Technische Universität München

Circuit Theory

Lecture Notes 2024

Dr.-Ing. Michael Joham

Fakultät für Elektrotechnik und Informationstechnik
Professur Methoden der Signalverarbeitung
Univ.-Prof. Dr.-Ing. Wolfgang Utschick

Contents

Preface

Circuit Theory is a fundamental discipline of electrical engineering and was, besides founding the physical priciples, one of the first research fields of electrical engineering.

Our daily life is hard to imagine without Circuit Theory. Many seemingly natural things are based on it, e.g., computers, mobile phones, energy provision, instrumentation, or control. Even in areas which are not apparently connected with Circuit Theory, like automobiles, Circuit Theory plays a central role.

Firstly, the Kirchhoff's laws are discussed and basic two-terminal circuit elements are introduced. Secondly, circuit elements with more than two terminals are introduced and operational amplifiers are investigated as an example. Afterwards, the nodal analysis, which is a simple method for circuit analysis, is introduced. In a later part of the lecture, dynamic circuits are discussed where depending on the number of reactive elements, first-order and second-order circuits are considered. Finally, the complex phasor analysis is introduced that is an important tool for the analysis of circuits with sinusoidal excitations.

The lecture manuscript at hand should motivate the students and help them to understand the main concepts of Circuit Theory. To that end, principles of linear algebra and calculus are employed.

Michael Joham

Chapter 1

Kirchhoff's Laws

Circuits consist of interconnected circuit elements, each of which exhibits a number of accessible *terminals (nodes)*.

1.1 Reference Directions

Voltages between two terminals (nodes) have an associated *polarity*. The same is true for *currents*, which flow into or out of circuit elements via the terminals. These *voltage-* and *current reference directions* are highlighted by arrows (e.g., see Fig. 1.1).

The voltage- and current reference directions (arrows) define the orientation of the *measuring instruments* (volt and ampere meters, respectively) which are used in a physical circuit to measure voltages and currents. The reference directions together with the sign of the measured quantities determine the actual direction of the flow of electric charges (currents) and, accordingly, the

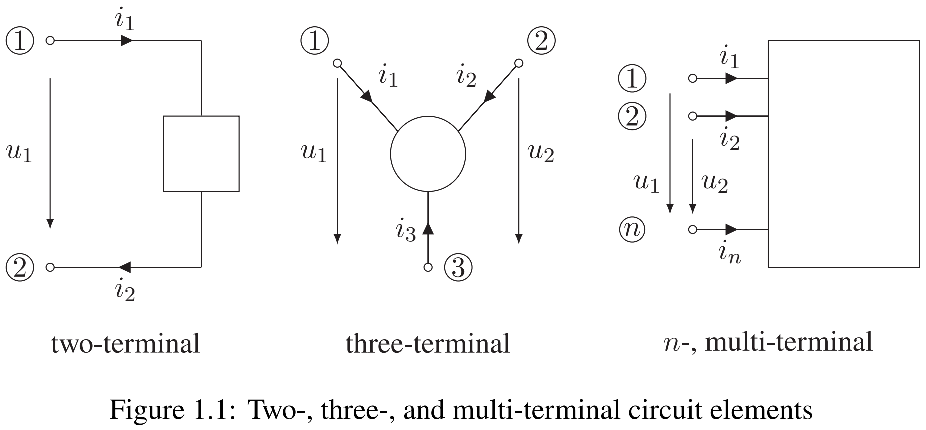


Figure 1.1: Two-, three-, and multi-terminal circuit elements

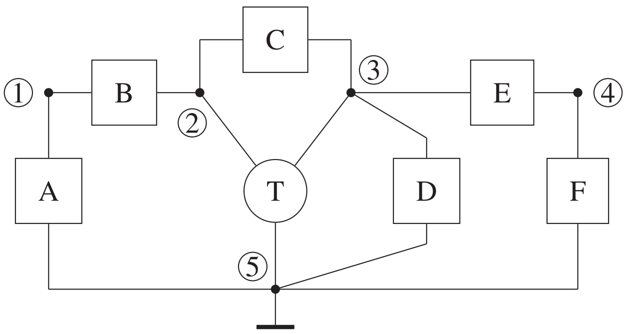


Figure 1.2: Network with two- and three-terminals with five nodes

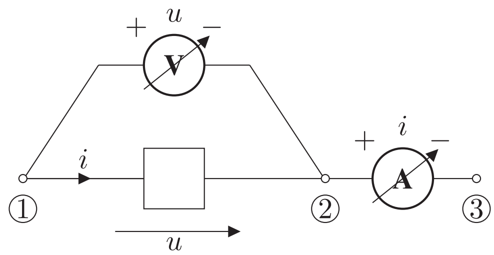


Figure 1.3: Reference direction and measurement

electric potential difference (voltages). If the sign of the measured magnitude is positive, the actual direction and reference (arrow) direction coincide (see Fig. 1.3).

1.2 Ports

Two terminals form a *terminal-pair* or a (one-)port if the two currents at the terminals have the same magnitude but flow in the opposite direction. When the directions of the current and voltage arrows match at a terminal-pair, we say that we have chosen *associated reference directions*.

If the $2n$ terminal currents of a $2n$-terminal are equal and opposite, it is referred to as an n-port and the pair-wise equal currents are the port currents. An outer (snubber) circuit can be chosen such that a three-terminal constitutes a two-port (see Fig. 1.4).

1.3 Kirchhoff's Current Law (KCL)

Kirchhoff's current law (KCL) expresses the law of electric charge conservation in the context of lumped circuits. To this end, we consider a Gaussian (two-sided closed) surface with an "inside" and "outside" that does not cut apart any circuit element (but only connecting wires, see Fig. 1.5). Due to the conservation of charge, the charge inside the Gaussian surface is constant. Hence, the algebraic sum of all currents leaving the closed surface is required to vanish at all

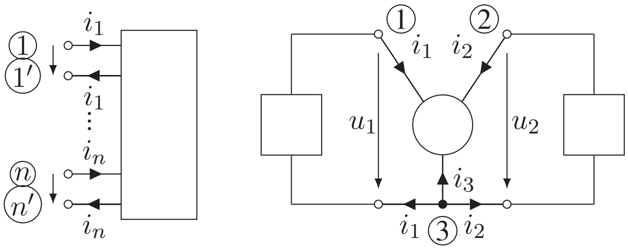


Figure 1.4: Multi-ports and multi-terminals

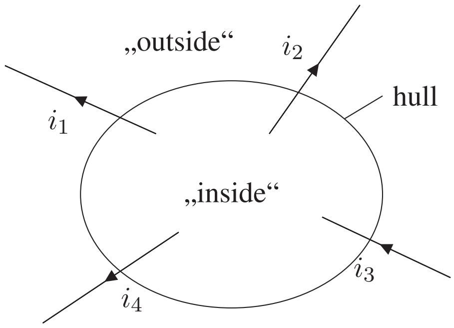


Figure 1.5: Gaussian hull and currents

times. Thereby, outgoing currents are added and ingoing currents are subtracted (added with a negative sign).

Choosing a Gaussian surface which encloses a single node only, yields the elementary node law.

Axiom 1. KCL
For all lumped circuits and at all times, the algebraic sum of the currents leaving any node is equal to zero, i.e.,

$$\sum_{j \in \text{branches connected to node}} i_j(t) = 0. \tag{1.1}$$

If the Gaussian surface includes more than one node (e.g., two nodes of a two-terminal), it is referred to as a *supernode*. As an example, consider the Gaussian surface $\mathcal{H}_1$ in Fig. 1.6, which encompasses the nodes ①, ②, and ③. Note that the KCL remains valid in this case. Moreover, Fig. 1.6 shows that a two-terminal is always a one-port. In fact, the current leaving the closed surface $\mathcal{H}_2$ via node ⑤ is equal and opposite to the current leaving $\mathcal{H}_2$ via node ④. Thus, the two-terminal constitutes a one-port.

1.4 Kirchhoff's Voltage Law (KVL)

Similar to the case of static electric fields in space, a scalar value, the electric potential, can be assigned to each node (point) of the circuit under the lumped-circuit assumption. Up to an

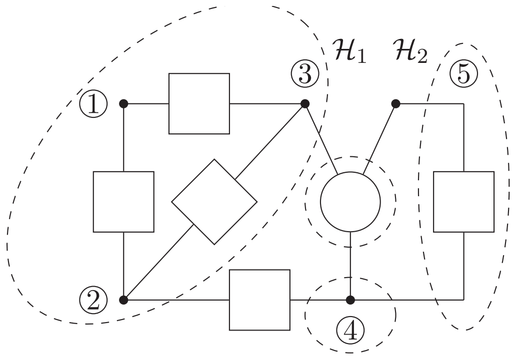


Figure 1.6: Nodes and supernodes

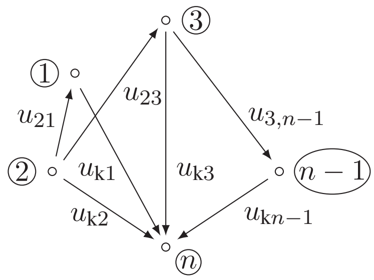


Figure 1.7: Node potentials, node voltages

additive constant, the electric potential is well defined for all nodes. The electric potential of an arbitrary datum node is defined to be zero in order to fix this constant. The node with the highest node number is very often chosen to be the datum (reference) node (e.g., see Fig. 1.7).

Axiom 2. KVL
In a connected circuit with lumped elements, the voltage between a pair of nodes α and β equals the difference of the potentials of these nodes (node-to-node voltage), i.e.,

$$u_{\alpha\beta} = u_{\mathrm{k}\alpha} - u_{\mathrm{k}\beta} \tag{1.2}$$

Using such pairs of nodes α and β a *closed node sequence* can be defined (e.g., the sequence α and β, β and η, η and λ, λ and α which is determined by the node sequence α, β, η, λ, α, see Fig. 1.8). A closed node sequence is called a *loop* if it is possible to move along it by traversing two-terminal elements only.

Hence, we obtain following alternaive form of *Kirchhoff's voltage law.*

Corollary 1. KVL
In a connected circuit of lumped elements, the algebraic sum of all node-to-node voltages around any closed node sequence is equal to zero for all times, i.e.,

$$\sum_{j \in \text{branches of loop}} u_j(t) = 0. \tag{1.3}$$

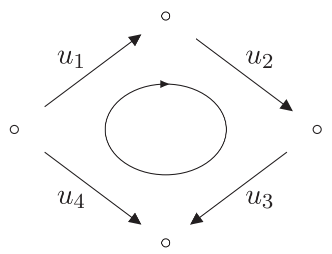


Figure 1.8: Loop and voltages. Voltages are added up in accordance with the reference directions, i.e., $u_1(t) + u_2(t) + u_3(t) - u_4(t) = 0, \forall\, t$

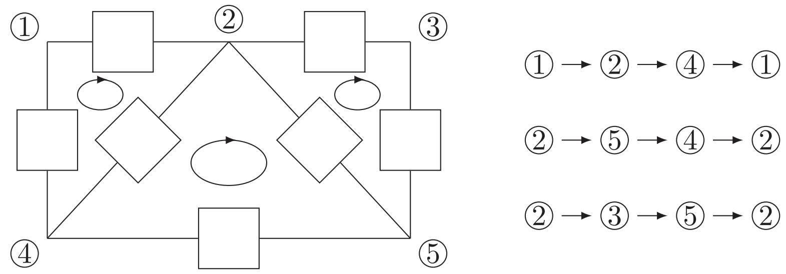


Figure 1.9: Closed node sequence as loop

Kirchhoff's voltage law is a formulation of the law of induction for lumped circuits. The lumped-circuit assumption implies that no time varying magnetic flux occurs outside of the circuit elements.

The KVL holds true for all closed node sequences as well as for loops. All closed node sequences in Fig. 1.9 are loops.

1.5 Number of Linearly Independent Kirchhoff's Equations

Consider a circuit with n nodes and b branches. The number of linearly independent KCL equations is given by

$$n - 1. \tag{1.4}$$

As the number of linearly independent KVL equations is

$$b - (n - 1) \tag{1.5}$$

the overall number of equations delivered by the Kirchhoff's laws is equal to the number of branches b.

The number of unkown quantities in a circuit with b branches is $2b$, one voltage and one current per branch. Therefore, the Kirchhoff's laws give half of the equations necessary to analyze a circuit.

Chapter 2

Resistive One-Ports

A *resistive one-port* or *resistive two-terminal* is a one-port, where the relation $\mathcal{F}(t)$ at time t between the node variables, i.e., the *operating variables* $u(t)$ and $i(t)$, depends only on the instant of time t, but not on the previous history (the past values of the operating variables).[1] Fig. 2.1 shows the element symbol of a resistive one-port $\mathcal{F}(t)$ and the reference directions (arrows) of the voltage and current.

If a current $i(t)$ and a voltage $u(t)$ can occur at the resistive element $\mathcal{F}$ at the same time, the pair $(u(t), i(t))$ is denoted as a (valid) *operating point* of $\mathcal{F}(t)$ at time t. For fixed t, all such pairs constitute a subset $\mathcal{F}(t)$ of the u-i plane $\mathcal{F}_{ui}$, a so-called operating point characteristic, where

$$\mathcal{F}_{ui} = \left\{ (u,i) \,\middle|\, \frac{u}{\mathrm{V}} \in \mathbb{R} \text{ und } \frac{i}{\mathrm{A}} \in \mathbb{R} \right\}. \tag{2.1}$$

$\mathcal{F}_{ui}$ exhibits the structure of a two-dimensional real vector space and it is often considered as a vector space. The operating points of its characteristic are then written as vectors

$$\mathcal{F}_{ui} = \left\{ \begin{bmatrix} u \\ i \end{bmatrix} \,\middle|\, \frac{u}{\mathrm{V}} \in \mathbb{R} \text{ und } \frac{i}{\mathrm{A}} \in \mathbb{R} \right\}. \tag{2.2}$$

The set of operating points $\mathcal{F}(t)$ is called the *characteristic* (of the one-port $\mathcal{F}$) *at time* t:

$$\mathcal{F}(t) = \{(u(t), i(t)) | (u(t), i(t)) \text{ is operating point of } \mathcal{F} \text{ at time } t\}. \tag{2.3}$$

The application of the same symbol $\mathcal{F}$ for the characteristic and the element already indicates that we do not need to differentiate between a circuit element and its characteristic in Circuit Theory.

[1] There exist also so called *dynamic circuit elements* whose behavior is also influenced by port currents and port voltages at previous instants of time, like inductors and capacitors, see Chapter 6.

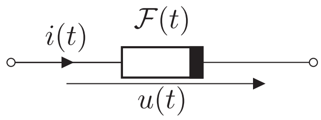


Figure 2.1: Element symbol of a resistive one-port $\mathcal{F}$

If the characteristic is the same for all time instants, the one-port is called *time-invariant* (contrary to *time-varying*). In this case, no reference to the time has to be made, and we simply say: The *characteristic* $\mathcal{F}$ of the one-port $\mathcal{F}$ is the set of all its operating points:

$$\mathcal{F} = \{(u,i) | (u,i) \text{ is operating point of } \mathcal{F}\} . \tag{2.4}$$

In circuit theory, the considered characteristics have to be formally treated as definitions, e.g., $\mathcal{F}$ can also be a discrete set. The direct correspondence to electric devices or ports of larger physical circuits provides the connection to the application in practice. However, an explicit correspondence is neither necessary for a consistent development of the theory, nor is it possible to describe a real circuit element exactly and completely using a mathematical model.

In fact, circuit theory can be developed independently of the properties of the physical electric circuit elements (to a large extent). Hence, circuit theory remains applicable as the technology progresses. The focus may change over time, but once proved, results remain timelessly valid.

2.1 Explicit Representation Form

The purely topological approach for modeling a resistive element as a set of operating points is very general but difficult to incorporate in circuit analysis. Fortunately, most characteristics can be modeled accurately with an algebraic equation. Thus, calculations can be carried out easily.

Exemplarily, the following characteristic of the one-port $\mathcal{G}$ will be considered

$$\frac{i}{i_0} = \arctan \frac{u}{u_0} \tag{2.5}$$

with the constants i_0 and u_0 (see Fig. 2.2).

In an *explicit representation*, one of the operating variables is expressed as a function of the other. The argument of this function is the *controlling* variable. For a given choice of this controlling quantity, the explicit form is unique (if it exists). For the case of a one-port, there are two possible explicit forms.

On the one hand, it may be possible to express the current i as a function of the voltage u, that is,

$$i = g_{\mathcal{F}}(u). \tag{2.6}$$

Then $\mathcal{F}$ is referred to as *voltage-controlled* and $g_{\mathcal{F}}$ is its *voltage-controlled representation*.

On the other hand, it may also be possible to express u as a function of the current i, i.e.,

$$u = r_{\mathcal{F}}(i) \tag{2.7}$$

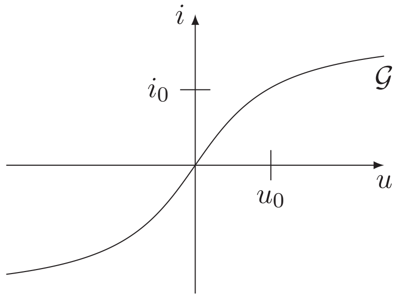


Figure 2.2: Characteristic of the one-port $\mathcal{G}$

and $\mathcal{F}$ is called *current-controlled* and $r_{\mathcal{F}}$ is its *current-controlled representation*.

The voltage-controlled form of $\mathcal{G}$ is given by the function

$$g_{\mathcal{G}}(u) = i_0 \arctan \frac{u}{u_0}$$

and its current-controlled form is given by the function

$$r_{\mathcal{G}}(i) = u_0 \tan \frac{i}{i_0}.$$

Here, the domain of definition of the current i has to be restricted

$$i \in \left(-\frac{\pi}{2} i_0, +\frac{\pi}{2} i_0\right).$$

2.2 Properties of Resistive One-Ports

Many resistive one-ports fulfill additional conditions. In the following, some of these special properties will be presented in a topological (set) notation as well as in a algebraic (formula) notation.

2.2.1 Bilateral Property

A resistive one-port $\mathcal{F}$ is called *bilateral* or *non-polarized* (contrary to *unilateral* and *polarized*), if its characteristic is symmetric with respect to the origin of the u-i plane, i.e.,

$$\forall (u, i) \in \mathcal{F} : (-u, -i) \in \mathcal{F} \tag{2.8}$$

or equivalently,

$$\begin{aligned} g(-u) &= -g(u) \\ r(-i) &= -r(i). \end{aligned} \tag{2.9}$$

This means that both terminals of the resistive one-port can be exchanged without modifying its behavior. For this reason, symmetric element symbols are used to represent bilateral ports.

2.2.2 Power

The *power* $p(t)$ which is *consumed* by the resistor $\mathcal{F}$ and transformed into heat, i.e., $p(t) > 0$, is given by

$$p(t) = u(t)i(t). \tag{2.10}$$

Analogously, *power* is *delivered* from the resistor $\mathcal{F}$ to the remainder of the circuit, if $p(t) < 0$ holds true.

$\mathcal{F}$ is said to be *active*, if there exists at least one operating point for which it delivers power

$$(\exists (u,i) \in \mathcal{F} : ui < 0) \iff \mathcal{F} \text{ is active} \tag{2.11}$$

and *passive*, if this is not the case

$$(\forall (u,i) \in \mathcal{F} : ui \geq 0) \iff \mathcal{F} \text{ is passive.} \tag{2.12}$$

The characteristic of a passive resistive one-port lies completely in the closed first and third quadrants of the u-i plane.

Moreover, $\mathcal{F}$ is *lossless* (non-dissipative), if the power is equal to $0\,\mathrm{W}$ for all operating points, that is,

$$(\forall (u,i) \in \mathcal{F} : ui = 0) \iff \mathcal{F} \text{ is lossless} \tag{2.13}$$

and *lossy* (dissipative), if this is not the case, i.e.,

$$(\exists (u,i) \in \mathcal{F} : ui \neq 0) \iff \mathcal{F} \text{ is lossy.} \tag{2.14}$$

Lossless resistors are passive as well and their characteristic curve lies on the coordinate axes.

2.2.3 Source-Free

A resistive one-port is said to be *source-free* if its operating point characteristic contains the origin, i.e.,

$$(0,0) \in \mathcal{F} \iff \mathcal{F} \text{ is source-free.} \tag{2.15}$$

2.2.4 Duality

Consider a constant R_d with the unit $\frac{\mathrm{V}}{\mathrm{A}}$. Two resistive one-ports $\mathcal{F}$ and $\mathcal{F}^\mathrm{d}$ are said to be *dual to one another* with respect to the *duality constant* R_d, if:

$$\forall \left(u^\mathrm{d}, i^\mathrm{d}\right) = \left(R_\mathrm{d} i, \frac{1}{R_\mathrm{d}} u\right) : \quad (u,i) \in \mathcal{F} \iff \left(u^\mathrm{d}, i^\mathrm{d}\right) \in \mathcal{F}^\mathrm{d}. \tag{2.16}$$

Hence, the characteristic of $\mathcal{F}^\mathrm{d}$ can be obtained from the one of $\mathcal{F}$ (and vice-versa) by exchanging u with i and by correctly applying the duality constant.

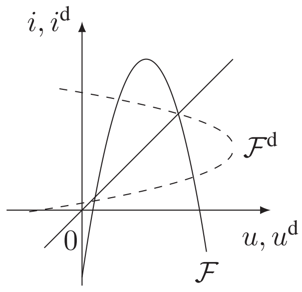


Figure 2.3: Two dual characteristics

If the one-port $\mathcal{F}$ is voltage-controlled, i.e.,

$$i = g_{\mathcal{F}}(u)$$

the replacements $R_{\mathrm{d}} i^{\mathrm{d}}$ for u and $\frac{u^{\mathrm{d}}}{R_{\mathrm{d}}}$ for i lead to

$$\frac{1}{R_{\mathrm{d}}}\, u^{\mathrm{d}} = g_{\mathcal{F}}(R_{\mathrm{d}} i^{\mathrm{d}}).$$

Therefore, the current-controlled representation of the dual one-port $\mathcal{F}^{\mathrm{d}}$ reads as

$$u^{\mathrm{d}} = R_{\mathrm{d}} g_{\mathcal{F}}(R_{\mathrm{d}} i^{\mathrm{d}}) = r^{\mathrm{d}}_{\mathcal{F}}(i^{\mathrm{d}}).$$

In a similar way, the current-controlled representation of $\mathcal{F}$

$$u = r_{\mathcal{F}}(i)$$

leads to the voltage-controlled representation of the dual one-port $\mathcal{F}^{\mathrm{d}}$

$$i^{\mathrm{d}} = \frac{1}{R_{\mathrm{d}}}\, r_{\mathcal{F}}\left(\frac{1}{R_{\mathrm{d}}}\, u^{\mathrm{d}}\right) = g^{\mathrm{d}}_{\mathcal{F}}(u^{\mathrm{d}}).$$

If the u-i characteristic is scaled, such that the straight line $u = R_{\mathrm{d}} i$ becomes an angle bisector of the first quadrant, then $\mathcal{F}$ is the mirror image of $\mathcal{F}^{\mathrm{d}}$ with respect to this line (see Fig 2.3).

Many properties of a one-port $\mathcal{F}$ have an analogous meaning for its dual one-port $\mathcal{F}^{\mathrm{d}}$.

- If $\mathcal{F}$ is voltage-controlled, then $\mathcal{F}^{\mathrm{d}}$ is current-controlled.
- If $\mathcal{F}$ current-controlled, then $\mathcal{F}^{\mathrm{d}}$ is voltage-controlled.
- If $\mathcal{F}$ is bilateral, then $\mathcal{F}^{\mathrm{d}}$ is bilateral as well.
- If $\mathcal{F}$ is passive or active, then the same holds for $\mathcal{F}^{\mathrm{d}}$.
- If $\mathcal{F}$ is lossless or lossy, then the same holds for $\mathcal{F}^{\mathrm{d}}$.

- If $\mathcal{F}$ is time-varying or time-invariant, then the same holds for $\mathcal{F}^{\mathrm{d}}$.

The concept of duality will be applied to whole circuits later on. It allows for significant savings in time, because only one of the dual circuits has to be analyzed. The obtained results can simply be assigned to the respective dual circuit.

2.3 Strictly Linear One-Ports

A resistive one-port $\mathcal{F}$ is said to be *strictly linear* when

$$\forall k \in \mathbb{R}, (u, i) \in \mathcal{F}: \quad (ku, ki) \in \mathcal{F} \tag{2.17}$$

$$\forall (u_1, i_1), (u_2, i_2) \in \mathcal{F}: \quad (u_1 + u_2, i_1 + i_2) \in \mathcal{F}. \tag{2.18}$$

Choosing $k = 0$ and $k = -1$ in (2.17), it becomes obvious that all strictly linear resistors are source-free and bilateral.

A resistive one-port is said to be *strictly linear* if its characteristic is a subspace of the u-i plane $\mathcal{F}_{ui}$.

As Kirchhoff's laws are strictly linear as well (see Chapter 1). the strictly linear circuit elements are very important. There exist one-ports which have an associated vector subspace of dimension zero, one, or two, that are treated in the following subsections.

2.3.1 Nullator

The *nullator* is a strictly linear circuit element whose characteristic is a vector subspace of the u-i plane of dimension zero. Therefore, it consists only of one point, namely the origin

$$\mathcal{F}_0 = \{(0, 0)\}. \tag{2.19}$$

The following two equations provide an equivalent representation

$$\begin{aligned} u &= 0 \\ i &= 0 \end{aligned} \tag{2.20}$$

In addition to the fundamental properties which all strictly linear circuit elements have, the nullator is lossless and self-dual. Fig 2.4 shows its characteristic together with its element symbol.

The nullator serves as a component in many highly idealized models of electric devices. In particular, it gives a very good representation of the input port of an operational amplifier in the linear region (as will be shown in Chapter 4).

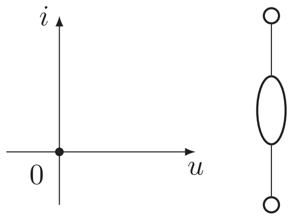


Figure 2.4: Characteristic and element symbol of the nullator

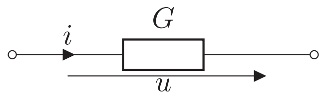


Figure 2.5: Element symbol of a strictly linear resistor

2.3.2 Strictly Linear Resistor

Strictly linear resistors are very important in practice. Their element symbol is depicted in Fig. 2.5. Their one-dimensional characteristic is a straight line passing through the origin, the *straight-line characteristic*.

The voltage-controlled representation of a strictly linear resistor is a simple proportionality:

$$i = Gu \tag{2.21}$$

the so-called *Ohm's law*. The proportionality factor G, which is equal to the slope of this straight line in the u-i plane, is the so called *conductance*. The conductance can be calculated by means of an arbitrary operating point $(u, i) \neq (0, 0)$ of the characteristic

$$G = \frac{i}{u} \tag{2.22}$$

and it is calculated in the unit of *Siemens*, which has the symbol S

$$1\,\mathrm{S} = \frac{1\,\mathrm{A}}{1\,\mathrm{V}}. \tag{2.23}$$

Analogously, the current-controlled representation is of the form

$$u = Ri \tag{2.24}$$

an alternative form of Ohm's law. Note that the resistance R is reciprocal to the conductance, i.e.,

$$R = \frac{1}{G} = \frac{u}{i} \tag{2.25}$$

for $(u, i) \neq (0, 0)$. The resistance's unit is *Ohm* (being reciprocal to Siemens) and its formula symbol is Ω

$$1\,\Omega = \frac{1\,\mathrm{V}}{1\,\mathrm{A}}. \tag{2.26}$$

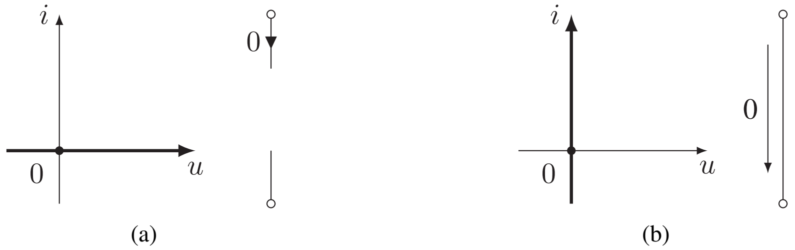


Figure 2.6: Characteristic and element symbols of (a) open circuit and (b) short circuit

In general, the element symbol of strictly linear resistors is labeled as conductance or resistance in circuit diagrams. Either the unit or the formula letter (R or G) indicate which is meant.

The dual element of a strictly linear resistor with conductance G is a strictly linear resistor with resistance

$$R^{\mathrm{d}} = R_{\mathrm{d}}^2 G. \tag{2.27}$$

The dual element of a strictly linear resistor with resistance R is a strictly linear resistor with conductance

$$G^{\mathrm{d}} = \frac{R}{R_{\mathrm{d}}^2}. \tag{2.28}$$

2.3.2.1 Open Circuits and Short Circuits

Two important special cases are the *trivial straight-line characteristics*. Both are lossless.

The *open circuit* in Fig. 2.6a has conductance $0\,\mathrm{S}$ and is defined by

$$i = 0. \tag{2.29}$$

The open circuit is not current-controlled.

The *short circuit* in Fig. 2.6b is dual to the open circuit and, consequently, it does not have a voltage-controlled representation. The short circuit's resistance is $0\,\Omega$ and its explicit form is given by

$$u = 0. \tag{2.30}$$

The convenient reciprocity relation between R and G in (2.25) is still applicable, if one defines the slope ∞ for perpendicular characteristics (only for this special purpose!):

$$0 := \frac{1}{\infty} \quad \text{and} \quad \infty := \frac{1}{0}. \tag{2.31}$$

Accordingly, the resistance $\infty\,\Omega$ is assigned to the open circuit and the conductance $\infty\,\mathrm{S}$ is assigned to the short circuit.

Figure 2.7: Characteristics of an (a) Ohmic and (b) negative resistor

2.3.2.2 Ohmic Resistors

A passive strictly linear resistor is said to be an *Ohmic resistor* (ideal resistor). Its characteristic satisfies

$$\infty \geq G \geq 0 \quad \text{or} \quad \infty \geq R \geq 0. \tag{2.32}$$

The limit cases short circuit and open circuit are even lossless.

The model of Ohmic resistors applies to many electric *loads*. Two-terminals which solely consume electric energy and transform it into heat or into another non-electric form of energy.

2.3.2.3 Negative Resistors

Strictly linear resistors which are not Ohmic, but which are active, are said to be *negative resistors*, because they fulfill

$$-\infty < G < 0 \quad \text{or} \quad -\infty < R < 0. \tag{2.33}$$

A typical characteristic is shown in Fig. 2.7b.

2.3.3 Norator

Finally, the *norator* is the strictly linear resistive two-terminal whose characteristic is two-dimensional and, therefore covers the whole u-i plane

$$\mathcal{F}_\infty = \mathcal{F}_{ui}. \tag{2.34}$$

An equation is not necessary for its description because all operating points are valid.

The norator is active and is dual to itself. Fig. 2.8 depicts its characteristic and its element symbol.

The norator is often used jointly with the nullator in idealized models. In particular, it serves as a model for the output port of an operational amplifier in the linear region.

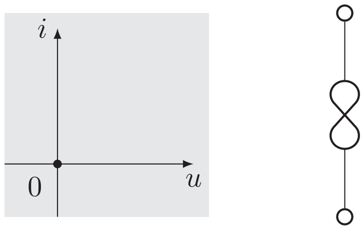


Figure 2.8: Characteristic and element symbol of the norator

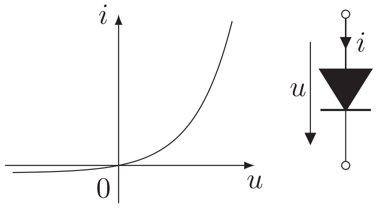


Figure 2.9: Characteristic and element symbol of a pn-junction diode

2.4 pn-Junction Diode

The simplest diode is the *pn(-junction) diode*, which is shown in Fig. 2.9 together with its u-i characteristic.

A considerable current flows only if the voltage u is positive. Then, the diode is *forward biased* and a (large) current flows in the *forward direction*. On the contrary, if the diode is *reversed biased*, only an extremely small *reverse saturation current* I_s flows in the *reverse direction*.

Both the voltage-controlled and current-controlled representations of the pn-diode exist. The characteristic of a pn-diode obeys the equations

$$\begin{aligned} i &= I_\mathrm{s}\left(\mathrm{e}^{u/U_\mathrm{T}} - 1\right) \\ u &= U_\mathrm{T} \ln\left(\frac{i}{I_\mathrm{s}} + 1\right) \end{aligned} \tag{2.35}$$

where I_s is the *reverse saturation current* of the diode and U_T is its *thermal voltage* defined as

$$U_\mathrm{T} = \frac{\mathrm{k_B} T}{\mathrm{q_e}}$$

with the Boltzmann's constant $\mathrm{k_B} = 1.38065 \times 10^{-23}\,\mathrm{J/K}$, the temperature T in Kelvins, and the charge $\mathrm{q_e} = 1.602 \times 10^{-19}$ As of an electron. For example, $U_\mathrm{T} = 25\,\mathrm{mV}$ and $U_\mathrm{T} = 26\,\mathrm{mV}$ for a temperature of 17 °C (290.15 K) and 29 °C (302.15 K), respectively. In the following, we will not take into account the dependence of the characteristic on the temperature. Instead, we use $U_\mathrm{T} = 25\,\mathrm{mV}$ which is a good approximation at room temperature. The reverse saturation current of a typical diode is of the order of pA (pico-Amperes).

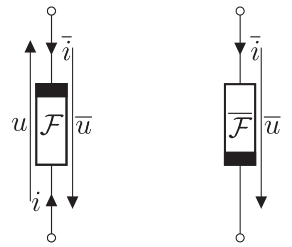


Figure 2.10: A resistive one-port $\mathcal{F}$ with two opposite reference directions and its polarity reversal $\overline{\mathcal{F}}$

2.5 Basic One-Port Circuits

Prior to analyzing a circuit and deriving a complete system of equations, it makes sense to simplify the circuit as much as possible by combining connected two-terminal resistors.
In this section, we introduce the basic circuits, which behave like single one-ports.

2.5.1 Polarity Reversal

The characteristic of a resistive one-port $\mathcal{F}$ is fixed for a given reference direction. The orientation is indicated by the element symbol as, for example, in Fig. 2.1. However, sometimes it is more appropriate to use opposite reference directions. In this case, a new characteristic $\bar{\mathcal{F}}$ is introduced with respect to this reversed reference direction. $\bar{\mathcal{F}}$ is said to be the *polarity reversal* of $\mathcal{F}$.

The left circuit element in Fig. 2.10 shows a one-port $\mathcal{F}$. Now, we take the left one-port as the right one-port in Fig. 2.10 and describe it with the variables $\bar{u}$ and $\bar{i}$. In fact, Kirchhoff's laws provide a simple connection between these two reference directions

$$\begin{aligned} \bar{u} &= -u \\ \bar{i} &= -i \end{aligned}$$

The following substitution, which can be interpreted as a rotation of the u-i plane by $180°$, delivers the characteristic of the reversed one-port

$$(u, i) \in \mathcal{F} \Longleftrightarrow (\bar{u}, \bar{i}) \in .\bar{\mathcal{F}} \tag{2.36}$$

Corresponding substitutions exist for the explicit representation forms, too. For voltage-controlled and current-controlled representations, we obtain

$$\begin{aligned} g_{\bar{\mathcal{F}}}(\bar{u}) &= -g_{\mathcal{F}}(-\bar{u}) \\ r_{\bar{\mathcal{F}}}(\bar{i}) &= -r_{\mathcal{F}}(-\bar{i}) \end{aligned} \tag{2.37}$$

respectively. In fact, this discussion justifies the definitions of Subsection 2.2.1. A one-port is bilateral if it is invariant under polarity reversal.

2.5.2 Parallel-Connection Circuit

The *parallel-connection circuit* of two resistive elements $\mathcal{F}_1$ and $\mathcal{F}_2$ (see Fig. 2.11a) is equivalent to a single resistive one-port $\mathcal{G}$, whose characteristic will now be derived.

The application of Kirchhoff's laws yields

$$\begin{aligned} i &= i_1 + i_2 \\ u_1 &= u_2 = u. \end{aligned}$$

Obviously, the operating points of $\mathcal{F}_1$ and $\mathcal{F}_2$ must exhibit the same voltage u, for instance (u, i_1) and (u, i_2). This gives an operating point of the parallel connection $\mathcal{G}$

$$(u, i) = (u, i_1 + i_2)$$

A simpler formula is possible, if voltage-controlled representations of $\mathcal{F}_1$ and $\mathcal{F}_2$ exist, i.e.,

$$\begin{aligned} i_1 &= g_{\mathcal{F}_1}(u) \\ i_2 &= g_{\mathcal{F}_2}(u). \end{aligned}$$

In this case, $\mathcal{G}$ is voltage-controlled as well, and the total current i equals

$$i = i_1 + i_2 = g_{\mathcal{F}_1}(u) + g_{\mathcal{F}_2}(u) = g_{\mathcal{G}}(u).$$

Consequently, the voltage-controlled representation of the parallel connection is the sum of the voltage-controlled forms of the single resistors and $g_{\mathcal{G}}$ is given by

$$g_{\mathcal{G}} = g_{\mathcal{F}_1} + g_{\mathcal{F}_2}. \tag{2.38}$$

This corresponds to a simple addition of the functions in the u-i plane. In fact, this graphical approach of finding the one-port $\mathcal{G}$ is possible even if not more than one of the resistive one-ports is not voltage-controlled.

2.5.3 Series Connection

Likewise, the *series-connection circuit* of the resistive one-ports $\mathcal{F}_1$ and $\mathcal{F}_2$ in Fig. 2.11b is equivalent to a single resistive one-port $\mathcal{G}$.

At this point, we can skip a complete analysis, as the serial connection is dual to the parallel connection. By exchanging current and voltage, KCL and KVL, voltage-controlled and current controlled representations, etc., and accounting for the duality constant, the findings of the parallel-connection circuit translate into the series-connection circuit. Nevertheless, the results could also be derived analogously to the previous subsection.

If the current-controlled representations of $\mathcal{F}_1$ and $\mathcal{F}_2$ exist, i.e.,

$$\begin{aligned} u_1 &= r_{\mathcal{F}_1}(i) \\ u_2 &= r_{\mathcal{F}_2}(i) \end{aligned}$$

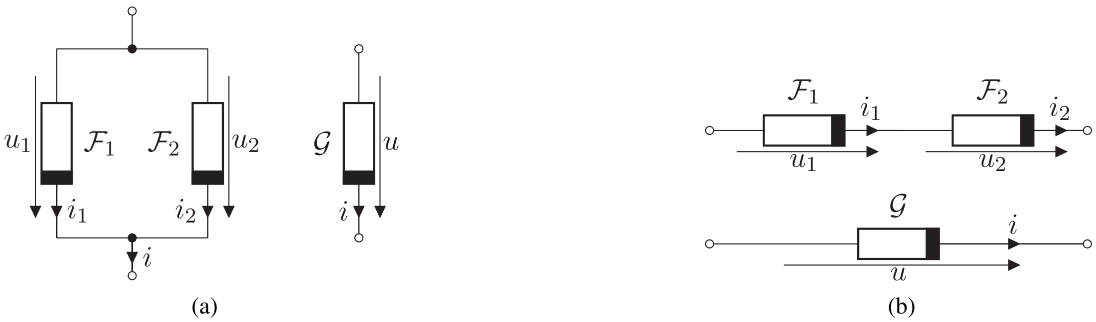


Figure 2.11: The combination of two resistors in (a) parallel or (b) parallel connection

respectively, the total voltage can be expressed as

$$u = u_1 + u_2 = r_{\mathcal{F}_1}(i) + r_{\mathcal{F}_2}(i).$$

Again, the sum of the current-controlled forms of the single resistors gives the current controlled representation of $\mathcal{G}$, i.e.,

$$r_{\mathcal{G}} = r_{\mathcal{F}_1} + r_{\mathcal{F}_2}. \tag{2.39}$$

In summary, the series connection of current-controlled resistive elements corresponds to a simple addition of the functions in the i-u plane.

2.5.4 Connection of Strictly Linear Resistors

The parallel and serial connection of strictly linear resistors is particularly simple and occurs frequently. Thereby, it is common to use a short notation.

Let the *parallel sum (sum of reciprocals)* of two real numbers a and b be defined as the real number $a \| b$:

$$\frac{1}{a \parallel b} = \frac{1}{a} + \frac{1}{b}. \tag{2.40}$$

$$a \parallel b = \frac{ab}{a+b}. \tag{2.41}$$

Among the binary arithmetic operations, the multiplication takes precedence over the parallel sum and the parallel sum has precedence over the addition, i.e.,

$$a \parallel b + c = (a \parallel b) + c \tag{2.42}$$

$$a \parallel bc = a \parallel (bc). \tag{2.43}$$

It is easy to verify that the parallel sum is commutative and associative, that is

$$a \parallel b = b \parallel a \tag{2.44}$$

$$a \parallel (b \parallel c) = (a \parallel b) \parallel c. \tag{2.45}$$

Therefore, the definition can be expanded to more than two operands, for instance,

$$a \parallel b \parallel c = a \parallel (b \parallel c) = \frac{abc}{ab + bc + ca}. \tag{2.46}$$

2.5.4.1 Parallel Connection of Strictly Linear Resistors

Due to

$$\begin{aligned} i_1 &= G_1 u \\ i_2 &= G_2 u \end{aligned} \tag{2.47}$$

the parallel connection of strictly linear resistors with conductances G_1 and G_2 is given by

$$i = i_1 + i_2 = (G_1 + G_2)u = Gu$$

leading to the result that the characteristic of this parallel-connection circuit is again a strictly linear resistor with conductance

$$G = G_1 + G_2. \tag{2.48}$$

In terms of the respective resistances we have

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Hence, the total resistance of the parallel-connection circuit is the parallel sum of the partial resistances, i.e.,

$$R = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2}. \tag{2.49}$$

Due to the proportionality of all currents with the voltage u [see Eqn. (2.47)], we find the current divider rule since

$$\begin{aligned} \frac{i_1}{i_2} &= \frac{G_1 u}{G_2 u} = \frac{G_1}{G_2} = \frac{R_2}{R_1} \\ \frac{i_1}{i} &= \frac{G_1}{G_1 + G_2} = \frac{R_1 \parallel R_2}{R_1} = \frac{R_2}{R_1 + R_2} \\ \frac{i}{i_2} &= \frac{G_1 + G_2}{G_2} = \frac{R_2}{R_1 \parallel R_2} = \frac{R_1 + R_2}{R_1}. \end{aligned}$$

The ratio of two currents in a parallel connection of two strictly linear resistors is equal to the ratio of the corresponding conductances.

2.5.4.2 Serial Connection of Strictly Linear Resistors

Analogously, the series-connection circuit of two strictly linear resistors with resistances R_1 and R_2 forms a strictly linear resistor with resistance

$$R = R_1 + R_2. \tag{2.50}$$

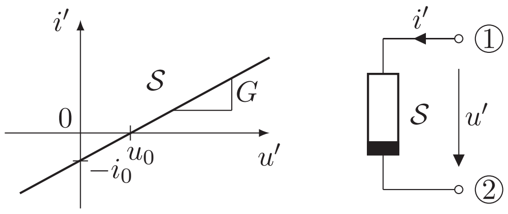


Figure 2.12: Characteristic of the linear source $\mathcal{S}$

The total conductance of the series-connection circuit is the parallel sum of the partial conductances

$$G = G_1 \parallel G_2 = \frac{G_1 G_2}{G_1 + G_2}. \tag{2.51}$$

Similar to the current divider rule for the parallel connection, the voltage divider rule for the series connection of strictly linear resistors can be derived. Since $u_1 = R_1 i$, $u_2 = R_2 i$, and $u = Ri$, we obtain

$$\begin{aligned}
\frac{u_2}{u_1} &= \frac{R_2 i}{R_1 i} = \frac{R_2}{R_1} = \frac{G_1}{G_2} \\
\frac{u_1}{u} &= \frac{R_1}{R_1 + R_2} = \frac{G_1 \parallel G_2}{G_1} = \frac{G_2}{G_1 + G_2} \\
\frac{u}{u_2} &= \frac{R_1 + R_2}{R_2} = \frac{G_2}{G_1 \parallel G_2} = \frac{G_1 + G_2}{G_1}.
\end{aligned}$$

The ratio of two voltages of the series connection of strictly linear resistors is equal to the ratio of the corresponding resistances.

2.6 Linear Sources

Section 2.3 dealt with strictly linear characteristics which take the form of a vector subspace of the u-i plane. *Linear characteristics* have a similar meaning in circuit theory. A resistive one-port is said to be *linear* if its characteristic is an affine subspace of the u-i plane.

A linear characteristic becomes strictly linear when it is shifted to the origin. This is the reason why strictly linear elements are also referred to as *linear* and *source-free*. Circuit elements which are not linear, for instance diodes, are said to be *non-linear*.

At this point, we will only discuss characteristics of one dimension, i.e., affine (straight) lines in the u-i plane. Note that any affine line that does not pass through the origin is active. Therefore, these linear two-terminals are frequently called, and used as, sources.

In this section, the port variables of sources will be denoted as primed variables u' and i'. This is preparation for Section 2.8.

A *linear source* is a resistive two-terminal element, whose u-i characteristic, a straight line, does not pass through the origin. An example is provided in Fig. 2.12.

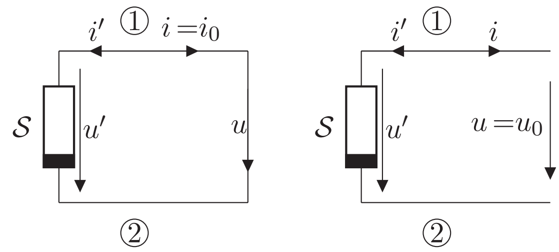


Figure 2.13: Connection of a linear source to a short circuit and an open circuit

Considering the characteristic of a linear source shown in Fig. 2.12 leads to following voltage-controlled representation of a linear source

$$i' = Gu' - i_0 \tag{2.52}$$

with the slope G and the i'-intercept $-i_0$. The u'-intercept is denoted as u_0. With the connections depicted in Fig. 2.13, these intercepts can be found.

