Topic 3
Multivariable Functions & Partial Derivatives

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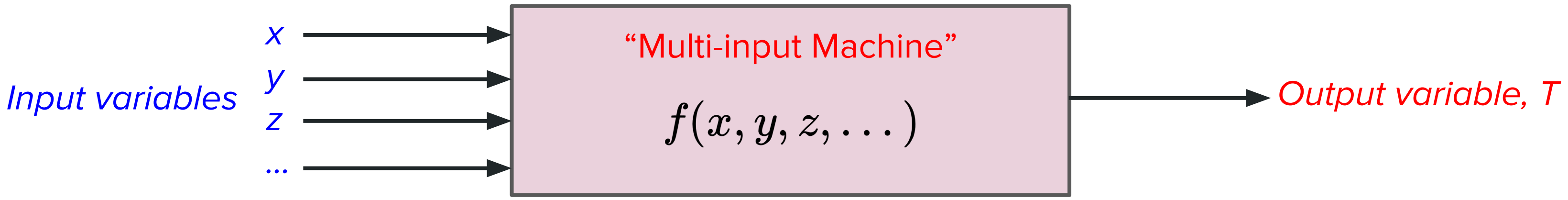
Outline

- A Multivariable Function
- Level Sets
- Limits & Continuity
- Partial Derivatives
- Linear Approximation & Tangent Planes
- Total Differential & Chain Rule
- Differentiability

Concept of a Multivariable Function

A multivariable function is a function that takes in **multiple** inputs and produces an output. The ‘machine’ perspective is shown below.



For example, the temperature (T) in a room can be a function of space (x, y, z) and time (t), i.e.

$$T = f(x, y, z, t)$$

Another example is the altitude (z) of a terrain as a function of location (x, y), i.e.

$$z = f(x, y)$$

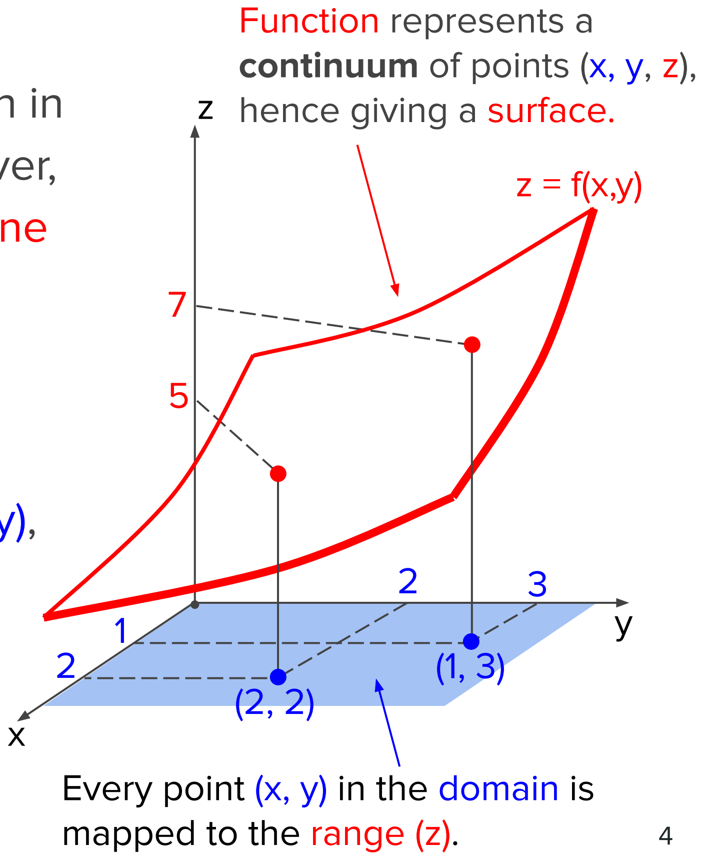
Graphing a Function of Two Variables

A function of two variables can be represented in a graph in a similar manner as with single-variable functions. However, since there are two independent inputs, say, x & y, and one output, say z, there is a z-value for every ordered pair of inputs (x, y) in the domain of the function. Hence, the function looks like a surface as shown.

It is also common to express (x, y) as a vector, say **x** = (x, y), such that

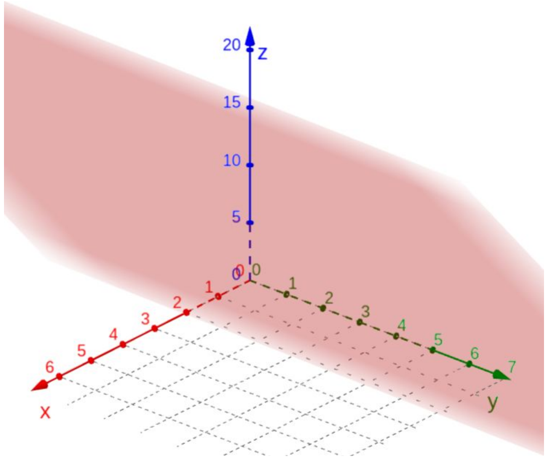
$$z = f(\mathbf{x}) = f(x, y)$$

Therefore, for a function of n inputs, **x** would be n-dimensional input vector.

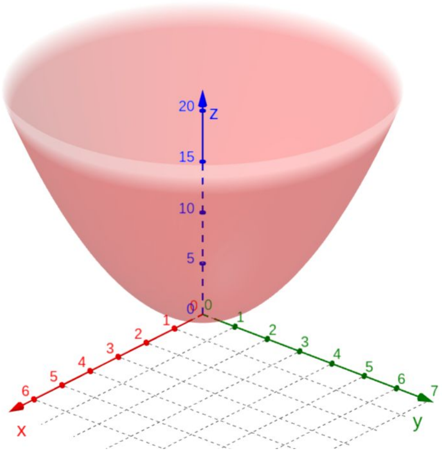


Some Functions of Two Variables

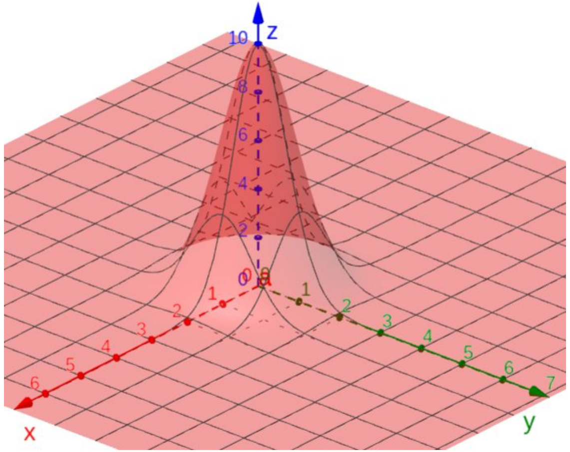
Similar to how single-variable functions can give different curves, functions of two variables give different surfaces when plotted. Same examples are shown below. Use https://www.geogebra.org/3d?lang=en to plot f(x,y).



$$f(x,y) = 2x - y + 5$$



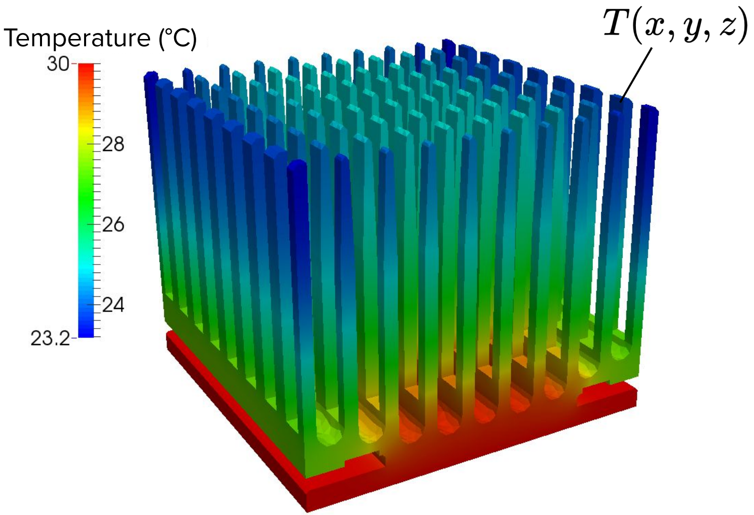
$$f(x,y) = x^2 + y^2$$



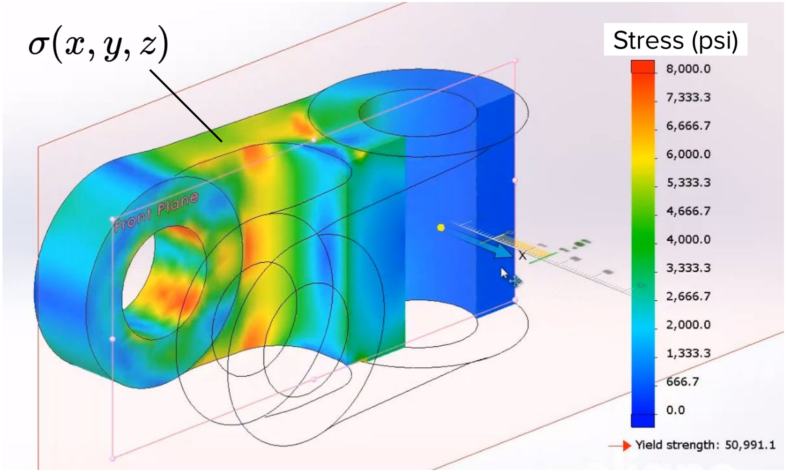
$$f(x,y) = 10e^{-(x^2+y^2)}$$

A Function of Three Variables

While a function of three variables such as T = f(x, y, z) cannot be plotted as a geometrical feature since there are four variables, it can be (partly) represented in a colour plot as shown in the examples below.



