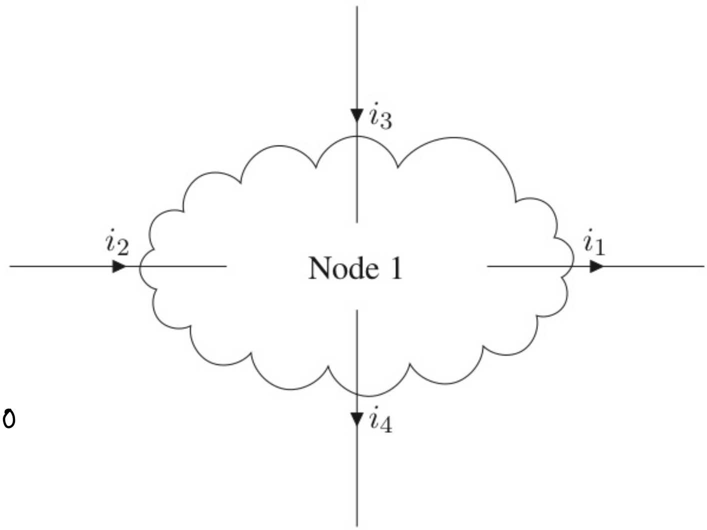
1.1 Kirchhoff's Laws

a) Formulate the Kirchhoff's current law for the depicted node.

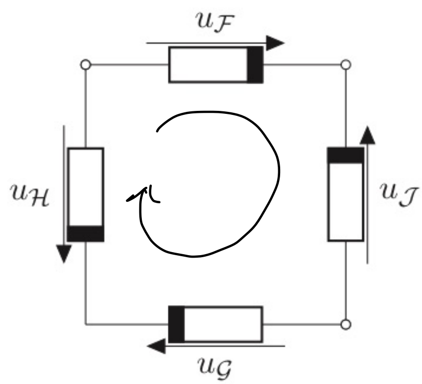


$i_2 + i_3 - i_1 - i_4 = 0$

b) Give the Kirchhoff's current law in general form. $\sum_{j \in \text{branches connected to node}} i_j(t) = 0$

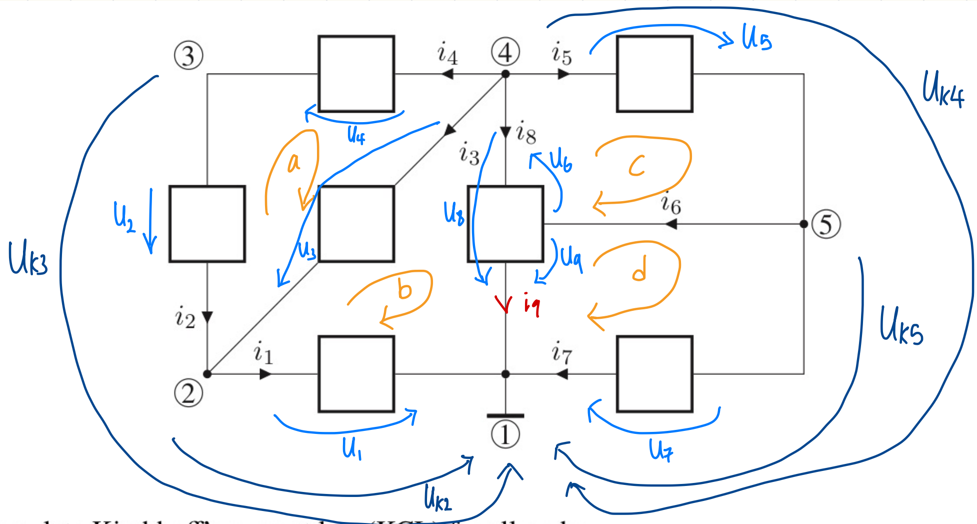
c) Formulate the Kirchhoff's voltage law for the depicted network.

$U_F + U_G - U_J - U_H = 0$



d) Give the Kirchhoff's voltage law in general form. $\sum_{j \in \text{branches of loops}} U_j(t) = 0$

1.2



a) Formulate Kirchhoff's current law (KCL) for all nodes.

b) Find the Kirchhoff's voltage law (KVL) equations for some loops.

c) Give the alternative form of KVL by expressing all branch voltages depending on the node voltages.

a) KCL①: $i_1 + i_7 + i_9 = 0$

KCL②: $i_1 - i_2 - i_3 = 0$

KCL③: $i_2 - i_4 = 0$

KCL④: $i_3 + i_4 + i_5 + i_8 = 0$

⑤: $-i_5 + i_6 + i_7 = 0$

b) KVL ⓐ: $-U_2 - U_4 + U_3 = 0$

ⓑ: $-U_3 + U_8 - U_1 = 0$

ⓒ: $U_5 + U_6 = 0$

ⓓ: $-U_9 + U_7 = 0$

c) $U_1 = U_{k2} - \cancel{U_{k1}}^{0} = U_{k2}$

$U_2 = U_{k3} - U_{k2}$

$U_3 = U_{k4} - U_{k2}$

$U_4 = U_{k4} - U_{k3}$

$U_5 = U_{k4} - U_{k5}$

$U_6 = U_{k5} - U_{k4}$

$U_7 = U_{k5} - \overset{0}{\cancel{U_{k4}}} = U_{k5}$

$U_8 = U_{k4}$

$U_9 = U_{k5}$

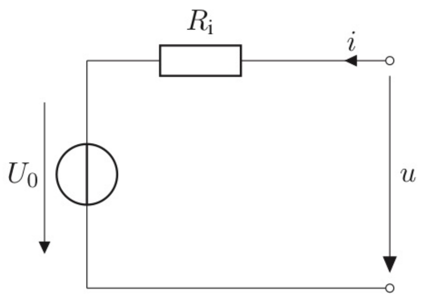
2.1 Linear Sources and Source Transform

a) What is the difference between general linear sources and independent sources? Depict the difference graphically.

b) How are the *internal resistance* and *internal conductance* of linear sources defined?

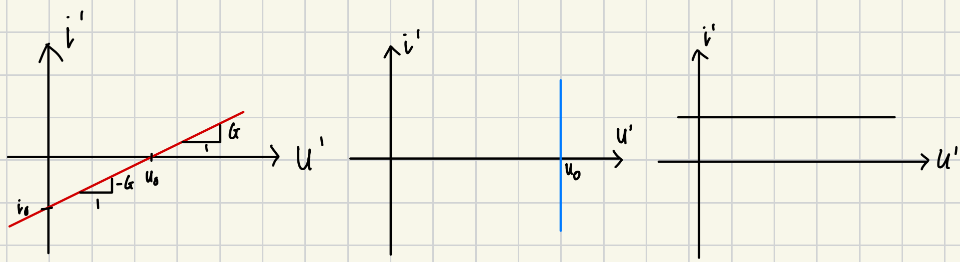
By source transform, a linear voltage source combined with an internal resistance can be equivalently transformed to a current source combined with an internal conductance.

Hence, consider the following circuit.



c) Draw the equivalent circuit with a current source and internal conductance. Give all parameters depending on U_0 and R_i.

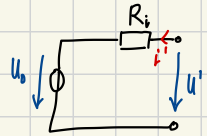
a)



b) $R = \frac{U_0}{i_0}$

$G = \frac{1}{R} = \frac{i_0}{U_0}$

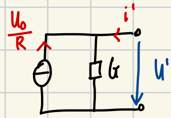
c)



KVL: $R_i' + U_0 - U' = 0$

$u' = R_i' + U_0$

$i' = \frac{1}{R_i} u' - \frac{U_0}{R_i}$



$G = \frac{1}{R_i}$

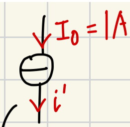
2.2 Resistive Circuits and Duality

a) Apply the duality transform to an independent current source $I_0 = 1\,\mathrm{A}$ with the duality constant $R_\mathrm{d} = 2\,\Omega$.

Let the duality constant R_d be given.

b) Find the element dual to a strictly linear resistor with conductance G depending on G and R_d.

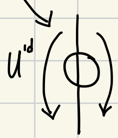
a)



$R_d = 2\Omega$

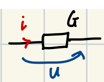
$i' = I_0 \xrightarrow{R_d} \frac{1}{R_d} u'^d = I_0$

$u'^d = R_d I_0$



$R_d I_0 = 2V$

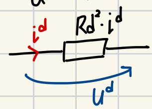
b)



Ohm: $i = Gu$

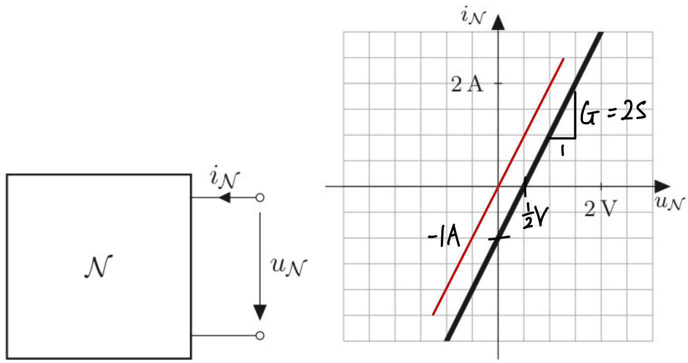
$\xrightarrow{R^d}$ $\frac{1}{R_d} u^d = G R_d i^d$

$u^d = R_d^2 i^d$



2.7 Source Transform

Given is the linear one-port $\mathcal{N}$ whose characteristic is depicted in the following diagram.



a) The circuit can be represented by two different equivalent circuit diagrams which only contain two network elements. Give the two equivalent circuit diagrams with the element values.

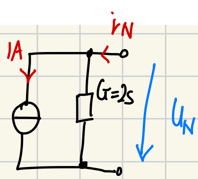
Now the following internal structur of $\mathcal{N}$ is given.

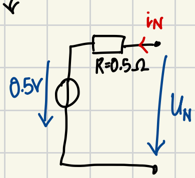
$$G = \frac{1A}{0.5V} = 2S$$

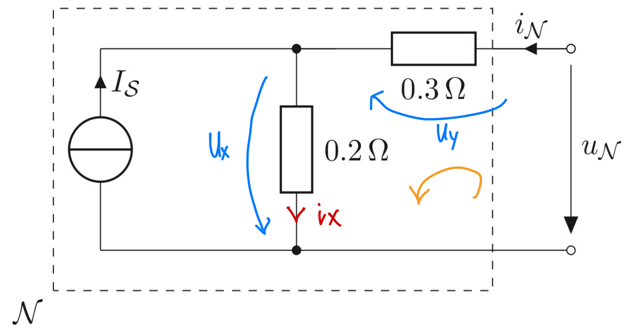
$$R = \frac{1}{G} = 0.5\Omega$$

$$i_N = 2S\, u_N - 1A$$

$$U_N = 0.5\Omega\, i_N + 0.5V$$







b) Determine the source current I_S. (Hint: With the help of the characteristic, find out the behaviour of the circuit if it is connected to an open circuit.)

KCL : $I_s + i_N - i_x = 0$

KVL : $-U_N + U_y + U_x = 0$

$U_y = i_x \cdot 0.3\Omega$, $U_x = i_x \cdot 0.2\Omega$

$i_x = I_s + i_N$

$U_x = 0.2\Omega\,(I_s + i_N)$

$U_N = U_y + U_x$

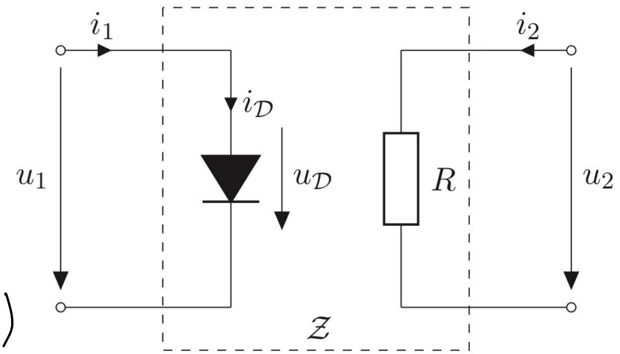
$= 0.3\Omega\, i_N + 0.2\Omega\, I_s + 0.2\, i_N$

$= 0.5\Omega\, i_N + 0.2\Omega\, I_s$

$0.5V = 0.2\Omega \cdot I_s \rightarrow I_s = 2.5A$

3.2 Representation of a Non-Linear Two-Port

Consider the following two-port $\mathcal{Z}$.



$$i_D = I_S\left(e^{\frac{U_D}{U_T}} - 1\right)$$

a) Can the two-port $\mathcal{Z}$ be represented by a matrix?

non-linear two port $\Rightarrow$ no matrix represent

b) Give the current-controlled representation of $\mathcal{Z}$.

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = f\left(\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}\right)$$

Ohm: $U_2 = R i_2$

$$\frac{i_D}{I_S} = e^{\frac{U_D}{U_T}} - 1$$

$$U_D = U_T \ln\left(\frac{i_D}{I_S} + 1\right)$$

Kirchhoff:

$U_D = U_1$, $i_D = i_1$

$$U_1 = U_T \ln\left(\frac{i_1}{I_S} + 1\right)$$

$$u_1 = r_1(i_1, i_2) = U_T \ln\left(\frac{i_1}{I_S}+1\right)$$

$$u_2 = r_2(i_1, i_2) = R i_2$$

c) Determine the hybrid representation of $\mathcal{Z}$.

hybrid repr.:

$$\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \underline{h}\left(\begin{bmatrix} i_1 \\ u_2 \end{bmatrix}\right)$$

$$u_1 = h_1(i_1, u_2) = U_T \ln\left(\frac{i_1}{I_S}+1\right)$$

$$i_2 = h_2(i_1, u_2) = \frac{1}{R} u_2$$

d) Find the inverse hybrid representation of $\mathcal{Z}$.

$$\begin{bmatrix} i_1 \\ u_2 \end{bmatrix} = \underline{h}'\left(\begin{bmatrix} u_1 \\ i_2 \end{bmatrix}\right)$$

$$i_1 = h_1'(u_1, i_2) = I_s\left(e^{\frac{u_1}{U_T}} - 1\right)$$

$$u_2 = h_2'(u_1, i_2) = R i_2$$

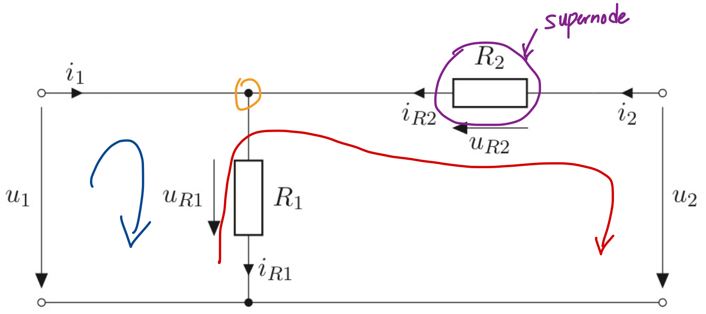
e) Determine the transmission representation of $\mathcal{Z}$.

$$\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = \underline{a}\left(\begin{bmatrix} u_2 \\ -i_2 \end{bmatrix}\right)$$

transmission representation does not exist, since $u_2 = R i_2$ is independent of u_1 & i_1.

3.3 Two-Port Matrices

Given is the following strictly linear two-port comprising two Ohmic resistors.



Use in all calculations the conductances $G_1 = \frac{1}{R_1}$ and $G_2 = \frac{1}{R_2}$ if it allows simpler formulas and results.

a) Formulate all elementary relations for the interior of the two-port.

Now determine the six two-port matrices of the two-port.

KCL: $i_1 + i_{R2} - i_{R1} = 0$

KCL: $i_{R2} - i_2 = 0$

KVL: $-U_{R1} - U_{R2} + U_2 = 0$

KVL: $-U_1 + U_{R1} = 0$

Ohm: $U_{R1} = R_1 \cdot i_{R1}$

$U_{R2} = R_2 \cdot i_{R2}$

b) conductance matrix $\boldsymbol{G}$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underset{\sim}{G} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underset{\sim}{G} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

$$i_1 = g_{11} u_1 + g_{12} u_2$$

$$i_2 = g_{21} u_1 + g_{22} u_2$$

Ohm: $i_{R1} = G_1 \cdot U_{R1}$

$i_{R2} = G_2 \cdot U_{R2}$

KCL: $i_{R2} = i_2$

KVL: $U_{R1} = U_1$

$i_{R1} = G_1 \cdot u_1$

$i_2 = G_2 \cdot U_{R2}$

From KCL: $i_{R1} = i_1 + i_{R2}$

$= i_1 + i_2$

$i_1 + i_2 = G_1 \cdot u_1$

From KVL : $U_{R2} = U_2 - U_{R1}$

$= U_2 - U_1$

$i_2 = G_2 (u_2 - u_1)$

$i_1 + G_2 (u_2 - U_1) = G_1 \cdot U_1$

$i_1 = (G_1 + G_2) u_1 - G_2 u_2$

$$\underset{\sim}{G} = \begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 \end{bmatrix}$$

c) resistance matrix $\boldsymbol{R}$

$\underline{u} = \underset{\sim}{R}\, \underline{i} \longrightarrow \underline{i} = \underset{\sim}{R}^{-1} \underline{u} = \underset{\sim}{G} u$

$U_1 = r_{11} i_1 + r_{12} i_2$

$U_2 = r_{21} i_1 + r_{22} i_2$

$U_1 = R_1 i_{R1}$

$= R_1 i_1 + R_1 i_2$

$$\underset{\sim}{R}^{-1} = \begin{bmatrix} R_1 & R_1 \\ R_1 & R_1 + R_2 \end{bmatrix}^{-1}$$

$$= \frac{1}{\det \underset{\sim}{R}} \begin{bmatrix} R_1 + R_2 & -R_1 \\ -R_1 & R_1 \end{bmatrix}$$

$$= \frac{1}{R_1 (R_1 + R_2) - R_1^2} \begin{bmatrix} R_1 + R_2 & -R_1 \\ -R_1 & R_1 \end{bmatrix}$$

$R_1 R_2$

$$U_2 = U_1 + U_{R2}$$
$$= U_1 + R_2 i_2$$
$$= R_1 i_1 + (R_1+R_2) i_2$$

$$\underset{\sim}{R} = \begin{bmatrix} R_1 & R_1 \\ R_1 & R_1+R_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cancel{R_1}}{\cancel{R_1} R_2} + \frac{\cancel{R_2}}{R_1 \cancel{R_2}} & -\frac{\cancel{R_1}}{\cancel{R_1} R_2} \\ -\frac{\cancel{R_1}}{\cancel{R_1} R_2} & \frac{\cancel{R_1}}{\cancel{R_1} R_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{R_2} + \frac{1}{R_1} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} \end{bmatrix} = \underset{\sim}{G}$$

d) hybrid matrix ***H***

$$\begin{bmatrix} U_1 \\ i_2 \end{bmatrix} = \underset{\sim}{H} \begin{bmatrix} i_1 \\ U_2 \end{bmatrix}$$

$$U_1 = h_{11} i_1 + h_{12} U_2$$
$$i_2 = h_{21} i_1 + h_{22} U_2$$

$$i_1 + i_2 = G_1 U_1$$
$$i_2 = G_2 (U_2 - U_1)$$
$$i_2 = G_1 U_1 - i_1$$
$$G_1 U_1 - i_1 = G_2 U_2 - G_2 U_1$$
$$(G_1+G_2) U_1 = i_1 + G_2 U_2$$
$$U_1 = \frac{1}{G_1+G_2} i_1 + \frac{G_2}{G_1+G_2} U_2$$

$$i_2 = G_2 \left(U_2 - \frac{1}{G_1+G_2} i_1 - \frac{G_2}{G_1+G_2} U_2 \right)$$

$$\frac{G_1+G_2}{G_1+G_2}$$

$$= \boxed{\frac{G_1 G_2}{G_1+G_2}} U_2 - \frac{G_2}{G_1+G_2} i_1$$

$$\Downarrow$$

$$G_1 \parallel G_2 = \left(\frac{1}{G_1} + \frac{1}{G_2} \right)^{-1}$$

$$\underset{\sim}{H} = \begin{bmatrix} \frac{1}{G_1+G_2} & \frac{G_2}{G_1+G_2} \\ \frac{G_1 G_2}{G_1+G_2} & -\frac{G_2}{G_1+G_2} \end{bmatrix}$$

e) inverse hybrid matrix $\boldsymbol{H}'$.

$$\underset{\sim}{H}' = \frac{1}{r_{11}} \begin{bmatrix} 1 & -r_{12} \\ r_{21} & \det \underset{\sim}{R} \end{bmatrix}$$

Pg 42

$$\det \underset{\sim}{R} = R_1 R_2$$

$$\underset{\sim}{R} = \begin{bmatrix} R_1 & R_1 \\ R_1 & R_1 + R_2 \end{bmatrix}$$

$$\underset{\sim}{H}' = \frac{1}{R_1} \begin{bmatrix} 1 & -R_1 \\ R_1 & R_1 R_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{R_1} & -1 \\ 1 & R_2 \end{bmatrix}$$

f) transmission matrix $\boldsymbol{A}$.

$$\underset{\sim}{A} = \frac{1}{g_{21}} \begin{bmatrix} -g_{22} & -1 \\ -\det \underset{\sim}{G} & -g_{11} \end{bmatrix}$$

$$\underset{\sim}{G} = \begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 \end{bmatrix}$$

$$\det \underset{\sim}{G} = G_2 (G_1 + G_2) - G_2^2$$

$$= G_1 G_2$$

$$\underset{\sim}{A} = \frac{1}{-G_2} \begin{bmatrix} -G_2 & -1 \\ -G_1 G_2 & -G_1 - G_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{G_2} \\ G_1 & 1 + \frac{G_1}{G_2} \end{bmatrix}$$

g) inverse transmission matrix $\boldsymbol{A}'$.

$$\underset{\sim}{A}' = \frac{1}{\det \underset{\sim}{A}} \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix}$$

$$\det A = \left(1 + \frac{G_1}{G_2}\right) - \frac{G_1}{G_2} = 1$$

$$\underset{\sim}{A}' = \begin{bmatrix} 1+\frac{G_1}{G_2} & \frac{1}{G_2} \\ G_1 & 1 \end{bmatrix}$$

For strictly linear two-ports, the different elements of the two-port matrices can be measured with the help of a particular connection of the two-port. Compare the results with those of above method.

