:

4.35 In the manuscript you'll find an overview of different possibilities of representing relations. If you want to automate a task using a computer program, an important step to take is to think about the structure of your task and to choose an appropriate data structure.

Determine the required amount of memory space for each of the possible representations named in the subsequent list. The basic set of relation R is set A. Use the following variables:

- $a = |A|$
- $r = |R|$
- b = required number of bits for storing one element of A
- i = number of bits required for storing a pointer

4.35.a List of ordered pairs

4.35.b Adjacency matrix

4.35.c Incidence matrix

4.35.d Chained successor list

4.35.e Successor table

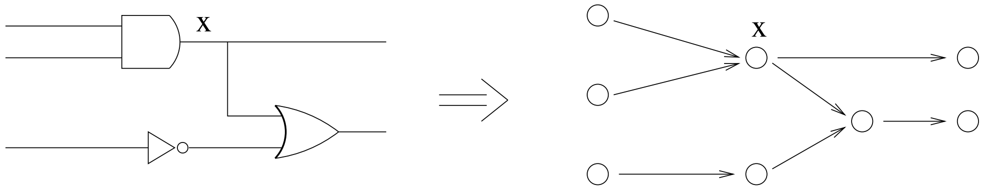
Compare the results in a table and mark those representations which are very memory efficient (+) or inefficient (-) if used in the three subsequent cases.

4.35.f $a = r = b = i$

4.35.g $a \gg r = b = i$

4.35.h $r \gg a = b = i$

4.36 In the event driven digital logic simulation, a structure graph is being used:



If the signal changes at output X, the states of all gates connected to X, i.e. of all succeeding nodes of X, have to be recalculated. The successor degree of each node is usually bounded (fan-out limitation).

The structure graph can be represented by a relation R over A^2 with $|A| = n$ and $\underset{x \in A}{\forall} d^+(x) \leq c$. This relation shall now be processed in a computer. Here, the required memory space $S(n)$ for storing the relation as well as the computation time $R(n)$ are of interest.

Determine the order of required memory space and computation time $O(S(n))$ and $O(R(n))$ for the following data structures:

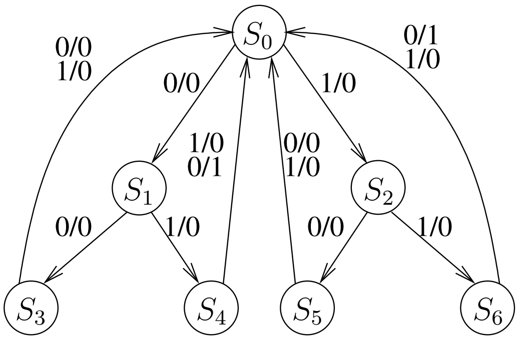
4.36.a List of ordered pairs

4.36.b Adjacency matrix

4.36.c Successor table

5. Finite State Machines

5.1 Consider the state transition graph:



5.1.a Determine the state output table.

5.1.b Minimize the number of states.

5.1.c Draw the minimized state transition graph and determine the minimized state output table.

5.1.d Determine a coding of the states and compute the state transition function and the output function.

5.1.e Draw the circuit implementation of the automaton.

Solutions

1. Propositional Logic

1.1 With '... *x and not(y)* ...' the propositional form $f(x,y) \iff x \wedge \neg y$ is implemented. With double negation (A9) and De Morgan (A10) $f(x,y)$ can be transformed as follows:

$$f(x,y) \iff x \wedge \neg y \iff \neg\neg(x \wedge \neg y) \iff \neg(\neg x \vee y).$$

Thus, you can write: '*If not(not(x) or y) then ...*'.

1.2 To save paperwork, the result of each operation has been written below the respective logic symbol in the following truth tables. The value pattern of the whole propositional form can hence be found below the "innermost" logic symbol.

1.2.a $A \iff a \wedge (b \vee c)$

a	b	c	a	$\wedge$	$(b \vee c)$
f	f	f		**f**	f
f	f	t		**f**	t
f	t	f		**f**	t
f	t	t		**f**	t
t	f	f		**f**	f
t	f	t		**t**	t
t	t	f		**t**	t
t	t	t		**t**	t

1.2.b $B \iff (a \wedge b) \vee (a \wedge c)$

a	b	c	$(a \wedge b)$	$\vee$	$(a \wedge c)$
f	f	f	f	**f**	f
f	f	t	f	**f**	f
f	t	f	f	**f**	f
f	t	t	f	**f**	f
t	f	f	f	**f**	f
t	f	t	f	**t**	t
t	t	f	t	**t**	f
t	t	t	t	**t**	t

1.2.c $C \iff ((a \vee b) \wedge (b \vee c)) \wedge (c \vee a)$

a	b	c	$((a \vee b)$	$\wedge$	$(b \vee c))$	$\wedge$	$(c \vee a)$
f	f	f	f	f	f	**f**	f
f	f	t	f	f	t	**f**	t
f	t	f	t	t	t	**f**	f
f	t	t	t	t	t	**t**	t
t	f	f	t	f	f	**f**	t
t	f	t	t	t	t	**t**	t
t	t	f	t	t	t	**t**	t
t	t	t	t	t	t	**t**	t

1.2.d $D \iff ((a \wedge b) \vee (b \wedge c)) \vee (c \wedge a)$

a	b	c	$((a \wedge b)$	$\vee$	$(b \wedge c))$	$\vee$	$(c \wedge a)$
f	f	f	f	f	f	**f**	f
f	f	t	f	f	f	**f**	f
f	t	f	f	f	f	**f**	f
f	t	t	f	t	t	**t**	f
t	f	f	f	f	f	**f**	f
t	f	t	f	f	f	**t**	t
t	t	f	t	t	f	**t**	f
t	t	t	t	t	t	**t**	t

1.2.e $E \iff \neg a \longrightarrow (b \vee c)$

a	b	c	$\neg a$	$\longrightarrow$	$(b \vee c)$
f	f	f	t	**f**	f
f	f	t	t	**t**	t
f	t	f	t	**t**	t
f	t	t	t	**t**	t
t	f	f	f	**t**	f
t	f	t	f	**t**	t
t	t	f	f	**t**	t
t	t	t	f	**t**	t

1.2.f $F \iff \neg a \longleftrightarrow (b \vee c)$

a	b	c	$\neg a$	$\longleftrightarrow$	$(b \vee c)$
f	f	f	t	**f**	f
f	f	t	t	**t**	t
f	t	f	t	**t**	t
f	t	t	t	**t**	t
t	f	f	f	**t**	f
t	f	t	f	**f**	t
t	t	f	f	**f**	t
t	t	t	f	**f**	t

1.2.g $G \iff (a \longleftrightarrow b) \longleftrightarrow c$

a	b	c	$(a \longleftrightarrow b)$	$\longleftrightarrow$	c
f	f	f	t	**f**	f
f	f	t	t	**t**	t
f	t	f	f	**t**	f
f	t	t	f	**f**	t
t	f	f	f	**t**	f
t	f	t	f	**f**	t
t	t	f	t	**f**	f
t	t	t	t	**t**	t

1.2.h $H \iff a \longleftrightarrow (b \longleftrightarrow c)$

a	b	c	a	$\longleftrightarrow$	$(b \longleftrightarrow c)$
f	f	f	f	$\mathbf{f}$	t
f	f	t	f	$\mathbf{t}$	f
f	t	f	f	$\mathbf{t}$	f
f	t	t	f	$\mathbf{f}$	t
t	f	f	t	$\mathbf{t}$	t
t	f	t	t	$\mathbf{f}$	f
t	t	f	t	$\mathbf{f}$	f
t	t	t	t	$\mathbf{t}$	t

$\boxed{\textbf{1.3}}$ **1.3.a** Truth tables:

I

a	b	c	$(\neg a \wedge a) \vee (a \not\longleftrightarrow (b \wedge c))$
f	f	f	f
f	f	t	f
f	t	f	f
f	t	t	t
t	f	f	t
t	f	t	t
t	t	f	t
t	t	t	f

II

a	b	c	$[(a \longleftrightarrow b) \longrightarrow b] \wedge (a \not\longleftrightarrow a)$
f	f	f	f
f	f	t	f
f	t	f	f
f	t	t	f
t	f	f	f
t	f	t	f
t	t	f	f
t	t	t	f

III

a	b	c	$\neg\,[(c \vee \neg a) \wedge (a \vee \neg b \vee \neg c)]$
f	f	f	f
f	f	t	f
f	t	f	f
f	t	t	t
t	f	f	t
t	f	t	f
t	t	f	t
t	t	t	f

The value patterns of the propositional forms I–III are different.

1.3.b $E[I] = \{(f,t,t),(t,f,f),(t,f,t),(t,t,f)\}$
$E[II] = \emptyset$
$E[III] = \{(f,t,t),(t,f,f),(t,t,f)\}$

1.4 First, we define propositional variables $a, b, c, d \iff$ "Miller invites Anne / Betty / Charlotte / Dana."
Thus, the three prerequisites can be formalised as follows:

$$\begin{array}{lcl} V_1 & \iff & \neg(a \wedge d) \\ V_2 & \iff & c \longrightarrow d \\ V_3 & \iff & b \longrightarrow a \end{array}$$

Since all three prerequisites have to be fulfilled, we obtain the following propositional form $V \iff V_1 \wedge V_2 \wedge V_3$.

Truth table:

a	b	c	d	$\neg(a \wedge d)$	$c \longrightarrow d$	$b \longrightarrow a$	V
f	f	f	f	t	t	t	t
f	f	f	t	t	t	t	t
f	f	t	f	t	f	t	f
f	f	t	t	t	t	t	t
f	t	f	f	t	t	f	f
f	t	f	t	t	t	f	f
f	t	t	f	t	f	f	f
f	t	t	t	t	t	f	f
t	f	f	f	t	t	t	t
t	f	f	t	f	t	t	f
t	f	t	f	t	f	t	f
t	f	t	t	f	t	t	f
t	t	f	f	t	t	t	t
t	t	f	t	f	t	t	f
t	t	t	f	t	f	t	f
t	t	t	t	f	t	t	f

On-set: $E = \{(f,f,f,f),(f,f,f,t),(f,f,t,t),(t,f,f,f),(t,t,f,f)\}$
Hence, Mr. Miller has the following alternatives: He could invite just Dana, Charlotte and Dana, just Anne or Anne and Betty (exclusive or, this time).

Alternative solution: Calculation of CDNF[V]

$$\begin{array}{rll}
V & \Longleftrightarrow & \neg(a \wedge d) \wedge (c \longrightarrow d) \wedge (b \longrightarrow a) \\
& \Longleftrightarrow & (\neg a \vee \neg d) \wedge (\neg c \vee d) \wedge (\neg b \vee a) \quad \text{A10, D1} \\
& \Longleftrightarrow & \big[(\neg a \wedge \neg c) \vee (\neg a \wedge d) \vee (\neg d \wedge \neg c)\big] \wedge (a \vee \neg b) \quad \text{A3, A8, A6} \\
& \Longleftrightarrow & (a \wedge \neg c \wedge \neg d) \vee (\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge d) \\
& & \vee(\neg b \wedge \neg c \wedge \neg d) \quad \text{A3, A8, A6} \\
& \Longleftrightarrow & (a \wedge b \wedge \neg c \wedge \neg d) \vee (a \wedge \neg b \wedge \neg c \wedge \neg d) \vee (\neg a \wedge \neg b \wedge \neg c \wedge d) \\
& & \vee(\neg a \wedge \neg b \wedge \neg c \wedge \neg d) \vee (\neg a \wedge \neg b \wedge c \wedge d) \quad \text{RRD, A4} \\
& \Longleftrightarrow & CDNF[V]
\end{array}$$

1.5 It is important, that "always" and "all the time" mean that there are people who never lie, and that there are people who always lie.

1.5.a
$$\begin{array}{rll}
I_1 & \Longleftrightarrow & (a \longrightarrow b) \wedge (\neg a \longrightarrow \neg b) \wedge \\
& & (b \longrightarrow (a \longrightarrow c)) \wedge (\neg b \longrightarrow \neg (a \longrightarrow c)) \\
& \Longleftrightarrow & (a \longleftrightarrow b) \wedge (b \longleftrightarrow (a \longrightarrow c))
\end{array}$$

1.5.b Truth table:

a	b	c	$a \longleftrightarrow b$	$a \longrightarrow c$	$b \longleftrightarrow (a \longrightarrow c)$	I_1
f	f	f	t	t	f	f
f	f	t	t	t	f	f
f	t	f	f	t	t	f
f	t	t	f	t	t	f
t	f	f	f	f	t	f
t	f	t	f	t	f	f
t	t	f	t	f	f	f
t	t	t	t	t	t	t

$E[I_1] = \{(t,t,t)\} = E[a \wedge b \wedge c]$

1.5.c $I_2 \Longleftrightarrow c \longleftrightarrow (c \wedge (b \longrightarrow \neg a))$

1.5.d

a	b	c	I_2
f	f	f	t
f	f	t	t
f	t	f	t
f	t	t	t
t	f	f	t
t	f	t	t
t	t	f	t
t	t	t	f

$E[I_2] = \{(f,f,f),(f,f,t),(f,t,f),(f,t,t),(t,f,f),(t,f,t),(t,t,f)\} =$
$E[\neg a \vee \neg b \vee \neg c]$

At least one of those three citizens lies all the time.

1.5.e
- Either the premise of disjoint set is untrue, so that there are also citizens who do sometimes lie and sometimes tell the truth,
- or the citizens A, B, C in the first interview are not the same as those in the second one.

$\boxed{\textbf{1.6}}$ **1.6.a**

$$
\begin{array}{rcl}
V_1 & \Longleftrightarrow & a \vee b \vee c \vee d \vee e \\
V_2 & \Longleftrightarrow & \neg[(a \wedge b \wedge c \wedge d \wedge e) \\
 & & \vee(a \wedge b \wedge c \wedge d \wedge \neg e) \vee (a \wedge b \wedge c \wedge \neg d \wedge e) \\
 & & \vee(a \wedge b \wedge \neg c \wedge d \wedge e) \vee (a \wedge \neg b \wedge c \wedge d \wedge e) \\
 & & \vee(\neg a \wedge b \wedge c \wedge d \wedge e) \vee (\neg a \wedge \neg b \wedge \neg c \wedge \neg d \wedge \neg e)] \\
B_1 & \Longleftrightarrow & \neg a \longrightarrow \neg b \wedge \neg c \\
B_2 & \Longleftrightarrow & \neg b \longrightarrow (a \wedge b \wedge c \wedge \neg d \wedge \neg e) \\
 & & \vee(a \wedge b \wedge d \wedge \neg c \wedge \neg e) \vee (a \wedge b \wedge e \wedge \neg c \wedge \neg d) \vee (a \wedge c \wedge d \wedge \neg b \wedge \neg e) \\
 & & \vee(a \wedge c \wedge e \wedge \neg b \wedge \neg d) \vee (a \wedge d \wedge e \wedge \neg b \wedge \neg c) \vee (b \wedge c \wedge d \wedge \neg a \wedge \neg e) \\
 & & \vee(b \wedge c \wedge e \wedge \neg a \wedge \neg d) \vee (b \wedge d \wedge e \wedge \neg a \wedge \neg c) \vee (c \wedge d \wedge e \wedge \neg a \wedge \neg b) \\
B_3 & \Longleftrightarrow & \neg c \longrightarrow (d \wedge (a \longrightarrow e)) \\
B_4 & \Longleftrightarrow & \neg d \longrightarrow a \wedge b \\
B_5 & \Longleftrightarrow & \neg e \longrightarrow a \longleftrightarrow (b \wedge c \wedge d)
\end{array}
$$

1.6.b

a	b	c	d	e	V_1	V_2	B_1	B_2	B_3	B_4	B_5	V	B	B_6	R
f	f	f	f	f	f	f	t	f	f	f	f	f	f	f	f
f	f	f	f	t	t	t	t	f	f	f	t	t	f	t	f
f	f	f	t	f	t	t	t	f	t	t	f	t	f	t	f
f	f	f	t	t	t	t	t	f	t	t	t	t	f	t	f
f	f	t	f	f	t	t	f	f	t	f	f	t	f	f	f
f	f	t	f	t	t	t	f	f	t	f	t	t	f	t	f
f	f	t	t	f	t	t	f	f	t	t	f	t	f	t	f
f	f	t	t	t	t	t	f	t	t	t	t	t	f	t	f
f	t	f	f	f	t	t	f	t	f	f	f	t	f	f	f
f	t	f	f	t	t	t	f	t	f	f	t	t	f	t	f
f	t	f	t	f	t	t	f	t	t	t	f	t	f	t	f
f	t	f	t	t	t	t	f	t	t	t	t	t	f	t	f
f	t	t	f	f	t	t	f	t	t	f	f	t	f	f	f
f	t	t	f	t	t	t	f	t	t	f	t	t	f	t	f
f	t	t	t	f	t	t	f	t	t	t	t	t	f	t	f
f	t	t	t	t	t	f	f	t	t	t	t	f	f	t	f
t	f	f	f	f	t	t	t	f	f	f	t	t	f	f	f
t	f	f	f	t	t	t	t	f	f	f	t	t	f	t	f
t	f	f	t	f	t	t	t	f	f	t	t	t	f	f	f

a	b	c	d	e	V_1	V_2	B_1	B_2	B_3	B_4	B_5	V	B	B_6	R
t	f	f	t	t	t	t	t	t	t	t	t	t	t	t	t
t	f	t	f	f	t	t	t	f	t	f	t	t	f	f	f
t	f	t	f	t	t	t	t	t	t	f	t	t	f	t	f
t	f	t	t	f	t	t	t	t	t	t	t	t	t	f	f
t	f	t	t	t	t	f	t	f	t	t	t	f	f	t	f
t	t	f	f	f	t	t	t	t	f	t	t	t	f	f	f
t	t	f	f	t	t	t	t	t	f	t	t	t	f	t	f
t	t	f	t	f	t	t	t	t	f	t	t	t	f	f	f
t	t	f	t	t	t	f	t	t	t	t	t	f	t	t	f
t	t	t	f	f	t	t	t	t	t	t	t	t	t	f	f
t	t	t	f	t	t	f	t	t	t	t	t	f	t	t	f
t	t	t	t	f	t	f	t	t	t	t	f	f	f	f	f
t	t	t	t	t	t	f	t	t	t	t	t	f	t	t	f

$$E[V \wedge B] = \{(t,f,f,t,t),(t,f,t,t,f),(t,t,t,f,f)\}$$

1.6.c $B_6 \iff \neg e \longrightarrow \neg a \wedge d$

$E[R] = \{(t,f,f,t,t)\}$
A, D and E are the murderers.

$\boxed{\textbf{1.7}}$

$$\begin{array}{rcl}
A & \iff & (a \longleftrightarrow b) \wedge \neg c \\
 & \iff & [(a \wedge b) \vee (\neg a \wedge \neg b)] \wedge \neg c \\
 & \iff & (a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge \neg c) \\
B & \iff & a \wedge (b \not\longleftrightarrow c) \wedge b \\
 & \iff & [(b \wedge \neg c) \vee (\neg b \wedge c)] \wedge a \wedge b \\
 & \iff & (a \wedge b \wedge b \wedge \neg c) \vee (a \wedge b \wedge \neg b \wedge c) \\
 & \iff & (a \wedge b \wedge \neg c) \\
C & \iff & t \not\longleftrightarrow [c \vee \neg ((a \wedge b) \vee \neg(a \vee b))] \\
 & \iff & \neg [c \vee \neg ((a \wedge b) \vee \neg(a \vee b))] \\
 & \iff & \neg c \wedge [(a \wedge b) \vee \neg(a \vee b)] \\
 & \iff & \neg c \wedge [(a \wedge b) \vee (\neg a \wedge \neg b)] \\
 & \iff & (a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge \neg c)
\end{array}$$

The propositional forms A and C are equivalent.

$\boxed{\textbf{1.8}}$ **1.8.a** Calculation of the $CDNF[A_1]$:

$$\begin{array}{lcll}
A_1 & \Longleftrightarrow & [a \longleftrightarrow\!\!\!\!\!\!/\;\; (b \longrightarrow c)] \vee \neg(a \longrightarrow \neg b) & \\
& \Longleftrightarrow & a \wedge \neg(b \longrightarrow c) \vee \neg a \wedge (b \longrightarrow c) \vee a \wedge b & \text{B9} \\
& \Longleftrightarrow & a \wedge b \wedge \neg c \vee \neg a \wedge (\neg b \vee c) \vee a \wedge b & \text{D1} \\
& \Longleftrightarrow & a \wedge b \wedge \neg c \vee \neg a \wedge \neg b \vee \neg a \wedge c \vee a \wedge b & \text{A3} \\
& \Longleftrightarrow & (a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c) & \\
& & \vee(\neg a \wedge b \wedge c) \vee (a \wedge b \wedge c) & \text{RRD, A5} \\
& \Longleftrightarrow & CDNF[A_1] &
\end{array}$$

Calculation of the $CCNF[A_1]$:

$$\begin{array}{lcll}
A_1 & \Longleftrightarrow & [a \longleftrightarrow\!\!\!\!\!\!/\;\; (b \longrightarrow c)] \vee \neg(a \longrightarrow \neg b) & \\
& \Longleftrightarrow & [a \vee (b \longrightarrow c)] \wedge [\neg a \vee \neg(b \longrightarrow c)] \vee a \wedge b & \text{B10, D2} \\
& \Longleftrightarrow & [(a \vee \neg b \vee c) \wedge (\neg a \vee b \wedge \neg c)] \vee a \wedge b & \text{D1, D2} \\
& \Longleftrightarrow & (a \vee \neg b \vee c) \wedge t \wedge t \wedge [b \vee (\neg a \vee b) \wedge (\neg a \vee \neg c)] & \text{A3, A8} \\
& \Longleftrightarrow & (a \vee \neg b \vee c) \wedge (\neg a \vee b) & \text{A6, A3, A5} \\
& \Longleftrightarrow & (a \vee \neg b \vee c) \wedge (\neg a \vee b \vee c) \wedge (\neg a \vee b \vee \neg c) & \text{RRC} \\
& \Longleftrightarrow & CCNF[A_1] &
\end{array}$$

1.8.b Calculation of the $CDNF[A_2]$:

$$\begin{array}{lcll}
A_2 & \Longleftrightarrow & (a \longleftrightarrow\!\!\!\!\!\!/\;\; b \longrightarrow c) \wedge (a \longrightarrow b \wedge \neg c) & \\
& \Longleftrightarrow & [(a \longleftrightarrow b) \vee c] \wedge (\neg a \vee b \wedge \neg c) & \text{D1, B11} \\
& \Longleftrightarrow & [(a \wedge b) \vee (\neg a \wedge \neg b) \vee c] \wedge [\neg a \vee (b \wedge \neg c)] & \text{C9} \\
& \Longleftrightarrow & f \vee (\neg a \wedge \neg b) \vee (\neg a \wedge c) \vee (a \wedge b \wedge \neg c) \vee f \vee f & \text{A3, A8, A4} \\
& \Longleftrightarrow & (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c) & \\
& & \vee(\neg a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c) & \text{A6, RRD, A4} \\
& \Longleftrightarrow & CDNF[A_2] &
\end{array}$$

Calculation of the $CCNF[A_2]$:

$$\begin{array}{lcll}
A_2 & \Longleftrightarrow & (a \longleftrightarrow\!\!\!\!\!\!/\;\; b \longrightarrow c) \wedge (a \longrightarrow b \wedge \neg c) & \\
& \Longleftrightarrow & [(a \longleftrightarrow b) \vee c] \wedge (\neg a \vee b \wedge \neg c) & \text{D1, B11} \\
& \Longleftrightarrow & [(a \vee \neg b) \wedge (\neg a \vee b) \vee c] \wedge (\neg a \vee b \wedge \neg c) & \text{C10} \\
& \Longleftrightarrow & [(a \vee \neg b \vee c) \wedge (\neg a \vee b \vee c)] \wedge (\neg a \vee b) \wedge (\neg a \vee \neg c) & \text{A3} \\
& \Longleftrightarrow & (a \vee \neg b \vee c) \wedge (\neg a \vee b \vee c) \wedge (\neg a \vee b \vee \neg c) & \\
& & \wedge(\neg a \vee \neg b \vee \neg c) & \text{RRC, A5} \\
& \Longleftrightarrow & CCNF[A_2] &
\end{array}$$

1.8.c Calculation of the $CDNF[A_3]$:

$$\begin{array}{lll}
A_3 & \Longleftrightarrow & [a \longrightarrow (b \longleftrightarrow c)] \wedge (\neg a \longrightarrow b) \\
& \Longleftrightarrow & [\neg a \vee (b \wedge c) \vee (\neg b \wedge \neg c)] \wedge (a \vee b) \quad \text{D1, C9} \\
& \Longleftrightarrow & \neg a \wedge (a \vee b) \vee (a \vee b) \wedge (b \wedge c) \vee (a \vee b) \wedge (\neg b \wedge \neg c) \quad \text{A3} \\
& \Longleftrightarrow & (\neg a \wedge b) \vee (a \wedge b \wedge c) \vee (b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \quad \text{A3, A4, A8, A6} \\
& \Longleftrightarrow & (\neg a \wedge b) \vee (b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \quad \text{A5} \\
& \Longleftrightarrow & (\neg a \wedge b \wedge c) \vee (\neg a \wedge b \wedge \neg c) \vee (a \wedge b \wedge c) \\
& & \vee (a \wedge \neg b \wedge \neg c) \quad \text{RRD, A4} \\
& \Longleftrightarrow & CDNF[A_3]
\end{array}$$

Calculation of the $CCNF[A_3]$:

$$\begin{array}{lll}
A_3 & \Longleftrightarrow & [a \longrightarrow (b \longleftrightarrow c)] \wedge (\neg a \longrightarrow b) \\
& \Longleftrightarrow & [\neg a \vee (b \vee \neg c) \wedge (\neg b \vee c)] \wedge (a \vee b) \quad \text{D1, C10} \\
& \Longleftrightarrow & [(\neg a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee c)] \wedge (a \vee b) \quad \text{A3} \\
& \Longleftrightarrow & (\neg a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee c) \wedge (a \vee b \vee c) \wedge (a \vee b \vee \neg c) \quad \text{RRC} \\
& \Longleftrightarrow & CCNF[A_3]
\end{array}$$

$\boxed{\textbf{1.9}}$ Laws of subjunction:
Proof using equivalence transformations.