Temperature function in a heat sink
(https://quickersim.com/cfdtoolbox/heat-transfer/)

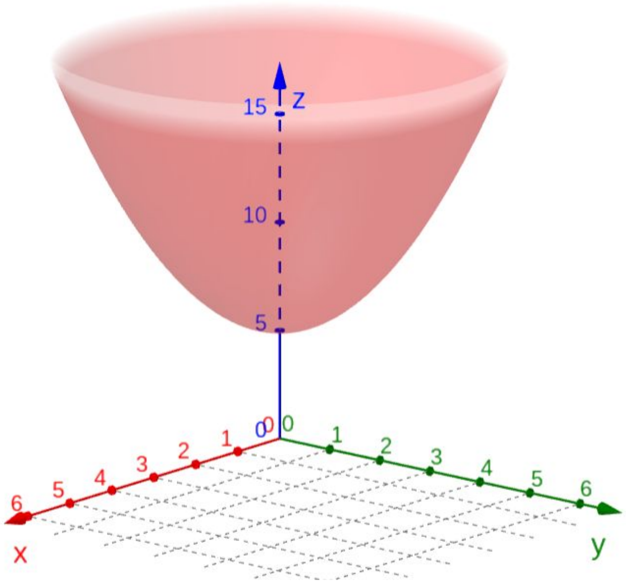


Stress function in a joint knuckle
(https://blogs.solidworks.com/tech/2020/01/introduction-to-solidworks-simulation-finite-element-analysis.html)

Domain & Range

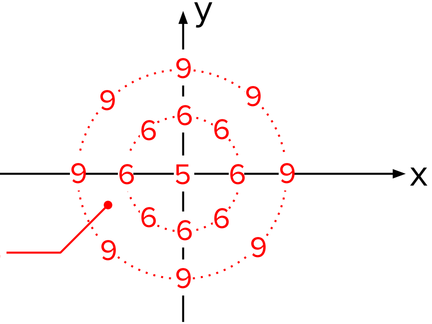
The domain and range of a multivariable function has a similar meaning to that of single-variable functions. For example, the function

$$z = f(x, y) = x^2 + y^2 + 5$$



has a domain of $\mathbb{R}^2$ ($-\infty < x < \infty$, $-\infty < y < \infty$) and a range of $[5, \infty)$.

Because the above multivariable function returns a scalar at every point (x, y) in the domain, it is also called a scalar field (field of scalars).



Every point in the xy plane has a z value, hence giving a field of scalars.

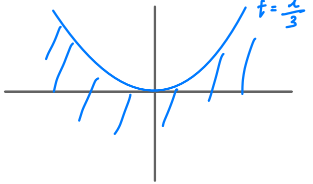
Domain & Range

Exercise: Determine the domain and range of the following functions. Verify in Geogebra if possible.

a) $y(x,t) = 1 - \sqrt{\overbrace{x^2 - 3t}^{\geq 0}}$

$x^2 - 3t \geq 0 \rightarrow \underbrace{x^2 \geq 3t}_{t \leq \frac{x^2}{3}}$ (Domain)

Range is $y \leq 1$ or $(-\infty, 1]$



b) $q(\theta, t) = 2\sin\left(t + \theta\right)$

Range is $[-2, 2]$

$\sin \underbrace{t+\theta}_{} \in \mathbb{R}$, so $t \in \mathbb{R}$ & $\theta \in \mathbb{R}$ Hence domain is $\mathbb{R}^2$

c) $T(x,y,z) = \dfrac{1}{\underbrace{x^2 + y^2 + z^2}_{}}$ (Describe this 3D temperature field.)

Since $x^2+y^2+z^2 \neq 0$, domain is $\{\mathbb{R}^3 \mid (x,y,z) \neq (0,0,0)\}$

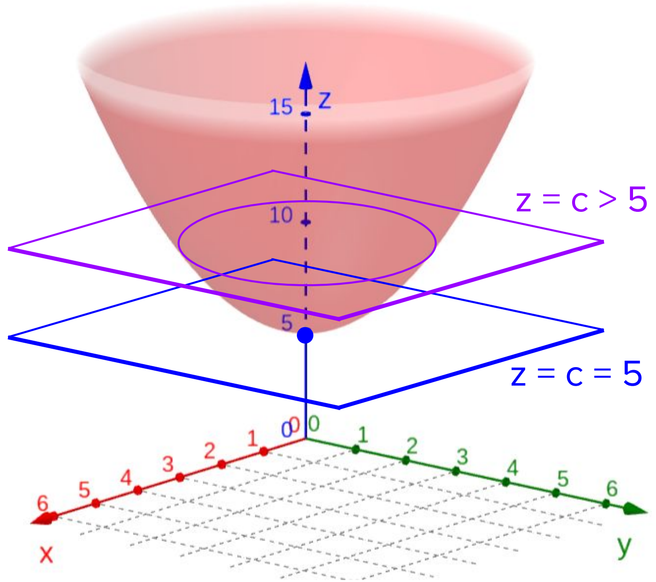
Since $x^2 + y^2 + z^2 > 0$, range is $T > 0$

The temperature is lower the further away from the origin.

A set can be a point, a curve, a surface etc.

Level Sets

$z = f(x,y) = C \to y = g(x)$ is a curve



A level set of a function is defined as

$$z = f(\mathbf{x}) = c$$

C must be in the range of f

where c is a constant in the range of the function. For z = f(x, y) that can be represented as a surface, a level set has the **same meaning as the contour lines of a terrain** as illustrated. For example, the level set

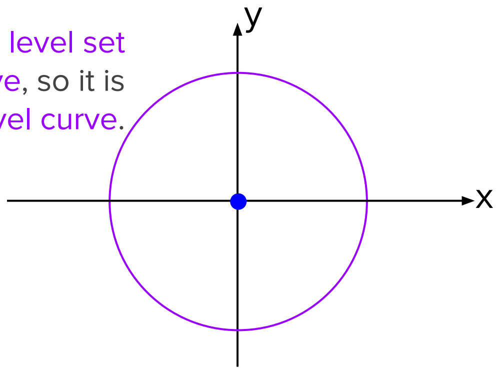
$$f(x,y) = x^2 + y^2 + 5 = 5 \to x^2 + y^2 = 0$$

represents a point at (0, 0). The level set

$$f(x,y) = x^2 + y^2 + 5 = c \to x^2 + y^2 = \overbrace{c - 5}^{R^2},\ c > 5$$

is a circle centered at (0, 0) with radius √(c-5).

Generally, the level set of f(x,y) is a curve, so it is also called a level curve.



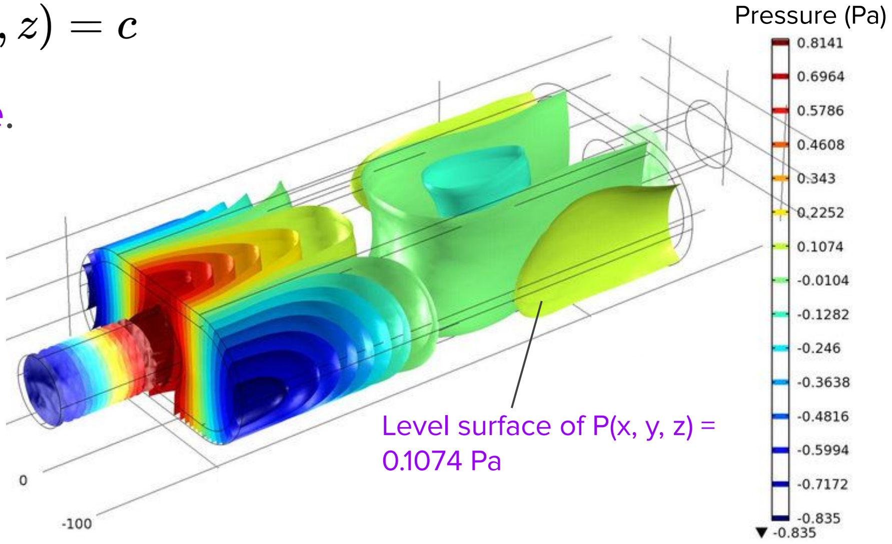
Level Sets

Similarly, the level sets for a function of three variables such as a pressure function

$$P = f(x, y, z) = c$$

represent surfaces of the same pressure value.