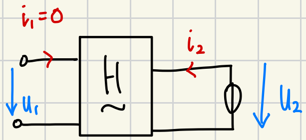
h) Derive the necessary connection to measure the different elements of the hybrid matrix of a strictly linear two-port. Find the hybrid matrix $\boldsymbol{H}$.

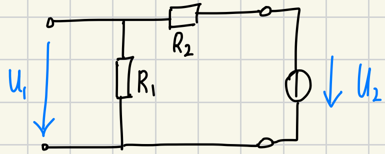
$$\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \underset{\sim}{H} \begin{bmatrix} i_1 \\ u_2 \end{bmatrix}$$

$$u_1 = h_{11} i_1 + h_{12} u_2$$
$$i_2 = h_{21} i_1 + h_{22} u_2$$

for h_{12} : $i_1 = 0 \rightarrow$ Open circuit

define U_2 by Voltage source





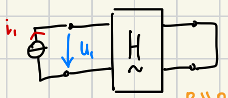
Voltage divider ule

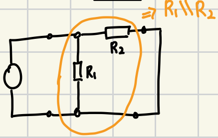
$$\frac{U_1}{U_2} = \frac{R_1}{R_1+R_2}$$

$$= \frac{1/G_1}{1/G_1 + 1/G_2} = \frac{G_2}{G_1+G_2}$$

for h_{11}: $U_2 = 0$, short circuit

define i_1 by current source





ohm: $U_1 = \underbrace{(R_2 \| R_2)}_{h_{11}} i_1$

$$R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

$$= \frac{1}{\frac{1}{R_2} + \frac{1}{R_1}}$$

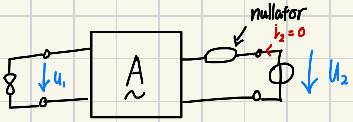
i) Derive the connections such that the different elements of the transmission matrix of a strictly linear two-port can be measured. Determine the transmission matrix $\boldsymbol{A}$.

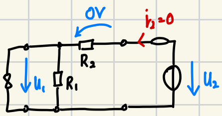
$$\begin{bmatrix} U_1 \\ i_1 \end{bmatrix} = \underset{\sim}{A} \begin{bmatrix} U_2 \\ -i_2 \end{bmatrix}$$

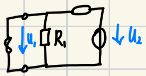
$$U_1 = a_{11} U_2 + a_{12}(-i_2)$$

$$i_1 = a_{21} U_2 + a_{22}(-i_2)$$

for a_{11} : $i_2 = 0 \rightarrow$ open circuit



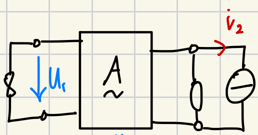


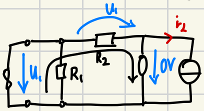


$\Rightarrow U_1 = U_2$

$a_{11} = 1$

for a_{12} : $U_2 = 0$





KVL : $-U_1 + U_{R2} + 0V = 0$

$\rightarrow U_{R2} = U_1 = R_2 \, i_{R2}$

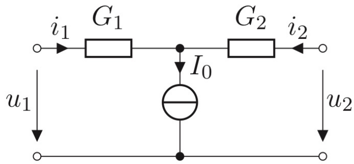
$= R_2 \cdot -i_2$

$U_1 = R_2 \cdot -i_2$

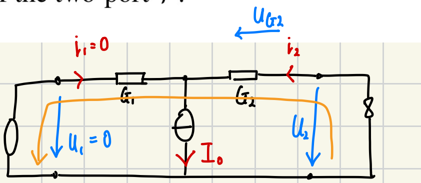
$\Rightarrow a_{12} = R_2$

3.4 Linear Two-Port   not strictly linear

We consider the following two-port $\mathcal{T}$ with $G_1 = 1\,\mathrm{S}$, $G_2 = 2\,\mathrm{S}$, and $I_0 = 4\,\mathrm{A}$.



a) Perform the necessary number of measurements to be able to formulate the parametric representation of the two-port $\mathcal{T}$.

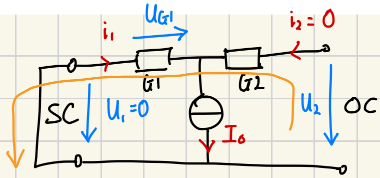


due to nullator: $i_2 = I_0$

Ohm : $U_{G2} = \frac{i_2}{G_2}$

KVL : $-U_2 + U_{G2} + 0V + 0V = 0$

$$U_2 = U_{G2} = \frac{I_0}{G_2}$$

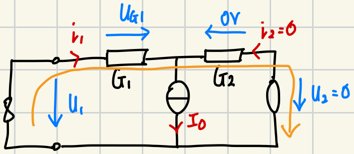


$i_1 = I_0$ (due to $i_2 = 0$)

$$U_{G1} = \frac{I_0}{G_1}$$

KVL: $-U_2 + 0V - U_{G1} + 0V = 0$

$$U_2 = -U_{G1} = -\frac{I_0}{G1}$$



$i_1 = I_0$

$U_{G1} = \frac{i_1}{G_1} = \frac{I_0}{G_1}$

KVL: $-U_1 + U_{G1} - 0 - 0 = 0$

$$U_1 = U_{G1} = \frac{I_0}{G_1}$$

3 measurements

$$\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ i_1^{(1)} \\ i_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0V \\ I_0/G_2 \\ 0A \\ I_0 \end{bmatrix} = \begin{bmatrix} 0V \\ 2V \\ 0A \\ 4A \end{bmatrix}$$

$$\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ i_1^{(2)} \\ i_2^{(2)} \end{bmatrix} = \begin{bmatrix} 4V \\ 0V \\ 4A \\ 0A \end{bmatrix}$$

$$\begin{bmatrix} u_1^{(3)} \\ u_2^{(3)} \\ i_1^{(3)} \\ i_2^{(3)} \end{bmatrix} = \begin{bmatrix} 0V \\ -4V \\ 4A \\ 0A \end{bmatrix} - \begin{bmatrix} \underline{u}^{(3)} \\ \underline{i}^{(3)} \end{bmatrix}$$

$$\underset{\sim}{U} = \left[\underline{u}^{(1)} - \underline{u}^{(3)} , \; \underline{u}^{(2)} - \underline{u}^{(3)} \right]$$

$$= \begin{bmatrix} 0V & 4V \\ 6V & 4V \end{bmatrix}$$

$$\underset{\sim}{I} = \left[\underline{i}^{(1)} - \underline{i}^{(3)} , \; \underline{i}^{(2)} - \underline{i}^{(3)} \right]$$

$$= \begin{bmatrix} -4A & 0A \\ 4A & 0A \end{bmatrix}$$

$$\begin{bmatrix} \underline{u} \\ \underline{i} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{U} \\ \underset{\sim}{I} \end{bmatrix} \underline{c} + \begin{bmatrix} \underline{u}^{(3)} \\ \underline{i}^{(3)} \end{bmatrix}$$

b) Rearrange the parametric representation such that you obtain the explicit representation of the form $\boldsymbol{i} = \boldsymbol{G}\boldsymbol{u} + \boldsymbol{i}_G$.

$$\underline{u} = \underset{\sim}{U}\,\underline{c} + \underline{u}^{(3)}$$

$$\underline{i} = \underset{\sim}{I}\,\underline{c} + \underline{i}^{(3)}$$

$$\underline{c} = \underset{\sim}{U}^{-1} \left(\underline{u} - \underline{u}^{(2)} \right)$$

$$= \begin{bmatrix} 0V & 4V \\ 6V & 4V \end{bmatrix}^{-1} \left(\underline{u} - \begin{bmatrix} 0V \\ -4V \end{bmatrix} \right)$$

$$= \frac{1}{-24V^2} \begin{bmatrix} 4V & -4V \\ -6 & 0V \end{bmatrix} \left(\underline{u} - \begin{bmatrix} 0V \\ -4V \end{bmatrix} \right)$$

$$= \begin{bmatrix} -\frac{1}{6}\frac{1}{V} & \frac{1}{6}\frac{1}{V} \\ \frac{1}{4}\frac{1}{V} & 0\frac{1}{V} \end{bmatrix} \left(\underline{u} - \begin{bmatrix} 0V \\ -4V \end{bmatrix} \right)$$

$$\underline{i} = \underset{\sim}{I}\,\underset{\sim}{U}^{-1}\left(\underline{u} - \underline{u}^{(3)}\right) + \underline{i}^{(3)}$$

$$= \begin{bmatrix} -4A & 0A \\ 4A & 0A \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{6}\frac{1}{V} & \frac{1}{6}\frac{1}{V} \\ \frac{1}{4}\frac{1}{V} & 0\frac{1}{V} \end{bmatrix} \cdot \left(\underline{u} - \begin{bmatrix} 0V \\ -4V \end{bmatrix} \right) + \begin{bmatrix} 4A \\ 0A \end{bmatrix}$$

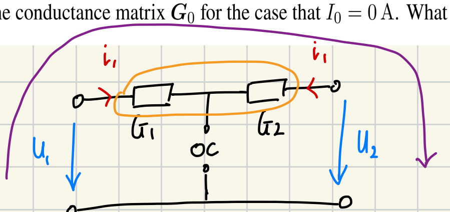
$$= \begin{bmatrix} \frac{2}{3}S & -\frac{2}{3}S \\ -\frac{2}{3}S & \frac{2}{3}S \end{bmatrix} \underline{u} \underbrace{- \begin{bmatrix} \frac{8}{3}A \\ -\frac{8}{3}A \end{bmatrix} + \begin{bmatrix} 4A \\ 0A \end{bmatrix}}_{\begin{bmatrix} \frac{4}{3}A \\ \frac{8}{3}A \end{bmatrix}}$$

$$\underline{i} = \underset{\sim}{G}\,\underline{u} + \underline{i}_G$$

$$\underset{\sim}{G} = \frac{1}{3}S \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\underline{i}_G = \frac{1}{3}A \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

c) Find the conductance matrix G_0 for the case that $I_0 = 0\,\mathrm{A}$. What do you observe?



$$G_{tot} = G_1 \parallel G_2 = \frac{G_1 G_2}{G_1 + G_2} = \frac{2}{3}S$$

$$\text{KVL: } -U_1 + U_{tot} + U_2 = 0$$

$$U_{tot} = U_1 - U_2$$

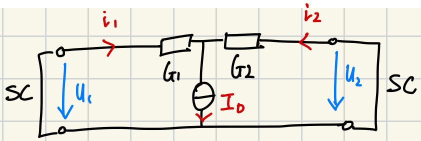
$$\text{ohm: } i_1 = G_{tot}\,U_{tot}$$

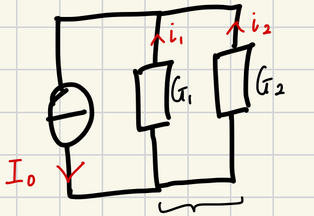
$$= \frac{2}{3}S\,(U_1 - U_2)$$

$$i_2 = -i_1 = -\frac{2}{3}S\,(u_1 - u_2)$$

$$\underset{\sim}{G_0} = \begin{bmatrix} \frac{2}{3}S & -\frac{2}{3}S \\ -\frac{2}{3}S & \frac{2}{3}S \end{bmatrix} = \underset{\sim}{G}$$

d) Determine the currents i_1 and i_2 for $I_0 = 4\,\mathrm{A}$ if the two ports are connected to short circuits. What do you observe?





$G_{12} = G_1 + G_2$

$$\frac{i_1}{I_0} = \frac{G_1}{G_{12}} = \frac{G_1}{G_1 + G_2} = \frac{1}{3}$$

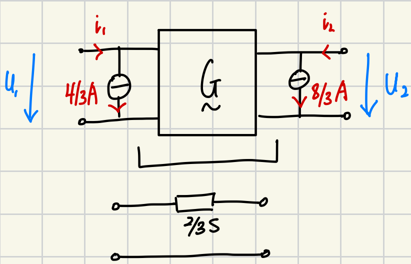
$I_0 = 4A$

$$i_1 = \frac{4}{3}A$$

$$\frac{i_2}{I_0} = \frac{G_2}{G_1 + G_2} = \frac{2}{3}$$

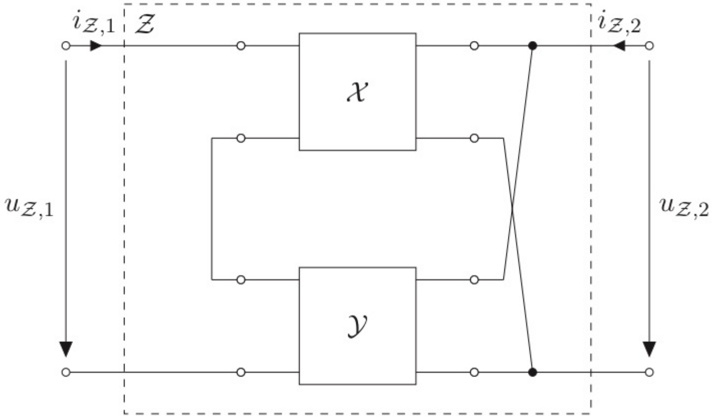
$$i_2 = \frac{2}{3} \cdot 4A = \frac{8}{3}A$$

e) Draw the equivalent circuit diagram of the circuit.



3.5 Series-Parallel Connection

Consider the following connection of two two-ports.



The two-ports $\mathcal{X}$ and $\mathcal{Y}$ are strictly linear two-ports with the hybrid matrices

$$\boldsymbol{H}_{\mathcal{X}} = \begin{bmatrix} 1\,\Omega & 0 \\ 0.5 & 1\,\mathrm{S} \end{bmatrix} \qquad \boldsymbol{H}_{\mathcal{Y}} = \begin{bmatrix} -1\,\Omega & 0.5 \\ -1.0 & -1\,\mathrm{S} \end{bmatrix}$$

a) What conditions must be fulfilled such that it is possible to obtain the hybrid matrix of $\mathcal{Z}$ from the given hybrid matrices?

port Conditions must be fulfilled

Current from first port of first terminal = 2nd port of second terminal

b) Assume that the necessary conditions are fulfilled and find the hybrid matrix of $\mathcal{Z}$ depending on $\boldsymbol{H}_{\mathcal{X}}$ and $\boldsymbol{H}_{\mathcal{Y}}$.

$$\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \underset{\sim}{H} \begin{bmatrix} i_1 \\ u_2 \end{bmatrix}$$

$$\underset{\sim}{H}_{\mathcal{Z}} = \underset{\sim}{H}_X + \underset{\sim}{H}_Y$$

$$= \begin{bmatrix} 0\Omega & 0.5 \\ -0.5 & 0S \end{bmatrix}$$

c) What type of two-port behaves identically to $\mathcal{Z}$? Give the parameter characterizing that two-port.

$u_1 = 0.5 u_2$

$i_2 = -0.5 i_1$

$\Rightarrow$ transformer with $n = 0.5$

NIC :

$u_1 = -k u_2$

$i_1 = -\frac{1}{k} i_2$

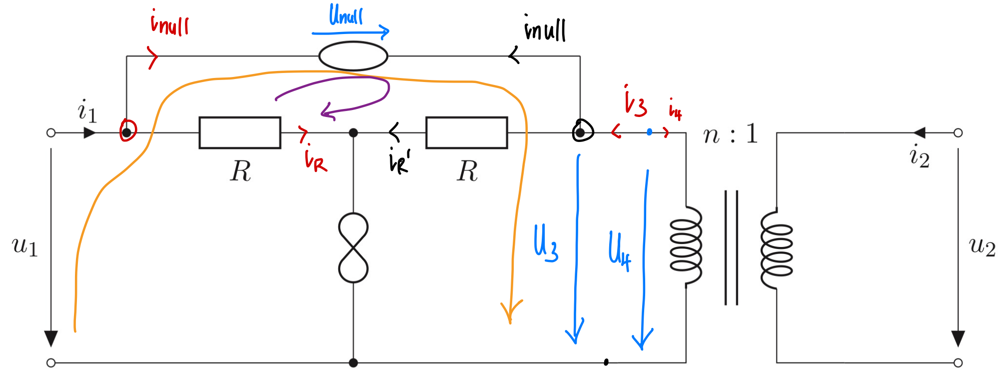
transformer :

$u_1 = n u_2$

$i_2 = -n i_1$

Pg 56

3.6



a) What is the connection between the NIC and the transformer?

KVL: $-U_1 + \cancelto{0}{U_{null}} + U_3 = 0$

$U_1 = U_3$

KCL: $i_1 - \cancelto{0}{i_{null}} - i_R = 0$

$i_1 = i_R$

KCL: $i_R' = i_3$

KVL: $-Ri_R + \cancelto{0}{U_{null}} + Ri_R' = 0$

$Ri_R = Ri_R'$, $i_R = i_R'$

$i_1 = i_3$

$\Rightarrow$ NIC with $k = -1$

$$\left.\begin{aligned} U_4 &= nU_2 \\ i_2 &= -ni_4 \end{aligned}\right\} \text{transformer}$$

Cascade Connection

$\Rightarrow$ transmission representative

b) Determine the transmission matrix $\boldsymbol{A}$ of the two-port. What type of two-port is it?

$$\underset{\sim}{A} = \underset{\sim}{A}_{NIC} \, \underset{\sim}{A}_{transf}$$

$$\underset{\sim}{A}_{NIC} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad \underset{\sim}{A}_{transf} = \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$$

$$\underset{\sim}{A} = \begin{bmatrix} n & 0 \\ 0 & -\frac{1}{n} \end{bmatrix} \rightarrow \begin{aligned} u_1 &= n u_2 \\ i_1 &= -\frac{1}{n}(-i_2) \\ &= \frac{1}{n}(i_2) \end{aligned}$$

$\rightarrow$ NIC with $K = -n$

c) Is the two-port reciprocal? $\rightarrow$ if got $\underset{\sim}{A}$ matrix, det $= 1$

$$\det \underset{\sim}{A} \overset{!}{=} 1$$

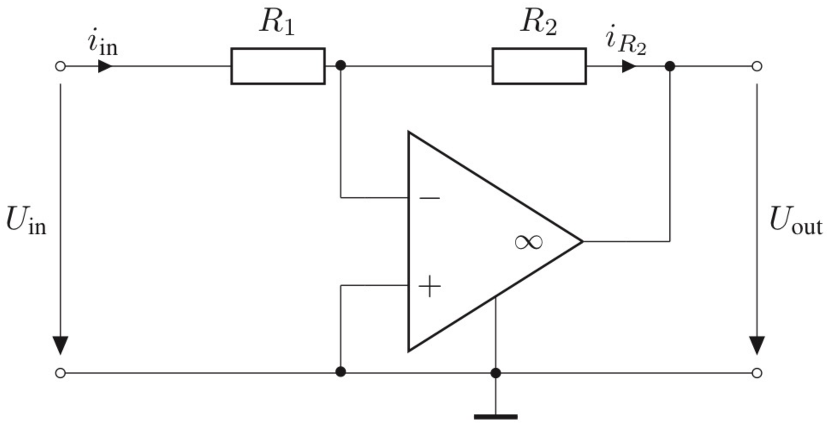
use representation found, do not convert

$$\det \begin{bmatrix} n & 0 \\ 0 & -\frac{1}{n} \end{bmatrix} = -1 \neq 1$$

$\Rightarrow$ not reciprocal

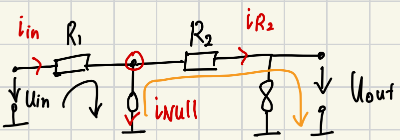
4.2 Amplifier Circuit

Consider the following circuit.