1.9.a D1: Definition of the subjunction

1.9.b D2:
$$\begin{array}{lll}
\neg(a \longrightarrow b) & \Longleftrightarrow & \neg(\neg a \vee b) \\
& \Longleftrightarrow & a \wedge \neg b
\end{array}$$

1.9.c D3:
$$\begin{array}{lll}
a \longrightarrow b & \Longleftrightarrow & \neg a \vee b \\
& \Longleftrightarrow & \neg(\neg b) \vee \neg a \\
& \Longleftrightarrow & \neg b \longrightarrow \neg a
\end{array}$$

1.9.d D4:
$$\begin{array}{lll}
a \longrightarrow t & \Longleftrightarrow & \neg a \vee t \\
& \Longleftrightarrow & t \\
f \longrightarrow a & \Longleftrightarrow & \neg f \vee a \\
& \Longleftrightarrow & t
\end{array}$$

1.9.e D5:
$$\begin{array}{lll}
t \longrightarrow a & \Longleftrightarrow & \neg t \vee a \\
& \Longleftrightarrow & a \\
a \longrightarrow f & \Longleftrightarrow & \neg a \vee f \\
& \Longleftrightarrow & \neg a
\end{array}$$

1.9.f D6:

$$\begin{aligned}
a \longrightarrow a &\iff \neg a \vee a \\
&\iff t \\
\neg a \longrightarrow a &\iff \neg(\neg a) \vee a \\
&\iff a \vee a \\
&\iff a \\
a \longrightarrow \neg a &\iff \neg a \vee \neg a \\
&\iff \neg a
\end{aligned}$$

1.9.g D7:

$$\begin{aligned}
a \longrightarrow (b \wedge c) &\iff \neg a \vee (b \wedge c) \\
&\iff (\neg a \vee b) \wedge (\neg a \vee c) \\
&\iff (a \longrightarrow b) \wedge (a \longrightarrow c) \\
(a \wedge b) \longrightarrow c &\iff \neg(a \wedge b) \vee c \\
&\iff \neg a \vee \neg b \vee c \\
&\iff \neg a \vee \neg b \vee c \vee c \\
&\iff (\neg a \vee c) \vee (\neg b \vee c) \\
&\iff (a \longrightarrow c) \vee (b \longrightarrow c)
\end{aligned}$$

1.9.h D8:

$$\begin{aligned}
a \longrightarrow (b \vee c) &\iff \neg a \vee b \vee c \\
&\iff \neg a \vee \neg a \vee b \vee c \\
&\iff (a \longrightarrow b) \vee (a \longrightarrow c) \\
(a \vee b) \longrightarrow c &\iff \neg(a \vee b) \longrightarrow c \\
&\iff (\neg a \wedge \neg b) \vee c \\
&\iff (\neg a \vee c) \wedge (\neg b \vee c) \\
&\iff (a \longrightarrow c) \wedge (b \longrightarrow c)
\end{aligned}$$

1.9.i D9:

$$\begin{aligned}
a \longrightarrow (b \longrightarrow c) &\iff \neg a \vee (\neg b \vee c) \\
&\iff \neg b \vee \neg a \vee c \\
&\iff b \longrightarrow (a \longrightarrow c) \\
a \longrightarrow (b \longrightarrow c) &\iff \neg a \vee (\neg b \vee c) \\
&\iff \neg(a \wedge b) \vee c \\
&\iff (a \wedge b) \longrightarrow c
\end{aligned}$$

1.9.j D10:

$$\begin{aligned}
a \longrightarrow (a \longrightarrow b) &\iff \neg a \vee \neg a \vee b \\
&\iff \neg a \vee b \\
&\iff a \longrightarrow b
\end{aligned}$$

1.9.k D11, D12:

$$\begin{aligned}
a \longrightarrow b &\iff \neg a \vee b \\
&\iff t \wedge (\neg a \vee (a \wedge b)) \\
&\iff (a \vee \neg(a \wedge b)) \wedge (\neg a \vee (a \wedge b)) \\
&\iff a \longleftrightarrow (a \wedge b) \\
&\iff \neg(a \not\longleftrightarrow (a \wedge b))
\end{aligned}$$

1.9.l D13:

$$\begin{aligned}
a \vee b &\iff (a \vee b) \wedge (\neg b \vee b) \\
&\iff (a \wedge \neg b) \vee b \\
&\iff \neg(\neg a \vee b) \vee b \\
&\iff (a \longrightarrow b) \longrightarrow b
\end{aligned}$$

1.9.m D14:
$$\begin{aligned} a \wedge b &\iff \neg(\neg a \vee \neg b) \\ &\iff \neg(a \longrightarrow \neg b) \end{aligned}$$

$\boxed{\textbf{1.10}}$ Laws using implication:

Transformation into tautologies by resolving the subjunctions

1.10.a E1:
$$\begin{aligned} f \longrightarrow a &\iff t \\ \neg f \vee a &\iff t \\ t \vee a &\iff t \\ t &\iff t \end{aligned}$$

1.10.b E2:
$$\begin{aligned} a \longrightarrow t &\iff t \\ \neg a \vee t &\iff t \\ t &\iff t \end{aligned}$$

1.10.c E3:
$$\begin{aligned} a \longrightarrow a &\iff t \\ \neg a \vee a &\iff t \\ t &\iff t \end{aligned}$$

1.10.d E4:
$$\begin{aligned} a \wedge b \longrightarrow a &\iff t \\ \neg a \vee \neg b \vee a &\iff t \\ t \vee \neg b &\iff t \\ t &\iff t \end{aligned}$$

1.10.e E5:
$$\begin{aligned} a \longrightarrow (a \vee b) &\iff t \\ \neg a \vee a \vee b &\iff t \\ t &\iff t \end{aligned}$$

1.10.f E6:
$$\begin{aligned} (a \wedge b) \longrightarrow (a \vee b) &\iff t \\ \neg(a \wedge b) \vee a \vee b &\iff t \\ \neg a \vee \neg b \vee a \vee b &\iff t \\ t &\iff t \end{aligned}$$

1.10.g E7:
$$\begin{aligned} \neg a \longrightarrow (a \longrightarrow b) &\iff t \\ a \vee \neg a \vee b &\iff t \\ t &\iff t \end{aligned}$$

1.10.h E8:
$$\begin{aligned} b \longrightarrow (a \longrightarrow b) &\iff t \\ \neg b \vee \neg a \vee b &\iff t \\ t &\iff t \end{aligned}$$

1.10.i E9:

$$\begin{aligned} a \wedge (a \longrightarrow b) \longrightarrow b &\iff t \\ a \wedge (\neg a \vee b) \longrightarrow b &\iff t \\ \neg(a \wedge b) \vee b &\iff t \\ \neg a \vee \neg b \vee b &\iff t \\ t &\iff t \end{aligned}$$

1.10.j E10:

$$\begin{aligned} \neg b \wedge (a \longrightarrow b) \longrightarrow \neg a &\iff t \\ \neg b \wedge (\neg b \longrightarrow \neg a) \longrightarrow \neg a &\iff t \\ \neg a \longrightarrow \neg a &\iff t \\ t &\iff t \end{aligned}$$

1.10.k E11:

$$\begin{aligned} (a \longrightarrow b) \wedge (b \longrightarrow c) \longrightarrow (a \longrightarrow c) &\iff t \\ (a \wedge \neg b) \vee (b \wedge \neg c) \vee \neg a \vee c &\iff t \\ \left[\underbrace{(a \vee \neg a)}_{t} \wedge (\neg b \vee \neg a)\right] \vee \left[(b \vee c) \wedge \underbrace{(\neg c \vee c)}_{t}\right] &\iff t \\ \neg b \vee \neg a \vee b \vee c &\iff t \\ t \vee \neg a \vee c &\iff t \\ t &\iff t \end{aligned}$$

1.10.l E12:

$$\begin{aligned} (a \longrightarrow b) \wedge (\neg a \longrightarrow b) \longrightarrow b &\iff t \\ \neg[(\neg a \vee b) \wedge (a \vee b)] \vee b &\iff t \\ \neg[b \vee (\neg a \wedge a)] \vee b &\iff t \\ \neg b \vee b &\iff t \\ t &\iff t \end{aligned}$$

1.10.m E13:

$$\begin{aligned} (a \longrightarrow b) \wedge (a \longrightarrow \neg b) \longrightarrow \neg a &\iff t \\ \neg[(\neg a \vee b) \wedge (\neg a \vee \neg b)] \vee \neg a &\iff t \\ \neg[\neg a \vee (b \wedge \neg b)] \vee \neg a &\iff t \\ a \vee \neg a &\iff t \\ t &\iff t \end{aligned}$$

1.10.n E14:

$$\begin{aligned} [(a \longrightarrow b) \wedge (c \longrightarrow d)] \longrightarrow [(a \wedge c) \longrightarrow (b \wedge d)] &\iff t \\ \neg\left[(\neg a \vee b) \wedge (\neg c \vee d)\right] \vee \neg(a \wedge c) \vee (b \wedge d) &\iff t \\ (a \wedge \neg b) \vee (c \wedge \neg d) \vee \neg a \vee \neg c \vee (b \wedge d) &\iff t \\ [(a \vee c) \wedge (a \vee \neg d) \wedge (\neg b \vee c) \wedge (\neg b \vee \neg d)] & \\ \vee \neg a \vee \neg c \vee (b \wedge d) &\iff t \\ (a \vee \neg a \vee \ldots) \wedge (a \vee \neg a \vee \ldots) \wedge (c \vee \neg c \ldots) \wedge & \\ (\neg b \vee \neg d \vee \neg a \vee \neg c \vee (b \wedge d)) &\iff t \\ \neg a \vee \neg c \vee ((\neg b \vee \neg d \vee b) \wedge (\neg b \vee \neg d \vee d)) &\iff t \\ \neg a \vee \neg c \vee t &\iff t \\ t &\iff t \end{aligned}$$

1.10.o E15:
$$\begin{aligned}
[(a \longrightarrow b) \wedge (c \longrightarrow d)] \longrightarrow [(a \vee c) \longrightarrow (b \vee d)] &\iff t\\
\neg((\neg a \vee b) \wedge (\neg c \vee d)) \vee \neg(a \vee c) \vee (b \vee d) &\iff t\\
(a \wedge \neg b) \vee (c \wedge \neg d) \vee (\neg a \wedge \neg c) \vee b \vee d &\iff t\\
[(a \vee c) \wedge (a \vee \neg d) \wedge (\neg b \vee c) \wedge (\neg b \vee \neg d)] &\\
\vee(\neg a \wedge \neg c) \vee b \vee d &\iff t\\
(a \vee \neg a \ldots) \wedge (a \vee \neg a \ldots) \wedge &\\
(c \vee \neg c \ldots) \wedge (\neg b \vee b \vee \ldots) &\iff t\\
t &\iff t
\end{aligned}$$

1.10.p E16:
$$\begin{aligned}
[(a \vee b) \wedge [(a \longrightarrow c) \wedge (b \longrightarrow c)]] \longrightarrow c &\iff t\\
\neg(a \vee b) \vee \neg[(\neg a \vee c) \wedge (\neg b \vee c)] \vee c &\iff t\\
(\neg a \wedge \neg b) \vee (a \wedge \neg c) \vee (b \wedge \neg c) \vee c &\iff t\\
[(\neg a \vee a) \wedge (\neg a \vee \neg c) \wedge (\neg b \vee a) \wedge (\neg b \vee \neg c)] &\\
\vee(b \vee \neg c) \vee c &\iff t\\
[(\neg a \vee \neg c \vee c) \wedge (\neg b \vee a \vee c) \wedge (\neg b \vee \neg c \vee c)] &\\
\vee(b \wedge \neg c) &\iff t\\
\neg b \vee a \vee c \vee (b \wedge \neg c) &\iff t\\
t &\iff t
\end{aligned}$$

1.10.q E17:
$$\begin{aligned}
[(a \vee b) \wedge [(a \longrightarrow c) \wedge (b \longrightarrow d)]] \longrightarrow c \vee d &\iff t\\
\neg(a \vee b) \vee \neg[(\neg a \vee c) \wedge (\neg b \vee d)] \vee c \vee d &\iff t\\
(\neg a \wedge \neg b) \vee (a \wedge \neg c) \vee (b \wedge \neg d) \vee c \vee d &\iff t\\
[(\neg a \vee a) \wedge (\neg a \vee \neg c) \wedge (\neg b \vee a) \wedge (\neg b \vee \neg c)] &\\
\vee(b \vee \neg d) \vee c \vee d &\iff t\\
[(\neg a \vee \neg c \vee c) \wedge (\neg b \vee a \vee c) \wedge (\neg b \vee \neg c \vee c)] &\\
\vee(b \wedge \neg d) \vee d &\iff t\\
\neg b \vee a \vee c \vee (b \wedge \neg d) \vee d &\iff t\\
t &\iff t
\end{aligned}$$

1.10.r E18:
$$\begin{aligned}
[(a \wedge b) \wedge [(a \longrightarrow c) \wedge (b \longrightarrow d)]] \longrightarrow c \wedge d &\iff t\\
\neg(a \wedge b) \vee \neg\left[(\neg a \vee c) \wedge (\neg b \vee d)\right] \vee (c \wedge d) &\iff t\\
\neg a \vee \neg b \vee (a \wedge \neg c) \vee (b \wedge \neg d) \vee (c \wedge d) &\iff t\\
\neg a \vee \neg b \vee (a \wedge \neg c) \vee &\\
[(b \vee c) \wedge (b \vee d) \wedge (\neg d \vee c) \wedge (\neg d \vee d)] &\iff t\\
\neg a \vee (a \wedge \neg c) \vee &\\
[(b \vee \neg b \vee c) \wedge (b \vee \neg b \vee d) \wedge (\neg b \vee \neg d \vee c)] &\iff t\\
\neg a \vee \neg c \vee \neg b \vee \neg d \vee d &\iff t\\
t &\iff t
\end{aligned}$$

1.10.s E19:
$$\begin{aligned}
[a \longrightarrow (b \longrightarrow c)] \longrightarrow [(a \longrightarrow b) \longrightarrow (a \longrightarrow c)] &\iff t\\
\neg a \vee (\neg b \vee c) \longrightarrow \neg(\neg a \vee b) \vee (\neg a \vee c) &\iff t\\
[a \wedge (b \wedge \neg c)] \vee (a \wedge \neg b) \vee \neg a \vee c &\iff t\\
(a \wedge b \wedge \neg c) \vee [(a \vee \neg a \vee c) \wedge (\neg b \vee \neg a \vee c)] &\iff t\\
(a \wedge b \wedge \neg c) \vee \neg b \vee \neg a \vee c &\iff t\\
(a \vee \neg b \vee \neg a \vee c) \wedge (b \vee \neg b \vee \neg a \vee c) \wedge &\\
(\neg c \vee \neg b \vee \neg a \vee c) &\iff t\\
t &\iff t
\end{aligned}$$

1.11 **1.11.a**

$$\begin{aligned}
(b \longrightarrow a) \wedge (b \longrightarrow \neg c) &\Longleftrightarrow \neg b \vee (a \wedge \neg c) \\
(\neg b \vee a) \wedge (\neg b \vee \neg c) &\Longleftrightarrow \neg b \vee (a \wedge \neg c) \\
\neg b \vee (a \wedge \neg c) &\Longleftrightarrow \neg b \vee (a \wedge \neg c) \\
&\Longleftrightarrow \neg(a \wedge \neg c) \longrightarrow \neg b \\
&\Longleftrightarrow (a \longrightarrow c) \longrightarrow \neg b
\end{aligned}$$

1.11.b **III:**

(1) b
(2) $(b \longrightarrow a)$
(3) $(b \longrightarrow \neg c) \Longrightarrow a \wedge \neg c$

---

(4) a	(1), (2), E9
(5) $\neg c$	(1), (3), E9
(6) $a \wedge \neg c$	(4), (5)

IV:

(1) c
(2) $(b \longrightarrow a)$
(3) $(b \longrightarrow \neg c) \Longrightarrow \neg b$

---

(4) $c \longrightarrow \neg b$	(3), D3
(5) $\neg b$	(1), (4), E9

1.11.c

a	b	c	$(b \longrightarrow a) \wedge (b \longrightarrow \neg c)$	$c \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c)$	$\neg a$
f	f	f	t	f	t
f	f	t	t	t	t
f	t	f	f	f	t
f	t	t	f	f	t
t	f	f	t	f	f
t	f	t	t	t	f
t	t	f	t	f	f
t	t	t	f	f	f

$E[c \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c)] = \{(f, f, t), (t, f, t)\}$

1.11.d $m = 6;\ 2^m - 1 = 2^6 - 1 = 63$

1.11.e $E[\neg a] = \{(f, f, f), (f, f, t), (f, t, f), (f, t, t)\}$

Assertion V invalid, since:
$E[c \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c)] \not\subseteq E[\neg a]$

1.12

I:

(1)	$l \longrightarrow s \wedge b$	$\Longrightarrow \neg s \longrightarrow \neg l$
(2)	$(l \longrightarrow s) \wedge (l \longrightarrow b)$	(1), D7
(3)	$l \longrightarrow s$	(2), E4
(4)	$\neg s \longrightarrow \neg l$	(3), D3

II:

(1)	$s \wedge b \longleftrightarrow l$	$\Longrightarrow \neg s \longrightarrow \neg l$
(2)	$(s \wedge b \longrightarrow l) \wedge (l \longrightarrow s \wedge b)$	(1), C11
(3)	$l \longrightarrow s \wedge b$	(2), E4
(4)	$\neg s \longrightarrow \neg l$	(3), D7, E4, D3 (see I)

III:

(1)	$\neg l$	
(2)	s	
(3)	$s \wedge b \longleftrightarrow l$	$\Longrightarrow \neg b$
(4)	$(s \wedge b \longrightarrow l) \wedge (l \longrightarrow s \wedge b)$	(3), C11
(5)	$s \wedge b \longrightarrow l$	(4), E4
(6)	$\neg l \longrightarrow \neg(s \wedge b)$	(5), D3
(7)	$\neg(s \wedge b)$	(6), (1), E9
(8)	$\neg s \vee \neg b$	(7), A10
(9)	$s \longrightarrow \neg b$	(8), D1
(10)	$\neg b$	(2), (9), E9

1.13 **1.13.a**

(1)	$\neg d$	
(2)	b	
(3)	$a \longrightarrow [d \longrightarrow (b \longrightarrow c)]$	$\Longrightarrow d \longrightarrow c$
(4)	$d \longrightarrow (a \longrightarrow (b \longrightarrow c))$	(3), D9
(5)	$(d \longrightarrow a) \longrightarrow [d \longrightarrow (b \longrightarrow c)]$	(4), E19
(6)	$d \longrightarrow a$	(1), E7
(7)	$d \longrightarrow (b \longrightarrow c)$	(6), (5), E9
(8)	$b \longrightarrow (d \longrightarrow c)$	(7), D9
(9)	$d \longrightarrow c$	(8), (2), E9

Alternative solution:

(4)	$d \longrightarrow c$	(1), E7

1.13.b

(1)	$\neg a \longrightarrow (b \vee d)$	
(2)	$a \longleftrightarrow c$	
(3)	$b \longrightarrow c$	

(4)	$d \longrightarrow c$	$\Longrightarrow c$
(5)	$(b \vee d) \longrightarrow c$	(3), (4), D8
(6)	$\neg a \longrightarrow c$	(1), (5), E11
(7)	$a \longrightarrow c$	(2), C11, E4
(8)	c	(6), (7), E12

1.13.c

(1)	$a \longrightarrow [c \longrightarrow ((d \longrightarrow b) \longrightarrow b)]$	
(2)	$\neg c \longrightarrow d$	
(3)	$\neg d$	$\Longrightarrow a \longrightarrow b$
(4)	$\neg d \longrightarrow c$	(2), D3
(5)	c	(3), (4), E9
(6)	$a \longrightarrow c$	(5), E8
(7)	$(a \longrightarrow c) \longrightarrow [a \longrightarrow ((d \longrightarrow b) \longrightarrow b)]$	(1), E19
(8)	$a \longrightarrow ((d \longrightarrow b) \longrightarrow b)$	(7), (6), E9
(9)	$(d \longrightarrow b) \longrightarrow (a \longrightarrow b)$	(8), D9
(10)	$d \longrightarrow b$	(3), E7
(11)	$a \longrightarrow b$	(9), (10), E9

1.13.d

(1)	$a \longleftrightarrow c$	$\Longrightarrow (b \longrightarrow a) \longleftrightarrow (b \longrightarrow c)$
(2)	$a \longrightarrow c$	(1), C11, E4
(3)	$c \longrightarrow a$	(1), C11, E4
(4)	$(b \longrightarrow a) \longrightarrow (b \longrightarrow c)$	(2), E20
(5)	$(b \longrightarrow c) \longrightarrow (b \longrightarrow a)$	(3), E20
(6)	$(b \longrightarrow a) \longleftrightarrow (b \longrightarrow c)$	(4), (5), C11

1.13.e

(1)	$(\neg a \longrightarrow \neg b) \longrightarrow c$	
(2)	$b \longrightarrow [(a \longrightarrow c) \longrightarrow \neg c]$	$\Longrightarrow b \not\longleftrightarrow c$
(3)	$(a \vee \neg b) \longrightarrow c$	(1), D1
(4)	$(a \longrightarrow c) \wedge (\neg b \longrightarrow c)$	(3), D8
(5)	$\neg b \longrightarrow c$	(4), E4
(6)	$a \longrightarrow c$	(4), E4
(7)	$(a \longrightarrow c) \longrightarrow (b \longrightarrow \neg c)$	(2), D9
(8)	$b \longrightarrow \neg c$	(6), (7), E9
(9)	$c \longrightarrow \neg b$	(8), D3
(10)	$c \longleftrightarrow \neg b$	(5), (9), C11
(11)	$b \not\longleftrightarrow c$	(10), C5, B11

alternative steps (7) and (8):

(7)	$b \longrightarrow [(a \wedge \neg c) \vee \neg c]$	(2), D1, A10
(8)	$b \longrightarrow \neg c$	(7), A5

1.13.f

(1)	$\neg a$	
(2)	$b \longrightarrow (c \longrightarrow a)$	
(3)	$\neg(c \longrightarrow b) \longrightarrow a$	$\Longrightarrow a \longleftrightarrow c$
(4)	$\neg a \longrightarrow (c \longrightarrow b)$	3, D3
(5)	$c \longrightarrow b$	1, 4, E9
(6)	$a \longrightarrow c$	1, E7
(7)	$c \longrightarrow (c \longrightarrow a)$	2, 5, E11
(8)	$c \longrightarrow a$	7, D10
(9)	$a \longleftrightarrow c$	6, 8, C11

Alternative:

(4)	$a \longrightarrow c$	1, E7
(5)	$\neg a \longrightarrow (c \longrightarrow b)$	3, D3
(6)	$\neg a \longrightarrow (\neg b \longrightarrow \neg c)$	5, D3
(7)	$\neg b \longrightarrow (\neg a \longrightarrow \neg c)$	6, D9
(8)	$\neg(\neg a \longrightarrow \neg c) \longrightarrow b$	7, D3
(9)	$\neg(\neg a \longrightarrow \neg c) \longrightarrow (c \longrightarrow a)$	2, 8, E11
(10)	$\neg(c \longrightarrow a) \longrightarrow (c \longrightarrow a)$	9, D3
(11)	$c \longrightarrow a$	10, D6
(12)	$a \longleftrightarrow c$	4, 11, B11

1.14 **1.14.a**

- **[I:]** $s \longrightarrow \neg p$
- **[II:]** $\neg s \longrightarrow h$
- **[III:]** $(\neg h \longrightarrow \neg p) \longrightarrow s$

1.14.b

(1)	$s \longrightarrow \neg p$	
(2)	$\neg s \longrightarrow h$	
(3)	$(\neg h \longrightarrow \neg p) \longrightarrow s$	$\Longrightarrow s$
(4)	$\neg h \longrightarrow s$	(2), D3
(5)	$\neg h \longrightarrow \neg p$	(4), (1), E11
(6)	s	(5), (3), E9

This means, the state secretary's decision was correct which could soon lead to his promotion.

1.15 The restriction to just a few laws for the proof had the following intentions:

- You get hints for how to solve the problems.
- You can't transform all of the expressions into the $\wedge, \vee, \neg$-system as in real life this could well lead to huge expressions which can hardly be solved.

1.15.a (1) $(b \vee c) \longrightarrow a$
(2) $a \vee b \vee c$ $\Longrightarrow \quad a$

(3) $\neg a \longrightarrow (b \vee c)$	(2), D1, A2
(4) $\neg a \longrightarrow \neg(b \vee c)$	(1), D3
(5) $[\neg a \longrightarrow (b \vee c)] \wedge [\neg a \longrightarrow \neg(b \vee c)]$	(3), (4)
(6) a	(5), E13

1.15.b (1) b
(2) $\neg a \longrightarrow \neg c$
(3) $a \longrightarrow \neg b$ $\Longrightarrow a \longleftrightarrow c$

(4) $b \longrightarrow \neg a$	(3), D3
(5) $\neg a$	(1), (4), E9
(6) $a \longrightarrow c$	(5), E7
(7) $c \longrightarrow a$	(2), D3
(8) $a \longleftrightarrow c$	(6), (7), C11

1.16 **1.16.a**

$$\begin{aligned}
P_1 &\iff r \longrightarrow s \\
P_2 &\iff r \vee p \vee w \\
P_3 &\iff \neg s \\
P_4 &\iff p \longrightarrow i \\
P_5 &\iff l \longrightarrow \neg i \\
P_6 &\iff l
\end{aligned}$$

1.16.b

(1)	$r \longrightarrow s$	
(2)	$r \vee p \vee w$	
(3)	$\neg s$	
(4)	$p \longrightarrow i$	
(5)	$l \longrightarrow \neg i$	
(6)	l	$\Longrightarrow w$
(7)	$\neg r$	(1), (3) E10
(8)	$\neg r \longrightarrow (p \vee w)$	(2) D1
(9)	$p \vee w$	(7), (8) E9
(10)	$\neg i$	(5), (6) E9
(11)	$\neg p$	(4), (10) E10
(12)	$\neg p \longrightarrow w$	(9), D1
(13)	w	(11), (12) E9

For comparison, here's Sherlock Holmes' chain of proof:

		law	commentary
(1)	$r \longrightarrow s$	premise	If it was robbery, something would have been taken.
(2)	$\neg s$	premise	Nothing was taken.
(3)	$\neg r$	(1), (2) E10	It was no robbery.
(4)	$\neg r \longrightarrow (p \vee w)$	premise, D1	If it was not robbery, it must have been politics or a woman.
(5)	$p \vee w$	(3), (4) E9	It was politics, or it was a woman.
(6)	$p \longrightarrow i$	premise	If it was politics, the assassin would have left immediately.
(7)	$l \longrightarrow \neg i$	premise	If the assassin left tracks all over the room, he cannot have left immediately.
(8)	l	premise	The assassin left tracks all over the room.
(9)	$\neg i$	(7), (8) E9	The assassin did not leave immediately.
(10)	$\neg p$	(6), (9) E10	It was not politics.
(11)	w	(5), (10) D1, E9	Consequently, it was a woman.

$\boxed{\textbf{1.17}}$ **1.17.a** $a \longleftrightarrow (a \wedge b) \iff a \longrightarrow b$ always holds (truth table). Assigned to propositional forms, this is the adjunctive form of the enhancement rule RI3:

$$A \Longrightarrow B \text{ IF AND ONLY IF } A \iff A \wedge B$$

Now, it is to be shown that:

$$\begin{aligned} (a \vee b) \wedge (\neg a \vee c) &\iff (a \vee \neg a) \wedge (\neg a \longrightarrow b) \wedge (a \longrightarrow c) && (A7) \\ &\Longrightarrow b \vee c && (E17) \end{aligned}$$

With the adjunctive form of RI3, $(a \vee b) \wedge (\neg a \vee c) \Longrightarrow (b \vee c)$ produces the solution $(a \vee b) \wedge (\neg a \vee c) \iff (a \vee b) \wedge (\neg a \vee c) \wedge (b \vee c)$.

1.17.b analogous to a.) $b \longrightarrow a \iff a \longleftrightarrow a \vee b$ always holds. Assigned to propositional forms, this is the disjunctive form of the enhancement rule RI3:

$$B \Longrightarrow A \text{ IF AND ONLY IF } A \iff A \vee B$$

Now, it is to be shown that $(b \wedge c) \Longrightarrow (a \wedge b) \vee (\neg a \wedge c)$. To prove this, the deductive proof scheme is applied:

(1)	$b \wedge c$	$\Longrightarrow (a \wedge b) \vee (\neg a \wedge c)$
(2)	b	(1), E4
(3)	c	(1), E4
(4)	$\neg a \vee b$	(2), E5
(5)	$a \vee c$	(3), E5
(6)	$(\neg a \vee a) \wedge (\neg a \vee b)$	(4), A8, A6
(7)	$(\neg a \vee a) \wedge (a \vee c)$	(5), A8, A6
(8)	$a \longrightarrow (a \wedge b)$	(6), A3, D1
(9)	$\neg a \longrightarrow (\neg a \wedge c)$	(7), A3, D1
(10)	$(a \vee \neg a) \longrightarrow (a \wedge b) \vee (\neg a \wedge c)$	(8), (9), E15
(11)	$(a \wedge b) \vee (\neg a \wedge c)$	(10), A8, D5

With the disjunctive form of RI3 and $b \wedge c \Longrightarrow (a \wedge b) \vee (\neg a \wedge c)$ follows $(a \wedge b) \vee (\neg a \vee c) \iff (a \wedge b) \vee (\neg a \vee c) \vee (b \wedge c)$.

1.17.c

Layer		
1	$A \iff$	$(a \wedge b) \vee (b \wedge \neg c) \vee \neg b \vee (\neg a \wedge c)$
2		$\vee a \vee (b \wedge c) \vee \neg c \vee (\neg a \wedge b)$
3		$\vee c \vee b \vee \neg a$
4		$\vee t$ (from $c \vee \neg c$ or $b \vee \neg b$)

$\boxed{\textbf{1.18}}$ **1.18.a**

$$A \iff (a \vee b) \wedge (\neg b \vee \neg c) \wedge (c \vee \neg d)$$

$$\wedge (a \vee \neg c) \wedge (\neg b \vee \neg d)$$

$\wedge(a \vee \neg d)$

No more conclusions to be drawn, A is a partly valid propositional form.

1.18.b $B \iff (\neg a \vee b) \wedge (a \vee \neg c) \wedge (c \vee \neg b) \wedge a \wedge (\neg a \vee \neg b)$

$\wedge \neg a$ (aus: $(\neg a \vee b) \wedge (\neg a \vee \neg b)$)

$\wedge f$

B is a contradiction.

1.18.c $C \iff a \vee (\neg a \wedge b) \vee (\neg a \wedge \neg c) \vee (\neg b \wedge c)$

$\vee b \vee \neg c \vee (\neg a \wedge c)$

$\vee c$

$\vee t$ (aus: $c \vee \neg c$)

C is a tautology.

1.18.d $D \iff \neg a \wedge (a \vee \neg b) \wedge (a \vee c) \wedge (b \vee \neg c)$

$\wedge \neg b \wedge c \wedge (a \vee \neg c)$

$\wedge \neg c$

$\wedge f$ (aus: $c \wedge \neg c$)

D is a contradiction.
Note: This follows directly from c), since $C \iff \neg D$!