In the case of the connection to a short circuit (see Fig. 2.13 on the left), $u = 0$ is enforced by the short circuit. From the KVL, we infer

$$u' = u = 0.$$

Therefore, we get from the characteristic in Fig. 2.12 ablesen that

$$i' = -i_0$$

and the KCL for node ① delivers for the current through the short circuit

$$i = -i' = i_0. \tag{2.53}$$

The negative axis intercept i_0 on the current-axis (see Fig. 2.12) is thus called the *short circuit current* of $\mathcal{S}$.

Analogously, connecting to an open circuit (see Fig. 2.13 on the right) the voltage u between the terminals of $\mathcal{S}$ equals to

$$u = u' = u_0. \tag{2.54}$$

The u'-intercept u_0 is called the *open circuit voltage* of $\mathcal{S}$.

Furthermore, the *internal conductance* G (the slope of the characteristic in the u-i-plane) and *internal resistance* R (which are the reciprocal to each other) are defined as

$$G = \frac{i_0}{u_0} \quad \text{and} \quad R = \frac{u_0}{i_0}.$$

The characteristic in Fig. 2.12 can be parametrized by u_0 and i_0 or u_0 and R or i_0 and G.

Figure 2.14: Characteristic and element symbol of a (a) current source and (b) voltage source

2.6.1 Independent Sources

Independent sources are linear sources whose characteristics are parallel to one of the axes of the u'-i' plane. Irrespective of all other parts of the circuit, an independent source enforces a defined current flow or rather a voltage between its terminals. In other words, the independent source *pre-determines* a current or a voltage of the circuit.

2.6.1.1 Current Sources

The current flowing through a *current source* $i' = i_S$ is independent of the voltage between its nodes. Current sources are voltage-controlled, active, unilateral, and linear. Fig. 2.14a shows the characteristic and element symbol of a constant current source.

The open circuit can also be interpreted as a *zero current source*, because no current flows through it at any time. The internal conductance and resistance of a current source is $0\,\mathrm{S}$ and $\infty\,\Omega$, respectively.

2.6.1.2 Voltage Sources

The voltage across the nodes of a *voltage source* $u' = u_S$ is independent of the current flowing through it. Voltage sources are dual to current sources with respect to an appropriately chosen duality constant (for instance, $R_\mathrm{d} = \frac{u_S}{i_S}$). Hence, they are current-controlled, active, unilateral, and linear. Fig. 2.14b shows the characteristic and element symbol of a constant voltage source.

The short circuit can be interpreted as a *zero voltage source*. The internal resistance and conductance of a voltage source are $0\,\Omega$ and $\infty\,\mathrm{S}$, respectively.

2.6.2 Internal Structure of Linear Sources

Linear sources, whose u'-i' characteristics are not parallel to one of the axes, can be replaced by one of the two equivalent circuit diagrams in Fig. 2.15. Thereby, the given magnitudes correspond to the open circuit voltage, the short circuit current, the internal resistance R, and the internal conductance G. The equivalence is easily demonstrated.

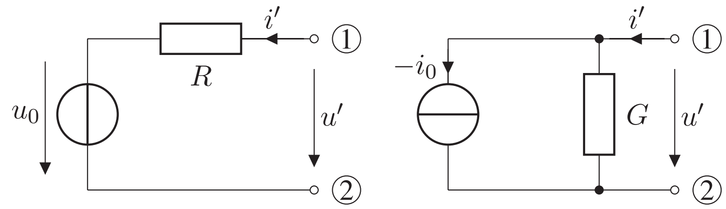


Figure 2.15: Internal structure of linear sources and source transformation

Applying KVL to the left circuit yields

$$u' - Ri' - u_0 = 0 \tag{2.55}$$

which is the current-controlled form of (2.52). Multiplying (2.55) with $-G$ and considering (2.6) gives

$$-Gu' + i' + i_0 = 0 \tag{2.56}$$

which is equivalent to (2.52). On the one hand, it is equivalent to equation (2.55), and on the other hand, it results from applying KCL to node ② of the right circuit of Fig. 2.15.

In practice, one chooses the more appropriate form of these two equivalent internal structures for the given circuit.

Another application consists in source transformation. If the *voltage source with internal resistance* on the left of Fig. 2.15 on the left is part of a larger circuit, it can be replaced by the *current source with internal conductance* on the right of Fig. 2.15 on the right. In order to preserve the properties of the circuit, the new parameters are chosen according to

$$i_{\mathcal{S}} = \frac{u_{\mathcal{S}}}{R} \quad \text{and} \quad G = \frac{1}{R}. \tag{2.57}$$

The reverse transformation is given by

$$u_{\mathcal{S}} = \frac{i_{\mathcal{S}}}{G} \quad \text{and} \quad R = \frac{1}{G}. \tag{2.58}$$

This so-called *source transformation*, which allows us to replace a circuit with an independent source by an equivalent source circuit, is useful for the simplification of circuits.

2.7 Piecewise Linear Resistors

Certain computations can be carried out efficiently by approximating a characteristic by *piecewise linear* characteristics, which are composed of linear straight line *segments*.

Each segment is linear and thus specified by a linear source characteristic. The restriction of its domain is mostly expressed by a *conditional inequality*.

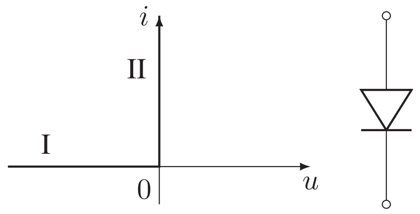


Figure 2.16: Characteristic and element symbol of the ideal diode

2.7.1 Piecewise Linear Diodes

Simple piecewise linear diode models are only made up of two line segments. The triangles in the symbols of these idealized diodes are not filled. As a generalization, the dual circuit elements will also be discussed in the following.

2.7.1.1 Ideal Diodes

The *ideal diode* is an idealized model for the forward and reverse direction of a pn-diode. Its element symbol and characteristic, which only consists of the negative part of the u-axis and the positive part of the i-axis, is shown in Fig. 2.16. Ideal diodes are lossless. Their dual element is simply an ideal diode with reversed polarity.

The representation of an ideal diode is composed of two line segments, i.e.,

$$i = 0 \quad \text{if } \; u \leq 0 \quad \text{(line segment I)} \tag{2.59}$$

$$u = 0 \quad \text{if } \; i \geq 0 \quad \text{(line segment II).} \tag{2.60}$$

Both segments contain the origin $(0, 0)$.

2.7.1.2 Concave Resistors

A *concave resistor* is defined by the pair of parameters (G, U). Fig. 2.17a shows its element symbol and characteristic, which only consists of the line segment $(-\infty, U)$ on the u-axis, the corner point $(U, 0)$ and the line segment with positive slope G in the u-i plane.

Fig. 2.17a depicts the characteristic and element symbol of a typical concave resistor. Concave resistors of the form $(G, 0\,\mathrm{V})$ or (G, U_{bp}) with $G > 0$ and *breakpoint voltage* $U_{\mathrm{bp}} > 0$ constitute a simple and good pn-diode model. For instance, $(\infty\,\mathrm{S}, 0.7\,\mathrm{V})$ usually is a very good approximation of a pn-diode for many practical applications. The ideal diode is the extreme case of a concave resistor with $(\infty\,\mathrm{S}, 0\,\mathrm{V})$.

The characteristic of a concave resistor can be realized by the series connection of an ideal diode and a linear source, as shown in Fig. 2.17b.

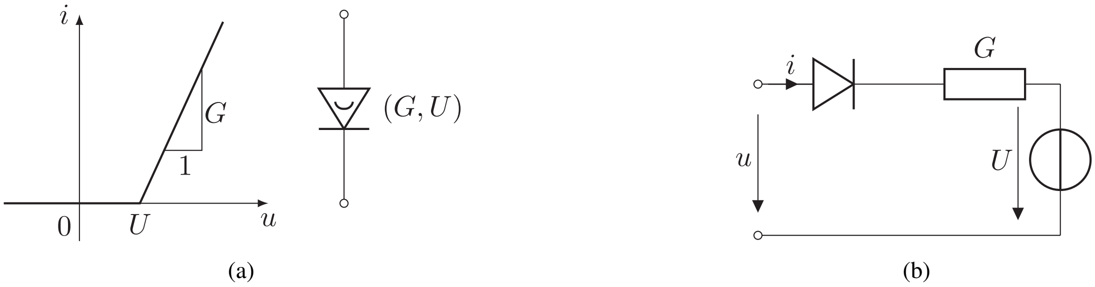


Figure 2.17: Characteristic and element symbol (a) of a concave resistor, internal structure of a concave resistor (b)

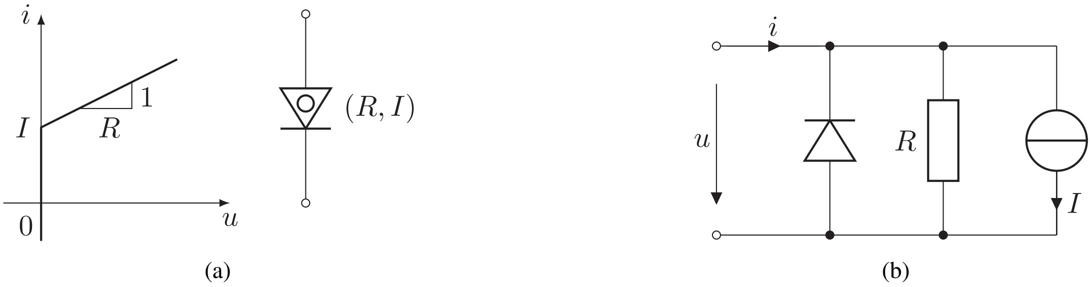


Figure 2.18: Characteristic and element symbol (a) of a convex resistor, internal structure of a convex resistor (b)

2.7.1.3 Convex Resistors

A *convex resistor* is defined by the pair of parameters (R, I). Its characteristic consists only of the line segment $(-\infty\,\Omega, I)$ on the i-axis, the corner point $(0, I)$ and the line segment with positive slope $1/R$. Consequently, it is dual to an appropriately chosen concave resistor

$$(R, I) \quad \text{is dual to} \quad (G, U) = \left(\frac{R}{R_\mathrm{d}^2}, R_\mathrm{d} I\right). \tag{2.61}$$

Fig. 2.18a depicts the characteristic and element symbol of a convex resistor and Fig. 2.18b illustrates a possible internal structure for it.

2.7.2 Real Negative Resistors

As discussed in Subsubsection 2.3.2.3, negative resistors are not realizable in practice. However, in Chapter 4, we will demonstrate how to generate the characteristics in Fig. 2.19 which exhibit a negative resistance in the middle region (line segment).

These two circuits are referred to as *real negative resistors*. The left voltage-controlled circuit has an *N-characteristic* while the right current-controlled one has an *S-characteristic*.

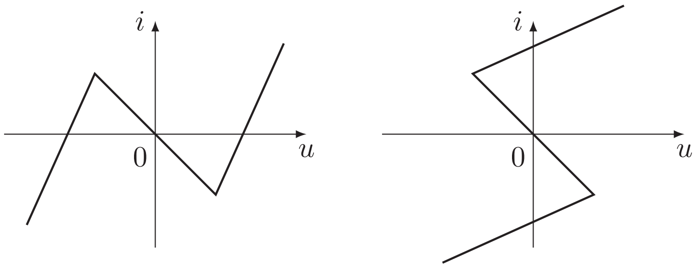


Figure 2.19: N- and S-characteristic of real negative resistors

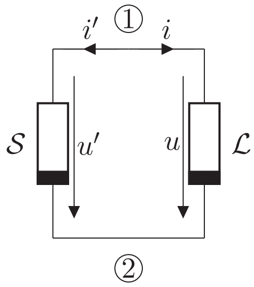


Figure 2.20: Circuit with linear source and resistive load

2.8 One-Port Circuits

Every real circuit fulfills a certain *task* in some *application*. In *information technology* (or *communications engineering*), the task is to transmit and/or process information signals. These information signals are part of the electrical signals, i.e., they are operating variables of the circuit and constitute the “interesting” alternating component.

The direct current (DC) component is important to guarantee the desired operational behavior of the circuit, i.e., for adjusting the operating point. However, it does not play a crucial role regarding the given task and application. The DC component can vanish too. Then, the operating point of the circuit lies in the origin of the u-i plane.

This separate consideration of time-dependent and constant signal components will now be discussed on the basis of the simplest useful connected circuit. The connection of a source one-port and a load one-port.

2.8.1 Operating Point

The direct current component of the operating variables is said to be the *operating point* of the circuit. Thereby, the implicit assumption is that both the source $\mathcal{S}$ and the load $\mathcal{L}$ in Fig. 2.20 are time-varying.

The source and load characteristics are given in the u'-i' plane and u-i plane, respectively.

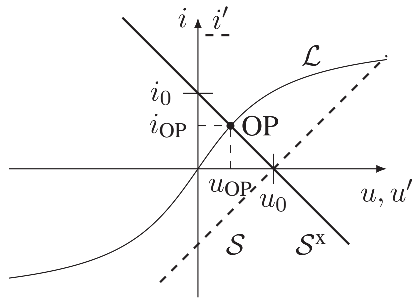


Figure 2.21: Graphical method of determining the operating points of the circuit in Fig. 2.20

In this circuit, a valid solution must satisfy the following conditions

$$(u', i') \in \mathcal{S} \quad \text{und} \quad (u, i) \in \mathcal{L}$$

and the Kirchhoff's laws

$$i = -i' \quad \text{and} \quad u = u'.$$

Apparently, it is advantageous to eliminate the variables u' and i', whose only purpose is to describe the source as a two-terminal (or one-port) with the usual associated reference directions. Instead, let us describe the source $\mathcal{S}$ by means of the operating variables u and i of the load $\mathcal{L}$.

This representation is referred to as the *external characteristic* $\mathcal{S}^{\mathrm{x}}$ of the source $\mathcal{S}$ and reads as

$$\mathcal{S}^{\mathrm{x}} = \big\{(u, i) | (u', i') = (u, -i) \in \mathcal{S}\big\} . \tag{2.62}$$

Graphically speaking, it is just a reflection of $\mathcal{S}$ w.r.t. the u-axis.

Hence, a solution to the circuit must fulfill the conditions

$$(u, i) \in \mathcal{S}^{\mathrm{x}} \quad \text{und} \quad (u, i) \in \mathcal{L}. \tag{2.63}$$

Because the external source characteristic is a restriction to the load, and because it defines a set of possible operating points for the circuit, it is also said to be the *load line*.

The solution set, the so-called *set of operating points*, is the intersection of $\mathcal{S}^{\mathrm{x}}$ and $\mathcal{L}$:

$$\mathcal{OP} = \mathcal{S}^{\mathrm{x}} \cap \mathcal{L}. \tag{2.64}$$

The easiest way is to draw both characteristics in the u-i plane and to determine their intersection graphically. Fig. 2.21 also shows the coordinate directions of u' and i'.

In this example, the intersection of $\mathcal{S}^{\mathrm{x}}$ and $\mathcal{L}$ yields a unique *operating point (OP)* for the circuit, the voltage and current pair $u_{\mathrm{OP}}, i_{\mathrm{OP}}$.

In fact, the set of operating points can be empty as well. This is the case if no pair of operating variables exists which is part of both, the source and load characteristic. As an example, consider the circuit in Fig. 2.22 that consists of an ideal diode and an independent voltage source. Obviously, the respective characteristics do not intersect.

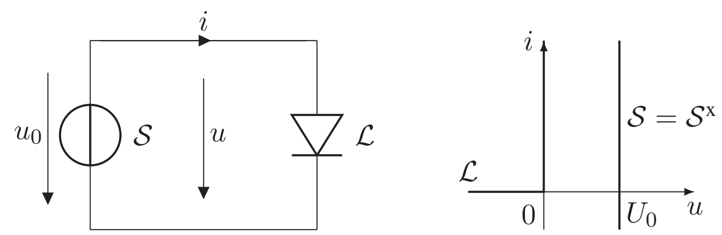


Figure 2.22: An over-idealized and thus contradictory circuit

This contradiction is caused by over-idealized modeling. However, if we take the internal resistance of the source into account, or if we replace the ideal diode by an pn-diode, the contradiction vanishes.

2.8.2 Large Signal Analysis

Contrary to the small signal analysis (see Section 2.8.4), the large signal analysis takes all currents and voltages as a whole, e.g., $i(t)$ and $u(t)$. Therefore, the nonlinearities of the different elements of the circuit are taken into account, e.g., $r_{\mathcal{F}}(i_{\mathcal{F}})$ or $g_{\mathcal{F}}(u_{\mathcal{F}})$, or the linearization thereof [see (2.66)].

In general, at least one operating variable is assumed to be predetermined. With regard to the circuit in Fig. 2.20 and the respective characteristics in Fig. 2.21, the open circuit voltage u_0 of the two-terminal source $\mathcal{S}$ is fixed.
The other operating variables u_{OP} and i_{OP} result from this preset for example. The concept to divide operating variables into a predetermined *input variable (excitation)*, or *cause*, and one or more dependent *output variables (responses)*, or *effects*, is very general. It is particularly useful for the description of a time-dependent signals. Thereby, the time-dependent signal is split into a constant and a time-varying component.

The large signal response to a time-varying excitation a can be written as

$$b(t) = b(a(t)) \tag{2.65}$$

with the mapping $b(a)$ that is a characteristic in the a-b-plane. This transfer function can be determined with the help of the elements and their connections in the given circuit.

2.8.3 Linearization

If the transfer characteristic $b(a)$ [see Eqn. (2.65)] is continuous and differentiable at the operating point, it can be *linearized*. Any continuous and differentiable function can be written as

$$b(a) = b(A) + k(a - A) + r(a, A)(a - A)$$

where $r(a, A)(a - A)$ denotes the residual error term of the linear approximation $b(A) + k(a - A)$. For the best linear approximation, the error term $r(a, A)(a - A)$ must converge to zero faster

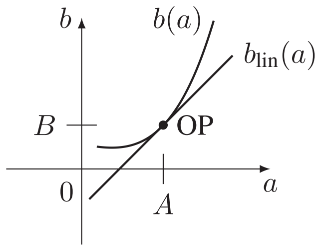


Figure 2.23: Linearization of the characteristic in the operating point

than $k(a-A)$. In other words, $r(a,A)$ must converge to zero when a converges to A. The slope k can be expressed as

$$k = \frac{b(a)-b(A)}{a-A} - r(a,A).$$

Note that the slope k is constant. Applying the limit $a \to A$ on both sides leads to

$$k = \lim_{a\to A} \frac{b(a)-b(A)}{a-A} - \lim_{a\to A} r(a,A).$$

For $\lim_{a\to A} r(a,A) = 0$, we must have that

$$k = \lim_{a\to A} \frac{b(a)-b(A)}{a-A} = \left.\frac{\mathrm{d}b(a)}{\mathrm{d}a}\right|_{a=A}.$$

In other words, the transfer characteristic $b(a)$ is replaced by the linear function

$$b_{\text{lin}}(a) = b(A) + \left.\frac{\mathrm{d}b_{\text{lin}}}{\mathrm{d}a}\right|_{a=A} (a-A) \tag{2.66}$$

which has the same function value and slope at the operating point:

$$b_{\text{lin}}(A) = b(A) \quad \text{und} \quad \left.\frac{\mathrm{d}b_{\text{lin}}(a)}{\mathrm{d}a}\right|_{a=A} = \left.\frac{\mathrm{d}b(a)}{\mathrm{d}a}\right|_{a=A}. \tag{2.67}$$

As demonstrated above, $b_{\text{lin}}(a)$ is the best linear approximation for $b(a)$ arround the operating point (see Fig. 2.23) with the transmission factor or gain $k = \left.\frac{\mathrm{d}b(a)}{\mathrm{d}a}\right|_{a=A}$. In other words, $b_{\text{lin}}(a)$ is the tangent of $b(a)$ at the operating point A.

In the following, the representation of a non-linear one-port $\mathcal{F}$ with the port quantities u and i is linearized. We assume that the one-port $\mathcal{F}$ is voltage- and also current-controlled. The current- and the voltage-controlled representations of $\mathcal{F}$ are

$$\begin{aligned} u &= r_{\mathcal{F}}(i) \\ i &= g_{\mathcal{F}}(u) \end{aligned} \tag{2.68}$$

respectively. Consider the trivial equation $i = i$. Therefore, $i = g_{\mathcal{F}}(r_{\mathcal{F}}(i))$ and $g_{\mathcal{F}}$ is the inverse function of $r_{\mathcal{F}}$ and vice versa. In particular, $U = r_{\mathcal{F}}(I)$ and $I = g_{\mathcal{F}}(U)$ in the operating point.

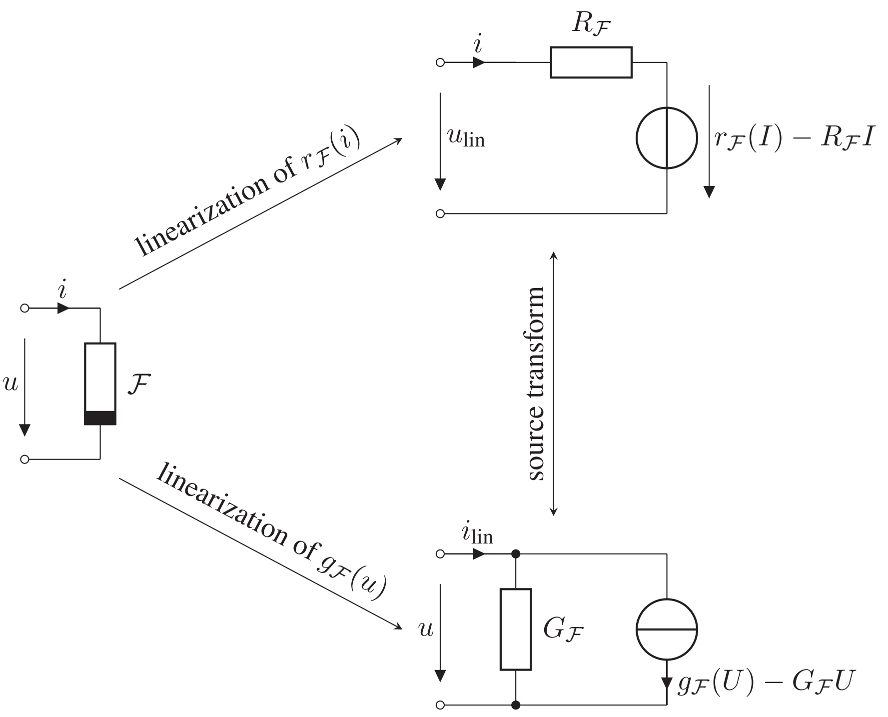


Figure 2.24: Linearization of the non-linear one-port $\mathcal{F}$

The linearization of the function $r_\mathcal{F}$ [cf. Eqn. (2.66)] in the operating point I leads to

$$\begin{aligned} u_{\text{lin}} &= r_{\mathcal{F},\text{lin}}(i) = r_\mathcal{F}(I) + \left.\frac{\mathrm{d}r_\mathcal{F}(i)}{\mathrm{d}i}\right|_{i=I}(i-I) \\ &= r_\mathcal{F}(I) - R_\mathcal{F}I + R_\mathcal{F}i \end{aligned} \tag{2.69}$$

with the small signal resistance $R_\mathcal{F} = \left.\frac{\mathrm{d}r_\mathcal{F}(i)}{\mathrm{d}i}\right|_{i=I}$. We observe that the linearization of $r_\mathcal{F}(i)$ is the sum of voltages and is represents thus a series connection. The first two terms are constant and stand for an independent voltage source. The third term is Ohm's law and represents a strictly linear resistor. The resulting equivalent circuit diagram is the linearization of $r_\mathcal{F}(i)$ depicted in Fig. 2.24.

Linearizing $g_\mathcal{F}$ [see Eqn. (2.66)] in the operating point $U = r_\mathcal{F}(I)$ yields

$$\begin{aligned} i_{\text{lin}} &= g_{\mathcal{F},\text{lin}}(u) = g_\mathcal{F}(U) + \left.\frac{\mathrm{d}g_\mathcal{F}(u)}{\mathrm{d}u}\right|_{u=U}(u-U) \\ &= g_\mathcal{F}(U) - G_\mathcal{F}U + G_\mathcal{F}u \end{aligned} \tag{2.70}$$

with the small signal conductance $G_\mathcal{F} = \left.\frac{\mathrm{d}g_\mathcal{F}(u)}{\mathrm{d}u}\right|_{u=U}$. Since the functions $g_\mathcal{F}$ and $r_\mathcal{F}$ are the inverse functions of each other, it must hold that

$$G_\mathcal{F} = \frac{1}{R_\mathcal{F}}.$$

The expression in Eqn. (2.70) is the sum of currents. Thus, it represents a parallel connection. The first two terms are constant. They can therefore be identified as independent current source. The last term is Ohm's law and represents a strictly linear resistor. The equivalent circuit for the linearization of $g_{\mathcal{F}}(u)$ can be found in Fig. 2.24.

Rearranging the linearization of $g_{\mathcal{F}}(u)$ in Eqn. (2.70) to get an expression for u gives

$$u = U - R_{\mathcal{F}} g_{\mathcal{F}}(U) + R_{\mathcal{F}} i_{\text{lin}}$$

where the equivalence $G_{\mathcal{F}} = \frac{1}{R_{\mathcal{F}}}$ has been employed. Taking additionally into account that $g_{\mathcal{F}}(U) = I$ and $U = r_{\mathcal{F}}(I)$, we obtain the expression in Fig. (2.69). Therefore, Eqn. (2.69) and Eqn. (2.70) represent the same straight line and the two corresponding equivalent circuits for $r_{\mathcal{F}}(i)$ and $g_{\mathcal{F}}(u)$ can be used and also source transform can be used.

2.8.4 Small Signal Analysis

In the following, we consider time-varying signals. Therefore, the excitation a and the response b are split into a constant and a time-varying part. Accordingly, the excitation a can be written as

$$a(t) = A + \Delta a(t) \tag{2.71}$$

with the constant *operating point* or *direct current component* A and the time-varying alternating current component $\Delta a(t)$ which is the *information carrying signal* or short *signal*. The response b corresponding to the excitation a [see Eqn. (2.65)] can be split in a similar way:

$$b(t) = B + \Delta b(t) \tag{2.72}$$

with the operating point B and the alternating current component $\Delta b(t)$.

Substituting the decomposition of the excitation of Eqn. (2.71) into the linear approximation [see Eqn. (2.66)] leads to

$$\begin{aligned} b_{\text{lin}}(a(t)) &= b_{\text{lin}}(A + \Delta a(t)) \\ &= b(A) + k(A + \Delta a(t) - A) = b(A) + k\Delta a(t). \end{aligned} \tag{2.73}$$

The linearized response of the circuit $b_{\text{lin}}(t)$ can thus be decomposed into an operating and a small signal component as in Eqn. (2.72), that is,

$$b_{\text{lin}}(a(t)) = B + \Delta b(t). \tag{2.74}$$

The time-invariant operating point component obeys the non-linear relationship

$$B = b(A). \tag{2.75}$$

However, the transform of the *small signals* is linear, i.e.,

$$\Delta b(t) = k\Delta a(t). \tag{2.76}$$

Consequently, a resistive circuit scales small signals with a constant factor and does not change the waveform. The name small signal comes from the restrictive assumption that these information carrying signal components are small enough such that the approximative linear representation can be used.

The small signal approximation is now found for the non-linear one-port $\mathcal{F}$ assuming the two explicit representations exist [see Eqn. (2.68)]. When linearizing the voltage-controlled representation $g_{\mathcal{F}}(u)$ [see Eqn. (2.70)], this linearization can be written as

$$u_{\text{lin}} = U + R_{\mathcal{F}}(i - I)$$

with $I = g_{\mathcal{F}}(U)$. The operation point of $\mathcal{F}$ is defined by U and I. Reformulating these expression leads to

$$u - U = R_{\mathcal{F}}(i - I).$$

With the decomposition into operating point and small signal component, i.e., $u(t) = U + \Delta u(t)$ and $i(t) = I + \Delta i(t)$, we can obtain the following relationship of the small signal components

$$\Delta u(t) = R_{\mathcal{F}} \Delta i(t).$$

In other words, the equivalent element for a non-linear one-port, when considering small signal analysis, is a strictly linear resistor with the small signal resistance $R_{\mathcal{F}} = \frac{\mathrm{d}r_{\mathcal{F}}(i)}{\mathrm{d}i}|_{i=I}$ or the small signal concuctance $G_{\mathcal{F}} = \frac{\mathrm{d}g_{\mathcal{F}}(u)}{\mathrm{d}u}|_{u=U}$. Addionally, it holds that $G_{\mathcal{F}} = \frac{1}{R_{\mathcal{F}}}$.

Consequently, we can draw the conclusion that for the small signal analysis, alls non-linear one-ports are replaced by their small signal resistances or conductances and constant sources are replaced by zero-sources (short circuits for constant voltage sources and open circuits for constant current sources). The resulting small signal equivalent circuit consists only of linear elements and is therefore easy to analyze.

Chapter 3

Resistive Two-Ports

When modeling electrical components by circuit elements, it is not always sufficient to have only two terminals, or a one-port. The obvious generalization lies in the transition from the one-port to the *two-port*. Fig. 3.1 shows a two-port with an external circuit that ensures the compliance with the port requirements.

In case this is not enforced by the special form of the external circuit, e.g., by connecting to two one-ports, it has to be clear through *a priori* knowledge about the internal structure of the two-port (e.g., using a transformer, see Subsection 3.5.3). Each of the terminal-pairs ①, ①′ and ②, ②′ constitute a port, thus the corresponding terminal currents in each case are equal and opposed. If this is not guaranteed, then the circuit element with four terminals depicted in Fig. 3.1 is a *four-terminal* and *not* a two-port. Only a two-port can be completely described with the four parameters u_1, u_2, i_1, i_2, but a "true" four-terminal cannot.[1] It should be noted that the two-port representation makes no statement at all about the voltage between the terminals that belong to different ports (for example ①′, ②′). A three-terminal can always be interpreted as a two-port and be described as such (see Fig. 3.2).

Similar to a resistive one-port, a two-port is also completely characterized by the relationship between the four parameters $u_1(t), u_2(t), i_1(t)$, and $i_2(t)$, and at the same time instant t.

The set of all feasible operating points, from which each can be regarded as the quadruple (u_1, u_2, i_1, i_2), describes a *characteristic* in the four dimensional *signal space*.

[1]Unfortunately, a two-port is not always clearly distinguished from a four-terminal in the older literature.

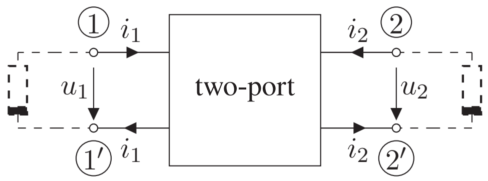


Figure 3.1: Two-Port with External Connections

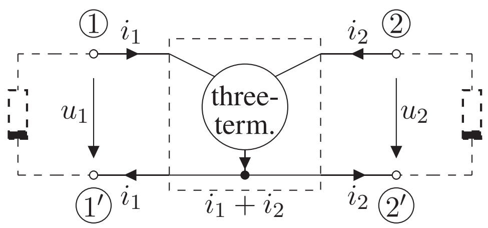


Figure 3.2: Three-Terminal Used as Two-Port

3.1 Representation Forms

The port voltages and port currents are combined pair-wisely in the column vectors

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \boldsymbol{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

respectively. The plain of the charestice $\mathcal{F}$ can be written as the set of zeros of a vector-valued constituting function, i.e.,

$$\boldsymbol{f}(\boldsymbol{u}, \boldsymbol{i}) = \boldsymbol{0} \tag{3.1}$$

where $\boldsymbol{f} = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$. Expressed element-wise, we have

$$\begin{aligned} f_1(u_1, u_2, i_1, i_2) &= 0 \\ f_2(u_1, u_2, i_1, i_2) &= 0. \end{aligned}$$

For the characteristic, we therefore get

$$\mathcal{F} = \{(\boldsymbol{u}, \boldsymbol{i}) | \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{i}) = \boldsymbol{0}\}. \tag{3.2}$$

The constituting function of the implicit representation (as it is the case for one-ports) is not unique. However, the set of zeros, that is, the characteristic, is unique.

Besides the implicit representation, a parametric representation can be found, where the operating points of the characteristic can be expressed with the help of the parameter pair $c_1, c_2 \in \mathbb{R}$ which can be comprised in the vector

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2.$$

Hence, the operating points have the form

$$\begin{bmatrix} \boldsymbol{u}(\boldsymbol{c}) \\ \boldsymbol{i}(\boldsymbol{c}) \end{bmatrix} \in \mathcal{F}. \tag{3.3}$$

All components of this vector can be written as functions, i.e.,

$$\begin{bmatrix} \boldsymbol{u}(\boldsymbol{c}) \\ \boldsymbol{i}(\boldsymbol{c}) \end{bmatrix} = \begin{bmatrix} u_1(c_1, c_2) \\ u_2(c_1, c_2) \\ i_1(c_1, c_2) \\ i_2(c_1, c_2) \end{bmatrix}.$$

Such a parametric representation form will be used to derive important properties of strictly linear two-ports (introduced in Subsection).

In addition to the implicit and parametric representations, two-ports can have an explicit representation. To this end, the function given in (3.1) is solved for two of the four port variables.

Therefore, we obtain six different explicit representations.

$i_1 = g_1(u_1, u_2)$ $i_2 = g_2(u_1, u_2)$	$\boldsymbol{i} = \boldsymbol{g}(\boldsymbol{u})$	voltage-controlled representation
$u_1 = r_1(i_1, i_2)$ $u_2 = r_2(i_1, i_2)$	$\boldsymbol{u} = \boldsymbol{r}(\boldsymbol{i})$	current-controlled representation
$u_1 = h_1(i_1, u_2)$ $i_2 = h_2(i_1, u_2)$	$\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \boldsymbol{h}\left(\begin{bmatrix} i_1 \\ u_2 \end{bmatrix}\right)$	hybrid representation
$i_1 = h'_1(u_1, i_2)$ $u_2 = h'_2(u_1, i_2)$	$\begin{bmatrix} i_1 \\ u_2 \end{bmatrix} = \boldsymbol{h}'\left(\begin{bmatrix} u_1 \\ i_2 \end{bmatrix}\right)$	inverse hybrid representation
$u_1 = a_1(u_2, -i_2)$ $i_1 = a_2(u_2, -i_2)$	$\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = \boldsymbol{a}\left(\begin{bmatrix} u_2 \\ -i_2 \end{bmatrix}\right)$	transmission representation
$u_2 = a'_1(u_1, -i_1)$ $i_2 = a'_2(u_1, -i_1)$	$\begin{bmatrix} u_2 \\ i_2 \end{bmatrix} = \boldsymbol{a}'\left(\begin{bmatrix} u_1 \\ -i_1 \end{bmatrix}\right)$	inverse transmission representation

The negative sign for the controlling current (i_2 or i_1) in the transmission representations is advantageous for the cascade connection of two-ports. The names of the different explicit representation forms will be clearer by the discussion of the connection of two-ports (see Section 3.6).

The six explicit representation do not always exist. It is possible that all explicit representations exist or maybe none of them (like for one-ports).

Employing the vector-based notation, all defined representation forms can also be used for multi-ports (for the transmission representations, the number of ports must be even).

3.1.1 Non-Linear Two-Port

Consider the particular example of a current-controlled representation of a non-linear two-port that reads as

$$\begin{aligned} u_1 &= r_1(i_1, i_2) = R_1 i_1 \\ u_2 &= r_2(i_1, i_2) = U_\mathrm{T} \ln\left(i_1^2 k R_1 + 1 + \frac{i_2}{I_\mathrm{s}}\right) \end{aligned} \tag{3.4}$$

with the temperature voltage U_T, the saturation current I_s, and the parameters R_1 and k. The according voltage-controlled representation is given by

$$\begin{aligned} i_1 &= g_1(u_1, u_2) = \frac{1}{R_1} u_1 \\ i_2 &= g_2(u_1, u_2) = I_\mathrm{s}\left(-k\frac{u_1^2}{R_1} - 1 + \mathrm{e}^{u_2/U_\mathrm{T}}\right). \end{aligned} \tag{3.5}$$

Both hybrid representations also exist, that is,

$$\begin{aligned} u_1 &= h_1(i_1, u_2) = R_1 i_1 \\ i_2 &= h_2(i_1, u_2) = I_\mathrm{s}\left(-k i_1^2 R_1 - 1 + \mathrm{e}^{u_2/U_\mathrm{T}}\right) \\ i_1 &= h_1'(u_1, i_2) = \frac{1}{R_1} u_1 \\ u_2 &= h_2'(u_1, i_2) = U_\mathrm{T} \ln\left(u_1^2 \frac{k}{R_1} + 1 + \frac{i_2}{I_\mathrm{s}}\right). \end{aligned}$$

However, both transmission representations do not exist for the considered non-linear two-port.

3.2 Strictly Linear Two-Ports

A resistive two-port is called *strictly linear* or *linear and source-free* when an arbitrary linear combination of two operating points is also an operating point, i.e., when

$$\begin{bmatrix} u_1{}^{(1)} \\ u_2{}^{(1)} \\ i_1{}^{(1)} \\ i_2{}^{(1)} \end{bmatrix} \in \mathcal{F} \quad \text{und} \quad \begin{bmatrix} u_1{}^{(2)} \\ u_2{}^{(2)} \\ i_1{}^{(2)} \\ i_2{}^{(2)} \end{bmatrix} \in \mathcal{F}: \qquad \begin{bmatrix} \alpha u_1{}^{(1)} + \beta u_1{}^{(2)} \\ \alpha u_2{}^{(1)} + \beta u_2{}^{(2)} \\ \alpha i_1{}^{(1)} + \beta i_1{}^{(2)} \\ \alpha i_2{}^{(1)} + \beta i_2{}^{(2)} \end{bmatrix} \in \mathcal{F},\ \alpha, \beta \in \mathbb{R}.$$

Thus, in a compact vector notation it holds that

$$\begin{bmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{i}^{(1)} \end{bmatrix} \in \mathcal{F} \quad \text{und} \quad \begin{bmatrix} \boldsymbol{u}^{(2)} \\ \boldsymbol{i}^{(2)} \end{bmatrix} \in \mathcal{F}: \qquad \alpha \begin{bmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{i}^{(1)} \end{bmatrix} + \beta \begin{bmatrix} \boldsymbol{u}^{(2)} \\ \boldsymbol{i}^{(2)} \end{bmatrix} \in \mathcal{F},\ \alpha, \beta \in \mathbb{R} \tag{3.6}$$

where the port voltage vector and the port current vector

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

have been employed.

Note that the currents and voltages of a strictly linear two-port fulfill the attributes of a (linear) vector space. But the "characteristic" of a linear two-port cannot be plotted anymore in a four dimensional signal space. However, some important attributes of this characteristic and some possibilities of algebraic representation can be derived by means of a comparison with a strictly linear one-port (e.g., the linear Ohmic resistor).

3.2.1 Nullspace and Range Space Representations

A two-port is based on the interconnection between four signals $[u_1, u_2, i_1, i_2]^{\mathrm{T}} = [\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{i}^{\mathrm{T}}]^{\mathrm{T}}$ which define a four-dimensional signal space. The solution of the representation obeys two linear independent restrictions. In the case of a strictly linear two-port, the characteristic is a two-dimensional subspace (a plane through the origin) of the four-dimensional signal space. Such a subspace can be interpreted as the nullspace of a linear mapping $[\boldsymbol{M}, \boldsymbol{N}]$, i.e.,

$$\mathcal{F} = \operatorname{null}\begin{bmatrix}\boldsymbol{M}, \boldsymbol{N}\end{bmatrix} = \left\{ \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} \middle| \begin{bmatrix}\boldsymbol{M}, \boldsymbol{N}\end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} = \boldsymbol{0} \right\} \tag{3.7}$$

with $\operatorname{rank}[\boldsymbol{M}, \boldsymbol{N}] = 2$. Expressing the matrix $[\boldsymbol{M}, \boldsymbol{N}]$ by its elements gives

$$\begin{bmatrix}\boldsymbol{M}, \boldsymbol{N}\end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & n_{11} & n_{12} \\ m_{21} & m_{22} & n_{21} & n_{22} \end{bmatrix}.$$

The representation $[\boldsymbol{M}, \boldsymbol{N}] \left[\begin{smallmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{smallmatrix}\right]$ corresponds to the constituting function of the implicit representation for a strictly linear two-port.

Another possibility for the representation is the parametric representation, in which the whole operating space is interpreted as the linear span of two linearly independent measurement vectors, i.e.,

$$\mathcal{F} = \left\{ \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} \middle| \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{i}^{(1)} \end{bmatrix} c_1 + \begin{bmatrix} \boldsymbol{u}^{(2)} \\ \boldsymbol{i}^{(2)} \end{bmatrix} c_2 = \begin{bmatrix} \boldsymbol{u}^{(1)} & \boldsymbol{u}^{(2)} \\ \boldsymbol{i}^{(1)} & \boldsymbol{i}^{(2)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},\ c_1, c_2 \in \mathbb{R} \right\}. \tag{3.8}$$

The two real parameters c_1 and c_2 in (3.8) substitute the numbers α and β from (3.6). Eq. (3.8) can be more compactly formulated by the introduction of the *basis matrix*. The basis matrix is constructed by arranging two linearly independent measurement vectors as columns:

$$\begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{(1)} & \boldsymbol{u}^{(2)} \\ \boldsymbol{i}^{(1)} & \boldsymbol{i}^{(2)} \end{bmatrix}.$$

The span of the column vectors of the basis matrix is also called the *range* or *image space* of the basis matrix:

$$\mathcal{F} = \text{range} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} = \left\{ \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} \middle| \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{c},\ \boldsymbol{c} \in \mathbb{R}^2 \right\} \tag{3.9}$$

with rank $\left[\begin{smallmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{smallmatrix}\right] = 2$.

Similarly, a basis matrix $\left[\begin{smallmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{smallmatrix}\right]$ can be converted through the multiplication with an invertible $\boldsymbol{W}^{-1}$ into another basis matrix

$$\begin{bmatrix} \boldsymbol{U}' \\ \boldsymbol{I}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{W}^{-1}$$

that possesses the same range space. The column vectors of the basis matrix $\left[\begin{smallmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{smallmatrix}\right]$ are a basis for the operating space.

Many attributes of a two-port can be identified by a basis matrix representation. Note additionally that the parametric representation with a basis matrix always exists and can be obtained by two linearly independent measurements.

3.2.2 Explicit Representation

Besides the parametric representation of strictly linear two-ports, there is a set of possible explicit representations which are found by resolving for 2 of the 4 quantities u_1, u_2, i_1, i_2. One has in total $\left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right) = 6$ possibilities to choose from and obtains, accordingly, the following 6 matrix equations:

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \boldsymbol{G} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} g_{11}u_1 + g_{12}u_2 \\ g_{21}u_1 + g_{22}u_2 \end{bmatrix} \tag{3.10}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} r_{11}i_1 + r_{12}i_2 \\ r_{21}i_1 + r_{22}i_2 \end{bmatrix} \tag{3.11}$$

$$\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} i_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} h_{11}i_1 + h_{12}u_2 \\ h_{21}i_1 + h_{22}u_2 \end{bmatrix} \tag{3.12}$$

$$\begin{bmatrix} i_1 \\ u_2 \end{bmatrix} = \boldsymbol{H}' \begin{bmatrix} u_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h'_{11}u_1 + h'_{12}i_2 \\ h'_{21}u_1 + h'_{22}i_2 \end{bmatrix} \tag{3.13}$$

$$\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} u_2 \\ -i_2 \end{bmatrix} = \begin{bmatrix} a_{11}u_2 - a_{12}i_2 \\ a_{21}u_2 - a_{22}i_2 \end{bmatrix} \tag{3.14}$$

$$\begin{bmatrix} u_2 \\ i_2 \end{bmatrix} = \boldsymbol{A}' \begin{bmatrix} u_1 \\ -i_1 \end{bmatrix} = \begin{bmatrix} a'_{11}u_1 - a'_{12}i_1 \\ a'_{21}u_1 - a'_{22}i_1 \end{bmatrix}. \tag{3.15}$$

These 6 two-port matrices have the names that are listed in Table 3.1. In the two transmission representations, it is important to pay attention because they either depend on $-i_2$ or $-i_1$!

$\boldsymbol{G}$	conductance matrix
$\boldsymbol{R}$	resistance matrix
$\boldsymbol{H}$	hybrid matrix
$\boldsymbol{H}'$	inverse hybrid matrix
$\boldsymbol{A}$	transmission matrix
$\boldsymbol{A}'$	inverse transmission matrix

Table 3.1: Names of Two-Matrices

The different explicit representation equations can be favorably applied to the interconnection of two-ports (see Section 3.6). All these two-port matrices can be calculated from the parametric representation, if the desired two-port matrix exists.

The conductance matrix $\boldsymbol{G}$ can be found via

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{c}$$

$$\boldsymbol{u} = \boldsymbol{U}\boldsymbol{c} \quad \text{and} \quad \boldsymbol{i} = \boldsymbol{I}\boldsymbol{c}$$

and if $\boldsymbol{U}^{-1}$ exists

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{U}^{-1}\boldsymbol{u} \\ \boldsymbol{i} &= \underbrace{\boldsymbol{I}\boldsymbol{U}^{-1}}_{\boldsymbol{G}}\boldsymbol{u}. \end{aligned}$$

The six two-port matrices do not always exist in contrast to the parametric representation. For example, there is no voltage-controlled representation with the conductance matrix $\boldsymbol{G}$ when the matrix $\boldsymbol{U}$ is not invertible. However, in case a two-port matrix exists, it is unique for a given two-port and there is the possibility to convert it into another two-port matrix according to Table 3.2.