In the example shown, a surface of one colour indicate the locations in a muffler where the **acoustic pressure is a constant value**. Such plots are called isosurface plots.



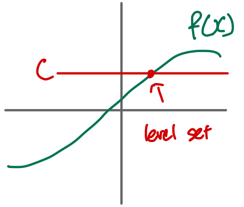
Source: https://www.researchgate.net/publication/283269341_Finite_Element_Analysis_of_Acoustic_Pressure_Levels_and_Transmission_Loss_of_a_Muffler

Can you explain why the level set of f(x, y, z) is a surface?

$f(x,y,z) = c \rightarrow z = c - g(x,y)$

$z = h(x,y)$ is a surface

Level Sets



$f(x) = c$ is a point, or points

Exercise: Describe the level set of a single variable function f(x). What is the general relationship between a function and its level sets dimensionally?

A level set is generally one dimension lower than its function

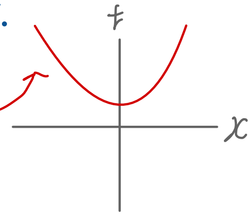
Exercise: Describe the level sets of each function below.

a) $y(x,t) = 1 - \sqrt{x^2 - 3t} = c \rightarrow \sqrt{x^2 - 3t} = 1 - c$

$x^2 - 3t = (1-c)^2$

$\underbrace{t = \frac{1}{3}\left[x^2 - (1-c)^2\right]}$

level curve is a parabola on the xt-plane



b) $T(x,y,z) = \dfrac{1}{x^2 + y^2 + z^2} = c$

$\downarrow$

$x^2 + y^2 + z^2 = \frac{1}{c} = R^2$

spherical surface of radius $\sqrt{\frac{1}{c}}$

ANS: a) Parabola: $t = ⅓[x^2 - (1-c)^2]$. b) Spherical surface: $x^2 + y^2 + z^2 = 1/c$.

Limits

Recall that the limit of a single variable function is defined as

$$\lim_{x \to a} f(x) = L$$

Similarly, the limit of a multivariable function is defined as

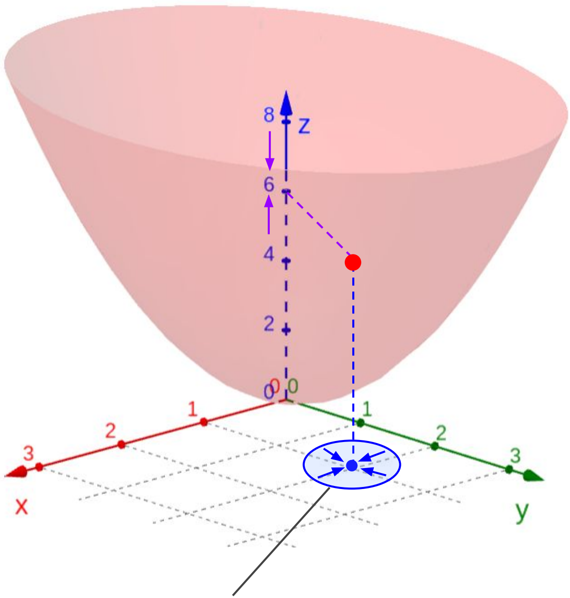
for f(x,y)

$$\lim_{\underbrace{\mathbf{x} \to \mathbf{a}}_{(x,y)\to(a,b)}} f(\mathbf{x}) = L$$

if the limit exists. Eg, for a function of two variables, we have

$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} 2x^2 + y^2 = 2(1^2) + 2^2 = 6$$

As illustrated, the limit is the value f(x, y) approaches as (x, y) approaches (1, 2) $\underbrace{\textbf{from all possible directions}}_{\infty \text{ directions}}$ (in the domain).



Graphically, the limit is the value that z = f(x, y) approaches when the disc closes in to (x, y) = (1, 2).

Nonexistent Limits

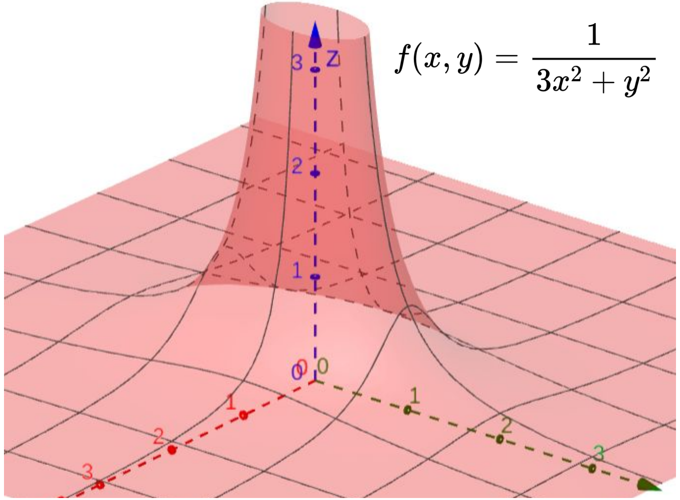
Similar to single-variable functions, the limit of a multivariable function can fail to exist, such as when

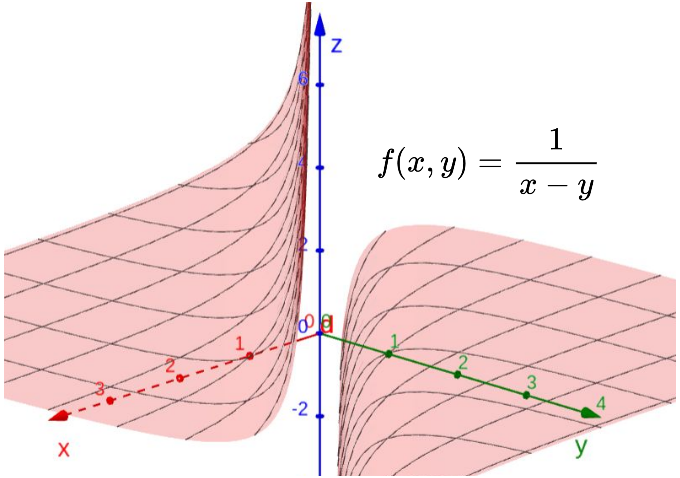
$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \pm\infty$$

For example, as illustrated,

$$\lim_{(x,y)\to(0,0)} \frac{1}{3x^2+y^2} = \infty \text{ (Limit D.N.E.)}$$

$$\lim_{(x,y)\to(a,a)} \frac{1}{x-y} = \pm\infty \text{ (Limit D.N.E.)}$$



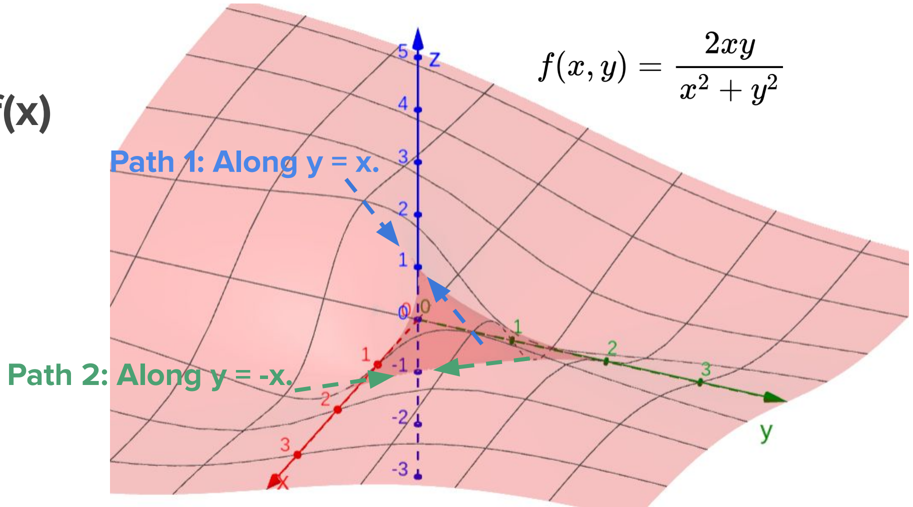


Nonexistent Limits

Another case of a nonexistent limit is when **f(x) approaches different limits along different paths**, i.e.

$$\lim_{\mathbf{x}\to\mathbf{a}\ (path\ 1)} f(\mathbf{x}) = L_1,$$

$$\lim_{\mathbf{x}\to\mathbf{a}\ (path\ 2)} f(\mathbf{x}) = L_2 \neq L_1$$



For example, the illustration shows that

$$\lim_{(x,y)\to(0,0)\ (y=x)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2}{x^2+x^2} = 1, \quad \lim_{(x,y)\to(0,0)\ (y=-x)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{-2x^2}{x^2+x^2} = -1$$

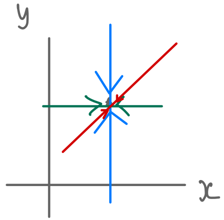
$$\to \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} \text{ D.N.E.}$$

Two-Path Test for *Nonexistent Limits Only

From the prior example, it implies that we can use non-similar limits along two different paths to show that a limit does not exist (two-path test). It is important to note that when the limits obtained along two (or any number of) paths are the same, it **does not mean that the limit exist, since there are infinite paths to a point**.