1.19

$$
\begin{aligned}
A \iff\ & (a \longrightarrow \neg b) \wedge (\neg a \longrightarrow \neg b) \wedge (\neg b \longrightarrow f) \wedge \\
& (b \longrightarrow \neg b) \wedge (b \longrightarrow f) \wedge \boldsymbol{(t \longrightarrow f)} \\
\iff\ & f
\end{aligned}
$$

$\boxed{\textbf{1.20}}$ **1.20.a**

$$\begin{array}{rcll}
\beta(p,a,a) & \Longleftrightarrow & (p \wedge a) \vee (\neg p \wedge a) & \\
& \Longleftrightarrow & (p \vee \neg p) \wedge a & \text{A3} \\
& \Longleftrightarrow & w \wedge a & \text{A8} \\
& \Longleftrightarrow & a & \text{A6}
\end{array}$$

1.20.b

$$\begin{array}{rcll}
\beta(p,w,f) & \Longleftrightarrow & (p \longrightarrow w) \wedge (\neg p \longrightarrow f) & \\
& \Longleftrightarrow & w \wedge p & \text{D4, D5} \\
& \Longleftrightarrow & p & \text{A6}
\end{array}$$

1.20.c

$$\begin{array}{rcll}
\beta(p,f,w) & \Longleftrightarrow & (p \longrightarrow f) \wedge (\neg p \longrightarrow w) & \\
& \Longleftrightarrow & \neg p \wedge w & \text{D4, D5} \\
& \Longleftrightarrow & \neg p & \text{A6}
\end{array}$$

1.20.d

$$\begin{array}{rcll}
\beta(w,a,b) & \Longleftrightarrow & (w \wedge a) \vee (\neg w \wedge b) & \\
& \Longleftrightarrow & a \vee f & \text{A6, A7} \\
& \Longleftrightarrow & a & \text{A6}
\end{array}$$

1.20.e

$$\begin{array}{rcll}
\beta(f,a,b) & \Longleftrightarrow & (f \wedge a) \vee (\neg f \wedge b) & \\
& \Longleftrightarrow & f \vee b & \text{A6, A7} \\
& \Longleftrightarrow & b & \text{A6}
\end{array}$$

1.20.f

$$\begin{array}{rcll}
p \wedge \beta(p,a,b) & \Longleftrightarrow & p \wedge [(p \wedge a) \vee (\neg p \wedge b)] & \\
& \Longleftrightarrow & (p \wedge a) \vee (f \wedge b) & \text{A3, A8} \\
& \Longleftrightarrow & \boxed{p \wedge a} & \text{A7} \\
& \Longleftrightarrow & (p \wedge a) \vee (\neg p \wedge f) & \text{A6, A7} \\
& \Longleftrightarrow & \beta(p,a,f) &
\end{array}$$

1.20.g

$$\begin{array}{rcll}
\neg p \wedge \beta(p,a,b) & \Longleftrightarrow & \neg p \wedge [(p \wedge a) \vee (\neg p \wedge b)] & \\
& \Longleftrightarrow & (f \wedge a) \vee (\neg p \wedge b) & \text{A3, A8} \\
& \Longleftrightarrow & \boxed{\neg p \wedge b} & \text{A7} \\
& \Longleftrightarrow & (p \wedge f) \vee (\neg p \wedge b) & \text{A6, A7} \\
& \Longleftrightarrow & \beta(p,f,b) &
\end{array}$$

1.20.h

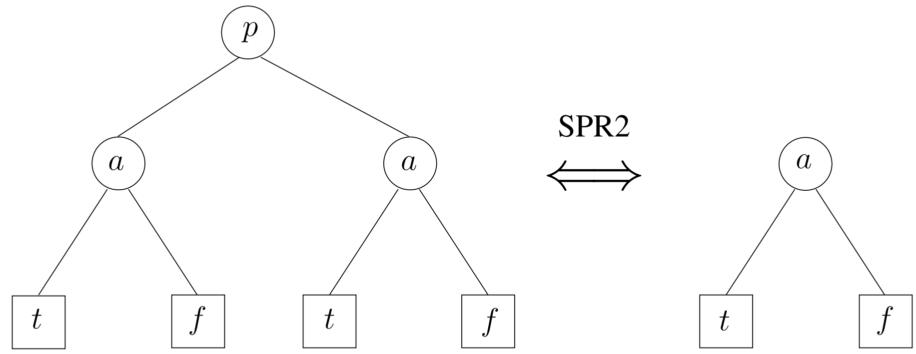
$$\begin{array}{rcll}
\beta(p,\beta(p,a,b),c) & \Longleftrightarrow & [p \wedge \beta(p,a,b)] \vee [\neg p \wedge c] & \\
& \Longleftrightarrow & (p \wedge a) \vee (\neg p \wedge c) & \text{(see f)} \\
& \Longleftrightarrow & \beta(p,a,c) &
\end{array}$$

1.20.i

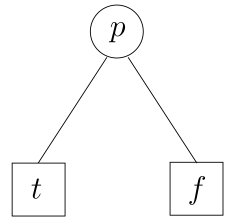
$$\begin{aligned}\beta(p,a,\beta(p,b,c)) &\iff [p\wedge a]\vee[\neg p\wedge\beta(p,b,c)]\\ &\iff (p\wedge a)\vee(\neg p\wedge c) && \text{(see g)}\\ &\iff \beta(p,a,c)\end{aligned}$$

1.20.j

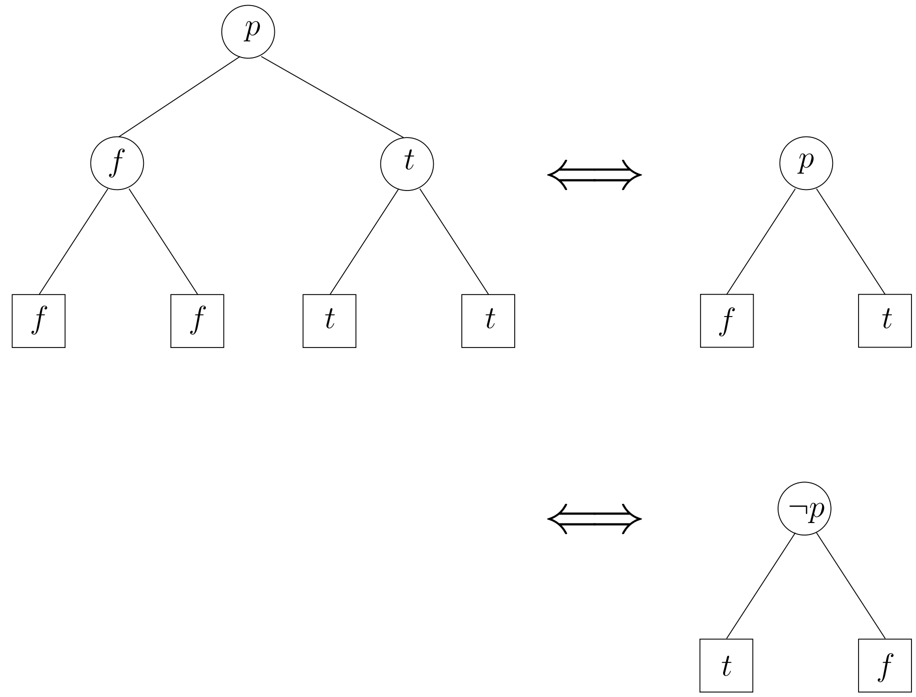
- $\beta(p,a,a) \iff a$



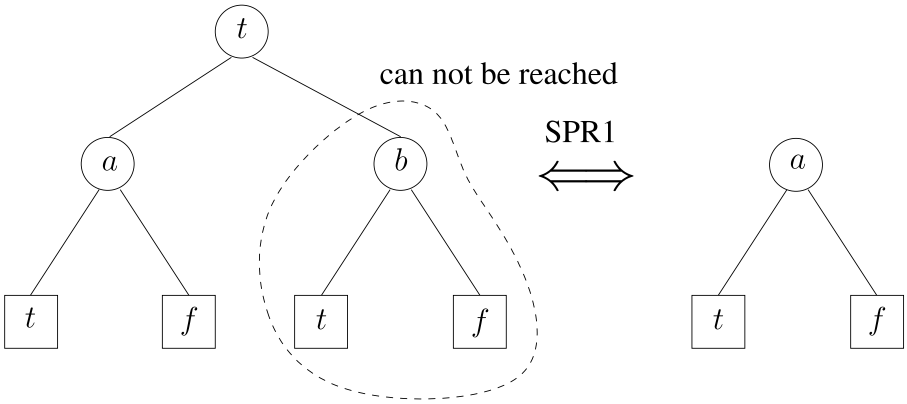
- $\beta(p,w,f) \iff p$



- $\beta(p,f,w) \iff \neg p$
  Observe, that the BDT on the left looks "strange", as it also includes non-terminal nodes with no decision variable but a truth value assigned to them.

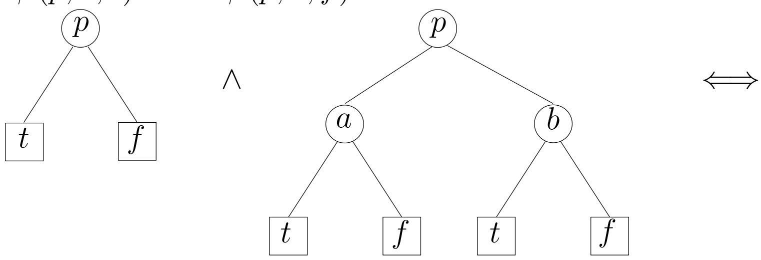


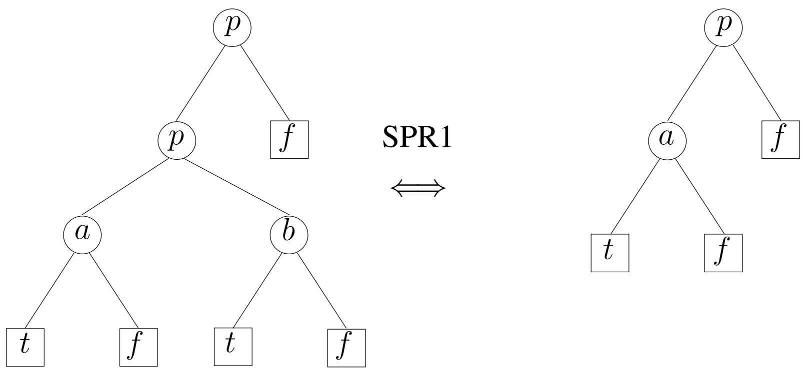
- $\beta(w, a, b) \iff a$
  Another "strange" BDT



- $\beta(f, a, b) \iff b$
  analogous

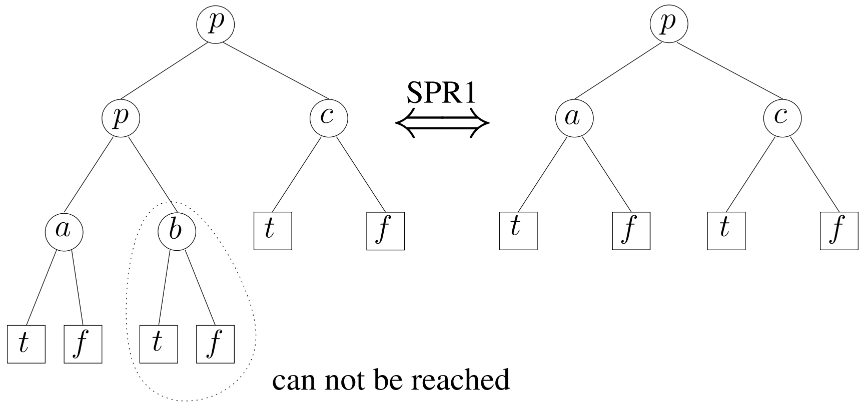
- $p \wedge \beta(p, a, b) \iff \beta(p, a, f)$



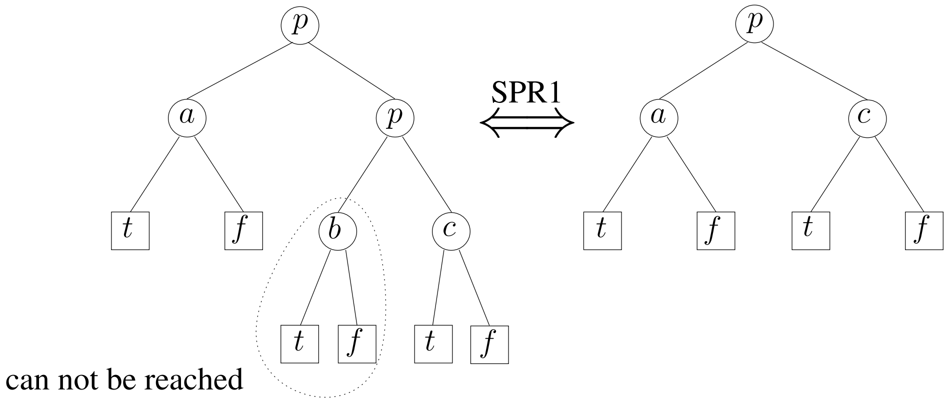


- $\neg p \wedge \beta(p, a, b) \iff \beta(p, f, b)$
  analogous

- $\beta(p, \beta(p, a, b), c) \iff \beta(p, a, c)$



- $\beta(p, a, \beta(p, b, c)) \iff \beta(p, a, c)$



$\boxed{\textbf{1.21}}$ **1.21.a** A has 3 t-assignments and 5 f-assignments.

$$\begin{aligned}
\mathrm{CDNF}[A] &\iff (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c) \\
\mathrm{CCNF}[A] &\iff (a \vee b \vee c) \wedge (\neg a \vee b \vee c) \wedge (a \vee b \vee \neg c) \\
&\qquad \wedge (a \vee \neg b \vee c) \wedge (a \vee \neg b \vee \neg c)
\end{aligned}$$

1.21.b

$$\mathrm{PNF}[A] \iff \beta(a, \beta(b, t, \beta(c, t, f)), f)$$

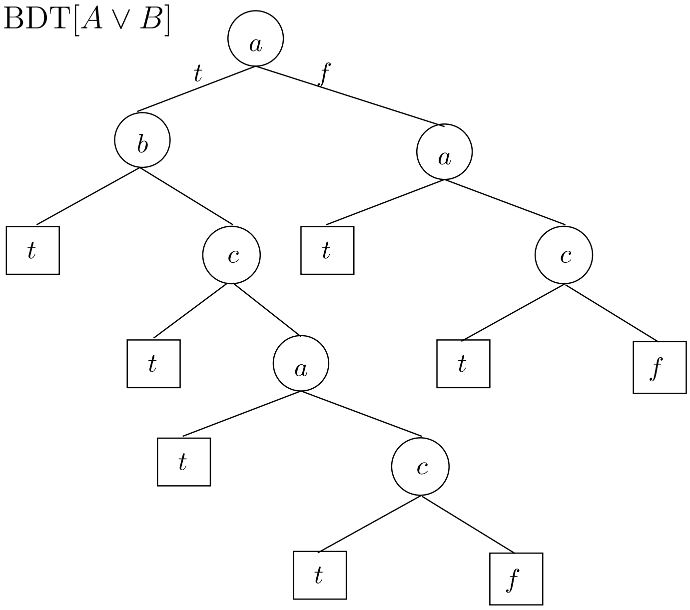
1.21.c

$$\begin{aligned}
A &\iff \beta(a, \beta(b, t, \beta(c, t, f)), f) \\
&\iff \beta(a, \beta(b, t, c), f) && \text{(BB1)} \\
&\iff \beta(a, b \vee c, f) && \text{(BB3)} \\
&\iff a \wedge (b \vee c) && \text{(BB3)} \\
&\iff (a \wedge b) \vee (a \wedge c) && \text{(A3)} \\
&\iff (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c) && \text{(RRD, A4)}
\end{aligned}$$

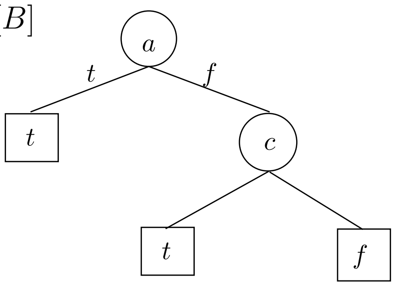
1.21.d Replace t-terminal nodes with f-terminal nodes and vice versa.

1.21.e First set up $\mathrm{BDT}[B]$, then $\mathrm{BDT}[A \vee B]$, finally reduce to $\mathrm{ROBDD}[A \vee B]$.

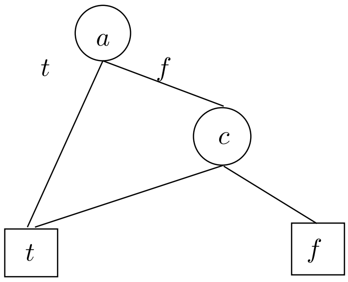
$\mathrm{BDT}[A \vee B]$



$\mathrm{BDT}[B]$



$\mathrm{ROBDD}[A \vee B]$

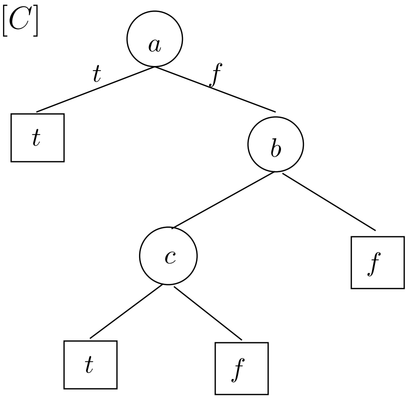


1.22 **1.22.a** Not all applied laws are given!

$$\begin{aligned}
C &\iff [a \longleftrightarrow\!\!\!\!\!\!/\;\; (b \wedge \neg c)] \longleftrightarrow [b \longrightarrow (a \wedge c)] \\
&\iff \beta(a, A(a \Leftrightarrow t), A(a \Leftrightarrow f)) && \text{(BO17)} \\
&\iff \beta(a, (\neg b \vee c) \longleftrightarrow (b \longrightarrow c), (b \wedge \neg c) \longleftrightarrow \neg b) && \text{(B5, A10, B4, D5)} \\
&\iff \beta(a, (b \longrightarrow c) \longleftrightarrow (b \longrightarrow c), (b \wedge \neg c) \longleftrightarrow \neg b) && \text{(D1)} \\
&\iff \beta(a, t, \beta(b, c, f)) && \text{(BO17, C8, C4)} \\
&\iff \beta(a, t, \beta(b, \beta(c, t, f), f)) && \text{(BB1)}
\end{aligned}$$

1.22.b

BDT$[C]$



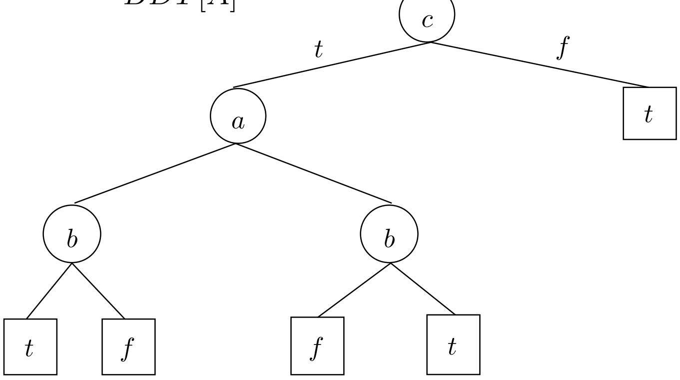
1.22.c

$$
\begin{aligned}
C &\iff [a \nleftrightarrow (b \wedge \neg c)] \longleftrightarrow [b \longrightarrow (a \wedge c)] \\
&\iff [a \wedge (C(a \Leftrightarrow t) \nleftrightarrow C(a \Leftrightarrow f))] \nleftrightarrow C(a \Leftrightarrow f) && \text{(ER4)} \\
&\iff [a \wedge (t \nleftrightarrow (b \wedge \neg c \longleftrightarrow \neg b))] \nleftrightarrow (b \wedge \neg c \longleftrightarrow \neg b) \\
&\iff a \nleftrightarrow a \wedge (b \wedge \neg c \longleftrightarrow \neg b) \nleftrightarrow (b \wedge \neg c \longleftrightarrow \neg b) && \text{(B3)} \\
&\iff b \wedge ((a \nleftrightarrow a \wedge c \nleftrightarrow c) \nleftrightarrow (a \nleftrightarrow a \wedge f \nleftrightarrow f)) \\
&\qquad \nleftrightarrow (a \nleftrightarrow a \wedge f \nleftrightarrow f) && \text{(ER4)} \\
&\iff b \wedge (a \nleftrightarrow a \wedge c \nleftrightarrow c \nleftrightarrow a) \nleftrightarrow a && \text{(B4, B6)} \\
&\iff a \wedge b \nleftrightarrow a \wedge b \wedge c \nleftrightarrow b \wedge c \nleftrightarrow a \wedge b \nleftrightarrow a && \text{(B3)} \\
&\iff a \wedge b \wedge c \nleftrightarrow b \wedge c \nleftrightarrow a && \text{(B4,B6)}
\end{aligned}
$$

$\boxed{\textbf{1.23}}$

$$
\begin{aligned}
A(a,b,c) &\iff \beta(c, (a \longleftrightarrow b) \vee a \wedge b, t) && \text{BO17} \quad \text{(D5, C12, C4, D4, A7)} \\
&\iff \beta\big(c, \beta(a, b, \neg b), t\big) && \text{BO17} \quad \text{(C4, A6, C5, A8, A6)} \\
PNF[A] &\iff \beta\Big(c, \beta\big(a, \beta(b,t,f), \beta(b,f,t)\big), t\Big) && \text{BB1}
\end{aligned}
$$

$BDT[A]$



$\boxed{\textbf{1.24}}$ $PNF[B] \iff \beta\Big(a, \beta\big(b, \beta(c,t,f), \beta(c,f,t)\big), \beta(b,t,f)\Big)$

$$
\begin{aligned}
B(a,b,c) &\iff \beta\Big(a, \beta\big(b, \beta(c,t,f), \beta(c,f,t)\big), \beta(b,t,f)\Big) & \\
B(a,b,c) &\iff \beta\big(a, \beta(b, c, \neg c), b\big) & \text{BB1} \\
B(a,b,c) &\iff \beta(a, b \longleftrightarrow c, b) & \text{BB5} \\
B(a,b,c) &\iff a \wedge (b \longleftrightarrow c) \vee \neg a \wedge b & \text{Def. } \beta\text{-Op.} \\
B(a,b,c) &\iff a \wedge [(b \wedge c) \vee (\neg b \wedge \neg c)] \vee \neg a \wedge b & \text{C9} \\
B(a,b,c) &\iff (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b) & \text{A3} \\
B(a,b,c) &\iff (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (\neg a \wedge b \wedge \neg c) & \text{RRD} \\
&\iff CDNF[B] &
\end{aligned}
$$

$\boxed{\textbf{1.25}}$ **1.25.a**

a	b	c	$a \vee \neg b \vee c$	$\neg a \wedge b \wedge \neg c$	$\vee$
f	f	f	t	f	t
f	f	t	t	f	t
f	t	f	f	t	t
f	t	t	t	f	t
t	f	f	t	f	t
t	f	t	t	f	t
t	t	f	t	f	t
t	t	t	t	f	t

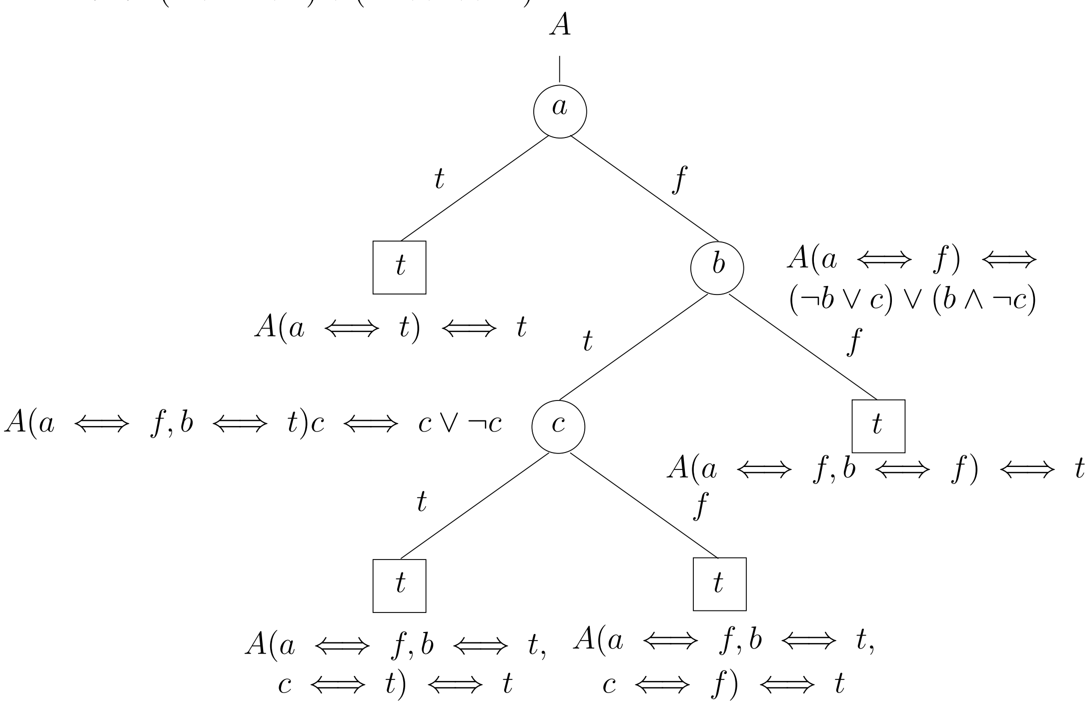
1.25.b

$$
\begin{aligned}
A &\iff a \vee \neg b \vee c \quad \vee \quad (\neg a \wedge b \wedge \neg c) \\
&\iff (a \vee \neg b \vee c \vee \neg a) \quad \wedge \quad (a \vee \neg b \vee c \vee b) \quad \wedge \quad (a \vee \neg b \vee c \vee \neg c) \\
&\iff t \quad \wedge \quad t \quad \wedge \quad t \\
&\iff t
\end{aligned}
$$

1.25.c

$$
\begin{aligned}
A &\iff E \vee \neg E \\
&\iff t
\end{aligned}
$$

1.25.d $A \iff (a \vee \neg b \vee c) \vee (\neg a \wedge b \wedge \neg c)$



All terminal nodes of the BDD carry the value t. Therefore all possible value assignments of A yield t, i.e., this propositional form is a tautology.

1.25.e $A \iff a \vee \neg b \vee c \vee (\neg a \wedge b \wedge \neg c)$

$\vee (b \wedge \neg c) \vee (\neg a \wedge \neg c) \vee (\neg a \wedge b)$

$\vee \neg c \vee b \vee \neg a$

$\vee t$ (e.g., from $a \vee \neg a$)

$\boxed{\textbf{1.26}}$ **1.26.a**

$$\begin{aligned} \text{dual}(A(a,b,c)) &\iff \text{dual}(\neg a \vee b \vee c) \\ &\iff \neg(a \vee \neg b \vee \neg c) \\ &\iff \neg a \wedge b \wedge c \end{aligned}$$

1.26.b dual(A) consists of one minterm, i.e., there are $2^3 - 1 = 7$ value assignments that falsify it. Thus, the number of non-trivial conclusions is: $l = 2^7 - 1 = 127$.