The Op-Amp is operated in the linear region.

a) What is the relationship between i_{in} and i_{R_2}?



a) KCL: $i_{in} - i_{R_2} - \cancel{i_{Null}}^{0} = 0$

$i_{R_2} = i_{in}$

b) Express u_{in} depending on i_{in}.

b) KVL: $-U_{in} + R_1 i_{in} + \cancel{U_{Null}}^{0} = 0$

$U_{in} = R_1 i_{in}$

c) Find u_{out} depending on i_{in} aus.

$$\text{KVL: } -\overset{0}{\cancel{U_{Null}}} + R_2 i_{R2} + U_{out} = 0$$

$$U_{out} = -R_2 i_{R2} = -R_2 i_{in}$$

d) Determine u_{in} depending on u_{out}.

$$U_{in} = -\frac{R_1}{R_2} U_{out}$$

e) What is the voltage gain ν of the circuit? What is the purpose of the circuit?

$$U_{out} = -\frac{R_2}{R_1} U_{in}$$

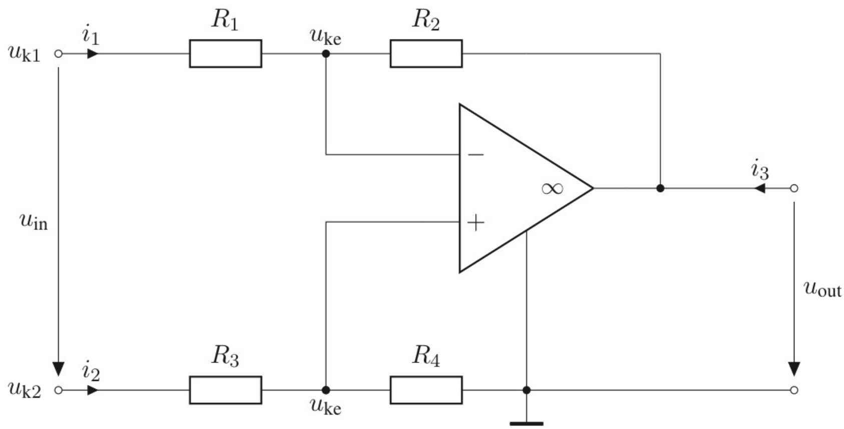
$$V = -\frac{R_2}{R_1}$$

Purpose $\Rightarrow$ amplify and invert voltage

inverting amplifier

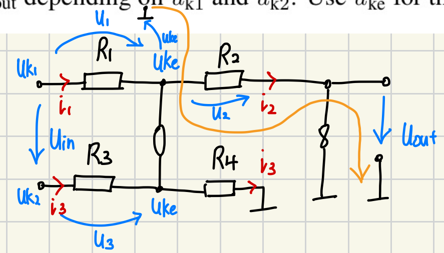
4.5 Difference Amplifier

The following circuit is a summer with an Op-Amp operated in the linear region.



This circuit is called potential difference amplifier if the output voltage u_{out} only depends on the difference $u_{in} = u_{k1} - u_{k2}$ of the two input potentials, i.e., $u_{out} = v_d u_{in}$.

a) Determine u_{out} depending on u_{k1} and u_{k2}. Use u_{ke} for the potential of the Op-Amp inputs.



a) KVL: $U_1 = U_{k1} - U_{ke}$

$\to$ KVL: $U_3 = U_{k2} - U_{ke}$

Ohm: $i_3 = \dfrac{U_3}{R_3}$

$\to U_{ke} = R_4 i_3 = \dfrac{R_4}{R_3} U_3$

$U_{ke} = \dfrac{R_4}{R_3}(U_{k2} - U_{ke})$

$U_{ke} + \dfrac{R_4}{R_3} \cdot U_{ke} = \dfrac{R_4}{R_3} \cdot U_{k2}$

$\dfrac{R_3 + R_4}{R_3} U_{ke} = \dfrac{R_4}{R_3} \cdot U_{k2}$

$U_{ke} = \dfrac{R_4}{R_3 + R_4} \cdot U_{k2}$

$U_1 = U_{k1} - U_{ke}$

$i_1 = \dfrac{U_1}{R_1} = \dfrac{U_{k1} - U_{ke}}{R_1}$

$U_2 = R_2 i_1 = \dfrac{R_2}{R_1}(U_{k1} - U_{ke})$

KVL: $-U_{ke} + U_2 + U_{out} = 0$

$U_{out} = -U_2 + U_{ke}$

$= -\dfrac{R_2}{R_1} U_{k1} + \dfrac{R_2}{R_1} U_{ke} + U_{ke}$

$= -\dfrac{R_2}{R_1} U_{k1} + \dfrac{R_1 + R_2}{R_1} U_{ke}$

$= -\dfrac{R_2}{R_1} U_{k1} + \dfrac{R_1 + R_2}{R_1} \cdot \dfrac{R_4}{R_3 + R_4} \cdot U_{k2}$

b) When behaves the circuit like a potential difference amplifer? Give the resulting difference gain v_d.

$$\frac{R_2}{R_1} = \frac{R_1+R_2}{R_1} \cdot \frac{R_4}{R_3+R_4}$$

$$= \frac{R_2}{R_1}\left(1+\frac{R_1}{R_2}\right) \cdot \frac{1}{1+R_3/R_4}$$

3

$$\Rightarrow \frac{R_1}{R_2} = \frac{R_3}{R_4}$$

$$\Rightarrow U_{out} = -\frac{R_2}{R_1}U_{k1} + \frac{R_2}{R_1}U_{k2}$$

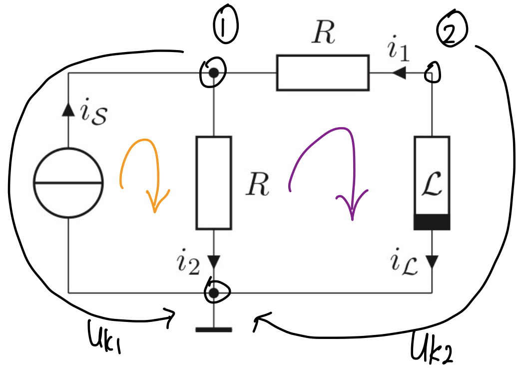
$$= -\frac{R_2}{R_1}(U_{k1} - U_{k2})$$

$\Rightarrow$ potential difference amp. $= -\frac{R_2}{R_1}$

v_g

5.1 Kirchhoff's Laws

Consider the following circuit with non-linear load $\mathcal{L}$.



a) What is the number of nodes and what is the number of branches of the circuit?

$b=4,\ n=3$

b) Give the number of linearly independent KCL equations. What is thus the size of the node incidence matrix $\boldsymbol{A}$?

$n-1 = 2$ KCL eqn

$\underset{\sim}{A} \in \{0,1,-1\}^{\underline{n-1 \times b}}$

$\downarrow$

2×4

c) Give the number of linearly independent KVL equations. What is therefore the size of the loop incidence matrix $\boldsymbol{B}$?

$b-(n-1) = 4-2 = 2$ KVL eqn

$\underset{\sim}{B} \in \{0,1,-1\}^{2\times 4}$

d) Find the loop incidence matrix $\boldsymbol{B}$ of the given circuit.

KVL : $U_s + U_2 = 0$

KVL : $-U_2 - U_1 + U_L = 0$

$\underline{u} = [u_1, u_2, u_s, u_L]^T$

$\underset{\sim}{B}\underline{u} = 0$

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

e) Determine the node incidence matrix $\boldsymbol{A}$ of the circuit.

KCL : $\underset{\sim}{A}\underline{i} = \underline{0}$

$\underline{i} = [i_1, i_2, i_s, i_L]^T$

$$\underset{\sim}{A} = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

f) What is the rank of $\boldsymbol{A}$, i.e., what is the number of linearly independent row vectors in $\boldsymbol{A}$? Justify your answer.

Based on the node incidence matrix $\boldsymbol{A}$, a different form of the KVL equations can be found.

$\operatorname{rank}(\underset{\sim}{A}) = 2$

two linearly independent row

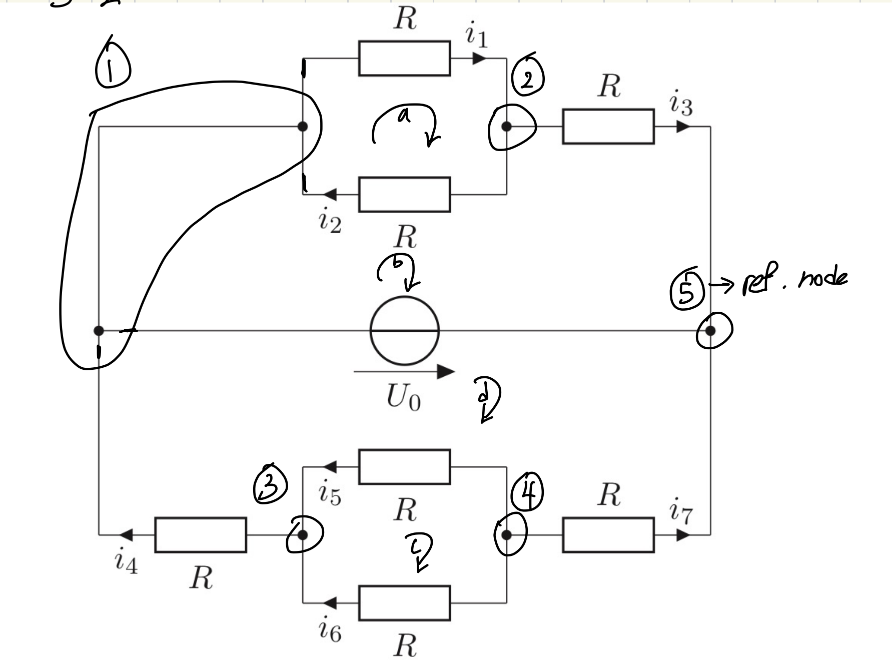
g) Formulate this alternative form of the KVL.

$\underline{u} = \underset{\sim}{A}^T \underline{u_k}$

$\underline{u_k} = [u_{k1}, u_{k2}]^T$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_s \\ u_L \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix} = \begin{bmatrix} u_{k2} - u_{k1} \\ u_{k1} \\ -u_{k1} \\ u_{k2} \end{bmatrix}$$

5-2



a) What is the number of nodes? Mark the nodes in the circuit.

b) Determine the node incidence matrix $\boldsymbol{A}$ of the circuit.

$b = 8!$ $\quad n = 5$

$\underline{i} = [i_0 \; i_1 \; i_2 \; i_3 \; i_4 \; i_5 \; i_6 \; i_7]$

$$\underset{\sim}{A} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

c) What is the number of linearly independent KVL equations?

$b-(n-1) = 4$ KVL eqn → holes in circuit

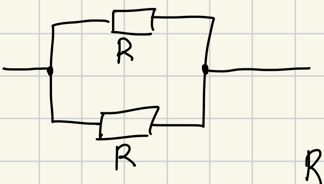
d) Find the loop incidence matrix $\boldsymbol{B}$.

$$\underset{\sim}{B}\,\underline{u} = \underline{0}$$

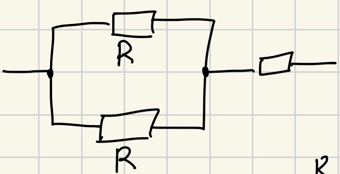
$$\underline{u} = \begin{bmatrix} u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{bmatrix}^T$$

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

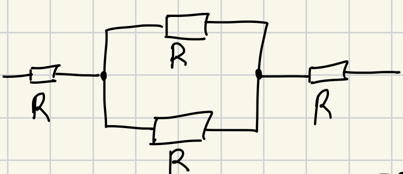
e) Combine the resistors to a single resistor R_{tot}. Draw the resulting equivalent circuit diagram.



$$R_{||} = R || R = \frac{RR}{R+R} = \frac{R}{2}$$

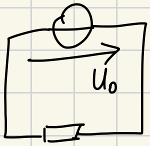


$$R + R||R = R + \frac{R}{2} = \frac{3R}{2}$$



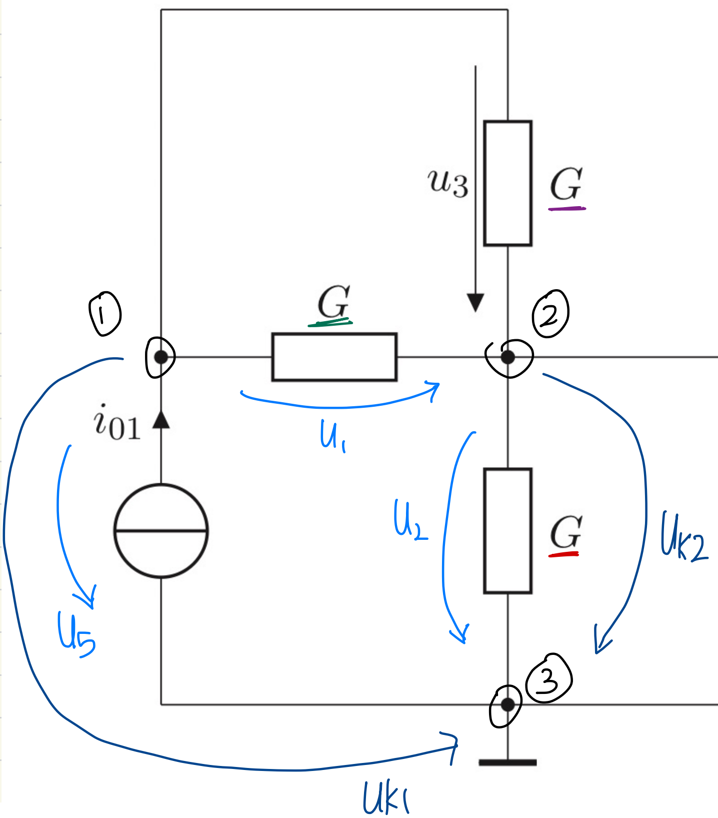
$$\frac{R}{2} + R + R = \frac{R}{2} + 2R = \frac{5R}{2}$$

$$R_{tot} = \frac{3R}{2} || \frac{5R}{2} = \frac{\frac{3R}{2} \cdot \frac{5R}{2}}{\frac{3R}{2} + \frac{5R}{2}} = \frac{\frac{15}{4}R^2}{4R} = \frac{15}{16}R$$



$R_{tot} = \frac{15}{16} R$

5.3



a) Label the branches by introducing branch voltages.

b) What is number of linearly independent KVL equations? Find the KVL equations in the form $\boldsymbol{B}\boldsymbol{u} = \mathbf{0}$.

$$b = 5 \quad n = 3$$

$$b - (n-1) = 5 - (3-1) = 3$$

$$\underset{\sim}{B}\,\underline{u} = 0$$

$$\underset{\sim}{B} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

c) What is the number of linearly independent KCL equations? Determine the KCL equations in the form $\boldsymbol{A}\boldsymbol{i} = \mathbf{0}$.

By Tellegen's law, $\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} = \mathbf{0}$.

d) Verify Tellegen's law.

$$n - 1 = 2 \text{ KCL eqns}$$

$$\underset{\sim}{A}\,\underline{i} = 0$$

$$\underset{\sim}{A} = \begin{array}{c} \begin{matrix} i_1 & i_2 & i_3 & i_4 & i_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

Tellegen: $\underset{\sim}{A}\,\underset{\sim}{B}^{T} = \underset{\sim}{0}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underset{\sim}{0} \Rightarrow \text{verified}$$

e) Find a representation of the circuit elements in the form $[\boldsymbol{M}, \boldsymbol{N}]\left[\begin{smallmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{smallmatrix}\right] = \boldsymbol{e}$.

$$\underset{\sim}{M}\,\underline{u} + \underset{\sim}{N}\,\underline{i} = \underline{e}$$

$$i_1 = u_1 G$$

$$-Gu_1 + i_1 = 0$$

$$-Gu_2 + i_2 = 0$$

$$-Gu_3 + i_3 = 0$$

$$i_4 = i_{02}$$

$$i_5 = -i_{01}$$

$$\underset{\sim}{M} = \begin{bmatrix} -G & & & & \\ & -G & & & \\ & & -G & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} \qquad \underset{\sim}{N} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$\underline{e} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ i_{02} \\ -i_{01} \end{bmatrix}$$

f) Give the condition for the circuit elements such that N is invertible.

Circuit elements are voltage controlled

$$\underset{\sim}{N}^{-1} = \underset{\sim}{1}$$

g) Collect the equations of the previous subproblems and formulate the tableau equation system.

$$\left[\begin{array}{ccccc:ccccc}
-1 & 0 & 1 & 0 & 0 & & & & & \\
1 & 1 & 0 & 0 & -1 & & & \underset{\sim}{0} & & \\
0 & -1 & 0 & 1 & 0 & & & & & \\
\hdashline
 & & \underset{\sim}{0} & & & 1 & 0 & 1 & 0 & 1 \\
 & & & & & -1 & 1 & -1 & 1 & 0 \\
\hdashline
-G & & & & & 1 & & & & \\
 & -G & & & & & 1 & & & \\
 & & -G & & & & & 1 & & \\
 & & & 0 & & & & & 1 & \\
 & & & & 0 & & & & & 1
\end{array}\right]
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i_{02} \\ -i_{01} \end{bmatrix}
\begin{matrix} \\ \\ \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} \underset{\sim}{e} \end{matrix}$$

$\underset{\sim}{B}$ $\underset{\sim}{A}$ $\underset{\sim}{M}$ $\underset{\sim}{N}$

h) Is the equation system in Subproblem g) uniquely solvable?

Yes, all rows are linearly independent

i) Formulate the nodal analysis for the circuit. To this end, mark the node voltages.

$$u_k = \begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix}$$

$$\begin{matrix} ① \\ ② \end{matrix}
\begin{bmatrix} G+G & -G-G \\ -G-G & G+G+G \end{bmatrix}
\begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix}
=
\begin{bmatrix} i_{01} \\ -i_{02} \end{bmatrix}
\begin{matrix} \leftarrow \text{flow in } \oplus \\ \leftarrow \text{flow out } \ominus \end{matrix}$$

single entry when element connected with reference

element connected with 2 node = 4 entries

2 $\oplus$ in main diagonal

2 $\ominus$ in other diagonal

$$\begin{bmatrix} 2G & -2G \\ -2G & 3G \end{bmatrix} \cdot \begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix} = \begin{bmatrix} i_{01} \\ -i_{02} \end{bmatrix}$$

j) Find u_3.

$$U_3 = U_{k1} - U_{k2}$$

$$\begin{bmatrix} 2G - 2G, & -2G + 3G \end{bmatrix} \cdot \begin{bmatrix} U_{k1} \\ U_{k2} \end{bmatrix} = i_{01} - i_{02}$$

$$U_{k2} = \frac{1}{G}\left(i_{01} - i_{02} \right)$$

$$\begin{bmatrix} 2G, -2G \end{bmatrix} \cdot \begin{bmatrix} U_{k1} \\ U_{k2} \end{bmatrix} = i_{01}$$

$$2G\,u_{k1} - 2G\,u_{k2} = i_{01}$$

$$2G\,u_{k1} - 2\left(i_{01} - i_{02} \right) = i_{01}$$

$$2G\,u_{k1} = 3i_{01} - 2i_{02}$$

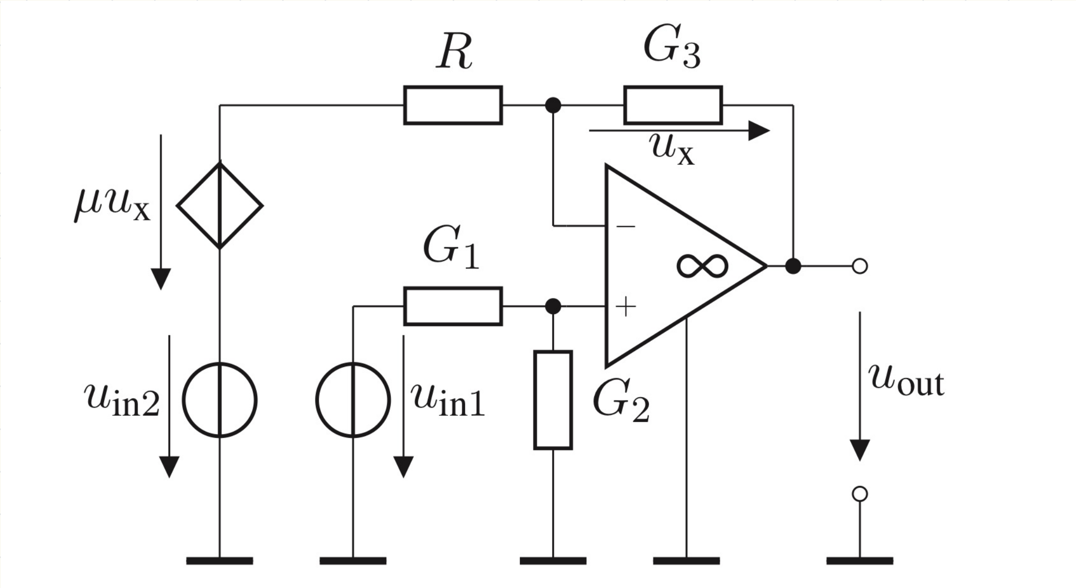
$$u_{k1} = \frac{3}{2G} i_{01} - \frac{1}{G} i_{02}$$

$$U_3 = u_{k1} - u_{k2}$$

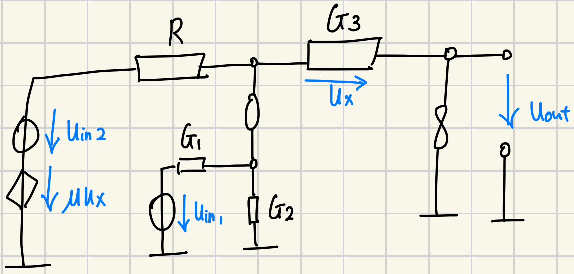
$$= \frac{3}{2G} i_{01} - \frac{1}{G} i_{02} - \frac{1}{G}\left(i_{01} - i_{02} \right)$$

$= \frac{1}{2G} i_{01} + 0$

5.4



a) Give the elements which are not voltage-controlled. Discuss how these elements are transformed to voltage-controlled elements.