Exercise: Determine if the limit below exists. Then verify it from the graph.

$$L = \lim_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} = \frac{0}{0} \text{ (indeterminate)}$$



Along $x=1$,

$$L_1 = \lim_{y\to 2} \frac{(0)(y-2)}{0^2+(y-2)^2} = \lim_{y\to 2} \frac{0}{(y-2)^2} = 0$$

Along $y=2$,

$$L_2 = \lim_{x\to 1} \frac{(x-1)(0)}{(x-1)^2+0} = \lim_{x\to 1} \frac{0}{(x-1)^2} = 0$$

Along $y=2x$

$$L_3 = \lim_{x\to 1} \frac{(x-1)(2x-2)}{(x-1)^2+(2x-2)^2} = \lim_{x\to 1} \frac{2(x-1)^2}{(x-1)^2+4(x-1)^2}$$

$$= \frac{2\cancel{(x-1)^2}}{5\cancel{(x-1)^2}}$$

$$= \frac{2}{5} \neq 0$$

Since the limits are different along different paths, the limits DNE

ANS: DNE.

Properties of Limits

Similar to that in single-variable functions, the limit operator for multivariable functions follows the following properties. Given

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L, \quad \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = K$$

a) $\lim_{\mathbf{x}\to\mathbf{a}}\{f(\mathbf{x}) \pm g(\mathbf{x})\} = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) \pm \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = L \pm K$

b) $\lim_{\mathbf{x}\to\mathbf{a}}\{f(\mathbf{x}) \cdot g(\mathbf{x})\} = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) \cdot \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = LK$

c) $\lim_{\mathbf{x}\to\mathbf{a}}\left\{\frac{f(\mathbf{x})}{g(\mathbf{x})}\right\} = \frac{\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})} = \frac{L}{K}$

d) $\lim_{\mathbf{x}\to\mathbf{a}}\left\{f(\mathbf{x})^{g(\mathbf{x})}\right\} = \lim_{\mathbf{x}\to\mathbf{a}} f(x)^{\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})} = L^K$

Limits

Always direct sub first

Exercise: Determine the following limits.

a) $\displaystyle\lim_{(x,y,z)\to(0,1,2)} \frac{2x^2+\ln y - z}{\sqrt{x+2z}}$

$= \dfrac{2(0)+\ln(1)-2}{\sqrt{0+2(2)}}$

$= \dfrac{0+0-2}{\sqrt{4}} = -1$

b) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{\sqrt{x-y}-1}{x-y-1} = \frac{0}{0}$

$\displaystyle\lim_{(x,y)\to(1,0)} \frac{\sqrt{x-y}-1}{x-y-1} \cdot \frac{\sqrt{x-y}+1}{\sqrt{x-y}+1}$

$= \displaystyle\lim_{(x,y)\to(1,0)} \frac{\cancel{(x-y-1)}}{\cancel{(x-y-1)}(\sqrt{x-y}+1)}$

$= \dfrac{1}{\sqrt{1-0}+1} = \dfrac{1}{2}$

c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x+y} = \frac{0}{0}$

Along $x=0$,

$\displaystyle\lim_{y\to 0} \frac{y}{0+y} = 1$

Along $y=0$,

$\displaystyle\lim_{x\to 0} \frac{0}{x+0} = 0 \neq 1$

$\therefore$ limit DNE

ANS: a) -1. b) 1/2. c) DNE.

Using Polar Coordinates to Evaluate Limits

In some cases, a conversion of the function in Cartesian coordinates to polar coordinates can assist in the limit evaluation. For example, the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$$

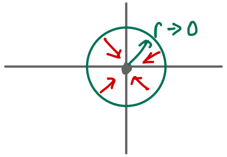
seems to converge to zero along multiple paths, i.e.

$$\lim_{(x,y)\to(0,0)\ (y=0)} \frac{x^2 y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0, \qquad \lim_{(x,y)\to(0,0)\ (x=0)} \frac{x^2 y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{y^2} = 0,$$

$$\lim_{(x,y)\to(0,0)\ (y=x)} \frac{x^2 y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3}{2x^2} = \lim_{(x,y)\to(0,0)} \frac{x}{2} = 0$$

But the **multiple-path test cannot be used to prove that a limit exists**.

Using Polar Coordinates to Evaluate Limits



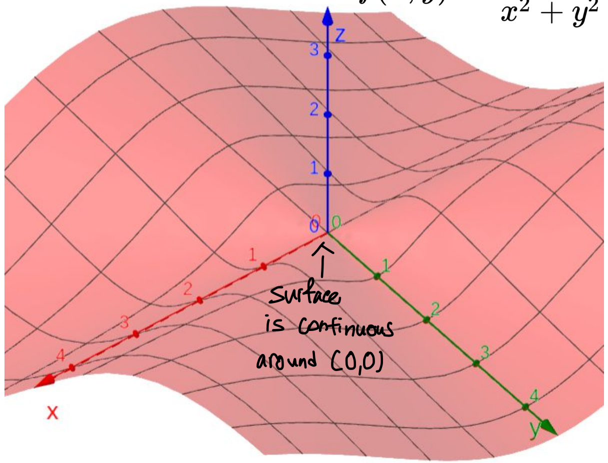
Now, if we substitute x = rcos(θ) and y = rsin(θ) to convert the function to polar coordinates, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{\underbrace{x^2+y^2}_{r^2(\sin^2\theta+\cos^2\theta)}} = \lim_{r\to0}\frac{r^2\cos^2\theta\cdot r\sin\theta}{r^2} = \lim_{r\to0} r\underbrace{\cos^2\theta\sin\theta}_{\text{bounded}} = 0$$

Take note that **as (x, y) approaches (0, 0) from all paths**, this means **r approaches 0**. Conceptually, this means a **circle with radius r shrinks to a point** at (0, 0).

The limit can be verified graphically as shown.

$$f(x,y) = \frac{x^2y}{x^2+y^2}$$



Using Polar Coordinates to Evaluate Limits

Exercise: Evaluate the limit below using polar coordinate substitution. Verify graphically.

$$\lim_{(x,y)\to(0,0)} \left\{ \frac{2x^3 + y^5}{\sqrt{x^2+y^2}} + 1 \right\}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$$= \lim_{r\to 0} \left\{ \frac{2r^3\cos^3\theta + r^5\sin^5\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} + 1 \right\}$$

$$= \lim_{r\to 0} \left\{ \frac{r^{\cancel{3}}(2r\cos^3\theta + r^2\sin^5\theta}{\cancel{r}} + 1 \right\} = 0 + 1$$

$$= 1$$

ANS: 1.

Continuity

The continuity requirements for a multivariable function is similar to those in a single-variable function, i.e.

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

which implies that the **limit exists** and **must be equal to the function value at x** = **a**. For a function of two variables, we have

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

which graphically means there is a continuous 'patch' of surface in the neighbourhood of (a, b). For example, the polynomial function

$$f(x,y) = x^2y + x + 4y + 3xy^2 + y + 2y^3$$

is **continuous in $\mathbb{R}^2$** because the above continuity relation is **satisfied for all (a, b)**.

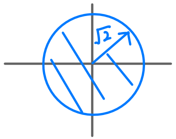
Continuity

Example: Determine the region where each function below is continuous.

a) $f(x,y) = \begin{cases} \sqrt{2-x^2-y^2}, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0) \end{cases}$

Domain of f is $2-x^2-y^2 \geq 0$

$\Rightarrow x^2+y^2 \leq 2$



Check : $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \sqrt{2-x^2-y^2} = \sqrt{2} \neq 2$

$\neq f(0,0)$

$f(x,y)$ is cont. in $x^2+y^2 \leq 2 \,/\, (x,y) \neq (0,0)$

b) $T(y,z) = e^{-z}[\cos(2y) + \sin(2y)]$

Check : $\lim\limits_{(y,z)\to(a,b)} T(y,z) = T(a,b)$

Since the exponential, Cosine & Sine functions are continuous everywhere

$\therefore$ $T(y,z)$ is continuous in $\mathbb{R}^2$.

ANS: a) On the disc $x^2 + y^2 \leq 2$, excluding (0, 0). b) $\mathbb{R}^2$ ($-\infty < y < \infty$, $-\infty < z < \infty$).

Continuity

Exercise: Where is each function below discontinuous?

a) $f(x,y) = \begin{cases} \dfrac{x+y}{2x-y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Domain is $2x - y \neq 0 \rightarrow y \neq 2x$

But since $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{0}{0}$, hence the limit might exist

Try along $x=0$, $\lim_{y\to 0} \frac{0+y}{0-y} = -1 \neq 0 \neq f(0,0)$

$f(x,y)$ is discontinuous at $(0,0)$ & along $y=2x$

However, for the limit to exist, all paths to (0,0) must give the limit. Yet, the path y = 2x cannot be taken, since it is not in the domain of f(x,y). So the limit DNE, and hence the f(x,y) is discontinuous at (0,0).

b) $g(x,y) = \begin{cases} \dfrac{xy^2}{4x^2+y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$

Domain is $\mathbb{R}^2 \mid (x,y) \neq (0,0)$

Check at $(0,0)$:

Along $x=0$, $\lim_{y\to 0} \frac{0\cdot y^2}{0+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0 \neq 1$
$\neq f(0,0)$

So $g(x,y)$ is discont. at $(0,0)$

ANS: a) Along y = 2x, including (0, 0). b) At (0, 0).