By an appropriate connection of the two-port, the elements of the representation matrices can be directly determined and adequately interpreted. Table 3.3 exemplarily shows the meaning of the elements of the matrices $\boldsymbol{G}$ and $\boldsymbol{R}$ (the elements of the other matrices can be interpreted analogously), where port 1 is called *input* and port 2 *output*.

3.3 Linear Two-Ports

The operating space of linear resistive two-ports that contain sources is not a plane through the origin, but an affine plane. The parametric representation reads as

$$\mathcal{F} = \left\{ \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} \middle| \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{c} + \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{i}_0 \end{bmatrix}, \boldsymbol{c} \in \mathbb{R}^2 \right\} = \text{range} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} + \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{i}_0 \end{bmatrix}. \tag{3.16}$$

	$\boldsymbol{R}$	$\boldsymbol{G}$	$\boldsymbol{H}$	$\boldsymbol{H}'$	$\boldsymbol{A}$	$\boldsymbol{A}'$
$\boldsymbol{R}$	$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$	$\frac{1}{\det \boldsymbol{G}} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix}$	$\frac{1}{h_{22}} \begin{bmatrix} \det \boldsymbol{H} & h_{12} \\ -h_{21} & 1 \end{bmatrix}$	$\frac{1}{h'_{11}} \begin{bmatrix} 1 & -h'_{12} \\ h'_{21} & \det \boldsymbol{H}' \end{bmatrix}$	$\frac{1}{a_{21}} \begin{bmatrix} a_{11} & \det \boldsymbol{A} \\ 1 & a_{22} \end{bmatrix}$	$\frac{1}{a'_{21}} \begin{bmatrix} a'_{22} & 1 \\ \det \boldsymbol{A}' & a'_{11} \end{bmatrix}$
$\boldsymbol{G}$	$\frac{1}{\det \boldsymbol{R}} \begin{bmatrix} r_{22} & -r_{12} \\ -r_{21} & r_{11} \end{bmatrix}$	$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$	$\frac{1}{h_{11}} \begin{bmatrix} 1 & -h_{12} \\ h_{21} & \det \boldsymbol{H} \end{bmatrix}$	$\frac{1}{h'_{22}} \begin{bmatrix} \det \boldsymbol{H}' & h'_{12} \\ -h'_{21} & 1 \end{bmatrix}$	$\frac{1}{a_{12}} \begin{bmatrix} a_{22} & -\det \boldsymbol{A} \\ -1 & a_{11} \end{bmatrix}$	$\frac{1}{a'_{12}} \begin{bmatrix} a'_{11} & -1 \\ -\det \boldsymbol{A}' & a'_{22} \end{bmatrix}$
$\boldsymbol{H}$	$\frac{1}{r_{22}} \begin{bmatrix} \det \boldsymbol{R} & r_{12} \\ -r_{21} & 1 \end{bmatrix}$	$\frac{1}{g_{11}} \begin{bmatrix} 1 & -g_{12} \\ g_{21} & \det \boldsymbol{G} \end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$	$\frac{1}{\det \boldsymbol{H}'} \begin{bmatrix} h'_{22} & -h'_{12} \\ -h'_{21} & h'_{11} \end{bmatrix}$	$\frac{1}{a_{22}} \begin{bmatrix} a_{12} & \det \boldsymbol{A} \\ -1 & a_{21} \end{bmatrix}$	$\frac{1}{a'_{11}} \begin{bmatrix} a'_{12} & 1 \\ -\det \boldsymbol{A}' & a'_{21} \end{bmatrix}$
$\boldsymbol{H}'$	$\frac{1}{r_{11}} \begin{bmatrix} 1 & -r_{12} \\ r_{21} & \det \boldsymbol{R} \end{bmatrix}$	$\frac{1}{g_{22}} \begin{bmatrix} \det \boldsymbol{G} & g_{12} \\ -g_{21} & 1 \end{bmatrix}$	$\frac{1}{\det \boldsymbol{H}} \begin{bmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11} \end{bmatrix}$	$\begin{bmatrix} h'_{11} & h'_{12} \\ h'_{21} & h'_{22} \end{bmatrix}$	$\frac{1}{a_{11}} \begin{bmatrix} a_{21} & -\det \boldsymbol{A} \\ 1 & a_{12} \end{bmatrix}$	$\frac{1}{a'_{22}} \begin{bmatrix} a'_{21} & -1 \\ \det \boldsymbol{A}' & a'_{12} \end{bmatrix}$
$\boldsymbol{A}$	$\frac{1}{r_{21}} \begin{bmatrix} r_{11} & \det \boldsymbol{R} \\ 1 & r_{22} \end{bmatrix}$	$\frac{1}{g_{21}} \begin{bmatrix} -g_{22} & -1 \\ -\det \boldsymbol{G} & -g_{11} \end{bmatrix}$	$\frac{1}{h_{21}} \begin{bmatrix} -\det \boldsymbol{H} & -h_{11} \\ -h_{22} & -1 \end{bmatrix}$	$\frac{1}{h'_{21}} \begin{bmatrix} 1 & h'_{22} \\ h'_{11} & \det \boldsymbol{H}' \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$	$\frac{1}{\det \boldsymbol{A}'} \begin{bmatrix} a'_{22} & a'_{12} \\ a'_{21} & a'_{11} \end{bmatrix}$
$\boldsymbol{A}'$	$\frac{1}{r_{12}} \begin{bmatrix} r_{22} & \det \boldsymbol{R} \\ 1 & r_{11} \end{bmatrix}$	$\frac{1}{g_{12}} \begin{bmatrix} -g_{11} & -1 \\ -\det \boldsymbol{G} & -g_{22} \end{bmatrix}$	$\frac{1}{h_{12}} \begin{bmatrix} 1 & h_{11} \\ h_{22} & \det \boldsymbol{H} \end{bmatrix}$	$\frac{1}{h'_{12}} \begin{bmatrix} -\det \boldsymbol{H}' & -h'_{22} \\ -h'_{11} & -1 \end{bmatrix}$	$\frac{1}{\det \boldsymbol{A}} \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix}$	$\begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix}$

Table 3.2: Conversion of Two-Port Matrices

$g_{11} = \left.\frac{i_1}{u_1}\right\|_{u_2=0}$	input conductance with short-circuited output	$r_{11} = \left.\frac{u_1}{i_1}\right\|_{i_2=0}$	input resistance with open-circuited output
$g_{22} = \left.\frac{i_2}{u_2}\right\|_{u_1=0}$	output conductance with short-circuited input	$r_{22} = \left.\frac{u_2}{i_2}\right\|_{i_1=0}$	output resistance with open-circuited input
$g_{21} = \left.\frac{i_2}{u_1}\right\|_{u_2=0}$	forward transfer conductance with short-circuited output	$r_{21} = \left.\frac{u_2}{i_1}\right\|_{i_2=0}$	forward transfer resistance with open-circuited output
$g_{12} = \left.\frac{i_1}{u_2}\right\|_{u_1=0}$	backward transfer conductance with short-circuited input	$r_{12} = \left.\frac{u_1}{i_2}\right\|_{i_1=0}$	backward transfer resistance with open-circuited input

Table 3.3: Interpretation of Some Two-Port Matrix Elements

Note that it is necessary to find three linearly independent measurements to set up the parametric representation in (3.16). One measurement is

$$\begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{i}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{i}^{(1)} \end{bmatrix}$$

and the other two measurements are collected in

$$\begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{(2)} - \boldsymbol{u}_0 & \boldsymbol{u}^{(3)} - \boldsymbol{u}_0 \\ \boldsymbol{i}^{(2)} - \boldsymbol{i}_0 & \boldsymbol{i}^{(3)} - \boldsymbol{i}_0 \end{bmatrix}.$$

The parametric representation (3.16) is a plane equation with the shift $\left[\begin{smallmatrix} \boldsymbol{u}_0 \\ \boldsymbol{i}_0 \end{smallmatrix}\right]$, whose choice is not unique, because it can be any point on the affine plane. Note that the parametric representation form always exists for linear two-ports.

Starting from the parametric representation, an explicit representation can be found, e.g., the voltage-controlled representation. To this end, let us rewrite the first equation of (3.16),

$$\boldsymbol{U}^{-1}\boldsymbol{u} = \boldsymbol{c} + \boldsymbol{U}^{-1}\boldsymbol{u}_0.$$

Substituting into the second equation yields

$$\boldsymbol{i} = \boldsymbol{I}\boldsymbol{U}^{-1}\boldsymbol{u} + \boldsymbol{i}_0 - \boldsymbol{I}\boldsymbol{U}^{-1}\boldsymbol{u}_0.$$

Therefore, we have obtained the voltage-controlled representation

$$\boldsymbol{i} = \boldsymbol{G}\boldsymbol{u} + \boldsymbol{i}_G$$

of the linear two-port with the conductance matrix $\boldsymbol{G} = \boldsymbol{I}\boldsymbol{U}^{-1}$ and the current source vector

$$\boldsymbol{i}_G = \boldsymbol{i}_0 - \boldsymbol{G}\boldsymbol{u}_0.$$

Similarly, the following representations can be found.

$$
\begin{aligned}
\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} &= \boldsymbol{G} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} i_{G1} \\ i_{G2} \end{bmatrix} \\
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \boldsymbol{R} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} u_{R1} \\ u_{R2} \end{bmatrix} \\
\begin{bmatrix} u_1 \\ i_2 \end{bmatrix} &= \boldsymbol{H} \begin{bmatrix} i_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} u_{H1} \\ i_{H2} \end{bmatrix} \\
\begin{bmatrix} i_1 \\ u_2 \end{bmatrix} &= \boldsymbol{H}' \begin{bmatrix} u_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} i_{H'1} \\ u_{H'2} \end{bmatrix} \\
\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} &= \boldsymbol{A} \begin{bmatrix} u_2 \\ -i_2 \end{bmatrix} + \begin{bmatrix} u_{A1} \\ i_{A1} \end{bmatrix} \\
\begin{bmatrix} u_2 \\ i_2 \end{bmatrix} &= \boldsymbol{A}' \begin{bmatrix} u_1 \\ -i_1 \end{bmatrix} + \begin{bmatrix} u_{A'2} \\ i_{A'2} \end{bmatrix} .
\end{aligned}
\tag{3.17}
$$

Every equation in (3.17) exhibits a term that represent the internal sources that disappear in the strictly linear case. Then, the equations (3.17) simplify to Eqns. (3.10)–(3.15). The equations (3.17) lead directly to equivalent circuits, in which the strictly linear two-ports are described by the matrices $\boldsymbol{G}$, $\boldsymbol{R}$, $\boldsymbol{H}$, $\boldsymbol{H}'$, $\boldsymbol{A}$, and $\boldsymbol{A}'$ The strictly linear part is complemented by *only two* independent sources (see Fig. 3.3).

These representations are important. For instance, in the treatment of non-autonomous dynamic circuits of second degree, where both ports are connected to reactances, it is useful to use the decomposition of (3.17). An additional significant practical application is the analysis of noisy circuits, where each noisy linear two-port is substituted by a noise-free linear two-port and two external noise sources.

3.4 Properties of Two-Ports

3.4.1 Power

The power, or rather energy distribution has great importance in physical systems because of the central law of energy conservation. Therewith, the electric networks act as converters between different forms of energy (electrical and mechanical energy, light or heat energy, ...).

3.4.1.1 Losslessness

A two-port is called *lossless* when the sum of the total (electrical) power absorbed in both ports vanishes for all time and for all operating points:

$$p_1(t) + p_2(t) = u_1(t)i_1(t) + u_2(t)i_2(t) = 0$$

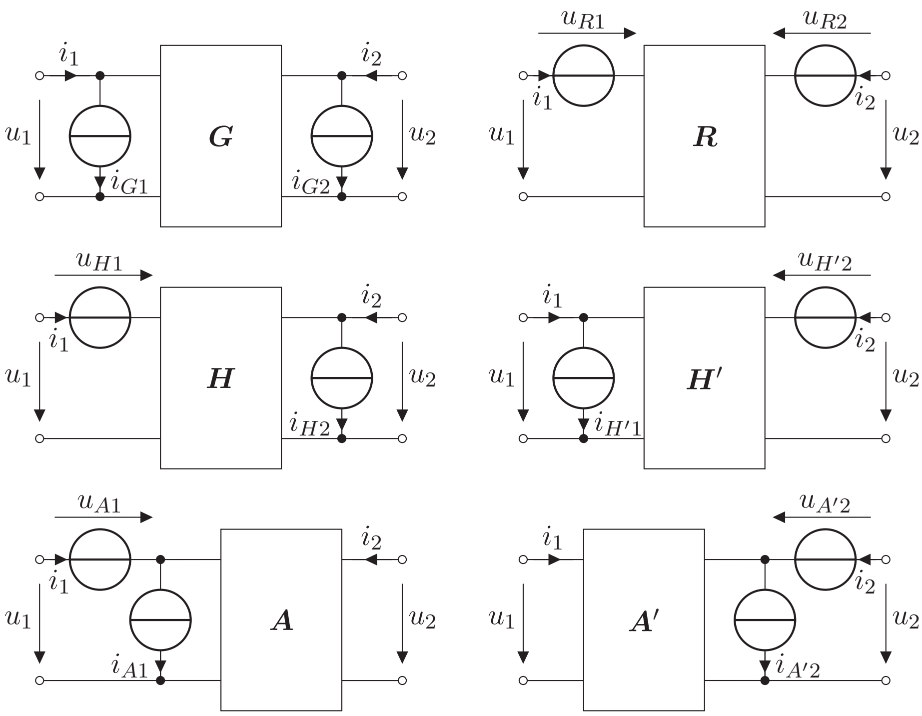


Figure 3.3: Decomposition of Linear Two-Ports

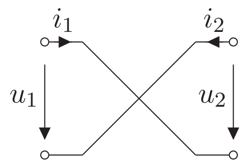


Figure 3.4: Lossless Polarity Inversion

or in compact form, we find following definition.

Definition 1. Losslessness

$$\forall \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} \in \mathcal{F} : \ \boldsymbol{u}^{\mathrm{T}} \boldsymbol{i} = 0. \tag{3.18}$$

In Fig. 3.4, a simple example is given for a lossless two-port. For strictly linear two-ports, the condition for losslessness can be derived not only pointwisely but for the whole basis matrix. With

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{c}$$

and $\boldsymbol{c} \in \mathbb{R}^2$, the condition for losslessness reads as

$$\forall \boldsymbol{c} \in \mathbb{R}^2 : \quad \boldsymbol{u}^{\mathrm{T}} \boldsymbol{i} = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{U}^{\mathrm{T}} \boldsymbol{I} \boldsymbol{c} = 0.$$

The decomposition of the matrix $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}$ into a symmetric and skew-symmetric part leads to

$$\boldsymbol{c}^{\mathrm{T}}\left[\frac{1}{2}\left(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}+\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}\right)\right]\boldsymbol{c}+\boldsymbol{c}^{\mathrm{T}}\left[\frac{1}{2}\left(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}-\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}\right)\right]\boldsymbol{c}=0. \tag{3.19}$$

Note that $\boldsymbol{A}=\frac{1}{2}(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}-\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U})$ is skew-symmetric, i.e., $\boldsymbol{A}=-\boldsymbol{A}^{\mathrm{T}}$. Thus, the second summand of equation (3.19) vanishes independently of the vector of parameters $\boldsymbol{c}$ because a quadratic form $h=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{c}$ with skew-symmetric $\boldsymbol{A}$ is always equal to zero. In our case,

$$\boldsymbol{A}=\frac{1}{2}\left(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}-\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}\right).$$

For the proof, note that the transposition does not change scalar quantities. With the help of the properties of transposed matrices and by the utilization of the skew-symmetry, the following equations can be set up:

$$h=h^{\mathrm{T}}=\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{c}\right)^{\mathrm{T}}=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}.$$

Due to the skew symmetry of $\boldsymbol{A}$,

$$h=-\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{c}=-h.$$

This result $h=-h$ proves that $h=0$. Therefore, $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{c}=0$ for skew-symmetric $\boldsymbol{A}$.

Whereas the second summand of Eq. (3.19) generally vanishes, the symmetric portion of $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}+\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}$ of $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}$ has to be identical to zero, so that the equation is valid for any $\boldsymbol{c}\in\mathbb{R}^2$, i.e.,

$$\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}+\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}=\boldsymbol{0}. \tag{3.20}$$

3.4.1.2 Passivity/Activity

In case the total absorbed power for all time and for all operating points is greater or equal to zero, the two-port is called *passive*, and by the exclusion of the equality sign, *strictly passive*.

Definition 2. Passivität eines Zweitors

$$\forall\begin{bmatrix}\boldsymbol{u}\\\boldsymbol{i}\end{bmatrix}\in\mathcal{F}:\boldsymbol{u}^{\mathrm{T}}\boldsymbol{i}\geq 0. \tag{3.21}$$

The passivity of a two-port does *not* indicate that the power is absorbed at *both* ports. Instead, the power has to be absorbed solely in the sum! In Fig. 3.5a, the passive two-port voltage divider is depicted.

In the case of a strictly linear two-port, we must have for passivity that $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}+\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}$ is non-negative definite.

A two-port is called *active* when it is not passive.

Definition 3. Activity of a Two-Port

$$\exists\begin{bmatrix}\boldsymbol{u}\\\boldsymbol{i}\end{bmatrix}\in\mathcal{F}:\boldsymbol{u}^{\mathrm{T}}\boldsymbol{i}<0. \tag{3.22}$$

The circuit in Fig. 3.5b depicts the active two-port amplifier (negative resistor). For a strictly linear two-port, activity leads to $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}+\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}$ being not non-negative definite.

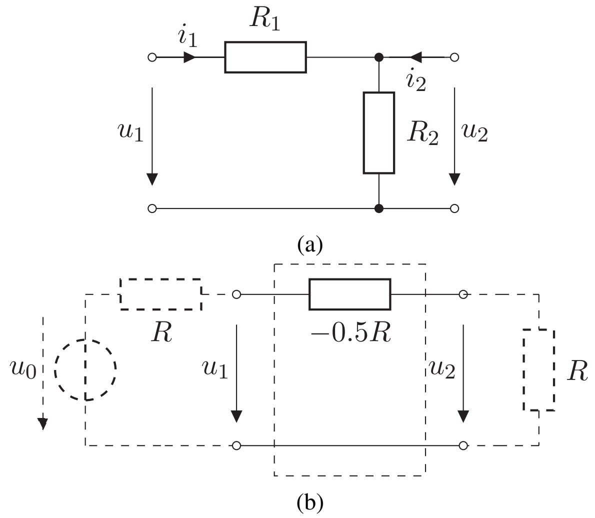


Figure 3.5: (a) Passive Two-Port Voltage Divider and (b) Active Two-Port Amplifier

3.4.2 Duality

The *dual two-port* with respect to the duality constant R_d is denoted as $\mathcal{F}^\mathrm{d}$ when the roles of current and voltage are interchanged (by the conversion with R_d), that is,

$$\begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix}^\mathrm{d} = \begin{bmatrix} R_\mathrm{d}\boldsymbol{I} \\ \frac{1}{R_\mathrm{d}}\boldsymbol{U} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & R_\mathrm{d}\mathbf{1} \\ \frac{1}{R_\mathrm{d}}\mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix}.$$

Therefore,

$$\boldsymbol{G}^\mathrm{d} = \frac{1}{R_\mathrm{d}^2}\boldsymbol{R} \quad \text{and} \quad \boldsymbol{R}^\mathrm{d} = R_\mathrm{d}^2\boldsymbol{G}.$$

3.4.3 Reversibility (Symmetry)

A two-port is called *reversible* or *symmetric* when the operating space remains unchanged through the interchange of the two ports:

$$\mathcal{F}^\mathrm{r} = \mathcal{F}. \tag{3.23}$$

With the permutation matrix

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \boldsymbol{P}^{-1}$$

whose inverse is $\boldsymbol{P}$, the reversion can be expressed as

$$\boldsymbol{u}^\mathrm{r} = \boldsymbol{P}\boldsymbol{u} \quad \text{and} \quad \boldsymbol{i}^\mathrm{r} = \boldsymbol{P}\boldsymbol{i}.$$

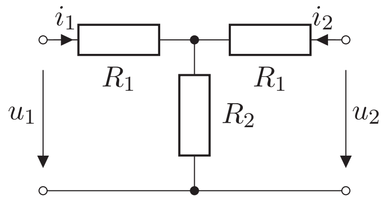


Figure 3.6: Symmetric Two-Port T-Network

Therefore, the resistance matrix of a reversed two-port is

$$\boldsymbol{R}^{\mathrm{r}} = \boldsymbol{P}\boldsymbol{R}\boldsymbol{P}$$

and the condition for symmetry is $\boldsymbol{R}^{\mathrm{r}} = \boldsymbol{R}$. In particular, the equivalent conditions for symmetry read as

$$\boldsymbol{G} = \boldsymbol{P}\,\boldsymbol{G}\,\boldsymbol{P}$$
$$\boldsymbol{R} = \boldsymbol{P}\,\boldsymbol{R}\,\boldsymbol{P}.$$

This means that the reversible two-port $\boldsymbol{G}$ and $\boldsymbol{R}$ (if they exist!) are invariant under column and row exchange. Moreover, in case a reversible two-port possesses both transmission representations, it holds that:

$$\boldsymbol{A} = \boldsymbol{A}'.$$

As an example for a symmetric two-port, consider the T-network in Fig. 3.6 with the resistance matrix

$$\boldsymbol{R}_{\text{T-network}} = \begin{bmatrix} R_1 + R_2 & R_2 \\ R_2 & R_1 + R_2 \end{bmatrix}.$$

Besides the obvious symmetry of the T-network in Fig. 3.6, $\boldsymbol{R}_{\text{T-network}}$ fulfills the condition for symmetry, namely, $\boldsymbol{R}_{\text{T-network}} = \boldsymbol{P}\boldsymbol{R}_{\text{T-network}}\boldsymbol{P}$.

3.4.4 Reciprocity of Linear Two-Ports

An analysis of the voltage divider in Fig. 3.5a leads to the voltage-controlled equation:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} R_1 + R_2 & R_2 \\ R_2 & R_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}.$$

Although the circuit does not exhibit any visible symmetry, both transfer resistances are the same. This property is called *reciprocity* or *transmission symmetry*. All the two-ports composed of resistors and transformers (and also capacitors and inductors) are reciprocal. Other transmission quantities are also pairwise identical in the case of reciprocal two-ports. The relation between cause and effect does not change if source (independent current and voltage source) and load

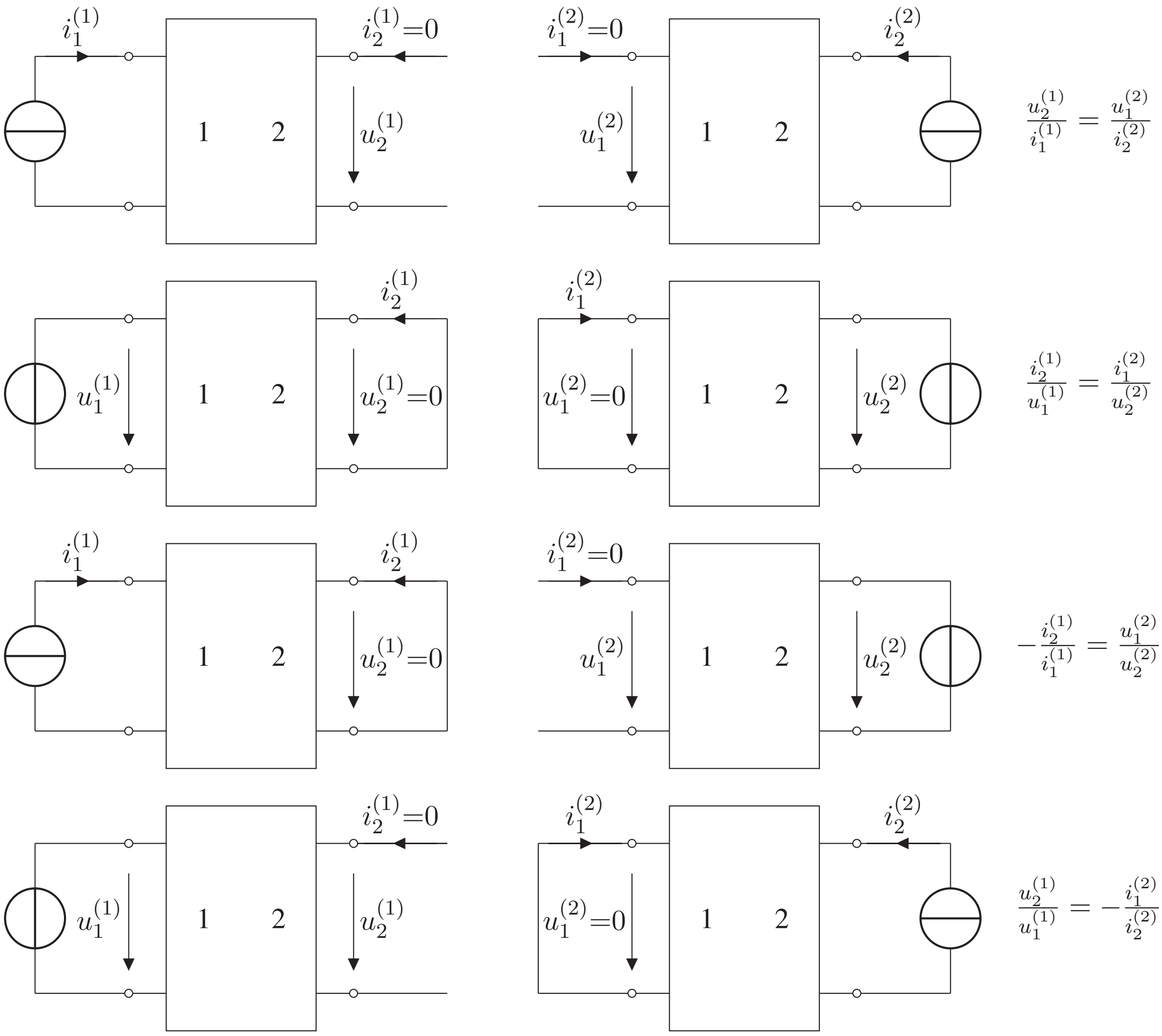


Figure 3.7: Reciprocity Relations

(zero source: open circuit or short circuit) are interchanged, as shown in Fig. 3.7, provided sources and zero sources on each port are of the same type. According to the four measurements depicted in Fig. 3.7, reciprocity holds if at least one of the following conditions is fulfilled

$$\begin{aligned} r_{21} &= r_{12} \\ g_{21} &= g_{12} \\ h_{21} &= -h_{12} \\ h'_{21} &= -h'_{12}. \end{aligned} \tag{3.24}$$

The reciprocity conditions for the entries of $\boldsymbol{G}$ and $\boldsymbol{R}$ can be concisely expressed as

$$\begin{aligned}\boldsymbol{G} &= \boldsymbol{G}^{\mathrm{T}}\\ \boldsymbol{R} &= \boldsymbol{R}^{\mathrm{T}}.\end{aligned}$$

With Table 3.2, it can be shown that additional conditions are

$$\begin{aligned}\det \boldsymbol{A} &= 1\\ \det \boldsymbol{A}' &= 1.\end{aligned}$$

Under the assumption that $\boldsymbol{R}$ exists, $\boldsymbol{U} = \boldsymbol{R}\boldsymbol{I}$ holds and therefore, $\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U} = \boldsymbol{I}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{I}$. If the corresponding two-port is reciprocal, $\boldsymbol{R}$ is symmetric, i.e., $\boldsymbol{R} = \boldsymbol{R}^{\mathrm{T}}$, and $\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}$ is also symmetric. Similarly, it can be shown that $\boldsymbol{I}^{\mathrm{T}}\boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}$ if the two-port is reciprocal and $\boldsymbol{G}$ exists. For existing $\boldsymbol{H}$, note that by using the partitionings

$$\boldsymbol{U} = \begin{bmatrix}\boldsymbol{u}_1^{\mathrm{T}}\\ \boldsymbol{u}_2^{\mathrm{T}}\end{bmatrix} \quad \text{und} \quad \boldsymbol{I} = \begin{bmatrix}\boldsymbol{i}_1^{\mathrm{T}}\\ \boldsymbol{i}_2^{\mathrm{T}}\end{bmatrix}$$

it follows that

$$\begin{bmatrix}\boldsymbol{u}_1^{\mathrm{T}}\\ \boldsymbol{i}_2^{\mathrm{T}}\end{bmatrix} = \boldsymbol{H}\begin{bmatrix}\boldsymbol{i}_1^{\mathrm{T}}\\ \boldsymbol{u}_2^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{i}_1^{\mathrm{T}}\\ \boldsymbol{u}_2^{\mathrm{T}}\end{bmatrix}.$$

For a reciprocal two-port, i.e., $h_{21} = -h_{12}$, we get

$$\begin{aligned}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I} &= [\boldsymbol{u}_1, \boldsymbol{u}_2]\begin{bmatrix}\boldsymbol{i}_1^{\mathrm{T}}\\ \boldsymbol{i}_2^{\mathrm{T}}\end{bmatrix} = \boldsymbol{u}_1\boldsymbol{i}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{i}_2^{\mathrm{T}}\\ &= (h_{11}\boldsymbol{i}_1 + h_{12}\boldsymbol{u}_2)\,\boldsymbol{i}_1^{\mathrm{T}} + \boldsymbol{u}_2\left(h_{21}\boldsymbol{i}_1^{\mathrm{T}} + h_{22}\boldsymbol{u}_2^{\mathrm{T}}\right)\\ &= h_{11}\boldsymbol{i}_1\boldsymbol{i}_1^{\mathrm{T}} + h_{22}\boldsymbol{u}_2\boldsymbol{u}_2^{\mathrm{T}} = \left(h_{11}\boldsymbol{i}_1\boldsymbol{i}_1^{\mathrm{T}} + h_{22}\boldsymbol{u}_2\boldsymbol{u}_2^{\mathrm{T}}\right)^{\mathrm{T}}\end{aligned} \tag{3.25}$$

which is symmetric, i.e., $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I} = \boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}$. Similarly, it is possible to infer $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I} = \boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}$ from reciprocity and existence of $\boldsymbol{H}'$. Therefore, we must have in general that $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}$ is symmetric, i.e.,

$$\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I} - \boldsymbol{I}^{\mathrm{T}}\boldsymbol{U} = \boldsymbol{0} \tag{3.26}$$

for a strictly linear two-port which is reciprocal. This condition is independent of the existence of the different two-port matrices and can be used to show reciprocity in general. An example for a reciprocal two-port is given in Fig. 3.8.

3.5 Special Two-Ports

For the modeling of practical, important two-ports or three terminal components, a series of special, idealized two-ports is required. The transfer characteristics are of highest importance, and are best described by the transmission matrix.

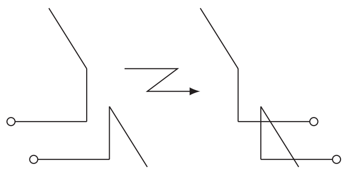
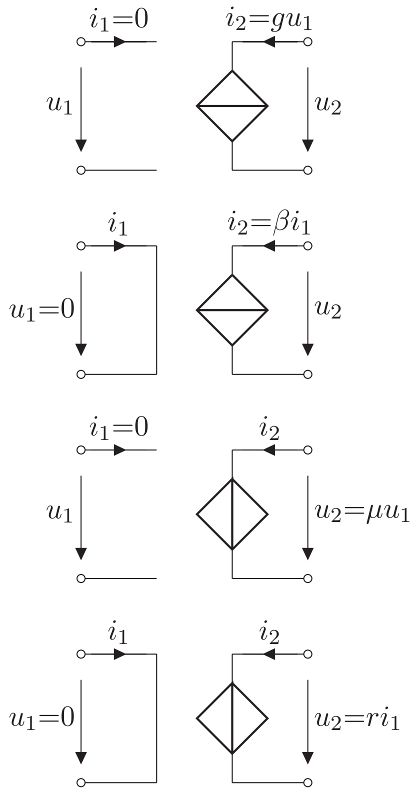


Figure 3.8: Reciprocal Two-Port Wireless Transmission (not resistive)

3.5.1 Controlled Sources



$$\boldsymbol{G} = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0 & -\frac{1}{g} \\ 0 & 0 \end{bmatrix}$$

VCCS (voltage controlled current source)

$$\boldsymbol{H} = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{\beta} \end{bmatrix}$$

CCCS (current controlled current source)

$$\boldsymbol{H}' = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} \frac{1}{\mu} & 0 \\ 0 & 0 \end{bmatrix}$$

VCVS (voltage controlled voltage source)

$$\boldsymbol{R} = \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ \frac{1}{r} & 0 \end{bmatrix}$$

CCVS (current controlled voltage source)

Figure 3.9: Controlled Sources and Their Matrix Representation

Controlled sources (see Fig. 3.9) are special two-ports consisting of two branches. One branch, the so-called *control branch*, is either an open circuit or a short circuit. The other branch, the so-called *controlled branch*, is an independent current or voltage source. From these different possibilities (two for the control branch, two for the controlled branch), a total of four different controlled sources results.

In general, an electrical isolation from control and controlled branch (as in the example of the opto-isolator) is assumed. But, in many cases, this electrical isolation is not important and

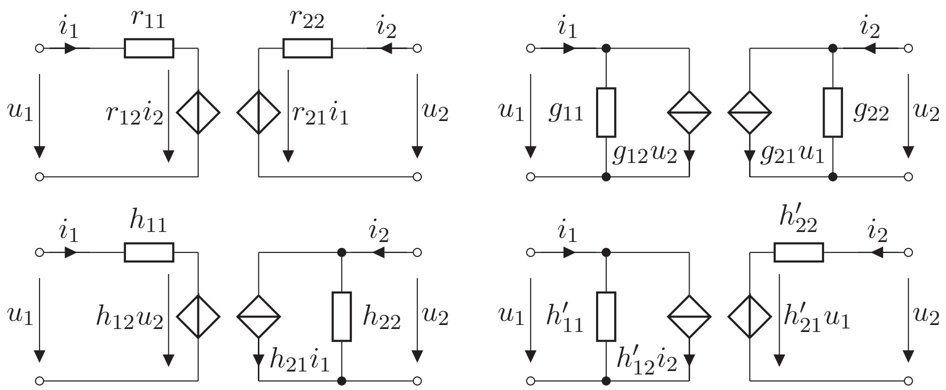


Figure 3.10: Equivalent Circuit Diagrams for Two-Ports with Two Controlled Sources

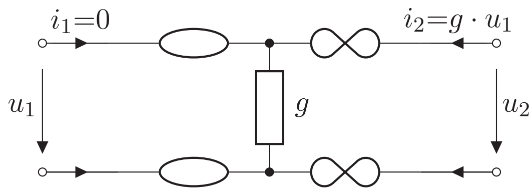


Figure 3.11: VCCS with Nullators and Norators

is compensated by an external connection (common reference terminal, the two-port is then a three-terminal). Also, the circuit realization of the controlled sources is easier if one common reference terminal (datum node) is assumed (see Chapter 4).

With the help of controlled sources, simple equivalent circuit diagrams can be specified from which parameters are directly given by the elements of the two-port matrices $\boldsymbol{G}$, $\boldsymbol{R}$, $\boldsymbol{H}$, $\boldsymbol{H}'$ (see Fig. 3.10).

Controlled sources can be realized with the help of nullators, norators, and Ohmic resistors. This is, for instance, illustrated in Fig. 3.11 for a VCCS.

Note that the controlled sources depicted in Fig. 3.9 are strictly linear.

Note that the controlled sources are neither passive nor reciprocal for non-zero gain factor, i.e., $g \neq 0$, $\beta \neq 0$, $\mu \neq 0$, or $r \neq 0$. For the trivial case of zero gain factor, the controlled sources reduce to trivial two-ports composed of open circuits or short circuits that are lossless and reciprocal. For example, the CCCS is a short circuit at port one and an open circuit at port two for $\beta = 0$. The corresponding hybrid 1 matrix is $\boldsymbol{H} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Thus, the CCCS with $\beta = 0$ is lossless (u_1 and i_2 are zero) and reciprocal ($h_{21} = -h_{12} = 0$).

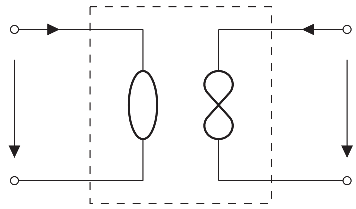


Figure 3.12: Equivalent Circuit Diagram of a Nullor

3.5.2 Nullor

The nullor is a strictly linear two-port with the following representation.

$$u_1 = 0\,\mathrm{V}$$
$$i_1 = 0\,\mathrm{A}.$$

No statements are given for u_2 and i_2. Thus,

$$u_2 = \text{arbitrary}$$
$$i_2 = \text{arbitrary}.$$

Therefore, it is clear that the two-port nullor can be substituted by the two one-ports nullator and norator (see Fig. 3.12). Out of the six two-port matrices, only the transmission matrix exists for the nullor

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \tag{3.27}$$

whose particular form is responsible for its name. At the same time, it can be recognized that each of the four different controlled sources (see Fig. (see Fig. 3.12) converges to the nullor when the associated control factor increases beyond all measure ($g, \beta, \mu, r \to \infty$). Note that the nullor as a two-port will be favorably employed as a linear equivalent circuit of the ideal operational amplifier (see Chapter 4).

3.5.3 Transformer

Any two-port that is lossless and reciprocal at the same time is an *ideal transformer*. If $\boldsymbol{R}$ or $\boldsymbol{G}$ exists, the two conditions lead to

$$\boldsymbol{R} = \mathbf{0}$$
$$\boldsymbol{G} = \mathbf{0}$$

respectively. Note that $\boldsymbol{R} = \mathbf{0}$ corresponds to two short circuits (CCVS with $r = 0$) and $\boldsymbol{G} = \mathbf{0}$ represents two open circuits (VCCS with $g = 0$). To obtain non-trivial cases for an ideal transformer, we must assume that $\boldsymbol{H}$ (or $\boldsymbol{H}'$) exists. For any reciprocal two-port whose hybrid matrix exists, (3.25) holds, i.e.,

$$\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I} = h_{11}\boldsymbol{i}_1\boldsymbol{i}_1^{\mathrm{T}} + h_{22}\boldsymbol{u}_2\boldsymbol{u}_2^{\mathrm{T}}.$$

Consequently, $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I} = \boldsymbol{I}^{\mathrm{T}}\boldsymbol{U}$. For losslessness, we must have that [see Eqn. (3.20)]

$$\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I} + \boldsymbol{I}^{\mathrm{T}}\boldsymbol{U} = \boldsymbol{0}.$$

Due to the symmetry of $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}$, this condition can be rewritten as $2\boldsymbol{U}^{\mathrm{T}}\boldsymbol{I} = \boldsymbol{0}$. Therefore, $h_{11} = h_{22} = 0$ is necessary for losslessness. Combining this result with the condition for reciprocity, i.e., $h_{21} = -h_{12}$, gives the hybrid matrix of an ideal transformer

$$\boldsymbol{H}_{\text{transf}} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix}. \tag{3.28}$$

Note that $\boldsymbol{H}_{\text{transf}}$ is skew-symmetric, i.e., $\boldsymbol{H}_{\text{transf}}^{\mathrm{T}} = -\boldsymbol{H}_{\text{transf}}$. Additionally, note that $\boldsymbol{H}_{\text{transf}} = \boldsymbol{0}$ for $n = 0$ (CCCS with $\beta = 0$) is a short circuit at port one and an open circuit at port 2. Alternatively, we can find the inverse hybrid matrix of an ideal transformer

$$\boldsymbol{H}'_{\text{transf}} = \begin{bmatrix} 0 & -\frac{1}{n} \\ \frac{1}{n} & 0 \end{bmatrix}. \tag{3.29}$$

Note that $\boldsymbol{H}'_{\text{transf}}$ is skew-symmetric, i.e., $\boldsymbol{H}'^{,\mathrm{T}}_{\text{transf}} = -\boldsymbol{H}'_{\text{transf}}$. For $n \to \infty$, $\boldsymbol{H}'_{\text{transf}} \to \boldsymbol{0}$ (VCVS with $\mu = 0$).

In scalar form, we get

$$\begin{aligned} u_1 - nu_2 &= 0 \\ ni_1 + i_2 &= 0. \end{aligned}$$

These equations illustrate that neither $\boldsymbol{G}$ nor $\boldsymbol{R}$ exist (except for the trivial cases $\boldsymbol{G} = \boldsymbol{0}$ and $\boldsymbol{R} = \boldsymbol{0}$). The transmission matrices of the transformer read as

$$\boldsymbol{A}_{\text{transf}} = \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}'_{\text{transf}} = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & n \end{bmatrix}. \tag{3.30}$$

Note that the ideal transformer is reversible for $n = \pm 1$.

The ideal transformer is the idealized model of a *real transformer* whose *transformation ratio* n is determined by the ratio of the primary to secondary number of turns. The constitutive equations of the implicit representation, as well as the derived two-port matrices are valid by definition for all time responses $u(t)$ and $i(t)$ (also for direct current or direct voltage!).

If the transformer is connected to a strictly linear resistor with resistance R (see Fig. 3.13), the resulting characteristic at port one is again a strictly linear resistor with resistanc3e

$$R_{\text{transf}} = \frac{u_1}{i_1} = n^2 R.$$

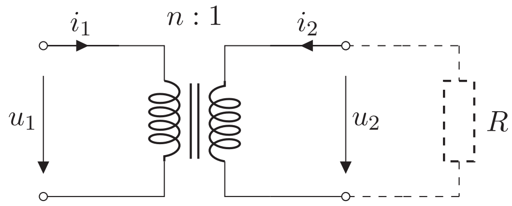


Figure 3.13: Transformer

3.5.4 Gyrator

In Subsection 3.5.3, we have derived the ideal transformer as the class of two-ports that are lossless and reciprocal at the same time. As a consequence, the conductance matrix $\boldsymbol{G}$ is the zero matrix or the resistance matrix $\boldsymbol{R}$ is the zero matrix, or both matrices do not exist. In other words, the conductance matrix and the resistance matrix do not exist at the same time for an ideal transformer.

In the following, we find the gyrator as the lossless two-port with the property that both, the conductance and the resistance matrix, exist. From the condition of losslessness in (3.20) and $\boldsymbol{U} = \boldsymbol{R}\boldsymbol{I}$, we find

$$\boldsymbol{I}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{I} + \boldsymbol{I}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{I} = \boldsymbol{0}$$

and infer that

$$\boldsymbol{R} = -\boldsymbol{R}^{\mathrm{T}}$$

must hold for losslessness, i.e., the resistance matrix $\boldsymbol{R}$ must be skew-symmetric and can be written as

$$\boldsymbol{R}_{\text{gyrator}} = \begin{bmatrix} 0 & -R_{\mathrm{d}} \\ R_{\mathrm{d}} & 0 \end{bmatrix}. \tag{3.31}$$

Note that $\det \boldsymbol{R}_{\text{gyrator}} = R_{\mathrm{d}}^2$ and that $\boldsymbol{R}_{\text{gyrator}}$ is invertible, i.e., $\boldsymbol{G}_{\text{gyrator}} = \boldsymbol{R}_{\text{gyrator}}^{-1}$ exists, as long as $R_{\mathrm{d}} \neq 0$ (but for $R_{\mathrm{d}} = 0$, we would have a transformer). The corresponding conductance matrix reads as

$$\boldsymbol{G}_{\text{gyrator}} = \boldsymbol{R}_{\text{gyrator}}^{-1} = \begin{bmatrix} 0 & G_{\mathrm{d}} \\ -G_{\mathrm{d}} & 0 \end{bmatrix} \tag{3.32}$$

with $G_{\mathrm{d}} = \frac{1}{R_{\mathrm{d}}}$. With Table 3.2, it can be easily seen that the hybrid matrices do not exist for the gyrator since $r_{11} = r_{22} = 0$. The transmission matrices of the gyrator are

$$\boldsymbol{A}_{\text{gyrator}} = \begin{bmatrix} 0 & R_{\mathrm{d}} \\ \frac{1}{R_{\mathrm{d}}} & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}'_{\text{gyrator}} = \begin{bmatrix} 0 & -R_{\mathrm{d}} \\ -\frac{1}{R_{\mathrm{d}}} & 0 \end{bmatrix}. \tag{3.33}$$

When the gyrator is connected to a resistive two-terminal $\mathcal{F}$ at port two (see Fig. 3.14), the two-terminal obtained at port one is

$$\mathcal{F}_{\text{Gyr}} = \left\{ (u_1, i_1) \,|(u_2, -i_2) = \left(i_1 R_{\mathrm{d}}, \frac{u_1}{R_{\mathrm{d}}} \right) \in \mathcal{F} \right\}.$$

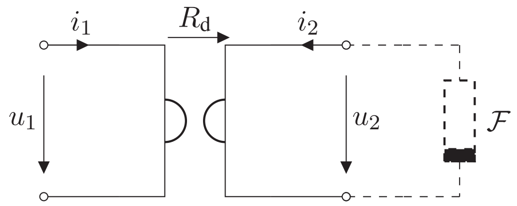


Figure 3.14: Gyrator

Therefore, the characteristic at port one is given by

$$\mathcal{F}_{\text{Gyr}} = \mathcal{F}^{\text{d}}$$

i.e., the one-port $\mathcal{F}^{\text{d}}$ dual to $\mathcal{F}$ w.r.t. the duality constant R_{d}.

If a two-port is connected to a gyrator at port one and to another gyrator at port two (both with the same gyrator resistance R_{d}), the new two-port is the dual two-port of the original two-port w.r.t. the duality constant R_{d}.

Based on above matrix representations $\boldsymbol{G}_{\text{gyrator}}$ and $\boldsymbol{R}_{\text{gyrator}}$, the gyrator can be realized by two VCCS or by two CCVS, respectively. The controlled sources can again be realized with the help of Ohmic resistors and nullors (thus, with operational amplifiers).