1.26.c Two non-trivial conclusions:

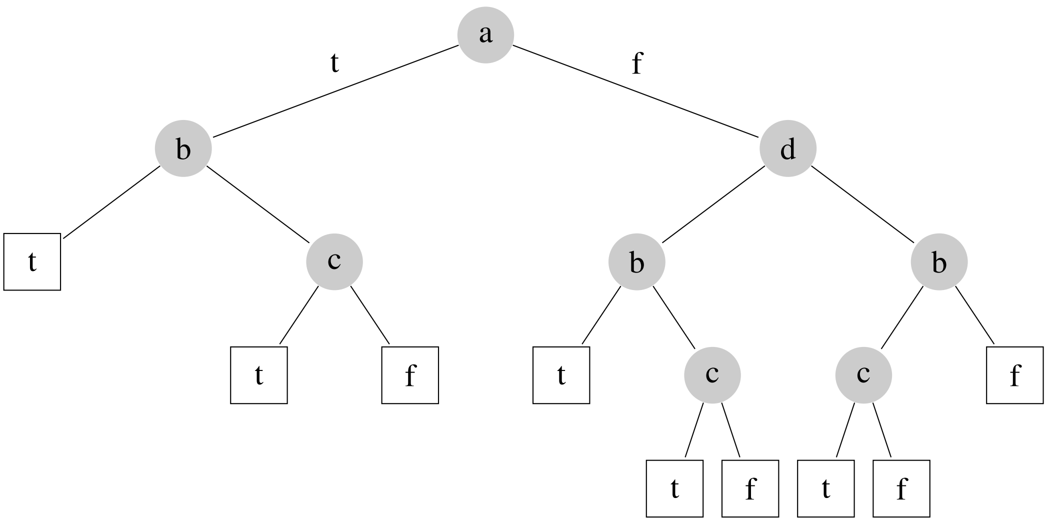
$S_1 \iff \neg a \wedge b \wedge c$

$S_2 \iff \neg a$

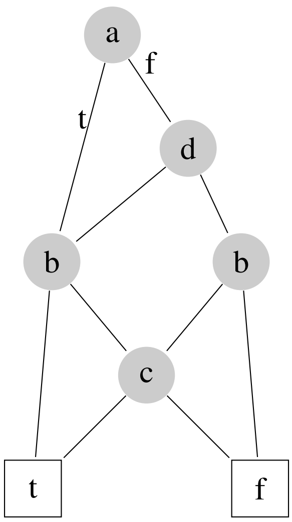
1.27 D is k-ary and from D there can be drawn n non-trivial conclusions.
$n = 2^m - 1$ (m: number value assignments that falsify D)
$2^k - m$ value assignments falsify $dual(D)$.

$$m_{dual(D)} = 2^k - m = 2^k - ld(n+1)$$
$$\Longrightarrow l = 2^{m_{dual(D)}} - 1 = 2^{(2^k - ld(n+1))} - 1 = \frac{2^{2^k}}{n+1} - 1$$

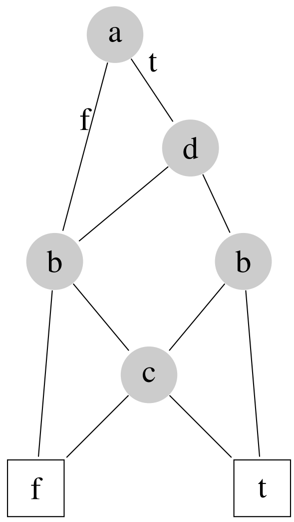
1.28 **1.28.a** ROBDT[A]:



ROBDD[A]:



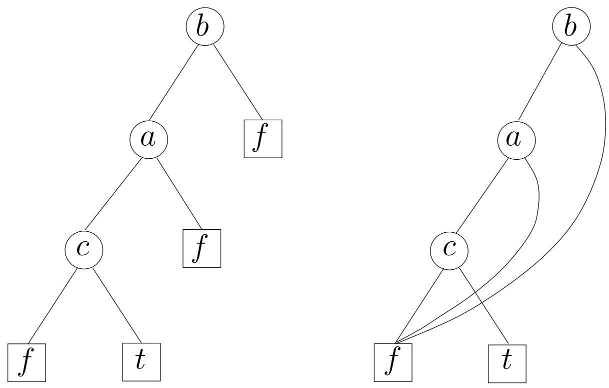
1.28.b ROBDD[dual(A)]:



1.29 **1.29.a** The BDT can be directly derived from the given propositional form A. Alternatively, it is also possible to transform the propositional form into β-form, first.

$$\begin{aligned}
A &\iff b \wedge (b \longrightarrow a) \wedge (b \longrightarrow \neg c) \\
&\iff \beta(b, a \wedge \neg c, f) \\
&\iff \beta(b, \beta(a, \neg c, f), f) \\
&\iff \beta(b, \beta(a, \beta(c, f, t), f), f)
\end{aligned}$$

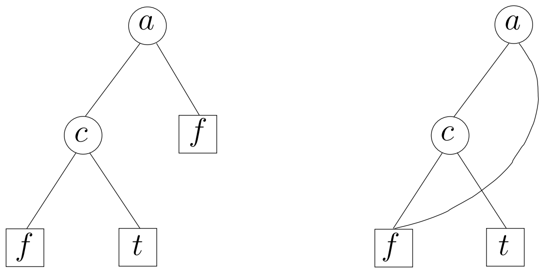
1.29.b



1.29.c Notation in β-form:

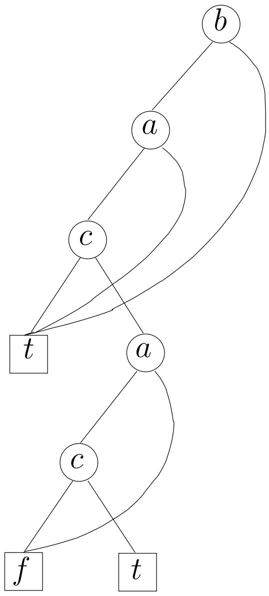
$$\begin{aligned}
B &\iff a \wedge \neg c \\
&\iff \beta(a, \neg c, f) \\
&\iff \beta(a, \beta(c, f, t), f)
\end{aligned}$$

1.29.d



1.29.e $\beta(A, B, t)$

1.29.f $\beta(b, \beta(a, \beta(c, t, \beta(a, \beta(c, f, t), f)), t), t)$



1.29.g $\beta(b, \beta(a, \beta(c, t, \beta(t, \beta(f, f, t), f)), t), t)$

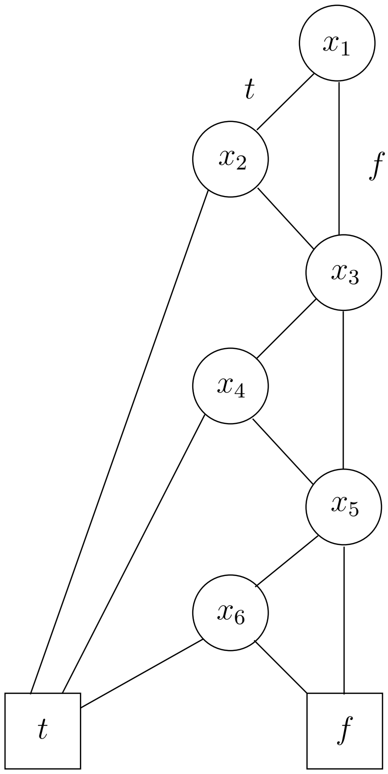
1.29.h

$$
\begin{array}{lll}
 & \beta(b, \beta(a, \beta(c, t, \beta(t, \beta(f, f, t), f)), t), t) & \\
\iff & \beta(b, \beta(a, \beta(c, t, \beta(t, t, f)), t), t) & \text{BO3} \\
\iff & \beta(b, \beta(a, \beta(c, t, t), t), t) & \text{BO3} \\
\iff & \beta(b, \beta(a, t, t), t) & \text{BO1} \\
\iff & \beta(b, t, t) & \text{BO1} \\
\iff & t & \text{BO1}
\end{array}
$$

1.29.i Therewith, the given implication has been proven to be a tautology.

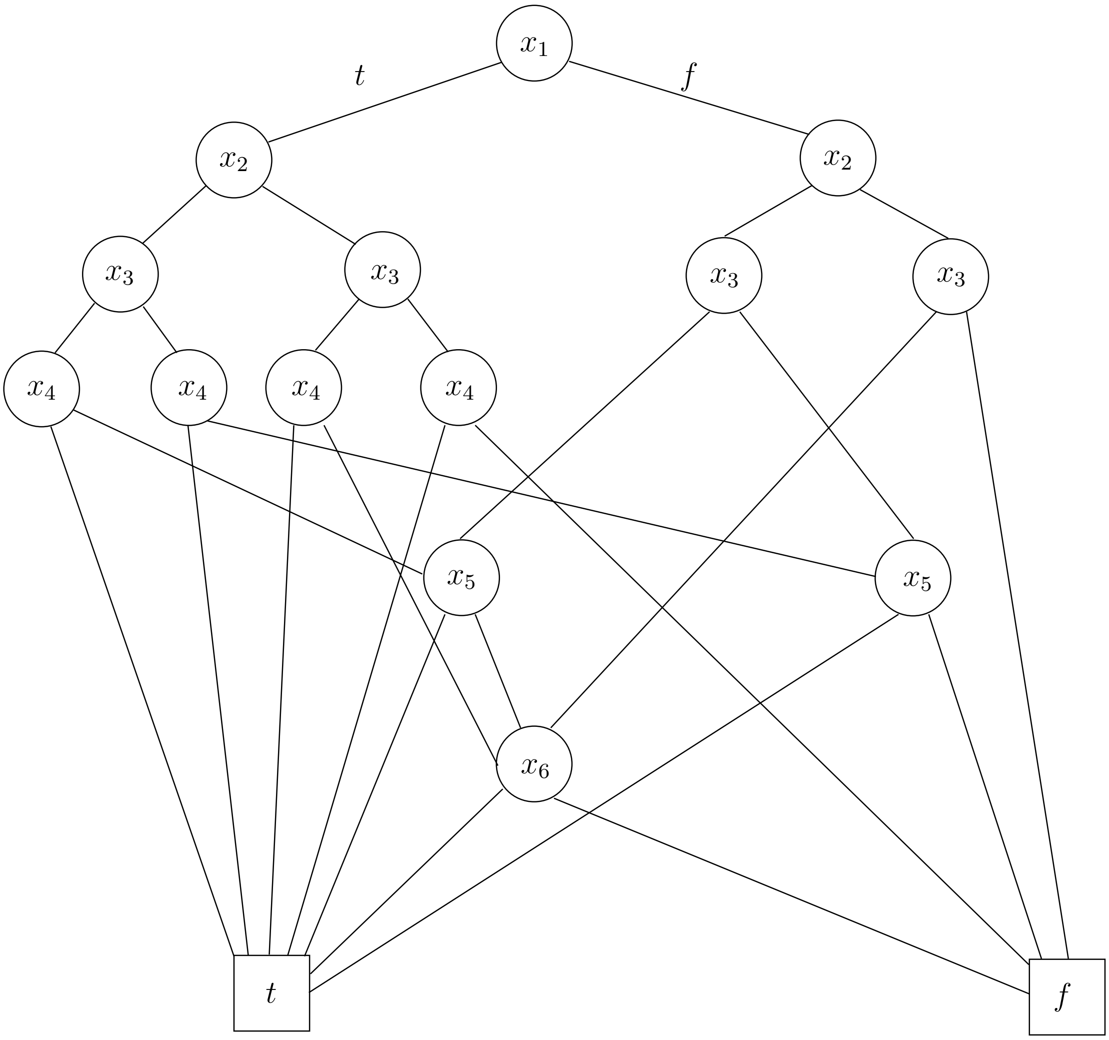
$\boxed{\textbf{1.30}}$ **1.30.a** The OBDT is not shown as it is too large. Therefore, just the ROBDD is shown.

$(x_1 \wedge x_2) \vee (x_3 \wedge x_4) \vee (x_5 \wedge x_6)$



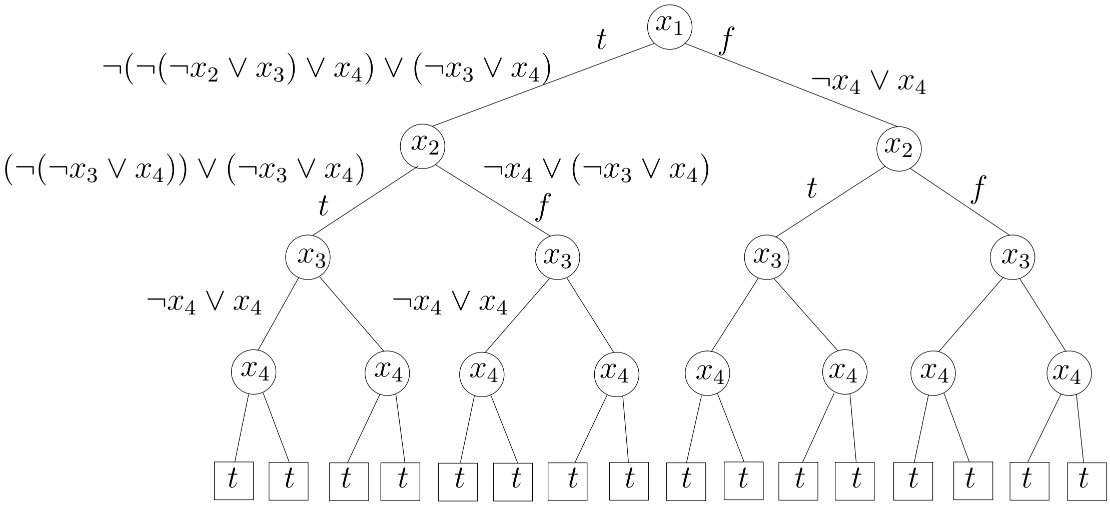
1.30.b The OBDT is not shown as it is too large. Therefore, just the ROBDD is shown.

$(x_1 \wedge x_4) \vee (x_2 \wedge x_5) \vee (x_3 \wedge x_6)$

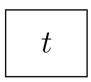


1.30.c $\neg(\neg(\neg(x_1 \wedge x_2) \vee x_3) \vee x_4) \vee (\neg(\neg x_1 \vee x_3) \vee x_4)$

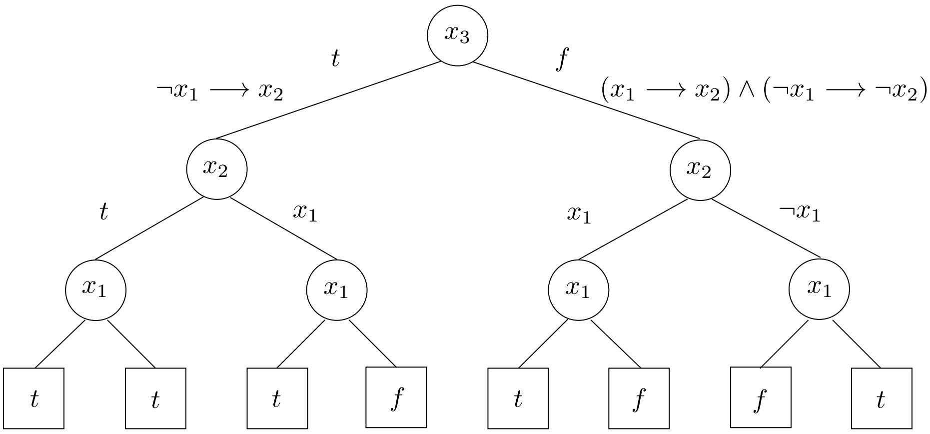
OBDT



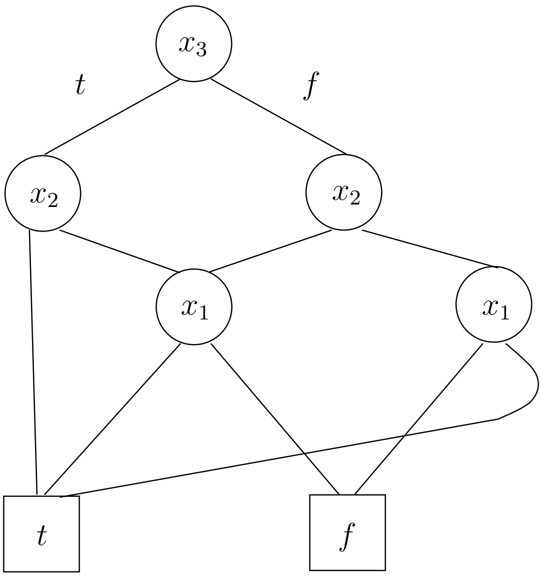
ROBDD



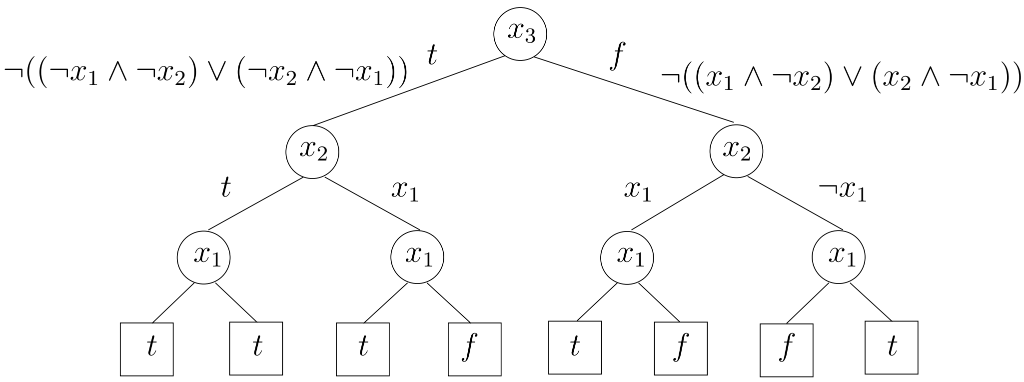
1.30.d OBDT:



ROBDD:



1.30.e $\neg(((x_1 \longleftrightarrow x_3) \wedge \neg x_2) \vee ((x_2 \longleftrightarrow x_3) \wedge \neg x_1))$



The propositonal forms of exercises d) and e) are identical.

$\boxed{\textbf{1.31}}$ **1.31.a** correct, as the set of premises $\mathcal{M}_1$ can be extended to $\mathcal{M}_2 = \mathcal{M}_1 \cup \mathcal{M}_2 \backslash \mathcal{M}_1$
Premise (a) can be traced back to premise (c), as $\mathcal{M}_1 \subseteq \mathcal{M}_2$ results in $U_1 \wedge \ldots \wedge U_m \implies V_1 \wedge \ldots \wedge V_n$ and finally in $E[U_1 \wedge \ldots \wedge U_m] \subseteq E[V_1 \wedge \ldots \wedge V_n]$

1.31.b wrong, choose $V_1, \ldots, V_n$ so that all premises are required to derive the propositional form A. If $V_1 \neq A$ and $U_1 = V_1$, then:

$$\begin{array}{lcl} V_1 \wedge \ldots \wedge V_n & \implies & U_1 \\ V_1 \wedge \ldots \wedge V_n & \implies & A \end{array} \qquad \text{but } U_1 \not\Longrightarrow A$$

Example: $V_1 = a \longrightarrow b, V_2 = a, U_1 = a \longrightarrow b, A = b$.
$V_1 \wedge V_2 \implies U_1$, $V_1 \wedge V_2 \implies A$ holds, but $U_1 \not\Longrightarrow A$

1.31.c correct; due to $V_1 \wedge \ldots \wedge V_n \implies A$, $E[V_1 \wedge \ldots \wedge V_n] \subseteq E[A]$ holds. Using premise (c) results in:
$E[U_1 \wedge \ldots \wedge U_m] \subseteq E[V_1 \wedge \ldots \wedge V_n] \subseteq E[A]$ and finally $U_1 \wedge \ldots \wedge U_m \implies A$

2. Predicate Logic

2.1 **2.1.a**

$$A_1 \iff \underset{x \in P}{\forall}(Bx \longrightarrow Gx)$$

$$A_2 \iff \underset{x \in P}{\forall}(Gx \leftrightarrow\!\!\!\!\!\!\;+ Ix)$$

$$A_3 \iff Bx_1$$

2.1.b Formalize the conclusions and prove them using the deductive proof scheme:
C_1 :

1)	$\underset{x \in P}{\forall}(Bx \longrightarrow Gx)$	
2)	$\underset{x \in P}{\forall}(Gx \not\leftrightarrow Ix)$	
3)	Bx_1	$\Longrightarrow \neg Ix_1$
4)	$Bx_1 \longrightarrow Gx_1$	$1, \mathrm{G1}$
5)	Gx_1	$3, 4, \mathrm{E9}$
6)	$Gx_1 \not\leftrightarrow Ix_1$	$2, \mathrm{G1}$
7)	$Gx_1 \longleftrightarrow \neg Ix_1$	$6, \mathrm{B11}, \mathrm{C5}$
8)	$Gx_1 \longrightarrow \neg Ix_1$	$7, \mathrm{C11}, \mathrm{E4}$
9)	$\neg Ix_1$	$5, 8, \mathrm{E9}$

C_2 :

1)	$\underset{x \in P}{\forall}(Bx \longrightarrow Gx)$	
2)	$\underset{x \in P}{\forall}(Gx \not\leftrightarrow Ix)$	$\Longrightarrow \neg \underset{x \in P}{\exists}(Bx \wedge Ix)$
3)	$\underset{x \in P}{\forall}[(Gx \longrightarrow \neg Ix) \wedge (\neg Ix \longrightarrow Gx)]$	$2, \mathrm{B11}, \mathrm{C5}, \mathrm{C11}$
4)	$\underset{x \in P}{\forall}(Gx \longrightarrow \neg Ix) \wedge \underset{x \in P}{\forall}(\neg Ix \longrightarrow Gx)$	$3, \mathrm{F5}$
5)	$\underset{x \in P}{\forall}(Gx \longrightarrow \neg Ix)$	$4, \mathrm{E4}$
6)	$\underset{x \in P}{\forall}[(Bx \longrightarrow Gx) \wedge (Gx \longrightarrow \neg Ix)]$	$1, 5, \mathrm{F5}$
7)	$\underset{x \in P}{\forall}(Bx \longrightarrow \neg Ix)$	$6, \mathrm{E11}$
8)	$\neg \underset{x \in P}{\exists} \neg(Bx \longrightarrow \neg Ix)$	$7, \mathrm{A9}, \mathrm{F1}$
9)	$\neg \underset{x \in P}{\exists}(Bx \wedge Ix)$	$8, \mathrm{D2}$

2.2 **2.2.a** $Sf \iff \underset{x_0 \epsilon D}{\forall} \underset{\varepsilon \epsilon R^+}{\forall} \underset{\delta \epsilon R^+}{\exists} \underset{x \in D}{\forall} K(|x_0 - x|)\, \delta \longrightarrow K(|f(x_0) - f(x)|)\varepsilon$

2.2.b $G f\, y_1 \iff \underset{\varepsilon \epsilon R^+}{\forall}\ \underset{x_0 \epsilon D}{\exists}\ \underset{x \in D}{\forall} K x_0\, x \longrightarrow K(|f(x) - y_1|)\, \varepsilon$

$\boxed{\textbf{2.3}}$ Backtransformation to propositional logic:

2.3.a F1:

$$\begin{array}{ll} \neg \underset{x \in \mathcal{M}}{\forall} Px & \iff \\ \neg\,(Px_1 \wedge Px_2 \wedge \ldots \wedge Px_m) & \iff \\ (\neg Px_1 \vee \neg Px_2 \vee \ldots \vee \neg Px_m) & \iff \\ \underset{x \in \mathcal{M}}{\exists} \neg Px & \end{array}$$

2.3.b F2: analogous to F1.

2.3.c F3:

$$\begin{array}{ll} \underset{x \in \mathcal{M}}{\forall}\ \underset{y \in \mathcal{N}}{\forall} Pxy & \iff \\ Px_1y_1 \wedge Px_1y_2 \wedge \ldots \wedge Px_1y_n \wedge Px_2y_1 \wedge \ldots \wedge Px_my_n & \iff \\ Px_1y_1 \wedge Px_2y_1 \wedge \ldots \wedge Px_my_1 \wedge Px_1y_2 \wedge \ldots \wedge Px_my_n & \iff \\ \underset{y \in \mathcal{N}}{\forall}\ \underset{x \in \mathcal{M}}{\forall} Pxy & \end{array}$$

2.3.d F4: analogous to F3.

2.3.e F5:

$$\begin{array}{ll} \underset{x \in \mathcal{M}}{\forall}(Px \wedge Qx) & \iff \\ (Px_1 \wedge Qx_1) \wedge (Px_2 \wedge Qx_2) \wedge \ldots \wedge (Px_m \wedge Qx_m) & \iff \\ Px_1 \wedge Qx_1 \wedge Px_2 \wedge Qx_2 \wedge \ldots \wedge Px_m \wedge Qx_m & \iff \\ Px_1 \wedge Px_2 \wedge \ldots \wedge Px_m \wedge Qx_1 \wedge Qx_2 \wedge \ldots \wedge Qx_m & \iff \\ \underset{x \in \mathcal{M}}{\forall} Px \wedge \underset{x \in \mathcal{M}}{\forall} Qx & \end{array}$$

2.3.f F6:

$$\begin{array}{ll} \underset{x \in \mathcal{M}}{\exists}(Px \vee Qx) & \iff \\ (Px_1 \vee Qx_1) \vee (Px_2 \vee Qx_2) \vee \ldots \vee (Px_m \vee Qx_m) & \iff \\ Px_1 \vee Qx_1 \vee \ldots & \iff \\ Px_1 \vee Px_2 \vee \ldots \vee Px_m \vee Qx_1 \vee Qx_2 \vee \ldots \vee Qx_m & \iff \\ (\underset{x \in \mathcal{M}}{\exists} Px) \vee (\underset{x \in \mathcal{M}}{\exists} Qx) & \end{array}$$

2.3.g F7:

$$\begin{array}{ll}
\forall_{x \in \mathcal{M}} (S \vee Qx) & \Longleftrightarrow \\
(S \vee Qx_1) \wedge (S \vee Qx_2) \wedge \ldots \wedge (S \vee Qx_m) & \Longleftrightarrow \\
S \vee (Qx_1 \wedge Qx_2 \wedge \ldots \wedge Qx_m) & \Longleftrightarrow \\
S \vee \forall_{x \in \mathcal{M}} Qx &
\end{array}$$

2.3.h F8: analogous to F6.

2.3.i F9, F10:

$$\begin{array}{ll}
\forall_{x \in \mathcal{M}} \exists_{y \in \mathcal{N}} (Px \wedge Qy) & \Longleftrightarrow \\
[(Px_1 \wedge Qy_1) \vee (Px_1 \wedge Qy_2) \vee \ldots \vee (Px_1 \wedge Qy_n)] \wedge & \\
[(Px_2 \wedge Qy_1) \vee \ldots] \wedge \ldots \wedge [\ldots \vee (Px_m \wedge Qy_1)] & \Longleftrightarrow \\
[Px_1 \wedge (Qy_1 \vee Qy_2 \vee \ldots \vee Qy_n)] \wedge & \\
[Px_2 \wedge (Qy_1 \vee \ldots)] \wedge \ldots \wedge [Px_m \wedge (Qy_1 \vee \ldots)] & \Longleftrightarrow \\
[Px_1 \wedge Px_2 \wedge Px_3 \wedge \ldots \wedge Px_m] \wedge & \\
[Qy_1 \vee Qy_2 \vee Qy_3 \vee \ldots \vee Qy_n] & \Longleftrightarrow \\
\forall_{x \in \mathcal{M}} Px \wedge \exists_{y \in \mathcal{N}} Qy & \Longleftrightarrow \\
[Qy_1 \wedge (Px_1 \wedge Px_2 \wedge \ldots \wedge Px_m)] \vee & \\
[Qy_2 \wedge (Px_1 \ldots)] \vee \ldots \vee [Qy_n \wedge (Px_1 \ldots)] & \Longleftrightarrow \\
[(Qy_1 \wedge Px_1) \wedge \ldots \wedge (Qy_1 \wedge Px_m)] \vee [\ldots] \vee \ldots & \Longleftrightarrow \\
\forall_{x \in \mathcal{M}} (Qy_1 \wedge Px) \vee \forall_{x \in \mathcal{M}} (Qy_2 \wedge Px) \vee \ldots & \Longleftrightarrow \\
\exists_{y \in \mathcal{N}} \forall_{x \in \mathcal{M}} (Qy \wedge Px) & \Longleftrightarrow \\
\exists_{y \in \mathcal{N}} \forall_{x \in \mathcal{M}} (Px \wedge Qy) &
\end{array}$$

Alternative solution:

$$\begin{array}{llll}
\forall_{x \in \mathcal{M}} \exists_{y \in \mathcal{N}} (Px \wedge Qy) & \Longleftrightarrow & \forall_{x \in \mathcal{M}} \left[Px \wedge \exists_{y \in \mathcal{N}} Qy \right] & \text{F8} \\
& \Longleftrightarrow & (\forall_{x \in \mathcal{M}} Px) \wedge (\exists_{y \in \mathcal{N}} Qy) & \text{F5, A4} \\
& \Longleftrightarrow & \exists_{y \in \mathcal{N}} \left[\left(\forall_{x \in \mathcal{M}} Px \right) \wedge Qy \right] & \text{F8} \\
& \Longleftrightarrow & \exists_{y \in \mathcal{N}} \forall_{x \in \mathcal{M}} (Px \wedge Qy) & \text{A4, F5}
\end{array}$$

2.3.j F11,12 : Analogous to F9 if transformed back to propositional logic.
Alternative solution:

$$\begin{aligned}
\forall_{x \in \mathcal{M}} \exists_{y \in \mathcal{N}} (Px \vee Qy) &\iff \forall_{x \in \mathcal{M}} \left[Px \vee \exists_{y \in \mathcal{N}} Qy \right] && \text{F6, A4} \\
&\iff (\forall_{x \in \mathcal{M}} Px) \vee (\exists_{y \in \mathcal{N}} Qy) && \text{F7} \\
&\iff \exists_{y \in \mathcal{N}} \left[Qy \vee \left(\forall_{x \in \mathcal{M}} Px \right) \right] && \text{A4, F6} \\
&\iff \exists_{y \in \mathcal{N}} \forall_{x \in \mathcal{M}} (Px \vee Qy) && \text{F7}
\end{aligned}$$

2.3.k F13:

$$\begin{array}{lr}
\exists_{x \in \mathcal{M}} (Px \longrightarrow Qx) & \iff \\
(Px_1 \longrightarrow Qx_1) \vee (Px_2 \longrightarrow Qx_2) \vee \ldots \vee (Px_m \longrightarrow Qx_m) & \iff \\
(\neg Px_1 \vee Qx_1) \vee (\neg Px_2 \vee Qx_2) \vee \ldots \vee (\neg Px_m \vee Qx_m) & \iff \\
(\neg Px_1 \vee \neg Px_2 \vee \ldots \vee \neg Px_m) \vee (Qx_1 \vee Qx_2 \vee \ldots \vee Qx_m) & \iff \\
\neg(Px_1 \wedge Px_2 \wedge \ldots \wedge Px_m) \vee (Qx_1 \vee Qx_2 \vee \ldots \vee Qx_m) & \iff \\
\neg(\forall_{x \in \mathcal{M}} Px) \vee \exists_{x \in \mathcal{M}} Qx & \iff \\
(\forall_{x \in \mathcal{M}} Px) \longrightarrow (\exists_{x \in \mathcal{M}} Qx) &
\end{array}$$

Alternative solution:

$$\begin{aligned}
\exists_{x \in \mathcal{M}} (Px \longrightarrow Qx) &\iff \exists_{x \in \mathcal{M}} (\neg Px \vee Qx) && \text{D1} \\
&\iff (\exists_{x \in \mathcal{M}} \neg Px) \vee (\exists_{x \in \mathcal{M}} Qx) && \text{F6} \\
&\iff (\neg \forall_{x \in \mathcal{M}} Px) \vee (\exists_{x \in \mathcal{M}} Qx) && \text{F1} \\
&\iff (\forall_{x \in \mathcal{M}} Px) \longrightarrow (\exists_{x \in \mathcal{M}} Qx) && \text{D1}
\end{aligned}$$

2.4 **2.4.a** (1) $\forall_{x \in \mathcal{G}} (Px \to Sx)$

(2) $\forall_{x \in \mathcal{G}} (Sx \to Qx)$

(3) $\exists_{x \in \mathcal{G}} Px \qquad \Longrightarrow \qquad \exists_{x \in \mathcal{G}} Qx$

---

(4) $\forall_{x \in \mathcal{G}} (Px \to Sx) \wedge \forall_{x \in \mathcal{G}} (Sx \to Qx)$ (1), (2)

(5) $\forall_{x \in \mathcal{G}} [(Px \to Sx) \wedge (Sx \to Qx)]$ (4), F5

(6) $\underset{x \in \mathcal{G}}{\forall}(Px \longrightarrow Qx)$ (5), E11

(7) $\underset{x \in \mathcal{G}}{\exists} Qx$ (3), (6), G12

Two possible alternatives of this alter step (7) and require an additional step (8) which features the use of a law from propositional logic.