Partial Derivatives

Recall that the derivative of a single-variable function is

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

which represents the instantaneous rate of change of a function w.r.t. the input x. For a multivariable function, say f(x, y), we can apply the same concept, but since there is **more than one input variable**, we can define a **'partial'** derivative

'del'

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

which represents the instantaneous change of f(x, y) w.r.t. **only** the input x. The derivative is called **'partial'** because it **does not consider changes of the function w.r.t. other input variables**. We use the letter '**∂**' instead of 'd' to denote **partial** derivatives.

Partial Derivatives

$f(x,y) = x^2 + xy + y^2$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy + y^2) = 2x + y + 0$

Graphically, the partial derivative represents the slope of the tangent of the intersection curve between the surface f(x, y) and a vertical plane at some y coordinate, as shown. For the f(x,y) shown,

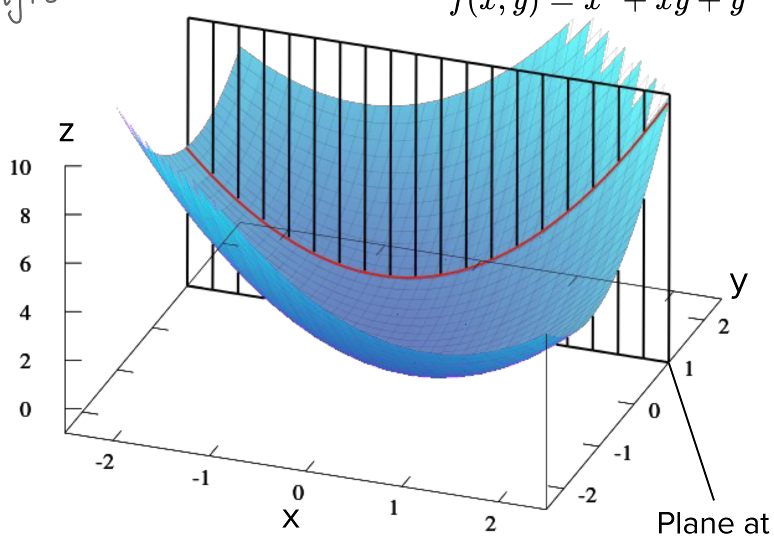
treat as constant

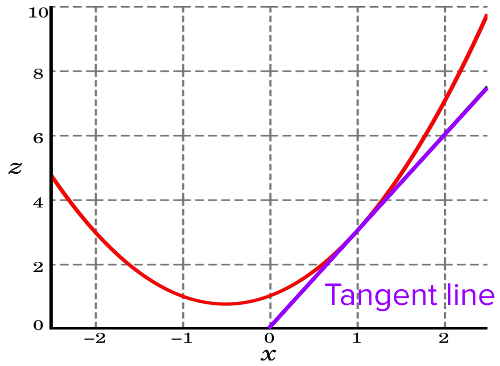
$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + (x+h)y + y^2 - \left[x^2 + xy + y^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + xy + hy + y^2 - \left[x^2 + xy + y^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 + hy}{h} = \lim_{h \to 0} 2x + h + y = 2x + y$$





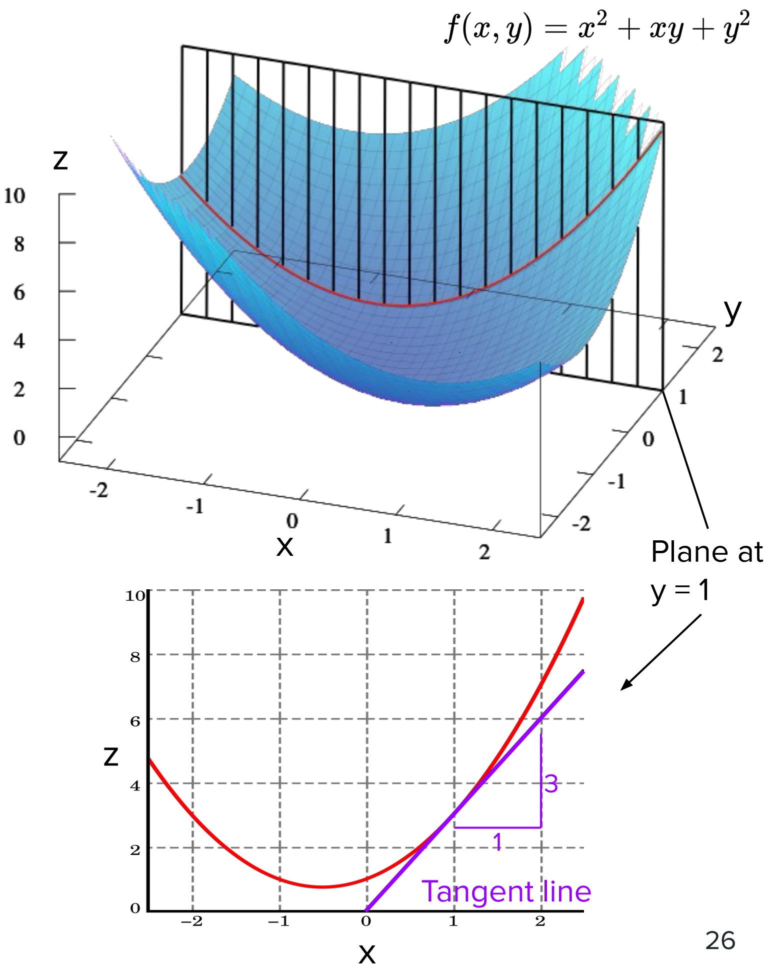
Partial Derivatives

The partial derivative $f_x(x, y) = 2x + y$ gives the gradient of f(x, y) w.r.t. x at any coordinate (x, y). For example, at (1, 1) shown in the graphs, the gradient is

$$f_x(1, 1) = 2(1) + 1 = 3$$

which agrees with the slope of the tangent line shown.

Now, notice that in the limit evaluation, there is **no change of the variable y**. This means one can differentiate f(x, y) w.r.t. x while **treating y as a constant** to obtain $f_x(x, y)$ using the table of derivatives.



Partial Derivatives

Example: Evaluate the partial derivatives of the function below directly. Verify $f_y(x, y)$ using the limit definition. Use the graph to illustrate the meaning of $f_y(x, y)$.

$$f(x, y) = x^2 + xy + y^2$$

$f_x(x,y) = 2x+y$, $f_y(x,y) = \frac{\partial}{\partial y}(x^2+xy+y^2)$

treat x as constant

$= 0 + x + 2y$

$= x + 2y$

$f(-1,y) = (-1)^2 + (-1)y + y^2$

$= y^2 - y + 1$

$= h(y)$

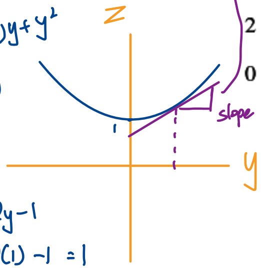
$h'(y) = 2y - 1$

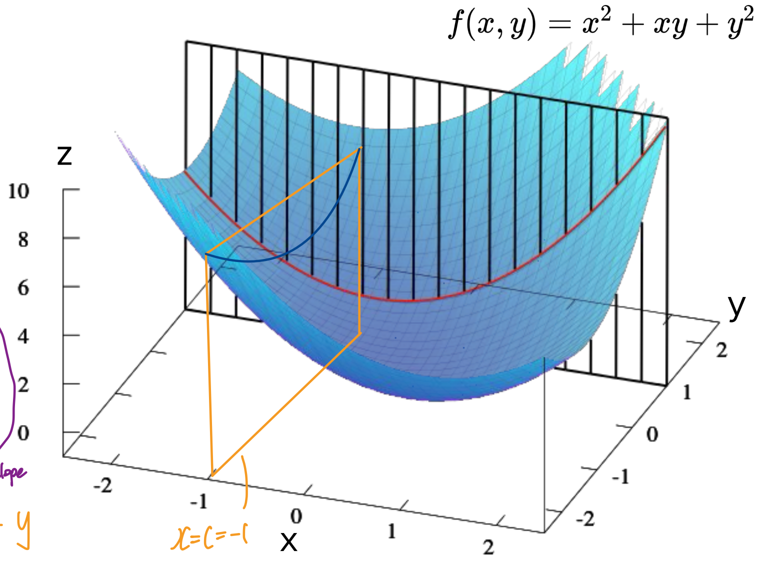
$h'(1) = 2(1) - 1 = 1$

Eg) $f_y(-1,1) = -1 + 2(1)$

$= 1$

same





ANS: $f_x(x, y) = 2x + y$. $f_y(x, y) = x + 2y$.