3.5.5 Negative-Immittance-Converter

The *negative-immittance-converter* (NIC) is characterized by

$$\begin{aligned} u_1 &= -k u_2 \\ i_1 &= -\frac{1}{k} i_2 \end{aligned}$$

with $k \in \mathbb{R}$.

The NIC is therefore neither voltage- nor current-controlled, i.e., neither the conductance matrix $\boldsymbol{G}$ nor the resistance matrix $\boldsymbol{R}$ do exist. The two-port matrices of the NIC are given by

$$\boldsymbol{H}_{\text{NIC}} = \begin{bmatrix} 0 & -k \\ -k & 0 \end{bmatrix} \qquad \boldsymbol{H}'_{\text{NIC}} = \begin{bmatrix} 0 & -\frac{1}{k} \\ -\frac{1}{k} & 0 \end{bmatrix} \tag{3.34}$$

$$\boldsymbol{A}_{\text{NIC}} = \begin{bmatrix} -k & 0 \\ 0 & \frac{1}{k} \end{bmatrix} \qquad \boldsymbol{A}'_{\text{NIC}} = \begin{bmatrix} -\frac{1}{k} & 0 \\ 0 & k \end{bmatrix}. \tag{3.35}$$

The NIC is active and symmetric for $|k| = 1$. In Fig. 3.15, possible realizations of the NIC are depicted.

Connecting the NIC to a resistive one-port $\mathcal{F}$ at port 2, we get at port 1

$$\mathcal{F}_{\text{NIC}} = \left\{ (u_1, i_1) \,|\, (u_2, -i_2) = \left(-\frac{u_1}{k}, i_1 k \right) \in \mathcal{F} \right\}.$$

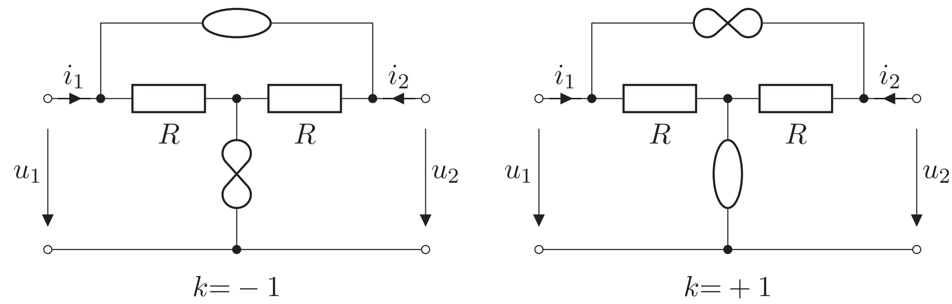


Figure 3.15: Nullor Realizations of a NIC with $k = \pm 1$

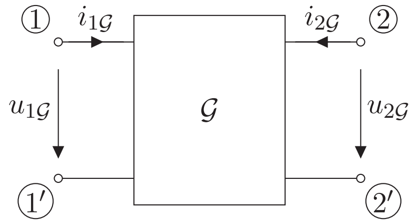


Figure 3.16: Equivalent Two-Port $\mathcal{G}$ of the Two-Port Connections

For $k = 1$, this is the characteristic of $\mathcal{F}$ mirrored w.r.t. the i_1-axis. For $k = -1$, it is the two-terminal mirrored w.r.t. the u_1-axis. If $\mathcal{F}$ lies entirely in the 1st and 3rd quadrants of the u-i-plane, then $\mathcal{F}_{\text{NIC}}$ lies entirely in the 2nd and 4th quadrants.

If $\mathcal{F}$ is an Ohmic resistor with slope $\frac{1}{R}$, then $\mathcal{F}_{\text{NIC}}$ is a negative resistance with the slope $-\frac{1}{R}$. From this property of converting a positive resistance into a negative resistance comes the name of the two-port. The word *immittance* is a combination of impedance and admittance, in our case, resistance and conductance.

3.6 Connection of Two-Ports

Analogous to the discussed interconnections for one-ports, one can also consider two suitable two-ports, $\mathcal{F}_1$ and $\mathcal{F}_2$, connected to each other again as a single "equivalent two-port" $\mathcal{G}$ (see Fig. 3.16).

There are in total five different fundamental possibilities to interconnect two two-ports that, as for one-ports, correspond to one of the two-port representation forms.

3.6.1 Parallel Connection

For the *parallel connection* depicted in Fig. 3.17, both, the input and the output ports, of the two two-ports are connected in parallel (similar to the parallel connection of one-ports).

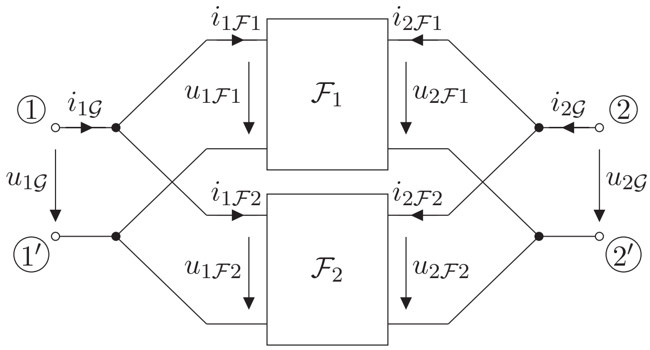


Figure 3.17: Parallel Connection of Two-Ports

By the Kirchhoff's voltage law, we get

$$u_{1\mathcal{G}} = u_{1\mathcal{F}1} = u_{1\mathcal{F}2}$$
$$u_{2\mathcal{G}} = u_{2\mathcal{F}1} = u_{2\mathcal{F}2}$$

that can be written in vector notation with the port voltage vectors $\boldsymbol{u}_\mathcal{G} = \left[\begin{smallmatrix} u_{1\mathcal{G}} \\ u_{2\mathcal{G}} \end{smallmatrix}\right]$, $\boldsymbol{u}_{\mathcal{F}1} = \left[\begin{smallmatrix} u_{1\mathcal{F}1} \\ u_{2\mathcal{F}1} \end{smallmatrix}\right]$, and $\boldsymbol{u}_{\mathcal{F}2} = \left[\begin{smallmatrix} u_{1\mathcal{F}2} \\ u_{2\mathcal{F}2} \end{smallmatrix}\right]$:

$$\boldsymbol{u}_\mathcal{G} = \boldsymbol{u}_{\mathcal{F}1} = \boldsymbol{u}_{\mathcal{F}2}. \tag{3.36}$$

Similarly, the Kirchhoff's current law leads to

$$i_{1\mathcal{G}} = i_{1\mathcal{F}1} + i_{1\mathcal{F}2}$$
$$i_{2\mathcal{G}} = i_{2\mathcal{F}1} + i_{2\mathcal{F}2}$$

or based on the port current vectors $\boldsymbol{i}_\mathcal{G} = \begin{bmatrix} i_{1\mathcal{G}} \\ i_{2\mathcal{G}} \end{bmatrix}$, $\boldsymbol{i}_{\mathcal{F}1} = \begin{bmatrix} i_{1\mathcal{F}1} \\ i_{2\mathcal{F}1} \end{bmatrix}$, and $\boldsymbol{i}_{\mathcal{F}2} = \begin{bmatrix} i_{1\mathcal{F}2} \\ i_{2\mathcal{F}2} \end{bmatrix}$,

$$\boldsymbol{i}_\mathcal{G} = \boldsymbol{i}_{\mathcal{F}1} + \boldsymbol{i}_{\mathcal{F}2} \tag{3.37}$$

Assuming that the port conditions for $\mathcal{F}_1$ and $\mathcal{F}_2$ are fulfilled, thus, the four-terminals can be interpreted as two-ports, and assuming that the two-ports are voltage-controlled, i.e.,

$$\boldsymbol{i}_{\mathcal{F}1} = \boldsymbol{g}_{\mathcal{F}1}(\boldsymbol{u}_{\mathcal{F}1})$$
$$\boldsymbol{i}_{\mathcal{F}2} = \boldsymbol{g}_{\mathcal{F}2}(\boldsymbol{u}_{\mathcal{F}2})$$

we find

$$\boldsymbol{i}_\mathcal{G} = \boldsymbol{i}_{\mathcal{F}1} + \boldsymbol{i}_{\mathcal{F}2} = \boldsymbol{g}_{\mathcal{F}1}(\boldsymbol{u}_\mathcal{G}) + \boldsymbol{g}_{\boldsymbol{F}2}(\boldsymbol{u}_\mathcal{G}) = \boldsymbol{g}_\mathcal{G}(\boldsymbol{u}_\mathcal{G})$$

which can be concisely be written as

$$\boldsymbol{g}_\mathcal{G} = \boldsymbol{g}_{\mathcal{F}1} + \boldsymbol{g}_{\mathcal{F}2}. \tag{3.38}$$

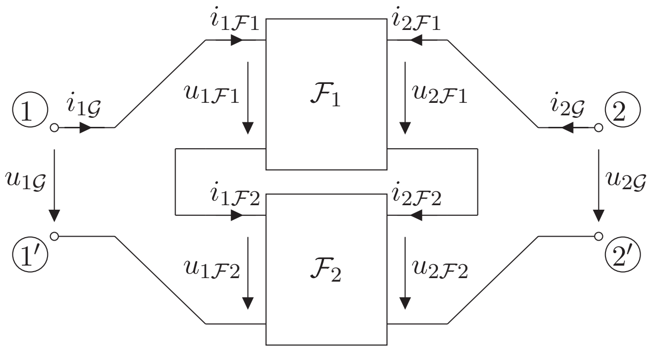


Figure 3.18: Series Connection of Two-Ports

For the strictly linear case, assume that the conductance matrices $\boldsymbol{G}_{\mathcal{F}1}$ and $\boldsymbol{G}_{\mathcal{F}2}$ exist. With

$$\boldsymbol{i}_{\mathcal{F}1} = \boldsymbol{G}_{\mathcal{F}1}\boldsymbol{u}_{\mathcal{F}1}$$
$$\boldsymbol{i}_{\mathcal{F}2} = \boldsymbol{G}_{\mathcal{F}2}\boldsymbol{u}_{\mathcal{F}2}$$

we can obtain with (3.37)

$$\boldsymbol{i}_{\mathcal{G}} = \boldsymbol{i}_{\mathcal{F}1} + \boldsymbol{i}_{\mathcal{F}2} = \boldsymbol{G}_{\mathcal{F}1}\boldsymbol{u}_{\mathcal{F}1} + \boldsymbol{G}_{\mathcal{F}2}\boldsymbol{u}_{\mathcal{F}2} = \boldsymbol{G}_{\mathcal{F}1}\boldsymbol{u}_{\mathcal{G}} + \boldsymbol{G}_{\mathcal{F}2}\boldsymbol{u}_{\mathcal{G}} = \boldsymbol{G}_{\mathcal{G}}\boldsymbol{u}_{\mathcal{G}}$$

The parallel connection of two strictly linear two-ports leads to the sum of the conductance matrices, that is,

$$\boldsymbol{G}_{\mathcal{G}} = \boldsymbol{G}_{\mathcal{F}1} + \boldsymbol{G}_{\mathcal{F}2}. \tag{3.39}$$

3.6.2 Series Connection

For the series connection in Fig. 3.18, the ports 1 and the ports 2 are connected in series.

The series connection of two two-ports is dual to the parallel connection of two two-ports, since in the vector-values Kirchhoff's equation currents and voltages have exchanged roles:

$$\boldsymbol{i}_{\mathcal{G}} = \boldsymbol{i}_{\mathcal{F}1} = \boldsymbol{i}_{\mathcal{F}2} \tag{3.40}$$
$$\boldsymbol{u}_{\mathcal{G}} = \boldsymbol{u}_{\mathcal{F}1} + \boldsymbol{u}_{\mathcal{F}2}. \tag{3.41}$$

To combine the two two-ports, the current-controlled representations are employed for the series connection. Assuming that the port conditions are fulfilled, we get [cf. (3.38)]

$$\boldsymbol{r}_{\mathcal{G}} = \boldsymbol{r}_{\mathcal{F}1} + \boldsymbol{r}_{\mathcal{F}2}. \tag{3.42}$$

For the strictly linear case, we obtain for the resistance matrix of $\mathcal{G}$

$$\boldsymbol{R}_{\mathcal{G}} = \boldsymbol{R}_{\mathcal{F}1} + \boldsymbol{R}_{\mathcal{F}2}. \tag{3.43}$$

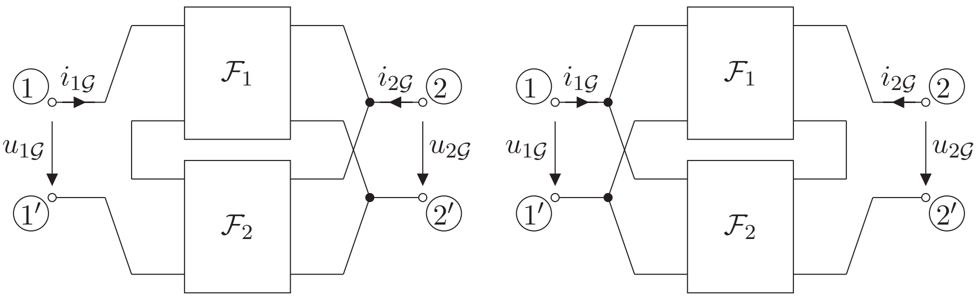


Figure 3.19: Series-Parallel- (left) und Parallel-Series-Connection (right) of Two Two-Ports

3.6.3 Hybrid Connections

In a *hybrid* or *mixed interconnection*, one of the port pairs is connected in series and the other pair is connected in parallel. In line with its unequivocal nomenclature, both performed interconnections are explicitly specified: first the one of the input, then the one of the output. For this reason there exist the following two hybrid connections:

series-parallel-connection The ports 1 are connected in series but the ports 2 are connected in parallel.

parallel-series-connection The ports 1 are connected in parallel but the ports 2 are connected in series.

Both cases are depicted in Fig. 3.19.

The treatment is in full analogy to the analysis of parallel and series circuits. The single substantial difference is that now "mixed" vectors of operating variables that contain both, voltage as well as current components, appear.

For the series-parallel connection, we get

$$\boldsymbol{h}_{\mathcal{G}} = \boldsymbol{h}_{\mathcal{F}1} + \boldsymbol{h}_{\mathcal{F}2} \tag{3.44}$$

and for the parallel-series connection, the inverse hybrid representations must be summed up, i.e.,

$$\boldsymbol{h}'_{\mathcal{G}} = \boldsymbol{h}'_{\mathcal{F}1} + \boldsymbol{h}'_{\mathcal{F}2} \tag{3.45}$$

For strictly linear two-ports, the series-parallel connection leads to

$$\boldsymbol{H}_{\mathcal{G}} = \boldsymbol{H}_{\mathcal{F}1} + \boldsymbol{H}_{\mathcal{F}2} \tag{3.46}$$

and the parallel-series connection to

$$\boldsymbol{H}'_{\mathcal{G}} = \boldsymbol{H}'_{\mathcal{F}1} + \boldsymbol{H}'_{\mathcal{F}2}. \tag{3.47}$$

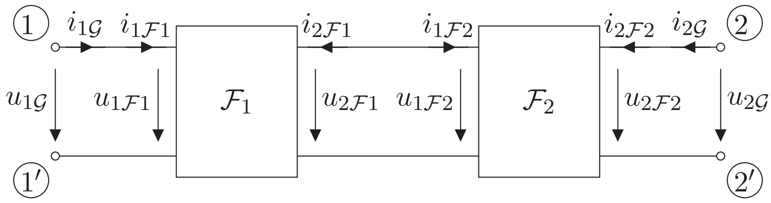


Figure 3.20: Cascade Connection of Two Two-Ports

3.6.4 Cascade Connection

The connection shown in Fig. 3.20 is called cascade connection. Because of Kirchhoff's current law and when the entire circuit is operated as a two-port, the port conditions of the circuit are automatically fulfilled.

An analysis is immediately possible with the transmission representations of the two two-ports and leads to

$$\begin{bmatrix} u_{1\mathcal{G}} \\ i_{1\mathcal{G}} \end{bmatrix} = \begin{bmatrix} u_{1\mathcal{F}1} \\ i_{1\mathcal{F}1} \end{bmatrix} = \boldsymbol{a}_{\mathcal{F}1}\left(\begin{bmatrix} u_{2\mathcal{F}1} \\ -i_{2\mathcal{F}1} \end{bmatrix}\right) = \boldsymbol{a}_{\mathcal{F}1}\left(\begin{bmatrix} u_{1\mathcal{F}2} \\ i_{1\mathcal{F}2} \end{bmatrix}\right) = \boldsymbol{a}_{\mathcal{F}1}\left(\boldsymbol{a}_{\mathcal{F}2}\left(\begin{bmatrix} u_{2\mathcal{G}} \\ -i_{2\mathcal{G}} \end{bmatrix}\right)\right)$$

due to Kirchhoff's current law applied to the connection of port two of $\mathcal{F}_1$ and port one of $\mathcal{F}_2$. Therefore,

$$\boldsymbol{a}_{\mathcal{G}} = \boldsymbol{a}_{\mathcal{F}1} \circ \boldsymbol{a}_{\mathcal{F}2}. \tag{3.48}$$

In the strictly linear case, we get

$$\boldsymbol{A}_{\mathcal{G}} = \boldsymbol{A}_{\mathcal{F}1}\boldsymbol{A}_{\mathcal{F}2} \tag{3.49}$$

with the transmission matrices $\boldsymbol{A}_{\mathcal{F}1}$ and $\boldsymbol{A}_{\mathcal{F}2}$ of the two two-ports.

Equivalently, the inverse transmission representation can be employed. Thus,

$$\boldsymbol{a}'_{\mathcal{G}} = \boldsymbol{a}'_{\mathcal{F}2} \circ \boldsymbol{a}'_{\mathcal{F}1} \tag{3.50}$$

$$\boldsymbol{A}'_{\mathcal{G}} = \boldsymbol{A}'_{\mathcal{F}2}\boldsymbol{A}'_{\mathcal{F}1}. \tag{3.51}$$

Chapter 4

Operational Amplifier

The term *operational amplifier* (Op-Amp) was coined in May 1947 [1] for high-quality differential amplifiers. These differential amplifiers had been developed for the implementation of analog arithmetic operations and were found to be applicable for the circuits (so-called analog computers) that serve for analog simulation of integro-differential equations.

4.1 Modelling

Real operational amplifiers are actually reactive elements. However, they can be regarded as resistive elements if the signals change slowly enough so that it is possible and useful to develop the idealized and purely resistive models which will be discussed in the following.

4.1.1 Idealized Non-Linear Model

By a simplification, one can reach the ideal operational amplifier whose element-symbol is denoted by ∞ (see Fig. 4.1a).

As shown in Fig. 4.1b, the transfer characteristic of the real Op-Amp, which is shifted to the origin by the eliminating the offset variables, is in addition approximated in the linear region by a vertical line that corresponds to the implementation of the limit $A_0 \to \infty$. The output voltage fulfills the following conditions over the three branches of this piecewise linear idealized characteristic:

$$\begin{aligned} u_2 &= -U_{\text{sat}} && \text{for} \quad u_\text{d} < 0 \Leftrightarrow u_{1+} < u_{1-} && \text{(I)} \\ |u_2| &\leq U_{\text{sat}} && \text{for} \quad u_\text{d} = 0 \Leftrightarrow u_{1+} = u_{1-} && \text{(II)} \\ u_2 &= U_{\text{sat}} && \text{for} \quad u_\text{d} > 0 \Leftrightarrow u_{1+} > u_{1-}. && \text{(III)} \end{aligned} \tag{4.1}$$

[1] In the paper “Analysis of Problems in Dynamics by Electronic Circuits” by J. R. Ragazzini, R. H. Randall and F. A. Russell, Proceedings of the IRE, 35(5), pp. 444–452, May 1947.

Figure 4.1: Symbol of an ideal Op-Amp (a) and its transfer characteristic (b)

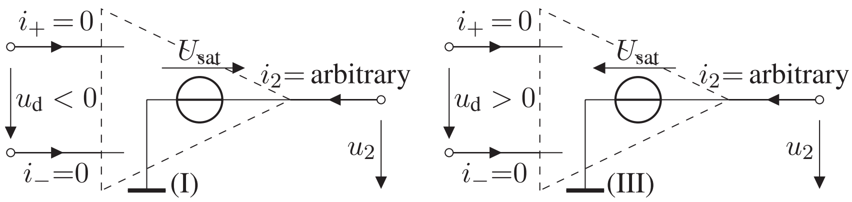


Figure 4.2: Equivalent circuit diagrams for an Op-Amp in saturation

4.1.2 Saturation Regions

In saturation, the current at port 1 of the Op-Amp is zero but the voltage at port 1 is different from zero. Therefore, the appropriate model for port 1 is an open circuit with the voltage u_d (negative for negative saturation and positive for positive saturation). As the output voltage is constant ($-U_{sat}$ for negative saturation and $+U_{sat}$ for positive saturation) irrespective of the output current, the appropriate model for port 2 is a voltage source. The resulting equivalent circuits are depicted in Fig. 4.2.

4.1.3 Nullor Model

The linear region is characterized by

$$u_d = 0.$$

Following the idealized model, it holds that

$$i_+ = i_- = 0.$$

Therefore, the port 1 of the idealized Op-Amp can be adequately be modelled by a nullator.

However, the process of the voltage transfer characteristic in the linear region does not allow any conclusion about the output voltage (apart from the little informative condition $|u_2| \leq U_{sat}$). Since no information about the output current can be drawn from the negligible small output resistance, the output port behaves like a norator.

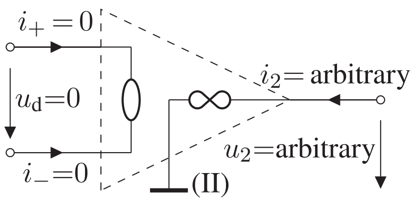


Figure 4.3: Nullor model of an ideal Op-Amp

Therefore, the operational amplifier can be described by a *nullor* in the linear region of the characteristic: a two-port where the input port is connected to a nullator and the output port to a norator (see Fig. 4.3).

Note that this nullor model is not applicable if the outer connection does not allow any feedback operation from the output variable-pair (u_2, i_2) to the input variable-pair $(u_\mathrm{d}, i_+ = -i_-)$. In such a case, it certainly means that within the frame of this connection the operational amplifier is no longer operated in the linear region, but rather in saturation. This implies an abandonment of the validity of the nullor-model.

Through the practical analysis of non-linear operational amplifier circuits, all solutions can be determined for each of the three operating regions using, in each case, suitable equivalent circuits. By means of the inequalities which characterize the characteristic branches, we can examine whether they also actually lie inside the provided operating region, and thus, also correspond to a solution of the real circuit.

4.2 Op-Amp Circuits

All Op-Amp circuits are in principle non-linear (mostly piecewise linear), if one does not particularly ensure that they are only operated in a linear region around the operating point. In many practical applications, the non-linear properties of the Op-Amp are used advantageously.

4.2.1 Inverting Amplifier

The circuit in Fig. 4.4 is the so-called inverting amplifier.

Linear analysis:

For this arrangement the following equations are written which are valid for the operation of the Op-Amp in the strictly linear region (branch II of the characteristic):

$$\begin{aligned}&u_1 - i_1 R_1 = 0\\&u_0 + i_0 R_0 = 0\\&i_1 = i_0.\end{aligned}$$

By elimination of the currents, we have

$$u_1 = -\frac{u_0}{R_0} R_1$$

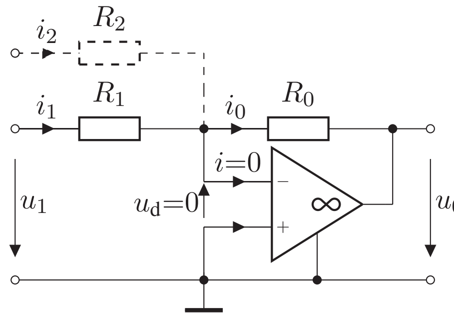


Figure 4.4: Inverting amplifier

The *voltage gain* v_u of the circuit accordingly is

$$v_u = \frac{u_0}{u_1} = -\frac{R_0}{R_1}. \tag{4.2}$$

If further coupling resistors R_2, R_3, ... (as shown by dashed lines in the figure) are added with an auxiliary current i_2, i_3, ... injected in the virtual ground-point in each case, the sum of these currents enters in the feedback-resistor R_0 and thus the output voltage reads as

$$u_0 = -\left(i_1 + i_2 + i_3 + \ldots\right) R_0.$$

Or, in terms of the input voltages $u_1, u_2, u_3 \ldots$ from each input to the ground:

$$u_0 = -\left(\frac{u_1}{R_1} + \frac{u_2}{R_2} + \frac{u_3}{R_3} + \ldots\right) R_0. \tag{4.3}$$

Thus, the output voltage of the inverting amplifier is the negative of a linear combination of the input currents or voltages with positive coefficients (all resistors are Ohmic!).

Non-linear analysis:

In the linear analysis, the polarity of the input port of the operational amplifier seems to have no influence for the moment, since the nullator, which is used to model the input, is non-polarized. However, the Op-Amp is overall a non-linear element, so that the existence of an operating point in the saturation regions must be examined as well.

Considering the boundary properties of the Op-Amp actually leads to two different transfer characteristics of the inverting amplifier. In Fig. 4.5 , the characteristic depicted on the left corresponds to the “correct” polarity of the Op-Amp input port which is shown in Fig. 4.4, while the one on the right is obtained with the “wrong” polarity of the input.

Although the characteristics are identical in the linear region, the distinct locations of the branches which belong to the boundary regions lead to the fact that only the output voltage in the characteristic shown on the left represents a bijective function of the input voltage, while the right one is ambiguous: for example, for $u_1 = 0$ there are altogether three possible output

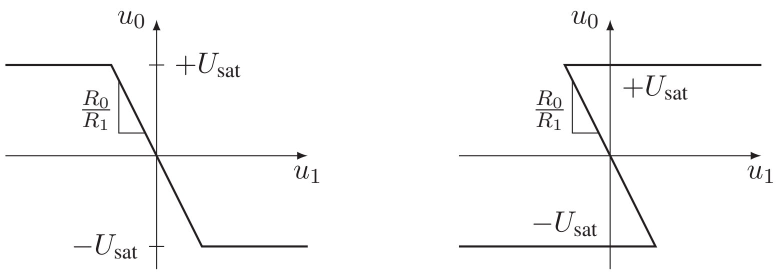


Figure 4.5: Characteristics of an inverting amplifier with correct and wrong polarity

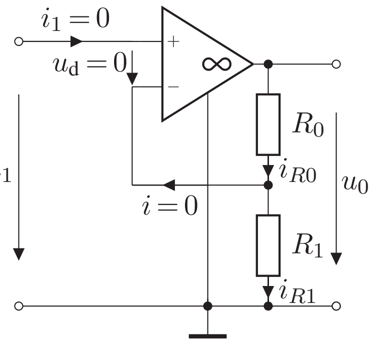


Figure 4.6: Non-inverting amplifier

voltages, namely $u_0 = +U_{\text{sat}}$, $u_0 = 0$, or $u_0 = -U_{\text{sat}}$. It is impossible to predict which one will actually occur.

If the wrongly polarized circuit (reversed input port) is constructed in a lab, the resulting measurement of the output voltage is always one of the voltages $+U_{\text{sat}}$ or $-U_{\text{sat}}$ and the corresponding solutions are *stable*. Note that the output voltage 0 will never be obtained and the corresponding operating point is unstable. Thus, the wrongly polarized inverting amplifier can not operate in the linear region at all!

4.2.2 Non-Inverting Amplifier

The circuit in Fig. 4.6) is a non-inverting amplifier.

In the linear region, the corresponding equations are apparently

$$
\begin{aligned}
&u_0 - i_{R0}R_0 - i_{R1}R_1 = 0 \\
&u_1 - i_{R1}R_1 = 0 \\
&i_{R0} = i_{R1}.
\end{aligned}
$$

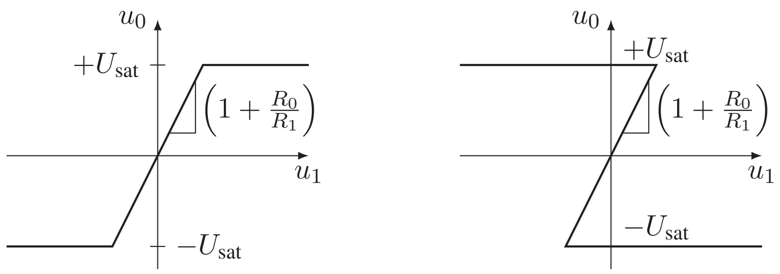


Figure 4.7: Characteristics of a non-inverting amplifier with correct and wrong polarity

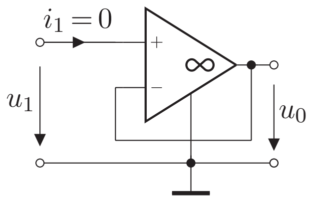


Figure 4.8: Voltage follower

Thus, the output voltage and voltage gain are:

$$\begin{aligned} u_0 &= i_{R1}(R_0 + R_1) = \frac{u_1}{R_1}(R_0 + R_1) \\ v_u &= \frac{u_0}{u_1} = 1 + \frac{R_0}{R_1}. \end{aligned} \tag{4.4}$$

Hence, the output voltage has the same sign as the input voltage.

By non-linear analysis, one obtains the characteristic for the correctly (as shown in Fig. 4.7) and wrongly polarized case.

4.2.2.1 Voltage Follower

A non-inverting amplifier with the gain $v_u = 1$ is named *voltage follower*, as the output always has the same voltage value as the input. The voltage follower does not load the source that is connected with its input ($i_1 = 0$), and it can supply any amount of current to a load which is connected to its output.

From Equation (4.4), the desired voltage gain $v_u = 1$ can be obtained through the limit $R_0 = 0$ and $R_1 \to \infty$, which corresponds to a replacement of R_0 by a short circuit and a replacement of R_1 by an open circuit. In this manner, the very simple circuit of the voltage follower in Fig. 4.6 can be obtained from the non-inverting amplifier (shown in Fig. 4.8).

4.2.3 Negative Immittance Converter (NIC)

An important application of the operational amplifier is the realization of a negative resistor by means of a negative-immittance converter. By operating the Op-Amp in the linear region, the

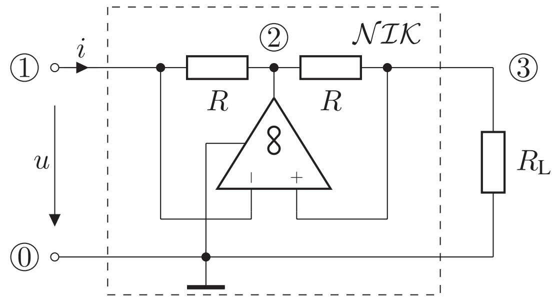


Figure 4.9: Realization of a negative resistor with an Op-Amp NIC

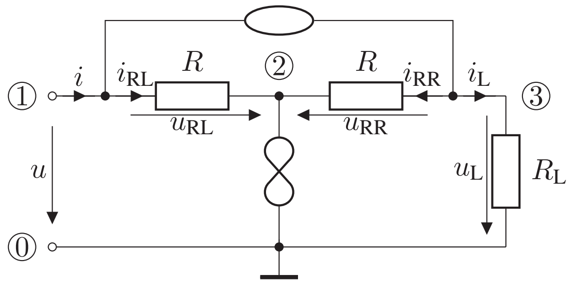


Figure 4.10: Nullor model for the NIC

characteristic of the circuit shown in Fig. 4.9 is a straight line with the negative resistance value $-R_{\mathrm{L}}$.

Linear analysis: Replacing the Op-Amp by the nullor model leads to the equivalent circuit in Fig. 4.10.

The behavior of this equivalent circuit is a consequence of the symmetry w.r.t. the nullor of the Op-Amp as will be seen in the following.

The nullator enforces that the voltages at the nodes ① and ③are the same:

$$u = u_{\mathrm{L}}.$$

Hence, for symmetry reasons, we have $u_{\mathrm{RR}} = u_{\mathrm{RL}}$. Thus, the values of the currents flowing through the two identical resistors R must be the same, i.e., $i_{\mathrm{RR}} = i_{\mathrm{RL}}$. Because no current flows through the nullator, it also holds that

$$i = i_{\mathrm{RL}}$$
$$i_{\mathrm{L}} = -i_{\mathrm{RR}}.$$

and $i = -i_{\mathrm{L}}$ With the Ohm's law for the load resistor R_{L}, i.e.,

$$u_{\mathrm{L}} = R_{\mathrm{L}} i_{\mathrm{L}}$$

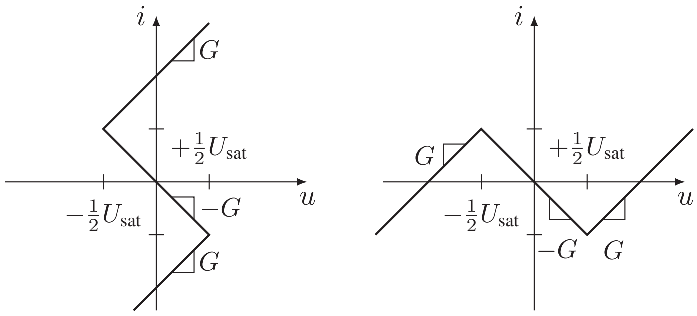


Figure 4.11: Characteristics of the Op-Amp realization of a negative resistor of S- and N-type

the input variables are connected via

$$u = -R_{\text{L}} i.$$

Since u and i are defined in terms of the associated reference direction, it follows that the input of the negative-immittance converter which is connected to an Ohmic load $R_{\text{L}} > 0$ behaves like a negative resistor with the negative resistance $-R_{\text{L}}$.

Non-linear analysis: For the NIC, we observe different behaviors depending on the polarity of the Op-Amp input port, namely the so-called S- and N-characteristic. Based on a discussion of the characteristics, we identify the so-called *short-circuit-unstable (open-circuit-stable)* and *open-circuit-unstable (short-circuit-stable)* negative resistors.

The following two-terminal characteristics are obtained through the analysis of the circuit (shown in Fig. 4.9) for a negative resistor using the idealized piecewise linear model and the connection on the output side with an ohmic conductance G.

The S-characteristic arises in polarity depicted above and the N-characteristic in the reversed one. For $i = 0$ (open circuit) and for $u = 0$ (short circuit) in each case there are again 3 possible operating points, which are not all stable.

4.2.4 Piecewise Linear Resistors

Using operational amplifiers and diodes, many piecewise linear resistors can be synthesized like the S- and N-characteristics of negative resistors as discussed in Subsection 4.2.3. Other examples are the elementary piecewise linear resistive two-terminals already discussed in Chapter 2, viz., the ideal diode, the concave resistors, and the convex resistors, which can all be constructed by combining diodes and Op-Amps.

4.2.4.1 Ideal Diodes

The circuit shown in Fig. 4.12 behaves w.r.t. its input terminal like an ideal diode in a very good approximation.

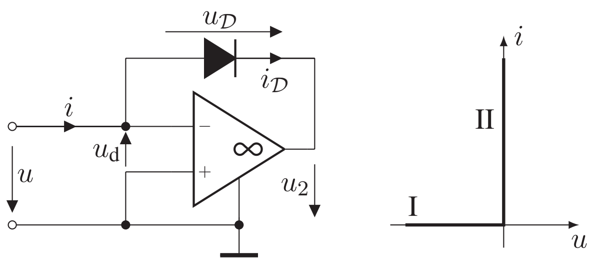


Figure 4.12: Op-Amp realization and characteristic of an ideal diode

With the aid of the piecewise linear Op-Amp model (for example, with $U_\mathrm{sat} = 12.5\,\mathrm{V}$) and the element equation

$$i_\mathcal{D} = I_\mathrm{s}\left(\exp\left(\frac{u_\mathcal{D}}{U_\mathrm{T}}\right) - 1\right)$$

for the pn-junction diode (e.g., with $U_\mathrm{T} = 25\,\mathrm{mV}$ and $I_\mathrm{s} = 1 \times 10^{-17}\,\mathrm{A}$), this circuit can be analyzed. To this end, the different cases corresponding to the three operating regions of the Op-Amp must be considered (see Fig. 4.1b).

Region I: Since $u_\mathrm{d} < 0$ or correspondingly $u > 0$, we have that $u_2 = -U_\mathrm{sat}$ and hence,

$$u_\mathcal{D} = -u_2 - u_\mathrm{d} > +U_\mathrm{sat}$$

Substituting into the element equation of the diode yields

$$\begin{aligned} i = i_\mathcal{D} &= I_\mathrm{s}\left(\exp\left(\frac{U_\mathrm{sat}}{U_\mathrm{T}}\right)\exp\left(\frac{u}{U_\mathrm{T}}\right) - 1\right) \\ &= 10^{-17}\,\mathrm{A}\cdot\left(\exp(+500)\exp\left(\frac{u}{U_\mathrm{T}}\right) - 1\right) \\ &\approx 10^{200}\,\mathrm{A}\cdot\exp\left(\frac{u}{U_\mathrm{T}}\right) \end{aligned}$$

which is totally unrealistic for the circuit in Fig. 4.12.

Region III: Since $u_\mathrm{d} > 0$ and thus, $u < 0$, we obtain $u_2 = U_\mathrm{sat}$ and the voltage drop over the diode is

$$u_\mathcal{D} = -u_2 - u_\mathrm{d} < -U_\mathrm{sat}$$

Plugging into the elment equation of the diode leads to

$$\begin{aligned} i = i_\mathcal{D} &= I_\mathrm{s}\left(\exp\left(-\frac{U_\mathrm{sat}}{U_\mathrm{T}}\right)\exp\left(\frac{u}{U_\mathrm{T}}\right) - 1\right) \\ &= 10^{-17}\,\mathrm{A}\cdot\left(\exp(-500)\exp\left(\frac{u}{U_\mathrm{T}}\right) - 1\right) \approx 0. \end{aligned}$$

The current in region III of the Op-Amp (region I of the characteristic in Fig. 4.12) is therefore equal to the negligible saturation current I_s of the diode.

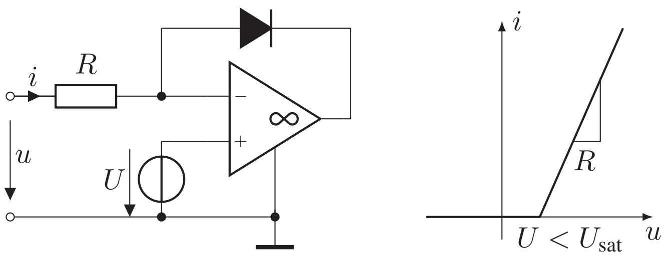


Figure 4.13: Op-Amp realization and characteristic of a concave resistor

Region II: In this region, the nullor model can be applied. However, the nullor model gives no information about u_2. Nevertheless, not all currents $i > I_{\text{s}}$ are possible due to the upper bound I_{max} for the output current of the Op-Amp. Hence, the range for the allowed currents is

$$I_{\text{max}} > i > -I_{\text{s}}$$

which is in a good approximation $i > 0$ as for an ideal diode. Consequently, we have found the region II of the characteristic in Fig. 4.12.

In order to realize an ideal diode with reversed polarity, it suffices to reverse the pn-diode in the circuit of Fig. 4.12.

4.2.4.2 Concave Resistors

Analogously to the realization of a concave resistor (which has been discussed in Subsection 2.7.1.2) as a serial connection of an Ohmic resistor, a voltage source, and an ideal diode, a practical equivalent Op-Amp circuit for a concave resistor is obtained as shown in Fig. 4.13.

The incorporation of the voltage source between the ground and the non-inverting input of the Op-Amp is advantageous because the usage of a voltage source is permitted, where one terminal is identical to the ground, e.g., a power supply, or which is sensitive to drawn currents, e.g., the output of a voltage divider.

4.2.4.3 Convex Resistors

Analogously, also an Op-Amp realization of a convex resistor can be constructed as depicted in Fig. 4.14.

As suggested above, the ideal diode should be included with reversed polarity by rotating the pn-diode.

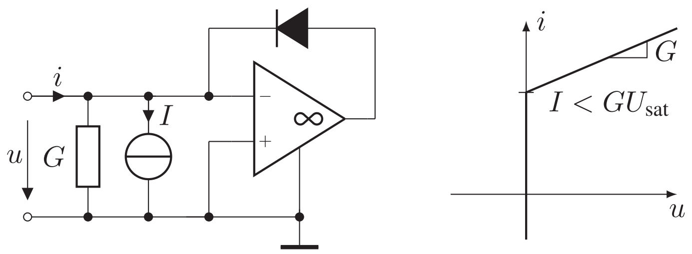


Figure 4.14: Op-Amp realization and characteristic of a convex resistor

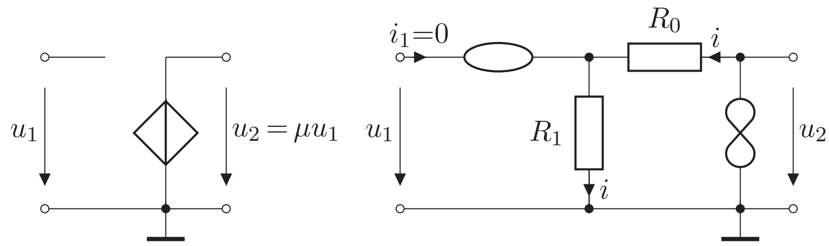


Figure 4.15: Nullor realization of a VCVS with transfer factor $\mu \geq 1$

4.3 Linear Op-Amp Circuits

If it is guaranteed by the outer connection that the operational amplifier is operated exclusively in the strictly linear region, i.e., region II in Fig. 4.1b, the Op-Amp can be modeled by one (strictly linear) nullor. The resuling circuit is called a linear Op-Amp circuit. In the following, only a few typical examples of linear Op-Amp circuits are considered among plenty of practically important circuits. By these examples, it is demonstrated that formulating the equations for analysis is simple due to the properties of the nullor.

4.3.1 Controlled Sources

Under the constraint that one terminal of the input or the output port must be constantly connected with the ground, there are altogether four types of controlled sources with Op-Amp realization: VCVS, CCVS, VCCS, and CCCS. For every controlled source, two Op-Amps are required in general. As the implementation of the VCCS and the CCCS is non-trivial also with Op-Amp circuits, we concentrate in the following on the VCVS and the CCVS.

4.3.1.1 Voltage Controlled Voltage Source (VCVS)

The nullor circuit shown on the right of Fig. 4.15 behaves like a voltage controlled voltage source (VCVS) with positive voltage transfer ratio $\mu = 1 + \frac{R_0}{R_1}$.

The ground reference of the norator allows this nullor circuit to be regarded as a non-inverting amplifier which is operated in the strictly linear region. This non-inverting amplifier can be used

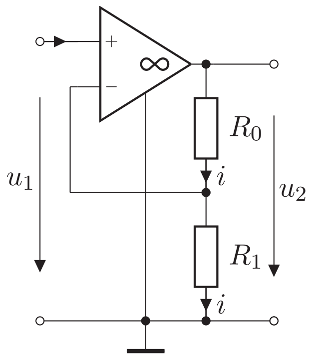


Figure 4.16: Op-Amp realization of a VCVS with $\mu \geq 1$

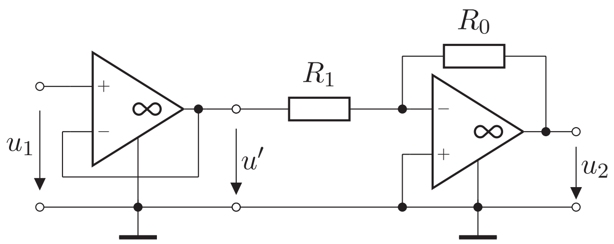


Figure 4.17: Op-Amp realization of a VCVS with $\mu < 0$

as an equivalent circuit for the voltage-controlled voltage source.

If a VCVS is realized with a negative voltage transfer ratio, i.e., $\mu < 0$, an inverting amplifier can be connected with a non-inverting amplifier or a voltage follower. The total voltage transfer factor will then be the product of the two voltage gains. For example, in the circuit shown in Fig. 4.17, the voltage follower at the input has a voltage gain of 1 and the voltage gain of the following inverting amplifier is $-\frac{R_0}{R_1}$. Therefore, the total voltage gain of the whole circuit is $\mu = \frac{u_2}{u_1} = -\frac{R_0}{R_1}$.

By including another inverting amplifier, it is possible to realize a voltage gain between zero and one, i.e., $\mu \in (0, 1)$. However, this solution for $0 < \mu < 1$ is not efficient because three Op-Amps are required. Instead, the cascade of a voltage follower, a voltage divider, and another voltage divider should be used to implement a voltage gain $0 < \mu < 1$ since only two Op-Amps are necessary.

4.3.1.2 Current Controlled Voltage Source (CCVS)

A CCVS with negative resistance r can be particularly easily realized (see Fig. 4.18). The corresponding Op-Amp circuit is the main part of an inverting amplifier (see Fig. 4.19).

In order to realize a positive resistance for the CCVS, i.e., $r > 0$, an CCVS with $r < 0$ is simply connected to an inverting amplifier.

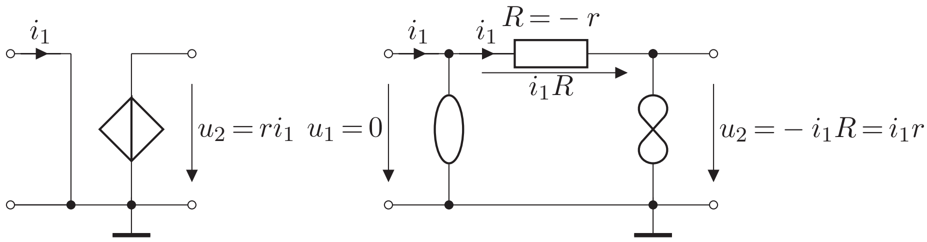


Figure 4.18: Nullor realization of a CCVS with $r < 0$

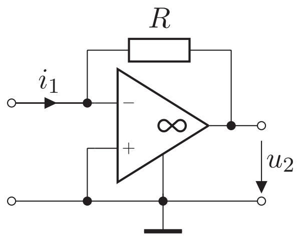


Figure 4.19: Op-Amp realization of a CCVS with $r < 0$

4.3.2 Gyrator

A gyrator can be obtained through the parallel connection of two VCCS or through the series connection of two CCVS. A third possibility of realization is based on the following factorization of its transmission matrix

$$\boldsymbol{A}_{\text{Gyr}} = \begin{bmatrix} 0 & R \\ \frac{1}{R} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & R \\ -\frac{1}{R} & 0 \end{bmatrix} = \boldsymbol{A}_{\text{NIC}} \cdot \boldsymbol{A}_{\text{NII}}.$$

This corresponds to the cascade connection of a NIC with $k = -1$ with a negative immittance inverter (NII) (see Fig. 4.20). An example of a corresponding Op-Amp circuit is shown in Fig. 4.21.