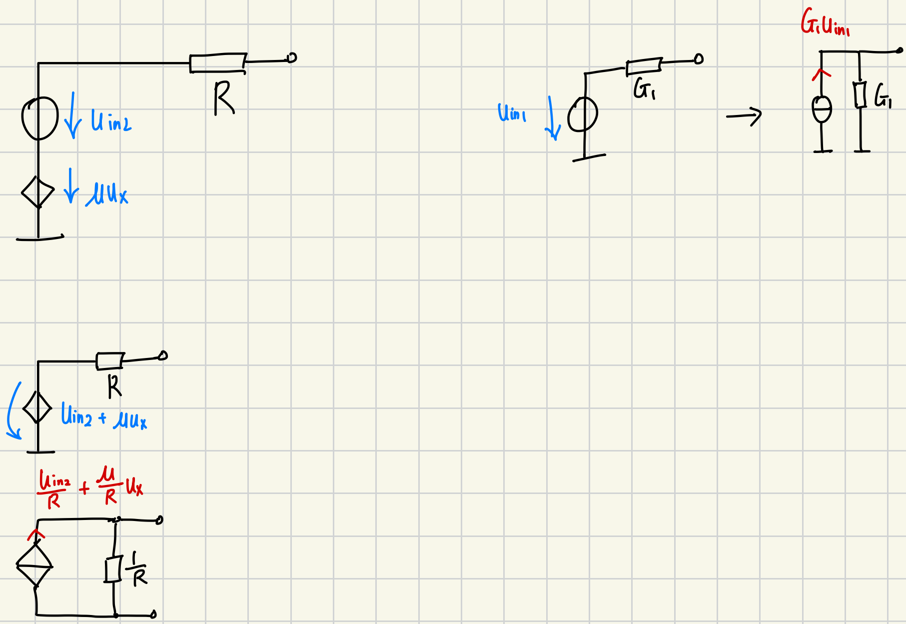
a) nullator, norator

$R \to \frac{1}{R}$

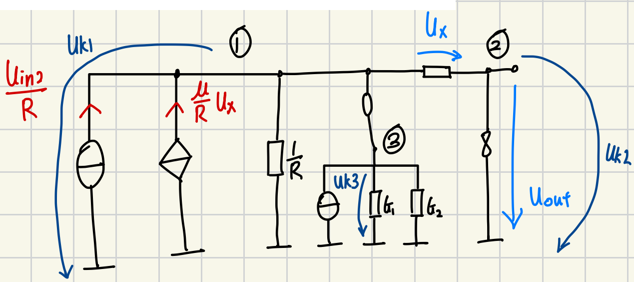
U_{in1}, U_{in2}, VCVS

→ source transform

b) Draw the equivalent circuit diagram where the non-voltage-controlled elements are replaced by voltage-controlled elements. Include the equivalent circuit diagram of the Op-Amp according to the linear region.



c) Number the nodes of the equivalent circuit diagram.



ignore nullator & norator → OC

d) Determine u_x and u_{out} depending on the node voltages.

$$U_x = U_{k1} - U_{k2}$$

e) Find the node conductance matrix G'_{k}, the node voltage vector u'_{k}, and the node current source vector i'_{q} ignoring the Op-Amp.

$$\begin{bmatrix} \frac{1}{R} + G_3 - \frac{\mu}{R} & -G_3 + \frac{\mu}{R} & \frac{1}{R} + G_3 - \frac{\mu}{R} \\ \cancel{-G_3} & \cancel{G_3} & \\ & & G_1 + G_2 \end{bmatrix} \begin{bmatrix} \cancel{u_{k1}} \\ \cancel{u_{k2}} \\ u_{k3} \end{bmatrix} = \begin{bmatrix} \frac{U_{in2}}{R} \\ \cancel{0} \\ G_1 U_{in1} \end{bmatrix}$$

Nullator between ① & ③

f) Incorporate the Op-Amp to obtain $\boldsymbol{G}_{\mathrm{k}}$, $\boldsymbol{u}_{\mathrm{k}}$, and $\boldsymbol{i}_{\mathrm{q}}$.

$$\begin{bmatrix} -G_3 + \frac{\mu}{R} & \frac{1}{R} + G_3 - \frac{\mu}{R} \\ 0 & G_1 + G_2 \end{bmatrix} \begin{bmatrix} u_{k2} \\ u_{k3} \end{bmatrix} = \begin{bmatrix} \frac{U_{in2}}{R} \\ G_1 U_{in1} \end{bmatrix}$$

g) Solve the resulting equation system.

2nd eqn.: $(G_1 + G_2) u_{k3} = G_1 U_{in1}$

$$u_{k3} = \frac{G}{G_1 + G_2} U_{in1}$$

1st eqn: $\left(-G_3 + \frac{\mu}{R}\right) u_{k2} + \left(\frac{1}{R} + G_3 - \frac{\mu}{R}\right) u_{k3} = \frac{U_{in2}}{R}$

$$\left(-G_3 + \frac{\mu}{R}\right) u_{k2} + \left(\frac{1}{R} + G_3 - \frac{\mu}{R}\right) \frac{G_1}{G_1 + G_2} U_{in1} = \frac{U_{in2}}{R}$$

$$u_{k2} = \frac{1}{\mu - G_3 R} \cdot U_{in2} - \frac{\left(\frac{1}{R} + G_3 - \frac{\mu}{R}\right) G_1}{G_1 + G_2} \frac{1}{\frac{\mu}{R} - G_3} U_{in1}$$

$$= \frac{1}{\mu - G_3 R} \cdot U_{in2} - \frac{(1 + RG_3 - \mu) G_1}{G_1 + G_2} \frac{1}{\mu - RG_3} U_{in1}$$

h) Determine u_{out} depending on u_{in1} and u_{in2}.

h)

$$U_{out} = U_{k2}$$

$$= \frac{1}{\mu - G_3 R} U_{in2} - \frac{(1 + RG_2 - \mu) G_1}{(G_1 + G_2)(\mu - RG_3)} U_{in1}$$

i) Is it possible to choose a value for the gain factor μ such that u_{out} is independent of u_{in2}? If yes, find the corresponding μ.

i) $\mu \to \infty$

$$U_{out} \to \frac{G_1}{G_1 + G_2} U_{in1}$$

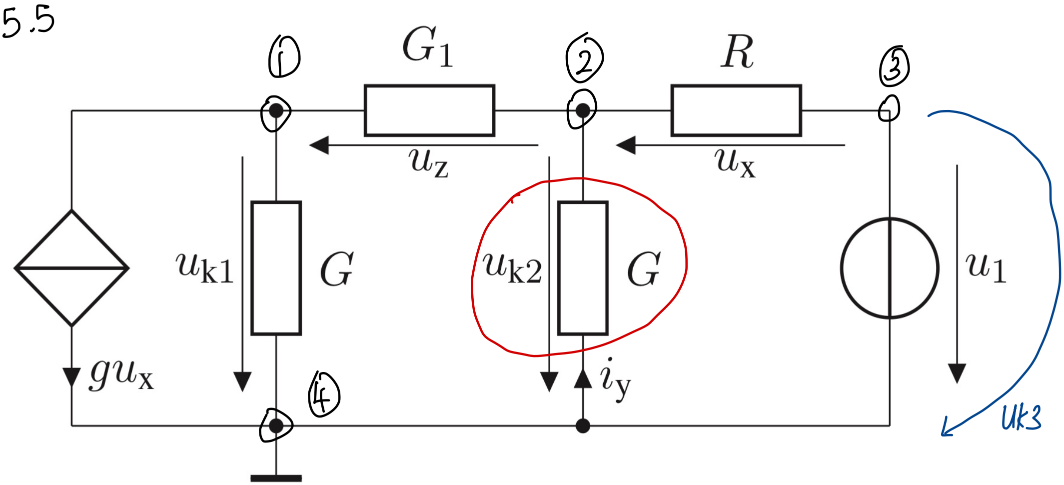
j) Is it possible to choose a value for the gain factor μ such that u_{out} is independent of u_{in1}? If yes, find the corresponding μ.

j) $1 + RG_3 - \mu = 0 \rightarrow U_{in1}$ disappear

$$\mu = 1 + RG_3$$

$$\rightarrow U_{out} = U_{in2}$$

5.5



a) Express u_x depending on the node voltages and u_1.

$U_x = U_{k3} - U_{k2}$

$U_{k3} = U_1$

$U_x = U_1 - U_{k2}$

b) Find i_y depending on the node voltages and u_1.

$i_G = G U_{k2}$

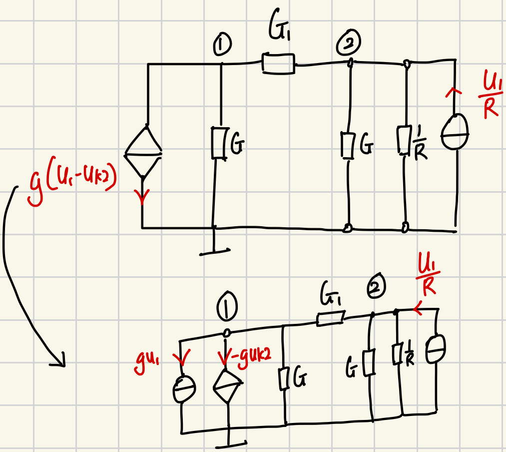
KCL $= i_G + i_y = 0$

$i_y = -i_G = -G U_{k2}$

c) Determine u_z depending on the node voltages and u_1.

$U_z = U_{k2} - U_{k1}$

d) Change the circuit such that the nodal analysis can be employed.



e) Formulate the nodal analysis.

$$\left[\begin{array}{c:c} G+G_1 & -G_1-g \\ \hdashline -G_1 & G+\frac{1}{R}+G_1 \end{array}\right] \left[\begin{array}{c} u_{k1} \\ \hdashline u_{k2} \end{array}\right] = \left[\begin{array}{c} -gu_1 \\ \hdashline \frac{u_1}{R} \end{array}\right]$$

f) Solve the resulting equation system.

$$-G_1 u_{k1} + \left(G + \frac{1}{R} + G_1\right) u_{k2} = \frac{u_1}{R}$$

$$u_{k1} = \frac{G+\frac{1}{R}+G_1}{G_1} u_{k2} - \frac{u_1}{RG_1}$$

$$① : \ (G+G_1) u_{k1} + (-G_1-g) u_{k2} = -gu_1$$

$$\frac{G+G_1}{G_1} \cdot \left(G + \frac{1}{R} + G_1\right) u_{k2} - \frac{G+G_1}{RG_1} u_1 + (-G_1-g) u_{k2} = -gu_1$$

$$\frac{(G+G_1)(G+\frac{1}{R}+G_1)-G_1(G_1+g)}{G_1}U_{k2} = \left(\frac{G_1+G}{RG_1}-g\right)U_1$$

$$\frac{R\cdot G^2+G\cancel{\frac{1}{R}}+\overbrace{RGG_1+RGG_1}^{2GG_1R}+G_1\cancel{\frac{1}{R}}-RgG_1}{\cancel{G_1}}U_{k2} = \frac{G_1+G-gRG_1}{\cancel{RG_1}}\cdot U_1$$

$$(R\cdot G^2+G+G_1+2GG_1R-gG_1R)\,U_{k2} = (G_1+G-gRG_1)\cdot U_1$$

$$U_{k2} = \frac{G_1+G-gRG_1}{RG^2+G+G_1+2GG_1R-gG_1R}U_1$$

$$U_{k1} = \frac{GR+1+G_1R}{G_1R}\cdot\frac{G+G_1-RG_1g}{RG^2+G+G_1+2GG_1R-gG_1R}U_1 - \frac{U_1}{RG_1}$$

g) What do you obtain for i_y and u_z?

$$i_y = -Gu_{k2}$$

$$= -G\frac{G+G_1-RG_1g}{RG^2+G+G_1+2GG_1R-gG_1R}u_1$$

$$U_Z = U_{k2} - U_{k1}$$

h) Can g chosen such that i_y is zero? If yes, give the according value of g.

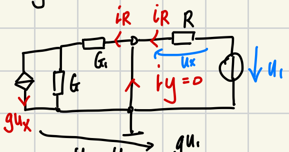
i) Can g chosen such that u_z is zero? If yes, give the according value of g.

$$i_y = 0$$

$$G+G_1-RG_1g = 0$$

$$g = \frac{G+G_1}{RG_1}$$

if $i_y = 0$,



$U_x = U_1$

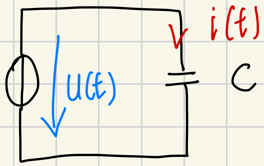
$i_R = \frac{U_1}{R} \quad \Big| \quad i_R = \frac{G_1}{G+G_1}gu_1$

6.1 Strictly Linear Reactances

Consider the voltage source $u(t) = U_0 \sin(\omega t)$.

Firstly, the voltage source is connected to a strictly linear capacitor C at time $t = 0$.

a) Find the current $i(t)$ flowing through the capacitor for $t \geq 0$.



$$i(t) = C\dot{u}(t)$$

$$= U_0 \cos(\omega t) \cdot \omega$$

b) What is the flux $\Phi(t)$ for $t \geq 0$ under the assumption that $\Phi(0) = \Phi_0$?

$$\Phi(t) = \Phi(0) + \int_0^t u(t')\,dt'$$

$$= \Phi_0 + \int_0^t U_0 \sin(\omega t')\,dt'$$

$$= \Phi_0 + U_0 \left[-\cos(\omega t)\frac{1}{\omega}\right]_0^t$$

$$\Phi(t) = \Phi_0 + U_0\left(-\cos(\omega t)\frac{1}{\omega} + 1\frac{1}{\omega}\right)$$

c) Determine the charge $q(t)$.

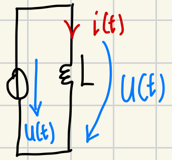
$$q(t) = Cu(t) = C\,U_0 \sin(\omega t)$$

d) What is therefore the initial condition $q(0)$?

$$q(0) = C U_0 \sin(w \cdot 0) = 0$$

Secondly, the voltage source is connected to a strictly linear inductor L at time $t = 0$.

e) Find the current $i(t)$ flowing through the inductor for $t \geq 0$ assuming that $i(t_0 = 0) = -\frac{U_0}{\omega L}\cos(\omega t_0)$.



$$\Phi = L \cdot i$$

$$\Phi(t) = \Phi_0 + \frac{1}{w} U_0 - \frac{U_0}{w}\cos(wt)$$

$$i(t) = \frac{1}{L}\Phi(t) = \frac{\Phi_0}{L} + \frac{U_0}{wL} - \frac{U_0}{wL}\cos(wt)$$

$$= \frac{\Phi_0}{L} + \frac{U_0}{wL} - \frac{U_0}{wL} = \frac{\Phi_0}{L} = -\frac{U_0}{wL}$$

f) Give the flux $\Phi(t)$ for $t \geq 0$.

$$\Phi(t) = \Phi_0 + \frac{U_0}{w} - \frac{U_0}{w}\cos(wt)$$
$$= -\frac{U_0}{w}\cos(wt)$$

alternatively: $U(t) = L\,\dot{i}(t)$

$$\dot{i}(t) = \frac{1}{L} U(t)$$
$$= \frac{U_0}{L}\sin(wt)$$
$$i(t) = \int_0^t \frac{U_0}{L}\sin(wt')\,dt'$$
$$= \frac{U_0}{L}\left[-\cos(wt')\cdot\frac{1}{w}\right]_0^t$$
$$= \frac{U_0}{wL} - \frac{U_0}{wL}\cos(wt)$$

g) Determine the charge $q(t)$ for $t \geq 0$ under the assumption that $q(0) = \frac{U_0}{\omega^2 L}$.

$$q(t) = q(0) + \int_0^t i(t')\,dt'$$

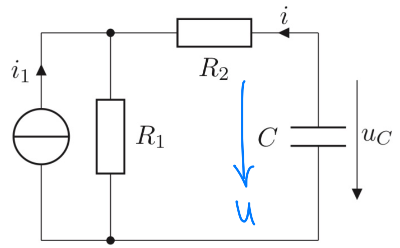
$$= \frac{U_0}{\omega^2 L} + \int_0^t -\frac{U_0}{\omega L}\cos(\omega t')\,dt'$$

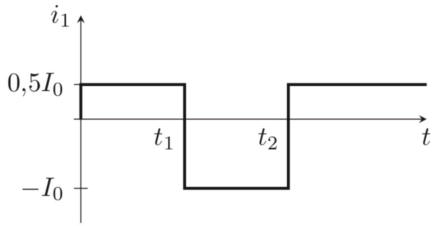
$$= \frac{U_0}{\omega^2 L} + \left[-\frac{U_0}{\omega L}\sin(\omega t')\frac{1}{\omega}\right]_0^t$$

$$= \frac{U_0}{\omega^2 L} - \frac{U_0}{\omega^2 L}\sin(\omega t)$$

7.2 RC-Circuit with Piecewise Linear Excitation

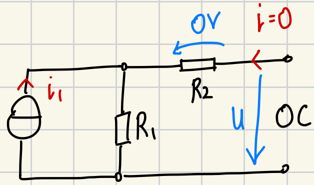
Consider the following circuit with piecewise linear excitation i_1.



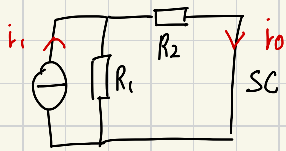


a) Find the representation of the linear part as a linear source.

Capacitor replaced with OC:



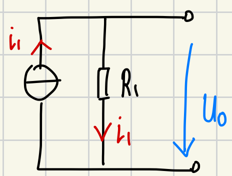
Capacitor replaced with SC:



Current div:

$$\frac{i_0}{i_1} = \frac{\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}} = \frac{R_1}{R_1+R_2}$$

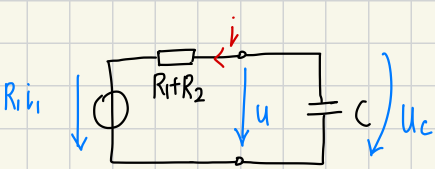
$$i_0 = \frac{R_1}{R_1+R_2} \cdot i_1$$



$U_0 = R_1 i_1$

internal resistance:

$$R_i = \frac{U_0}{i_0} = R_1+R_2$$



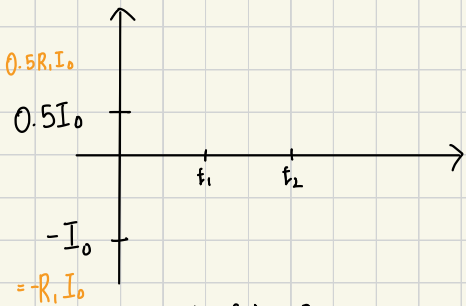
b) What is the time constant τ of the circuit?

$$\tau = R_i C$$
$$= (R_1 + R_2) C$$

c) Determine the voltage $u_C(t)$ for $t \geq 0$ when $u_C(0) = 0$ and where $t_1 = \tau$ and $t_2 = 2\tau$.

$$U_C(t) = R_1 i_1 + (U_{C0} - R_1 i_1) e^{-\frac{t - t_0}{\tau}}$$

$U_0 = R_1 i_1$



$$U_C(0) = 0$$

$$0 \leq t \leq t_1 : \quad U_C(t) = 0.5 R_1 I_0 (1 - e^{-\frac{t}{\tau}})$$

$$t = t_1 : \quad U_C(t_1) = 0.5 R_1 I_0 (1 - e^{-1})$$
$$\approx \frac{1}{3} R_1 I_0$$

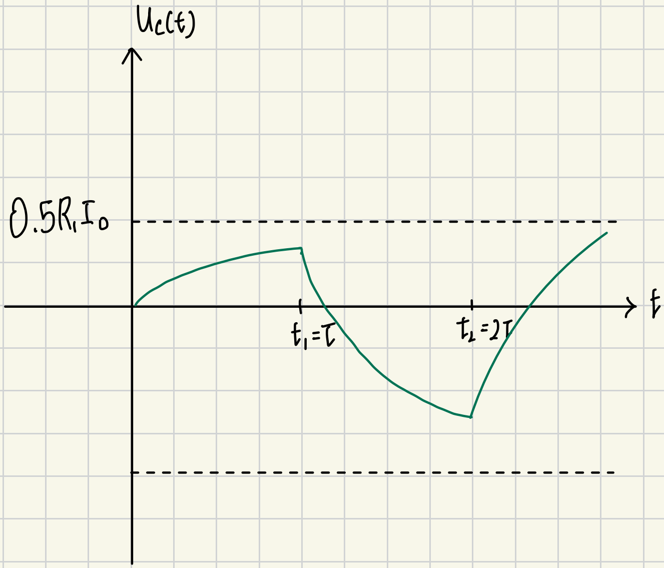
$$t_1 \leq t \leq t_2 : \quad U_C(t) = -R_1 I_0 + (\tfrac{1}{3} R_1 I_0 + R_1 I_0) e^{-\frac{t - t_1}{\tau}}$$
$$= -R_1 I_0 + \tfrac{4}{3} R_1 I_0 \cdot e^{-\frac{t_1}{\tau} + 1}$$

$$U_C(t_2) = -R_1 I_0 + \tfrac{4}{3} R_1 I_0 \tfrac{1}{3} = -\tfrac{5}{9} R_1 I_0$$

2τ

$$t \geq t_3 : \quad U_C(t) = 0.5 R_1 I_0 + (-\tfrac{5}{9} R_1 I_0 - 0.5 R_1 I_0) e^{-\frac{t - t_2}{\tau}}$$
$$= 0.5 R_1 I_0 - \tfrac{19}{18} R_1 I_0 \cdot e^{-\frac{t}{\tau} + 2}$$

d) Sketch $u_C(t)$.