Partial Derivatives

$\frac{d}{dx}(a^x) = a^{kx} \ln a (k)$

Exercise: Evaluate the partial derivatives of each function below directly.

a) $f(x,y) = xe^{xy} + \ln(xy)$

product rule

$f_x(x,y) = \left[(e^{xy} + x(ye^{xy})\right] + \frac{y}{xy}$

$= e^{xy}(1+xy) + \frac{1}{x}$

$f_y(x,y) = x(xe^{xy}) + \frac{x}{xy}$

$= x^2 e^{xy} + \frac{1}{y}$

b) $g(x,y) = \sin\left(\frac{x}{1+2y}\right)$

$g_x(x,y) = \cos\left(\frac{x}{1+2y}\right) \cdot \left(\frac{1}{1+2y}\right)$

$= \left(\frac{1}{1+2y}\right)\cos\left(\frac{x}{1+2y}\right)$

$g_y(x,y) = \cos\left(\frac{x}{1+2y}\right) \cdot \left(x\left[\frac{(-1)\cdot 2}{(1+2y)^2}\right]\right)$

$= \frac{-2x}{(1+2y)^2}\cos\left(\frac{x}{1+2y}\right)$

c) $h(x,y,z) = z^{xy}$

$h_x = \underbrace{z^{xy}}_{\text{like } a^{kx}} \ln z \,(y) = yz^{xy}\ln z$

$h_y = z^{xy} \ln z\,(x)$

$= xz^{xy}\ln z$

$h_z = xyz^{xy-1}$

ANS: a) $f_x = e^{xy}(1+xy) + 1/x,\ f_y = x^2e^{xy} + 1/y.$ b) $g_x = \frac{1}{1+2y}\cos\left(\frac{x}{1+2y}\right),\ g_y = \frac{-2x}{(1+2y)^2}\cos\left(\frac{x}{1+2y}\right).$

c) $h_x = yz^{xy}\ln z,\ h_y = xz^{xy}\ln z,\ h_z = xyz^{(xy-1)}.$

Partial Derivatives

Exercise: Evaluate the partial derivatives of the implicit function z = f(x, y) in the equation below.

$$x^3 + y^3 + z^3 + \overbrace{6xyz}^{6y(xz)} = 1 \quad - \quad Eq^n$$

$$\frac{\partial}{\partial x}\{Eq^n\} \rightarrow 3x^2 + 0 + 3z^2 . z_x + 6y[z + xz_x] = 0$$

$$z_x(3z^2 + 6xy) = -3x^2 - 6yz$$

$$z_x = \frac{-3x^2 - 6yz}{3z^2 + 6xy} = \frac{-x^2 - 2yz}{z^2 + 2xy}$$

$$\frac{\partial}{\partial y}\{Eq^n\} \rightarrow 0 + 3y^2 + 3z^2 . z_y + 6x[z + yz_y] = 0$$

$$z_y(3z^2 + 6xy) = -3y^2 - 6xz$$

$$z_y = \frac{-3y^2 - 6xz}{3z^2 + 6xy} = \frac{-y^2 - 2xz}{z^2 + 3xy}$$

ANS: $z_x = \dfrac{-x^2 - 2yz}{z^2 + 2xy}, \; z_y = \dfrac{-y^2 - 2xz}{z^2 + 2xy}.$

Higher-order Partial Derivatives

Just like single-variable functions, multivariable functions have higher-order (partial) derivatives. For example,

$$f(x,y) = x^3y^2 + \cos(xy) \rightarrow f_x(x,y) = 3x^2y^2 - y\sin(xy)$$
$$\rightarrow f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x(x,y) = 6xy^2 - y^2\cos(xy)$$

$$f(x,y) = x^3y^2 + \cos(xy) \rightarrow f_y(x,y) = 2x^3y - x\sin(xy)$$
$$\rightarrow f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} f_y(x,y) = 2x^3 - x^2\cos(xy)$$

Generally, the nth-order partial derivative is found by **differentiating** the (n-1)th-order partial derivative. Rules of differentiation are applied as usual.

Mixed Partial Derivatives

For multivariable functions, there are mixed partial derivatives as well. For example,

$$f(x,y) = x^3y^2 + \cos(xy) \rightarrow f_x(x,y) = 3x^2y^2 - y\sin(xy)$$
$$\rightarrow f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} f_x(x,y) = 6x^2y - [\sin(xy) + xy\cos(xy)]$$

$$f(x,y) = x^3y^2 + \cos(xy) \rightarrow f_y(x,y) = 2x^3y - x\sin(xy)$$
$$\rightarrow f_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} f_y(x,y) = 6x^2y - [\sin(xy) + xy\cos(xy)]$$

Notice that the mixed partial derivatives $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are **equal**. This is **always true** if $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are **continuous**, which is called **Clairaut's Theorem** (to be proven in Tutorial 4).

Higher-order Partial Derivatives

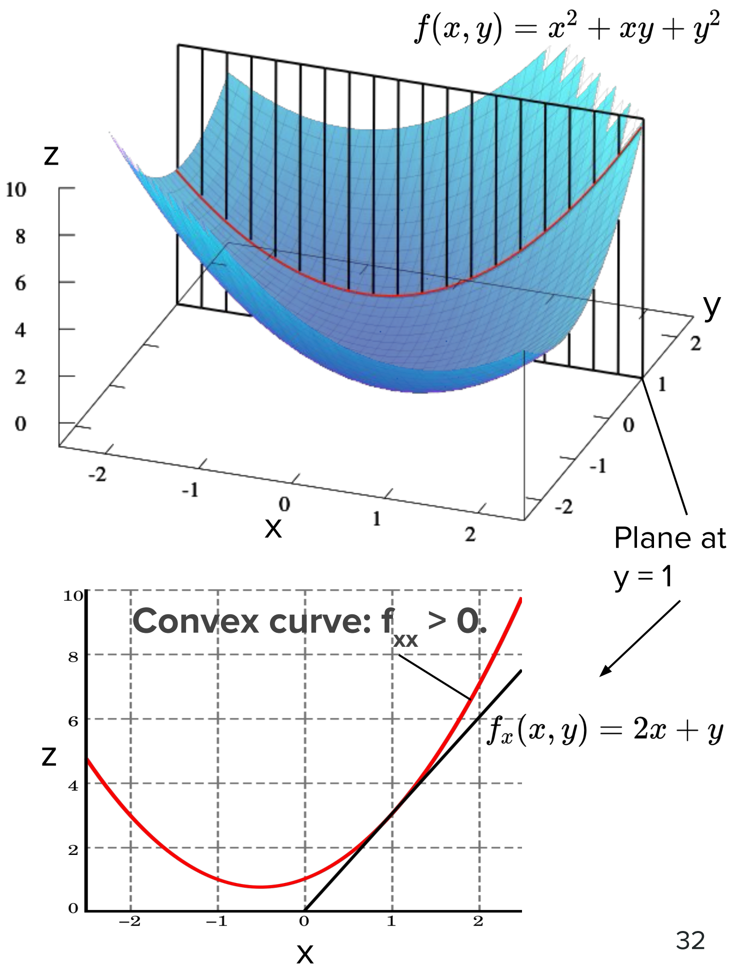
The higher-order derivatives in multivariable functions have similar meanings to those in single-variable functions. For example, for

$$f(x,y) = x^2 + xy + y^2$$

we have

$$f_x(x,y) = 2x + y, \;\; f_y(x,y) = 2y + x$$
$$\rightarrow f_{xx}(x,y) = 2 > 0, \;\; f_{yy}(x,y) = 2 > 0$$

which means the intersection curve is **convex (concave up)** at all plane sections at any y & x value. The slope is always increasing at a rate of 2 in both the x & y directions.