For the pairwise pooling of nullator and norator to a nullor, there are two possibilities. For the polarity of each Op-Amp input port there are two possibilities as well. From these total eight

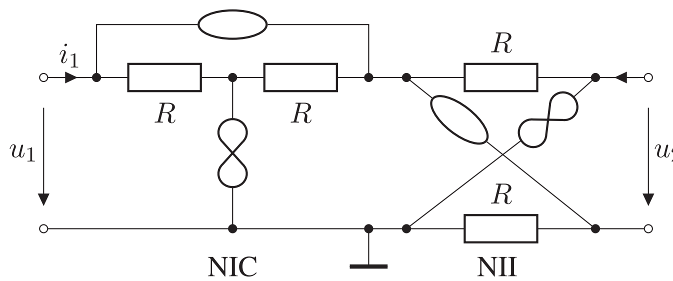


Figure 4.20: Gyrator consisting of a NIC and a NII connected in cascade

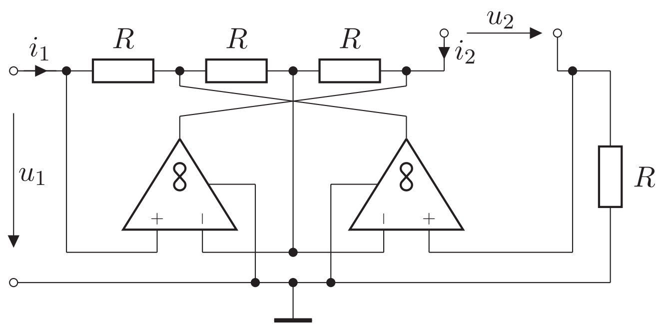


Figure 4.21: Op-Amp realization of the gyrator in Fig. 4.20

strictly linear nullor circuit equivalent circuits, the one represented here is the most qualified by the real Op-Amp for the realization of a gyrator.

Chapter 5

General Circuit Analysis

General circuit analysis techniques are systematic methods to symbolically or numerically compute the currents and the voltages in an electric circuit, depending on the interconnection structure and the element characteristics. By developing such methods, several aspects have to be taken into account:

- For computer-aided analysis, a formalized machine-accessible procedure is needed. Additionally, the underlying algorithm must be insensitive to rounding errors.
- For a manual symbolic analysis, a simple form consisting of a minimal number of equations is advantageous.

To develop general analysis techniques, a founded understanding of the interaction between the Kirchhoff's laws and the circuit elements is necessary. The first important step has been already done in Chapter 1 by introducing the Kirchhoff's laws.

In lumped circuits, the spatial configuration of the elements is irrelevant. Instead, only the circuitry itself matters. Therefore, we can separate the analysis of these Kirchhoff's networks into two steps:

- representation of the interconnection structure by Kirchhoff's laws and
- characterization of the circuit elements.

The representation of an circuits using incidence matrices reflects the interconnection structure and will be employed in this chapter. It enables the construction of tableau equation systems, which capture the Kirchhoff's laws and the element equations. When the Kirchhoff's voltage laws is incorporated in range space representation, we end up with the node analysis. By reducing the tableau equation system of the node analysis, such that it is an equation system only in the node voltages, the nodal analysis is formulated.

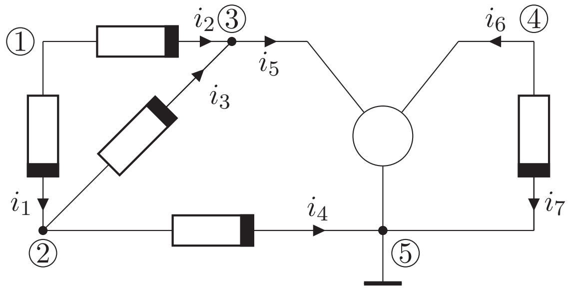


Figure 5.1: Sample circuit

5.1 Incidence Matrices

As discussed in Section 1.5, the number of linearly independent KVL equations is $b-(n-1)$ and the number of linearly independent KCL equations is $n-1$ for a circuit with n nodes and b branches.

In the following, a systematic method is introduced that delivers a unique form of the KCL in matrix-vector notation.

As the circuit comprises n nodes but only $n-1$ linearly independent KCL equations can be found, we formulate

- the KCL equations for all nodes except the reference node and
- give currents flowing out of the node a positive sign and currents flowing into the node a negative sign.

Then,

$$\boldsymbol{A}\boldsymbol{i} = \boldsymbol{0} \tag{5.1}$$

with the branch current vector

$$\boldsymbol{i} = [i_1, \ldots, i_b]^{\mathrm{T}} \in \mathbb{R}^b \tag{5.2}$$

and the node incidence matrix $\boldsymbol{A} \in \{-1,+1,0\}^{n-1\times b}$ comprising the coefficients for the $n-1$ linearly independent KCL equations. Following above convention, we have

$$a_{\beta\alpha} = \begin{cases} +1 & \text{if branch } \alpha \text{ leaves node } \beta \\ -1 & \text{if branch } \alpha \text{ enters node } \beta \\ 0 & \text{if branch } \alpha \text{ is not connected to node } \beta. \end{cases} \tag{5.3}$$

For example, the node incidence matrix for the circuit in Fig. 5.1 reads as

$$\boldsymbol{A} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix} \end{array} \tag{5.4}$$

where node ⑤ is ignored as it is the reference node. The node incidence matrix $\boldsymbol{A}$ represents the structure of the circuit. The KCL equation for the reference node is simply the negative sum of all the other KCL equations.

With the node incidence matrix from Eq. (5.4), we get the Kirchhoff's current law

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = $$

$$= \begin{bmatrix} i_1 & +i_2 & & & & & \\ -i_1 & & +i_3 & +i_4 & & & \\ & -i_2 & -i_3 & & +i_5 & & \\ & & & -i_4 & -i_5 & -i_6 & -i_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

As discussed in Section 1.5, the $n-1$ KCL equations are linearly independent.

When finding the KVL as in Eq. (1.3), the procedure is similar as described above. As there are $b-(n-1)$ linearly KVL equations for a circuit with n nodes and b branches, following form for the KVL can be obtained, i.e.,

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{0} \tag{5.5}$$

with the branch voltage vector

$$\boldsymbol{u} = [u_1, \dots, u_b]^{\mathrm{T}} \in \mathbb{R}^b \tag{5.6}$$

and the loop incidence matrix $\boldsymbol{B} \in \{-1, +1, 0\}^{b-(n-1)\times b}$ comprising the coefficients for the $b-(n-1)$ linearly independent KVL equations. For the circuit in Fig. 5.1, the loop incidence matrix reads as

$$\boldsymbol{B} = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

where the three loops were defined clock-wise. Note that $\boldsymbol{B}$ for Fig. 5.1 has got three rows whereas $\boldsymbol{A}$ has four rows. Therefore, the Kirchhoff's laws give overall seven equations (the same as the number of branches b).

Alternatively, the KVL can be found corresponding to Eqn. (1.2), that is, the branch voltages can be expressed depending on the node voltages $u_{\mathrm{k}1}, \dots, u_{\mathrm{k}(n-1)}$. As the branch voltages are the difference of the corresponding node voltages, e.g.,

$$u_{\gamma\delta} = u_{\mathrm{k}\gamma} - u_{\mathrm{k}\delta}$$

for the branch voltage $u_{\gamma\delta}$ of the branch starting in node γ and ending in node δ, we can define a matrix $\boldsymbol{M}$ whose elements are given by

$$m_{\alpha\beta} = \begin{cases} +1 & \text{branch } \alpha \text{ starts from node } \textcircled{\beta} \\ -1 & \text{branch } \alpha \text{ ends in node } \textcircled{\beta} \\ 0 & \text{branch } \alpha \text{ does not touch } \textcircled{\beta}. \end{cases}$$

Correspondingly, we obtain following form of the KVL:

$$\boldsymbol{u} = \boldsymbol{M}\boldsymbol{u}_{\mathrm{k}}$$

with the vector of node voltages $\boldsymbol{u}_{\mathrm{k}} = [u_{\mathrm{k}1}, \ldots, u_{\mathrm{k}(n-1)}]^{\mathrm{T}} \in \mathbb{R}^{n-1}$. Note that $m_{\alpha\beta} = a_{\beta\alpha}$. [cf. Eqn. (5.3)]. Therefore, it holds that $\boldsymbol{M} = \boldsymbol{A}^{\mathrm{T}}$. The node voltages $\boldsymbol{u}_{\mathrm{k}}$ are mapped to the branch voltages $\boldsymbol{u}$ by the transpose of the node incidence matrix, that is,

$$\boldsymbol{u} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{k}}. \tag{5.7}$$

In other words, the discussed method to formulate the KCL equations based on the node incidence matrix delivers also the basis matrix for above KVL depending on the node voltages. For example, the KVL based on the node voltages for the circuit in Fig. 5.1 reads as [the basis matrix is the transpose of node incidence matrix in Eq. (5.4)]

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_{\mathrm{k}1} \\ u_{\mathrm{k}2} \\ u_{\mathrm{k}3} \\ u_{\mathrm{k}4} \end{bmatrix}.$$

Thus, the node incidence matrix enables to formulate the KCL and the KVL

$$\boldsymbol{A}\boldsymbol{i} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{k}} \tag{5.8}$$

respectively, where the node incidence matrix $\boldsymbol{A}$ is found via the instructions given in Eqn. (5.3).

5.1.1 Tellegen's Theorem

Above results lead to a fundamental relation between the branch voltage and the branch current vector called *Tellegen's theorem*. Consider the inner product of the branch voltage and the branch current vectors, i.e., $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{i}$, and substitute the representation of the branch voltage vector depending on the node voltage vector [see Eqn. (5.8)] to obtain

$$\boldsymbol{u}^{\mathrm{T}}\boldsymbol{i} = \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{k}}\right)^{\mathrm{T}}\boldsymbol{i} = \boldsymbol{u}_{\mathrm{k}}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{i} = \boldsymbol{u}_{\mathrm{k}}^{\mathrm{T}}\boldsymbol{0} = 0 \tag{5.9}$$

where the KCL with the node incidence matrix was used [see also Eqn. (5.8)]. In other words, any valid voltage vector $\boldsymbol{u}$ fulfilling the KVL is perpenticular to any valid current vector $\boldsymbol{i}$ satisfying the KCL of the same circuit, i.e., $\boldsymbol{u} \perp \boldsymbol{i}$.

An alternative form for Tellegen's theorem can be written as

$$\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} = \mathbf{0} \quad \text{and} \quad \boldsymbol{B}\boldsymbol{A}^{\mathrm{T}} = \mathbf{0} \tag{5.10}$$

with the node incidence matrix $\boldsymbol{A}$ and the loop incidence matrix $\boldsymbol{B}$. Note that the second statement results from transposing the first one. It can be seen that the vector of coefficients for any of the $s = b - (n-1)$ KVL equations defined by the corresponding row of the node incidence matrix $\boldsymbol{B}$ is orthogonal to the coefficient vector for any of the $n-1$ KCL equations constituted by the node incidence matrix $\boldsymbol{A}$.

5.2 Tableau Equations

For a complete analysis of the circuit, the Kirchhoff's equations must be complemented by the element equations. We consider resistive one- and two-ports. The unknown branch voltage and branch current vector results from the intersection of the elements' characteristics with the Kirchhoff's laws.

For linear, resistive circuit elements, all constituting equations can be summarized in the form

$$\begin{bmatrix}\boldsymbol{M}, \boldsymbol{N}\end{bmatrix} \begin{bmatrix}\boldsymbol{u} \\ \boldsymbol{i}\end{bmatrix} = \boldsymbol{e}. \tag{5.11}$$

with the branch voltage vector $\boldsymbol{u}$, the branch vector $\boldsymbol{i}$, and the vector $\boldsymbol{e}$ comprising the excitations of the circuits (independent voltage and current sources).

Consider, for example, a linear source with the open-circuit voltage u_0 and the internal resistance R with the corresponding current-controlled representation

$$u = Ri + u_0.$$

This representation can be easily brought into the form of Eqn. (5.11), i.e.,

$$u - Ri = u_0.$$

If the circuit is composed only of one-ports, then $\boldsymbol{M}$ and $\boldsymbol{N}$ are diagonal matrices, and for two-ports $\boldsymbol{M}$ and $\boldsymbol{N}$ are block diagonal with 2×2 blocks.

The combination of the constituting equations [see Eqn. (5.11)] with the Kirchhoff's laws [see Eqns. (5.1) and (5.5)] is called a *tableau system of equations* and reads as

$$\begin{bmatrix}\boldsymbol{B} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A} \\ \boldsymbol{M} & \boldsymbol{N}\end{bmatrix} \begin{bmatrix}\boldsymbol{u} \\ \boldsymbol{i}\end{bmatrix} = \begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ \boldsymbol{e}\end{bmatrix} \tag{5.12}$$

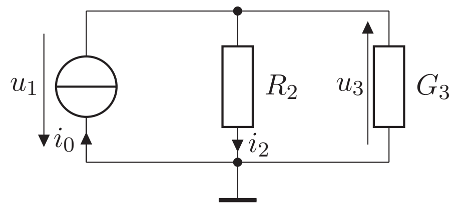


Figure 5.2: Circuit with two resistors

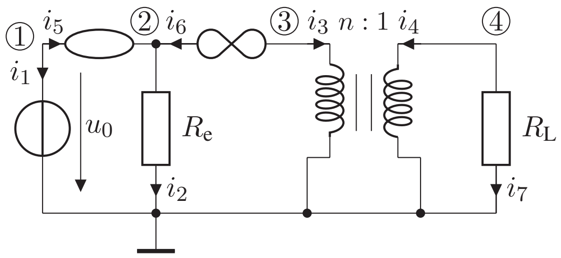


Figure 5.3: Amplifier circuit

comprising $2b$ equations for $2b$ unknowns (b branch voltages and b branch currents) since besides b Kirchhoff's equations, the network elements give additional b equations. In compact form, we have

$$\boldsymbol{T}\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{i}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{e}\end{bmatrix}$$

with the $2b \times 2b$ tableau matrix

$$\boldsymbol{T} = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}\\ \boldsymbol{M} & \boldsymbol{N}\end{bmatrix}.$$

For the circuit in Fig. 5.2 (with $b = 3$ branches and $n = 2$ nodes), we can formulate the ergeben sich die two ($s = b - (n-1) = 2$) KVL equations

$$\begin{aligned} u_1 - u_2 &= 0\\ u_2 + u_3 &= 0 \end{aligned}$$

and the single KCL-Gleichung ($n - 1 = 1$)

$$i_1 + i_2 - i_3 = 0.$$

The representation of the current source is

$$i_1 = -i_0$$

and the two resistors obey Ohm's law

$$\begin{aligned} u_2 &= R_2 i_2 \\ i_3 &= G_3 u_3 \end{aligned}$$

or alternatively,

$$\begin{aligned} u_2 - R_2 i_2 &= 0 \\ i_3 - G_3 u_3 &= 0. \end{aligned}$$

Combining the Kirchhoff's laws and the element equations in a single equation system yields the tableau equation system

$$\begin{bmatrix} 1 & -1 & 0 & & & \\ 0 & 1 & 1 & & & \\ & & & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -R_2 & 0 \\ 0 & 0 & -G_3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -i_0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\det(\boldsymbol{T}) = 1 + R_2 G_3$, the equation system can be solved as long as $R_2 G_3 \neq -1$. Excluding this special case makes sense because otherwise, $\frac{1}{R_2} = -G_3$ for $R_2 G_3 = -1$ and the parallel connection of the two resistors becomes an open circuit (total conductance is zero). This leads to a contradiction because the current i_0 would have to flow over the open circuit.

For another example to find the tableau equation, consider the circuit in Fig. 5.3. The corresponding tableau equation system can be expressed as

$$\begin{array}{c} a \\ b \\ c \\ ① \\ ② \\ ③ \\ ④ \\ \text{volt. s.} \\ R_\mathrm{e} \\ \text{transform} \\ \\ \text{nullator} \\ \\ \text{load res.} \end{array}
\begin{bmatrix}
-1 & 1 & 0 & 0 & 1 & 0 & 0 & & & & & & & \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & & & & & & & \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & & & & & & & \\
& & & & & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
& & & & & & & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\
& & & & & & & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
& & & & & & & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & & & & & & & 0 & & & & & & \\
& 1 & & & & & & & -R_\mathrm{e} & & & & & \\
& & 1 & -n & & & & & & 0 & 0 & & & \\
& & 0 & 0 & & & & & & n & 1 & & & \\
& & & & 1 & 0 & & & & & & 0 & 0 & \\
& & & & 0 & 0 & & & & & & 1 & 0 & \\
& & & & & & 1 & & & & & & & -R_\mathrm{L}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The system of tableau equations has a unique solution if and only if the determinant of the tableau matrix $\boldsymbol{T}$ is non-zero, i.e.,

$$\det \boldsymbol{T} \neq 0. \tag{5.13}$$

The tableau analysis is applicable without any limitation. It is appropriate for a computer-aided analysis when numerical methods are used that exploit the sparsity of $\boldsymbol{T}$.

5.2.1 Node Analysis

In the node analysis, the KCL is expressed again with the node incidence matrix [see Eqn. (5.1)] and the KVL is used where the branch voltages are expressed by the node voltages [cf. Eqn. (5.7)], i.e.,

$$-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{k}} + \boldsymbol{u} = \boldsymbol{0}.$$

Combining with the element equations [see Eqn. (5.11)]

$$\begin{bmatrix}\boldsymbol{M}, \boldsymbol{N}\end{bmatrix} \begin{bmatrix}\boldsymbol{u} \\ \boldsymbol{i}\end{bmatrix} = \boldsymbol{e} \tag{5.14}$$

gives the tableau equation system of the node analysis

$$\begin{bmatrix} -\boldsymbol{A}^{\mathrm{T}} & \boldsymbol{1}_b & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{0} & \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\mathrm{k}} \\ \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{e} \end{bmatrix} \tag{5.15}$$

where the KVL leads to b equations, the KCL to $n-1$ equations, and there are b element equations. Taking into account that the circuit has $n-1$ node voltages, there are overall $2b+n-1$ equations for $2b+n-1$ unknowns. Therefore, the equation system of the node analysis is well defined.

The node analysis has the advantage that the formulation is very easy after the node incidence matrix $\boldsymbol{A}$ has been found for the KCL as the basis matrix for the KVL is just the transpose of $\boldsymbol{A}$. Furthermore, note that the node analysis can be formulated for any circuit.

5.3 Nodal Analysis

For an analytical (symbolic) circuit analysis, it is appropriate to reduce the number of equations and unknowns of the tableau system by simple substitutions. If the circuit includes only voltage-controlled elements (that means, in particular, no voltage sources), this reduction becomes possible in the context of node analysis. To this end, we proceed as follows.

Starting from the tableau equation system of the node analysis [see Eqn. (5.15)], we can solve the element equations for the branch current vector

$$\boldsymbol{i} = -\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{u} + \boldsymbol{N}^{-1}\boldsymbol{e}.$$

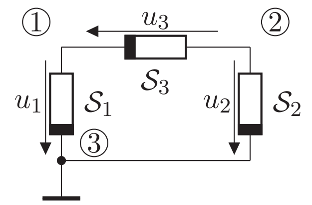


Figure 5.4: Circuit with three linear sources

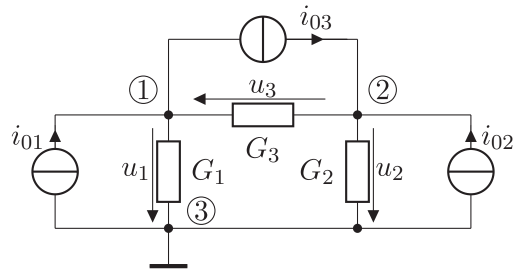


Figure 5.5: Circuit with three linear sources with voltage-controlled equivalent circuit diagrams

For this step, it is necessary that $\boldsymbol{N}$ is invertible and that condition implies that all network element must be voltage-controlled. With the branch conductance matrix $\boldsymbol{G} = -\boldsymbol{N}^{-1}\boldsymbol{M}$ and the branch current vector $\boldsymbol{i}_0 = \boldsymbol{N}^{-1}\boldsymbol{e}$, above equation can be equivalently be rewritten as

$$\boldsymbol{i} = \boldsymbol{G}\boldsymbol{u} + \boldsymbol{i}_0.$$

Remember that the KVL reads as $\boldsymbol{u} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{k}}$. Therefore,

$$\boldsymbol{i} = \boldsymbol{G}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{k}} + \boldsymbol{i}_0.$$

Substituting the branch current vector into the KCL yields

$$\begin{aligned} \boldsymbol{A}\boldsymbol{i} = \boldsymbol{A}\boldsymbol{G}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{k}} + \boldsymbol{A}\boldsymbol{i}_0 = \boldsymbol{0} \\ \boldsymbol{G}_{\mathrm{k}}\boldsymbol{u}_{\mathrm{k}} = \boldsymbol{i}_{\mathrm{q}} \end{aligned} \tag{5.16}$$

with $n-1 \times n-1$ node conductance matrix $\boldsymbol{G}_{\mathrm{k}} = \boldsymbol{A}\boldsymbol{G}\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{A}^{\mathrm{T}}$ and the $n-1$-dimensional node current source vector or short source vector $\boldsymbol{i}_{\mathrm{q}} = -\boldsymbol{A}\boldsymbol{N}^{-1}\boldsymbol{e}$.

As an example to derive the nodal analysis, we set up the tableau equations of the node analysis for the circuit in Fig. 5.4. For the linear sources, we employ the voltage-controlled representations (see Fig. 5.5).

To set up the node incidence matrix $\boldsymbol{A}$, we follow the strategy described in Section 5.1 and obtain

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The transposed node incidence matrix is then the basis matrix for the KVL [see Eq. (5.8)]. With the voltage-controlled representation for the linear source $\mathcal{S}_k$, $k = 1, 2, 3$, that is,

$$i_k = G_k u_k - i_{0k}$$

we can formulate the node analysis

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & & & \\ 0 & -1 & 0 & 1 & 0 & & & \\ 1 & -1 & 0 & 0 & 1 & & & \\ & & & & & 1 & 0 & -1 \\ & & & & & 0 & 1 & 1 \\ & & -G_1 & & & 1 & & \\ & & & -G_2 & & & 1 & \\ & & & & -G_3 & & & 1 \end{bmatrix} \begin{bmatrix} u_{\mathrm{k}1} \\ u_{\mathrm{k}2} \\ u_1 \\ u_2 \\ u_3 \\ i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -i_{01} \\ -i_{02} \\ -i_{03} \end{bmatrix}.$$

Therefore, the branch conductance matrix and the branch current source vector can be found

$$\boldsymbol{G} = -\boldsymbol{N}^{-1}\boldsymbol{M} = \begin{bmatrix} G_1 & & \\ & G_2 & \\ & & G_3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{i}_0 = \begin{bmatrix} -i_{01} \\ -i_{02} \\ -i_{03} \end{bmatrix}$$

respectively. Finally, we get the node conductance matrix

$$\boldsymbol{G}_{\mathrm{k}} = \boldsymbol{A}\boldsymbol{G}\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} G_1 + G_3 & -G_3 \\ -G_3 & G_2 + G_3 \end{bmatrix}$$

and the node current source vector

$$\boldsymbol{i}_{\mathrm{q}} = -\boldsymbol{A}\boldsymbol{i}_0 = \begin{bmatrix} i_{01} - i_{03} \\ i_{02} + i_{03} \end{bmatrix}.$$

This result shows that a conductance, e.g., G_1, connected to a node and the reference node, leads to a single entry in the diagonal entry of $\boldsymbol{G}_{\mathrm{k}}$ corresponding to the node (for G_1, node 1; therefore, the 1,1 element). In contrast, a conductance between two nodes (like G_3) gives four entries in $\boldsymbol{G}_{\mathrm{k}}$ (two positive terms in the corresponding diagonal entries and two negative terms in the corresponding off-diagonal entries).

If the current of a current source flows into the node (like i_{01} for node ①), it gives a positive term in the right-hand side of the KCL equation of that node and a negative term for currents flowing out of the node (like i_{03} for node ①).

With the node conductance matrix $\boldsymbol{G}_{\mathrm{k}}$ and the node current source vector $\boldsymbol{i}_{\mathrm{q}}$, the following equation system of the nodal analysis is constituted

$$G_1 u_{\mathrm{k}1} + G_3(u_{\mathrm{k}1} - u_{\mathrm{k}2}) = i_{01} - i_{03}$$
$$G_2 u_{\mathrm{k}2} + G_3(u_{\mathrm{k}2} - u_{\mathrm{k}1}) = i_{02} + i_{03}.$$

It can be seen that each of the two equations corresponds to the KCL of that node that was expressed with the node voltages.

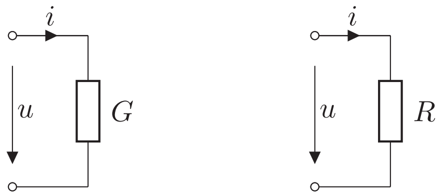


Figure 5.6: Strictly linear resistors

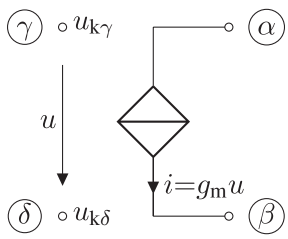


Figure 5.7: VCCS

5.4 Direct Formulation of Node Conductance Matrix

Remember that all elements must be voltage-controlled for the nodal analysis. Therefore, the set of possible elements is restricted to strictly linear resistors [see Fig. 5.6] with

$$i = Gu \quad \text{or} \quad i = \frac{1}{R}u \tag{5.17}$$

voltage-controlled current sources (VCCS) with conductance matrix

$$\boldsymbol{G}_{\text{VCCS}} = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} \tag{5.18}$$

or gyrators with conductance matrix

$$\boldsymbol{G}_{\text{gyr}} = \begin{bmatrix} 0 & G_\text{d} \\ -G_\text{d} & 0 \end{bmatrix}. \tag{5.19}$$

For instance, consider the VCCS as in Fig. 5.7. The controlling voltage is given by $u = u_{\text{k}\gamma} - u_{\text{k}\delta}$. Therefore, the controlled current is $i = g_\text{m}u = g_\text{m}u_{\text{k}\gamma} - g_\text{m}u_{\text{k}\delta}$. Accordingly, we get following terms in the KCL equations for node α and β:

$$\begin{aligned} \text{KCL}\,\alpha: &\ \cdots + g_\text{m}u_{\text{k}\gamma} \ \cdots - g_\text{m}u_{\text{k}\delta} \ \cdots = \cdots \\ \text{KCL}\,\beta: &\ \cdots - g_\text{m}u_{\text{k}\gamma} \ \cdots + g_\text{m}u_{\text{k}\delta} \ \cdots = \cdots \end{aligned}$$

since the current i flows out of α and into β. Accordingly, this VCCS leads to four entries in the node conductance matrix $\boldsymbol{G}_\text{k}$.

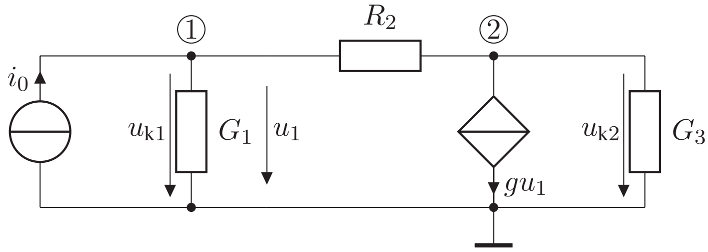


Figure 5.8: Sample circuit for direct formulation of nodal analysis

The direct formulation of the nodal analysis is now employed for the circuit in Fig. 5.8. The resistor with conductance G_1 is connected between node ① and reference node. Therefore, one positive term G_1 is included the $1,1$-element of $\boldsymbol{G}_\mathrm{k}$. Likewise, The resistor with conductance G_3 is connected between node ② and reference node. This gives one positive term G_3 in the $2,2$-element of $\boldsymbol{G}_\mathrm{k}$. The resistance R_2 corresponds to the current-controlled version of Ohm's law. Therefore, the resistor with resistance R_2, which is connected between the nodes ① and ②, leads to the positive term $\frac{1}{R_2}$ in the $1,1$- and the $2,2$-elements of $\boldsymbol{G}_\mathrm{k}$ and the negative term $-\frac{1}{R_2}$ in the $1,2$- and $2,1$-elements. Due to KVL, the controlling voltage u_1 is the same as $u_{\mathrm{k}1}$. Hence, the current of the VCCS is given by $gu_1 = gu_{\mathrm{k}1}$ leads to the single positive entry g in the $2,1$-element of $\boldsymbol{G}_\mathrm{k}$. Finally, the independent current source i_0 flows into node ① resulting in the positive term i_0 in the first element of $\boldsymbol{i}_\mathrm{k}$. In other words, we have found the nodal analysis of the circuit in Fig. 5.8, that is,

$$\begin{bmatrix} G_1 + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} + g & G_3 + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} u_{\mathrm{k}1} \\ u_{\mathrm{k}2} \end{bmatrix} = \begin{bmatrix} i_0 \\ 0 \end{bmatrix}.$$

5.5 Non-Voltage-Controlled Elements

Non-voltage controlled elements, e.g., independent voltage source, CCVS, VCVS, CCCS, cannot be directly included in the nodal analysis. However, that does not imply that the nodal analysis cannot be employed for circuits with such elements since all these elements can be transformed to voltage-controlled elements as will be demonstrated in the next subsections. For example, gyrators can be used to perform such a transform (see Subsection 5.5.3). Finally, the inclusion of nullors will be discussed in Subsection 5.5.4.

5.5.1 Source Transform

If an ideal voltage source (either independent or voltage-controlled) is connected in series to a strictly linear resistor, it is a linear source. As discussed in Subsection 2.6.2, this series connection can equivalently be replaced by the parallel connection of a current source (respectively independent or voltage-controlled) and a strictly linear resistor by source transformation. As

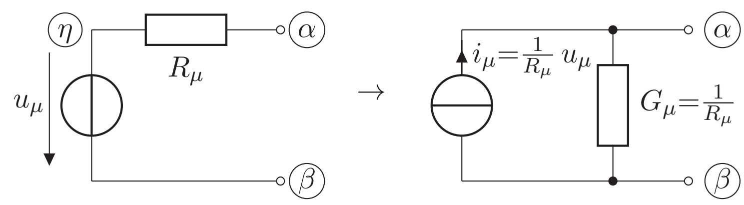


Figure 5.9: Source transform of an independent voltage source

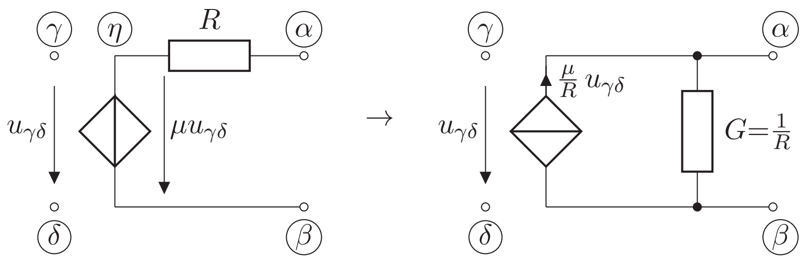


Figure 5.10: Source transform of a VCVS

both, the current source and the strictly linear resistor, are voltage-controlled, they can be incorporated in the nodal analysis. The source transformation is an especially attractive strategy as the 'internal' node η of the original linear source one-port disappears (see Fig. 5.9).

Contains the circuit controlled voltage sources, i.e., CCVS or VCVS, connected in series with strictly linear resistors, they can be transformed to controlled current sources (CCCS or VCCS) by source transform. In Fig. 5.10, a VCVS is transformed to a VCCS by source transform. Note that a VCCS can directly be incorporated in the nodal analysis (see Section 5.4). For the CCCS resulting from the source transform of a CCVS, see the next Subsection 5.5.2).

Note that the node η between the voltage source and the resistor disappears by the source transform. If the value $u_{\mathrm{k}\eta}$ is of interest, it has to be expressed with the help of the remaining node voltages, i.e., $u_{\mathrm{k}\eta} = u_{\mathrm{k}\beta} + u_\mu$ for the independent voltage source in Fig. 5.9 and $u_{\mathrm{k}\eta} = u_{\mathrm{k}\beta} + \mu(u_{\mathrm{k}\gamma} - u_{\mathrm{k}\delta})$ for the controlled voltages source in Fig. 5.10.

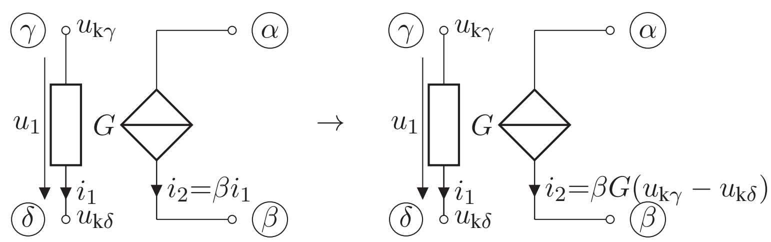


Figure 5.11: Application of Ohm's law to change the control

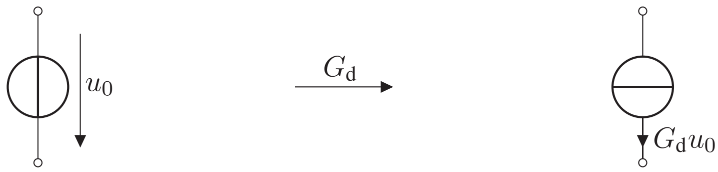


Figure 5.12: Dual transform of a voltage source

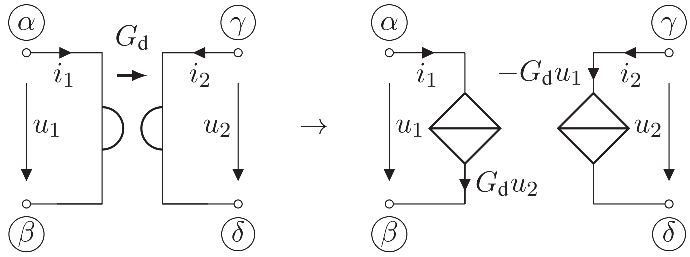


Figure 5.13: Gyrator equivalent circuit diagram with duality constant G_d based on two VCCS

5.5.2 Ohm's Law

If a controlled source is controlled by a current, which flows through a resistor, this current can be expressed by the node voltages based on Ohm's law (see Fig. 5.11 where a CCCS becomes a VCCS). As by Ohm's law, the control by a current is replaced by the control by node voltages, the inclusion in the nodal analysis is possible (for a CCVS, Ohm's law has to be used for the controlling current and the controlled source has to be source transformed, see Subsection 5.5.1).

5.5.3 Duality Transform

Any current-controlled port can be transformed into a voltage-controlled port by duality transform, e.g., an independent voltage source which is current-controlled can be transformed into a current source (voltage-controlled, see Fig. 5.12). Therefore, a voltage source results at port 1 when connecting a current source to port 2 of a gyrator (see Fig. 5.14).

Therefore, a gyrator must be connected to the non-voltage-controlled ports to end up with voltage-controlled ports. The voltage-controlled equivalent circuit diagram of a gyrator is depicted in Fig. 5.13 and Table 5.1 shows which entries in the node conductance matrix $\boldsymbol{G}_\mathrm{k}$ result from a gyrator. A duality transform of all non-voltage-controlled ports leads to an overall voltage-controlled one- or two-port which can be included in the nodal analysis.

In particular, when considering the other three types of controlled source (the already discussed VCCS is voltage-controlled) whose conductance matrix does not exist, one gyrator (VCVS and CCCS) or two gyrators (CCVS) can be used to end up with a VCCS (see Figs. 5.15–5.17). The inclusion of a VCCS in the node conductance matrix has been discussed in Section 5.4.

Note that the usage of a gyrator leads to an increase of the number of nodes (every gyrator one

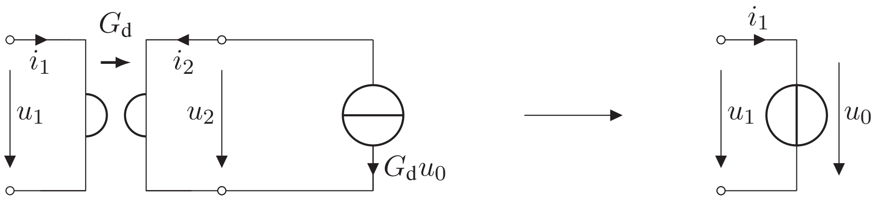


Figure 5.14: Dual transform of a current source by a gyrator

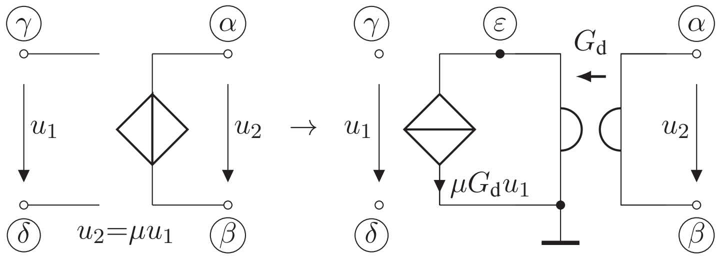


Figure 5.15: Dual conversion of the second port of a VCVS

additional node if the second terminal of port 2 is connected to the reference node). Therefore, using dual transform for a VCVS or a CCCS, increases the number of nodes by one and the for a CCVS, the number of nodes is increased by two.

In Fig. 5.16, a CCCS is depicted whose first port is not voltage-controlled (the short circuit is current-controlled). The short circuit can be obtained by a open circuit at port 2 of a gyrator. As can be seen in Fig. 5.16, the CCCS can therefore be replaced by a gyrator combined with a VCCS.

However, for a CCVS as in Fig. 5.17, two gyrators are necessary to replace the short circuit at port 1 of the CCVS by an open circuit and the controlled voltage source by a controlled current source at port 2.

To understand the representation of a transformer by the cascade connection of two gyrators as depicted in Fig. 5.18, remember the transmission representation of a transformer (see

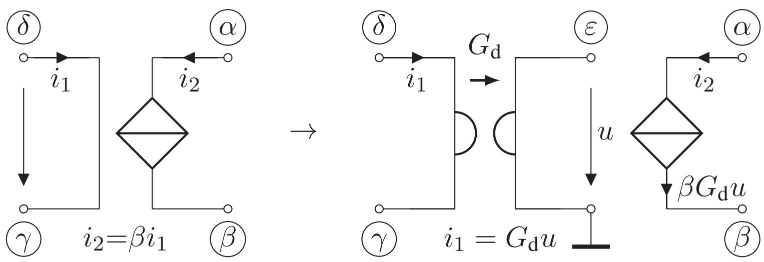
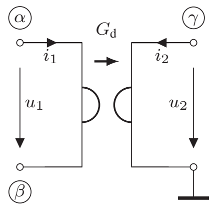


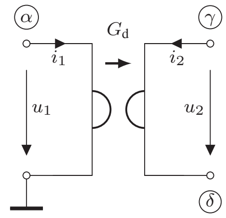
Figure 5.16: Dual conversion of the first port of a CCCS

$$
\boldsymbol{G}_\mathrm{k} = \begin{array}{c} \\ \textcircled{\alpha} \\ \textcircled{\beta} \\ \textcircled{\gamma} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \alpha & & \beta & & \gamma \end{array} \\
\begin{bmatrix}
 & & & & \vdots & \\
 & & & \cdots & G_\mathrm{d} & \cdots \\
 & & & & \vdots & \\
 & & & \cdots & -G_\mathrm{d} & \cdots \\
\vdots & & \vdots & & \vdots & \\
-G_\mathrm{d} & \cdots & G_\mathrm{d} & \cdots & & \\
\vdots & & \vdots & & &
\end{bmatrix}
\end{array}
$$



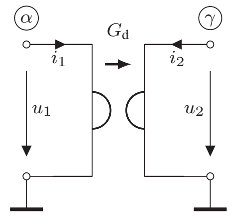
$$
\begin{aligned} i_1 &= G_\mathrm{d} u_{\mathrm{k}\gamma} \\ i_2 &= -G_\mathrm{d}(u_{\mathrm{k}\alpha} - u_{\mathrm{k}\beta}) \end{aligned}
$$

$$
\boldsymbol{G}_\mathrm{k} = \begin{array}{c} \\ \textcircled{\alpha} \\ \textcircled{\gamma} \\ \textcircled{\delta} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \alpha & & \gamma & & \delta \end{array} \\
\begin{bmatrix}
 & & & \vdots & & \vdots & \\
 & & \cdots & G_\mathrm{d} & \cdots & -G_\mathrm{d} & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
\cdots & -G_\mathrm{d} & \cdots & & & & \\
 & \vdots & & & & & \\
\cdots & G_\mathrm{d} & \cdots & & & & \\
 & \vdots & & & & &
\end{bmatrix}
\end{array}
$$



$$
\begin{aligned} i_1 &= G_\mathrm{d}(u_{\mathrm{k}\gamma} - u_{\mathrm{k}\delta}) \\ i_2 &= -G_\mathrm{d} u_{\mathrm{k}\alpha} \end{aligned}
$$

$$
\boldsymbol{G}_\mathrm{k} = \begin{array}{c} \\ \textcircled{\alpha} \\ \textcircled{\gamma} \end{array}
\begin{array}{c}
\begin{array}{ccc} \alpha & & \gamma \end{array} \\
\begin{bmatrix}
 & & & \vdots & \\
 & & \cdots & G_\mathrm{d} & \cdots \\
 & \vdots & & \vdots & \\
\cdots & -G_\mathrm{d} & \cdots & & \\
 & \vdots & & &
\end{bmatrix}
\end{array}
$$



$$
\begin{aligned} i_1 &= G_\mathrm{d} u_{\mathrm{k}\gamma} \\ i_2 &= -G_\mathrm{d} u_{\mathrm{k}\alpha} \end{aligned}
$$

Table 5.1: Gyrators in the node conductance matrix

Subsection 3.5.3)

$$
\boldsymbol{A}_\mathrm{Gyr1}\boldsymbol{A}_\mathrm{Gyr2} = \begin{bmatrix} 0 & \frac{1}{G_\mathrm{d}} \\ G_\mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{nG_\mathrm{d}} \\ nG_\mathrm{d} & 0 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} = \boldsymbol{A}_\mathrm{transf}.
$$

Therefore, all linear circuit elements, whose conductance matrix, resistance matrix, or any of its hybrid respresentations exists, can be replaced by the combination of voltage-controlled elements and gyrators. Following this strategy, this class of linear circuit elements can be made accessible for the reduced node analysis.

5.5.4 Inclusion of Nullors

The only missing linear elements that were introduced in the previous chapters are the nullator and the norator which constitute as a nullor the model of the ideal operational amplifier operated in the linear region. However, only the transmission representation of the nullor exists (see

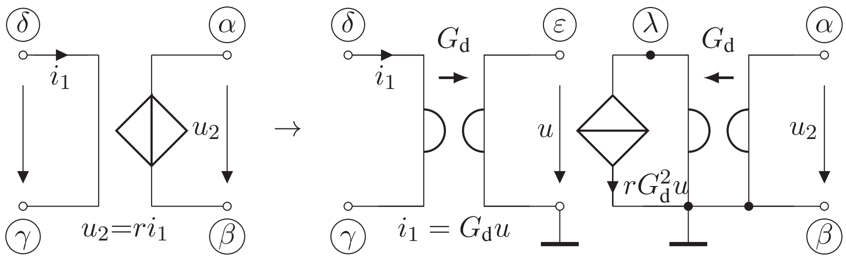


Figure 5.17: Dual conversion of a CCVS

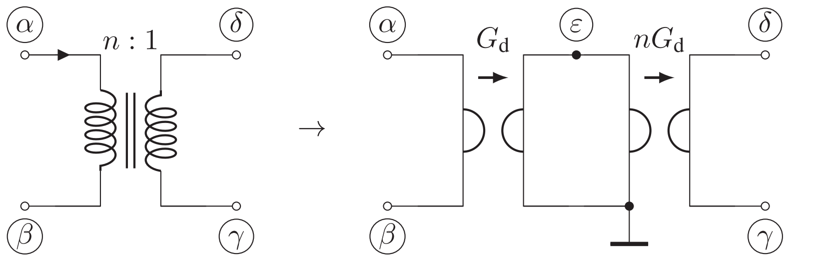


Figure 5.18: Equivalent circuit for a transformer for the nodal analysis

Subsection 3.5.2). Therefore, neither the resistance matrix, the conductance matrix, nor any of the hybrid matrices exist. In other words, the strategy based on gyrators discussed in Section 5.5.3 cannot be employed. Nevertheless, the inclusion of nullators and norators in the reduced node analysis is not difficult as described in the following.

In the first step, the node conductance matrix $\boldsymbol{G}_\mathrm{k}'$ and the node current source vector $\boldsymbol{i}_\mathrm{q}'$ of the circuit without nullors, i.e., nullators and norators are replaced by open circuits. The resulting matrix $\boldsymbol{G}_\mathrm{k}'$ and vector $\boldsymbol{i}_\mathrm{k}'$ will be modified in the next step.