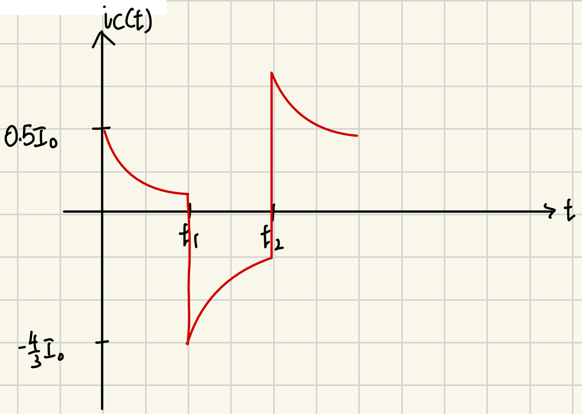
e) Find $i_C(t)$.

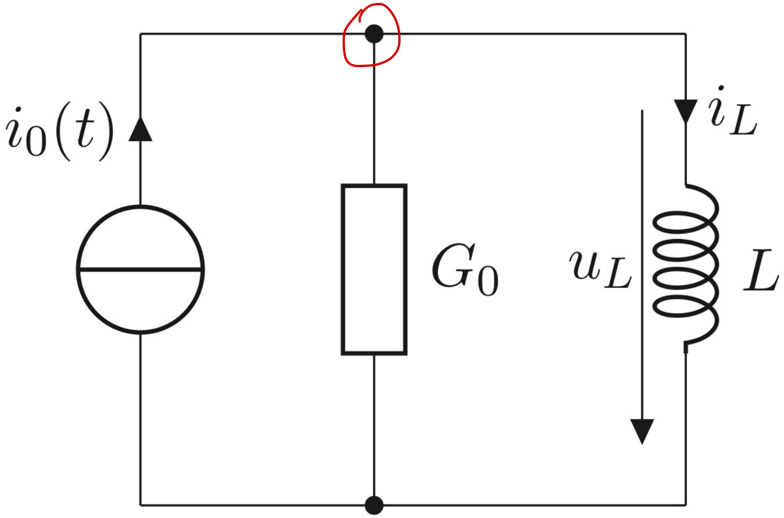
$0 \le t \le t_1$: $i_C(t) = C \cdot \dot{u}(t) = 0.5 I_0 e^{\frac{t}{\tau}}$

$t_1 \le t \le t_2$: $i_C(t) = -\frac{4}{3} I_0 e^{-\frac{t}{\tau}+1}$

$t \ge t_2$: $i_C(t) = \frac{19}{18} I_0 e^{-\frac{t}{\tau}+2}$

f) Sketch $i_C(t)$.





The excitation is given by

$$i_0(t) = \begin{cases} 0 & \text{for } t < 0, \\ I_0 \, \mathrm{e}^{-\frac{t}{G_0 L}} \cos(\omega t) & \text{for } t \geq 0. \end{cases}$$

a) Determine $i_L(t)$ for $t > 0$ assuming that $i_L(0) = I_0$. Use the abbreviation $\tau = G_0 L$.

KCL: $-i_0 + i_L + G_0 u_L = 0$

$$u_L = L \cdot \dot{i}_L$$

$$-i_0 + i_L + G_0 L \dot{i}_L = 0$$

$$\dot{i}_L = \frac{1}{G_0 L} i_L + \frac{1}{G_0 L} i_0$$

$$t \geq 0 : i_0(t) = I_0 e^{-\frac{1}{G_0 L}} \cos(\omega t)$$

$$\tau = G_0 L, \quad i_L(0) = I_0, t_0 = 0$$

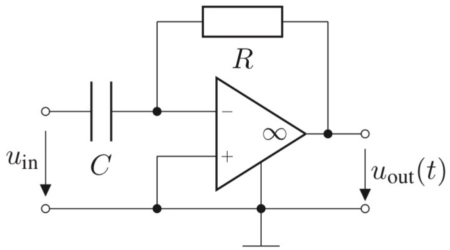
$$i(t) = I_0 e^{-\frac{t}{\tau}} + \int_{t_0=0}^{t} \frac{1}{\tau} e^{-\frac{t-t'}{\tau}} I_0 \cdot e^{-\frac{t'}{\tau}} \cos(\omega t') \, dt'$$

$$= I_0 e^{-\frac{t}{\tau}} + \int_0^t \frac{1}{\tau} e^{-\frac{t}{\tau}} \cdot I_0 \cos(\omega t') \, dt'$$

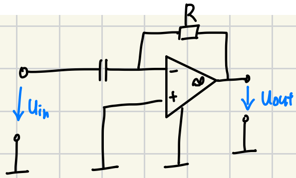
$$= I_0 e^{-\frac{t}{\tau}} + \frac{1}{\omega \tau} e^{-\frac{t}{\tau}} \cdot I_0 \sin(\omega t)$$

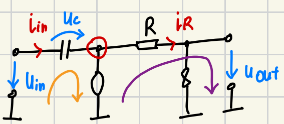
7.5 Integrator and Differentiator

Given is following circuit. The Op-Amp is operated in the linear region.



a) Find the output voltage $u_{\text{out}}(t)$ depending on the input voltage u_{in} for $t > t_0$.





KVL: $-U_{in} + U_c = 0$

$U_c = U_{in}$

Cap: $i_{in} = C\dot{U}_{in}$

KCL: $i_{in} - i_R = 0$

$i_R = i_{in} = C\dot{U}_{in}$

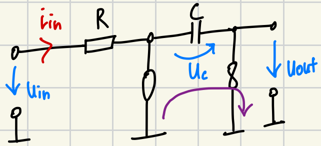
KVL: $Ri_R + U_{out} = 0$

$U_{out} = -Ri_R$

$= -RC\dot{U}_{in}$

$\Rightarrow$ differentiator

b) Exchange the capacitor and the resistor. Determine the output voltage $u_{\text{out}}(t)$ for $t > t_0$ and $u_C(t_0) = 0$.



$U_{in} = R i_{in}$

$i_{in} = \frac{1}{R} U_{in}$

Cap. : $i_{in} = C \dot{u}_C$

$\frac{1}{R} U_{in} = C \dot{u}_C$

$U_C(t) = \frac{1}{RC} \int U_{in}(t')\, dt'$

KVL: $U_C + U_{out} = 0$

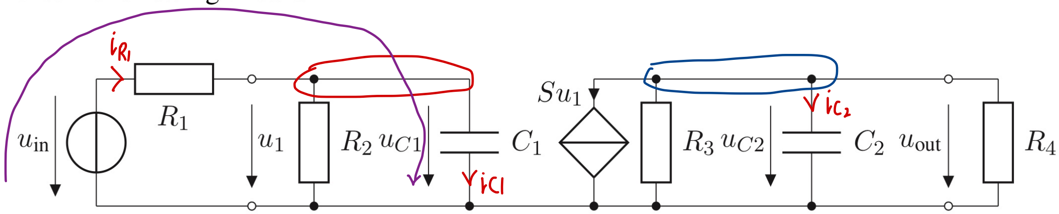
$U_{out}(t) = -U_C(t)$

$= -\frac{1}{RC} \int U_{in}(t')\, dt'$

$\Rightarrow$ integrator

8.1 Formulation of State Equations (1)

Given is following circuit.



a) Formulate the state equations. Give $\boldsymbol{x}$, $\boldsymbol{A}$, $\boldsymbol{b}$, and v.

$\underline{x} = [u_{C1}, u_{C2}]^T$

excitation ↓

$\underline{v = u_{in}}$

KCL: $-i_{R1} + i_{C1} + \frac{1}{R_2}u_{C1} = 0$

KVL: $-u_{in} + R_1 i_{R1} + u_{C1} = 0$

$$i_{R1} = \frac{u_{in} - u_{C1}}{R_1}$$

sub into KCL:

$$i_{C1} = -\frac{1}{R_2}u_{C1} + \frac{u_{in} - u_{C1}}{R_1}$$

$$= -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)u_{C1} + \frac{1}{R_1}u_{in}$$

KCL: $Su_1 + \frac{1}{R_3}\cdot u_{C2} + i_{C2} + \frac{1}{R_4}u_{out} = 0$

$$Su_{C1} + \frac{1}{R_3}u_{C2} + i_{C2} + \frac{1}{R_4}u_{C2} = 0$$

$$i_{C2} = -Su_{C1} - \left(\frac{1}{R_3} + \frac{1}{R_4}\right)u_{C2}$$

$$i_{C1} = C_1\dot{u}_{C1}, \quad i_{C2} = C_2\dot{u}_{C2}$$

$$\dot{u}_{C1} = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)u_{C1} + \frac{1}{R_1C_1}u_{in}$$

$$\dot{u}_{C2} = -\frac{S}{C_2}u_{C1} - \left(\frac{1}{R_3C_2} + \frac{1}{R_4C_2}\right)u_{C2}$$

$\dot{\underline{x}} = \underset{\sim}{A}\,\underline{x} + \underset{\sim}{B}\,\underline{v}$ ← search for u_{in}

$$\underset{\sim}{B} = \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix}$$

$$\begin{array}{c} U_{C1} \;\&\; U_{C2} \\ \downarrow \end{array}$$

$$\underset{\sim}{A} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & 0 \\ -\frac{S}{C_2} & -\left(\frac{1}{R_3C_2} + \frac{1}{R_4C_2}\right) \end{bmatrix}$$

b) Find the output equation. Give $\boldsymbol{c}^{\mathrm{T}}$ and d.

$$U_{out} = U_{C2} = [0, 1] \cdot \underline{X} + 0 \cdot v$$

$$\underline{c}^T = [0,1], \quad d = 0$$

$$y = U_{C2} = \underline{c}^T \underline{X} + dv$$

$$\Rightarrow d = 0$$

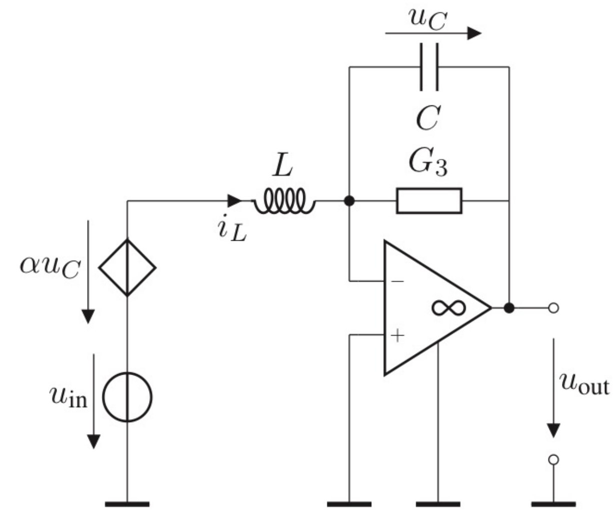
$$U_{C2} = \underline{c}^T \underline{X}$$

$$= \underbrace{\underline{[0, 1]^T}}_{\underline{c}^T} \cdot \underline{X}$$

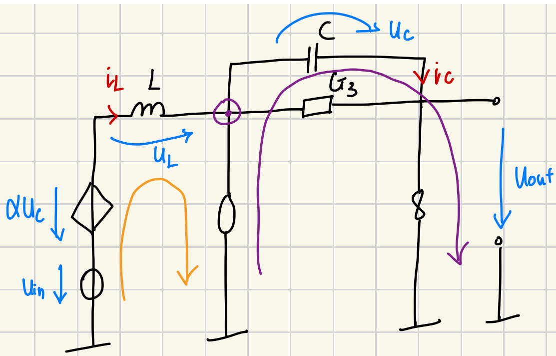
$$y = U_{C2} = 0 \cdot U_{C1} + 1 \cdot U_{C2}$$

8.2 Formulation of State Equations (2)

Given is following circuit with an Op-Amp operated in the linear region.



a) Formulate the state equations. Give $\boldsymbol{x}$, $\boldsymbol{A}$, $\boldsymbol{b}$, and v an.



inductor must find KVL with u_L

Capacitor find KCL with i_C

$v = u_{in}$

$\underline{x} = [i_L, u_c]^T$

$$KVL: \; -U_{in} - \alpha U_C + U_L = 0$$

$$U_L = \alpha U_C + U_{in}$$

$$KCL: \; -i_L + i_C + G_3 U_C = 0$$

$$i_C = i_L - G_3 U_C$$

$$U_L = L\dot{i}_L \;, \; i_C = C\dot{u}_C$$

$$\dot{i}_L = \frac{\alpha}{L} U_C + \frac{1}{L} U_{in}$$

$$\dot{u}_C = \frac{1}{C} i_L - \frac{G_3}{C} U_C$$

$$b = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \quad \underset{\sim}{A} = \begin{bmatrix} 0 & \frac{\alpha}{L} \\ \frac{1}{C} & -\frac{G_3}{C} \end{bmatrix}$$

b) Find the output equation. Give $\mathbf{c}^T$ and d.

$$KVL: \; U_C + U_{out} = 0$$

$$U_{out} = -U_C$$

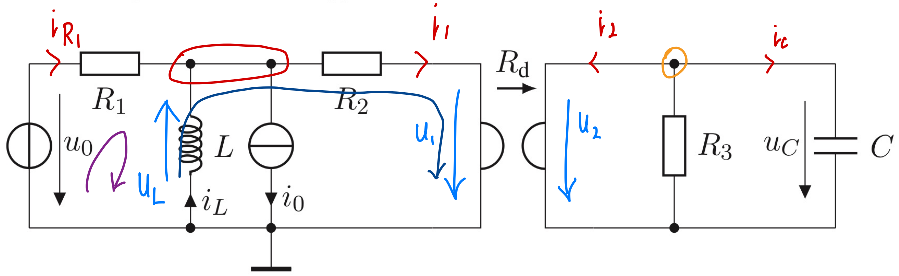
$$= [0, -1] \, \underline{x} + 0 \cdot U_{in}$$

$$C^T = [0, -1]$$

$$d = 0$$

8.3 Formulation of State Equations (3)

Given is the following circuit with a gyrator.



a) Find the parameters $\boldsymbol{x}$, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{v}$.

gyrator: $U_1 = R_d(-i_2)$

$U_2 = R_d i_1$

KCL $= -i_{R_1} - i_L + i_0 + i_1 = 0 \rightarrow -i_{R1} - i_L + i_0 + \frac{1}{R_d}U_c = 0$

$i_{R1} = -i_L + i_0 + \frac{1}{R_d}U_c$

KVL $= -U_0 + R_1 i_{R1} - U_L = 0$

$U_2 = U_c = R_d i_1$

$i_1 = \frac{1}{R_d}U_c$

$U_L = -U_0 + R_1 i_{R1}$

$$U_L = -U_0 + R_1\left(-i_L + i_0 + \frac{1}{R_d}U_c\right)$$

$$= -R_1 i_L + \frac{R_1}{R_d}U_c + R_1 i_0 - U_0$$

KCL: $i_2 + i_c + \frac{1}{R_3}U_c = 0$

$$-\frac{1}{R_d}U_1 + i_c + \frac{1}{R_3}U_c = 0$$

KVL: $U_L + U_1 + R_2 i_1 = 0$

$U_L + U_1 + \frac{R_2}{R_d}U_c = 0 \rightarrow U_1 = -U_L - \frac{R_2}{R_d}U_c$

$$\frac{1}{Rd}u_L + \frac{R_2}{R^2d}u_c + i_c + \frac{u_c}{R_3} = 0$$

Sub u_L from above:

$$-\frac{R_1}{Rd}i_L + \frac{R_1}{R^2d}u_c + \frac{R_1}{Rd}i_o - \frac{u_o}{Rd} + \frac{R_2}{R^2d}u_c + i_c + \frac{u_c}{R_3} = 0$$

$$i_c = -\left(\frac{R_1}{R^2d} + \frac{R_2}{R^2d} + \frac{1}{R_3}\right)u_c + \frac{R_1}{Rd}i_L - \frac{R_1}{Rd}i_o + \frac{u_o}{Rd}$$

$$\mathring{i}_L = -\frac{R_1}{L}i_L + \frac{R_1}{RdL}u_c + \frac{R_1}{L}i_o - \frac{1}{L}u_o$$

$$\mathring{u}_c = -\left(\frac{R_1}{R^2dC} + \frac{R_2}{R^2dC} + \frac{1}{R_3C}\right)u_c + \frac{R_1}{RdC}i_L - \frac{R_1}{RdC}i_o + \frac{1}{RdC}u_o$$

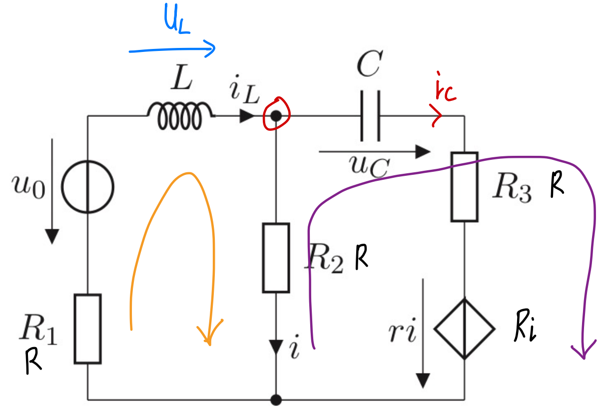
$$\underline{X} = \begin{bmatrix} i_L \\ u_c \end{bmatrix}, \quad \underset{\sim}{A} = \begin{bmatrix} -\frac{R_1}{L} & \frac{R_1}{RdL} \\ \frac{R_1}{RdC} & -\left(\frac{R_1}{R^2dC} + \frac{R_2}{R^2dC} + \frac{1}{R_3C}\right) \end{bmatrix}$$

$$\underline{v} = \begin{bmatrix} i_o \\ u_o \end{bmatrix}, \quad \underset{\sim}{B} = \begin{bmatrix} \frac{R_1}{L} & -\frac{1}{L} \\ -\frac{R_1}{RdC} & \frac{1}{RdC} \end{bmatrix}$$

$$\mathring{\underline{X}} = \underset{\sim}{A}\,\underline{X} + \underset{\sim}{B}\,\underline{v}$$

8.4 Formulation of State Equations (4)

Consider the following circuit with $R_1 = R_2 = R_3 = r = R$.



a) Formulate the state equations.

b) Give $\boldsymbol{x}$, $\boldsymbol{A}$, $\boldsymbol{b}$, and v.

KVL: $Ri_L - U_0 + U_L + Ri = 0$

KVL: $-Ri + U_C + Ri_C + Ri = 0 \rightarrow Ri_C = -U_C$

$RC\dot{U}_C = -U_C$

$\dot{U}_C = \frac{-1}{RC} U_C$

KCL: $i_L - i_C - i = 0$

$i_L + \frac{U_C}{R} - i = 0$

$\Rightarrow i = i_L + \frac{U_C}{R}$

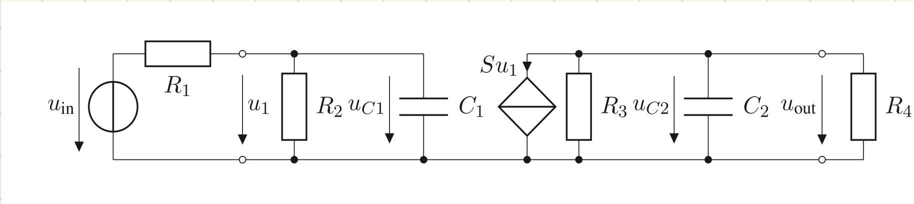
$Ri_L - U_0 + U_L + Ri_L + U_C = 0$

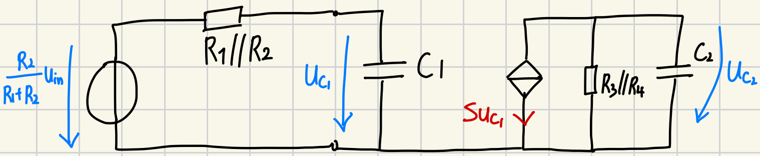
$U_L = -2Ri_L - U_C + U_0$

$\dot{i}_L = -\frac{2R}{L} i_L - \frac{1}{L} U_C + \frac{1}{L} U_0$

$$\underline{X} = \begin{bmatrix} u_c \\ i_L \end{bmatrix}, \quad \underset{\sim}{A} = \begin{bmatrix} -\frac{1}{RC} & 0 \\ -\frac{1}{L} & -\frac{2R}{L} \end{bmatrix}$$

$$\underline{V} = U_0, \quad \underset{\sim}{B} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$





$$\mathring{U}_{C_1} = -\frac{1}{(R_1\|R_2)C_1} U_{C_1} + \frac{1}{(R_2\|R_2)C_1} U_{in} \frac{R_2}{R_1+R_2}$$

$$= -\frac{1}{(R_1\|R_2)C_1} U_{C_1} + \frac{1}{R_1 C_1} U_{in}$$

$$\mathring{U}_{C2} = -\frac{1}{R_3\|R_4 C_2} U_{C2} - \frac{S}{C_2} U_{C_1}$$

8.5 Solution of Homogeneous State Equations (1)

Consider the circuit in Problem 8.1.