(7) $\underset{x \in \mathcal{G}}{\exists} Px \longrightarrow \underset{x \in \mathcal{G}}{\exists} Qx$ (6), G10

(8) $\underset{x \in \mathcal{G}}{\exists} Qx$ (3), (7), E9

(7) $\underset{x \in \mathcal{G}}{\exists} [Px \wedge (Px \longrightarrow Qx)]$ (3), (6), G6

(8) $\underset{x \in \mathcal{G}}{\exists} Qx$ (7), E9

The following version embarks on a different strategy and arrives at its goal after some predicative transformations and finally by the use of propositional laws:

(4) $\underset{x \in \mathcal{G}}{\exists} Px \longrightarrow \underset{x \in \mathcal{G}}{\exists} Sx$ (1), G10

(5) $\underset{x \in \mathcal{G}}{\exists} Sx \longrightarrow \underset{x \in \mathcal{G}}{\exists} Qx$ (2), G10

(6) $\underset{x \in \mathcal{G}}{\exists} Px \longrightarrow \underset{x \in \mathcal{G}}{\exists} Qx$ (4), (5), E11

(7) $\underset{x \in \mathcal{G}}{\exists} Qx$ (3), (6), E9

The last proof to be introduced, arrives very quickly at its goal by applying powerful laws of predicate logic:

(4) $\underset{x \in \mathcal{G}}{\exists} Sx$ (1), (3), G12

(5) $\underset{x \in \mathcal{G}}{\exists} Qx$ (2), (4), G12

2.4.b (1) $\underset{x \in \mathcal{G}}{\forall}(Sx \rightarrow Qx)$

(2) $\underset{x \in \mathcal{G}}{\exists}(Px \wedge Sx)$ $\Longrightarrow$ $\underset{x \in \mathcal{G}}{\exists}(Px \wedge Qx)$

---

(3) $\underset{x \in \mathcal{G}}{\exists} [Px \wedge (Sx \wedge (Sx \longrightarrow Qx))]$ (1), (2), G6, A2

(4) $\underset{x \in \mathcal{G}}{\exists} (Px \wedge Qx)$ (3), E9

An alternative - and not very useful - solution arises from the transformation of the subjunction $\longrightarrow$ into $\wedge$ and $\vee$:

(4) $\underset{x \in \mathcal{G}}{\exists} [Px \wedge Sx \wedge (\neg Sx \vee Qx)]$	(3), D1
(5) $\underset{x \in \mathcal{G}}{\exists} [(Px \wedge Sx \wedge \neg Sx) \vee (Px \wedge Sx \wedge Qx)]$	(4), A3
(6) $\underset{x \in \mathcal{G}}{\exists} (Px \wedge Sx \wedge Qx)$	(5), A8, A6
(7) $\left(\underset{x \in \mathcal{G}}{\exists} (Px \wedge Qx)\right) \wedge \left(\underset{x \in \mathcal{G}}{\exists} Sx\right)$	(6), G4
(8) $\underset{x \in \mathcal{G}}{\exists} (Px \wedge Qx)$	(6), E4

2.4.c (1) $\underset{x \in \mathcal{G}}{\forall} (Tx \longrightarrow \neg Px)$

(2) $\underset{x \in \mathcal{G}}{\forall} Tx \vee \neg \underset{x \in \mathcal{G}}{\exists} Qx \qquad \Longrightarrow \qquad \underset{x \in \mathcal{G}}{\forall} (Qx \rightarrow \neg Px)$

---

(3) $\underset{x \in \mathcal{G}}{\forall} Tx \vee \underset{x \in \mathcal{G}}{\forall} \neg Qx$	(2), F2
(4) $\underset{x \in \mathcal{G}}{\forall} (Tx \vee \neg Qx)$	(3), G5
(5) $\underset{x \in \mathcal{G}}{\forall} (Qx \longrightarrow Tx)$	(4), D1
(6) $\underset{x \in \mathcal{G}}{\forall} (Qx \longrightarrow Tx) \wedge \underset{x \in \mathcal{G}}{\forall} (Tx \longrightarrow \neg Px)$	(1), (5)
(7) $\underset{x \in \mathcal{G}}{\forall} [(Qx \longrightarrow Tx) \wedge (Tx \longrightarrow \neg Px)]$	(6), F5
(8) $\underset{x \in \mathcal{G}}{\forall} (Qx \longrightarrow \neg Px)$	(7), E11

In a second solution, the expressions are first dissolved into single predicates:

(3) $\underset{x \in \mathcal{G}}{\exists} Qx \longrightarrow \underset{x \in \mathcal{G}}{\forall} Tx$	(2), D1
(4) $\underset{x \in \mathcal{G}}{\forall} Tx \longrightarrow \underset{x \in \mathcal{G}}{\forall} \neg Px$	(1), G9
(5) $\underset{x \in \mathcal{G}}{\exists} Qx \longrightarrow \underset{x \in \mathcal{G}}{\forall} \neg Px$	(3), (4), E11
(6) $\neg \underset{x \in \mathcal{G}}{\exists} Qx \vee \underset{x \in \mathcal{G}}{\forall} \neg Px$	(5), D1
(7) $\underset{x \in \mathcal{G}}{\forall} \neg Qx \vee \underset{x \in \mathcal{G}}{\forall} \neg Px$	(6), F2
(8) $\underset{x \in \mathcal{G}}{\forall} (\neg Qx \vee \neg Px)$	(7), G5
(9) $\underset{x \in \mathcal{G}}{\forall} (Qx \longrightarrow \neg Px)$	(8), D1

2.4.d (1) $\neg \underset{x \in \mathcal{G}}{\exists} (Px \longrightarrow \neg Qx)$

(2) $\neg \underset{x \in \mathcal{G}}{\exists} Qx \vee \underset{x \in \mathcal{G}}{\forall} Sx \qquad \Longrightarrow \underset{x \in \mathcal{G}}{\exists} Sx$

---

(3)	$\underset{x \in \mathcal{G}}{\forall} \neg(Px \longrightarrow \neg Qx)$	(1), F2
(4)	$\underset{x \in \mathcal{G}}{\forall} \neg(\neg Px \vee \neg Qx)$	(3), D1
(5)	$\underset{x \in \mathcal{G}}{\forall} (Px \wedge Qx)$	(4), A10
(6)	$\underset{x \in \mathcal{G}}{\forall} Px \wedge \underset{x \in \mathcal{G}}{\forall} Qx$	(5), F5
(7)	$\underset{x \in \mathcal{G}}{\forall} Qx$	(6), E4
(8)	$\underset{x \in \mathcal{G}}{\forall} \neg Qx \vee \underset{x \in \mathcal{G}}{\forall} Sx$	(2), F2
(9)	$\underset{x \in \mathcal{G}}{\forall} (\neg Qx \vee Sx)$	(8), G5
(10)	$\underset{x \in \mathcal{G}}{\forall} (Qx \longrightarrow Sx)$	(9), D1
(11)	$\underset{x \in \mathcal{G}}{\forall} Sx$	(7), (10), G11
(12)	$\underset{x \in \mathcal{G}}{\exists} Sx$	(11), G2

2.4.e (1) $\neg \underset{x \in \mathcal{G}}{\exists} Qx$

(2) $\underset{x \in \mathcal{G}}{\forall} (Tx \longrightarrow \neg(Qx \longrightarrow Px)) \qquad \Longrightarrow \underset{x \in \mathcal{G}}{\forall} \neg Tx$

---

(3)	$\underset{x \in \mathcal{G}}{\forall} \neg Qx$	(1), F2
(4)	$\underset{x \in \mathcal{G}}{\forall} (Qx \longrightarrow Px)$	(3), E7
(5)	$\underset{x \in \mathcal{G}}{\forall} ((Qx \longrightarrow Px) \longrightarrow \neg Tx)$	(2), D3
(6)	$\underset{x \in \mathcal{G}}{\forall} \neg Tx$	(4), (5), G11

2.4.f (1) $\underset{x \in \mathcal{G}}{\exists} Px \longleftrightarrow \underset{x \in \mathcal{G}}{\exists} Qx$

(2) $\underset{x \in \mathcal{G}}{\exists} Qx \qquad \Longrightarrow \underset{x \in \mathcal{G}}{\exists} \neg Px$

---

(3)	$\left(\underset{x \in \mathcal{G}}{\exists} Px \vee \underset{x \in \mathcal{G}}{\exists} Qx\right) \wedge \left(\neg \underset{x \in \mathcal{G}}{\exists} Px \vee \neg \underset{x \in \mathcal{G}}{\exists} Qx\right)$	(1), B10
(4)	$\neg \underset{x \in \mathcal{G}}{\exists} Px \vee \neg \underset{x \in \mathcal{G}}{\exists} Qx$	(3), E4

(5) $\forall_{x \in \mathcal{G}} \neg Px \vee \forall_{x \in \mathcal{G}} \neg Qx$ (4), F2

(6) $\forall_{x \in \mathcal{G}} (\neg Px \vee \neg Qx)$ (5), G5

(7) $\forall_{x \in \mathcal{G}} (Px \longrightarrow \neg Qx)$ (6), D1

(8) $\forall_{x \in \mathcal{G}} (Qx \longrightarrow \neg Px)$ (7), D3

(9) $\exists_{x \in \mathcal{G}} \neg Px$ (2), (8), G12

alternative solution:

(3) $\left(\exists_{x \in \mathcal{G}} Px \longrightarrow \exists_{x \in \mathcal{G}} Qx\right) \longrightarrow \neg \left(\exists_{x \in \mathcal{G}} Qx \longrightarrow \exists_{x \in \mathcal{G}} Px\right)$ (1), B12

(4) $\exists_{x \in \mathcal{G}} Px \longrightarrow \exists_{x \in \mathcal{G}} Qx$ (2), E8

(5) $\neg \left(\exists_{x \in \mathcal{G}} Qx \longrightarrow \exists_{x \in \mathcal{G}} Px\right)$ (3), (4), E9

(6) $\exists_{x \in \mathcal{G}} Qx \wedge \neg \exists_{x \in \mathcal{G}} Px$ (5), D2

(7) $\neg \exists_{x \in \mathcal{G}} Px$ (6), E4

(8) $\forall_{x \in \mathcal{G}} \neg Px$ (7), F2

(9) $\exists_{x \in \mathcal{G}} \neg Px$ (8), G2

2.4.g (1) $\forall_{x \in \mathcal{G}} \neg Px \vee \neg \exists_{x \in \mathcal{G}} Tx$

(2) $\forall_{x \in \mathcal{G}} (\neg Px \longrightarrow Qx)$ $\Longrightarrow \forall_{x \in \mathcal{G}} (Tx \longrightarrow Qx)$

---

(3) $\forall_{x \in \mathcal{G}} \neg Px \vee \forall_{x \in \mathcal{G}} \neg Tx$ (1), F2

(4) $\forall_{x \in \mathcal{G}} (\neg Px \vee \neg Tx)$ (3), G5

(5) $\forall_{x \in \mathcal{G}} (Tx \longrightarrow \neg Px)$ (4), D1

(6) $\forall_{x \in \mathcal{G}} (Tx \longrightarrow \neg Px) \wedge \forall_{x \in \mathcal{G}} (\neg Px \longrightarrow Qx)$ (2), (5)

(7) $\forall_{x \in \mathcal{G}} [(Tx \longrightarrow \neg Px) \wedge (\neg Px \longrightarrow Qx)]$ (6), F5

(8) $\forall_{x \in \mathcal{G}} (Tx \longrightarrow Qx)$ (7), E11

$\boxed{2.5}$ **2.5.a**

$$\underbrace{\exists_{x \in \mathcal{G}} Px}_{A(\underline{x})} \wedge \left(\underbrace{\exists_{x \in \mathcal{G}} Px}_{A(\underline{x})} \longrightarrow \underbrace{\exists_{x \in \mathcal{G}} Qx}_{B(\underline{x})} \right) \Longrightarrow \underbrace{\exists_{x \in \mathcal{G}} Qx}_{B(\underline{x})}$$

Because of E9 and RT1, this is a valid implication.

2.5.b Let $x_1, x_2 \in \mathcal{G}$,

$Px_1 \iff t\,, Px_2 \iff f$

and $\forall_{x \in \mathcal{G}} (Qx \iff f)$.

Therefore

$$\underbrace{\exists_{x \in \mathcal{G}} Px}_{t \text{ cause of } x_1} \wedge \underbrace{\exists_{x \in \mathcal{G}} (Px \to Qx)}_{t \text{ cause of } x_2} \Longrightarrow \underbrace{\exists_{x \in \mathcal{G}} Qx}_{f}$$

is not a valid implication.

The following **alternative solution** uses a standard method and can therefore treat both subtasks analogous. In both cases, the proof is built on the transformation of an implication to a subjunction which is to be proven to be tautological or not.

2.5.a

$$\exists_{x \in \mathcal{G}} Px \wedge \left(\exists_{x \in \mathcal{G}} Px \longrightarrow \exists_{x \in \mathcal{G}} Qx \right) \overset{?}{\Longrightarrow} \exists_{x \in \mathcal{G}} Qx$$

$$\begin{aligned}
& \exists_{x \in \mathcal{G}} Px \wedge \left(\exists_{x \in \mathcal{G}} Px \longrightarrow \exists_{x \in \mathcal{G}} Qx \right) \longrightarrow \exists_{x \in \mathcal{G}} Qx \\
\iff\ & \exists_{x \in \mathcal{G}} Px \wedge \left(\neg \exists_{x \in \mathcal{G}} Px \vee \exists_{x \in \mathcal{G}} Qx \right) \longrightarrow \exists_{x \in \mathcal{G}} Qx \\
\iff\ & \left[\left(\exists_{x \in \mathcal{G}} Px \wedge \neg \exists_{x \in \mathcal{G}} Px \right) \vee \left(\exists_{x \in \mathcal{G}} Px \wedge \exists_{x \in \mathcal{G}} Qx \right) \right] \longrightarrow \exists_{x \in \mathcal{G}} Qx \\
\iff\ & \left(\exists_{x \in \mathcal{G}} Px \wedge \exists_{x \in \mathcal{G}} Qx \right) \longrightarrow \exists_{x \in \mathcal{G}} Qx \\
\iff\ & \neg \left(\exists_{x \in \mathcal{G}} Px \wedge \exists_{x \in \mathcal{G}} Qx \right) \vee \exists_{x \in \mathcal{G}} Qx \\
\iff\ & \neg \exists_{x \in \mathcal{G}} Px \vee \neg \exists_{x \in \mathcal{G}} Qx \vee \exists_{x \in \mathcal{G}} Qx \\
\iff\ & \left(\neg \exists_{x \in \mathcal{G}} Qx \vee \exists_{x \in \mathcal{G}} Qx \right) \vee \neg \exists_{x \in \mathcal{G}} Px \\
\iff\ & t \vee \neg \exists_{x \in \mathcal{G}} Px \\
\iff\ & t
\end{aligned}$$

Hence, the above implication is valid.

2.5.b

$$\underset{x\,\in\,\mathcal{G}}{\exists} Px \wedge \underset{x\,\in\,\mathcal{G}}{\exists}(Px \to Qx) \overset{?}{\Longrightarrow} \underset{x\,\in\,\mathcal{G}}{\exists} Qx$$

$$\begin{aligned}
& \underset{x\,\in\,\mathcal{G}}{\exists} Px \wedge \underset{x\,\in\,\mathcal{G}}{\exists}(Px \to Qx) \longrightarrow \underset{x\,\in\,\mathcal{G}}{\exists} Qx \\
\Longleftrightarrow\quad & \underset{x\,\in\,\mathcal{G}}{\exists} Px \wedge \underset{x\,\in\,\mathcal{G}}{\exists}(\neg Px \vee Qx) \longrightarrow \underset{x\,\in\,\mathcal{G}}{\exists} Qx \\
\Longleftrightarrow\quad & \underset{x\,\in\,\mathcal{G}}{\exists} Px \wedge \left(\underset{x\,\in\,\mathcal{G}}{\exists} \neg Px \vee \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right) \longrightarrow \underset{x\,\in\,\mathcal{G}}{\exists} Qx \\
\Longleftrightarrow\quad & \left[\left(\underset{x\,\in\,\mathcal{G}}{\exists} Px \wedge \underset{x\,\in\,\mathcal{G}}{\exists} \neg Px\right) \vee \left(\underset{x\,\in\,\mathcal{G}}{\exists} Px \wedge \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right)\right] \longrightarrow \underset{x\,\in\,\mathcal{G}}{\exists} Qx \\
\Longleftrightarrow\quad & \neg\left[\left(\underset{x\,\in\,\mathcal{G}}{\exists} Px \wedge \neg \underset{x\,\in\,\mathcal{G}}{\forall} Px\right) \vee \left(\underset{x\,\in\,\mathcal{G}}{\exists} Px \wedge \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right)\right] \vee \underset{x\,\in\,\mathcal{G}}{\exists} Qx \\
\Longleftrightarrow\quad & \left[\left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Px \vee \underset{x\,\in\,\mathcal{G}}{\forall} Px\right) \wedge \left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Px \vee \neg \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right)\right] \vee \underset{x\,\in\,\mathcal{G}}{\exists} Qx \\
\Longleftrightarrow\quad & \left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Px \vee \underset{x\,\in\,\mathcal{G}}{\forall} Px \vee \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right) \wedge \left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Px \vee \neg \underset{x\,\in\,\mathcal{G}}{\exists} Qx \vee \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right) \\
\Longleftrightarrow\quad & \left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Px \vee \underset{x\,\in\,\mathcal{G}}{\forall} Px \vee \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right) \wedge \left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Px \vee t\right) \\
\Longleftrightarrow\quad & \left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Px \vee \underset{x\,\in\,\mathcal{G}}{\forall} Px \vee \underset{x\,\in\,\mathcal{G}}{\exists} Qx\right)
\end{aligned}$$

If $\neg \underset{x\,\in\,\mathcal{G}}{\exists} Qx$ and $\underset{x\,\in\,\mathcal{G}}{\exists} Px$ hold, but $\neg \underset{x\,\in\,\mathcal{G}}{\forall} Px$, then this propositional form has the truth value f. So it is **not** a tautology but a partly valid proposition. Therefore, the examined implication is also invalid.

$\boxed{\textbf{2.6}}$ **2.6.a** Basis step:

$$P(1) \iff \underbrace{\sum_{k=1}^{1} \frac{1}{k(k+1)}}_{=\frac{1}{2}} = \frac{1}{1+1} \iff t$$

Induction step:

$$\begin{aligned}
P(n+1) &\iff \sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n+1}{n+2} \\
&\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2} \\
&\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{(n+1)^2-1}{(n+1)(n+2)} \\
&\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n^2+2n}{(n+1)(n+2)} \\
&\iff \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1} \\
&\iff P(n)
\end{aligned}$$

With $P(n) \iff P(n+1)$ it also holds that $P(n) \Longrightarrow P(n+1)$ (RI1).

Conclusion:

FROM $P(1) \iff t$ AND $\underset{n \in \mathbf{N}}{\forall} P(n) \longrightarrow P(n+1) \iff t$

FOLLOWS THAT $\underset{n \in \mathbf{N}}{\forall} P(n) \iff t$.

2.6.b Basis step:

$$P(1) \iff \underbrace{\left(\sum_{i=1}^{1} i\right)^2}_{=1} = \underbrace{\sum_{i=1}^{1} (i)^3}_{=1} \iff t$$

Induction step:

$$
\begin{aligned}
P(n+1) \quad &\iff \quad \left(\sum_{i=1}^{n+1} i\right)^2 = \sum_{i=1}^{n+1} i^3 \\
&\iff \quad \left(\sum_{i=1}^{n} i + (n+1)\right)^2 = \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&\iff \quad \left(\sum_{i=1}^{n} i\right)^2 + 2(n+1)\sum_{i=1}^{n} i + (n+1)^2 = \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&\iff \quad \left(\sum_{i=1}^{n} i\right)^2 + 2(n+1)\frac{n(n+1)}{2} + (n+1)^2 = \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&\iff \quad \left(\sum_{i=1}^{n} i\right)^2 + n(n+1)^2 + (n+1)^2 = \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&\iff \quad \left(\sum_{i=1}^{n} i\right)^2 + (n+1)(n+1)^2 = \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&\iff \quad \left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3 \\
&\iff \quad P(n)
\end{aligned}
$$

With $P(n) \iff P(n+1)$ it also holds that $P(n) \Longrightarrow P(n+1)$ (RI1).

Conclusion:

FROM $P(1) \iff t$ AND $\underset{n\,\in\,\mathbf{N}}{\forall} P(n) \longrightarrow P(n+1) \iff t$

FOLLOWS THAT $\underset{n\,\in\,\mathbf{N}}{\forall} P(n) \iff t$.

2.6.c Basis step:

$$P(0) \iff \underbrace{\sum_{k=0}^{0} a^{0-k} \cdot b^k}_{=1} = \underbrace{\frac{a-b}{a-b}}_{=1} \iff t$$

Induction step:

$$\begin{aligned}
P(n+1) &\iff \sum_{k=0}^{n+1} a^{n+1-k}\cdot b^k = \frac{a^{n+2}-b^{n+2}}{a-b}\\
&\iff b^{n+1}+\sum_{k=0}^{n} a^{n+1-k}\cdot b^k = \frac{a^{n+2}-b^{n+2}}{a-b}\\
&\iff b^{n+1}+\sum_{k=0}^{n} a\cdot a^{n-k}\cdot b^k = \frac{a^{n+2}-b^{n+2}}{a-b}\\
&\iff b^{n+1}+a\sum_{k=0}^{n} a^{n-k}\cdot b^k = \frac{a^{n+2}-b^{n+2}}{a-b}\\
&\iff \frac{b^{n+1}}{a}+\sum_{k=0}^{n} a^{n-k}\cdot b^k = \frac{a^{n+2}-b^{n+2}}{a(a-b)}\\
&\iff \sum_{k=0}^{n} a^{n-k}\cdot b^k = \frac{a^{n+2}-b^{n+2}}{a(a-b)}-\frac{b^{n+1}(a-b)}{a(a-b)}\\
&\iff \sum_{k=0}^{n} a^{n-k}\cdot b^k = \frac{a^{n+2}-b^{n+2}-ab^{n+1}+b^{n+2}}{a(a-b)}\\
&\iff \sum_{k=0}^{n} a^{n-k}\cdot b^k = \frac{a^{n+1}-b^{n+1}}{a-b}\\
&\iff P(n)
\end{aligned}$$

With $P(n) \iff P(n+1)$ it also holds that $P(n) \Longrightarrow P(n+1)$ (RI1).

Conclusion:

FROM $P(0) \iff t$ AND $\underset{n\,\in\,\mathbf{N}_0}{\forall} P(n) \longrightarrow P(n+1) \iff t$

FOLLOWS THAT $\underset{n\,\in\,\mathbf{N}_0}{\forall} P(n) \iff t$.

2.6.d Basis step:

$$P(1) \iff 1+2\le 3 \iff t$$

Induction step:

$$\begin{aligned}
P(n+1) &\iff 1+2(n+1)\le 3^{n+1}\\
&\iff 1+2n+2\le 3\cdot 3^n\\
&\iff \underbrace{1+2n}_{a}+\underbrace{2}_{b}\le\underbrace{3^n}_{c}+\underbrace{2\cdot 3^n}_{d}
\end{aligned}$$

The implication $a\le c\wedge b\le d \Longrightarrow a+b\le c+d$ holds.

For this exercise, this means:

$$\underbrace{1+2n\le 3^n}_{P(n)}\wedge\underbrace{2\le 2\cdot 3^n}_{w}\Longrightarrow\underbrace{1+2n+2\le 3^n+2\cdot 3^n}_{P(n+1)}$$

With A6: $P(n)\Longrightarrow P(n+1)$

Conclusion:
FROM $P(1) \iff t$ AND $\underset{n \in \mathbf{N}}{\forall} P(n) \longrightarrow P(n+1) \iff t$
FOLLOWS THAT $\underset{n \in \mathbf{N}}{\forall} P(n) \iff t.$

2.6.e Basis step:

$$P0 \iff \left(1+x^{(2^0)}\right) = \frac{1-x^2}{1-x} \iff 1+x = 1+x \iff t$$

Induction step:

$$\begin{aligned}
P(n+1) \quad &\iff \quad \prod_{i=0}^{n+1}\left(1+x^{(2^i)}\right) = \frac{1-x^{(2^{n+2})}}{1-x} \\
&\iff \quad \left(1+x^{(2^{n+1})}\right)\prod_{i=0}^{n}\left(1+x^{(2^i)}\right) = \frac{1-x^{(2^{n+1}\cdot 2)}}{1-x} \\
&\iff \quad \prod_{i=0}^{n}\left(1+x^{(2^i)}\right) = \frac{1-\left(x^{(2^{n+1})}\right)^2}{(1-x)\left(1+x^{2^{(n+1)}}\right)} \\
&\iff \quad \prod_{i=0}^{n}\left(1+x^{2^i}\right) = \frac{1-x^{2^{n+1}}}{1-x} \\
&\iff \quad Pn
\end{aligned}$$

With $P(n) \iff P(n+1)$ it also holds that $P(n) \Longrightarrow P(n+1)$ (RI1).
Conclusion:
FROM $P0 \iff t$ AND $\underset{n \in \mathbb{N}_0}{\forall} Pn \longrightarrow P(n+1) \iff t$
FOLLOWS THAT $\underset{n \in \mathbb{N}_0}{\forall} Pn \iff t.$

2.6.f Basis step:

$$P2 \iff \sum_{k=1}^{1}(x_k - x_{k+1}) = x_1 - x_2 \iff x_1 - x_2 = x_1 - x_2 \iff t$$

Induction step:

$$\begin{aligned}
P(n+1) \quad &\iff \quad \sum_{k=1}^{n}(x_k - x_{k+1}) = x_1 - x_{n+1} \\
&\iff \quad \left[\sum_{k=1}^{n-1}(x_k - x_{k+1})\right] + x_n - x_{n+1} = x_1 - x_{n+1} \\
&\iff \quad \sum_{k=1}^{n-1}(x_k - x_{k+1}) = x_1 - x_n \\
&\iff \quad Pn
\end{aligned}$$

With $P(n) \iff P(n+1)$ it also holds that $P(n) \Longrightarrow P(n+1)$ (RI1).

Conclusion:

FROM $P2 \iff t$ AND $\underset{x \in \mathbb{N} \wedge n \geq 2}{\forall} Pn \longrightarrow P(n+1) \iff t$

FOLLOWS THAT $\underset{x \in \mathbb{N} \wedge n \geq 2}{\forall} Pn \iff t.$

$\boxed{\textbf{2.7}}$

(1)	$\underset{x \in \mathcal{M}}{\exists} \left[Px \wedge \underset{y \in \mathcal{M}}{\forall} (Py \longrightarrow (x=y))\right]$	$\Longrightarrow \underset{y \in \mathcal{M}}{\forall} \left[Py \longrightarrow \underset{x \in \mathcal{M}}{\exists} (x=y)\right]$
(2)	$\left(\underset{x \in \mathcal{M}}{\exists} Px\right) \wedge \underset{x \in \mathcal{M}}{\exists} \underset{y \in \mathcal{M}}{\forall} [Py \longrightarrow (x=y)]$	(1), G4
(3)	$\underset{x \in \mathcal{M}}{\exists} \underset{y \in \mathcal{M}}{\forall} [Py \longrightarrow (x=y)]$	(2), E4
(4)	$\underset{y \in \mathcal{M}}{\forall} \underset{x \in \mathcal{M}}{\exists} [Py \longrightarrow (x=y)]$	(3), auxiliary rule
(5)	$\underset{y \in \mathcal{M}}{\forall} \underset{x \in \mathcal{M}}{\exists} [\neg Py \vee (x=y)]$	(4), D1
(6)	$\underset{y \in \mathcal{M}}{\forall} \left[\neg Py \vee \underset{x \in \mathcal{M}}{\exists} (x=y)\right]$	(5), F6, A4
(7)	$\underset{y \in \mathcal{M}}{\forall} \left[Py \longrightarrow \underset{x \in \mathcal{M}}{\exists} (x=y)\right]$	(6), D1

$\boxed{\textbf{2.8}}$

$$P(k) \iff 2\prod_{i=1}^{k} \frac{2i}{2i+1} = \int_0^{\pi} \sin^{2k+1}(x)dx$$

Basis step:

$$P(1) \iff \underbrace{2\prod_{i=1}^{1} \frac{2i}{2i+1}}_{=2} = \underbrace{\int_0^{\pi} \sin(x)dx}_{=2} \iff t$$

Induction step:

$$\begin{aligned}
P(k+1) \quad &\iff \quad 2\prod_{i=1}^{k+1}\frac{2i}{2i+1} = \int_0^{\pi} \sin^{2k+3}(x)dx \\
&\iff \quad 2\prod_{i=1}^{k}\frac{2i}{2i+1}\cdot\frac{2k+2}{2k+3} = \frac{2k+2}{2k+3}\int_0^{\pi} \sin^{2k+1}(x)dx \\
&\iff \quad 2\prod_{i=1}^{k}\frac{2i}{2i+1} = \int_0^{\pi} \sin^{2k+1}(x)dx \\
&\iff \quad P(k)
\end{aligned}$$

With $P(k) \iff P(k+1)$ it also holds that $P(k) \Longrightarrow P(k+1)$ (RI1).