The inclusion of a nullator between the nodes α and β enforces that both node voltages are the same, i.e.,

$$u_{\mathrm{k}\alpha} = u_{\mathrm{k}\beta}.$$

For the equation system $\boldsymbol{G}_\mathrm{k}'\boldsymbol{u}_\mathrm{k}' = \boldsymbol{i}_\mathrm{q}'$ this implies that for every equation like for line j

$$y_{j1}'u_{\mathrm{k}1} + \cdots + \underbrace{y_{j\alpha}'u_{\mathrm{k}\alpha} + y_{j\beta}'u_{\mathrm{k}\beta}}_{(y_{j\alpha}'+y_{j\beta}')u_{\mathrm{k}\alpha}} + \cdots = i_{\mathrm{q}j}$$

the coefficients for $u_{\mathrm{k}\alpha}$ and $u_{\mathrm{k}\beta}$ can be combined. Consequently, $y_{\ell\alpha} = y_{\ell\alpha}' + y_{\ell\beta}'$, $\ell = 1, \ldots, n-1$. In other words, the columns α and β of $\boldsymbol{G}_\mathrm{k}'$ are added and the node voltage vector is shortened by crossing out the element $u_{\mathrm{k}\beta}$. If the node β is the reference node, column α of $\boldsymbol{G}_\mathrm{k}$ completely drops out and also the entry $u_{\mathrm{k}\alpha}$ of $\boldsymbol{u}_\mathrm{k}$. The procedure is performed for every nullator of the circuit.

The consequence of a norator between the nodes γ and δ is that the KCLs for the two nodes γ and δ must be combined to a supernode for γ and δ, since the current i' of the

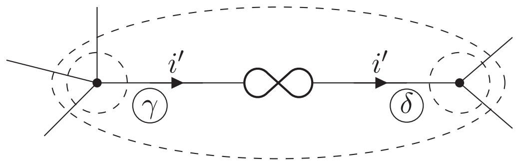


Figure 5.19: Super node around a norator

norator can have any value (see Fig. 5.19). That consideration is equivalent to the addition of the corresponding KCL equations, therefore,

$$\begin{aligned}
y'_{\gamma 1} u_{\mathrm{k}1} + \cdots + y'_{\gamma, n-1} u_{\mathrm{k}, n-1} &= i_{\mathrm{q}\gamma} - i' \\
y'_{\delta 1} u_{\mathrm{k}1} + \cdots + y'_{\delta, n-1} u_{\mathrm{k}, n-1} &= i_{\mathrm{q}\delta} + i' \\
\left(y'_{\gamma 1} + y'_{\delta 1}\right) u_{\mathrm{k}1} + \cdots + \left(y'_{\gamma, n-1} + y'_{\delta, n-1}\right) u_{\mathrm{k}, n-1} &= i_{\mathrm{q}\gamma} + i_{\mathrm{q}\delta}.
\end{aligned}$$

In other words, the rows γ and δ of $\boldsymbol{G}'_{\mathrm{k}}$ are added and the node current source vector is shortened by the combination of two currents, i.e., $i_{\mathrm{q}\gamma} + i_{\mathrm{q}\delta}$. In Op-Amp circuits, the norator is connected to the reference node with one terminal. The procedure to include the norator is simplified since the corresponding row of $\boldsymbol{G}'_{\mathrm{k}}$ and $\boldsymbol{i}'_{\mathrm{q}}$ just has to be cancelled.

Note that every nullator reduces the number of unknowns by one and every norator reduces the number of equations by one. Therefore, if always pairs of nullators and norators (nullors) are part of the circuit, the resulting equation system is well defined.

Chapter 6

Reactive Circuit Elements

The resistive circuits, we have analyzed so far—linear or non-linear—can be described with algebraic equations.

However, many important practical structures cannot be appropriately modeled using resistive circuits. That is true whenever we deal with structures that have the ability to *store energy* (mechanical analogies are a mass in motion or a spring under tension, for example). This inherent deficiency is corrected by adding two new classes of circuit elements, namely *capacitive* and *inductive* elements. The generic term for both is *reactive circuit element*.

In order to model virtually all important structures and phenomena, it suffices to consider reactive one-ports that are embedded in resistive circuits. Thus, we will discuss them first and in detail.

Even for the easiest instance of a linear time-invariant capacitor or inductor, it is no longer possible to represent the characteristic of a reactive element within the u-i-plane. Since reactive elements are not characterized by a relation between $i(t)$ and $u(t)$ at the same time in contrast to resistive one-ports, no u-i-representation exists.

In order to obtain a convenient representation in form of a characteristic, we introduce two new variables, viz., *charge* and *flux*, in addition to terminal voltage and current. By means of these additional variables, we can formulate a characteristic representation for appropriately chosen planes again Note that for non-linear and time-variant energy-storing one-ports such a characteristic can also be found.

The *charge* q and the *flux* Φ of a one-port are respectively defined through integration of the corresponding terminal current and voltage over time, based on a *reference time* t_0:

$$q(t) = q(t_0) + \int_{t_0}^{t} i(\tau)\,\mathrm{d}\tau, \quad \Phi(t) = \Phi(t_0) + \int_{t_0}^{t} u(\tau)\,\mathrm{d}\tau. \tag{6.1}$$

If the integrals also exist for $t_0 \to -\infty$, and if we can assume $q(-\infty) = 0$ resp. $\Phi(-\infty) = 0$,

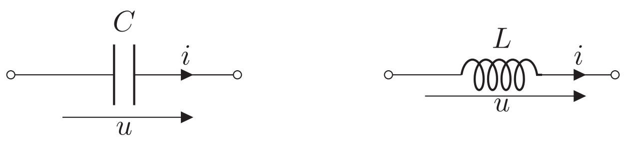


Figure 6.1: Element symbols of capacitors and inductors

then the integrals reduce to

$$q(t) = \int_{-\infty}^{t} i(\tau)\,\mathrm{d}\tau, \quad \Phi(t) = \int_{-\infty}^{t} u(\tau)\,\mathrm{d}\tau. \tag{6.2}$$

Current and voltage cannot be measured for $t_0 \to -\infty$. Hence, the integrals (6.2) cannot actually be evaluated. However, the convenient computation using (6.2) simply means setting the initial values of charge and flux (which are of no importance to the electrical behavior) to zero for some reference time t_0 that is sufficiently long in the past.

The corresponding SI units are called *Coulomb* C and *Weber* Wb:

$$1\,\mathrm{C} = 1\,\mathrm{As}, \quad 1\,\mathrm{Wb} = 1\,\mathrm{Vs}. \tag{6.3}$$

On the contrary, by differentiation[1]

$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} = \dot{q}(t), \qquad u(t) = \frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \dot{\Phi}(t). \tag{6.4}$$

Note that the abbreviation $\dot{x}(t)$ is used for $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ in the following.

By means of the two variables q and Φ, the representation of energy-storing one-ports may be defined and put into an algebraic framework.

A capacitor is defined by its relationship between the voltage u and charge q, i.e.,

$$q(t) = Cu(t) \quad \text{and} \quad u(t) = \frac{1}{C}\,q(t) \tag{6.5}$$

with the capacitance C whose SI unit is *Farrad* F. Due to Eqn. (6.4), the current of the capacitor is given by

$$i(t) = \dot{q}(t) = C\dot{u}(t). \tag{6.6}$$

Similarly, an inductor is defined by its relationship between the current i and the flux Φ, that is,

$$\Phi = Li(t) \quad \text{and} \quad i(t) = \frac{1}{L}\,\Phi(t) \tag{6.7}$$

where the inductance L has the SI unit *Henry* H. Therefore, the voltage of an inductor can be written as [see Eqn. (6.4)]

$$u(t) = \dot{\Phi}(t) = L\dot{i}(t). \tag{6.8}$$

The element symbols of a capacitor and an inductor are depicted in Fig. 6.1.

[1]For the existence of these differentials, it suffices that the derivatives exist for all but a finite number of times t

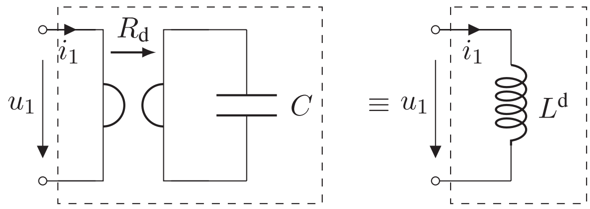


Figure 6.2: Dual conversion of a capacitor by a gyrator

6.1 Duality

To be able to apply the duality introduced in Subsection 2.2.4 for resistive one-ports also to reactive one-ports, it is necessary to explore the consequences of a duality transform (exchange of the roles of voltages and currents) for the charge q and the flux Φ. For resistive elements it holds that

$$u \xrightarrow{R_\mathrm{d}} R_\mathrm{d} i^\mathrm{d} \quad \text{and} \quad i \xrightarrow{R_\mathrm{d}} \frac{1}{R_\mathrm{d}} u^\mathrm{d}$$

and therefore,

$$R \xrightarrow{R_\mathrm{d}} G^\mathrm{d} = \frac{R}{R_\mathrm{d}^2} \quad \text{and} \quad G \xrightarrow{R_\mathrm{d}} R^\mathrm{d} = R_\mathrm{d}^2 G.$$

These supplements are applied also for $q(t)$ and $\Phi(t)$, i.e.,

$$\begin{aligned} q(t) &= \int_{-\infty}^{t} i(\tau)\,\mathrm{d}\tau \xrightarrow{R_\mathrm{d}} \int_{-\infty}^{t} \frac{u^\mathrm{d}(\tau)}{R_\mathrm{d}}\,\mathrm{d}\tau = \frac{1}{R_\mathrm{d}} \Phi^\mathrm{d}(t) \\ \Phi(t) &= \int_{-\infty}^{t} u(\tau)\,\mathrm{d}\tau \xrightarrow{R_\mathrm{d}} \int_{-\infty}^{t} R_\mathrm{d} i^\mathrm{d}(\tau)\,\mathrm{d}\tau = R_\mathrm{d} q^\mathrm{d}(t). \end{aligned} \tag{6.9}$$

The characteristics of a capacitor and an inductor resulting from the duality transform read as

$$\begin{aligned} q(t) = Cu(t) &\xrightarrow{R_\mathrm{d}} \frac{1}{R_\mathrm{d}} \Phi^\mathrm{d}(t) = C R_\mathrm{d} i^\mathrm{d}(t) \\ \Phi(t) = Li(t) &\xrightarrow{R_\mathrm{d}} R_\mathrm{d}\, q^\mathrm{d}(t) = L \frac{1}{R_\mathrm{d}} u^\mathrm{d}(t) \end{aligned} \tag{6.10}$$

respectively. Consequently,

$$\begin{aligned} C &\xrightarrow{R_\mathrm{d}} L^\mathrm{d} = R_\mathrm{d}^2 C \\ L &\xrightarrow{R_\mathrm{d}} C^\mathrm{d} = \frac{1}{R_\mathrm{d}^2} L. \end{aligned} \tag{6.11}$$

In other words, capacitors and inductors are dual to each other. Therefore, a gyrator can be used to transform a capacitor into its dual inductor (see Fig. 6.2).

Any property that can be implemented with dynamic circuits may also be realized using a single type of reactive elements. Such a realization is possible because every type can always be transformed to another type by means of a gyrator. Thus, the used representation is usually restricted to capacitive one-ports in the following. The relations for inductive one-ports may be derived analogously via duality.

6.2 Properties of Reactances

Many properties of reactive one-ports can be defined in accordance with the properties of resistive one-ports, that were analyzed in Section 2.2.

6.2.1 Continuity

The terminal voltage $u_C(t)$ of a linear and time-invariant capacitive one-port is *continuous* in the open interval $(t_0, t_0 + \Delta t)$ if the current waveform $i_C(t)$ is bounded.

Besides the definition of continuity, we only require the properties of integration to prove this statement. We have

$$\begin{aligned}
&u_C(t_0 + \Delta t) - u_C(t_0) = \frac{1}{C} \int_{t_0}^{t_0+\Delta t} i_C(\tau)\, \mathrm{d}\tau \\
&I_0 = \max_{t \in (t_0, t_0+\Delta t)} |i_C(t)| \Rightarrow |u_C(t_0 + \Delta t) - u_C(t_0)| \leq \left| \frac{1}{C} I_0 \Delta t \right| \\
&\lim_{\Delta t \to 0} \left| \frac{1}{C} I_0 \Delta t \right| = 0 \Rightarrow \lim_{\Delta t \to 0} |u_C(t_0 + \Delta t) - u_C(t_0)| = 0.
\end{aligned} \tag{6.12}$$

Consequently, u_C is continuous in t_0, since the right- and left-sided limits are equal to the value of the function.

The continuity law requires that the absolute value $|i_C(t)|$ is bounded. When using idealized models, one may encounter circuits for which this boundedness condition is not satisfied.

6.2.2 Energy

The instantaneous power of a one-port is given by

$$p(t) = u(t)i(t)$$

and the work necessary to change the state of a capacitor from t_1 to t_2 is therefore,

$$W_C(t_1, t_2) = \int_{t_1}^{t_2} p(t)\, \mathrm{d}t = \int_{t_1}^{t_2} u(t)i(t)\, \mathrm{d}t. \tag{6.13}$$

As the current of a capacitor can be expressed as $i(t) = C\dot{u}(t)$, the work can be written as

$$W_C(t_1, t_2) = \int_{t_1}^{t_2} u(t)C\dot{u}(t)\, \mathrm{d}t = \int_{u(t_1)}^{u(t_2)} Cu\, \mathrm{d}u = \frac{C}{2}\, (u^2(t_2) - u^2(t_1)). \tag{6.14}$$

With the states at time t_1 and t_2, viz., $u_1 = u(t_1)$ and $u_2 = u(t_2)$, respectively, and accordingly, $q_1 = Cu_1$ and $q_2 = Cu_2$, the work reads as

$$W_C(u_1, u_2) = \frac{C}{2}\, (u_2^2 - u_1^2) = \frac{1}{2C}\, (q_2^2 - q_1^2). \tag{6.15}$$

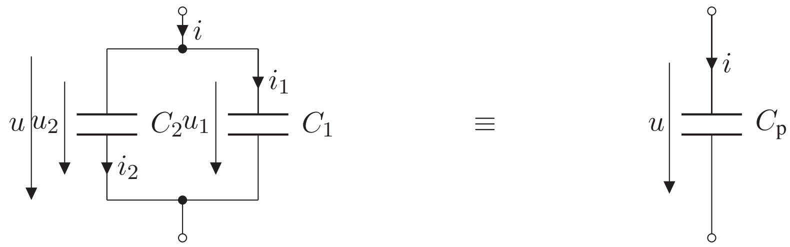


Figure 6.3: Parallel connection of two capacitors

In other words, the work is independent of the times t_1, t_2 and only depends on the states at the beginning and the end. This observation motivates to define the energy stored in the capacitor

$$E_C(u) = W(0, u) = \frac{C}{2}\, u^2 \tag{6.16}$$

where the energy is the work needed to leave the zero state to reach the state u.

Likewise, the work and energy for an inductor are given by

$$\begin{aligned} W_L(i_1, i_2) &= \frac{L}{2}\, (i_2^2 - i_1^2) \\ E_L(i) &= \frac{L}{2}\, i^2 \end{aligned} \tag{6.17}$$

respectively.

6.3 Connection of Reactances

Connecting two capacitors in parallel (see Fig. 6.3) results in

$$u(t) = u_1(t) = u_2(t)$$

and due to the KCL,

$$i(t) = i_1(t) + i_2(t).$$

Integrating the last equation gives

$$q(t) = q_1(t) + q_2(t)$$

and with the characteristics of the two capacitors

$$q(t) = C_1 u_1(t) + C_2 u_2(t) = (C_1 + C_2) u(t).$$

Therefore, the overall capacitance of the parallel connection reads as

$$C_\mathrm{p} = C_1 + C_2. \tag{6.18}$$

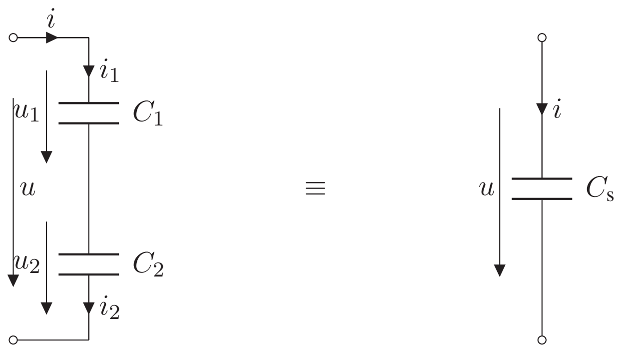


Figure 6.4: Series connection of two capacitors

The dual setup is the series connection of two inductors with inductances L_1 and L_2. The resulting overall inductance is given by

$$L_\mathrm{s} = L_1 + L_2. \tag{6.19}$$

When connecting two capacitors in series (see Fig. 6.4), we get

$$i(t) = i_1(t) = i_2(t)$$

and

$$u(t) = u_1(t) + u_2(t).$$

Integrating the expression for the currents gives

$$q(t) = q_1(t) = q_2(t)$$

and with the characteristics of the two capacitors,

$$u(t) = \frac{1}{C_1}\, q_1(t) + \frac{1}{C_2}\, q_2(t).$$

As the charges are all the same, the overall capacitance results from

$$\frac{1}{C_\mathrm{s}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{6.20}$$

or equivalently,

$$C_\mathrm{s} = C_1 \,\|\, C_2. \tag{6.21}$$

Likewise, the parallel connection of two inductors has the overall inductance

$$L_\mathrm{p} = L_1 \,\|\, L_2. \tag{6.22}$$

Chapter 7

Linear First-Order Circuits

A circuit is said to be *dynamic* if it includes one or multiple capacitors and/or one or multiple inductors. In general, dynamic circuits can be described by means of differential equations. The simplest dynamic circuits are first-order circuits. They contain either one capacitor or one inductor, and their behavior is governed by a first-order differential equation.

The circuit is composed of a reactive element and an arbitrary number of resistive elements. In this section, we assume all elements to be linear.

Due to the linearity of the resistive part of the circuit, it can be interpreted as a linear source (employing open circuit and short circuit measurements). Therefore, it is possible to obtain circuits as depicted in Fig. 7.1. We follow the convention that the reference directions of the reactances are chosen such that the state variable shows in the same direction as the corresponding port variable of the resistive one-port.

With the representations of the reactances, i.e., $i_C(t) = C\dot{u}_C(t)$ and $u_L(t) = L\dot{i}_L(t)$, and the suitable Kirchhoff's laws for these simple circuits, we obtain the linear first-order differential equations

$$\begin{aligned} &i_C(t)R + u_C(t) = u_0(t) && u_L(t)G + i_L(t) = i_0(t) \\ &\dot{u}_C(t) = -\frac{1}{RC}u_C(t) + \frac{1}{RC}u_0(t) && \dot{i}_L(t) = -\frac{1}{GL}i_L(t) + \frac{1}{GL}i_0(t). \end{aligned} \tag{7.1}$$

The equations in Eqn. (7.1) exhibit the general form

$$\dot{x}(t) = Ax(t) + Bv(t) \tag{7.2}$$

which is called state equation. The parameter A is the scalar special case of a state matrix and B is the scalar special case of an input matrix (see Section 8.1). The state variable is x i.e., u_C or i_L, and v is the excitation, that is, u_0 or i_0.

Our objective is to calculate the evolution of the state variable x for $t \geq t_0$, given an initial value $x_0 = x(t_0)$.

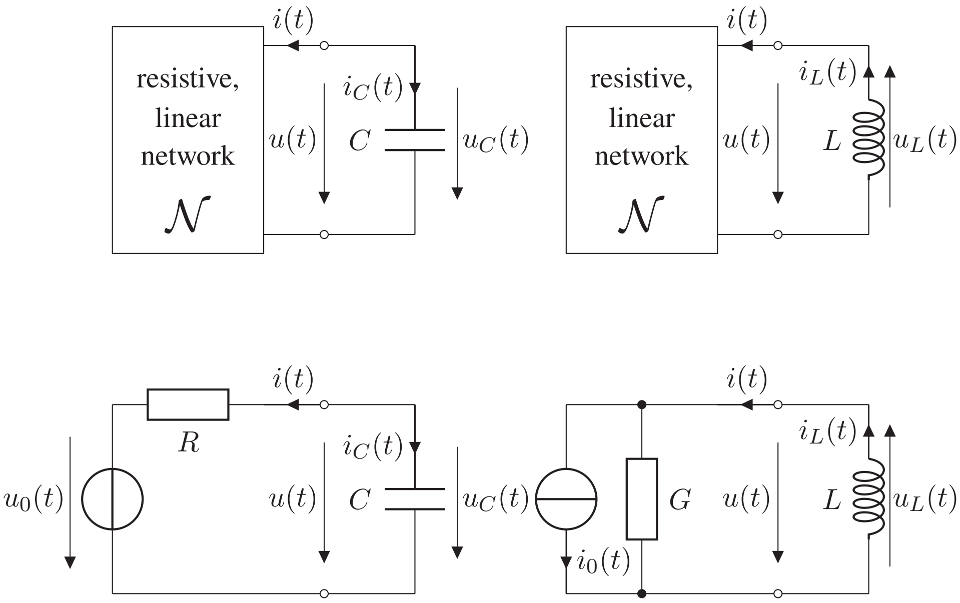


Figure 7.1: Replacing the resistive one-port by a linear source equivalent circuit diagram

7.1 General Excitation

Comparing the general form for the state equation in (7.2) with the results in (7.1), we find that

$$B = -A = \frac{1}{\tau}$$

with $\tau = RC$ for the first-order circuit with capacitor and $\tau = GL$ for that with inductor. For the case with capacitor, the excitation is $v(t) = u_0(t)$ and with the inductor, $v(t) = i_0(t)$. Therefore, the general state equation from (7.2) can be rewritten as

$$\dot{x}(t) = -\frac{1}{\tau}\, x(t) + \frac{1}{\tau}\, v(t). \tag{7.3}$$

In the following derivation, no restriction is imposed on the excitation $v(t)$. The homogeneous differential equation corresponding to (7.3) reads as

$$\dot{x}(t) = -\frac{1}{\tau}\, x(t). \tag{7.4}$$

Due to $\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$, an alternative form for the homogeneous differential equation is

$$\frac{1}{x}\, \mathrm{d}x = -\frac{1}{\tau}\, \mathrm{d}t.$$

Integrating on both sides yields

$$\ln(x_{\text{hom}}) = -\frac{t}{\tau} + C_{\text{int}}$$

with the integration constant C_{int}. Therefore, the solution to the homogeneous differential equation can be expressed as

$$x_{\text{hom}}(t) = C\,\mathrm{e}^{-\frac{t}{\tau}}$$

where $C = \mathrm{e}^{C_{\text{int}}}$. By variation of constants, the solution to (7.3) can be found based on the homogeneous solution $x_{\text{hom}}(t)$. To this end, C is assumed to be also a function of t, i.e.,

$$x(t) = C(t)\,\mathrm{e}^{-\frac{t}{\tau}}\,. \tag{7.5}$$

With the product rule and by substituting this form for $x(t)$ into (7.3), we obtain

$$\dot{C}(t)\,\mathrm{e}^{-\frac{t}{\tau}} - C(t)\,\frac{1}{\tau}\,\mathrm{e}^{-\frac{t}{\tau}} = -\frac{1}{\tau}\,C(t)\,\mathrm{e}^{-\frac{t}{\tau}} + \frac{1}{\tau}\,v(t).$$

As expected, the homogeneous part cancels out resulting in

$$\dot{C}(t)\,\mathrm{e}^{-\frac{t}{\tau}} = \frac{1}{\tau}\,v(t).$$

Solving for $\dot{C}(t)$ and after integration, we obtain

$$\begin{aligned} C(t) &= \int_{-\infty}^{t} \frac{1}{\tau}\,\mathrm{e}^{\frac{t'}{\tau}}\,v(t')\,\mathrm{d}t' \\ &= C_0 + \int_{t_0}^{t} \frac{1}{\tau}\,\mathrm{e}^{\frac{t'}{\tau}}\,v(t')\,\mathrm{d}t' \end{aligned}$$

where $C_0 = \int_{-\infty}^{t_0} \frac{1}{\tau}\,\exp(-\frac{t'}{\tau})v(t')\,\mathrm{d}t'$ and the assumption was made for the second line that $t \geq t_0$. Substituting into (7.5) yields

$$x(t) = C_0\,\mathrm{e}^{-\frac{t}{\tau}} + \mathrm{e}^{-\frac{t}{\tau}} \int_{t_0}^{t} \frac{1}{\tau}\,\mathrm{e}^{\frac{t'}{\tau}}\,v(t')\,\mathrm{d}t'$$

With the initial condition $x(t_0) = x_0$, we get

$$C_0 = x_0\,\mathrm{e}^{\frac{t_0}{\tau}}$$

and thus, the general solution to (7.3) reads as

$$x(t) = x_0\,\mathrm{e}^{-\frac{t-t_0}{\tau}} + \int_{t_0}^{t} \frac{1}{\tau}\,\mathrm{e}^{-\frac{t-t'}{\tau}}\,v(t')\,\mathrm{d}t'. \tag{7.6}$$

Accordingly, the general solutions to (7.1) can be decomposed into two parts, one depending on the initial condition and one depending on the excitation:

$$\begin{aligned} u_C(t) &= u_C(t_0)\,\mathrm{e}^{-\frac{t-t_0}{RC}} + \int_{t_0}^{t} \frac{1}{RC}\,u_0(t')\,\mathrm{e}^{-\frac{t-t'}{RC}}\,\mathrm{d}t' \\ i_L(t) &= i_L(t_0)\,\mathrm{e}^{-\frac{t-t_0}{GL}} + \int_{t_0}^{t} \frac{1}{GL}\,i_0(t')\,\mathrm{e}^{-\frac{t-t'}{GL}}\,\mathrm{d}t' \end{aligned} \tag{7.7}$$

where $\tau = RC$ and $\tau = GL$, respectively. The first term of each equation's right-hand side in Eqn. (7.7) is the part of the solution depending on the initial condition, i.e., the part to which the solution reduces when the excitation is set to zero (*zero-input response*). Similarly, the second term is the part of the solution resulting from the excitation, i.e., the part to which the solution reduces when the intitial condition is set to zero (*zero-state response*).

7.2 Constant Excitation

In case of a constant excitation, i.e., $u_0(t) = U_0 =$ const. or $i_0(t) = I_0 =$ const., Eqn. (7.3) simplifies to

$$\dot{x}(t) = -\frac{1}{\tau}\, x(t) + \frac{1}{\tau}\, x_\infty \tag{7.8}$$

with $\tau = RC$ or $\tau = GL$ and $v = x_\infty = U_0$ or $x_\infty = I_0$.

Incorporating the special structure of the differential equation Eqn. (7.8) into the general solution Eqn. (7.6) yields

$$x(t) = x_\infty + (x_0 - x_\infty)\, \mathrm{e}^{-\frac{t-t_0}{\tau}}, \ \forall t \geq t_0. \tag{7.9}$$

Here, $x_0 = x(t_0)$ is called the inital state, $x_\infty = x(t_\infty)$ is the equilibrium state (fixed point), and τ is the time constant.

The sign of the time constant τ is crucial for the qualitative behavior of the solution (7.9).

Stable Case

First, the so-called stable case is considered, when $\tau > 0$.

In order to be able to sketch the time function of the solution fast and accurately, the following properties are helpful.

- Consider the linearization, i.e., the tangent, of $x(t)$ at t_0,

$$\begin{aligned} x_{\text{lin}}(t) &= x(t_0) + \left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=t_0} (t - t_0) \\ &= x_0 + \frac{x_\infty - x_0}{\tau}(t - t_0). \end{aligned}$$

  Since $x_{\text{lin}}(t_0 + \tau) = x_\infty$, the tangent to $x(t)$ at the time t_0 passes through the points (t_0, x_0) and $(t_0 + \tau, x_\infty)$.

- After one time constant τ, that is,

$$\begin{aligned} x(t_0 + \tau) &= x_\infty + (x_0 - x_\infty)\, \mathrm{e}^{-\frac{t_0+\tau-t_0}{\tau}} \\ &= x_\infty + (x_0 - x_\infty)\, 0.37 \\ &\approx x_\infty + (x_0 - x_\infty)\left(1 - \frac{2}{3}\right) \\ &= x_0 + (x_\infty - x_0)\, \frac{2}{3} \end{aligned}$$

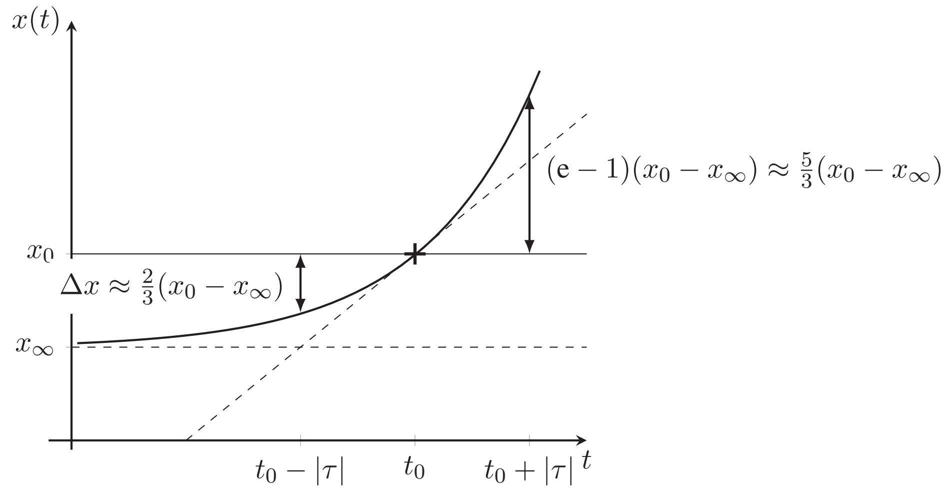


Figure 7.2: General solution for the unstable case ($\tau < 0$)

$x(t)$ has already moved 63 percent (about two third) of $x_\infty - x_0$ away from x_0 towards x_∞. von $x_\infty - x_0$ in Richtung x_∞ bewegt.

- For $t \to \infty$, $x(t)$ approaches the final value x_∞. After about seven time constants τ, this value is virtually reached, i.e., the error is smaller than $10^{-3}|x_0 - x(t_\infty)|$.

Unstable Case

In the second, the so-called unstable, case, when $\tau < 0$, the quantity $x(t) - x_\infty$ (the difference between the state variable and the equilibrium state) grows exponentially, i.e., for

$$x_0 > x_\infty : \lim_{t\to\infty} x(t) \to \infty,$$

or equivalently,

$$x_0 < x_\infty : \lim_{t\to\infty} x(t) \to -\infty.$$

However, it also holds that

$$\lim_{t\to-\infty} x(t) = x_\infty.$$

Again, a list of helpful properties can be made.

- The tangent to $x(t)$ at the initial state x_0 passes through the points (t_0, x_0) and $(t_0 - |\tau|, x_\infty)$.

- After one time constant $|\tau|$, when

$$\begin{aligned} x(t_0 + |\tau|) &= x_\infty + (x_0 - x_\infty)\, \mathrm{e}^{-\frac{t_0+|\tau|-t_0}{\tau}} \\ &= x_\infty + (x_0 - x_\infty)\, 2.72 \\ &\approx x_\infty + (x_0 - x_\infty) \left(1 + \frac{5}{3}\right) \\ &= x_0 + (x_0 - x_\infty)\, \frac{5}{3} \end{aligned}$$

  the state variable $x(t)$ has changed by 1.72 (or about five third) times $x_0 - x_\infty$. hat sich $x(t)$

- If we run back in time, an equilibrium state x_∞ is virtually reached after $|7\tau|$, i.e., the error is smaller than $10^{-3}\, |x_0 - x(t_\infty)|$.

7.3 Piecewise Constant Excitation

If the m_u voltage and the m_i current sources in the resistive circuit are only piecewise-constant for $t \geq t_0$, the time interval $t \leq t_0 < t_\infty$ is partitioned into subintervals $[t_j, t_{j+1}]$ such that the excitations do not change during each interval.

The solution for the state variable can be found as in the previous section, where the value of the state variable $u_C(t)$ [or $i_L(t)$] at the end of an interval is equal to the initial state for the following interval (assuming that $R \neq 0$ and $G \neq 0$, respectively). The branch voltages and currents of the resistive circuit elements will be discontinuous in general.

An interesting special case of a piecewise-constant excitation which is of practical importance, is the excitation shaped as an impulse (see Fig. 7.3):

$$v(t) = \begin{cases} x_{\mathcal{S}} & \text{for } 0 \leq t \leq t_1 \\ 0 & \text{else.} \end{cases} \tag{7.10}$$

Assuming the initial state is zero, i.e., $x_0 = x(0)$ and with $x_\infty = x_{\mathcal{S}}$, we get for $0 \leq t \leq t_1$,

$$\begin{aligned} x(t) &= x_\infty + (x_0 - x_\infty)\, \mathrm{e}^{-\frac{t-t_0}{\tau}} \\ &= x_{\mathcal{S}} - x_{\mathcal{S}}\, \mathrm{e}^{-\frac{t}{\tau}} \qquad \text{for } 0 \leq t \leq t_1. \end{aligned} \tag{7.11}$$

Due to the continuity of the state variable $x(t)$, the value of $x(t)$ does not change when the excitation $v(t)$ jumps at $t = t_1$. Thus, $x_0 = x(t_1) = x_{\mathcal{S}}(1 - \mathrm{e}^{-\frac{t_1}{\tau}})$ and $x_\infty = 0$ for $t \geq t_1$ leading to

$$x(t) = x_{\mathcal{S}} \left(1 - \mathrm{e}^{-\frac{t_1}{\tau}}\right) \mathrm{e}^{-\frac{t-t_1}{\tau}} \qquad \text{for } t \geq t_1. \tag{7.12}$$

The resulting response to the impulsive excitation (see Fig. 7.3) is depicted in Fig. 7.4.

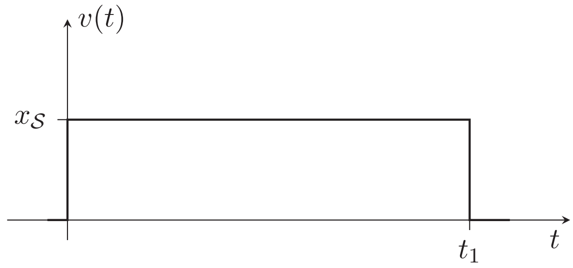


Figure 7.3: Excitation with impulse shape

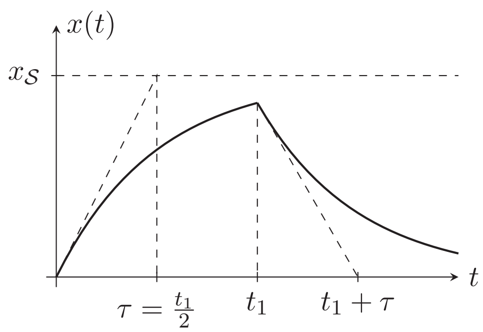


Figure 7.4: Response to impulsive excitation

Chapter 8

Linear Second-Order Circuits

Generally, *second-order dynamic circuits* contain two reactances (inductive or capacitive) and can be described by a second-order scalar differential equation using one variable. On the other hand, they can also be described by a system of two uncoupled first-order scalar differential equations, called state equations, using two variables which are denoted as *state variables*. In the following, primarily the latter formulation, applying state equations, will be used. Reasons for this are:

- The state equations completely describe the energy-storage of the circuit. The scalar differential equation initially describes the behavior of one state variable.
- The state equations can easily be derived using the methods for first-order circuits and the formulation procedure can be easily generalized beyond the case of second-order circuits. The matrix-vector notation provides a uniform mathematical formulation.

These reasons are also applicable for the simpler case of linear dynamic circuits. Furthermore, it is very difficult, perhaps even impossible, to formulate a higher-order differential equation for nonlinear circuits. On the other hand, the state equations can be easily formulated (even by inspection) for a general class of circuits. Unlike for first-order circuits, all essential effects can be studied and assessed for dynamic second-order circuits. Therefore, it is very useful to study second-order circuits in more detail.

8.1 Formulation of State Equations

At first, let us consider the simple example of a linear time-invariant second-order circuit (Fig. 8.1), which will be analyzed as before.

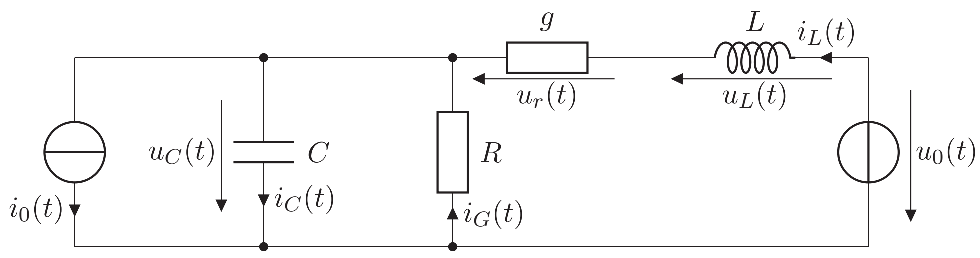


Figure 8.1: Example for a second-order circuit

As usual, Kirchoff's laws can be applied, i.e.,

$$i_0 = -i_C + i_G + i_L, \quad u_0 = u_L + u_r + u_C$$

and the representations of the elements read as

$$u_r = \frac{1}{g} i_L, \quad u_L = L\frac{\mathrm{d}}{\mathrm{d}t} i_L, \quad i_G = -\frac{1}{R} u_C, \quad i_C = C\frac{\mathrm{d}}{\mathrm{d}t} u_C.$$

An appropriate reformulation yields

$$i_0 = -C\frac{\mathrm{d}}{\mathrm{d}t} u_C - \frac{1}{R} u_C + i_L, \quad u_0 = L\frac{\mathrm{d}}{\mathrm{d}t} i_L + \frac{1}{g} i_L + u_C.$$

The system of state equations with the state variables $u_C(t)$ and $i_L(t)$ can be written as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u_C(t) &= -\frac{1}{RC} u_C(t) + \frac{1}{C} i_L(t) - \frac{1}{C} i_0(t), \\ \frac{\mathrm{d}}{\mathrm{d}t} i_L(t) &= -\frac{1}{L} u_C(t) - \frac{1}{gL} i_L(t) + \frac{1}{L} u_0(t). \end{aligned} \tag{8.1}$$

This is a system of linear first-order differential equations with constant coefficients.

In the following, we will formulate the state equations for the case of second-order, and we will solve them. The solutions will be shown and interpreted within the state space for the autonomous case.

Based on the system of state equations, the second-order scalar differential equation can be derived. The system of state equations Eqn. (8.1) in general is of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}v(t) \tag{8.2}$$

with the state vector $\boldsymbol{x} = [x_1, x_2]^{\mathrm{T}}$, $\dot{\boldsymbol{x}}$ is its derivative w.r.t. time, $\boldsymbol{A}$ is the state matrix, $\boldsymbol{b}$ the input vector, and $v(t)$ is the excitation.

Additionally, this system of state equation can be complemented with an *output equation*

$$y(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(t) + d\, v(t) \tag{8.3}$$

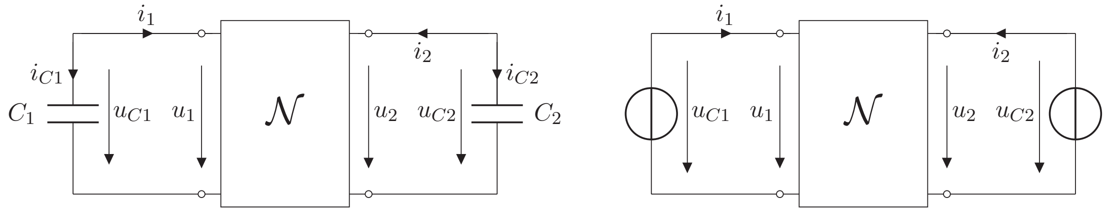


Figure 8.2: Resistive equivalent circuit for a circuit with two capacitors

where $\boldsymbol{c}^{\mathrm{T}} = [c_1, c_2]$ is the output vector and d is the feedthrough. Here, a dynamic second-order circuit with one excitation $v(t)$ and one output signal $y(t)$ was assumend (single-input single-output system).

Considering a number of inputs and outputs leads to the system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{v}(t), \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{v}(t), \end{aligned} \tag{8.4}$$

where $\boldsymbol{B} \in \mathbb{R}^{2\times k}$ is the input matrix, $\boldsymbol{C} \in \mathbb{R}^{j\times 2}$ is the output matrix, and $\boldsymbol{D} \in \mathbb{R}^{j\times k}$ the feedthrough matrix for a multiple-input multiple-output system.

Comparing Eqn. (8.4) with the analysis of the example given in Fig. 8.1, we can derive Eqn. (8.1), and thus we get:

$$\begin{aligned} x_1 &= u_C(t), & x_2 &= i_L(t), \\ v_1 &= i_0(t), & v_2 &= u_0(t), \\ \boldsymbol{A} &= \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{1}{gL} \end{bmatrix}, & \boldsymbol{B} &= \begin{bmatrix} -\frac{1}{C} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}. \end{aligned}$$

In order to write the state equations for a dynamic second-order circuit using the formulation given in Eqns. (8.2), (8.3), and accordingly (8.4), we need to draw the circuit in a resistive two-port and a reactive circuit element. Here, the reactive one-ports could be two capacitors (cf. Fig. 8.2), or two inductors, or one capacitor and one inductor. In general, the resistive two-port will be linear, but not source-free.

By duality, it becomes clear that the possible solutions for the state vector are not qualitatively different, because in each case we obviously can transform a reactive element of one type into the other dual type using a gyrator. The gyrator is then incorporated into the resistive circuit $\mathcal{N}$. Thus, only the characteristics of $\mathcal{N}$ have been modified. Therefore, we will first assess which properties have to apply to $\mathcal{N}$ in order to enable a formulation of a second-order dynamic circuit as given in Eqns. (8.2), (8.3), and accordingly (8.4).

8.2 Solution of State Equation

As we have previously shown, how the state equations can be established for all second-order circuits and that for all these systems of state equations an appropriate realization can be found, we will now assess the solutions for these systems of equations. We get the solutions by a transformation of the primary system of differential equations into a appropriate dual *normal form*.

8.2.1 General Solution

For first-order circuits, we can formulate following general form of the state equation [cf. Eqn. (7.2)]

$$\dot{x}(t) = Ax(t) + Bv(t)$$

where $A = -\frac{1}{\tau}$ and $B = \frac{1}{\tau}$. Its solution can be written as

$$x(t) = \exp(A(t-t_0))x_0 + \int_{t_0}^{t} \exp(A(t-t'))Bv(t')\,\mathrm{d}t'.$$

This result for the first-order case can be generalized to second-order circuits by introducing matrix-vector notation. Therefore,

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}(t-t_0))\boldsymbol{x}_0 + \int_{t_0}^{t} \exp(\boldsymbol{A}(t-t'))\boldsymbol{B}\boldsymbol{v}(t')\,\mathrm{d}t' \tag{8.5}$$

with the inital condition $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$, is the solution of the state equation in Eqn. (8.4). Here, we have employed the matrix-valued exponential function defined by the series

$$\exp(\boldsymbol{A}t) = \boldsymbol{1} + \boldsymbol{A}\frac{t}{1!} + \boldsymbol{A}^2\frac{t^2}{2!} + \cdots + \boldsymbol{A}^n\frac{t^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}\boldsymbol{A}^k \tag{8.6}$$

By this definition, it is apparent that the matrix-valued exponential function exhibits similar properties as the scalar exponential function, e.g.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(\boldsymbol{A}t) = \boldsymbol{A}\exp(\boldsymbol{A}t) = \exp(\boldsymbol{A}t)\boldsymbol{A}. \tag{8.7}$$

This property helps to understand the correctness of the solution in Eqn. (8.5) by substituting into Eqn. (8.4).

The solution in Eqn. (8.5), just as the state equations in Eqn. (8.4), are suitable also for circuits with order larger than two due to the employed vector-matrix notation.

The expression in Eqn. (8.5) falls apart into two parts. The first term $\exp(\boldsymbol{A}(t-t_0))\boldsymbol{x}_0$ is the zero-input response, i.e., the response of the circuit if the excitation vanishes. The second term is the result of a convolution integral and is called the zero-state response, that is, the response if the initial state of the circuit vanishes.

8.2.2 Autonomous Case

After having derived the solution of the system in Eqn. (8.5) depending on the general excitation $\boldsymbol{v}(t)$, we can discuss the special case of a constant excitation. With $\boldsymbol{v}(t) = \boldsymbol{v}_0$, Eqn. (8.5) becomes

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}(t - t_0))\boldsymbol{x}_0 + \int_{t_0}^{t} \exp(\boldsymbol{A}(t - t'))\boldsymbol{B}\boldsymbol{v}_0 \,\mathrm{d}t'.$$

Since $\frac{\mathrm{d}}{\mathrm{d}t} \exp(\boldsymbol{A}t) = \boldsymbol{A} \exp(\boldsymbol{A}t) = \exp(\boldsymbol{A}t)\boldsymbol{A}$ [see Eqn. (8.7)], we find for the integral

$$\int \exp(\boldsymbol{A}t)\,\mathrm{d}t = \exp(\boldsymbol{A}t)\boldsymbol{A}^{-1} + C. \tag{8.8}$$

Therefore, the solution for the autonomous case reads as

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}(t - t_0))\boldsymbol{x}_0 + [\boldsymbol{1} - \exp(\boldsymbol{A}(t - t_0))] \left(-\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{v}_0\right). \tag{8.9}$$

The solution in this form only exists if the state matrix $\boldsymbol{A}$ is invertable.