Given are the following element values: $R_1 = 1/3\,\Omega$, $R_2 = 1\,\Omega$, $R_3 = 2\,\Omega$, $R_4 = 2\,\Omega$, $C_1 = 1\,\text{F}$, $C_2 = 1\,\text{F}$, $S = -3\,\text{S}$, and $u_{\text{in}} = 0\,\text{V}$.

a) Find the eigenvalues of the state matrix.

$$R_1 \| R_2 = \frac{1}{3}\Omega \,//\, 1\Omega$$

$$= \frac{\frac{1}{3}\cdot 1}{\frac{1}{3}+1} = \frac{1}{3}\,\frac{1}{4/3}$$

$$= \frac{1}{4}\Omega$$

$$R_3 // R_4 = 1\,\Omega$$

$$\mathring{U}_{C_1} = -4\frac{1}{s} U_{C_1}$$

$$\mathring{U}_{C_2} = - -1\frac{1}{s} U_{C_2} + 3\frac{1}{s} U_{C_1}$$

$$\underset{\sim}{A} = \begin{bmatrix} -4 & 0 \\ 3 & -1 \end{bmatrix} \frac{1}{s}$$

$$\lambda_1 = -4\frac{1}{s}, \quad \lambda_2 = -1\frac{1}{s}$$

$$(\underset{\sim}{A} - \lambda_1 \underset{\sim}{1})\, q_1 = 0$$

$$\begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} q_1 = \underline{0}$$

$$q_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \frac{1}{s}$$

$$\begin{bmatrix} -3 & 0 \\ 3 & 0 \end{bmatrix} q_2 = \underline{0}$$

$$q_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \frac{1}{s}$$

c) Give the resulting general solution and the general solution for $u_{C1}(t_0) = 1\,\mathrm{V}$ and $u_{C2} = -1.5\,\mathrm{V}$.

$$\underline{X}(t) = \xi_{01} e^{\lambda_1 t} q_1 + \xi_{02} e^{\lambda_2 t} q_2$$

$$= \xi_{01} e^{-4t/s} \begin{bmatrix} 3 \\ -3 \end{bmatrix} + \xi_{02} e^{-t/s} \begin{bmatrix} 0 \\ -3 \end{bmatrix} \frac{1}{s}$$

$$\underline{X}(0) = \begin{bmatrix} 1\mathrm{V} \\ -1.5\mathrm{V} \end{bmatrix} = \xi_{01} \begin{bmatrix} 3 \\ -3 \end{bmatrix} \frac{1}{s} + \xi_{02} \begin{bmatrix} 0 \\ -3 \end{bmatrix} \frac{1}{s}$$

$$1\mathrm{V} = \xi_{01}\, 3\frac{1}{s} \rightarrow \xi_{01} = \frac{1}{3}\mathrm{Vs}$$

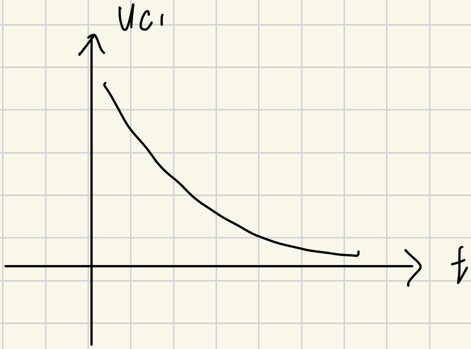
$$-1.5\mathrm{V} = \frac{1}{3}\mathrm{Vs}(-3)\frac{1}{s} + \xi_{02}(-3)\frac{1}{s}$$

$$-0.5\mathrm{V} = \xi_{02}(-3)\frac{1}{s}$$

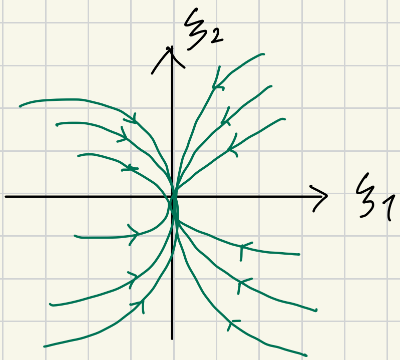
$$\rightarrow \xi_{02} = \frac{1}{6}\mathrm{Vs}$$

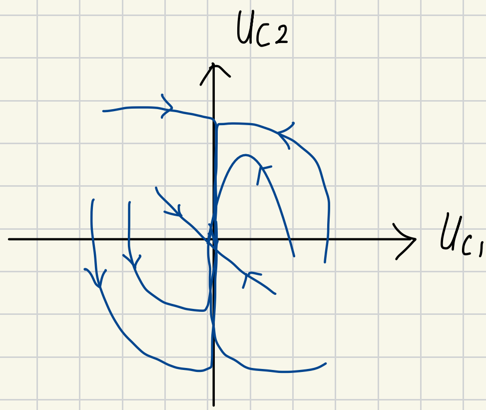
$$\underline{X}(t) = 1V e^{-4t/5} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2} V e^{-t/5} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

d) Draw $u_{C1}(t)$ and $u_{C2}(t)$.



e) Sketch the resulting phase portrait in the u_{C1}-u_{C2}-plane.





a) $R_1 \parallel R_2 = \frac{1}{4}\,\Omega$

$R_3 \parallel R_4 = 1\,\Omega$

$C_1 = C_2 = 1F$

$S = 3S$

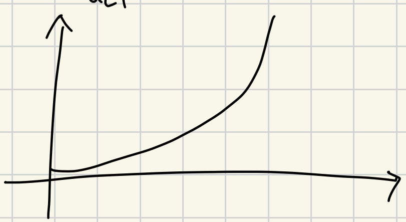
$$\mathring{U}_{C1} = 4\tfrac{1}{S}U_{C1}$$

$$\mathring{U}_{C2} = 1\tfrac{1}{S}U_{C2} + 3\tfrac{1}{S}U_{C1}$$

$$\underset{\sim}{A} = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}$$

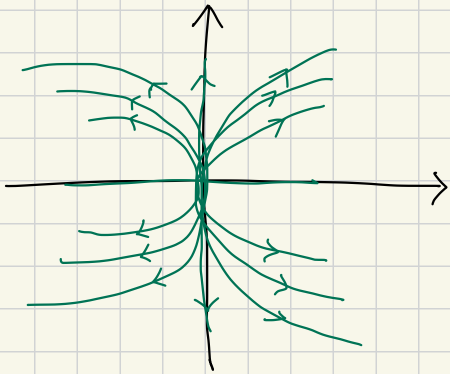
b) c) same as 8.5

d) U_{C1}



t

e)



8.7

a) $R_1 \| R_2 = 1\Omega$

$R_3 \| R_4 = 1\Omega$

$C_1 = 1F$

$C_2 = -1F$

$S = 3S$

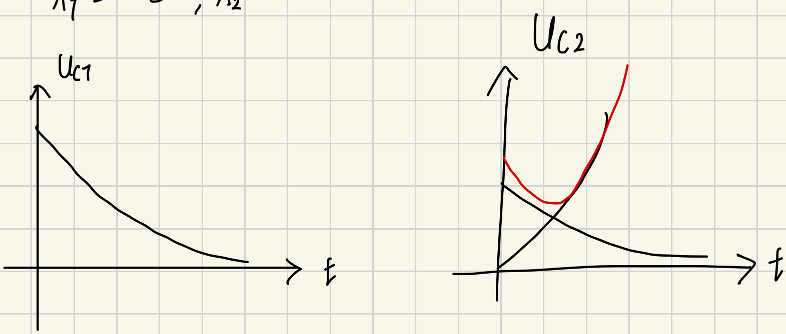
$\mathring{U}_{C1} = -1\frac{1}{s} U_{C1}$

$\mathring{U}_{C2} = 1\frac{1}{s} U_{C2} + 3\frac{1}{s} U_{C1}$

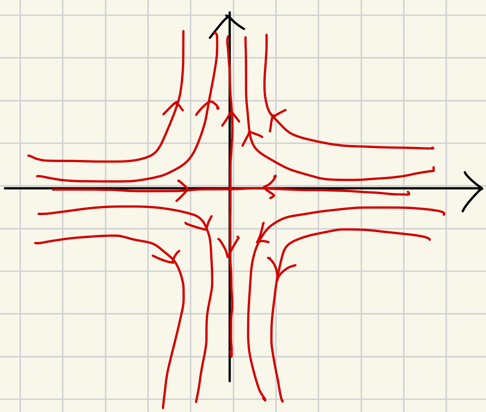
$$\underset{\sim}{A} = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} \frac{1}{s}$$

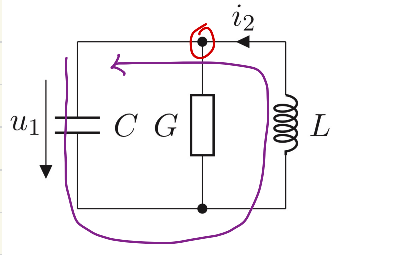
$\lambda_1 = -1\frac{1}{s}$, $\lambda_2 = +1\frac{1}{s}$

d)



e)





a) Formulate the state equation. Determine the state matrix $\boldsymbol{A}$ for $G = 4\,\mathrm{S}$, $C = 1\,\mathrm{F}$, and $L = 1/3\,\mathrm{H}$.

$$\underline{X} = [u_1, i_2]^T$$

$$\text{KCL: } C\dot{u}_1 + Gu_1 - i_2 = 0$$

$$\text{KVL: } u_1 + L\dot{i}_2 = 0$$

$$\dot{u}_1 = -\frac{G}{C}u_1 + \frac{1}{C}i_2$$

$$\dot{i}_2 = -\frac{1}{L}u_1$$

$$\dot{u}_1 = -4\frac{1}{s}u_1 + \frac{1}{1F}i_2$$

$$\dot{i}_2 = -\frac{3}{H}u_1$$

$$\dot{\underline{x}} = \underset{\sim}{A}\,\underline{x}$$

$$\underset{\sim}{A} = \begin{bmatrix} -4\frac{1}{s} & \frac{1}{1F} \\ -3\frac{1}{H} & 0 \end{bmatrix}$$

b) Find and sketch the solution for the initial conditions $u_1(t_0) = 1\,\mathrm{V}$ and $i_2(t_0) = -1\,\mathrm{A}$.

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{1}) = (-4-\lambda)(-\lambda) + 3\frac{1}{s^2}$$

$$= \lambda^2 + 4\lambda\frac{1}{s} + 3\frac{1}{s^2} = 0$$

$$\lambda_{1/2} = -2 \pm \sqrt{4-3}$$

$$= -2 \pm 1$$

$$\lambda_1 = -1\frac{1}{s}, \quad \lambda_2 = -3\frac{1}{s}$$

$$(A-\lambda_1 \underset{\sim}{1})\underset{\sim}{q}_1 = \begin{bmatrix} -3\frac{1}{s} & \frac{1}{1F} \\ -3\frac{1}{H} & 1\frac{1}{s} \end{bmatrix} \underset{\sim}{q}_1 = \underline{0}$$

$$\underset{\sim}{q}_1 = \begin{bmatrix} 1S \\ 3F \end{bmatrix}$$

$$\begin{bmatrix} -1\frac{1}{s} & \frac{1}{1F} \\ -3\frac{1}{H} & 3\frac{1}{s} \end{bmatrix} \underset{\sim}{q}_2 = \underline{0}$$

$$\underset{\sim}{q}_2 = \begin{bmatrix} 1S \\ 1F \end{bmatrix}$$

$$\underline{X}(t) = \zeta_{01} e^{-\frac{t}{s}} \begin{bmatrix} 1s \\ 3F \end{bmatrix} + \zeta_{02} e^{-\frac{3t}{s}} \begin{bmatrix} 1s \\ 1F \end{bmatrix}$$

$$\times \left(\frac{-1F}{1s}\right)$$

$t_0 = 0$:

$$\underline{X}(0) = \begin{bmatrix} 1V \\ -1A \end{bmatrix} = \zeta_{01} \begin{bmatrix} 1s \\ 3F \end{bmatrix} + \zeta_{02} \begin{bmatrix} 1s \\ 1F \end{bmatrix}$$

$$\frac{1F}{1s} = 1S$$

$$-1A = \zeta_{01}\, 1F - \zeta_{02}\, 1F$$

$$-1A = \zeta_{01}\, 3F + \zeta_{02}\, 1F$$

$$-2A = \zeta_{01}\, 2F \longrightarrow 2S \cdot s$$

Siemens — seconds

$$\zeta_{01} = -1\frac{V}{s}$$

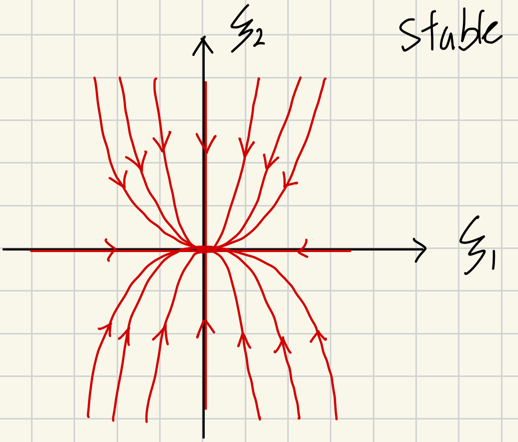
$$\zeta_{01}\, 1S + \zeta_{02}\, 1s = 1V$$

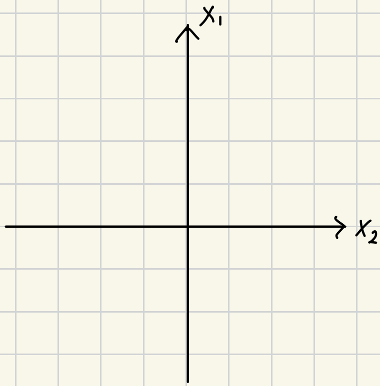
$$\zeta_{02} = 2\frac{V}{s}$$

$$\underline{X}(t) = 1\frac{V}{s} e^{-\frac{t}{s}} \begin{bmatrix} 1s \\ 3F \end{bmatrix} + 2\frac{V}{s} e^{-\frac{3t}{s}} \begin{bmatrix} 1s \\ 1F \end{bmatrix}$$

c) Sketch corresponding phase portrait in the x_1-x_2-plane.

stable node





$$\lambda_2 < \lambda_1 < 0$$

d) Determine and sketch the solution for $G = -4\,\mathrm{S}$, $C = 1\,\mathrm{F}$, and $L = 1/3\,\mathrm{H}$. The initial conditions are $u_1(t_0) = 1\,\mathrm{V}$ and $i_2(t_0) = -1\,\mathrm{A}$.

$$4\frac{1}{S}u_1 + \frac{1}{1F}i_2 = \dot{u}_1$$

$$-\frac{3}{H}u_1 = \dot{i}_2$$

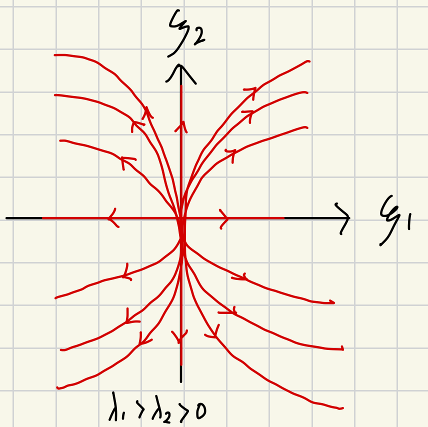
$$\underset{\sim}{A} = \begin{bmatrix} 4\frac{1}{S} & \frac{1}{1F} \\ -3\frac{1}{H} & 0 \end{bmatrix}$$

$$\lambda^2 - 4\lambda\frac{1}{S} + 3\frac{1}{S^2} = 0$$

$$\lambda_{1/2} = +2 \pm \sqrt{1}$$

$$\lambda_1 = 3\frac{1}{S}$$
$$\lambda_2 = +1\frac{1}{S}$$

e) Sketch the corresponding phase portrait in the x_1-x_2-plane.



$\lambda_1 > \lambda_2 > 0$

Unstable nod

f) Find and sketch the solution for $G = 1\,\mathrm{S}$, $C = 1\,\mathrm{F}$, and $L = -1/2\,\mathrm{H}$. The initial conditions are $u_1(t_0) = -1\,\mathrm{V}$ and $i_2(t_0) = -1.5\,\mathrm{A}$.

$$\underset{\sim}{A} = \begin{bmatrix} -1\frac{1}{S} & \frac{1}{1F} \\ 2\frac{1}{H} & 0 \end{bmatrix}$$

$$\lambda^2 + \lambda\frac{1}{S} - 2\frac{1}{S^2} = 0$$

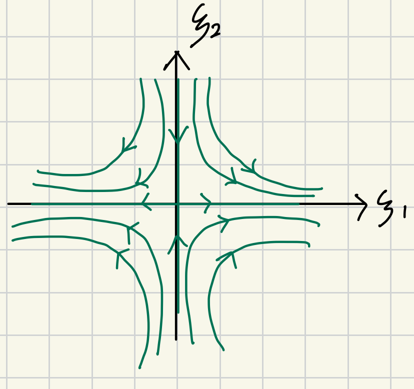
$$\lambda_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2}$$

$$= -\frac{1}{2} \pm \frac{3}{2}$$

$\lambda_1 = 1\frac{1}{S}$

$\lambda_2 = -2\frac{1}{S}$

g) Sketch the corresponding phase portrait in the x_1-x_2-plane.



Saddle point

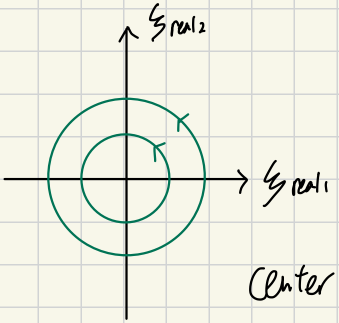
h) Determine and sketch the solution for $G = 0\,\mathrm{S}$, $C = 1\,\mathrm{F}$, and $L = 1\,\mathrm{H}$. The initial conditions are $u_1(t_0) = 1\,\mathrm{V}$ and $i_2(t_0) = 0\,\mathrm{A}$.

$$\underset{\sim}{A} = \begin{bmatrix} 0 & \frac{1}{1F} \\ -\frac{1}{1H} & 0 \end{bmatrix}$$

$$\lambda^2 + \frac{1}{s^2} = 0$$

$$\lambda_{1/2} = \pm j \frac{1}{s}$$

i)



Center

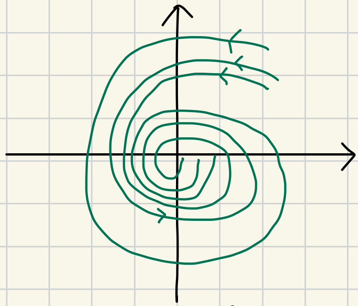
j) Find and sketch the solution for $G = 2\,\mathrm{S}$, $C = 1\,\mathrm{F}$, and $L = 1/2\,\mathrm{H}$. The initial conditions are $u_1(t_0) = -2\,\mathrm{V}$ and $i_2(t_0) = -2\,\mathrm{A}$.

$$\underset{\sim}{A} = \begin{bmatrix} -2\frac{1}{s} & 1\frac{1}{F} \\ -2\frac{1}{F} & 0 \end{bmatrix}$$

$$\lambda^2 + 2\lambda\frac{1}{s} + 2\frac{1}{s^2} = 0$$

$$\lambda_{1/2} = -1 \pm \sqrt{1-2}$$

$$= -1 \pm j$$



stable focus

k) Sketch the corresponding phase portrait in the $\xi_{\text{real}1}$-$\xi_{\text{real}2}$-plane.

Gegeben seien nun die Elementwerte $G = -2$ S, $C = 1$ F, and $L = 1/2$ H.

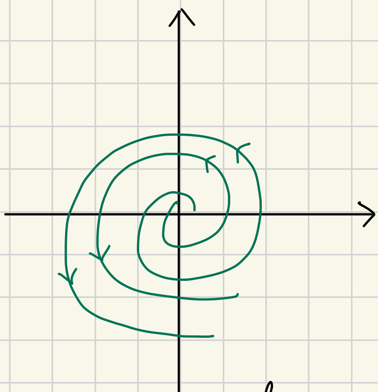
$$\underset{\sim}{A} = \begin{bmatrix} 2\frac{1}{S} & 1\frac{1}{F} \\ -2\frac{1}{H} & 0 \end{bmatrix}$$

$$\lambda^2 - 2\frac{1}{S}\lambda + 2\frac{1}{S^2} = 0$$

$$\lambda_{1/2} = 1 \pm \sqrt{-1}$$

$$= 1 \pm j$$

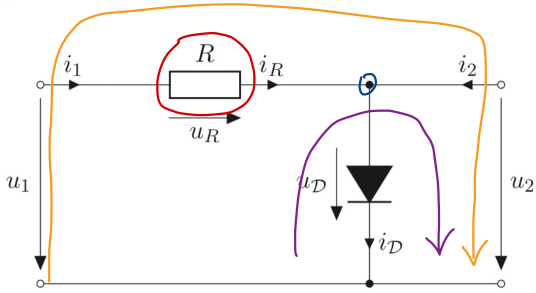
$$\lambda_1 = 1+j \quad , \quad \lambda_2 \; 1-j$$



unstable focus

3.1 Representation Forms of Two-Ports

Given is the following non-linear two-port consisting of an Ohmic resistor and a pn-diode.