Conclusion:

FROM $P(1) \iff t$ AND $\underset{k\,\in\,\mathbf{N}}{\forall} P(k) \longrightarrow P(k+1) \iff t$

FOLLOWS THAT $\underset{k\,\in\,\mathbf{N}}{\forall} P(k) \iff t$.

3. Sets

$\boxed{\textbf{3.1}}$

$$\begin{aligned}
A &= \{\{\}, \{\emptyset\}, \{b\}, \{a,c\}, \{a,b,c\}\}\\
B &= \{\{c\}, \{a,b\}, \{a,c\}\}\\
M_1 &= G \setminus \overline{A} = G \cap A = A\\
M_2 &= \{\{\}, \{b\}\}\\
M_3 &= \{(\{\}, \{a,b\}), (\{\}, \{a,c\}), (\{\emptyset\}, \{c\}), (\{b\}, \{c\})\}\\
M_4 &= P(\{\{a,c\}\}) = \{\emptyset, \{\{a,c\}\}\}
\end{aligned}$$

$\boxed{\textbf{3.2}}$ **3.2.a** $E[A] \cap E[B] \subseteq (E[A] \cap E[C]) \cup (E[B] \cap \overline{E}[C])$

3.2.b $\overline{E[A] \bigtriangleup E[B]} = (E[A] \cup \overline{E}[B]) \cap (\overline{E}[A] \cup E[B])$

3.2.c $E[A] \subseteq E[B] \Longrightarrow E[A] \cup E[C] \subseteq E[B] \cup E[C]$

$\boxed{\textbf{3.3}}$ **3.3.a Premises:**

A is a tautology: $E[A] = G \Longrightarrow \overline{E}[A] = \emptyset$
$E[B] \neq \emptyset$

Conclusions:

$$\begin{aligned}
E[A \longrightarrow B] &\overset{?}{\neq} \emptyset \iff\\
E[\neg A \vee B] &\overset{?}{\neq} \emptyset \iff\\
\overline{E}[A] \cup E[B] &\overset{?}{\neq} \emptyset \iff\\
E[B] &\overset{?}{\neq} \emptyset
\end{aligned}$$

As $E[B] \neq \emptyset$ corresponds to the premise, this proposition describes a tautology.

3.3.b Same proof as above:

$\vdots$
$E[B] \overset{?}{=} G$
As $E[B] \neq \emptyset$ holds, but not necessarily $E[B] = G$, this proposition is not a tautology.

3.3.c Premises:

$E[B] \neq \emptyset \Longrightarrow \overline{E}[B] \neq G$

Conclusions:

$$\begin{array}{lcl} E\,[B \longrightarrow (B \wedge \neg B)] & \stackrel{?}{=} & G \\ E\,[\neg B \vee (B \wedge \neg B)] & \stackrel{?}{=} & G \\ E\,[(\neg B \vee B) \wedge (\neg B \vee \neg B)] & \stackrel{?}{=} & G \\ E[\neg B] & \stackrel{?}{=} & G \\ \overline{E}[B] & \stackrel{?}{=} & G \end{array}$$

From the premise, we know that this is not the case indeed. Therefore, this proposition is not a tautology.

$\boxed{3.4}$ **3.4.a Premises:**

B is a contradiction: $E[B] = \emptyset$

A is not a tautology: $E[A] \neq G \Longrightarrow \overline{E}[A] \neq \emptyset$

Conclusions:

$$\begin{array}{rlcl} & E[A \longrightarrow B] & \neq & \emptyset \Longleftrightarrow \\ & E[\neg A \vee B] & \neq & \emptyset \Longleftrightarrow \\ & \overline{E}[A] \cup E[B] & \neq & \emptyset \Longleftrightarrow \\ \Longrightarrow & \overline{E}[A] \cup E[B] & \neq \emptyset & \end{array}$$

Therefore, this proposition is true, whereas the statement on B is superfluous.

3.4.b B is not a contradiction: $E[B] \neq \emptyset \Longleftrightarrow \overline{E}[B] \neq G$

$(A \vee B) \longrightarrow A$ is a tautology:

$$\begin{array}{ll} E[(A \vee B) \longrightarrow A] & = G \Longleftrightarrow \\ E[\neg(A \vee B) \vee A] & = G \Longleftrightarrow \\ E[(\neg A \wedge \neg B) \vee A] & = G \Longleftrightarrow \\ E[(\neg A \vee A) \wedge (\neg B \vee A)] & = G \Longleftrightarrow \\ E[(\neg B \vee A)] & = G \Longleftrightarrow \\ \overline{E}[B] \cup E[A] & = G \Longleftrightarrow \end{array}$$

As $\overline{E}[B] \neq G$ holds, this proposition is not fulfilled for every possible A. For $E[A] = \emptyset$ for example, it is surely not fulfilled.

$\boxed{\textbf{3.5}}$

$$(A \longrightarrow B) \longleftrightarrow C \Longrightarrow A$$

$$\begin{aligned}
& \mathbf{E}[(A \longrightarrow B) \longleftrightarrow C] \subseteq \mathbf{E}[A] \\
\Longleftrightarrow\ & \mathbf{E}[\neg(A \longrightarrow B) \not\longleftrightarrow C] \subseteq \mathbf{E}[A] \\
\Longleftrightarrow\ & \mathbf{E}[\neg(A \longrightarrow B)] \triangle \mathbf{E}[C] \subseteq \mathbf{E}[A] \\
\Longleftrightarrow\ & (\mathbf{E}[A] \cap \overline{\mathbf{E}}[B]) \triangle \mathbf{E}[C] \subseteq \mathbf{E}[A] \\
\Longleftrightarrow\ & \{(1,0,0),(1,1,0)\} \triangle \mathbf{E}[C] \subseteq \mathbf{E}[A] \\
\Longleftrightarrow\ & \{(0,1,0),(1,0,0)\} \subseteq \{(0,1,0),(1,0,0),(1,1,0)\} \\
\Longleftrightarrow\ & t
\end{aligned}$$

$\boxed{\textbf{3.6}}$

$$\begin{aligned}
A &\iff (\neg a \wedge \neg b) \vee (a \wedge b) \\
B &\iff (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)
\end{aligned}$$

$\boxed{\textbf{3.7}}$ **3.7.a**

$$\begin{array}{rrcll}
& (A\triangle C)\setminus D & \subseteq & \overline{\left[\overline{(A\setminus D)}\setminus(C \cap \overline{D})\right] \cap B} & \\
\Longleftrightarrow & [(A \cup C) \cap \overline{(A \cap C)}]\setminus D & \subseteq & \overline{\overline{(A\setminus D)}\setminus(C \cap \overline{D})} \cup \overline{B} & \text{J9, H10} \\
\Longleftrightarrow & (A \cup C) \cap \overline{(A \cap C)} \cap \overline{D} & \subseteq & \overline{\overline{\overline{(A \cap \overline{D})} \cap \overline{(C \cap \overline{D})}}} \cup \overline{B} & \text{J3} \\
\Longleftrightarrow & (A \cup C) \cap \overline{(A \cap C)} \cap \overline{D} & \subseteq & (A \cap \overline{D}) \cup (C \cap \overline{D}) \cup \overline{B} & \text{H10, H9} \\
\Longleftrightarrow & (A \cup C) \cap \overline{(A \cap C)} \cap \overline{D} & \subseteq & [(A \cup C) \cap \overline{D}] \cup \overline{B} & \text{H3} \\
\Longleftrightarrow & (A \cup C) \cap \overline{(A \cap C)} \cap \overline{D} \cap B & \subseteq & (A \cup C) \cap \overline{D} & \text{J32} \\
\Longleftrightarrow & & t & & \text{J21}
\end{array}$$

3.7.b

$$
\begin{array}{lrcll}
 & \overline{(A\backslash B)}\cap[A\backslash(B\cap C)] & \subseteq & A\backslash C & \\
\iff & \overline{(A\cap\overline{B})}\cap A\cap\overline{(B\cap C)} & \subseteq & A\cap\overline{C} & \text{J3} \\
\iff & \overline{\overline{(A\cap\overline{B})}\cap A\cap\overline{(B\cap C)}}\cup(A\cap\overline{C}) & = & G & \text{J29} \\
\iff & (A\cap\overline{B})\cup\overline{A}\cup(B\cap C)\cup(A\cap\overline{C}) & = & G & \text{H10} \\
\iff & \overline{B}\cup\overline{A}\cup(B\cap C)\cup(A\cap\overline{C}) & = & G & \text{H3, H8, H6} \\
\iff & \overline{B}\cup(B\cap C)\cup\overline{A}\cup(A\cap\overline{C}) & = & G & \text{H1} \\
\iff & \overline{B}\cup C\cup\overline{A}\cup\overline{C} & = & G & \text{H3, H8, H6} \\
\iff & G & = & G & \text{H8} \\
\iff & & t & & \text{J1}
\end{array}
$$

3.7.c

$$
\begin{array}{lrcll}
 & (A\triangle B) & \subseteq & A & \\
\iff & [(A\cap\overline{B})\cup(\overline{A}\cap B)] & \subseteq & A & \text{J11} \\
\iff & [(A\cap\overline{B})\cup(\overline{A}\cap B)] & \subseteq & A\cup A & \text{H4} \\
\iff & [(A\cap\overline{B})\cup(\overline{A}\cap B)]\cap\overline{A} & \subseteq & A & \text{J32} \\
\iff & \overline{A}\cap B & \subseteq & A & \text{H3, H8, H6} \\
\iff & B & \subseteq & A\cup A & \text{J32} \\
\iff & B & \subseteq & A & \text{H4}
\end{array}
$$

3.8 **3.8.a** This is law J35.

3.8.b Implication b) is not valid.
Counter example:
With $A = B = C = \emptyset$, the left side becomes $[\emptyset \triangle \mathcal{G} \cup \emptyset = \mathcal{G}] \wedge [\emptyset \triangle \mathcal{G} \neq \emptyset] \iff t$.
The right side becomes $\emptyset \cap \emptyset \neq \emptyset \iff f$.
Thus, we obtain $t \Longrightarrow f$, which is not a valid implication.

3.8.c (1) $A \subseteq B$

(2) $\overline{B} \cup A = G \qquad \Longrightarrow B \setminus A = \emptyset$

---

(3) $B \cap \overline{A} = \emptyset$ (2), J29

(4) $B \setminus A = \emptyset$ (3), J3

3.8.d (1) $G \triangle A = \emptyset$

(2) $B \subseteq C \qquad \Longrightarrow B \subseteq (A \cap C)$

(3) $\overline{A} = \emptyset$	(1), J15
(4) $A = G$	(3), J31
(5) $C = A \cap C$	(4), H6
(6) $C \subseteq A \cap C$	(5), J28, E4
(7) $B \subseteq A \cap C$	(2), (6), J34

3.8.e (1) $\overline{A} \cup \overline{B} = G$

(2) $\overline{A} \subseteq \overline{C} \qquad \Longrightarrow C \setminus \overline{B} = \emptyset$

(3) $B \subseteq \overline{A}$	(1), J29
(4) $B \subseteq \overline{C}$	(2), (3), J34
(5) $B \cap C = \emptyset$	(4), J29
(6) $C \setminus \overline{B} = \emptyset$	(5), J3

3.8.f (1) $A \triangle B = \emptyset \qquad \Longrightarrow \overline{B} \subseteq \overline{A} \wedge B \setminus A = \emptyset$

(2) $A = B$	(1), J25
(3) $A \subseteq B$	(2), J28, E4
(4) $\overline{B} \subseteq \overline{A}$	(3), J31
(5) $B \cap \overline{A} = \emptyset$	(2), H8
(6) $B \setminus A = \emptyset$	(5), J3

$\boxed{3.9}$ **3.9.a** $\underset{x \in \mathcal{G}}{\forall}(Px \longrightarrow Sx) \wedge \underset{x \in \mathcal{G}}{\forall}(Qx \longrightarrow Tx) \Longrightarrow \underset{x \in \mathcal{G}}{\forall}[(Px \vee Qx) \longrightarrow (Sx \vee Tx)]$

3.9.b $\underset{x \in \mathcal{G}}{\forall}[\neg Px \vee (Qx \wedge Sx)] \wedge \underset{x \in \mathcal{G}}{\exists}[\neg(Px \vee Qx) \longleftrightarrow Sx] \Longrightarrow \underset{x \in \mathcal{G}}{\exists}(Qx \wedge Sx)$

3.9.c $\mathcal{A} \subseteq \mathcal{B} \wedge \bar{\mathcal{B}} \cup \mathcal{A} = \mathcal{G} \Longrightarrow \underbrace{\mathcal{B} \backslash \mathcal{A} = \emptyset}_{\mathcal{B} \cap \bar{\mathcal{A}} = \emptyset}$

$$\underset{x \in \mathcal{G}}{\forall}(Px \longrightarrow Qx) \wedge \underset{x \in \mathcal{G}}{\forall}(\neg Qx \vee Px) \Longrightarrow \neg \underset{x \in \mathcal{G}}{\exists}(Qx \wedge \neg Px)$$

3.9.d $\underbrace{\mathcal{G}\triangle\mathcal{A}=\emptyset}_{\bar{\mathcal{A}}=\emptyset}\wedge\mathcal{B}\subseteq\mathcal{C}\Longrightarrow\mathcal{B}\subseteq(\mathcal{C}\cap\mathcal{A})$

$$\underset{x\,\in\,\mathcal{G}}{\forall}Px\wedge\underset{x\,\in\,\mathcal{G}}{\forall}(Qx\longrightarrow Sx)\Longrightarrow\underset{x\,\in\,\mathcal{G}}{\forall}[Qx\longrightarrow(Sx\wedge Px)]$$

3.9.e $(\bar{\mathcal{A}}\cup\bar{\mathcal{B}}=\mathcal{G})\wedge(\bar{\mathcal{A}}\subseteq\bar{\mathcal{C}})\Longrightarrow\underbrace{\mathcal{C}\backslash\bar{\mathcal{B}}=\emptyset}_{\mathcal{C}\cap\mathcal{B}=\emptyset}$

$$\left(\underset{x\,\in\,\mathcal{G}}{\forall}(\neg Px\vee\neg Qx)\right)\wedge\underset{x\,\in\,\mathcal{G}}{\forall}(\neg Px\longrightarrow\neg Sx)\Longrightarrow\neg\underset{x\,\in\,\mathcal{G}}{\exists}Sx\wedge Qx$$

alternative solution:

$$\underset{x\,\in\,\mathcal{G}}{\forall}(Qx\longrightarrow\neg Px)\wedge\underset{x\,\in\,\mathcal{G}}{\forall}(\neg Px\longrightarrow\neg Sx)\Longrightarrow\underbrace{\underset{x\,\in\,\mathcal{G}}{\forall}(Qx\longrightarrow\neg Sx)}_{\underset{x\,\in\,\mathcal{G}}{\forall}(\neg Qx\vee\neg Sx)}$$

3.9.f $\underbrace{(\mathcal{A}\triangle\mathcal{B})=\emptyset}_{\mathcal{A}=\mathcal{B}}\Longrightarrow(\bar{\mathcal{B}}\subseteq\bar{\mathcal{A}})\wedge(\underbrace{\mathcal{B}\backslash\mathcal{A}=\emptyset}_{\mathcal{B}\cap\bar{\mathcal{A}}=\emptyset})$

$$\left(\underset{x\,\in\,\mathcal{G}}{\forall}(Px\longleftrightarrow Qx)\right)\Longrightarrow\left(\underset{x\,\in\,\mathcal{G}}{\forall}(\neg Qx\longrightarrow\neg Px)\right)\wedge\left(\neg\underset{x\,\in\,\mathcal{G}}{\exists}(Qx\wedge\neg Px)\right)$$

alternative solution:

$$\underset{x\,\in\,\mathcal{G}}{\forall}(Px\longleftrightarrow Qx)\Longrightarrow\left(\underset{x\,\in\,\mathcal{G}}{\forall}(Px\longrightarrow Qx)\right)\wedge\underbrace{\left(\underset{x\,\in\,\mathcal{G}}{\forall}(\neg Qx\vee Px)\right)}_{\underset{x\,\in\,\mathcal{G}}{\forall}(Qx\longrightarrow Px)}$$

$\boxed{\textbf{3.10}}$ **3.10.a** $(A\subseteq C)\wedge(C\subseteq B)\wedge(A\neq\emptyset)\quad\Longrightarrow B\neq\emptyset$

3.10.b $(C\subseteq B)\wedge(A\cap C\neq\emptyset)\quad\Longrightarrow A\cap B\neq\emptyset$

3.10.c $(D\subseteq\overline{A})\wedge(D=G\vee B=\emptyset)\quad\Longrightarrow B\subseteq\overline{A}$

3.10.d $(A\cap B=G)\wedge(B=\emptyset\vee C=G)\quad\Longrightarrow C\neq\emptyset$

3.10.e $(B=\emptyset)\wedge(D\subseteq B\cap\overline{A})\quad\Longrightarrow\overline{D}=G$

3.10.f $(A\neq\emptyset\longleftrightarrow B\neq\emptyset)\wedge(B\neq\emptyset)\quad\Longrightarrow A\neq G$

3.10.g $(A = \emptyset \vee D = \emptyset) \wedge (\overline{A} \subseteq B) \qquad \Longrightarrow D \subseteq B$

$\boxed{\textbf{3.11}}$ **3.11.a**

(1^P)	$\underset{x\,\in\,G}{\forall}(Px \wedge Rx) \vee \underset{x\,\in\,G}{\forall} Qx$	$\Longrightarrow \underset{x\,\in\,G}{\exists}(Px \vee Qx)$
(2^P)	$[\underset{x\,\in\,G}{\forall} Px \wedge \underset{x\,\in\,G}{\forall} Rx] \vee \underset{x\,\in\,G}{\forall} Qx$	(1^P), F5
(3^P)	$[\underset{x\,\in\,G}{\forall} Px \vee \underset{x\,\in\,G}{\forall} Qx] \wedge [\underset{x\,\in\,G}{\forall} Rx \vee \underset{x\,\in\,G}{\forall} Qx]$	(2^P), A3
(4^P)	$\underset{x\,\in\,G}{\forall} Px \vee \underset{x\,\in\,G}{\forall} Qx$	(3^P), E4
(5^P)	$\underset{x\,\in\,G}{\forall}(Px \vee Qx)$	(4^P), G5
(6^P)	$\underset{x\,\in\,G}{\exists}(Px \vee Qx)$	(5^P), G1

3.11.b $(P \cap R = G) \vee (Q = G) \Longrightarrow P \cup Q \neq \emptyset$

3.11.c

(1^M)	$(P \cap R = G) \vee (Q = G)$	$\Longrightarrow P \cup Q \neq \emptyset$
(2^M)	$(P \cap R \subseteq G) \vee (Q = G)$	(1^M), J28, A3, E4
(3^M)	$(P \subseteq P) \vee (Q = G)$	J20, E5
(4^M)	$((P \cap R) \cup P \subseteq G \cup P) \vee (Q = G)$	$(2^M, 3^M)$, A3, J35, E15
(5^M)	$(G \cup P \subseteq (P \cap R) \cup P) \vee (Q = G)$	$(1^M, 3^M)$, J28, E4, J35
(6^M)	$((P \cap R) \cup P = G \cup P) \vee (Q = G)$	$(4^M, 5^M)$ A3, J28
(7^M)	$(P = G) \vee (Q = G)$	(6^M), H5, H7
(8^M)	$(P \cup Q = G \cup Q) \vee (Q \cup P = G \cup P)$	(7^M), J28, J20, J35
(9^M)	$(P \cup Q = G) \vee (P \cup Q = G)$	(8^M), H7
(10^M)	$(P \cup Q = G)$	(9^M), A4
(11^M)	$(P \cup Q \neq \emptyset)$	(10^M)

3.11.d The pairs of lines (4^P) and (7^M), (5^P) and (10^M), (6^P) and (11^M) can each be transformed into each other (cf. manuscript, sets (10)). Idea and method leading to the solution are identical for each pair.
Note: All laws of the predicate logic (featuring predicates with just one subject variable) can also be formulated as laws with sets.

3.12

$$A = B \Longrightarrow A \cup C = B \cup C \text{ IFF } \underset{x \in G}{\forall}(Px \longleftrightarrow Qx) \Longrightarrow \underset{x \in G}{\forall}[(Px \vee Sx) \longleftrightarrow (Qx \vee Sx)]$$

1)	$A = B$	$\Longrightarrow A \cup C = B \cup C$
2)	$(A \subseteq B) \wedge (B \subseteq A)$	$1, \mathrm{J28}$
3)	$(A \subseteq B) \wedge (C \subseteq C)$	$2, \mathrm{E4}, \mathrm{A6}, \mathrm{J20}$
4)	$(B \subseteq A) \wedge (C \subseteq C)$	$2, \mathrm{E4}, \mathrm{A6}, \mathrm{J20}$
5)	$A \cup C \subseteq B \cup C$	$3, \mathrm{J35}$
6)	$B \cup C \subseteq A \cup C$	$4, \mathrm{J35}$
7)	$A \cup C = B \cup C$	$5, 6, \mathrm{J28}$

1)	$\underset{x \in G}{\forall}(Px \longleftrightarrow Qx)$	$\Longrightarrow \underset{x \in G}{\forall}[(Px \vee Sx) \longleftrightarrow (Qx \vee Sx)]$
2)	$\underset{x \in G}{\forall}(Px \longrightarrow Qx) \wedge (Qx \longrightarrow Px)$	$1, \mathrm{C11}$
3)	$\underset{x \in G}{\forall}(Px \longrightarrow Qx)$	$2, \mathrm{F5}, \mathrm{E4}$
4)	$\underset{x \in G}{\forall}(Qx \longrightarrow Px)$	$2, \mathrm{F5}, \mathrm{E4}$
5)	$\underset{x \in G}{\forall}[(Px \vee Sx) \longrightarrow (Qx \vee Sx)]$	$3, \mathrm{E15}$
6)	$\underset{x \in G}{\forall}[(Qx \vee Sx) \longrightarrow (Px \vee Sx)]$	$4, \mathrm{E15}$
7)	$\underset{x \in G}{\forall}[(Px \vee Sx) \longleftrightarrow (Qx \vee Sx)]$	$5, 6, \mathrm{F5}, \mathrm{C11}$

3.13

$$V_1 \iff \neg \underset{x \in G}{\exists}(Qx \wedge \neg Sx) \iff B \cap \overline{C} = \emptyset$$

$$V_2 \iff \underset{x \in G}{\forall}(\neg Sx \not\longleftrightarrow Tx) \vee Tx \iff \big(\overline{C} \triangle D\big) \cup D = G$$

$$V_3 \iff \neg \underset{x \in G}{\exists}(Px \wedge \neg Qx \wedge \neg Sx) \iff A \cap \overline{B} \cap \overline{C} = \emptyset$$

$$S \iff \underset{x \in G}{\forall}(Px \vee Qx \longrightarrow Tx) \iff A \cup B \subseteq D$$

Deductive proof scheme:

1)	$A \cap \overline{B} \cap \overline{C} = \emptyset$	
2)	$B \cap \overline{C} = \emptyset$	
3)	$\big(\overline{C} \triangle D\big) \cup D = G$	$\Longrightarrow A \cup B \subseteq D$
4)	$\big(\overline{C} \cap \overline{D}\big) \cup (C \cap D) \cup D = G$	3, J11
5)	$\big(\overline{C} \cap \overline{D}\big) \cup D = G$	4, H5
6)	$\big(\overline{C} \cup D\big) \cap \big(\overline{D} \cup D\big) = G$	5, H3
7)	$C \subseteq D$	6, H6, H8, J29
8)	$B \subseteq C$	2, J29
9)	$A \cap \overline{B} \subseteq C$	1, J29
10)	$A \cap \overline{B} \cup B \subseteq C \cup C$	8, 9, J35
11)	$A \cup B \subseteq C$	10, H3, H8, H6
12)	$A \cup B \subseteq D$	11, 7, J34

3.14 **3.14.a** $P(M) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
$|P(M)| = 2^{|M|} = 2^3 = 8$

3.14.b $|P(M)^2| = |P(M)|^2 = 8^2 = 64$

3.14.c

$$\begin{aligned} N = \; & \{(x,y) | ((x,y) \in P(M)^2) \wedge (x \subset y)\} = \\ & \{(\{\}, \{1\}), (\{\}, \{2\}), (\{\}, \{3\}), \\ & (\{\}, \{1,2\}), (\{\}, \{1,3\}), (\{\}, \{2,3\}), (\{\}, \{1,2,3\}), \\ & (\{1\}, \{1,2\}), (\{1\}, \{1,3\}), (\{1\}, \{1,2,3\}), \\ & (\{2\}, \{1,2\}), (\{2\}, \{2,3\}), (\{2\}, \{1,2,3\}) \\ & (\{3\}, \{1,3\}), (\{3\}, \{2,3\}), (\{3\}, \{1,2,3\}) \\ & (\{1,2\}, \{1,2,3\}), (\{1,3\}, \{1,2,3\}), (\{2,3\}, \{1,2,3\})\} \end{aligned}$$

3.15 **3.15.a** It is to be shown, that

$N \subseteq M \Longrightarrow P(N) \subseteq P(M)$

as well as

$N \subseteq M \Longleftarrow P(N) \subseteq P(M)$

holds.

"$\Longrightarrow$":

(1) $N \subseteq M$	$\Longrightarrow P(N) \subseteq P(M)$
(2) $\underset{K \in P(N)}{\forall} K \subseteq N$	Def. power set
(3) $\underset{K \in P(N)}{\forall} N \subseteq M$	(1), A4

(4) $\underset{K \in P(N)}{\forall} (K \subseteq N) \wedge (N \subseteq M) \wedge (K \subseteq M)$	(2), (3), F5, J34, RI3
(5) $\underset{K \in P(N)}{\forall} K \subseteq M$	(4), F5, E4
(6) $\underset{K \in P(N)}{\forall} K \in P(M)$	(5), Def. power set
(7) $P(N) \subseteq P(M)$	(6), Def. $\subseteq$

Particular to this solution is to relate the universal quantor to the power set of N.

"$\Longleftarrow$":
Assume, G_p is the universal set of $P(M)$ and $P(N)$.

(1) $P(N) \subseteq P(M)$	$\Longrightarrow N \subseteq M$
(2) $N \subseteq N$	J20
(3) $N \in P(N)$	(2), Def. power set
(4) $\underset{K \in G_p}{\forall} K \in P(N) \longrightarrow K \in P(M)$	(1), Def. $\subseteq$
(5) $N \in P(N) \longrightarrow N \in P(M)$	(4), G1
(6) $N \in P(M)$	(3), (5), E9
(7) $N \subseteq M$	(6), Def. power set

3.15.b For any K, it is to be shown that: $K \in P(M) \cup P(N) \Longrightarrow K \in P(M \cup N)$. Assume, the universal set of M, N and K is G.

$$
\begin{aligned}
& K \in P(M) \cup P(N) \\
\Longleftrightarrow\quad & K \in P(M) \vee K \in P(N) \\
\Longleftrightarrow\quad & K \subseteq M \vee K \subseteq N \\
\Longleftrightarrow\quad & \left(\underset{x \in G}{\forall} x \in K \longrightarrow x \in M\right) \vee \left(\underset{x \in G}{\forall} x \in K \longrightarrow x \in N\right) \\
\overset{\text{G5}}{\Longrightarrow}\quad & \underset{x \in G}{\forall} [(x \in K \longrightarrow x \in M) \vee (x \in k \longrightarrow x \in N)] \\
\overset{\text{D8}}{\Longleftrightarrow}\quad & \underset{x \in G}{\forall} [x \in K \longrightarrow (x \in M \vee x \in N)] \\
\Longleftrightarrow\quad & \underset{x \in G}{\forall} [x \in K \longrightarrow x \in (M \cup N)] \\
\Longleftrightarrow\quad & K \subseteq M \cup N \\
\Longleftrightarrow\quad & K \in P(M \cup N)
\end{aligned}
$$

Since the implication is valid for all $K \in G$,
$P(M) \cup P(N) \subseteq P(M \cup N)$ holds as well.

3.15.c Here it is to examine, if the direction of the implication can be reversed in the example above. That this is not possible, can easily be seen, as law G5 has just been used. But it can also be proven by giving a counter-example:

sets $M = \{1\}$
and $N = \{2\}$
have the power sets $P(M) = \{\emptyset, \{1\}\}$
and $P(N) = \{\emptyset, \{2\}\}$.
This yields $P(M \cup N) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
and $P(M) \cup P(N) = \{\emptyset, \{1\}, \{2\}\}$.

The relation $P(M \cup N) \subseteq P(M) \cup P(N)$ is therefore not valid.

3.15.d According to law J28 it is to be shown that
$P(M) \cup P(N) \subseteq P(M \cup N)$
as well as
$P(M \cup N) \subseteq P(M) \cup P(N)$
holds.

In the predecessing subtask, it was shown however, that the second relation is not valid. Therefore, the proposition to be proven here is false.