Mixed Partial Derivatives

For mixed partial derivatives, they describe how the slope in one direction changes w.r.t. **another direction**. For example, for

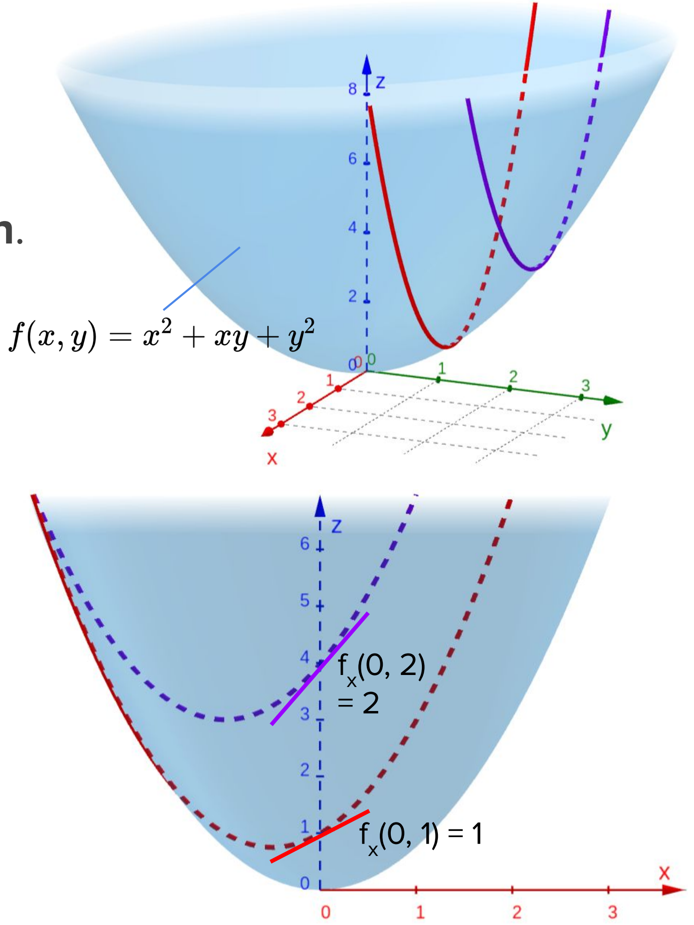
$$f(x,y) = x^2 + xy + y^2$$

$$\rightarrow f_x(x,y) = 2x + y, \quad f_y(x,y) = 2y + x$$

$$\rightarrow f_{xy}(x,y) = f_{yx}(x,y) = 1$$

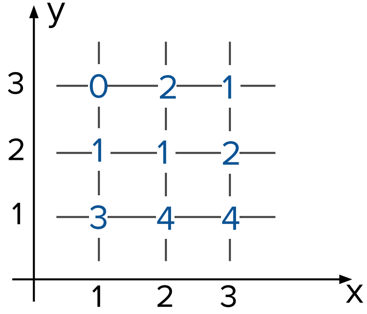
This means the slope in the x-direction **increases at a rate of 1 in the y-direction**. It also means the slope in the y-direction **increases at the rate of 1 in the x-direction**.

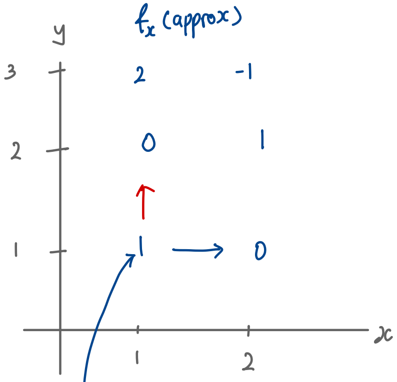
Intersection curves with planes at y = 1, y = 2.



Higher-order Partial Derivatives

Exercise: The grid below represents a region of f(x, y) where the function values are indicated at each coordinate point. Estimate f_{xx}, f_{yy}, f_{xy} and f_{yx} at (1, 1). Hint: Estimate the slope values in the x & y directions first.

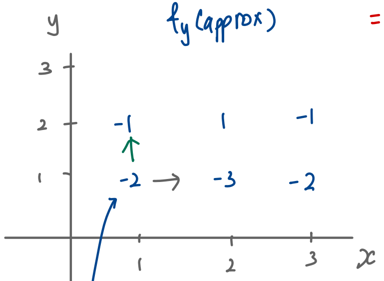




$f_x(1,1) \approx \frac{4-3}{2-1} = 1$

$f_{xx}(1,1) \approx \frac{\Delta f_x}{\Delta x} \approx \frac{0-1}{1} = -1$

$f_{xy}(1,1) \approx \frac{\Delta f_x}{\Delta y} \approx \frac{0-1}{1} = -1$



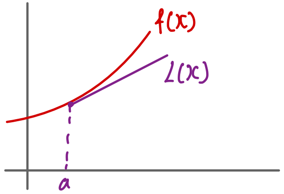
$f_y(1,1) \approx \frac{\Delta f}{\Delta y} = \frac{1-3}{2-1} = -2$

$f_{yy}(1,1) \approx \frac{\Delta f_y}{\Delta y} = \frac{-1-(-2)}{1} = 1$

$f_{yx}(1,1) \approx \frac{\Delta f_y}{\Delta x} = \frac{-3-(-2)}{1} = -1$ $(= f_{xy})$

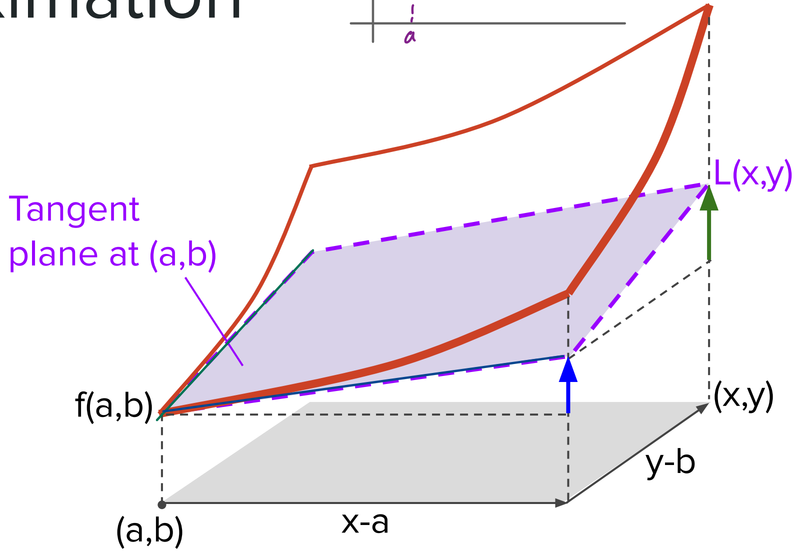
ANS: At (1, 1), $f_{xx} \approx -1$, $f_{yy} \approx 1$, $f_{xy} = f_{yx} \approx -1$.

Tangent Plane & Linear Approximation



For single-variable functions, recall that the linear approximation is defined as

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$



where L(x) represents the tangent line to the curve of f(x) at x = a. Analogously, we can also define the linear approximation for multivariable functions like f(x, y) as

$$f(x,y) \approx L(x,y) = f(a,b) + \underbrace{f_x(a,b)(x-a)}_{\substack{\text{Approx. change in f(x,y)} \\ \text{due to change in x.}}} + \underbrace{f_y(a,b)(y-b)}_{\substack{\text{Approx. change in f(x,y)} \\ \text{due to change in y.}}}$$

where L(x, y) here represents the tangent plane to the surface of f(x, y) at (a, b).

Total Differential

Now, in a region **infinitesimally close** to a point (x, y), we can imagine that the surface f(x, y) is **'coincidental'** with its tangent plane at (x, y). Hence we have

$$f(x+\underbrace{dx}_{dx\to 0}, y+\underbrace{dy}_{dy\to 0}) = f(x,y) + f_x(x,y)dx + f_y(x,y)dy$$

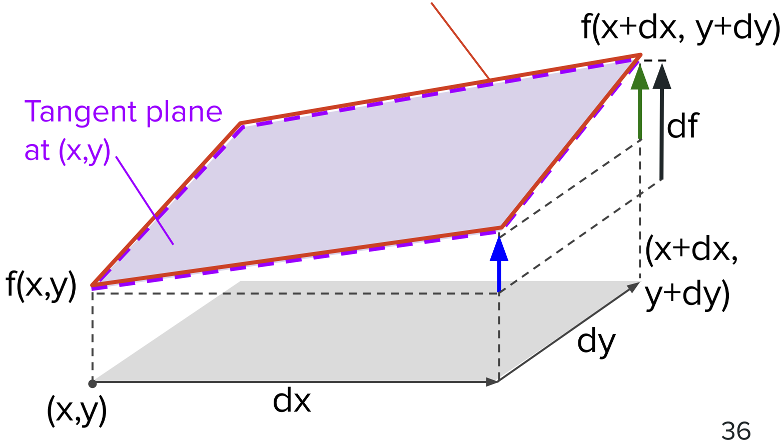
Then, the total differential is defined as

like total change

$$\begin{aligned} df &= f(x+dx, y+dy) - f(x,y) \\ &= \underbrace{f_x(x,y)dx}_{\text{Change in f(x,y) due to infinitesimal change in x.}} + \underbrace{f_y(x,y)dy}_{\text{Change in f(x,y) due to infinitesimal change in y.}} \end{aligned}$$

which represents the total change in f(x, y) when inputs x & y change infinitesimally. Notice the **total change is a simple sum of partial changes**.

f(x,y) approaches the tangent plane as dx ➔ 0, dy ➔ 0.



Chain Rule

Recall that for single-variable functions, such as f(x) where x = x(t), the chain rule for evaluating f'(t) is

$$f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = f'(x)\, x'(t)$$

For multivariable functions, such as f(x, y) where x = x(t) and y = y(t), the chain rule applies in a similar manner. By **'dividing' the total differential df by dt**, we get

$$f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = \underbrace{f_x(x, y)\, x'(t)}_{\text{ROC in f due to ROC in x}} + \underbrace{f_y(x, y)\, y'(t)}_{\text{ROC in f due to ROC in y}}$$

which means the **total rate of change (ROC)** in f(x, y) is the **sum of partial ROCs** due to each input variable x & y changing w.r.t. t.