Use the conductance $G = \frac{1}{R}$ in case that the resulting expression can be simplified.

a) Find u_R, u_D, i_R, and i_D depending on the port quantities.

KVL: $-U_1 + U_R + U_2 = 0$

$$U_R = U_1 - U_2$$

KVL: $-U_D + U_2 = 0$

$$U_D = U_2$$

KCL: $-i_1 + i_R = 0$

$$i_R = i_1$$

KCL: $i_D - i_1 - i_2 = 0$

$$i_D = i_1 + i_2$$

b) Give the representations for the two one-ports included in the given two-port, both in voltage- and in current-controlled form.

$$U_R = R i_R$$

$$i_R = G U_R$$

$$i_D = I_S \left(e^{\frac{U_D}{U_T}} - 1\right)$$

or

$$U_D = U_T \ln\left(\frac{i_D}{I_S} + 1\right)$$

Now derive the following six explicit representations of the two-port.

c) Voltage-controlled representation: $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \boldsymbol{g}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$

$$U_R = U_1 - U_2$$

$$i_R = G U_R$$

$$i_1 = G U_1 - G U_2$$

$$i_D = i_1 + i_2$$

$$i_2 = i_D - i_1$$

$$U_D = U_2$$

$$i_D = I_s\left(e^{\frac{U_2}{U_T}} - 1\right)$$

$$i_2 = I_s\left(e^{\frac{U_2}{U_T}} - 1\right) - G U_1 + G U_2$$

$$g(U_1, U_2) = \begin{bmatrix} G U_1 - G U_2 \\ I_s\left(e^{\frac{U_2}{U_T}} - 1\right) - G U_1 + G U_2 \end{bmatrix}$$

d) Current-controlled representation: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \boldsymbol{r}\left(\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}\right)$

$$i_D = i_1 + i_2$$

$$U_D = U_T \ln\left(\frac{i_1 + i_2}{I_S} + 1\right) = U_2$$

$$U_1 = U_R + U_2 = R i_1 + U_2$$

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} R i_1 + U_T \ln\left(\frac{i_1 + i_2}{I_S} + 1\right) \\ U_T \ln\left(\frac{i_1 + i_2}{I_S} + 1\right) \end{bmatrix}$$

e) Hybrid representation: $\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \boldsymbol{h}\left(\begin{bmatrix} i_1 \\ u_2 \end{bmatrix}\right)$

$$U_1 = U_R + U_2$$
$$U_1 = Ri_1 + U_2$$

$$i_2 = i_D - i_1$$
$$= I_S(e^{\frac{U_2}{U_T}} - 1) - i_1$$

f) Transmission representation: $\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = \boldsymbol{a}\left(\begin{bmatrix} u_2 \\ -i_2 \end{bmatrix}\right)$

$$U_D = U_2$$
$$i_D = I_S(e^{\frac{U_2}{U_T}} - 1)$$

$$i_1 = i_D - i_2$$
$$= I_S(e^{\frac{U_2}{U_T}} - 1) + (-i_2)$$

$$U_1 = U_R + U_2$$
$$= Ri_1 + U_2$$

$$U_1 = RI_S(e^{\frac{U_2}{U_T}} - 1) + R(-i_2) + U_2$$

g) Inverse transmission representation: $\begin{bmatrix} u_2 \\ i_2 \end{bmatrix} = \boldsymbol{a}'\left(\begin{bmatrix} u_1 \\ -i_1 \end{bmatrix}\right)$

$$U_2 = U_1 - U_R$$
$$= U_1 + R(-i_1)$$

$$i_2 = i_D - i_1$$
$$i_D = I_S(e^{\frac{U_2}{U_T}} - 1)$$
$$= I_S(e^{\frac{U_1 + R(-i_1)}{U_T}} - 1)$$

$$i_2 = I_S\left(e^{\frac{u_1 + R(-i_1)}{U_T}} - 1\right) + (-i_1)$$

h) Inverse hybrid representation: $\begin{bmatrix} i_1 \\ u_2 \end{bmatrix} = \boldsymbol{h}'\left(\begin{bmatrix} u_1 \\ i_2 \end{bmatrix}\right)$

$$i_1 = i_R = G U_R$$

$$U_R = U_1 - U_2$$

$$i_1 = G U_1 - G U_2$$

$$U_2 = U_T \ln\left(\frac{i_D}{I_S} + 1\right)$$

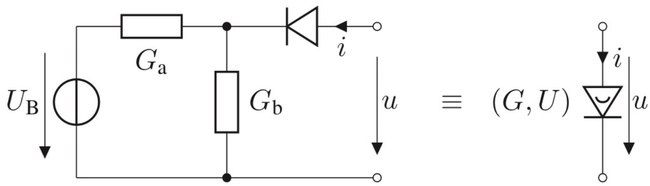
$$i_D = i_1 + i_2$$

$$U_2 = U_T \ln\left(\frac{i_1 + i_2}{I_S} + 1\right)$$

$$i_1 = G U_1 - G \cdot U_T \ln\left(\frac{i_1 + i_2}{I_S} + 1\right)$$

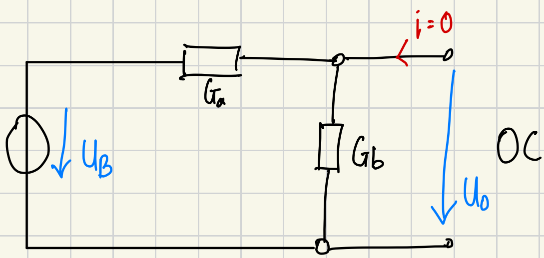
2.8 Realization of a Piecewise Linear Resistor

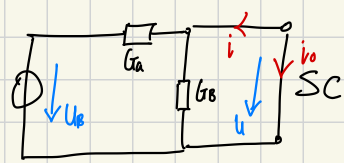
The following circuit $\mathcal{S}$ realizes a concave resistor.



Firstly, the parameters G and U of the equivalent concave resistor are determined depending on the parameters U_B, G_a, and G_b of $\mathcal{S}$.

a) Find the breakthrough point of the piecewise linear circuit $\mathcal{S}$. To this end, assume that the diode is also operated in its breakthrough point.





SC: $U = 0$

$i_0 = G_a U_B$

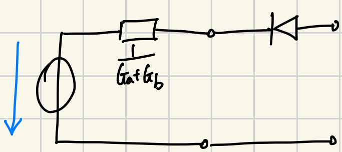
$$R_{inf} = \frac{U_0}{i_0} = \frac{1}{G_a + G_b}$$

Voltage divider:

$$\frac{U_0}{U_B} = \frac{1/G_b}{1/G_a + 1/G_b} = \frac{G_a}{G_a + G_b}$$

$$U_0 = \frac{G_a}{G_a + G_b} U_B$$

$U_0 = \frac{G_a}{G_a+G_B} U_B$

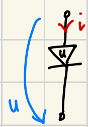


||

a) $U_0 = \frac{G_a}{G_a + G_b} U_B$

b) $G = G_a + G_b$

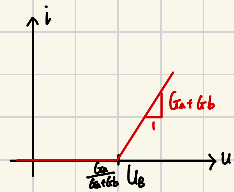
c)



$\left(G_a + G_b, \frac{G_a}{G_a+G_b} U_B \right)$

b) Determine the slope G of the right segment of the characteristic of $\mathcal{S}$ if the diode is operate in the conductive region.

c) Draw the characteristic of $\mathcal{S}$ in the u-i-plane and label the axes.



d) Express G_a and G_b depending on the parameters U and G. These equations constitute the *design rule* for $\mathcal{S}$.

d) $U = \frac{G_a}{G} U_B$

$G_a = G \cdot \frac{U}{U_B}$

$G = G_a + G_b$

$G_b = G - G_a$

$= G - G \cdot \frac{U}{U_B}$

$= G\left(1 - \frac{U}{U_B}\right)$

e) What are the conditions for G and U such that the circuit $\mathcal{S}$ is equivalent to the concave resistor (G, U) can be realized on a chip?

$G_a \geq 0$

$U \geq 0$

$G \geq 0$

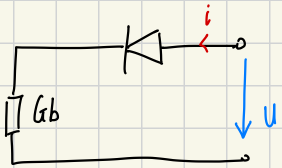
$1 - \frac{U}{U_B} \geq 0$

$U \leq U_B$

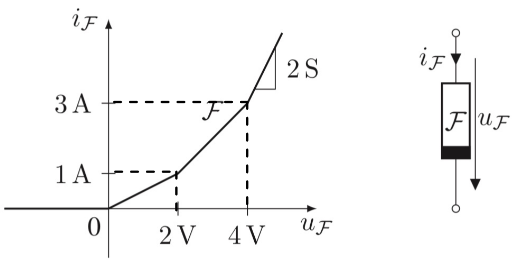
f) Simplify $\mathcal{S}$ and the design rule for the case that $U = 0$.

$G_a = 0$

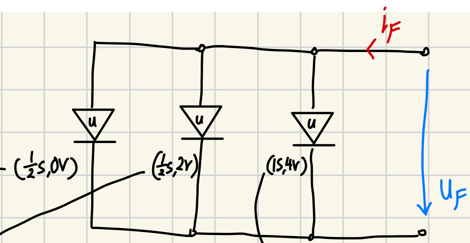
$G_b = G$



The piecewise linear resistor $\mathcal{F}$ with following characteristic is now realized.



g) Implement $\mathcal{F}$ as the parallel connection of three concave resistors $(G_j, U_j), j \in \{1, 2, 3\}$ and give the corresponding parameter values.



h) Realize $\mathcal{F}$ with ideal diodes, Ohmic resistors, and the voltage source $U_B = 10\,\mathrm{V}$. Draw the complete circuit with all element values.

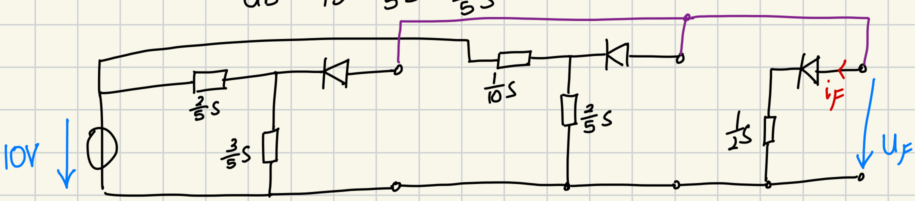
$G_b = \frac{1}{2}S$, $G_a = 0$, $U = 0V$

$G_a = \frac{1}{2}S \cdot \frac{2V}{10V} = \frac{1}{10}S$

$G_b = \frac{1}{2}S - \frac{1}{10}S = \frac{2}{5}S$

$G_a = 1S \cdot \frac{4V}{10V} = \frac{2}{5}S$

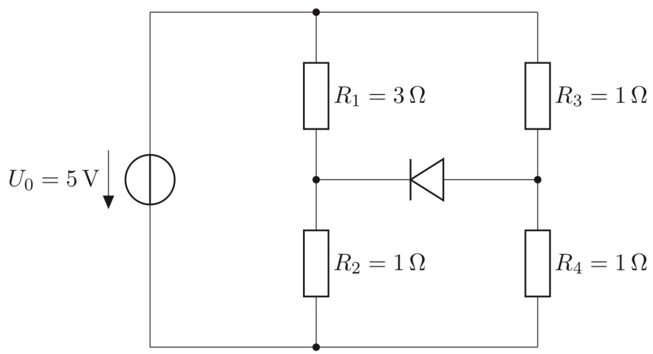
$G_b = 1S - \frac{2}{5}S = \frac{3}{5}S$



$(1S, 4V)$

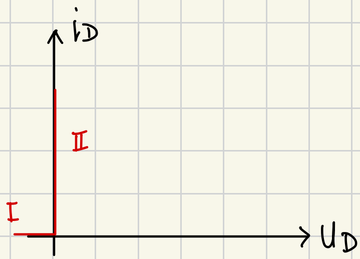
2.3

ideal diode



a) Find the operating point of the circuit assuming that the diode is operated in the cut-off region.

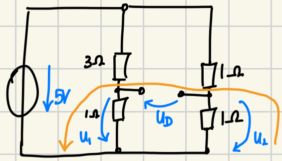
b) Determine the operating point of the circuit assuming that the diode is operated in the conductive region.



a) diode is in region I

$i_D = 0$, $U_D < 0$

$\Rightarrow$ OC



KVL: $-U_2 + U_D + U_1 = 0$

$U_D = U_2 - U_1$

Use voltage divider for U_1, U_2

$$U_1 = \frac{1\Omega}{1\Omega + 3\Omega} \cdot 5V = \frac{5}{4}V$$

$$U_2 = \frac{1\Omega}{1\Omega + 1\Omega} \cdot 5V = \frac{5}{2}V$$

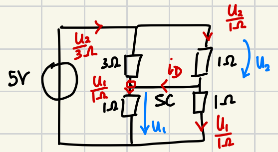
$$U_D = \frac{5}{2}V - \frac{5}{4}V = \frac{5}{4}V > 0$$

Contradiction

b) diode : II

$U_D = 0,\ i_D > 0$

$\Rightarrow$ SC

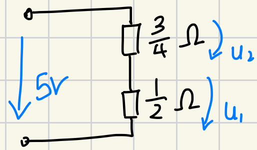


KVL : $\frac{U_1}{1\Omega} - \frac{U_2}{3\Omega} - i_D = 0$

$i_D = \frac{U_1}{1\Omega} - \frac{U_2}{3\Omega}$

upper case : $3\Omega \,//\, 1\Omega = \frac{3\cdot 1}{3+1} = \frac{3}{4}\Omega$

$1\Omega \,//\, 1\Omega = \frac{1\cdot 1}{1+1} = \frac{1}{2}\Omega$



$$\frac{U_1}{5V} = \frac{\frac{1}{2}\Omega}{\frac{3}{4}\Omega + \frac{1}{2}\Omega} = \frac{\frac{1}{2}}{\frac{5}{4}} = \frac{2}{5}$$

$U_1 = 2V$

$$\frac{U_2}{5V} = \frac{\frac{3}{4}\Omega}{\frac{5}{4}\Omega}$$

$U_2 = 3V$

$$i_D = \frac{U_1}{1\Omega} - \frac{U_2}{3\Omega}$$

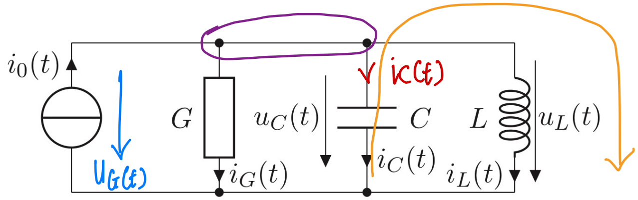
$= 2A - 1A$

$= 1A > 0$ ✓

$\rightarrow$ ideal diode is conductive

9.1 RLC-Resonant Circuit

Consider the following parallel RLC-resonant circuit with a sinusoidal $i_0(t) = I_{0m}\cos(\omega t + \pi)$.



a) Formulate the state equation for this circuit. Give $\boldsymbol{A}$.

$$\underline{x}(t) = [u_C(t),\ i_L(t)]^T$$

$$v(t) = i_0(t)$$

$$\dot{\underline{x}}(t) = \underset{\sim}{A}\,\underline{x}(t) + \underline{b}\,v(t)$$

KVL: $-u_C(t) + u_L(t) = 0$

$$u_L(t) = u_C(t)$$

$$L\dot{i}_L(t) = u_C(t)$$

$$\dot{i}_L(t) = \frac{u_C(t)}{L}$$

KCL: $-i_0(t) + i_G(t) + i_L(t) = 0$

Ohm: $i_G(t) = G u_C(t)$

$$i_C(t) = -G u_C(t) - i_L(t) + i_0(t)$$

$$\dot{u}_C(t) = -\frac{G}{C}u_C(t) - \frac{1}{C}i_L(t) + \frac{1}{C}i_0(t)$$

$$\underset{\sim}{A} = \begin{bmatrix} -\frac{G}{C} & -\frac{1}{C} \\ \frac{1}{L} & +0 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

b) What are the eigenvalues of $\boldsymbol{A}$?

$$\text{def}(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = \text{def} \begin{bmatrix} -\frac{G}{C} - \lambda & -\frac{1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix}$$

$$= \lambda^2 + \frac{G}{C}\lambda + \frac{1}{LC}\lambda = 0$$

$$\lambda_{1/2} = -\frac{G}{2C} \pm \sqrt{\left(\frac{G}{2C}\right)^2 - \frac{1}{LC}}$$

$$= -\frac{G}{2C} \pm \frac{G}{2C}\sqrt{1 - \frac{4C^2}{GLC}}$$

$$x^2 + px + q = 0$$

$$x_{1/2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$ax^2 + bx + c = 0$$

$$x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

c) Give the relationship between $i_L(t)$ and $u_C(t)$.

$$U_C(t) = U_L(t)$$

$$U_C(t) = L\,\dot{i}_L(t)$$

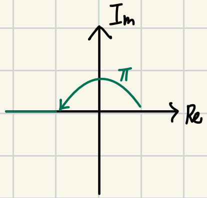
d) Express the phasor I_L depending on the phasor U_C corresponding to $u_C(t)$.

$$U_C = j\omega L I_L$$

$$I_L = \frac{U_C}{j\omega L}$$

e) Determine the phasor I_G depending on U_C.

$$I_G = G U_C$$



f) Give I_0 corresponding to $i_0(t)$.

$$I_0 = I_{0m} e^{j\pi}$$

$$= -I_{0m}$$

g) Find I_0 depending on U_C.

$$I_C = j\omega C \cdot U_C$$

KCL : $-I_0 + I_G + I_C + I_L = 0$

$$I_0 = G U_C + j\omega C U_C + \frac{1}{j\omega L} U_C$$

$$= \left(G + j\omega C + \frac{1}{j\omega L}\right) U_C$$

h) What is, therefore, U_C depending on I_0?

$$U_C = \frac{1}{G + j\omega C + \frac{1}{j\omega L}} I_0$$

$$= \frac{j\omega L}{1 + Gj\omega L + (j\omega)^2 LC}$$

i) Determine the angular frequencies ω such that U_C is real-valued.

$$\frac{X}{Y} = \frac{X \cdot Y^*}{Y \cdot Y^*} = \frac{X \cdot Y^*}{|Y|^2}$$

$$U_C = \frac{j\omega L}{1 + j\omega GL + (j\omega)^2 LC} \cdot \frac{1 - \omega^2 LC - j\omega GL}{1 - \omega^2 LC - j\omega GL} I_0$$

$$= \frac{\omega^2 GL^2 + j\omega L(1 - \omega^2 LC)}{\underbrace{(1 - \omega^2 LC)^2 + (\omega GL)^2}_{\in \mathbb{R}}} \underbrace{(-I_{0m})}_{\in \mathbb{R}}$$

for $U_C \in \mathbb{R}$: $\omega L(1 - \omega^2 LC) = 0$

I) $\omega = 0$

II) $1 - \omega^2 LC = 0$

$$\omega = \frac{1}{\sqrt{LC}}$$

III) $\omega \to \infty$

j) Give the values of U_C for these angular frequencies.

$$U_C(j0) = 0$$

$$U_C\left(j\frac{1}{\sqrt{LC}}\right) = \frac{\frac{1}{LC} GL^2}{\left(G\sqrt{\frac{L}{C}}\right)^2} (-I_{0m}) = \frac{\frac{GL}{C}}{G^2 \cdot \frac{L}{C}} (-I_{0m})$$

$$= -\frac{1}{G} I_{0m}$$

$$U_C(j\infty) = 0$$

zero for low/high frequencies, non-zero for in between

Thus, this circuit is a bandbass and one of these values for U_C corresponds to the maximum $U_{C,\max}$ of $|U_C|$.

k) Perform a dual transform of the given circuit with the duality constant R_d. Draw the resulting circuit.

$$U_{C,\max} = \frac{1}{G} I_{om}$$

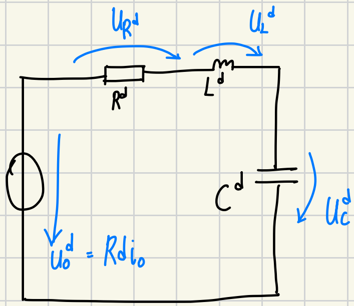
parallel connections $\xrightarrow{R_d}$ series connection

current source $\xrightarrow{R_d}$ voltage source

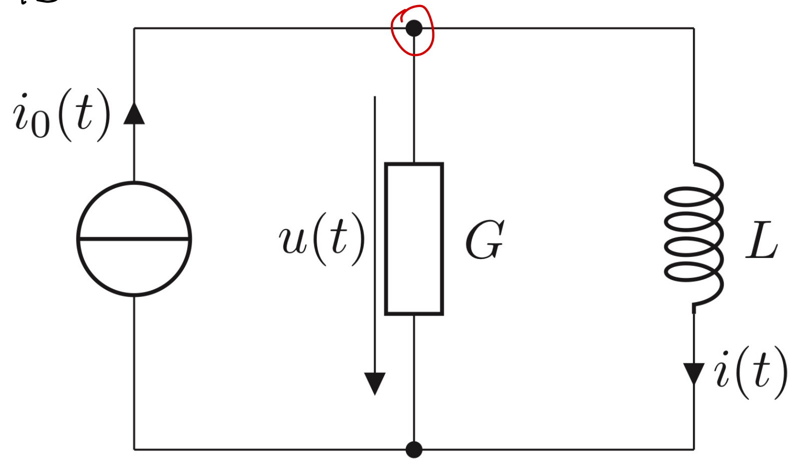
$$G \xrightarrow{R_d} R^d = R_d^2 G$$

$$C \xrightarrow{R_d} L^d = R_d^2 C$$

$$L \xrightarrow{R_d} C^d = \frac{1}{R_d^2} L$$



9.3



a) Give the phasor I_0 for $i_0(t) = \hat{I}_0 \cos(\omega t)$ and also for $i_0(t) = \hat{I}_0 \sin(\omega t)$.