To be able to understand this result for $\boldsymbol{x}(t)$ better, we find the equilibrium of the autonomous state equation [cf. Eqn. (8.4)]

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{v}_0. \tag{8.10}$$

In the equilibrium, nothing changes. Thus, any derivative w.r.t. time is zero and we get

$$\boldsymbol{A}\boldsymbol{x}_\infty + \boldsymbol{B}\boldsymbol{v}_0 = \boldsymbol{0}.$$

Assuming the inverse of $\boldsymbol{A}$ exists, the equilibrium reads as

$$\boldsymbol{x}_\infty = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{v}_0.$$

Based on this result, we understand that Eqn. (8.9) can be rewritten as

$$\boldsymbol{x}(t) = \boldsymbol{x}_\infty + \exp(\boldsymbol{A}(t - t_0))[\boldsymbol{x}_0 - \boldsymbol{x}_\infty]. \tag{8.11}$$

This expression is a straightforward generalization of the solution in Eqn. (7.9) for first-order circuits.

By substitution, Eqn. (8.10) can be transformed into a homogeneous state equation. Let us set

$$\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{x}_\infty$$

with $\boldsymbol{x}_\infty = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{v}_0$. Since $\dot{\boldsymbol{x}}'(t) = \dot{\boldsymbol{x}}(t)$, we obtain the homogeneous state equation i.e., Eqn. (8.4) with $\boldsymbol{v}(t) = \boldsymbol{0}$,

$$\dot{\boldsymbol{x}}'(t) = \boldsymbol{A}(\boldsymbol{x}'(t) - \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{v}_0) + \boldsymbol{B}\boldsymbol{v}_0 = \boldsymbol{A}\boldsymbol{x}'(t)$$

whose solution is

$$\boldsymbol{x}'(t) = \exp(\boldsymbol{A}(t - t_0))\boldsymbol{x}'(t_0)$$

where $\boldsymbol{x}'(t_0) = \boldsymbol{x}(t_0) - \boldsymbol{x}_\infty = \boldsymbol{x}_0 - \boldsymbol{x}_\infty$. Eqn. (8.11) follows from the back substitution of this solution with $\boldsymbol{x}(t) = \boldsymbol{x}'(t) + \boldsymbol{x}_\infty$.

8.2.3 Homogeneous State Equations

The homogeneous case corresponds to the investigation of the zero-input response, i.e., the component of the response for vanishing excitation. The general response for the case, where the excitation is non-zero, has been discussed in Subsection 8.2.1. For the special case of an autonomous circuit, see Subsection 8.2.2 where it is also shown that the state equation of an autonomous circuit can be transformed to homogeneous form by substitution.

When the excitation $\boldsymbol{v}(t)$ is switched off in Eqn. (8.4), we find the homogeneous state equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \tag{8.12}$$

whose solution is [cf. Eqn. (8.5)]

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}(t - t_0))\boldsymbol{x}_0. \tag{8.13}$$

This expression contains the initial state $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ and the matrix-valued exponential function which has been defined in Eqn. (8.6). With the derivative of the matrix-valued exponential function [see Eqn. (8.7)], it can easily be seen that the expression for $\boldsymbol{x}(t)$ in Eqn. (8.13) is the actual solution to the state equation in Eqn. (8.12).

In the following subsections, the solutions for different constellation regarding the eigenvalues of the state matrix $\boldsymbol{A}$ are discussed. The eigenvalues of the state matrix $\boldsymbol{A}$ are the roots of its characteristic polynomial. Hence, we must solve

$$\det(\boldsymbol{A} - \lambda\boldsymbol{1}) = 0. \tag{8.14}$$

For second-order circuits, this leads to

$$\det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

or

$$\lambda^2 - T\lambda + \varDelta = 0$$

with the trace T and the determinant $\varDelta$ of the state matrix $\boldsymbol{A}$

$$\begin{aligned} T &= \operatorname{tr} \boldsymbol{A} = a_{11} + a_{22} \\ \varDelta &= \det \boldsymbol{A} = a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

Therefore, we get for the eigenvalues

$$\lambda_{1,2} = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - \varDelta}. \tag{8.15}$$

For $\frac{T^2}{4} \geq \varDelta$, the eigenvalues are real-valued and for $\frac{T^2}{4} < \varDelta$, the eigenvalues are complex conjugate. If $\frac{T^2}{4} = \varDelta$, the two eigenvalues are the same since $\lambda_{1,2} = \frac{T}{2}$.

For the case of two different eigenvalues, the homogeneous state equation system can be diagonalized (normal form). When the two eigenvalues are the same, the state matrix cannot be diagonalized in most cases. Therefore, the Jordan normal form must be employed. Finally, for the case, where the two eigenvalues are complex conjugate, we will discuss the transform to the real-valued normal form.

8.2.4 Normal Form

When $\lambda_1 \neq \lambda_2$, the eigenvalue decomposition (EVD) of the state matrix $\boldsymbol{A}$ exists, i.e.,

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{-1} \tag{8.16}$$

with the diagonal matrix comprising the eigenvalues

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

and the modal matrix

$$\boldsymbol{Q} = [\boldsymbol{q}_1, \boldsymbol{q}_2]$$

with the two eigenvectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. An eigenvector $\boldsymbol{q}_k$ and the corresponding eigenvalue λ_k fulfill following fundamental relationship

$$\boldsymbol{A}\boldsymbol{q}_k = \lambda_k \boldsymbol{q}_k$$

which can be rewritten as

$$(\boldsymbol{A} - \lambda_k \mathbf{1})\boldsymbol{q}_k = \mathbf{0}.$$

This equation delivers the trivial solution $\boldsymbol{q}_k = \mathbf{0}$ in general. However, if the matrix $\boldsymbol{A} - \lambda_k \mathbf{1}$ is rank-deficient, any $\boldsymbol{q}_k \neq \mathbf{0}$ out of the nullspace of $\boldsymbol{A} - \lambda_k \mathbf{1}$ is a non-trivial solution of this condition and therefore, an eigenvector corresponding to the eigenvalue λ_k. The requirement that the matrix $\boldsymbol{A} - \lambda_k \mathbf{1}$ is rank-deficient leads to the condition for λ_k that

$$\det(\boldsymbol{A} - \lambda_k \mathbf{1}) = 0.$$

Consequently, every eigenvalue must be a root of the characteristic polynomial [see Eqn. (8.14)]. After the eigenvalue λ_k has been found, the eigenvector can be obtained via

$$(\boldsymbol{A} - \lambda_k \mathbf{1})\boldsymbol{q}_k = \mathbf{0}$$

that is, the eigenvector $\boldsymbol{q}_k$ must be orthogonal to the two rows of $\boldsymbol{A} - \lambda_k \mathbf{1}$. Since the matrix $\boldsymbol{A} - \lambda_k \mathbf{1} = \begin{bmatrix} \boldsymbol{t}_1^{\mathrm{T}} \\ \boldsymbol{t}_2^{\mathrm{T}} \end{bmatrix}$ is rank-deficient by definition, the two rows $\boldsymbol{t}_1^{\mathrm{T}}$ and $\boldsymbol{t}_2^{\mathrm{T}}$ are linearly dependent.

Using a non-trivial row, i.e., $\boldsymbol{t}_\ell^{\mathrm{T}} \neq \boldsymbol{0}^{\mathrm{T}}$, helps to determine the eigenvector since $\boldsymbol{t}_\ell \perp \boldsymbol{q}_k$ must be fulfilled. Based on the notation with components, that is, $\boldsymbol{t}_\ell^{\mathrm{T}} = [t_{1,\ell}, t_{2,\ell}]$, it follows

$$\boldsymbol{q}_k = \begin{bmatrix} q_{1,k} \\ q_{2,k} \end{bmatrix} = \begin{bmatrix} t_{2,\ell} \\ -t_{1,\ell} \end{bmatrix}.$$

If the two eigenvalues are different, i.e., $\lambda_1 \neq \lambda_2$, the two linearly independent eigenvectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ can be found. Combining the two fundamental relationships

$$\begin{aligned} \boldsymbol{A}\boldsymbol{q}_1 &= \boldsymbol{q}_1\lambda_1 \\ \boldsymbol{A}\boldsymbol{q}_2 &= \boldsymbol{q}_2\lambda_2 \end{aligned}$$

in one equation system, i.e.,

$$\begin{aligned} \boldsymbol{A}[\boldsymbol{q}_1, \boldsymbol{q}_2] &= [\boldsymbol{q}_1, \boldsymbol{q}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ \boldsymbol{A}\boldsymbol{Q} &= \boldsymbol{Q}\boldsymbol{\Lambda} \end{aligned}$$

we get the EVD $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{-1}$ [see Eqn. (8.16)]. Note that the EVD only exists surely for different eigenvalues. Substituting the EVD into the homogenous state equation Eqn. (8.12) yields

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{-1}\boldsymbol{x}(t).$$

Multiplying this differential equation by $\boldsymbol{Q}^{-1}$ leads to the state equation in normal form

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{\Lambda}\boldsymbol{\xi}(t) \tag{8.17}$$

based on the substitution

$$\boldsymbol{\xi}(t) = \boldsymbol{Q}^{-1}\boldsymbol{x}(t). \tag{8.18}$$

By the transform to normal form, the second-order circuit has been split into two separate first-order circuits. However, these circuits might not be realizable. When Eqn. (8.15) gives complex conjugate eigenvalues, the realization of Eqn. (8.17) necessitates complex-valued element values!

However, Eqn. (8.17) can be used for finding the solution of the state equation because it falls apart into two first-order differential equations:

$$\begin{aligned} \dot{\xi}_1(t) &= \lambda_1\xi_1(t) \\ \dot{\xi}_2(t) &= \lambda_2\xi_2(t) \end{aligned}$$

whose solutions we know from the discussion of first-order circuits (see Section 7.1). The initial state $\boldsymbol{x}(t_0)$ of the original system can be transformed to the initial state of the normal form by Eqn. (8.18), i.e.,

$$\boldsymbol{\xi}_0 = \begin{bmatrix} \xi_{01} \\ \xi_{02} \end{bmatrix} = \boldsymbol{\xi}(t_0) = \boldsymbol{Q}^{-1}\boldsymbol{x}(t_0) = \boldsymbol{Q}^{-1}\boldsymbol{x}_0.$$

Therefore, the solutions of the decoupled differential equations can be written as

$$\xi_1(t) = \mathrm{e}^{\lambda_1(t-t_0)}\,\xi_{01}$$
$$\xi_2(t) = \mathrm{e}^{\lambda_2(t-t_0)}\,\xi_{02}.$$

Combining these solutions gives the solution to the state equation in normal form

$$\boldsymbol{\xi}(t) = \begin{bmatrix} \exp(\lambda_1(t-t_0)) & 0 \\ 0 & \exp(\lambda_2(t-t_0)) \end{bmatrix} \boldsymbol{\xi}_0 = \begin{bmatrix} \exp(\lambda_1(t-t_0))\xi_{01} \\ \exp(\lambda_2(t-t_0))\xi_{02} \end{bmatrix}. \tag{8.19}$$

Since this solution must have the form of the solution in Eqn. (8.13), we have

$$\boldsymbol{\xi}(t) = \exp(\boldsymbol{\Lambda}(t-t_0))\boldsymbol{Q}^{-1}\boldsymbol{x}_0.$$

Transforming the result back to the original system by Eqn. (8.18) gives

$$\boldsymbol{x}(t) = \boldsymbol{Q}\boldsymbol{\xi}(t) = \boldsymbol{Q}\exp(\boldsymbol{\Lambda}(t-t_0))\boldsymbol{Q}^{-1}\boldsymbol{x}_0$$

and we can observe by comparing to Eqn. (8.13) that

$$\exp(\boldsymbol{A}t) = \boldsymbol{Q}\exp(\boldsymbol{\Lambda}t)\boldsymbol{Q}^{-1} = \boldsymbol{Q}\begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix}\boldsymbol{Q}^{-1}.$$

In other words, the matrix-valued exponential function be found with the help of the EVD of the matrix.

The solution in Eqn. (8.19) can be transformed back to the original state vector by reversing Eqn. (8.18) and we obtain following alternative expression

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{Q}\boldsymbol{\xi}(t) = [\boldsymbol{q}_1, \boldsymbol{q}_2]\begin{bmatrix} \exp(\lambda_1(t-t_0))\xi_{01} \\ \exp(\lambda_2(t-t_0))\xi_{02} \end{bmatrix} \\ \boldsymbol{x}(t) &= \boldsymbol{q}_1\,\mathrm{e}^{\lambda_1(t-t_0)}\,\xi_{01} + \boldsymbol{q}_2\,\mathrm{e}^{\lambda_2(t-t_0)}\,\xi_{02}. \end{aligned} \tag{8.20}$$

We observe that the solution of the homogeneous state equation in Eqn. (8.12) can be expressed as the superposition of two components for the case of two different eigenvalues of the state matrix $\boldsymbol{A}$. Each of the two components shows in the direction of the corresponding eigenvector and is proportional to an exponential function with the product of the eigenvalue and time in the exponent.

8.2.5 Jordan Normal Form

Suppose that the state matrix $\boldsymbol{A}$ has got two identical eigenvalues $\lambda_1 = \lambda_2 = \alpha$. Except for the trivial case that $\boldsymbol{A}$ is already diagonal, i.e.,

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \boldsymbol{\Lambda}, \quad \boldsymbol{Q} = \boldsymbol{1}. \tag{8.21}$$

the eigenvalue decomposition (EVD) of $\boldsymbol{A}$ does not exist, that is, $\boldsymbol{A}$ is defective and not diagonalizable. The non-existence of the EVD of $\boldsymbol{A}$ follows from the fact that only a single eigenvector $\boldsymbol{q}_1$ can be found for the two identical eigenvalues with

$$[\boldsymbol{A} - \alpha \mathbf{1}]\boldsymbol{q}_1 = \mathbf{0}.$$

Therefore, the modal matrix $\boldsymbol{Q}$ does not exist.

Therefore, we employ the Jordan normal form (Jordan canonical form) which exists for any λ_1 and λ_2. The state matrix of the Jordan form reads as

$$\boldsymbol{J} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}. \tag{8.22}$$

Firstly, we define a non-singular transformation matrix $\boldsymbol{Q}' = [\boldsymbol{q}_1', \boldsymbol{q}_2']$ such that

$$\boldsymbol{A}\boldsymbol{Q}' = \boldsymbol{Q}'\boldsymbol{J}. \tag{8.23}$$

Equating the two columns on both sides and employing the definition of the Jordan matrix yields

$$\begin{aligned} \boldsymbol{A}\boldsymbol{q}_1' &= \lambda \boldsymbol{q}_1' \\ \boldsymbol{A}\boldsymbol{q}_2' &= \boldsymbol{q}_1' + \lambda \boldsymbol{q}_2'. \end{aligned}$$

In other words, the first column is the eigenvector $\boldsymbol{q}_1$ of $\boldsymbol{A}$. The second column of $\boldsymbol{Q}'$ can be found via the generalized eigenvalue problem

$$[\boldsymbol{A} - \lambda \mathbf{1}]\,\boldsymbol{q}_2' = \boldsymbol{q}_1.$$

The equation has a solution although $\boldsymbol{A} - \lambda\mathbf{1}$ is rank-deficient since the right-hand side is equal to the eigenvector of $\boldsymbol{A}$ that spans the range space of $\boldsymbol{A}$. Based on Eqn. (8.23), it holds that

$$\boldsymbol{A} = \boldsymbol{Q}'\boldsymbol{J}\boldsymbol{Q}'^{-1}$$

with $\boldsymbol{Q}' = [\boldsymbol{q}_1, \boldsymbol{q}_2']$. Substituting this decomposition of the state matrix $\boldsymbol{A}$ into the state equation in Eqn. (8.12) and setting $\boldsymbol{\xi}'(t) = \boldsymbol{Q}'^{-1}\boldsymbol{x}(t)$ leads to the state equation in Jordan normal form

$$\dot{\boldsymbol{\xi}}'(t) = \boldsymbol{J}\boldsymbol{\xi}'(t) \tag{8.24}$$

or equivalently,

$$\begin{aligned} \dot{\xi}_1'(t) &= \lambda \xi_1'(t) + \xi_2'(t) \\ \dot{\xi}_2'(t) &= \lambda \xi_2'(t). \end{aligned} \tag{8.25}$$

The initial state $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ of the original system can be transformed to the initial state of the Jordan normal form by the usual transform

$$\boldsymbol{\xi}_0' = \begin{bmatrix} \xi_{01}' \\ \xi_{02}' \end{bmatrix} = \boldsymbol{\xi}'(t_0) = \boldsymbol{Q}'^{-1}\boldsymbol{x}_0.$$

Therefore, the solution to the second state equation in Eqn. (8.25) reads as

$$\xi_2'(t) = \mathrm{e}^{\lambda(t-t_0)}\,\xi_{02}'.$$

Plugging this result for $\xi_2'(t)$ into the first state equation in Eqn. (8.25) and solving the resulting differential equation yields

$$\xi_1'(t) = \mathrm{e}^{\lambda(t-t_0)}\,\xi_{01}' + (t-t_0)\,\mathrm{e}^{\lambda(t-t_0)}\,\xi_{02}'.$$

The correctness of this expression can be validated by substituting into Eqn. (8.25). Thus, the solution in Jordan normal form reads as

$$\boldsymbol{\xi}'(t) = \begin{bmatrix} \xi_1'(t) \\ \xi_2'(t) \end{bmatrix} = \begin{bmatrix} \exp(\lambda(t-t_0))\xi_{01}' + (t-t_0)\exp(\lambda(t-t_0))\xi_{02}' \\ \exp(\lambda(t-t_0))\xi_{02}' \end{bmatrix}. \tag{8.26}$$

Comparing to the general solution in the Jordan normal form

$$\boldsymbol{\xi}'(t) = \exp(\boldsymbol{J}(t-t_0))\boldsymbol{\xi}_0'$$

we can infer that

$$\exp(\boldsymbol{J}t) = \begin{bmatrix} \exp(\lambda t) & t\exp(\lambda t) \\ 0 & \exp(\lambda t) \end{bmatrix}.$$

By the back transformation

$$\boldsymbol{x}(t) = \boldsymbol{Q}'\boldsymbol{\xi}'(t)$$

we finally obtain the solution for the original system

$$\boldsymbol{x}(t) = \boldsymbol{q}_1\left(\mathrm{e}^{\lambda(t-t_0)}\,\xi_{01}' + (t-t_0)\,\mathrm{e}^{\lambda(t-t_0)}\,\xi_{02}'\right) + \boldsymbol{q}_2'\,\mathrm{e}^{\lambda(t-t_0)}\,\xi_{02}'. \tag{8.27}$$

This expression shows the peculiarity of the case with two identical eigenvalues with a non-diagonal state matrix $\boldsymbol{A}$. Besides the otherwise typical expression $\exp(\lambda t)$, the solution comprises a term in the form $t\exp(\lambda t)$. Such a component is impossible for first-order circuits.

8.2.6 Real-Valued Normal Form

For conjugate complex eigenvalues $\lambda_{1,2} = \alpha \pm \mathrm{j}\,\beta$, there exists besides the complex-valued diagonal normal form

$$\boldsymbol{\Lambda} = \begin{bmatrix} \alpha + \mathrm{j}\,\beta & 0 \\ 0 & \alpha - \mathrm{j}\,\beta \end{bmatrix} \tag{8.28}$$

also a *real-valued normal form*, that is,

$$\boldsymbol{\Lambda}_{\text{real}} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{8.29}$$

In the following, we assume that $\beta > 0$ and define $\lambda = \alpha + \mathrm{j}\,\beta$.

The eigenvectors corresponding the complex conjugate eigenvalues $\lambda_1 = \lambda,\ \lambda_2 = \lambda^*$ with

$$\boldsymbol{A}\boldsymbol{q}_1 = \lambda \boldsymbol{q}_1, \quad \boldsymbol{A}\boldsymbol{q}_2 = \lambda^* \boldsymbol{q}_2,$$

are also complex conjugate

$$\boldsymbol{q}_1 = \boldsymbol{q} = \boldsymbol{q}_{\mathrm{r}} + \mathrm{j}\,\boldsymbol{q}_{\mathrm{i}}, \quad \boldsymbol{q}_2 = \boldsymbol{q}^*.$$

Now we split the complex system of equations $\boldsymbol{A}\boldsymbol{q} = \lambda\boldsymbol{q}$ or equivalently,

$$\boldsymbol{A}\,(\boldsymbol{q}_{\mathrm{r}} + \mathrm{j}\,\boldsymbol{q}_{\mathrm{i}}) = (\alpha + \mathrm{j}\,\beta)\,(\boldsymbol{q}_{\mathrm{r}} + \mathrm{j}\,\boldsymbol{q}_{\mathrm{i}})$$

into two decoupled real-valued equations. We get

$$\begin{aligned} \boldsymbol{A}\boldsymbol{q}_{\mathrm{r}} &= \alpha\boldsymbol{q}_{\mathrm{r}} - \beta\boldsymbol{q}_{\mathrm{i}}, \\ -\boldsymbol{A}\boldsymbol{q}_{\mathrm{i}} &= -\beta\boldsymbol{q}_{\mathrm{r}} - \alpha\boldsymbol{q}_{\mathrm{i}}. \end{aligned}$$

The two vectors $\boldsymbol{A}\boldsymbol{q}_{\mathrm{r}}$ and $-\boldsymbol{A}\boldsymbol{q}_{\mathrm{i}}$ are comprised in one matrix and we obtain

$$\boldsymbol{A}[\boldsymbol{q}_{\mathrm{r}}, -\boldsymbol{q}_{\mathrm{i}}] = [\boldsymbol{q}_{\mathrm{r}}, -\boldsymbol{q}_{\mathrm{i}}] \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

or shorter

$$\boldsymbol{A}\boldsymbol{Q}_{\mathrm{real}} = \boldsymbol{Q}_{\mathrm{real}}\boldsymbol{\Lambda}_{\mathrm{real}}.$$

The matrix $\boldsymbol{Q}_{\mathrm{real}} = [\boldsymbol{q}_{\mathrm{r}}, -\boldsymbol{q}_{\mathrm{i}}]$ is alway invertible since the real and imaginary parts of a complex eigenvector for complex conjugate eigenvalues are never linearly dependent—otherwise, we would have $\boldsymbol{q} = \gamma\boldsymbol{q}_{\mathrm{r}}$ with $\gamma \in \mathbb{C}$ and $\lambda \in \mathbb{R}$, contraticting the assumption of complex eigenvalues and eigenvectors.

The modal matrix $\boldsymbol{Q} = [\boldsymbol{q}, \boldsymbol{q}^*]$ leads to the complex-valued normal form

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{\Lambda}\boldsymbol{\xi}(t)$$

with the diagonal state matrix $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda, \lambda^*)$ and $\boldsymbol{Q}_{\mathrm{real}} = [\boldsymbol{q}_{\mathrm{r}}, -\boldsymbol{q}_{\mathrm{i}}]$ gives the real-valued normal form

$$\dot{\boldsymbol{\xi}}_{\mathrm{real}}(t) = \boldsymbol{\Lambda}_{\mathrm{real}}\boldsymbol{\xi}_{\mathrm{real}}(t)$$

with the non-diagonal but real-valued state matrix $\boldsymbol{\Lambda}_{\mathrm{real}} = \left[\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right]$. The solutions of the state system in real-valued normal form can be found from the solutions of the complex-valued normal form by appropriate coordinate transform. Due to the according definitions, we have

$$\boldsymbol{Q} = [\boldsymbol{q}_{\mathrm{r}} + \mathrm{j}\,\boldsymbol{q}_{\mathrm{i}}, \boldsymbol{q}_{\mathrm{r}} - \mathrm{j}\,\boldsymbol{q}_{\mathrm{i}}] = [\boldsymbol{q}_{\mathrm{r}}, -\boldsymbol{q}_{\mathrm{i}}] \begin{bmatrix} 1 & 1 \\ -\mathrm{j} & \mathrm{j} \end{bmatrix}.$$

When the state vectors of the complex-valued and real-valued normal forms are transformed back to the original state space, we must obtain the same state vector $\boldsymbol{x}(t)$. Therefore, we obtain the following relationship between the state vectors $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_{\text{real}}$

$$\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{\xi} = \boldsymbol{Q}_{\text{real}}\boldsymbol{\xi}_{\text{real}}.$$

The combination of the last two results leads to

$$\boldsymbol{\xi}_{\text{real}} = \boldsymbol{Q}_{\text{real}}^{-1}\boldsymbol{Q}\boldsymbol{\xi} = \begin{bmatrix} 1 & 1 \\ -\mathrm{j} & \mathrm{j} \end{bmatrix}\boldsymbol{\xi}. \tag{8.30}$$

Therefore, we find based on the two complex-valued state variables $\xi_1 = \xi$ and $\xi_2 = \xi^*$ that

$$\begin{aligned}
\xi_{\text{real1}} &= \xi_1 + \xi_2 = \xi + \xi^* = 2\,\mathrm{Re}(\xi), \\
\xi_{\text{real2}} &= -\,\mathrm{j}\,(\xi_1 - \xi_2) = \frac{1}{\mathrm{j}}\,(\xi - \xi^*) = 2\,\mathrm{Im}(\xi).
\end{aligned}$$

Assuming that the initial state of the complex-valued normal form is

$$\boldsymbol{\xi}_0 = \boldsymbol{\xi}(t_0) = \boldsymbol{Q}^{-1}\boldsymbol{x}_0 = [\boldsymbol{q}, \boldsymbol{q}^*]^{-1}\boldsymbol{x}_0$$

with $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ and

$$\boldsymbol{\xi}_0 = \begin{bmatrix} \xi_0 \\ \xi_0^* \end{bmatrix} = \begin{bmatrix} \gamma + \mathrm{j}\,\delta \\ \gamma - \mathrm{j}\,\delta \end{bmatrix} \qquad \gamma, \delta \in \mathbb{R}.$$

Consequently, we obtain the real-valued solutions

$$\begin{aligned}
\xi_{\text{real1}} &= \exp((\alpha + \mathrm{j}\,\beta)t)\xi_0 + \exp((\alpha - \mathrm{j}\,\beta)t)\xi_0^* = 2\exp(\alpha t)\,(\gamma\cos\beta t - \delta\sin\beta t) \\
\xi_{\text{real2}} &= -\,\mathrm{j}\,(\exp((\alpha + \mathrm{j}\,\beta)t)\xi_0 - \exp((\alpha - \mathrm{j}\,\beta)t)\xi_0^*) = 2\exp(\alpha t)\,(\delta\cos\beta t + \gamma\sin\beta t)
\end{aligned} \tag{8.31}$$

with the real-valued initial states

$$\boldsymbol{\xi}_{\text{real0}} = \begin{bmatrix} \xi_{\text{real01}}(t) \\ \xi_{\text{real02}}(t) \end{bmatrix} = \begin{bmatrix} \xi_0 + \xi_0^* \\ -\,\mathrm{j}\,(\xi_0 - \xi_0^*) \end{bmatrix} = \begin{bmatrix} 2\gamma \\ 2\delta \end{bmatrix}. \tag{8.32}$$

The real-valued normal form is useful for the two-dimensional depiction of the phase portrait.

8.3 Phase Portraits

For a two-dimensional system, only a limited number of actually distinct and important classes of qualitative behavior exist. These classes of qualitative behavior will be discussed and derived in the following.

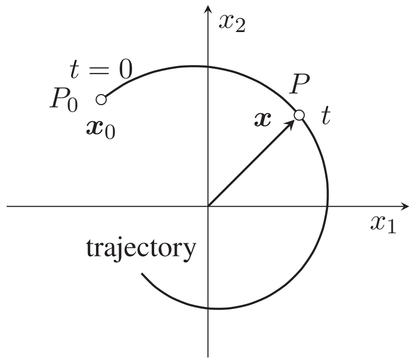


Figure 8.3: Trajectory for the initial state $\boldsymbol{x}_0$

8.3.1 Trajectories

The behavior of a dynamic second-order circuit can be demonstratively illustrated within the state space by drawing the curve (*trajectory*), which shows the state vectors with respect to time t, starting from the initial state $\boldsymbol{x}_0$ (or $\boldsymbol{\xi}_0$) (cf. Fig. 8.3). The set of all trajectories with different initial states defines the *phase portrait*. It is clear that only a few trajectories can be drawn.

The behavior of the circuit depends on its dimensioning and is uniquely defined by the eigenvalues and the bearing of the the eigenvectors within the x_1-x_2-plane. Depending on the eigenvalues λ_1 and λ_2 within the λ-plane, we get the case-by-case study which is shown in the following Table 8.1.

The eleven cases can be depicted in a $\mathit{\Delta}$-T-diagram (see Fig. 8.4) with $\mathit{\Delta} = \det(\boldsymbol{A})$ and $T = \operatorname{tr}(\boldsymbol{A})$. However, we will only discuss the practically most important cases in the following.

8.3.2 Focii (Cases 1 and 3)

For complex conjugate eigenvalues with $\alpha \neq 0$, we get a *focus* in the phase portrait. For the real-valued normal form, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_{\text{real1}}(t) = \alpha\xi_{\text{real1}}(t) - \beta\xi_{\text{real2}}(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_{\text{real2}}(t) = \beta\xi_{\text{real1}}(t) + \alpha\xi_{\text{real2}}(t).$$

With

$$2\delta = k\sin\theta, \quad 2\gamma = k\cos\theta$$

the corresponding solution Eqn. (8.31) can be expressed as

$$\xi_{\text{real1}}(t) = \exp(\alpha t)\left(k\cos\theta\cos(\beta t) - k\sin\theta\sin(\beta t)\right) = k\exp(\alpha t)\cos(\beta t + \theta),$$
$$\xi_{\text{real2}}(t) = \exp(\alpha t)\left(k\sin\theta\cos(\beta t) + k\cos\theta\sin(\beta t)\right) = k\exp(\alpha t)\sin(\beta t + \theta).$$

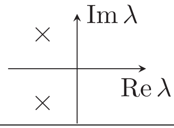
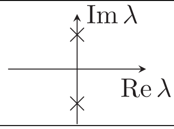
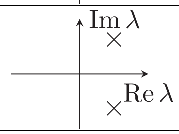
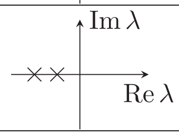
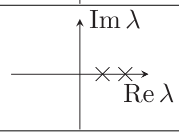
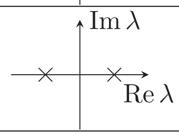
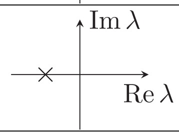
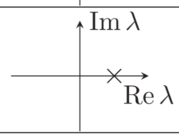
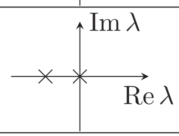
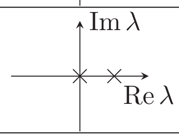
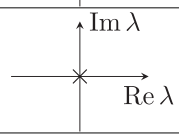
Case	Δ	T	λ_1, λ_2	λ-Plane	$\boldsymbol{\Lambda}, \boldsymbol{\Lambda}', \boldsymbol{J}$	ν
1		< 0	$\alpha \pm \mathrm{j}\,\beta$ $\quad \alpha < 0$, $\beta \in \mathbb{R}_+$		$\begin{bmatrix} \alpha + \mathrm{j}\,\beta & 0 \\ 0 & \alpha - \mathrm{j}\,\beta \end{bmatrix}$ $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$	0
2	$> \left(\frac{T}{2}\right)^2$	$= 0$	$\alpha \pm \mathrm{j}\,\beta$ $\quad \alpha = 0$, $\beta \in \mathbb{R}_+$			
3		> 0	$\alpha \pm \mathrm{j}\,\beta$ $\quad \alpha > 0$, $\beta \in \mathbb{R}_+$			
4	$< \left(\frac{T}{2}\right)^2$	< 0	$\alpha_1 \neq \alpha_2$ $\quad \alpha_1 < 0$, $\alpha_2 < 0$		$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$	0
5	<u>und</u> > 0	> 0	$\alpha_1 \neq \alpha_2$ $\quad \alpha_1 > 0$, $\alpha_2 > 0$			
6	< 0		$\operatorname{sgn} \alpha_1 \neq \operatorname{sgn} \alpha_2$			
7		< 0	$\alpha_1 = \alpha_2 = \alpha$, $\alpha < 0$		$\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$	0
8	$= \left(\frac{T}{2}\right)^2$	> 0	$\alpha_1 = \alpha_2 = \alpha$, $\alpha > 0$			
9		< 0	$\alpha_1 = 0\ \alpha_2 < 0$		$\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}$	
10	$= 0$	> 0	$\alpha_1 = 0\ \alpha_2 > 0$			$\neq 0$
11		$= 0$	$\alpha_1 = \alpha_2 = 0$		$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	

Table 8.1: Different cases for a second-order system

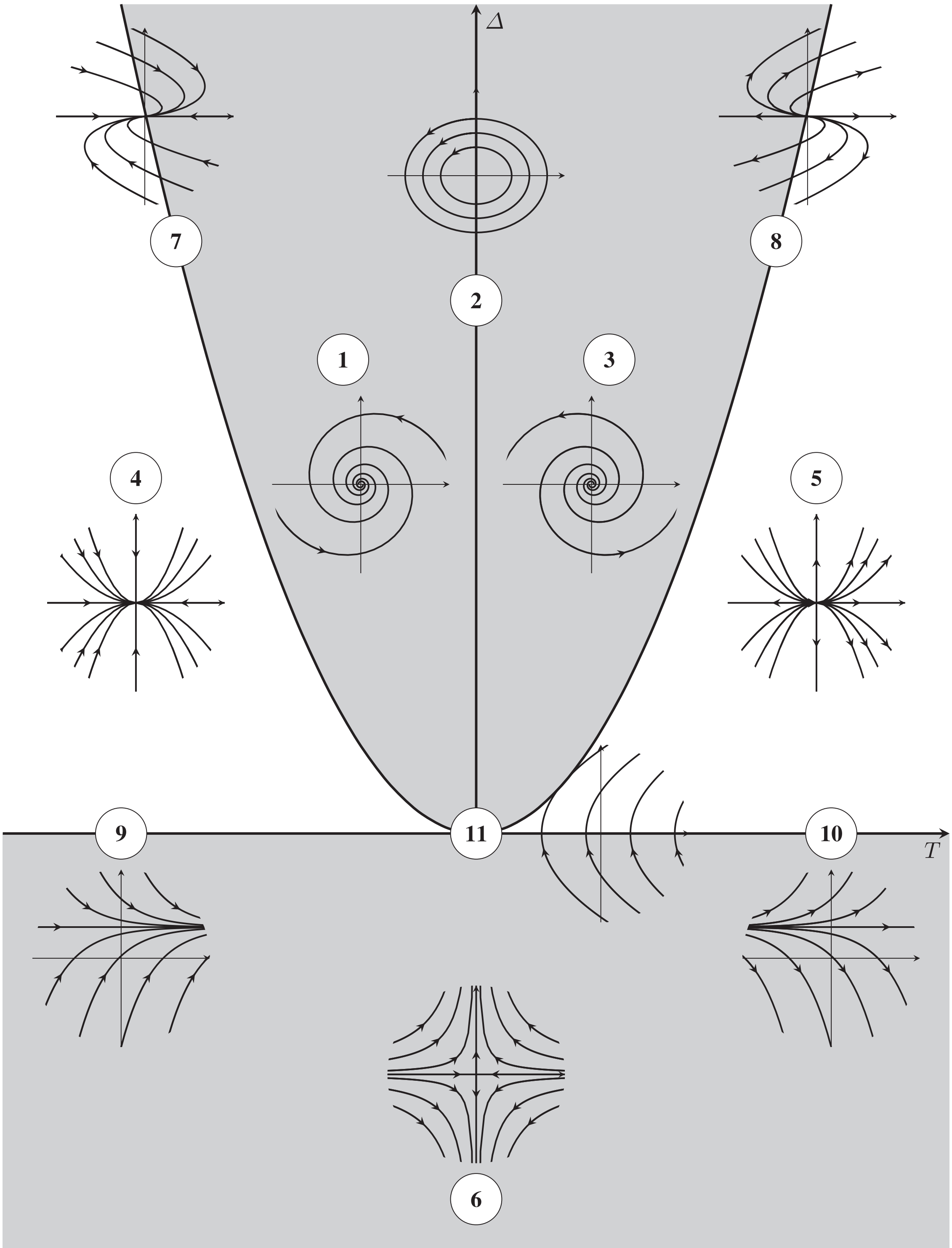


Figure 8.4: Phase portraits for the different cases in the Δ-T-Diagram

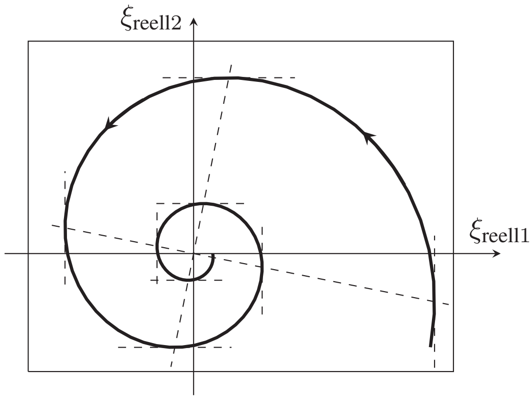


Figure 8.5: Phase portrait for a stable focus

When the solution is transformed to polar coordinates (radius ρ and angle ϕ) and then the time t is eliminated, we get:

$$\phi = \arctan \frac{\xi_{\text{real2}}(t)}{\xi_{\text{real1}}(t)} = \beta t + \theta \Rightarrow t = \frac{\phi - \theta}{\beta},$$
$$\rho = \sqrt{\xi^2_{\text{real1}}(t) + \xi^2_{\text{real2}}(t)} = k \exp(\alpha t) = k \exp(\alpha(\phi - \theta)/\beta).$$

This is the the equation of a logarithmic spiral, which for $\alpha < 0$ (Case 1) is turning counterclockwise and which tends exponentially (by definition $\beta > 0$) to zero (equilibrium state). For $t \to \infty$, all trajectories tend towards the point $\boldsymbol{\xi}_{\text{real}} = \mathbf{0}$. This phase portrait thus is called a *stable focus* (Fig. 8.5).

For $\alpha > 0$ (Case 3), we can derive the trajectories by mirroring the curves of Fig. 8.5 at a line through the initial value and the origin. The trajectories will again turn counterclockwise. The point $\boldsymbol{\xi}_{\text{real}} = \mathbf{0}$ is the equilibrium state towards which all the trajectories are tending for $t \to -\infty$. This phase portrait is called a *unstable focus*.

The trajectories of the underlying circuit can be derived by a *similarity transformation* towards the direction of the eigenvectors $\boldsymbol{q}_{\text{r}}$ and $-\boldsymbol{q}_{\text{i}}$ with

$$\boldsymbol{x}_{\text{real}} = \boldsymbol{Q}_{\text{real}} \boldsymbol{\xi}_{\text{real}}$$

and an additional coordinate shift around $\boldsymbol{x}_{\infty}$ (see Fig. 8.6). For the real-valued normal form, the eigenvectors are represented by the axes of the coordinate system.

8.3.3 Centers (Case 2)

For solely imaginary eigenvalues ($\alpha = 0$), we get a *center* as a phase portrait. Based on the definition of the trajectories in polar coordinates as given in Subsection 8.3.2, we get for $\alpha = 0$:

$$\rho(\phi) = k = \text{const.}$$

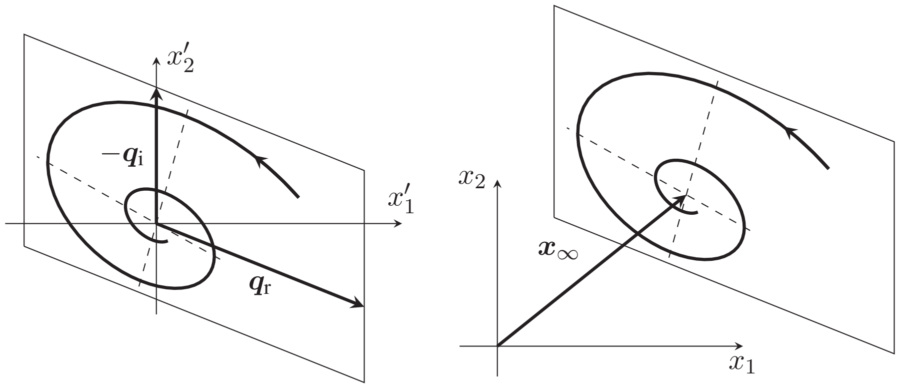


Figure 8.6: Similarity transformation (left-hand side) and coordinate shift (right-hand side) of a stable focus

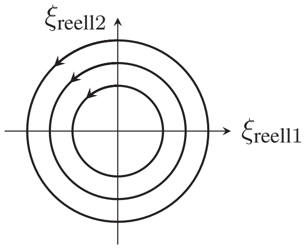


Figure 8.7: Phasenportrait für einen Wirbelpunkt

The trajectories are circles (or with uneven scaling of the axes, ellipses) around the origin (see Fig. 8.7).

8.3.4 Nodes (Cases 4 and 5)

In case of real-valued different eigenvalues with equal sign, we get a phase portrait of a *node*. For $\lambda_1, \lambda_2 < 0$ (Case 4), we get for the normal form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_1(t) = \lambda_1\xi_1(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_2(t) = \lambda_2\xi_2(t),$$

with the solution

$$\xi_1(t) = \exp(\lambda_1 t)\xi_{01},$$
$$\xi_2(t) = \exp(\lambda_2 t)\xi_{02}.$$

By the elimination of the time t, we obtain

$$t = \frac{1}{\lambda_1}\ln\left(\frac{\xi_1}{\xi_{01}}\right) \Rightarrow \xi_2 = \xi_{02}\left|\frac{\xi_1}{\xi_{01}}\right|^{\frac{\lambda_2}{\lambda_1}}.$$

The phase portrait is depicted in Fig. 8.8 for different trajectories. For Fig. 8.8, it is assumed that $|\lambda_1| < |\lambda_2|$, i.e., the first eigenfrequency $|\lambda_1|$ is low (*slow*) and $|\lambda_2|$ is high (*fast*). Therefore, λ_1 is called the *slow eigenvalue* and λ_2 is the *fast eigenvalue*.

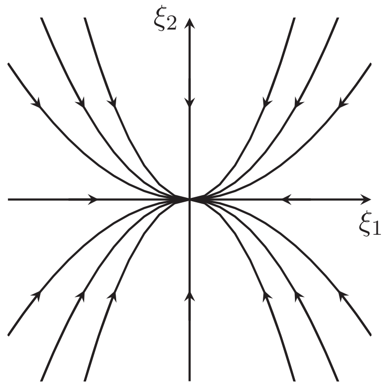


Figure 8.8: Phase portrait for a stable node with $|\lambda_1| < |\lambda_2|$

The trajectories for increasing t-values nestle to the coordinate direction of the low natural frequency (in Fig. 8.8 the direction given by ξ_1). They come from the coordinate direction given by the higher normal frequency (in Fig. 8.8, the direction given by ξ_2). The equilibrium state $\boldsymbol{\xi} = \mathbf{0}$ is stable: All trajectories expand towards it for $t \to \infty$. This is a *stable node*.

For positive eigenvalues (Case 5), we qualitatively get the same phase portrait, but the opposite direction of flow of the trajectories. The equilibrium state $\boldsymbol{\xi} = \mathbf{0}$ is unstable. All trajectories expand towards it for $t \to -\infty$. This is an *unstable node*.

8.3.5 Saddle Points (Case 6)

For real-valued and unequal eigenvalues with different sign $\text{sgn}\alpha_1 \neq \text{sgn}\alpha_2$ (Case 6), generally we get the equivalent solution for ξ_1 and ξ_2, as for Case 4 and Case 5. However, because of the different signs of α_1 and α_2, we have:

$$\xi_2 = \xi_{02} \left| \frac{\xi_1}{\xi_{01}} \right|^{-\left|\frac{\lambda_2}{\lambda_1}\right|} .$$

The trajectories are hyperbolas (see Fig. 8.9).

The trajectories come from the coordinate direction related to the negative eigenvalue ($t \to -\infty$) and for $t \to \infty$ they nestle to the coordinate direction related to the positive eigenvalue. The equilibrium $\boldsymbol{\xi} = \mathbf{0}$ is unstable and is called a *saddle point*. The phase portrait is called *saddle*.

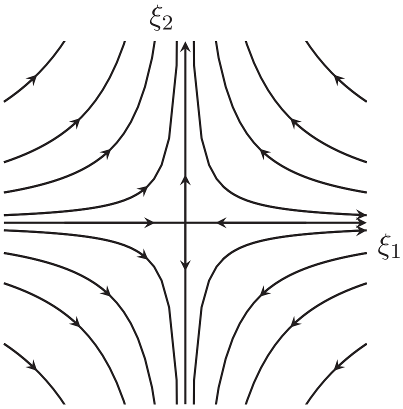


Figure 8.9: Phase portrait for a saddle point with $\lambda_1 < 0, \lambda_2 > 0$

8.3.6 Degenerate Cases (Cases 7–11)

The Cases 7 to 11 are called degenerate since either both non-zero eigenvalues are identical, one eigenvalue is zero, or both eigenvalues are zero. When the two eigenvalues are the same (Cases 7, 8, and 11), the solution can be obtained by using the Jordan normal form. For two non-zero but identical eigenvalues, the *critically damped* oscillation results with the equilibrium lying in the origin of the ξ_1'-ξ_2'-plane.

The other degenerate cases are a consequence of a singular state matrix $\boldsymbol{A}$. Due to the singularity of $\boldsymbol{A}$, the circuits have got an infinite number of equilibria for the homogeneous setup, i.e., when all the sources are zero. However, when the sources of dynamic circuit with only one zero eigenvalue are non-zero, the instability of these circuits becomes evident even for constant sources (i.e., autonomous dynamic systems). Systems with two zero eigenvalues are obviously unstable even for the homogeneous setup.

Chapter 9

Complex Phasor Analysis

In this chapter, the analysis of linear time-invariant dynamic circuits driven by *sinusoidal sources* will be discussed. We are interested in determining the solution of such circuits in the steady (stationary) state, i.e., after the effects of the *transients* have faded away depending on the initial conditions.

Even though it seems that we are restricting ourselves to a very special operation, this case is of extreme practical importance. Many circuits and systems have sinusoidal excitations (like the sinusoidal voltage of the power plug) that are at least approximately in the steady state.