3.15.e For an arbitrary K it has to be shown that: $K \in P(M) \cap P(N) \iff K \in P(M \cap N)$
Assume, the universal set of M, N and K is G.

$$
\begin{aligned}
& K \in P(M) \cap P(N) \\
\iff \quad & K \in P(M) \wedge K \in P(N) \\
\iff \quad & K \subseteq M \wedge K \subseteq N \\
\iff \quad & \left(\underset{x \in G}{\forall} x \in K \longrightarrow x \in M\right) \wedge \left(\underset{x \in G}{\forall} x \in K \longrightarrow x \in N\right) \\
\overset{\text{F5}}{\iff} \quad & \underset{x \in G}{\forall} \left[(x \in K \longrightarrow x \in M) \wedge (x \in k \longrightarrow x \in N)\right] \\
\overset{\text{D7}}{\iff} \quad & \underset{x \in G}{\forall} \left[x \in K \longrightarrow (x \in M \wedge x \in N)\right] \\
\iff \quad & \underset{x \in G}{\forall} \left[x \in K \longrightarrow x \in (M \cap N)\right] \\
\iff \quad & K \subseteq M \cap N \\
\iff \quad & K \in P(M \cap N)
\end{aligned}
$$

Since this equivalency is valid for all $K \in G$,
$P(M) \cap P(N) = P(M \cap N)$ holds as well.

4. Relations

4.1 **4.1.a**

$$
\begin{aligned}
& R \subseteq S^{-1} \cup (T \setminus S) \\
\iff & \forall_{(x,y) \in A^2} xRy \longrightarrow ySx \vee x(T \setminus S)y \\
\iff & \forall_{(x,y) \in A^2} xRy \longrightarrow ySx \vee x(T \cap \overline{S})y \\
\iff & \forall_{(x,y) \in A^2} xRy \longrightarrow ySx \vee xTy \wedge \neg xSy
\end{aligned}
$$

4.1.b

$$
\begin{aligned}
& R \setminus (S^2 \setminus T) \subseteq R \triangle \overline{S} \\
\iff & \forall_{(x,y) \in A^2} x\,[R \setminus (S^2 \setminus T)]\,y \longrightarrow x(R \triangle \overline{S})y \\
\iff & \forall_{(x,y) \in A^2} x\left[R \cap \overline{(S^2 \cap \overline{T})}\right] y \longrightarrow (xRy \not\longleftrightarrow x\overline{S}y) \\
\iff & \forall_{(x,y) \in A^2} xRy \wedge x(\overline{S^2} \cup T)y \longrightarrow (xRy \longleftrightarrow xSy) \\
\iff & \forall_{(x,y) \in A^2} xRy \wedge \left[\neg\left(\exists_{z \in A} xSz \wedge zSy\right) \vee xTy\right] \longrightarrow (xRy \longleftrightarrow xSy)
\end{aligned}
$$

4.2 **4.2.a**

$$
\begin{aligned}
& \forall_{(x,y) \in A^2} xRy \longleftrightarrow \left(\exists_{z \in A}(xSz \wedge zSy)\right) \wedge \neg xTy \\
\iff & \forall_{(x,y) \in A^2} xRy \longleftrightarrow xS^2y \wedge x\overline{T}y \\
\iff & \forall_{(x,y) \in A^2} xRy \longleftrightarrow x(S^2 \cap \overline{T})y \\
\iff & \forall_{(x,y) \in A^2} xRy \longleftrightarrow x(S^2 \setminus T)y \\
\iff & R = S^2 \setminus T, \qquad R,\, S,\, T,\, \subseteq A^2
\end{aligned}
$$

4.2.b

$$
\begin{aligned}
& \exists_{(x,y) \in A^2} \left[(xRy \vee xSy) \not\longleftrightarrow \exists_{z \in A}(xRz \wedge zSy)\right] \\
\iff & \exists_{(x,y) \in A^2} [x(R \cup S)y \not\longleftrightarrow x(R \circ S)y\,] \\
\iff & R \cup S \neq R \circ S, \qquad R,\, S,\, T,\, \subseteq A^2
\end{aligned}
$$

$\boxed{\textbf{4.3}}$ **4.3.a** x is initial $\iff \Gamma^-(x) = \emptyset$

$\Gamma^-(a) = \{d\}, \quad \Gamma^-(b) = \{b, e\}, \quad \Gamma^-(c) = \emptyset, \quad \Gamma^-(d) = \{a\}, \quad \Gamma^-(e) = \{c, d\}$

Node c is initial.

x is terminal $\iff \Gamma^+(x) = \emptyset$

$\Gamma^+(a) = \{d\}, \quad \Gamma^+(b) = \{b\}, \quad \Gamma^+(c) = \{e\}, \quad \Gamma^+(d) = \{a, e\}, \quad \Gamma^+(e) = \{b\}$

No node is terminal.

4.3.b x is reconvergence point $\iff d^-(x) \geq 2$

$d^-(a) = 1, \quad d^-(b) = 2, \quad d^-(c) = 0, \quad d^-(d) = 1, \quad d^-(e) = 2$

Nodes b and e are reconvergence points.

x is branch point $\iff d^+(x) \geq 2$

$d^+(a) = 1, \quad d^+(b) = 1, \quad d^+(c) = 1, \quad d^+(d) = 2, \quad d^+(e) = 1$

Node d is branch point.

4.3.c $R \not\subseteq \overline{I}$, since $(b, b) \in R$.

Thus, R has loops.

$\boxed{\textbf{4.4}}$ K3

$$\begin{array}{rcl} A \times (B \cap C) & = & (A \times B) \cap (A \times C) \\ x \in A \wedge (y \in (B \cap C)) & \iff & [(x, y) \in (A \times B)] \wedge [(x, y) \in (A \times C)] \\ x \in A \wedge y \in B \wedge y \in C & \iff & (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) \\ & \iff & x \in A \wedge y \in B \wedge y \in C \end{array}$$

K4

$$\begin{array}{rcl} (A \cup B) \times C & = & (A \times C) \cup (B \times C) \\ x \in (A \cup B) \wedge y \in C & \iff & [(x, y) \in (A \times C)] \vee [(x, y) \in (B \times C)]) \\ (x \in A \vee x \in B) \wedge y \in C & \iff & (x \in A \wedge y \in C) \vee (x \in B \wedge y \in C) \\ & \iff & (x \in A \vee x \in B) \wedge y \in C \end{array}$$

K5

$$\begin{array}{rcl} (A \cap B) \times C & = & (A \times C) \cap (B \times C) \\ x \in (A \cap B) \wedge y \in C & \iff & [(x, y) \in (A \times C)] \wedge [(x, y) \in (B \times C)] \\ x \in A \wedge x \in B \wedge y \in C & \iff & (x \in A \wedge y \in C) \wedge (x \in B \wedge y \in C) \\ & \iff & x \in A \wedge x \in B \wedge y \in C \end{array}$$

$\boxed{\textbf{4.5}}$ **4.5.a**

$$\begin{array}{rl} R = \{ & (U, O), (O, U), (O, L), (O, K), (O, M), (L, O), (H, K), (H, S), (K, O), \\ & (K, H), (K, M), (M, O), (M, K), (M, S), \\ & (S, H), (S, M)\} \end{array}$$

R is symmetric and irreflexive.

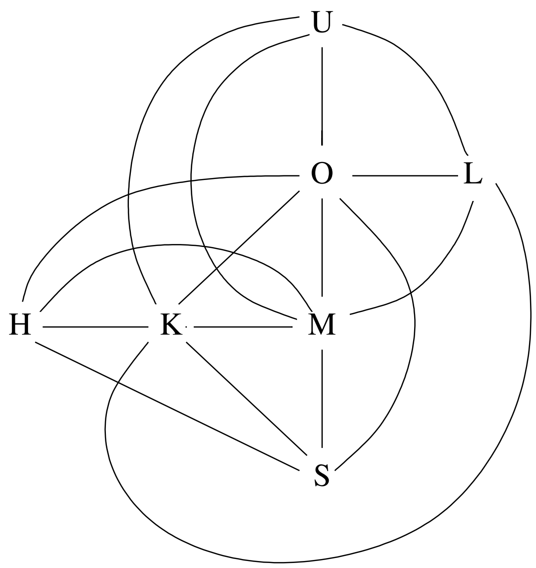
4.5.b

$$
\begin{aligned}
R' = \{ \quad & (U,U),(U,L),(U,K),(U,M),\\
& (O,O),(O,H),(O,M),(O,S),(O,K),\\
& (L,L),(L,U),(L,K),(L,M),\\
& (H,H),(H,O),(H,M)\\
& (K,K),(K,U),(K,L),(K,M),(K,S),(K,O)\\
& (M,M),(M,U),(M,L),(M,H),(M,K),(M,O)\\
& (S,S),(S,O),(S,K)\}
\end{aligned}
$$

R' is reflexive and symmetric

4.5.c S describes all stations to be reached with a so called “Kurzstreckenfahrkarte”, i.e., they are no more than two S- or U-Bahn stations away.

4.5.d



in addition, all nodes have loops

4.5.e

$$
M_R = \begin{array}{c} \\ U \\ O \\ L \\ H \\ K \\ M \\ S \end{array}
\begin{array}{c}
\begin{array}{ccccccc} U & O & L & H & K & M & S \end{array}\\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 1 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

$$M_{R'} = \begin{array}{c} \\ U \\ O \\ L \\ H \\ K \\ M \\ S \end{array} \begin{array}{c} \begin{array}{ccccccc} U & O & L & H & K & M & S \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$M_{S} = \begin{array}{c} \\ U \\ O \\ L \\ H \\ K \\ M \\ S \end{array} \begin{array}{c} \begin{array}{ccccccc} U & O & L & H & K & M & S \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

4.6 **4.6.a** symmetric:

$$\begin{aligned} R = R^{-1} &\iff \underset{x,y\in A}{\forall}[xRy \longleftrightarrow yRx] \\ &\iff \underset{x,y\in A}{\forall}[(xRy \longrightarrow yRx) \wedge (yRx \longrightarrow xRy)] && \text{C11} \\ &\iff \underset{x,y\in A}{\forall}(xRy \longrightarrow yRx) && \text{A4} \end{aligned}$$

4.6.b antisymmetric:

$$\begin{aligned} R \cap R^{-1} \subseteq I &\iff \underset{x,y\in A}{\forall}[(x,y) \in R\cap R^{-1} \longrightarrow (x,y)\in I] \\ &\iff \underset{x,y\in A}{\forall}[xRy \wedge yRx \longrightarrow x = y] \end{aligned}$$

4.6.c asymmetric:

$$\begin{aligned} R \cap R^{-1} = \emptyset &\iff \neg \underset{x,y\in A}{\exists}(xRy \wedge yRx) \\ &\iff \underset{x,y\in A}{\forall} \neg xRy \vee \neg yRx && \text{F2, A10} \\ &\iff \underset{x,y\in A}{\forall} xRy \longrightarrow \neg yRx && \text{D1} \end{aligned}$$

4.6.d transitive:

$$\begin{aligned} R^2 \subseteq R &\iff \forall_{x,z \in A}[(\exists_{y \in A} xRy \wedge yRz) \longrightarrow xRz] \\ &\iff \forall_{x,z \in A}[(\forall_{y \in A} \neg xRy \vee \neg yRz) \vee xRz] && \text{D1, F2, A10} \\ &\iff \forall_{x,y,z \in A}[\neg xRy \vee \neg yRz] \vee xRz] && \text{F7} \\ &\iff \forall_{x,y,z \in A}[(xRy \wedge yRz) \longrightarrow xRz] && \text{A10, D1} \end{aligned}$$

4.6.e intransitive:

$$\begin{aligned} R^2 \subseteq \overline{R} &\iff \forall_{x,z \in A}[\exists_{y \in A}(xRy \wedge yRz) \longrightarrow \neg xRz] \\ &\iff \forall_{x,z \in A}[(\forall_{y \in A} \neg xRy \vee \neg yRz) \vee \neg xRz] && \text{D1, F2, A10} \\ &\iff \forall_{x,y,z \in A}[\neg xRy \vee \neg yRz \vee \neg xRz] && \text{F7} \\ &\iff \forall_{x,y,z \in A}[(xRy \wedge yRz) \longrightarrow \neg xRz] && \text{A10, D1} \end{aligned}$$

4.6.f connex:

$$\begin{aligned} \overline{R} \cap \overline{R^{-1}} = \emptyset &\iff \neg \exists_{x,y \in A}[\neg xRy \wedge \neg yRx] \\ &\iff \forall_{x,y \in A}[xRy \vee yRx] && \text{F2, A10} \end{aligned}$$

4.6.g semiconnex:

$$\begin{aligned} \overline{R} \cap (\overline{R})^{-1} \subseteq I &\iff \forall_{x,y \in A}[\neg xRy \wedge \neg yRx \longrightarrow x = y] \\ &\iff \forall_{x,y \in A}[\neg(xRy \vee yRx) \longrightarrow x = y] && \text{A10} \\ &\iff \forall_{x,y \in A}[x \neq y \longrightarrow xRy \vee yRx] && \text{D3} \end{aligned}$$

4.7 **4.7.a** connex

4.7.b antisymmetric

4.7.c reflexive

4.7.d intransitive

4.7.e symmetric

4.7.f semiconnex

4.7.g irreflexive

4.7.h asymmetric

4.7.i transitive

$\boxed{\textbf{4.8}}$ **4.8.a** irreflexive

4.8.b irreflexive
symmetric
intransitive (in a polygamous society, which allows mixed marriage to men as well as to women, this does not hold.)

4.8.c irreflexive
intransitive
asymmetric
antisymmetric

4.8.d irreflexive
symmetric
semiconnex

4.8.e reflexive
antisymmetric (if two different human beings can not have exactly the same height)
transitive
connex
semiconnex

$\boxed{\textbf{4.9}}$ **4.9.a** adjacency matrix M = [m_{xy}] with

$$\underset{x \in A}{\forall} \ \underset{y \in A}{\forall} [xRy \longrightarrow m_{xy} = 1] \wedge [\neg xRy \longrightarrow m_{xy} = 0]$$
$$\Longleftrightarrow \underset{x \in A}{\forall} \ \underset{y \in A}{\forall} \beta(xRy, m_{xy} = 1, m_{xy} = 0)$$

4.9.b reflexive: $\underset{x}{\forall} m_{xx} = 1$

irreflexive: $\underset{x}{\forall} m_{xx} = 0$

symmetric: $\underset{x}{\forall}\underset{y}{\forall}(m_{xy} = 1 \longrightarrow m_{yx} = 1)$

antisymmetric: $\underset{x}{\forall}\underset{y}{\forall}(m_{xy} = 1 \wedge m_{yx} = 1 \longrightarrow x = y)$

asymmetric: $\underset{x}{\forall}\underset{y}{\forall}(\neg(m_{xy} = 1) \vee (\neg m_{yx} = 1))$

transitive: $\underset{x}{\forall}\underset{y}{\forall}\underset{z}{\forall}((m_{xy} = 1) \wedge (m_{yz} = 1) \longrightarrow (m_{xz} = 1))$

intransitive: $\underset{x}{\forall}\underset{y}{\forall}\underset{z}{\forall}((m_{xy} = 1) \wedge (m_{yz} = 1) \longrightarrow \neg(m_{xz} = 1))$

connex: $\underset{x}{\forall}\underset{y}{\forall}((m_{xy} = 1) \vee (m_{yx} = 1))$

semiconnex: $\underset{x}{\forall}\underset{y}{\forall}(x \neq y \longrightarrow ((m_{xy} = 1) \vee (m_{yx} = 1)))$

$\boxed{\textbf{4.10}}$ **4.10.a** $R : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$ $\qquad R^{-1} : \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$

4.10.b $R \circ R^{-1} : \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$

Interpretation:

$$\begin{aligned} R \circ R^{-1} &= \{(x,y)| \underset{z \in M}{\exists}(xRz \wedge zR^{-1}y)\}_{K^2} \\ &= \{(x,y)| \underset{z \in M}{\exists}(xRz \wedge yRz)\}_{K^2} \end{aligned}$$

$xR \circ R^{-1}y \iff x$ and y share common successors R

$R^{-1} \circ R : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$

Interpretation:

$$\begin{aligned} R^{-1} \circ R &= \{(x,y)| \underset{z \in K}{\exists}(xR^{-1}z \wedge zRy)\}_{M^2} \\ &= \{(x,y)| \underset{z \in K}{\exists}(zRx \wedge zRy)\}_{M^2} \end{aligned}$$

$$xR^{-1} \circ Ry \iff x \text{ and } y \text{ share common predecessors in } R$$

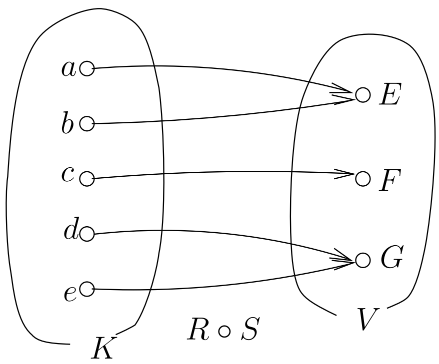
Properties:

$$\begin{aligned}
I_K \subseteq R \circ R^{-1} &\iff R \text{ is total} \\
&\iff R^{-1} \text{ is surjective} \\
I_M \subseteq R^{-1} \circ R &\iff R \text{ is surjective} \\
&\iff R^{-1} \text{ is total} \\
R^{-1} \circ R \subseteq I_M &\iff R \text{ is functional} \\
&\iff R^{-1} \text{ is injective}
\end{aligned}$$

R is total and functional $\iff$ R is a function
R is a function and R is surjective $\iff$ R is a surjection

4.10.c Arrow diagram:

$G = (K \cup V, R \circ S)$:



$$R \circ S : \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix};$$

Properties:

$$\begin{aligned}
\underset{x \in K}{\forall} d^+(x) = 1 &\iff R \circ S \text{ is total and functional} \\
&\iff R \circ S \text{ is a function} \\
\underset{y \in V}{\forall} d^-(y) \geq 1 &\iff R \circ S \text{ is surjective}
\end{aligned}$$

4.10.d Composition of two functional relations:

$$\begin{aligned}
R \subseteq K \times M; \quad R^{-1} \circ R \subseteq I_M &\iff R \text{ is functional} \\
S \subseteq M \times V; \quad S^{-1} \circ S \subseteq I_V &\iff S \text{ is functional}
\end{aligned}$$

To be shown:

$$(R \circ S)^{-1} \circ (R \circ S) \subseteq I_V$$

Proof:

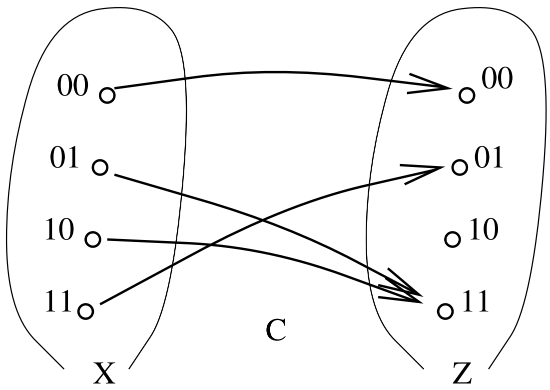
$$\begin{array}{rcl}(R\circ S)^{-1}\circ(R\circ S) & \overset{M6}{=} & S^{-1}\circ R^{-1}\circ R\circ S\\ & \overset{NR}{\subseteq} & S^{-1}\circ I_M\circ S\\ & = & S^{-1}\circ S\\ & \subseteq & I_V\end{array}$$

Auxiliary calculation:
To be shown: $A\subseteq B\Longrightarrow C\circ A\circ D\subseteq C\circ B\circ D$

$$\begin{array}{lcl}(A\subseteq B)\wedge(D\subseteq D) & \overset{M7}{\Longrightarrow} & A\circ D\subseteq B\circ D\\ (C\subseteq C)\wedge(A\circ D\subseteq B\circ D) & \overset{M7}{\Longrightarrow} & C\circ A\circ D\subseteq C\circ B\circ D\end{array}$$

4.10.e $xR\circ Sy\iff$ "x is child of a mother, who has children with father y."
As $R\circ S$ is functional, this means: "x is child of father y."

$\boxed{\textbf{4.11}}$ **4.11.a** Arrow diagram:



C is a (total) function $\iff C\overline{I}_Z=\overline{C}$

$$\overline{I}_Z=\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix};\ C\overline{I}_Z=\begin{pmatrix}0&1&1&1\\1&1&1&0\\1&1&1&0\\1&0&1&1\end{pmatrix};\ \overline{C}=\begin{pmatrix}0&1&1&1\\1&1&1&0\\1&1&1&0\\1&0&1&1\end{pmatrix};$$

Hence, C is a function.
In the arrow diagram, every node of the domain has exactly one outgoing edge.
In the adjacency matrix, every line contains exactly one 1.

4.11.b Block L: $y_1\iff x_1\wedge x_2$; $y_2\iff x_1\vee x_2$
Truth table:

x_1	x_2	y_1	y_2
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

$$L=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix};$$

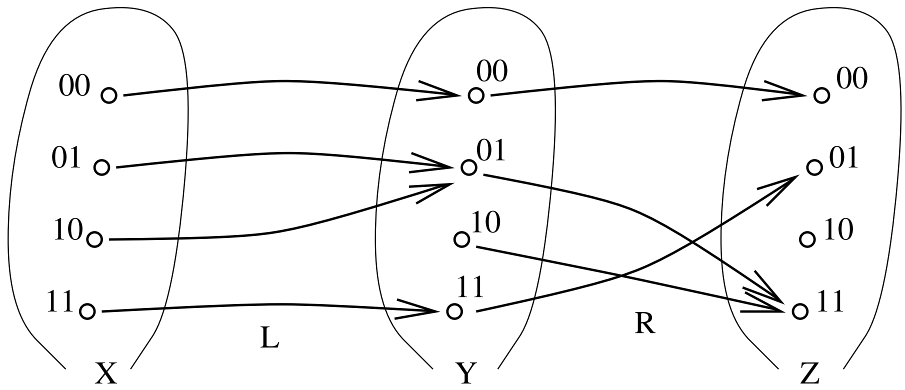
Block R: $z_1\iff y_1\not\leftrightarrow y_2$; $z_2\iff y_1\vee y_2$
Truth table:

y_1	y_2	z_1	z_2
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

We have: $L \circ R = C$.

Arrow diagram:



4.11.c Determination of relation R_{max}:

$$\begin{array}{lrcll} 1) & L \circ R & = & C & \\ 2) & L \circ R & \subseteq & C & \text{J28, E4} \\ 3) & L^{-1} \circ \overline{C} & \subseteq & \overline{R} & \text{M8} \\ 4) & R & \subseteq & \overline{L^{-1} \circ \overline{C}} & \text{J31} \\ & R_{max} & = & \overline{L^{-1} \circ \overline{C}} & \end{array}$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \ \overline{C} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}; \ L^{-1}\overline{C} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix};$$

$$R_{max} = \overline{L^{-1}\overline{C}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

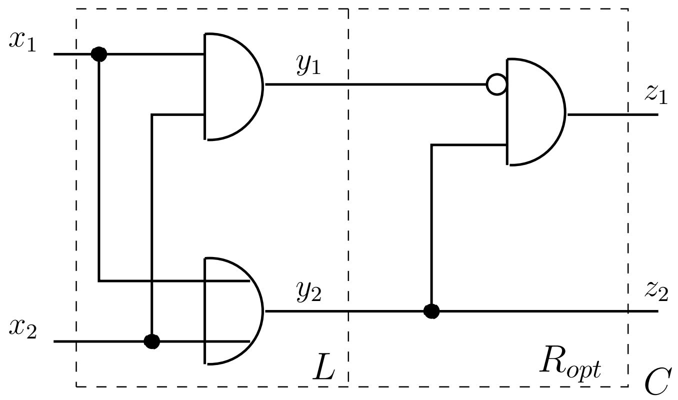
Truth table:

y_1	y_2	z_1	z_2
0	0	0	0
0	1	1	1
1	0	$\boxed{0}$	$\boxed{0}$
1	1	0	1

$$R_{opt} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

For $(y_1, y_2) = (1, 0)$, all possible output value assignments ($(0, 0), (0, 1), (1, 0)$ or $(1, 1)$) are permitted. Choosing $(z_1, z_2) = (0, 0)$, yields: $z_1 \iff \neg y_1 \wedge y_2$; $z_2 \iff y_2$.

Circuit:



4.11.d Determination of relation L_{max}:

1)	$L \circ R$	$=$	C	
2)	$L \circ R$	$\subseteq$	C	J28, E4
3)	$\overline{C} \circ R^{-1}$	$\subseteq$	$\overline{L}$	M8
4)	L	$\subseteq$	$\overline{\overline{C} \circ R^{-1}}$	J31

$$L_{max} = \overline{\overline{C} \circ R^{-1}}$$

$$\overline{C} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix};\ R^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix};\ \overline{C}R^{-1} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix};$$

$$L_{max} = \overline{\overline{C}R^{-1}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

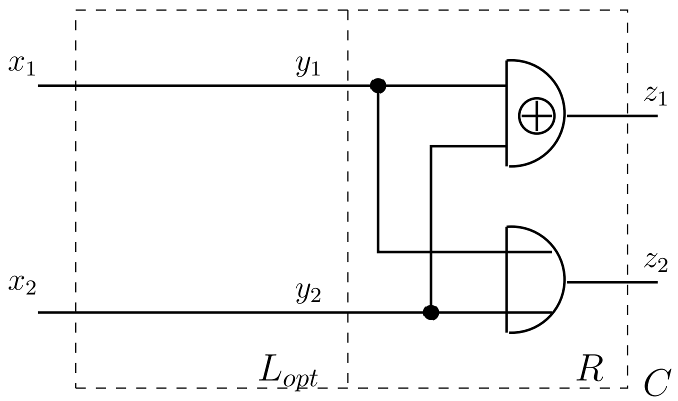
Truth table:

x_1	x_2	y_1	y_2
0	0	0	0
0	1	$\boxed{0}$	$\boxed{1}$
1	0	$\boxed{1}$	$\boxed{0}$
1	1	1	1

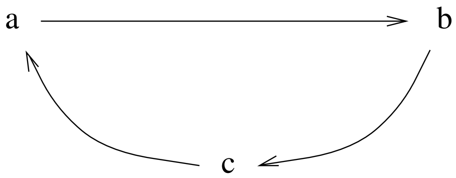
$$L_{opt} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

For $(x_1, x_2) = (0, 1)$ and $(x_1, x_2) = (1, 0)$, Y can be assigned to $(y_1, y_2) = (0, 1)$ and $(y_1, y_2) = (1, 0)$. Choosing $(y_1, y_2) = (0, 1)$ at $(x_1, x_2) = (0, 1)$ and $(y_1, y_2) = (1, 0)$ at $(x_1, x_2) = (1, 0)$, yields: $y_1 \iff x_1$; $y_2 \iff x_2$.

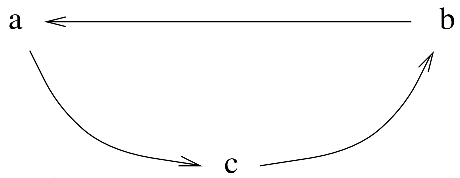
Circuit:



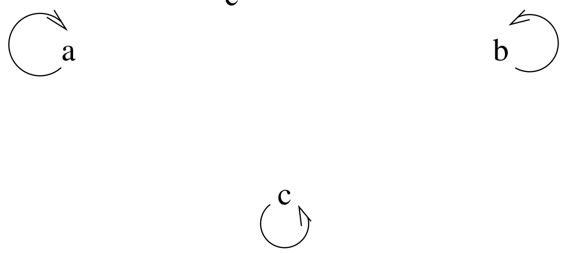
4.12 **4.12.a**



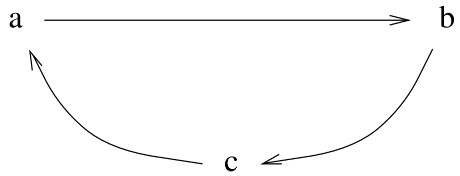
4.12.b $R_1^2 = \{(a,c),(b,a),(c,b)\}$



$R_1^3 = \{(a,a),(b,b),(c,c)\}$



$R_1^4 = \{(a,b),(b,c),(c,a)\}$

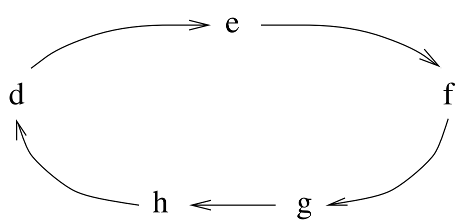


$R_1^4 = R^1$; $R_1^5 = R_1^2$; $R_1^6 = R_1^3$;
$p_1 = 3$

4.12.c $M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

4.12.d $M_1^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$; $\qquad M_1^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

4.12.e G_2:



$M_2 : \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

4.12.f $M_2^2 : \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix};$ $M_2^3 : \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix};$

$M_2^4 : \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix};$ $M_2^5 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$

$M_2^6 : \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix};$

$M_2^6 = M_2^1; M_2^7 = M_2^2; \ldots$

$p_2 = 5$

4.12.g $p = LCM(3,5) = 15$

4.12.h $R : \begin{pmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & R_2 \end{pmatrix};$

$A_1 \cap A_2 = \emptyset$. Therefore, in this case, $\nu_{max} = \max(|A_1|, |A_2|) = 5$ holds.