KCL: $i_0(t) - Gu(t) - i(t) = 0$

a) $i_0(t) = \hat{I}_0 \cos(\omega t)$

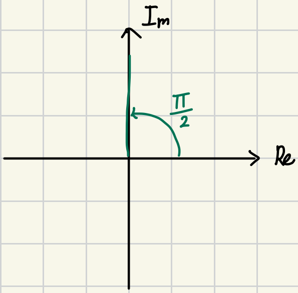
$I_0 = \hat{I}_0$

$i_0(t) = \hat{I}_0 \sin(\omega t)$

$= \hat{I}_0 \cos(\omega t + \frac{\pi}{2})$

$I_0 = \hat{I}_0 e^{j\frac{\pi}{2}}$

$= jI_0$



b) What ist the relationship between $i(t)$ and $u(t)$?

$U(t) = L\dot{i}(t)$

c) Express this relationship for the phasors I and U.

$U = j\omega L I$

d) Based on the Kirchhoff's laws and Ohm's law, give I depending on U and I_0.

from KCL:

$$\underline{I} = I_0 - Gu$$

e) Find the phasor I depending on I_0.

$$U = j\omega L(I_0 - Gu)$$

$$U + j\omega GLu = j\omega L I_0$$

$$U = \frac{j\omega L}{1 + j\omega GL} I_0$$

$$I = I_0 - \frac{j\omega LG}{1 + j\omega LG} I_0 = \frac{1}{1 + j\omega LG} I_0$$

In the following, $i_0(t)$ is the input to the circuit and $i(t)$ its output.

f) Show that the circuit is a lowpass, i.e., signals $i_0(t)$ with low frequency can pass and portions with high frequency are suppressed. To this end, investigate $|\frac{I}{I_0}|$ for $\omega = 0$, $\omega = \frac{1}{GL}$, and $\omega \to \infty$.

$$\left|\frac{I}{I_0}\right| = \frac{1}{\sqrt{1 + (\omega LG)^2}}$$

$$\omega = 0: \left|\frac{I}{I_0}\right| = 1$$

$$\omega = \frac{1}{GL}: \left|\frac{I}{I_0}\right| = \frac{1}{\sqrt{2}}$$

$$\omega \to \infty: \left|\frac{I}{I_0}\right| \to 0$$

$\to$ Lowpass

g) Determine the phasor I for $I_0 = \frac{1}{\sqrt{2}}(1+\mathrm{j})\hat{I}_1$ and $\omega = \frac{1}{GL}$.

$$I_0 = \frac{1}{\sqrt{2}}(1+j)\hat{I}_1$$

$$\omega = \frac{1}{GL}$$

$$I = \frac{1}{1+j} \cdot \frac{1}{\sqrt{2}}(1+j)\hat{I}_1$$

$$= \frac{\hat{I}_1}{\sqrt{2}}$$

h) Find I for $I_0 = (1-0.5\,\mathrm{j})\hat{I}_2$ and $\omega = \frac{2}{GL}$.

$$I_0 = \left(1-\frac{1}{2}j\right)\hat{I}_2 \qquad \omega = \frac{2}{GL}$$

$$I = \frac{1}{1+2j}\left(1-\frac{1}{2}j\right)\hat{I}_2$$

$\downarrow$ complex conjugate

$$= \frac{1-j^2}{1+4}\left(1-\frac{1}{2}j\right)\hat{I}_2$$

$$= \frac{1-2j-\frac{1}{2}j-1}{5}\hat{I}_2 = -\frac{1}{2}j\hat{I}_2$$

i) Give $i(t)$ for both cases.

$$i(t) = Re\{ I e^{j\omega t} \}$$

$$\text{I)}\ i(t) = Re\left\{ \frac{\hat{I}_1}{\sqrt{2}} e^{j\frac{1}{GL}t} \right\}$$

euler's rule, cos as (I) is in Re

$$= \frac{\hat{I}_1}{\sqrt{2}} \cos\left(\frac{1}{GL}t\right)$$

$$\text{II)}\ i(t) = Re\left\{ -\frac{1}{2} j \hat{I}_2 e^{j\frac{2}{GL}t} \right\}$$

$$= Re\left\{ -\frac{1}{2} j \hat{I}_2 \left(\cos\left(\frac{2}{GL}t\right) + j\sin\left(\frac{2}{GL}t\right) \right) \right\}$$

$$= \frac{1}{2}\hat{I}_2 \sin\left(\frac{2}{GL}t\right)$$

$$j \cdot j = (\sqrt{-1})^2 = -1$$

$$\therefore\ -1 \cdot -\frac{1}{2} = \frac{1}{2}$$

use sin as (II) in Im

Now the input current is given by

$$i_0(t) = \hat{I}_1 \cos\left(\frac{1}{GL}t + \frac{\pi}{4}\right) + \sqrt{1.25}\hat{I}_2 \cos\left(\frac{2}{GL}t + \arctan(-0.5)\right).$$

j) Find $i(t)$ resulting from this particular $i_0(t)$.

$$\tan(\varphi) = -0.5 = \frac{Im}{Re} = -\frac{1}{2}$$

$$\rightarrow Re = 2$$

$$Im = -1$$

$$\Rightarrow \underbrace{X \cdot (2-j)}$$

$$|2-j| = \sqrt{4-(-1)}$$

$$= \sqrt{5} \cdot \sqrt{1.25} = \sqrt{6.25} = 2.5 = 2$$

Phasor:

$$\sqrt{0.25} = \frac{1}{2}$$

$$\frac{5}{2}\hat{I}_2 \cdot (2-j) \frac{1}{\sqrt{5}} \sqrt{1.25} \approx \underbrace{\hat{I}_2 \left(1 - j\frac{1}{2}\right)}_{\text{from h)}}$$

$$\text{Phasor} = \hat{I}_1 e^{j\frac{\pi}{4}}$$

$$= \hat{I}_1 \left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \right)$$

$$= \underbrace{\hat{I}_0 \frac{1}{\sqrt{2}} (1+j)}_{\text{from g)}}$$

$$i(t) = \frac{\hat{I}_1}{\sqrt{2}} \cos\left(\frac{1}{GL} t\right) + \frac{1}{2} \hat{I}_2 \sin\left(\frac{2}{GL} t\right)$$

k) Assuming that $i_G(t) = Gu(t)$, investigate $|\frac{I_G}{I_0}|$ for $\omega = 0$, $\omega = \frac{1}{GL}$, and $\omega \to \infty$.

$$I_G = GU_G$$

$$= j\omega LGI$$

$$= \frac{j\omega LG}{1 + j\omega LG} I_0$$

$$\left|\frac{I_G}{I_0}\right| = \frac{\omega LG}{\sqrt{1 + (\omega LG)^2}}$$

$$\omega = 0 : \left|\frac{I_G}{I_0}\right| = 0$$

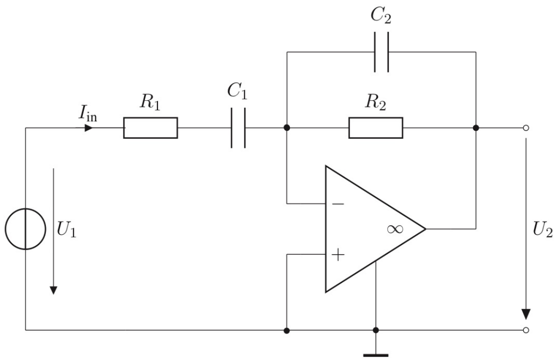
$$\omega = \frac{1}{LG} : \left|\frac{I_G}{I_0}\right| = \frac{1}{\sqrt{2}}$$

$$\omega \to \infty : \left|\frac{I_G}{I_0}\right| \to 1$$

→ highpass

9.4 Transfer Function

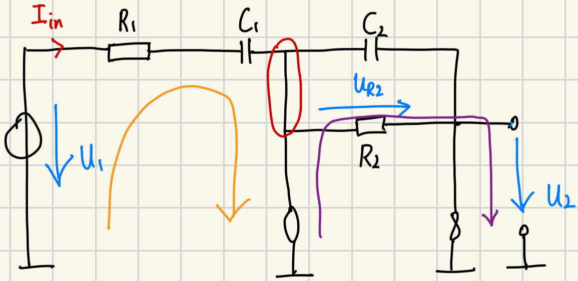
Consider the following circuit with ideal Op-Amp, the resistors R_1 and R_2, and the capacitors C_1 and C_2.
The Op-Omp is operated in the linear region.



a) Determine the transfer function $H(\mathrm{j}\,\omega) = U_2/U_1$ by formulating the KVL and KCL equations taking into account the properties of the ideal Op-Amp (nullor).

Set $R_1 = R$, $R_2 = 10R$, $C_1 = C$, and $C_2 = 10C$. Also use $RC = 1$.

$$H(j\omega) = \frac{U_2}{U_1}$$



$$\text{KVL}: -U_1 + R_1 I_{in} + \frac{1}{j\omega C_1} I_{in} = 0$$

$$\left(R_1 + \frac{1}{j\omega C_1}\right) I_{in} = U_1$$

$$I_{in} = \frac{j\omega C_1}{1 + j\omega R_1 C_1} U_1$$

$$\text{KCL}: I_{in} - \frac{1}{R_2} U_{R2} - j\omega C_2 U_{R2} = 0$$

$$I_{in} = \left(\frac{1}{R_2} + j\omega C_2\right) U_{R2}$$

$$U_{R2} = -\frac{R_2}{1 + j\omega R_2 C_2} I_{in}$$

KVL: $U_{R2} + U_2 = 0$

$$U_2 = -U_{R2}$$

$$= -\frac{R_2}{1+j\omega R_2 C_2} \frac{j\omega C_1}{1+j\omega R_1 C_1} U_1$$

$$H(j\omega) = -\frac{j\omega R_2 C_1}{(1+j\omega R_2 C_2)(1+j\omega R_1 C_1)}$$

b) Investigate $H(\mathrm{j}\omega)$ at $\omega = 0$, $\omega = \frac{1}{10}$, and $\omega \to \infty$.

$$H(j\omega) = -\frac{j\omega 10RC}{(1+j\omega 100RC)(1+j\omega RC)}$$

$$= -\frac{j\omega \cdot 10}{(1+j\omega 100)(1+j\omega)}$$

$\omega = 0$: $H(j0) = 0$

$\omega = \frac{1}{10}$: $H(j\frac{1}{10}) = \frac{-j}{(1+j10)(1+j\frac{1}{10})} = \frac{-j}{1+j10+j\frac{1}{10}-1} = \frac{-j}{j\frac{101}{10}}$

$\omega \to \infty$: $H(j\omega) \to 0$

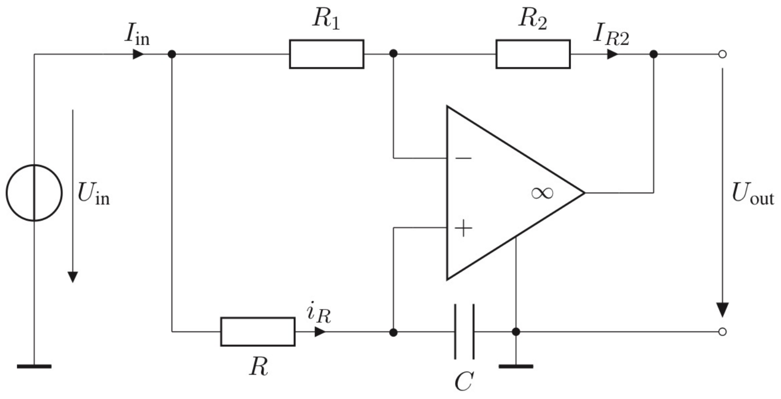
$$= -\frac{10}{101}$$

c) What can the circuit be used for?

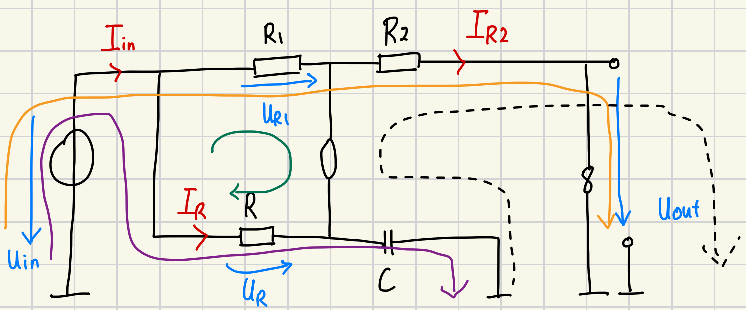
bandpass

9.5 Transfer Function, Allpass

Consider the circuit below. The Op-Omp is operated in the linear region.



a) Determine the transfer function $H(j\omega) = U_{out}/U_{in}$.



$$\text{KVL: } -U_{in} + RI_R + \underbrace{\frac{1}{j\omega C}}_{\text{impedence of Capacitor}} I_R = 0$$

$$\Rightarrow U_{in} = \left(R + \frac{1}{j\omega C}\right) I_R$$

$$\text{KVL : } -U_{in} + R_1 I_{R2} + R_2 I_{R2} + U_{out} = 0$$

$$\text{KVL : } U_{R1} - U_R = 0$$

$$U_{R1} = U_R$$

$$R I_R = R_1 I_{R2}$$

$$I_{R2} = \frac{R}{R_1} I_R$$

$$\text{KVL}: \quad -\frac{1}{j\omega C} I_R + R_2 I_{R2} + U_{out} = 0$$

$$\left(\frac{R}{R_1} R_2 - \frac{1}{j\omega C}\right) I_R + U_{out} = 0$$

$$U_{out} = -\left(\frac{R}{R_1} R_2 - \frac{1}{j\omega C}\right) I_R$$

$$U_{in} = \left(R + \frac{1}{j\omega C}\right) I_R$$

$$H(j\omega) = \frac{U_{out}}{U_{in}}$$

$$= \frac{-\left(\frac{R}{R_1} R_2 - \frac{1}{j\omega C}\right)}{R + \frac{1}{j\omega C}}$$

b) Find R_1 and R_2 such that the circuit is an allpass ($|H(\mathrm{j}\,\omega)| = 1$).

$$|H(j\omega)| = \left| \frac{-1 + \frac{R}{R_1} R_2 j\omega C}{1 + j\omega RC} \right|$$

$$= \sqrt{\frac{1 + \left(\omega \frac{R}{R_1} R_2 C\right)^2}{1 + (\omega RC)^2}} \overset{!}{=} 1$$

$$\cancel{1+} \left(\cancel{\omega} \frac{R}{R_1} R_2 \cancel{C}\right)^{\cancel{2}} = \cancel{1+} \left(\cancel{\omega} R \cancel{C}\right)^{\cancel{2}}$$

$$\frac{\cancel{R}}{R_1} R_2 = \cancel{R}$$

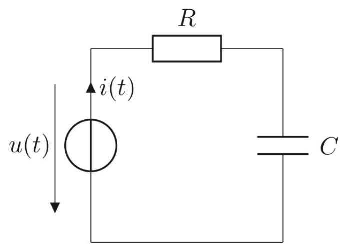
$$\frac{R_2}{R_1} = 1$$

R_2 & R_1 must be the same

purposes of allpass is to change phasors

9.6 Complex Power

Consider the following circuit with $u(t) = \hat{U}\cos(\omega t)$.



a) Give the complex phasor U corresponding to $u(t)$ depending on R, C, and the phasor I.

$i(t) = \mathrm{Re}\{I e^{j\omega t}\}$ $\quad u(t) = \hat{U}\cos(\omega t)$

$u(t) = \mathrm{Re}\{U e^{j\omega t}\}$

$$Z_{overall} = R + \frac{1}{j\omega C}$$

$$U = Z_{overall} \cdot I$$

$$= \left(R + \frac{1}{j\omega C}\right) I$$

b) Determine the complex power P that is consumed by the circuit depending on U.

$$P = \frac{1}{2} U I^*$$

$$= \frac{1}{2} U \cdot \frac{-j\omega C U^*}{1 - j\omega RC}$$

$$= \frac{1}{2} \frac{-j\omega C}{1 - j\omega RC} |U|^2$$

c) Give the average power P_{W} and the blind power P_{B} depending on U.

$$P = P_W + jP_B$$

$$P = \frac{1}{2} \frac{-j\omega C(1+j\omega RC)}{1+(\omega RC)^2} |U|^2$$

$$= \frac{1}{2} \frac{\omega^2 RC^2 - j\omega C}{1+(\omega RC)^2} |U|^2$$

$$P_W = \frac{1}{2} \frac{\omega^2 RC^2}{1+(\omega RC)^2} |U|^2$$

$$P_B = \frac{1}{2} \frac{-j\omega C}{1+(\omega RC)^2} |U|^2$$

d) Find the apparent power $S = |P|$.

$$S = \frac{\sqrt{(\omega^2 RC^2)^2 + (\omega C)^2}}{1+(\omega RC)^2} |U|^2$$

e) What is the energy that is delivered by the source in every period?

$$P_W = \frac{1}{T} \underbrace{\int_0^T p(t)\,dt}_{E}$$

$$E = T \cdot P_W$$

$$= \frac{T}{2} \frac{\omega^2 RC^2}{1+(\omega RC)^2} |U|^2$$

angular freq. : $\omega = \frac{2\pi}{T} \rightarrow T = \frac{2\pi}{\omega}$

$$E = \pi \cdot \frac{\omega RC^2}{1+(\omega RC)^2} |U|^2$$

f) Find the angular frequency ω such that this energy is maximized. Give the maximum energy E_{max}.

$$\frac{dE}{d\omega} = \pi |U|^2 \cdot \left(\frac{RC^2 (1+(\omega RC)^2)}{(1+(\omega RC)^2)^2} - \frac{2\omega (RC)^2 \omega RC^2}{(1+(\omega RC)^2)^2} \right) = 0$$

$$\cancel{RC^2}(1+(\omega RC)^2) - 2\omega (RC)^2 \omega R\cancel{C^2} = 0$$

$$1+(\omega RC)^2 - 2(\omega RC)^2 = 0$$

$$1-(\omega RC)^2 = 0$$

$$1 = \omega RC$$

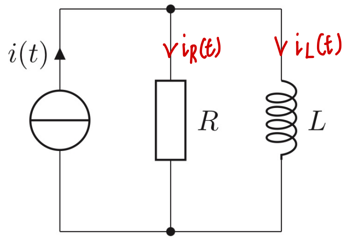
$$\omega = \frac{1}{RC}$$

$$E_{max} = \pi \frac{C}{1+(1)^2}$$

$$= \frac{\pi}{2} C$$

9.7 Complex Power

Consider the following circuit with $i(t) = \hat{I}\cos(\omega t + \varphi)$.



a) Give the current phasor I corresponding to $i(t)$.

$$I = \hat{I} e^{j\varphi}$$

$$i(t) = \mathrm{Re}\{I e^{j\omega t}\}$$

b) Determine the impedance Z of the parallel connection of R and L.

$$Z = Z_R \,//\, Z_L$$

$$= R \,//\, j\omega L$$

$$= \frac{R \cdot j\omega L}{R + j\omega L}$$

c) Find the complex power of the circuit.

$$U = ZI$$

$$P = \frac{1}{2} U I^*$$

$$= \frac{1}{2} Z I \cdot I^*$$

$$= \frac{1}{2} \frac{j\omega L R}{R + j\omega L} |I|^2$$

d) How much energy is dissipated by the circuit in every period?

$$P = P_W + jP_B$$

$$= \frac{1}{2} \frac{j\omega L R (R - j\omega L)}{R^2 + (\omega L)^2} |I|^2$$

$$= \frac{1}{2} \frac{\omega^2 L^2 R + j\omega L R^2}{R^2 + (\omega L)^2} |I|^2$$

$$\Rightarrow P_W = \frac{1}{2} \frac{\omega^2 L^2 R}{R^2 + (\omega L)^2} |I|^2$$

$$= \frac{1}{T} E, \quad T = \frac{2\pi}{\omega},$$

$$E = T P_W$$

$$= \pi \cdot \frac{\omega L^2 R}{R^2 + (\omega L)^2} \cdot |I|^2$$

e) Find the angular frequency ω to maximize the energy. What is the value of the maximum energy?

$$\frac{dE}{d\omega} = \pi \cdot \frac{\cancel{L^2 R}(R^2 + (\omega L)^2) - 2\omega L^2 \cdot \omega \cancel{L^2 R}}{(R^2 + (\omega L)^2)^2} |I|^2 = 0$$

$$R^2 + \omega^2 L^2 - 2\omega^2 L^2 = 0$$

$$R^2 - \omega^2 L^2 = 0$$

$$\omega = \frac{R}{L}$$

$$E_{max} = \pi \frac{R^2 L}{R^2 + R^2} |I|^2 = \frac{\pi}{2} L |I|^2$$