Hence, a very efficient technique will be developed which is based on representing sinusoidal signals (currents and voltages) with the aid of so-called *phasors*. This technique is known as *phasor analysis*. By considering complex signals, the technique allows to treat dynamic circuits by employing a very simple extension of the previously introduced resistive circuits analysis.

9.1 Sinusoidal Input and Steady State

Sinusoidal signals or sinusoids are functions of the following form

$$a(t) = A_\mathrm{m}\cos(\omega t + \alpha) \qquad \forall t \in (-\infty, \infty) \tag{9.1}$$

where A_m is the amplitude, ω is the angular frequency in rad/s and α is the initial phase in rad, with A_m, ω, $\alpha \in \mathbb{R}$. The corresponding ordinary frequency $f = \frac{\omega}{2\pi}$ is measured in Hertz (cycles per second).

For all independent sources of the circuit, it must hold that their voltages and currents have the form of Eqn. (9.1), i.e., they are sinusodial with the angular frequency ω. Under the assumption of a linear circuit (with strictly linear resistors, nullators, norators, linear sources, linear capacitors, linear inductors, and linear two-ports) that all currents and voltages of the whole circuit also exhibit the form of Eqn. (9.1).

This observation is a consequence of the inherent linearity of the Kirchhoff's equations and the assumed linearity of the elements constituting the circuit. To illustrate this, we consider a strictly linear resistor which obeys the Ohm's law

$$u(t) = Ri(t).$$

If the current is sinusoidal [cf. Eqn. (9.1)], that is,

$$i(t) = I_\mathrm{m}\cos(\omega t + \beta)$$

then, the voltage reads as

$$u(t) = RI_\mathrm{m}\cos(\omega t + \beta) = U_\mathrm{m}\cos(\omega t + \beta)$$

with $U_\mathrm{m} = RI_\mathrm{m}$. Therefore, the linear Ohm's law does not change the signal form (still sinusoidal with angular frequency ω), only the amplitude is altered. This result generalizes to resistive two-ports.

When investigating the derivative of $a(t)$ w.r.t. time, as the voltage of a capacitor or the current of an inductor are differentiated, we yield [cf. Eqn. (9.1)]

$$\dot{a}(t) = \frac{\mathrm{d}a(t)}{\mathrm{d}t} = -\omega A_\mathrm{m}\sin(\omega t + \alpha) = \omega A_\mathrm{m}\cos(\omega t + \gamma)$$

with $\gamma = \alpha + \frac{\pi}{2}$. Hence, differentiating w.r.t. time does not change the pricipal form as in Eqn. (9.1). Only the amplitude and the phase are altered.

In other words, linear elements to not change the sinewave form and the angular frequency ω. Merely the amplitude and the phase can vary.

For the Kirchhoff's laws, the same is true. For example, consider a KVL equation like (for a series connection)

$$u_\mathrm{tot}(t) = u_1(t) + u_2(t)$$

with

$$\begin{aligned} u_1(t) &= U_\mathrm{1m}\cos(\omega t) \\ u_2(t) &= U_\mathrm{2m}\cos(\omega t + \delta). \end{aligned}$$

Since $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ [to separate the phases from $\cos(\omega t)$ and $\sin(\omega t)$] and $\sin^2(\varphi) + \cos^2(\varphi) = 1$ [to get rid of $\sin(\varphi)$ and $\cos(\varphi)$], the total voltage can be written as [cf. Eqn. (9.1)]

$$u_\mathrm{tot}(t) = U_\mathrm{tot}\cos(\omega t + \varphi)$$

with

$$\begin{aligned} U_\mathrm{tot}^2 &= U_\mathrm{1m}^2 + U_\mathrm{2m}^2 + 2U_\mathrm{1m}U_\mathrm{2m}\cos(\delta) \\ \tan(\varphi) &= \frac{U_\mathrm{2m}\sin(\delta)}{U_\mathrm{1m} + U_\mathrm{2m}\cos(\delta)}. \end{aligned}$$

Consequently, the linear elements and also the linear Kirchhoff's equations do not change the principal properties sine waveform and angular frequency ω.

As a resultat of above discussion we can state the following. A linear circuit with resistive but also reactive elements whose independent sources are all sinusoidal with the same angular frequency ω, has branch currents and branch voltages which are all sinusoidal with the angular frequency ω. The different currents and voltages of the circuit only differ in their amplitudes and angles.

Due to the assumed linearity of the circuit, the superposition priciple can be employed. Therefore, the different voltages and currents of the circuit can be obtained depending on the different sources and in the end, the actual values can be found as the sum of the different portions.

This observation eliminates the restriction that all sources need to have the same angular frequency. For example, let the circuit have two independent sources with the angular frequencies ω_1 and ω_2. Thus, every quantity (branch voltage or branch current) of the circuit can be written as (according to the superposition principle)

$$b(t) = B_{1\mathrm{m}} \cos(\omega_1 t + \beta_1) + B_{2\mathrm{m}} \cos(\omega_2 t + \beta_2).$$

The first portion is a consequence of the source with the angular frequency ω_1 and the second portion is due to the source with the angular frequency ω_2.

Finally, we must note that the calculation of the sinusoidal quantities are founded on the assumption that the circuit is in steady state. The discussion of the transient response of the circuit is out of the scope of this lecture.

9.2 Complex Phasors

To every sinusoidal signal

$$a(t) = A_{\mathrm{m}} \cos(\omega t + \alpha)$$

with the amplitude $A_{\mathrm{m}} > 0$, the angular frequency ω, and the phase α, we can associate a complex number A, the phasor

$$A = A_{\mathrm{m}}\, \mathrm{e}^{\mathrm{j}\,\alpha}\,. \tag{9.2}$$

Note that in Eqn. (9.2), the phasor A is given in polar form. Therefore, its magnitude is the amplitude $A_{\mathrm{m}} = \sqrt{AA^*}$ and α is the phase of the complex number. An important point is that the assignment from sinusoidal signals to complex phasors expressed in Eq. (9.2) is unique.

The real-valued sinusoidal signal can be obtained from the phasor as follows

$$a(t) = \mathrm{Re}\left\{A\, \mathrm{e}^{\mathrm{j}\,\omega t}\right\}. \tag{9.3}$$

This relationship of the sinusoidal time function $a(t)$ and the corresponding phasor A can be shown with the help of Euler's rule $\mathrm{e}^{\mathrm{j}\,\varphi} = \cos(\varphi) + \mathrm{j}\sin(\varphi)$:

$$\begin{aligned}\mathrm{Re}\left\{A\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\} &= \mathrm{Re}\left\{A_\mathrm{m}\,\mathrm{e}^{\mathrm{j}(\omega t+\alpha)}\right\} \\ &= \mathrm{Re}\left\{A_\mathrm{m}\left[\cos(\omega t+\alpha) + \mathrm{j}\sin(\omega t+\alpha)\right]\right\} = A_\mathrm{m}\cos(\omega t+\alpha) = a(t).\end{aligned}$$

In order to employ the previously defined phasors to represent sinusoidal signals in the analysis of dynamic circuits, we formulate three lemmas.

Lemma 1: Uniqueness
Two sinusoidal signals $a(t)$ and $b(t)$ are equal if and only if the corresponding phasors A and B are equal, i.e.,

$$\forall t: \quad a(t) = b(t) \qquad \Leftrightarrow \qquad A = B. \tag{9.4}$$

Für den Beweis nutzen wir die Definition der Zeiger $A = A_\mathrm{m}\,\mathrm{e}^{\mathrm{j}\,\alpha}$ and $B = B_\mathrm{m}\,\mathrm{e}^{\mathrm{j}\,\beta}$, that is,

$$\begin{aligned}&\forall t: \quad a(t) = b(t) \\ \Leftrightarrow\quad &\forall t: \quad A_\mathrm{m}\cos(\omega t+\alpha) = B_\mathrm{m}\cos(\omega t+\beta) \\ \Leftrightarrow\quad &A_\mathrm{m} = B_\mathrm{m} \quad \text{und} \quad \alpha = \beta \\ \Leftrightarrow\quad &A = A_\mathrm{m}\,\mathrm{e}^{\mathrm{j}\,\alpha} = B_\mathrm{m}\,\mathrm{e}^{\mathrm{j}\,\beta} = B.\end{aligned}$$

Lemma 2: Linearity
The phasor of a linear combination of sinusoidal signals is equal to the same linear combination of the phasors which represent the individual phasors.

$$y(t) = \sum_{\ell=1}^{L} a_\ell x_\ell(t) \qquad \Leftrightarrow \qquad Y = \sum_{\ell=1}^{L} a_\ell X_\ell. \tag{9.5}$$

With the definition of the phasors [see Eqn. (9.2)] and the linearity of the real-part operator, it holds that

$$\begin{aligned}&y(t) = \sum_{\ell=1}^{L} a_\ell x_\ell(t) \\ \Leftrightarrow\quad &\mathrm{Re}\left\{Y\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\} = \sum_{\ell=1}^{L} a_\ell\,\mathrm{Re}\left\{X_\ell\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\} \\ \Leftrightarrow\quad &\mathrm{Re}\left\{Y\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\} = \mathrm{Re}\left\{\sum_{\ell=1}^{L} a_\ell X_\ell\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\} \\ \Leftrightarrow\quad &Y = \sum_{\ell=1}^{L} a_\ell X_\ell.\end{aligned}$$

Lemma 3: Differentiation
The phasor of a sinusoidal signal $A_\mathrm{m}\cos(\omega t+\alpha)$ is equal to A if and only if $\mathrm{j}\,\omega A$ is the phasor of its derivative, i.e. the phasor of the derivative of $A_\mathrm{m}\cos(\omega t+\alpha)$, i.e.,

$$b(t)=\dot{a}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\,a(t) \qquad \Leftrightarrow \qquad B=\mathrm{j}\,\omega A. \tag{9.6}$$

For the proof, we consider the derivative of a function $z(t)=x(t)+\mathrm{j}\,y(t)\in\mathbb{C}$ and $x,y\in\mathbb{R}$ together with eh linearity of the real-part operator, that is,

$$\begin{aligned}
& b(t)=\dot{a}(t)\\
\Leftrightarrow\quad & \operatorname{Re}\left\{B\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\}=\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\left\{A\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\}\\
\Leftrightarrow\quad & \operatorname{Re}\left\{B\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\}=\operatorname{Re}\left\{\frac{\mathrm{d}}{\mathrm{d}t}A\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\}\\
\Leftrightarrow\quad & \operatorname{Re}\left\{B\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\}=\operatorname{Re}\left\{A\,\mathrm{j}\,\omega\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\}\\
\Leftrightarrow\quad & B=\mathrm{j}\,\omega A.
\end{aligned}$$

Due to this poperty, $\mathrm{j}\,\omega$ is often called a differential operator. A consequence of Lemma 3 Differentiation is that any linear differential equation can be written as a polynomial in $\mathrm{j}\,\omega$, e.g., the differential equation

$$a_3\,\dddot{x}(t)+a_2\ddot{x}(t)+a_1\dot{x}(t)+a_0x(t)=w(t)$$

with sinusoidal $x(t)=\operatorname{Re}\{X\,\mathrm{e}^{\mathrm{j}\,\omega t}\}$ and $w(t)=\operatorname{Re}\{W\,\mathrm{e}^{\mathrm{j}\,\omega t}\}$, can be expressed as

$$\begin{aligned}
a_3(\mathrm{j}\,\omega)^3X+a_2(\mathrm{j}\,\omega)^2X+a_1\,\mathrm{j}\,\omega X+a_0X&=W\\
\left(a_3(\mathrm{j}\,\omega)^3+a_2(\mathrm{j}\,\omega)^2+a_1\,\mathrm{j}\,\omega+a_0\right)X&=W
\end{aligned}$$

with the phasors X and W. In other words, we obtain an algebraic linear equation where the coefficent for the phasor X is a cubic polynomial in $\mathrm{j}\,\omega$.

With these three lemmas, the basis is set for the usage of phasors for the steady-state analysis of dynamic circuits. Besides the unique assignment of phasors to the physical sinusoidal temporal response, it has been shown that the linearity allows the same superposition of signals in temporal but also phasor domain. Since the representation of dynamic circuits is also based on the derivative of signals (voltages for capacitors and currents for inductors), the simple rule for phasors to incorporate derivatives of time functions is valuable.

Due to assumed linearity of the considered circuits, the possible one-ports are nullators, norators, independent voltage or current sources, strictly linear resistors, capacitors, and inductors. For a strictly linear resistor with resistance R as in Fig. 9.1, the Ohm's law

$$u_R(t)=Ri_R(t)$$

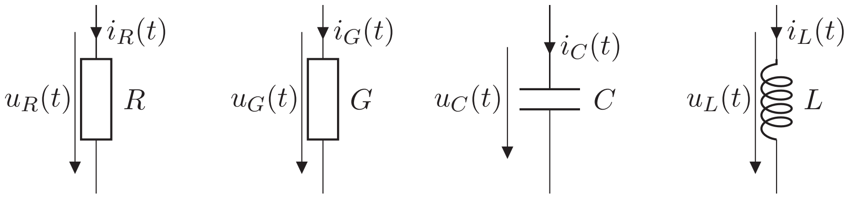


Figure 9.1: Resistor, conductor, capacitor, and inductor

is valid, where it is assumed that both, $u_R(t)$ and $i_R(t)$, are sinusoidal. With the Lemmas 1 and 2, it can be shown that for the corresponding phasor the relationship

$$U_R = RI_R$$

holds. Likewise, the phasors of a strictly linear resistor with conductance G (see Fig. 9.1) fulfill

$$I_G = GU_G.$$

The capacitor C (see Fig. 9.1) with the sinusoidal quantities $u_C(t)$ and $i_C(t)$ is represented by

$$i_C(t) = C\dot{u}_C(t).$$

By Lemma 3 on differentiation, this relationship translates to

$$I_C = \mathrm{j}\,\omega C U_C$$

for the corresponding phasors. Finally, for the inductor, we have (see Fig. 9.1)

$$u_L(t) = L\dot{i}_L(t).$$

For sinusoidal $u_L(t)$ and $i_L(t)$, the resulting expression for the corresponding phasors reads as

$$U_L = \mathrm{j}\,\omega L I_L.$$

9.2.1 Reactive Sample Circuit

The steady-state response of the simple second-order circuit depicted in Fig. 9.2 will be computed. The excitations $i_0(t)$ and $u_0(t)$ are sinusodial with $\alpha_i = 0$ and $\alpha_u = \pi$:

$$\begin{aligned} i_0(t) &= I_{0\mathrm{m}} \cos(\omega t + \alpha_i) = I_{0\mathrm{m}} \cos \omega t, \\ u_0(t) &= U_{0\mathrm{m}} \cos(\omega t + \alpha_u) = -U_{0\mathrm{m}} \cos \omega t. \end{aligned} \tag{9.7}$$

Therefore, the corresponding phasors read as

$$\begin{aligned} I_0 &= I_{0\mathrm{m}} \exp(\mathrm{j}\,\alpha_i) = I_{0\mathrm{m}} \\ U_0 &= U_{0\mathrm{m}} \exp(\mathrm{j}\,\alpha_u) = -U_{0\mathrm{m}}. \end{aligned}$$

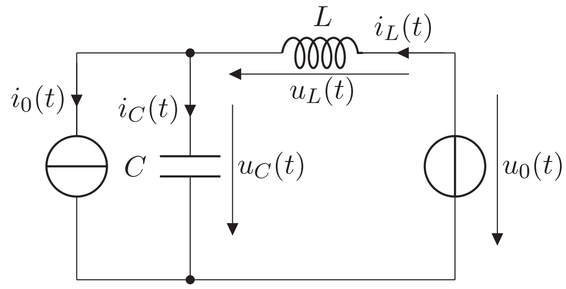


Figure 9.2: Example for a second-order dynamic circuit

Using Kirchhoff's current and voltage laws and based on the given phasor assignment, we can write due to three lemmas that

$$\begin{aligned} I_0 &= -I_C + I_L, & U_0 &= U_L + U_C, \\ U_L &= L\,\mathrm{j}\,\omega I_L, & I_C &= C\,\mathrm{j}\,\omega U_C, \\ I_0 &= -\mathrm{j}\,\omega C U_C + I_L, & U_0 &= \mathrm{j}\,\omega L I_L + U_C. \end{aligned}$$

For $i_L(t)$ and $u_C(t)$, we get for the corresponding phasors that

$$\begin{aligned} I_L = I_{L\mathrm{m}} \exp(\mathrm{j}\,\beta_i) &= \frac{I_0 + \mathrm{j}\,\omega C U_0}{1-\omega^2 LC} = \frac{I_{0\mathrm{m}} - \mathrm{j}\,\omega C U_{0\mathrm{m}}}{1-\omega^2 LC} \\ U_C = U_{C\mathrm{m}} \exp(\mathrm{j}\,\beta_u) &= \frac{U_0 - \mathrm{j}\,\omega L I_0}{1-\omega^2 LC} = \frac{-U_{0\mathrm{m}} - \mathrm{j}\,\omega L I_{0\mathrm{m}}}{1-\omega^2 LC}. \end{aligned} \tag{9.8}$$

From this we obtain the parameters of the complex phasors

$$\begin{aligned} I_{L\mathrm{m}} &= \sqrt{I_L I_L^*} = \frac{\sqrt{I_{0\mathrm{m}}^2 + (\omega C U_{0\mathrm{m}})^2}}{|1-\omega^2 LC|} \\ U_{C\mathrm{m}} &= \sqrt{U_C U_C^*} = \frac{\sqrt{U_{0\mathrm{m}}^2 + (\omega L I_{0\mathrm{m}})^2}}{|1-\omega^2 LC|} \end{aligned} \tag{9.9}$$

$$\begin{aligned} \beta_i &= \arctan\left(\frac{\mathrm{Im}\,\{I_L\}}{\mathrm{Re}\,\{I_L\}}\right) = \arctan\left(-\frac{\omega C U_{0\mathrm{m}}}{I_{0\mathrm{m}}}\right) \\ \beta_u &= \arctan\left(\frac{\mathrm{Im}\,\{U_C\}}{\mathrm{Re}\,\{U_C\}}\right) = \arctan\left(+\frac{\omega L I_{0\mathrm{m}}}{U_{0\mathrm{m}}}\right) + \pi. \end{aligned} \tag{9.10}$$

Note that the correction by π is necessary for β_u since the real part of U_C is negative. Given these phasors, the respective sinusoidal signals are given as follows

$$\begin{aligned} i_L(t) &= \mathrm{Re}\,\{I_L \exp(\mathrm{j}\,\omega t\}) = \mathrm{Re}\,\{I_{L\mathrm{m}} \exp(\mathrm{j}\,\beta_i)\exp(\mathrm{j}\,\omega t)\} = I_{L\mathrm{m}}\cos(\omega t + \beta_i) \\ u_C(t) &= \mathrm{Re}\,\{U_C \exp(\mathrm{j}\,\omega t)\} = \mathrm{Re}\,\{U_{C\mathrm{m}} \exp(\mathrm{j}\,\beta_u)\exp(\mathrm{j}\,\omega t)\} = U_{C\mathrm{m}}\cos(\omega t + \beta_u). \end{aligned} \tag{9.11}$$

This result exactly represents the steady-state component of the general response of the circuit, which can be derived with a transient analysis.

In general, we are not interested in the detailed representation of the time response given in Eqn. (9.11). We know beforehand that the response is a sinusoidal signal. What is really important is the amplitude and the phase of the signals, i.e., the values computed in Eqns. (9.9) and (9.10). These two functions of ω indicate how we can compute the amplitude and phase of the signals u_C and i_L from the amplitude and phase of the source.

Instead of using a transient analysis, this result can be derived in much simpler way by employing the analysis based on the phasor description. However, this simplification provides the solution *only* in the *steady-state*.

9.3 Network Equations and Phasors

Now the three lemmas of the phasor description, applied in the previous example, can be applied in general to all linear circuits.

9.3.1 Kirchhoff's Laws in Phasor Representation

Applying the uniqueness and linearity lemmas yields for the KCL,

$$\boldsymbol{A}\boldsymbol{i} = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{A}\boldsymbol{I} = \boldsymbol{0}, \tag{9.12}$$

with

$$\boldsymbol{i} = \mathrm{Re}\left\{\boldsymbol{I}\exp\left(\mathrm{j}\,\omega t\right)\right\}$$

and also for the KVL,

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{k}} = \boldsymbol{u} \quad \Leftrightarrow \quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}_{\mathrm{k}} = \boldsymbol{U}, \tag{9.13}$$

with

$$\boldsymbol{u}_{\mathrm{k}} = \mathrm{Re}\left\{\boldsymbol{U}_{\mathrm{k}}\exp\left(\mathrm{j}\,\omega t\right)\right\}, \quad \boldsymbol{u} = \mathrm{Re}\left\{\boldsymbol{U}\exp\left(\mathrm{j}\,\omega t\right)\right\}.$$

Alternatively, the KVL can be expressed based on the loop incidence matrix, i.e.,

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{B}\boldsymbol{U} = \boldsymbol{0}. \tag{9.14}$$

Here, $\boldsymbol{A}$ and $\boldsymbol{B}$ are the node incidence and the loop incidence matrix, respectively. Furthermore, we employed the node voltage vector $\boldsymbol{u}_{\mathrm{k}}$, the branch voltage vector $\boldsymbol{u}$, and the branch current vector $\boldsymbol{i}$.

9.3.2 Network Elements in Phasor Representation

For a circuit with linear resistive elements, the represenation of the elements can be written as

$$\boldsymbol{M}\boldsymbol{u} + \boldsymbol{N}\boldsymbol{i} = \boldsymbol{e}.$$

If the circuit also contains reactive elements, viz., capacitors with

$$i_C(t) = C\dot{u}_C(t)$$

or inductors with

$$u_L(t) = L\dot{i}_L(t)$$

then the element equations must be extended by terms with the derivatives of the voltages and currents. Therefore, the element equations for a linear circuit with resistive and reactive elements reads as

$$\boldsymbol{M}_1 \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{M}_0 \boldsymbol{u} + \boldsymbol{N}_1 \frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t} + \boldsymbol{N}_0 \boldsymbol{i} = \boldsymbol{e}.$$

With the three lemmas for complex phasors (uniqueness, linearity, and differentiation), we obtain

$$(\boldsymbol{M}_1 \,\mathrm{j}\,\omega + \boldsymbol{M}_0)\,\boldsymbol{U} + (\boldsymbol{N}_1 \,\mathrm{j}\,\omega + \boldsymbol{N}_0)\,\boldsymbol{I} = \boldsymbol{E}. \tag{9.15}$$

Thus, the resulting tableau equation system for phasors is

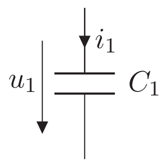
$$\begin{bmatrix} \boldsymbol{B} & 0 \\ 0 & \boldsymbol{A} \\ \boldsymbol{M}(\mathrm{j}\,\omega) & \boldsymbol{N}(\mathrm{j}\,\omega) \end{bmatrix} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{E} \end{bmatrix} \tag{9.16}$$

with $\boldsymbol{M}(\mathrm{j}\,\omega) = \boldsymbol{M}_0 + \mathrm{j}\,\omega \boldsymbol{M}_1$ and $\boldsymbol{N}(\mathrm{j}\,\omega) = \boldsymbol{N}_0 + \mathrm{j}\,\omega \boldsymbol{N}_1$. Note that this is—as in the purely resistive case—an algebraic equation system with constant (assuming $\omega = \mathrm{const}$) but complex-valued coefficients.

When the solutions for different frequencies of the excitation should be determined, the equation system in Eqn. (9.16) must be solved for the different values of ω. This motivates the term frequency response.

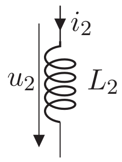
Example

branch 1:



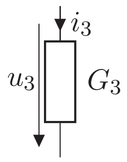
$i_1 = C_1 \frac{\mathrm{d}}{\mathrm{d}t} u_1 \quad \Leftrightarrow \quad \mathrm{j}\,\omega C_1 U_1 - I_1 = 0$

branch 2:



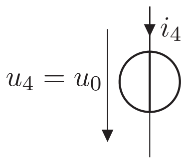
$u_2 = L_2 \frac{\mathrm{d}}{\mathrm{d}t} i_2 \quad \Leftrightarrow \quad U_2 - \mathrm{j}\,\omega L_2 I_2 = 0$

branch 3:

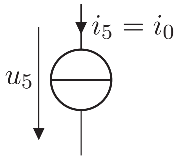


$i_3 = G_3 u_3 \quad \Leftrightarrow \quad G_3 U_3 - I_3 = 0$

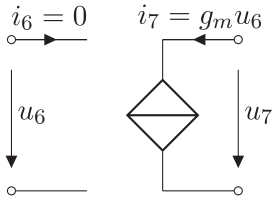
branch 4:



$u_4 = u_0 \quad \Leftrightarrow \quad U_4 = U_0$

branch 5: 

$i_5 = i_0 \quad \Leftrightarrow \quad I_5 = I_0$

branches 6 und 7: 

VCCS (voltage-controlled current source)

$i_6 = 0 \quad \Leftrightarrow \quad I_6 = 0$

$i_7 = g_m u_6 \quad \Leftrightarrow \quad g_m U_6 - I_7 = 0$

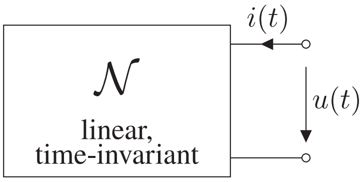
The different element equations can be collected in the equation system with the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ U_0 \\ I_0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathrm{j}\,\omega C_1 & & & & & & \\ & 1 & & & & & \\ & & G_3 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & g_m & 0 \end{bmatrix}}_{\boldsymbol{M}=\boldsymbol{M}_0+\mathrm{j}\,\omega\boldsymbol{M}_1} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & & & & & & \\ & -\mathrm{j}\,\omega L_2 & & & & & \\ & & -1 & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & 0 & -1 \end{bmatrix}}_{\boldsymbol{N}=\boldsymbol{N}_0+\mathrm{j}\,\omega\boldsymbol{N}_1} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix}.$$

Apparently, these matrices can be split according to $\boldsymbol{M} = \boldsymbol{M}_0 + \mathrm{j}\,\omega\boldsymbol{M}_1$ and $\boldsymbol{N} = \boldsymbol{N}_0 + \mathrm{j}\,\omega\boldsymbol{N}_1$ with

$$\boldsymbol{M}_0 = \begin{bmatrix} 0 & & & & & & \\ & 1 & & & & & \\ & & G_3 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & g_m & 0 \end{bmatrix} \qquad \boldsymbol{N}_0 = \begin{bmatrix} -1 & & & & & & \\ & 0 & & & & & \\ & & -1 & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{M}_1 = \begin{bmatrix} C_1 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & 0 & 0 \end{bmatrix} \qquad \boldsymbol{N}_1 = \begin{bmatrix} 0 & & & & & & \\ & -L_2 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & 0 & 0 \end{bmatrix}.$$



$$u(t) = \mathrm{Re}\{U e^{\mathrm{j}\,\omega t}\}$$
$$i(t) = \mathrm{Re}\{I e^{\mathrm{j}\,\omega t}\}$$

Figure 9.3: Linear, time-invariant two-pole

9.4 Network Functions

The term network function is used for the quotient of two complex phasors. If the two quantities belong to the same pair of terminals (port current and port voltage), a two-pole function is considered. In the case that the two phasors correspond to two different branches of the circuit, the quotient is a transfer function.

9.4.1 Two-Pole Function

For the two-pole depicted in Fig. 9.3, based on the phasors $U(\mathrm{j}\,\omega)$ and $I(\mathrm{j}\,\omega)$ corresponding to the time functions $u(t)$ and $i(t)$, respectively, two-pole functions can be defined.

The impedance (complex resistor) of a two-pol $\mathcal{N}$ is defined as the function

$$Z(\mathrm{j}\,\omega) = \frac{U(\mathrm{j}\,\omega)}{I(\mathrm{j}\,\omega)}. \tag{9.17}$$

Similarly, the admittance (complex conductance) reads as

$$Y(\mathrm{j}\,\omega) = \frac{I(\mathrm{j}\,\omega)}{U(\mathrm{j}\,\omega)} = \frac{1}{Z(\mathrm{j}\,\omega)}. \tag{9.18}$$

The generic term for impedance and admittance is the coinage immitance.

For the impedances of the elements in Fig. 9.1, it holds that

$$Z_R(\mathrm{j}\,\omega) = R \qquad Z_G(\mathrm{j}\,\omega) = \frac{1}{G} \qquad Z_C(\mathrm{j}\,\omega) = \frac{1}{\mathrm{j}\,\omega C} \qquad Z_L(\mathrm{j}\,\omega) = \mathrm{j}\,\omega L \tag{9.19}$$

and the corresponding admittaces read as

$$Y_R(\mathrm{j}\,\omega) = \frac{1}{R} \qquad Y_G(\mathrm{j}\,\omega) = G \qquad Y_C(\mathrm{j}\,\omega) = \mathrm{j}\,\omega C \qquad Y_L(\mathrm{j}\,\omega) = \frac{1}{\mathrm{j}\,\omega L}. \tag{9.20}$$

Note that circuits with immittances can be treated like circuits with strictly linear, resistive one-ports, when taking into account the basic rules for the calculation with complex numbers.

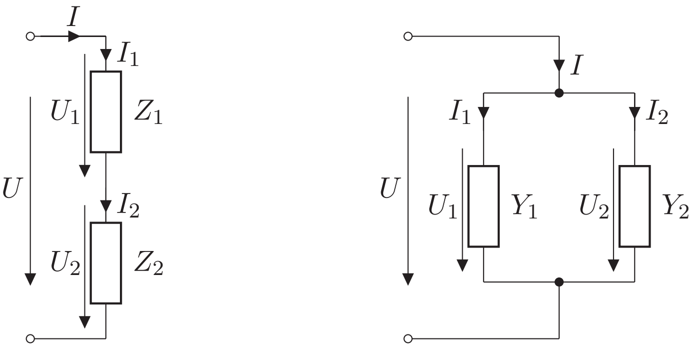


Figure 9.4: Series connection (left) and parallel connection (right) of immitances

The series and parallel connection of immitances are shown in Fig. 9.4. For the series connection, we get

$$
\begin{aligned}
U(\mathrm{j}\,\omega) &= U_1(\mathrm{j}\,\omega) + U_2(\mathrm{j}\,\omega) && \text{(KVL)}\\
I(\mathrm{j}\,\omega) &= I_1(\mathrm{j}\,\omega) = I_2(\mathrm{j}\,\omega) && \text{(KCL)}\\
U_1(\mathrm{j}\,\omega) &= Z_1(\mathrm{j}\,\omega)I_1(\mathrm{j}\,\omega) \qquad U_2(\mathrm{j}\,\omega) = Z_2(\mathrm{j}\,\omega)I_2(\mathrm{j}\,\omega) && \text{(Elementgleichungen)}\\
Z(\mathrm{j}\,\omega) &= Z_1(\mathrm{j}\,\omega) + Z_2(\mathrm{j}\,\omega) && \text{(Gesamtimpedanz)}\\
Y(\mathrm{j}\,\omega) &= \frac{1}{Z(\mathrm{j}\,\omega)} = Y_1(\mathrm{j}\,\omega) \parallel Y_2(\mathrm{j}\,\omega) && \text{(Gesamtadmittanz)}
\end{aligned}
$$

where $Y_1(\mathrm{j}\,\omega) = \frac{1}{Z_1(\mathrm{j}\,\omega)}$ and $Y_2(\mathrm{j}\,\omega) = \frac{1}{Z_2(\mathrm{j}\,\omega)}$. The total impedance of a series connection is the sum of the partial impedances. For the parallel connection, we find

$$
\begin{aligned}
U(\mathrm{j}\,\omega) &= U_1(\mathrm{j}\,\omega) = U_2(\mathrm{j}\,\omega) && \text{(KVL)}\\
I(\mathrm{j}\,\omega) &= I_1(\mathrm{j}\,\omega) + I_2(\mathrm{j}\,\omega) && \text{(KCL)}\\
I_1(\mathrm{j}\,\omega) &= Y_1(\mathrm{j}\,\omega)U_1(\mathrm{j}\,\omega) \qquad I_2(\mathrm{j}\,\omega) = Y_2(\mathrm{j}\,\omega)U_2(\mathrm{j}\,\omega) && \text{(Elementgleichungen)}\\
Y(\mathrm{j}\,\omega) &= Y_1(\mathrm{j}\,\omega) + Y_2(\mathrm{j}\,\omega) && \text{(Gesamtadmittanz)}\\
Z(\mathrm{j}\,\omega) &= Z_1(\mathrm{j}\,\omega) \parallel Z_2(\mathrm{j}\,\omega) && \text{(Gesamtimpedanz)}
\end{aligned}
$$

where $Z_1(\mathrm{j}\,\omega) = \frac{1}{Y_1(\mathrm{j}\,\omega)}$ and $Z_2(\mathrm{j}\,\omega) = \frac{1}{Y_2(\mathrm{j}\,\omega)}$. The total admittance of a parallel connection is the sum of the partial admittances.

By applying the aforementioned rules for impedances and admittances, it is possible to combine also complicated two-terminal circuits and replace it by a single equalent two-terminal.

9.4.2 Transfer Functions

By rewriting the state equation (8.2), i.e., $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}v(t)$, with complex phasors, we can obtain

$$\mathrm{j}\,\omega \boldsymbol{X}(\mathrm{j}\,\omega) = \boldsymbol{A}\boldsymbol{X}(\mathrm{j}\,\omega) + \boldsymbol{b}V(\mathrm{j}\,\omega)$$

where $\boldsymbol{x}(t) = \mathrm{Re}\{\boldsymbol{X}(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t}\}$, $v(t) = \mathrm{Re}\{V(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t}\}$. In addition to above state equation, we rewrite the output equation (8.3) by employing complex phasors, that is,

$$Y(\mathrm{j}\,\omega) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{X}(\mathrm{j}\,\omega) + dV(\mathrm{j}\,\omega) \tag{9.21}$$

with $y(t) = \mathrm{Re}\{Y(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t}\}$. Solving the state equation for the state vector yields

$$\boldsymbol{X}(\mathrm{j}\,\omega) = (\mathrm{j}\,\omega\mathbf{1} - \boldsymbol{A})^{-1}\boldsymbol{b}V(\mathrm{j}\,\omega).$$

By substituting this result into the output equation (9.21), we obtain the following expression for the output signal depending on the input signal

$$Y(\mathrm{j}\,\omega) = \boldsymbol{c}^{\mathrm{T}}(\mathrm{j}\,\omega\mathbf{1} - \boldsymbol{A})^{-1}\boldsymbol{b}V(\mathrm{j}\,\omega) + dV(\mathrm{j}\,\omega).$$

This result motivates the definition of the transfer function

$$H(\mathrm{j}\,\omega) = \frac{Y(\mathrm{j}\,\omega)}{V(\mathrm{j}\,\omega)} = \boldsymbol{c}^{\mathrm{T}}(\mathrm{j}\,\omega\mathbf{1} - \boldsymbol{A})^{-1}\boldsymbol{b} + d. \tag{9.22}$$

To understand the principal properties of the transfer function $H(\mathrm{j}\,\omega)$, we have to use the following alternative expression for the inverse of matrix, i.e.,

$$\boldsymbol{F}^{-1} = \frac{\mathrm{adj}(\boldsymbol{F})}{\det(\boldsymbol{F})}.$$

Every element of the adjugate $\mathrm{adj}(\boldsymbol{F})$ is the co-factor of $\boldsymbol{F}$ that is obtained by removing the corresponding row and column and finding the determinant of the resulting matrix.

Accordingly, the elements of $(\mathrm{j}\,\omega\mathbf{1} - \boldsymbol{A})^{-1}$ are rational functions in $\mathrm{j}\,\omega$, that is, fractions of polynomials in $\mathrm{j}\,\omega$. All the denominators are $\det(\mathrm{j}\,\omega\mathbf{1} - \boldsymbol{A})$ which, after the substitution $\mathrm{j}\,\omega \to p \in \mathbb{C}$, becomes the characteristic polynomial $\det(p\mathbf{1} - \boldsymbol{A})$ of $\boldsymbol{A}$. Remember that the roots of the characteristic polynomial are the eigenvalues of $\boldsymbol{A}$. In other words, based on the substitution $\mathrm{j}\,\omega \to p \in \mathbb{C}$, the roots of the denominator polynomial of $H(p)$ help to explore the stability of the circuit. If all roots of the denominator polynomial have negative real parts, the circuit is stable. If not, the circuit is unstable.

This discussion highlights the usefulness of the transfer function $H(\mathrm{j}\,\omega)$ to find the overall response of a circuit in time domain. Firstly, the steady-state response results from transforming the output phasors, i.e.,

$$y_{\text{steady}}(t) = \mathrm{Re}\{Y(\mathrm{j})\mathrm{e}^{\mathrm{j}\,\omega t}\} = \mathrm{Re}\{H(\mathrm{j}\,\omega)V(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t}\}. \tag{9.23}$$

Secondly, with the substution $\mathrm{j}\,\omega \rightarrow p$, the eigenvalues $p_{\infty,1}, \ldots, p_{\infty,n}$ of the state matrix $\boldsymbol{A}$ can be found where n is the order of the circuit (number of reactive elements). It can be shown that the transient response, i.e., the reaction of the circuit to switching on the sources, has a similar form as the zero-input response, that is,

$$y_{\text{trans}}(t) = \sum_{k=1}^{n} y_{0,k} \mathrm{e}^{p_{\infty,k} t} \tag{9.24}$$

with the eigenvalues $p_{\infty,1}, \ldots, p_{\infty,n}$ of $\boldsymbol{A}$. The overall response of the circuit is the superposition of the transient and the steady-state response, i.e.,

$$y(t) = y_{\text{trans}}(t) + y_{\text{steady}}(t) = \sum_{k=1}^{n} y_{0,k} \mathrm{e}^{p_{\infty,k} t} + \mathrm{Re}\{H(\mathrm{j}\,\omega) V(\mathrm{j}\,\omega) \mathrm{e}^{\mathrm{j}\,\omega t}\}. \tag{9.25}$$

9.5 Energy and Power

In this section, we will consider the *power calculation* with *sinusoidal sources* based on the complex phasor representation. To this end let us denote the *instantaneous power* as

$$p(t) = u(t) i(t). \tag{9.26}$$

Accordingly, the energy of one period T is given by

$$E = \int_0^T u(t) i(t) \,\mathrm{d}t$$

and the average power for periodic voltages and currents is

$$P_{\mathrm{W}} = \frac{1}{T} \int_0^T u(t) i(t) \,\mathrm{d}t \tag{9.27}$$

where T is the period of the periodic time functions. For a signal with angular frequency ω, the period is $\frac{2\pi}{\omega}$. This result for the average power P_{W} is the basic definition. However, we will discuss a more convenient way to find it.

Since the instantaneous power and the average power are computed as the product of two sinusoidal signals, we cannot make use of the three lemmas for the phasors. Therefore, we need another approach in order to analyze how the combination of the voltage and current phasors affect the desired power value.

To this end, consider the power which is dissipated as heat in a resistance driven by a sinusoidal input. For

$$\begin{aligned} u(t) &= U_{\mathrm{m}} \cos(\omega t + \alpha) \\ i(t) &= I_{\mathrm{m}} \cos(\omega t + \alpha) = \frac{u(t)}{R} = \frac{U_{\mathrm{m}}}{R} \cos(\omega t + \alpha) \end{aligned}$$

we obtain for the instantaneous power

$$p(t) = u(t)i(t) = \frac{U_\mathrm{m}^2}{R}\cos^2(\omega t + \alpha) = I_\mathrm{m}^2 R\cos^2(\omega t + \alpha)$$

and, therefore, for the average power

$$P_\mathrm{W} = \frac{1}{T}\frac{U_\mathrm{m}^2}{R}\int_0^T \cos^2(\omega t + \alpha)\,\mathrm{d}t = \frac{1}{T}I_\mathrm{m}^2 R\int_0^T \cos^2(\omega t + \alpha)\,\mathrm{d}t \tag{9.28}$$

with $T = \frac{2\pi}{\omega}$. It holds that

$$\frac{1}{T}\int_0^T \cos^2(\omega t + \alpha)\,\mathrm{d}t = \frac{1}{T}\frac{1}{2}\int_0^T [1 + \cos(2(\omega t + \alpha))]\,\mathrm{d}t = \frac{1}{2T}\,[t]_0^T = \frac{1}{2}.$$

Consequently,

$$P_\mathrm{W} = \frac{U_\mathrm{m}^2}{2R} = \frac{I_\mathrm{m}^2 R}{2}. \tag{9.29}$$

By introducing the concept of *effective values* or *root means square (RMS) values*,

$$U_\mathrm{eff} = \frac{U_\mathrm{m}}{\sqrt{2}} \qquad I_\mathrm{eff} = \frac{I_\mathrm{m}}{\sqrt{2}} \tag{9.30}$$

as the voltage and current value which as DC voltage and DC current produce the same amount of power which is dissipated as heat in a resistance: $\frac{1}{R}\,U_\mathrm{eff}^2$ and RI_eff^2.

Since we are interested in the power calculation based only on the combination between the voltage and current phasors, the computed power cannot depend on the arbitrary initial phases α_U and α_I, but only on the relative phase between the voltage and current phasors. Thus, we need the definition of the complex power

$$\begin{aligned} P &= \frac{1}{2}UI^* = \frac{1}{2}\,U_\mathrm{m}\exp(\mathrm{j}\,\alpha_U)I_\mathrm{m}\exp(-\,\mathrm{j}\,\alpha_I) \\ &= \frac{1}{2}\,U_\mathrm{m}I_\mathrm{m}\exp(\mathrm{j}(\alpha_U - \alpha_I)). \end{aligned} \tag{9.31}$$

Computing the average power for the sinusoidal voltage $u(t) = U_\mathrm{m}\cos(\omega t + \alpha_U)$ and current $i(t) = I_\mathrm{m}\cos(\omega t + \alpha_I)$ yields

$$\begin{aligned} P_\mathrm{W} &= \frac{1}{T}\int_0^T u(t)i(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T U_\mathrm{m}I_\mathrm{m}\cos(\omega t + \alpha_U)\cos(\omega t + \alpha_I)\,\mathrm{d}t \\ &= \frac{1}{T}U_\mathrm{m}I_\mathrm{m}\int_0^T \frac{1}{2}\,[\cos(\alpha_U - \alpha_I) + \cos(2\omega t + \alpha_U + \alpha_I)]\,\mathrm{d}t \\ &= \frac{U_\mathrm{m}I_\mathrm{m}}{2}\cos(\alpha_U - \alpha_I). \end{aligned} \tag{9.32}$$

Therefore, we can observe that

$$\mathrm{Re}\,\{P\} = \mathrm{Re}\left\{\frac{1}{2}UI^*\right\} = P_\mathrm{W}. \tag{9.33}$$

The imaginary part

$$\mathrm{Im}\{P\} = \mathrm{Im}\left\{\frac{1}{2}UI^*\right\} = P_\mathrm{B} \tag{9.34}$$

is called the blind (or reactive) power.

Although the reactive power does not contribute to the power which is dissipated as heat, it is of essential importance in power engineering. The sum

$$P = P_\mathrm{W} + \mathrm{j}\,P_\mathrm{B} = \frac{1}{2}\,UI^* \tag{9.35}$$

is known as complex *power*. Its magnitude is the *apparent power*

$$S = |P|. \tag{9.36}$$

For an impedance $Z(\mathrm{j}\,\omega)$, we have that

$$U(\mathrm{j}\,\omega) = Z(\mathrm{j}\,\omega)I(\mathrm{j}\,\omega) = [R(\omega) + \mathrm{j}\,X(\omega)]I(\mathrm{j}\,\omega)$$

with the real part resistance $R(\omega)$ and the imaginary part reactance $X(\omega)$ of $Z(\mathrm{j}\,\omega)$. Therefore, the corresponding complex power reads as

$$P(\mathrm{j}\,\omega) = \frac{1}{2}\,U(\mathrm{j}\,\omega)I^*(\mathrm{j}\,\omega) = \frac{1}{2}\,Z(\mathrm{j}\,\omega)I(\mathrm{j}\,\omega)I^*(\mathrm{j}\,\omega) = [R(\omega) + \mathrm{j}\,X(\omega)]\,\frac{|I(\mathrm{j}\,\omega)|^2}{2}. \tag{9.37}$$

The average power of the impedance $Z(\mathrm{j}\,\omega)$ is given by

$$P_\mathrm{W}(\omega) = R(\omega)\,\frac{|I(\mathrm{j}\,\omega)|^2}{2}$$

and the blind power is

$$P_\mathrm{B}(\omega) = X(\omega)\,\frac{|I(\mathrm{j}\,\omega)|^2}{2}.$$

For an admittance $Y(\mathrm{j}\,\omega)$, we similarly get

$$I(\mathrm{j}\,\omega) = Y(\mathrm{j}\,\omega)U(\mathrm{j}\,\omega) = [G(\omega) + \mathrm{j}\,B(\omega)]U(\mathrm{j}\,\omega)$$

with the conductance $G(\omega)$ and the susceptance $B(\omega)$. Accordingly, the complex power can be written as

$$P(\mathrm{j}\,\omega) = \frac{1}{2}\,U(\mathrm{j}\,\omega)I^*(\mathrm{j}\,\omega) = \frac{1}{2}\,Y^*(\mathrm{j}\,\omega)U(\mathrm{j}\,\omega)U^*(\mathrm{j}\,\omega) = [G(\omega) - \mathrm{j}\,B(\omega)]\,\frac{|U(\mathrm{j}\,\omega)|^2}{2}. \tag{9.38}$$

with the average power

$$P_\mathrm{W}(\omega) = G(\omega)\,\frac{|U(\mathrm{j}\,\omega)|^2}{2}$$

and the blind or reactive power

$$P_\mathrm{B}(\omega) = -B(\omega)\,\frac{|U(\mathrm{j}\,\omega)|^2}{2}.$$

Note that the reactive power does not contribute to the power which is dissipated as heat. However, to reduce possible losses in the power network, the usual question is how to change the circuit such that the blind power becomes zero.