$\boxed{\textbf{4.13}}$ **4.13.a**

$r(R) : \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $s(R) : \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$t(R) = \bigcup_{\nu = 1} 3R^\nu$ $R^1 = R;$ $R^2 : \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$ $R^3 : \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$t(R) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

4.13.b

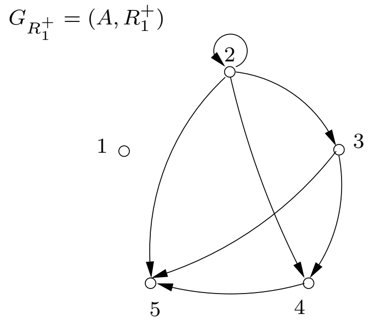
$$r(R):\begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}\qquad s(R):\begin{pmatrix}0&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{pmatrix}$$

$$t(R):\begin{pmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

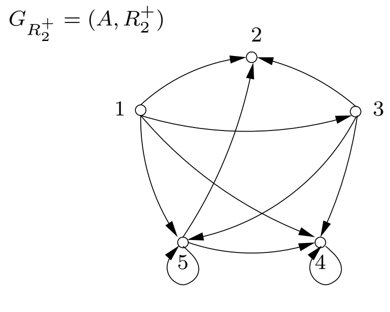
4.13.c

$$s(R):\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&1\\0&0&0&1&0\end{pmatrix}\qquad t(R):\begin{pmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}$$

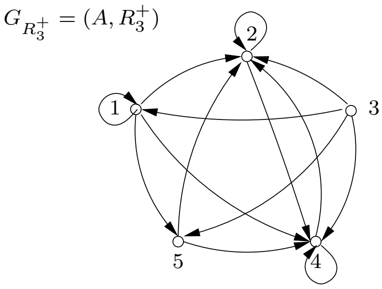
$\boxed{\textbf{4.14}}$ **4.14.a**



4.14.b



4.14.c



4.15

x	$\Gamma^-(x) \times \Gamma^+(x)$
a	$\{c\} \times \{c\} = \{(c,c)\}$
b	$\{b,d\} \times \{b\} = \{(b,b),(d,b)\}$
c	$\{a,c\} \times \{a,c,e\} = \{(a,a),(a,c),(a,e),(c,a),(c,c),(c,e)\}$
d	$\{e\} \times \{b\} = \{(e,b)\}$
e	$\{a,c\} \times \{b,d\} = \{(a,b),(a,d),(c,b),(c,d)\}$

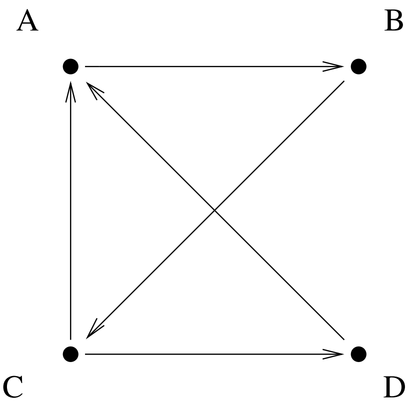
$R^+ = R \cup \{(c,c),(a,a),(a,e),(e,b),(a,b),(a,d),(c,b),(c,d)\} =$
$\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(c,a),(c,b),(c,c),(c,d),(c,e),(d,b),(e,b),(e,d)\}$

If we do not consider pairs (x,x) which do not produce any new pairs anyway, the table contains fewer entries.
For completeness, all pairs that were produced by calculating $\Gamma^-(x) \times \Gamma^+(x)$ have been listed in this subtask.

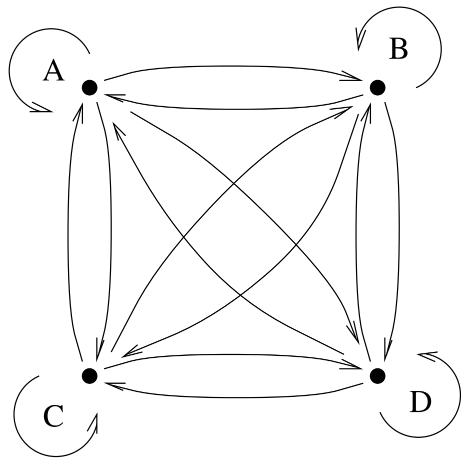
4.16 **4.16.a** The probabilities are shown in the following table:

	A	B	C	D
A	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{1}{3}$
B	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{2}$
C	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{2}{3}$
D	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

4.16.b $R = \{$(A,B), (B,C), (C,A), (C,D), (D,A)$\}$



4.16.c $t(R)$ is equivalent to the universal relation over the set $\{A, B, C, D\}$.

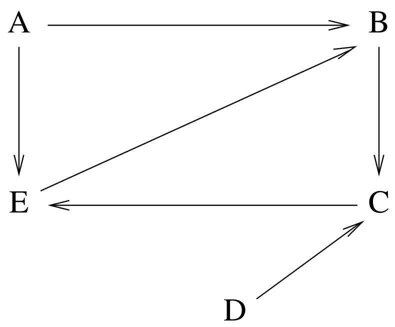


4.16.d As $R \neq t(R)$, R is obviously not transitive.

4.16.e Transitivity would mean, that there was a “best dice” in the game.

$\boxed{\textbf{4.17}}$ **4.17.a** $\rightsquigarrow = \{(A, B), (A, E), (B, C), (C, E), (D, C), (E, B)\}$

$G(P, \rightsquigarrow)$:



4.17.b No, since there's no element in P with a loop.

4.17.c Yes, $P_3(B, C, E, B)$ is a directed path.
The procedures B,C and E are transitively recursive.

4.17.d

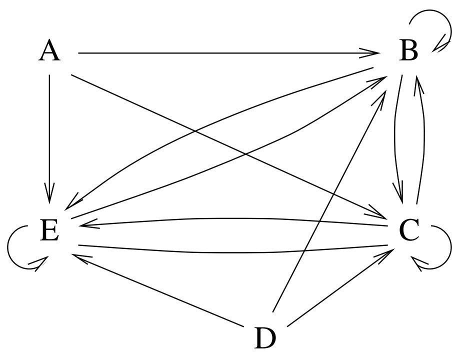
$$M_{\rightsquigarrow} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix};$$

$$M^2_{\rightsquigarrow} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}; \qquad M^3_{\rightsquigarrow} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$M^4_{\rightsquigarrow} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} = M^1_{\rightsquigarrow}$$

$$\rightsquigarrow^+ = \bigcup_{\nu=1}^{|P|} \rightsquigarrow^{\nu}; \qquad \bigcup_{\nu=1}^{|P|} M^{\nu}_{\rightsquigarrow} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix};$$

$G(P, \rightsquigarrow^+)$:



4.17.e Recursive procedures contain a loop. Therefore, B, C and E are recursive.

4.17.f It is no longer possible to distinguish between directly and transitively recursive procedures.

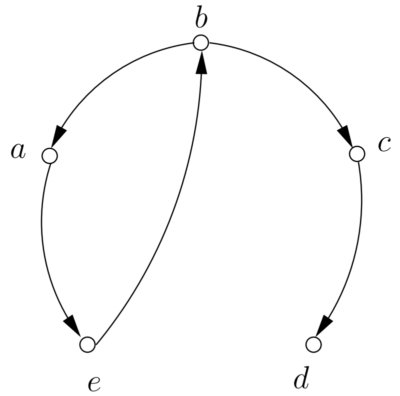
4.17.g $M_{\rightsquigarrow} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

$V_C = M_{\rightsquigarrow} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad C$ has predecessors B and D.

$N_C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \cdot M_{\rightsquigarrow} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad C$ has successor E.

$\boxed{\textbf{4.18}}$

$G_R = (A, R)$



Nodes a and c are not mutually accessible, since there is no path from c to a in direction of arrow. But there exists a path from a to c, thus, the two nodes are one-sided accessible, and therefore also indirectly accessible and connectable (see collection of formulae, chap. 4.5).

Nodes a and e are mutually accessible, since a can be reached from e, as well as e can be reached from a. Hence, these two nodes are also indirectly accessible and connectable.

From the arrow diagram, it can be seen that all nodes can be reached from a, b and e, but not from c and d. Thus, a, b and e are roots in G: $W(G) = \{a, b, e\}$.

For checking purposes: $\underset{\sim}{R}^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

The calculation of $\underset{\sim}{R}^*$ is indeed costly, but the connected components are easier to see.

4.19 **4.19.a**

$$\underset{\sim}{S^*} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

To determine the nodes that are mutually accessible, the following relation S' will be determined: $S' := S^* \cap (S^*)^{-1}$

$$\underset{\sim}{S'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

By definition, every node is accessible from itself, which is also reflected in $\underset{\sim}{S'}$. In addition, c and d are mutually accessible.

4.19.b "x and y have common successors in S" $\iff$ $(x, y) \in S \circ S^{-1}$

Hence, $T = S \circ S^{-1}$:

$$\underset{\sim}{T} = \underset{\sim}{S} \cdot \underset{\sim}{S}^T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Thus, a and b, a and d, as well as c and d have common successors.

4.20

$$\underset{\sim}{T^+} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad \underset{\sim}{T^*} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

T does have cycles, since $T^+ \not\subseteq \overline{I}$.

Relation $T' := T^* \cap (T^*)^{-1}$ contains all pairs of elements of B that are mutually accessible in T:

$$\underset{\sim}{T'} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consequently, a and d, b and d, as well as c and d are not mutually accessible.

4.21

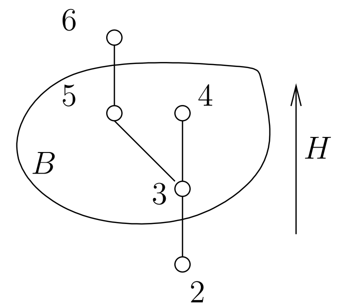
	partial order	strict order	total order	well order
$(N,<)$		*		
$(N,\leq)$	*		*	*
$(Z,\leq)$	*		*	
$(R,\leq)$	*		*	
$(P(N),\subset)$		*		
$(P(N),\subseteq)$	*			
$(P(\{a\}),\subseteq)$	*		*	*
$(P(\emptyset),\subseteq)$	*		*	*

Reasonable relations:

1. R is partial order $\Longrightarrow \neg$ (R is strict order)
2. R is strict order $\Longrightarrow \neg$ (R is partial order)
3. R is total order $\Longrightarrow$ R is partial order
4. $\neg$ (R is partial order) $\Longrightarrow \neg$ (R is total order)
5. R is well order $\Longrightarrow$ R is totale partial order
6. $\neg$ (R is totale partial order) $\Longrightarrow \neg$ (R is well order)

4.22 **4.22.a**

$$S=\{(2,2),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),\\(3,6),(4,4),(5,5),(5,6),(6,6)\}$$



4.22.b $B\subseteq A$ with $B=\{3,4,5\}$.

ub(B) = $\bigcap\limits_{x\in B}\Gamma^+(x)=\{3,4,5,6\}\cap\{4\}\cap\{5,6\}=\emptyset$

lb(B) = $\bigcap\limits_{x\in B}\Gamma^-(x)=\{2,3\}$

grt(B) = $B\cap$ ub(B) = $\emptyset$
lst(B) = $B\cap$ lb(B) = $\{3,4,5\} \cap \{2,3\} = \{3\}$
lub(B) = lst(ub(B)) = lst($\emptyset$) = $\emptyset$
glb(B) = grt(lb(B)) = grt($\{2,3\}$) = $\{3\}$
max(B) = $B \cap [\bigcap_{x\in B} \overline{\Gamma^-(x) \setminus \{x\}}] = B \cap \overline{\{2\}} \cap \overline{\{2,3\}} \cap \overline{\{2,3\}} = \{4,5\}$
min(B) = $\{3\}$

4.23 **4.23.a** The diagram is a Hasse diagram, if the saying “is subordinate to” is being formalized in a way, so that the emerging relation is an order relation. Therefore, the following must hold:

- Everyone is subordinate to him- or herself (reflexivity).
- No two different people are subordinate to each other (antisymmetry).
- The subordinate of my subordinate is also my subordinate (transitivity).

4.23.b ub$(B_1) = \{h, c, d, a\}$. This corresponds to the set of common supervisors of B_1.

4.23.c lb$(B_2) = \{f, l, m\}$. This corresponds to the set of all common subordinates of B_2.

4.23.d $\max(B)$ is the set of employees in B, who have no supervisors in B. grt(B) in contrast, is the set of employees in B, who are supervisors of all others in B.
$\max(B_3) = \{d\}$, $\max(B_4) = \{c, e\}$, grt$(B_3) = \{d\}$, grt$(B_4) = \{\}$.
B_3 is the preferred alternative, since there is definitely a supervisor in B_3, namely d.

4.23.e ub$(B_5) = \{c, a\}$
lub$(B_5) =$ lst(ub$(B_5)) = \{c\}$
Assuming that the boss a does not want to join the group himself, c is the one who has to accompany the group into the dark arctic winter.

4.24 **4.24.a** true: R is reflexive, antisymmetric and transistive
Proof for reflexivity:

$$\begin{aligned} \text{R is reflexive} \quad &\iff \quad \underset{(x,y)\,\in\, A}{\forall} (x,y)T(x,y) \\ &\iff \quad \underset{(x,y)\,\in\, A}{\forall} (x \le x) \wedge (y \le y) \\ &\iff \quad \underset{(x,y)\,\in\, A}{\forall} t \wedge t \iff t \end{aligned}$$

Proofs for antisymmetry and transitivity are similar.

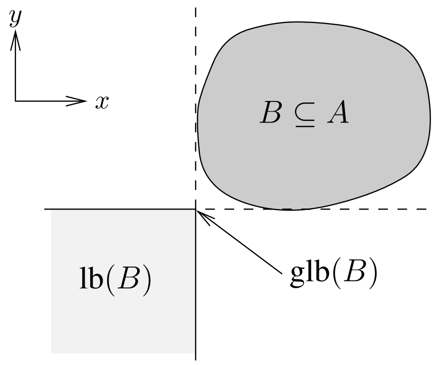
4.24.b false

Counter-proof for connexity with the tuples $(1, -1)$ and $(-1, 1)$:

$$\begin{aligned}
& \neg((x_1 \leq x_2) \wedge (y_1 \leq y_2) \vee (x_2 \leq x_1) \wedge (y_2 \leq y_1)) \\
\iff & \neg((1 \leq -1) \wedge (-1 \leq 1) \vee (-1 \leq 1) \wedge (1 \leq -1) \iff t \\
\iff & \neg[(x_1, y_1)T(x_2, y_2) \vee (x_2, y_2)T(x_1, y_1)] \\
\overset{G1}{\iff} & \underset{\substack{(x_1,y_1) \\ (x_2,y_2)} \in A}{\exists} \neg[(x_1, y_1)T(x_2, y_2) \vee (x_2, y_2)T(x_1, y_1)] \\
\overset{F1}{\iff} & \neg \underset{\substack{(x_1,y_1) \\ (x_2,y_2)} \in A}{\forall} [(x_1, y_1)T(x_2, y_2) \vee (x_2, y_2)T(x_1, y_1)] \\
\iff & \text{T is not connex}
\end{aligned}$$

4.24.c false (T is not a total order).

4.24.d true. Example:



4.25 **4.25.a**

To be proven:

$$b \in \text{grt}(B) \Longrightarrow b \in \max(B)$$

Proof using the deductive proof scheme:

1)	$\underset{x \in B}{\forall} x \preceq b \quad \Longrightarrow \underset{x \in B}{\forall}[(b \preceq x) \longrightarrow (x = b)]$	
2)	$\underset{x \in B}{\forall}[(x \preceq b) \vee (x = b)]$	1) (A5)
3)	$\underset{x \in B}{\forall}[\neg(x \succ b) \vee (x = b)]$	2) $\succ$ is complement of $\preceq$
4)	$\underset{x \in B}{\forall}[\neg(b \prec x) \vee (x = b)] \wedge [\neg(x = b) \vee (x = b)]$	3) (A6)
5)	$\underset{x \in B}{\forall}[(\neg(b \prec x) \wedge \neg(x = b)) \vee (x = b)]$	4) (A3)
6)	$\underset{x \in B}{\forall}[\neg((b \prec x) \vee (x = b)) \vee (x = b)]$	5) (A10)
7)	$\underset{x \in B}{\forall}[\neg(b \preceq x) \vee (x = b)]$	6) union of strict order $\prec$ and identity $=$ yields parital order $\preceq$
8)	$\underset{x \in B}{\forall}[(b \preceq x) \longrightarrow (x = b)]$	7) (D1)

4.25.b

Assume a and b with $a, b \in \text{lub}(B)$.
As $b \in \text{ub}(B)$ and $a \in \text{lub}(B)$, $a \preceq b$ must hold.
As $a \in \text{ub}(B)$ and $b \in \text{lub}(B)$, $b \preceq a$ must hold.
As R is an order relation and therefore antisymmetric,

$$(a \preceq b) \wedge (b \preceq a) \Longrightarrow a = b$$

holds.

$\boxed{\textbf{4.26}}$ **4.26.a** $G = (A, R)$ is either a clique (completely intermeshed graph), or it decomposes into several sugraphs, each representing a clique without connecting edges between them.

4.26.b

$$A = \bigcup_{i=1} nA_{ci} \;\text{ and }\; \underset{i,j,i \neq j}{\forall}(A_{ci} \cap A_{cj}) = \emptyset$$

$$\underset{i=1\ldots n}{\forall} R_{ci} = A_{ci}^2$$

$$R = \bigcup_{i=1} nR_{ci} \;\text{ and }\; \underset{i,j,i \neq j}{\forall}(R_{ci} \cap R_{cj}) = \emptyset$$

4.27 **4.27.a**

$$R = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad R^0 = I$$

$$R^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad R^3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R^4 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad R^5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

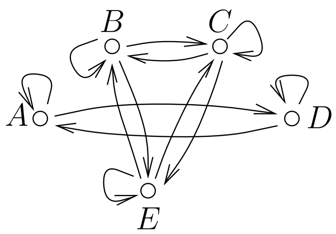
$$R^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

4.27.b

$$R^{*-1} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{EQ}(R^*) = R^* \cap R^{*-1} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\text{after re-sorting: EQ}(R^*) = \begin{array}{c} \\ A \\ D \\ B \\ C \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & D & B & C & E \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

4.27.c



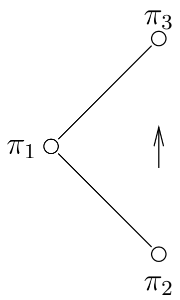
4.28 **4.28.a** $R_1 = \{(a,a),(a,b),(a,c),(b,b),(b,a),(b,c),(c,c),(c,a),(c,b),(d,d)\}$

4.28.b $R_2 = \{(a,a),(b,b),(c,c),(d,d)\} = I$

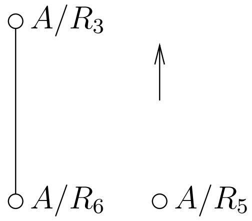
4.28.c

$$\begin{aligned} R_3 = \{&(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),\\ &(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d)\} = A \times A \end{aligned}$$

4.28.d



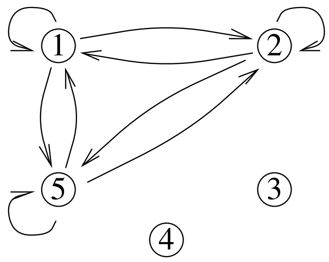
4.29 **4.29.a**



4.29.b $\hat{\pi}(A/R_3, A/R_6) = A/R_6$
$\check{\pi}(A/R_3, A/R_6) = A/R_3$
$\hat{\pi}(A/R_3, A/R_5) = A/R_{15}$
$\check{\pi}(A/R_3, A/R_5) = A/R_1$

4.29.c $\hat{\pi}(A/R_j, A/R_k) = A/R_{LCM(j,k)}$ and $\check{\pi}(A/R_j, A/R_k) = A/R_{GCD(j,k)}$.

4.30 Did you really solve this problem yourself? If not, check the validity of this "proof" using the example relation below. Otherwise you can check if you had the right idea by taking a close look to this relation.
$A = \{1,2,3,4,5\}$



4.31

1)	$WV \subseteq WV$	J20
2)	$W^{-1}\overline{WV} \subseteq \overline{V}$	$1, \mathrm{M8}$
3)	$U^{-1} \subseteq U^{-1}$	J20
4)	$U^{-1}W^{-1}\overline{WV} \subseteq U^{-1}\overline{V}$	$2, 3, \mathrm{M7}$
5)	$\overline{U^{-1}\overline{V}} \subseteq \overline{U^{-1}W^{-1}\overline{WV}}$	$4, \mathrm{J31}$
6)	$\overline{U^{-1}\overline{V}} \subseteq \overline{(WU)^{-1}\overline{WV}}$	$5, \mathrm{M6}$

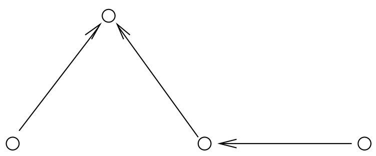
4.32

	$str(V) \cap \overline{V} \cap \overline{V}^{-1} \subseteq I$	
$\iff$	$sr(V) \cap \overline{V} \cap \overline{V}^{-1} \subseteq I$	$V = t(V)$, N6
$\iff$	$sr(V) \cap \overline{V \cup V^{-1}} \subseteq I$	H10
$\iff$	$sr(V) \cap \overline{s(V)} \subseteq I$	N19
$\iff$	$rs(V) \cap \overline{s(V)} \subseteq I$	N6
$\iff$	$(s(V) \cup I) \cap \overline{s(V)} \subseteq I$	N18
$\iff$	$I \cap \overline{s(V)} \subseteq I$	$\mathrm{H3, H8, H6}$
$\iff$	t	J21

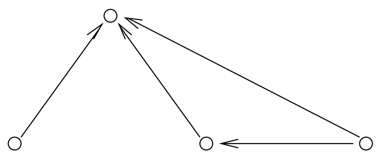
4.33

1)	$I \subseteq V$	
2)	$V^2 \subseteq V$	$\Longrightarrow (V = V^2) \wedge (V \cap \overline{V^2} = \emptyset)$
3)	$V \subseteq V$	J20
4)	$V \subseteq V^2$	$1, 3, \mathrm{M7}$
5)	$V = V^2$	$2, 4, \mathrm{J28}$
6)	$V \cap \overline{V^2} = \emptyset$	$4, \mathrm{J29}$

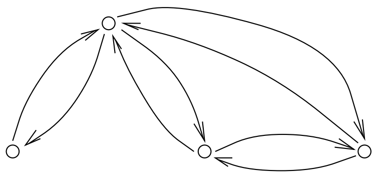
$\boxed{\textbf{4.34}}$ Counter-example:



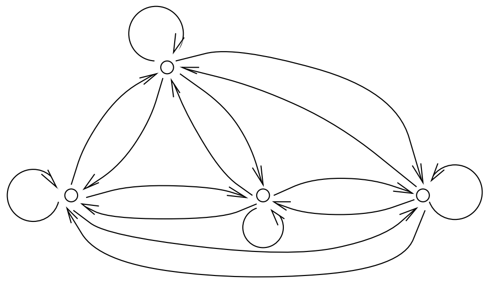
R



$t(R)$



$st(R)$



$tst(R)$

$\boxed{\textbf{4.35}}$ **4.35.a** $2rb$

4.35.b a^2

4.35.c $2ar$ (every element requires 2 bits, as it can obtain values -1, 0 or 1.)

4.35.d $(a + r) \cdot (b + i)$

4.35.e $(a + r) \cdot 2 \cdot \max(b, i)$

4.35.f

Representation	$a = r = b = i$	$a \gg r = b = i$	$r \gg a = b = i$
list of ordered pairs	$2a^2$	$2r^2$ +	$2ar$
adjacency matrix	a^2 +	a^2 –	a^2 +
incidence matrix	$2a^2$	$2ar$	$2ar$
successor list	$4a^2$ –	$2(ar + r^2)$	$2(ar + a^2)$ –
successor table	$4a^2$ –	$2(ar + r^2)$	$2(ar + a^2)$ –

$\boxed{\textbf{4.36}}$ **4.36.a** The number of bits required to store one element of A is assumed to be i.

$$S(n) = \sum_{x \in A} d^+(x) \cdot 2 \cdot i.$$

$\underset{x \in A}{\forall} d^+(x) \leq c$ results in:

$$S(n) \leq c \cdot n \cdot 2 \cdot i,$$

$$O(S(n)) = O(n).$$

For computation time applies:

$$R(n) \sim c \cdot n,$$

$$O(R(n)) = O(n).$$

Every element from the list has to be examined.

4.36.b

$$S(n) = n^2,$$

$$O(S(n)) = O(n^2).$$

$$R(n) \sim n,$$

$$O(R(n)) = O(n).$$

The successors of each element are listed in one row of the adjacency matrix. All elements of this line have to be examined.

4.36.c The number of bits to store an element of A is assumed to be equal to the number of bits required by a pointer (=i).

$$\begin{aligned} S(n) &= (n + \sum_{x \in A} d^+(x)) \cdot 2 \cdot i. \\ &= \sum_{x \in A} (d^+(x) + 1) \cdot 2 \cdot i. \end{aligned}$$

$\underset{x \in A}{\forall} d^+(x) \leq c$ results in

$$S(n) \leq (n \cdot c + n) \cdot 2 \cdot i,$$

$$O(S(n)) = O(n).$$

The computation time for proceeding the successors of a node x is proportional to the successor degree $d^+(x)$ ($d^+(x) \leq c$). This leads to

$$O(R(n)) = O(1).$$

In principle, a chained list of length c is being scanned.

5. Finite State Machines

5.1 **5.1.a** State output table:

μ	$x=0$	$x=1$
S_0	$S_1/0$	$S_2/0$
S_1	$S_3/0$	$S_4/0$
S_2	$S_5/0$	$S_6/0$
S_3	$S_0/0$	$S_0/0$
S_4	$S_0/1$	$S_0/0$
S_5	$S_0/0$	$S_0/0$
S_6	$S_0/1$	$S_0/0$

5.1.b Minimization of states

1-equivalence

μ	$x=0$	$x=1$	
S_0	$S_1/0^A$	$S_2/0^A$	A
S_1	$S_3/0^A$	$S_4/0^B$	
S_2	$S_5/0^A$	$S_6/0^B$	
S_3	$S_0/0^A$	$S_0/0^A$	
S_5	$S_0/0^A$	$S_0/0^A$	
S_4	$S_0/1^A$	$S_0/0^A$	B
S_6	$S_0/1^A$	$S_0/0^A$	

2-equivalence

μ	$x=0$	$x=1$	
S_0	$S_1/0^B$	$S_2/0^B$	A
S_3	$S_0/0^A$	$S_0/0^A$	
S_5	$S_0/0^A$	$S_0/0^A$	
S_1	$S_3/0^A$	$S_4/0^C$	B
S_2	$S_5/0^A$	$S_6/0^C$	
S_4	$S_0/1^A$	$S_0/0^A$	C
S_6	$S_0/1^A$	$S_0/0^A$	

3-equivalence

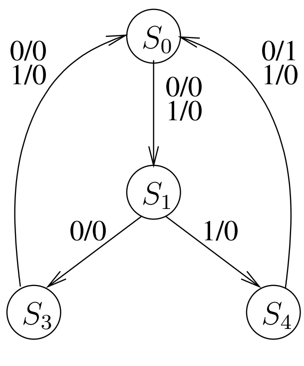
μ	$x=0$	$x=1$	
S_0	$S_1/0$	$S_2/0$	A
S_3	$S_0/0$	$S_0/0$	B
S_5	$S_0/0$	$S_0/0$	
S_1	$S_3/0$	$S_4/0$	C
S_2	$S_5/0$	$S_6/0$	
S_4	$S_0/1$	$S_0/0$	D
S_6	$S_0/1$	$S_0/0$	

Equivalence: $S_3 \sim S_5$, $S_1 \sim S_2$, $S_4 \sim S_6$

5.1.c Minimized table:

μ	$x=0$	$x=1$
S_0	$S_1/0$	$S_1/0$
S_3	$S_0/0$	$S_0/0$
S_1	$S_3/0$	$S_4/0$
S_4	$S_0/1$	$S_0/0$

Minimized automaton:



5.1.d State coding: $S_0 \triangleq 00 \triangleq \overline{s}_1 \cdot \overline{s}_2$
$S_1 \triangleq 01 \triangleq \overline{s}_1 \cdot s_2$
$S_3 \triangleq 10 \triangleq s_1 \cdot \overline{s}_2$
$S_4 \triangleq 11 \triangleq s_1 \cdot s_2$

x	state	next state	y
0	S_0	S_1	0
1	S_0	S_1	0
0	S_1	S_3	0
1	S_1	S_4	0
0	S_3	S_0	0
1	S_3	S_0	0
0	S_4	S_0	1
1	S_4	S_0	0

x	$s_1 s_2$	$z_1 z_2$	y
0	0 0	0 1	0
1	0 0	0 1	0
0	0 1	1 0	0
1	0 1	1 1	0
0	1 0	0 0	0
1	1 0	0 0	0
0	1 1	0 0	1
1	1 1	0 0	0

$y = \overline{x} \cdot s_1 \cdot s_2$
$z_1 = \overline{x} \cdot \overline{s}_1 \cdot s_2 + x \cdot \overline{s}_1 \cdot s_2 = \overline{s}_1 \cdot s_2;$
$z_2 = \overline{x} \cdot \overline{s}_1 \cdot \overline{s}_2 + x \cdot \overline{s}_1 \cdot \overline{s}_2 + x \cdot \overline{s}_1 \cdot s_2 = \overline{s}_1 \cdot \overline{s}_2 + x \cdot \overline{s}_1$

5.1.e