Chain Rule

If the input variables are x = x(s, t) & y = y(s, t) in f(x,y), then **'dividing' the total differential df by ds and dt** respectively yields the chain rule(s)

$$f_s(s,t) = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = \underbrace{f_x(x,y)\, x_s(s,t)}_{\text{Partial ROC in f due to partial ROC in x (w.r.t. s)}} + \underbrace{f_y(x,y)\, y_s(s,t)}_{\text{Partial ROC in f due to partial ROC in y (w.r.t. s)}}$$

$$f_t(s,t) = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = \underbrace{f_x(x,y)\, x_t(s,t)}_{\text{Partial ROC in f due to partial ROC in x (w.r.t. t)}} + \underbrace{f_y(x,y)\, y_t(s,t)}_{\text{Partial ROC in f due to partial ROC in y (w.r.t. t)}}$$

The general principle in applying the chain rule to differentiate any multivariable function is to **sum up all partial ROCs w.r.t. each input parameter** (s, t, etc).

Chain Rule

$f(s,t) = g(x(s,t),\, y(s,t))$

Example: Determine $f_s(s, t)$ and $f_t(s, t)$ for the functions defined below.

$$g\ \cancel{f}(x,y) = x^2y^3,\quad x(s,t) = e^{st},\quad y(s,t) = t\sin s$$

$$f_s = g_x \cdot x_s + g_y \cdot y_s = 2xy^3 \cdot te^{st} + 3x^2y^2 \cdot t\cos s$$

$$= 2e^{st}(t^3\sin^3 s) \cdot te^{st} + 3e^{2st}t^2\sin^2 s \cdot t\cos s$$

$$= t^3e^{2st}\sin^2 s\,(2t\sin s + 3\cos s)$$

$$f_t = g_x \cdot x_t + g_y \cdot y_t = 2xy^3 \cdot se^{st} + 3x^2y^2 \cdot \sin s$$

$$= 2e^{st}(t^3\sin^3 s) \cdot se^{st} + 3e^{2st}t^2\sin^2 s \cdot \sin s$$

$$= t^2e^{2st}\sin^3 s\,(2st + 3)$$

ANS: $f_s(s,t) = t^3 e^{2st}\sin^2 s \cdot (2t\sin s + 3\cos s),\ f_t(s,t) = t^2 e^{2st}\sin^3 s \cdot (2st + 3).$

Chain Rule

$f(t) = g(x(t), y(t), z(t))$

Exercise: Write down the chain rule for the functions defined below and evaluate f'(t). Verify the chain rule by differentiating f(t) directly.

$$g\ \cancel{f}(x,y,z) = \frac{xy}{z}, \quad x(t) = t^2, \quad y(t) = \ln t, \quad z(t) = 1/t$$

$$f'(t) = g_x \cdot x_t + g_y \cdot y_t + g_z \cdot z_t$$

$$= \frac{y}{z} \cdot 2t + \frac{x}{z} \cdot \frac{1}{t} + \frac{-xy}{z^2} \cdot -\frac{1}{t^2}$$

$$= \frac{\ln t}{1/t} \cdot 2t + \frac{t^2}{1/t} \cdot \frac{1}{t} + \frac{-(t^2)(\ln t)}{(1/t)^2} \cdot -\frac{1}{t^2}$$

$$= 2t^2 \ln t + t^2 + t^2 \ln t$$

$$= t^2 (3 \ln t + 1)$$

ANS: $f'(t) = t^2(3\ln t + 1)$

Differentiability of a Multivariable Function

Recall a single-variable function is differentiable if it is a smooth curve, which means a **tangent line can be drawn** anywhere on the curve. Similarly, for a multivariable function such as f(x, y) to be differentiable, it must be smooth surface, which means the **tangent plane can be drawn** anywhere on the surface.

Mathematically, differentiability conditions are that **partial derivatives must exist and be continuous**. Differentiability also implies continuity, but **not vice-versa**.

Example: Determine the region where the function below is differentiable. Verify with its graph.

$$f(x, y) = \left(x^2 + 2y^2\right)^{1/3}$$

$f_x(x,y) = \frac{1}{3}(x^2+2y^2)^{-2/3} \cdot (2x)$

$= \dfrac{2x}{3(x^2+2y^2)^{2/3}}$

Notice that f(x,y) is Continuous in R^2

$f_y(x,y) = \frac{1}{3}(x^2+2y^2)^{-2/3} \cdot (4y)$

$= \dfrac{4y}{3(x^2+2y^2)^{2/3}}$

f_x & f_y are continuous everywhere except possibly at (0,0)

ANS: $\mathbb{R}^2$ | (x, y) ≠ (0, 0).

Check at $(x,y) = (0,0)$

$$\lim_{(x,y)\to(0,0)} f_x$$

Along path $x=0$, $L_1 = \lim_{y\to 0} \frac{2(0)}{3(0+2y^2)^{2/3}} = 0$

Along path $y=0$, $L_2 = \lim_{x\to 0} \frac{2x}{3(x^2+0)^{2/3}} = \lim_{x\to 0} \frac{2x}{3x^{4/3}}$

$= \lim_{x\to 0} \frac{2}{3x^{1/3}}$ DNE

since $\lim_{(x,y)\to(0,0)} f_x$ DNE, $f_x(x,y)$ is discont at $(0,0)$

so $f(x,y)$ is not differentiable at $(0,0)$

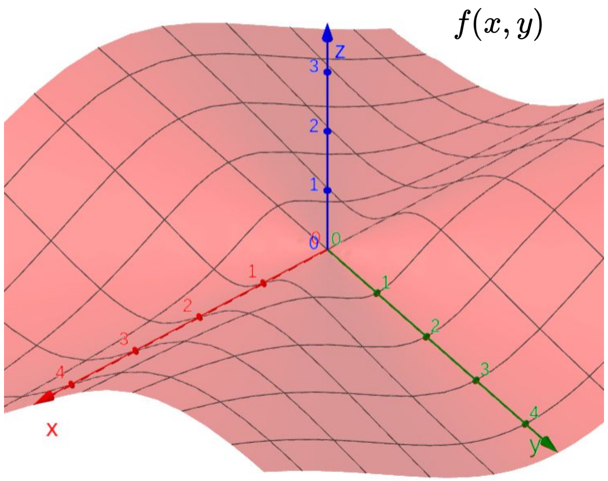
Differentiability of a Multivariable Function

It is important to note that even though a multivariable function can have continuity and existence of partial derivatives, it **might not** be differentiable as shown by the example below.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

From an earlier slide, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0 = f(0,0)$$



which means f(x,y) is continuous everywhere, including at (0, 0). The graph shows this clearly.

Differentiability of a Multivariable Function

And partial derivatives at (0, 0) can be evaluated to be

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0/h^2-0}{h}=0,$$
$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0/h^2-0}{h}=0$$

So partial derivatives exists at (0, 0). We can find that

$$f_x(x,y)=\begin{cases}\dfrac{2xy^3}{\left(x^2+y^2\right)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases},\quad f_y(x,y)=\begin{cases}\dfrac{x^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Now, we need to check **continuity of the partial derivatives**.

Differentiability of a Multivariable Function

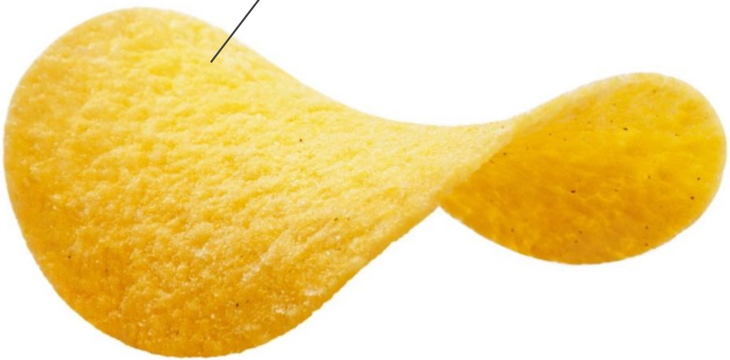
Along y = x, we get

$$\lim_{(x,y)\to(0,0)\ (y=x)} f_x(x,y) = \lim_{x\to 0} \frac{2x^4}{\left(x^2+x^2\right)^2} = \frac{1}{2} \neq f_x(0,0)$$

which is **discontinuous** at (0, 0). This means f(x,y) is **not differentiable** at (0, 0), which can somewhat be observed from the **'seemingly non-smooth'** point at (0, 0) from the graph. The same conclusion can be reached from the discontinuity of $f_y(0, 0)$, such as from the path y = 0.

Hence, continuity of a multivariable function and existence of partial derivatives does not guarantee differentiability (smoothness). **Partial derivatives must also be continuous**.

$$f(x, y) = ax^2 - by^2$$



End of Topic 3

Almost everything can be described by a multivariable function. Even a potato chip.