Introduction sets

- Deductive Proof scheme
  - Similar to propositional & predicate logic
  - Laws H, J

- Power set of M : $P(M)$
  - Contains all subsets of M
  - always contains $\{\}$, M
  - example : $M = \{\emptyset, \{a\}\}$

    $P(M) = \{\emptyset, \{\emptyset, \{a\}\}, \{\emptyset\}, \{\{a\}\}\}$

$\emptyset$; $|\emptyset| = 0$

$\{\emptyset\}$; $|\{\emptyset\}| = 1$

$P(\emptyset) = \{\emptyset\}$

$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

On-Sets vs Propositional Logic

Equivalence and Implications in propositional logic can be represented by equality- and subset relations with the of the propositional forms

example : $A \wedge B \Rightarrow C$

$E[A \wedge B] \subseteq E[C]$

$E[A] \cap E[B] \subseteq E[C]$

3.1 Given are the sets A, B and the universal set G:

$$\begin{aligned} A &= \{\{\}, \{\emptyset\}, \{b\}, \{a,c\}, \{a,b,c\}\} \\ B &= \{\{c\}, \{a,b\}, \{a,c\}\} \\ \text{with: } A, B &\subseteq G \end{aligned}$$

0 1 1 2 3

1 2 2

green, Cardinality (no. of elements in a set)

Give the following sets in explicite notation:

$$\begin{aligned} M_1 &= G \setminus \overline{A} \\ M_2 &= \left\{ X \in A \,\middle|\, \underset{Y \in B}{\exists} X \subset Y \right\} \\ M_3 &= \{(X,Y) \in A \times B \,|\, |X| + |Y| = 2\} \\ M_4 &= P(A \cap B) \end{aligned}$$

$$M_1 = G \setminus \overline{A} \overset{J3}{=} G \cap A \overset{H6}{=} A$$

$$M_2 = \{X \in A \,|\, \underset{Y \in B}{\exists} X \subset Y\} = \{\{\}, \{b\}\}$$

$$M_3 = \{(X,Y) \in A \times B \,|\, |X| + |Y| = 2\}$$

$$= \{ (\{\}, \{a,b\}), (\{\}, \{a,c\}), (\{\emptyset\}, \{c\}), (\{b\}, \{c\}) \}$$

$$M_4 = P(A \cap B) = P(\{\underbrace{\{a,c\}}\}) = \{\emptyset, \{\{a,c\}\}\}$$

$A, B \subseteq G$

$\emptyset \subseteq A$, for all fs A

$\emptyset \subset B$, for all sets B except $\emptyset$

$\emptyset \not\subset \emptyset$; $\emptyset \subseteq \emptyset$

$\{b\} \subset \{a,b\}$

$\{a,c\} \not\subset \{a,c\}$

$\{a,c\} \subseteq \{a,c\}$

3.2 Consider the following two laws in propositional logic and the rule of compatibility with three arbitrary propositional forms A, B and C. Name the corresponding relationships between the on-sets $E[A]$, $E[B]$ and $E[C]$.

3.2.c IF $A \Longrightarrow B$, THEN $A \vee C \Longrightarrow B \vee C$

$E[A] \subseteq E[B] \Rightarrow E[A] \cup E[C] \subseteq E[B] \cup E[C]$

$\boxed{\textbf{3.5}}$ Given is the following implication (IMP) and the corresponding on-set of A, B and C:

$$(A \longrightarrow B) \longleftrightarrow C \Longrightarrow A \qquad \text{(IMP)}$$

$$\begin{aligned}
\mathbf{E}[A] &= \{(0,1,0),(1,0,0),(1,1,0)\} \\
\mathbf{E}[B] &= \{(0,0,1),(0,1,0),(0,1,1),(1,1,1)\} \\
\mathbf{E}[C] &= \{(0,1,0),(1,1,0)\}
\end{aligned}$$

The universal set is $G = \{0,1\}^3$. Show that the implication IMP is valid for the given sets.

$(A \to B) \leftrightarrow C \Rightarrow A$

$\} X \Rightarrow Y$ IFF

$E[X] \subseteq E[Y] \Leftrightarrow t$

$E[(A \to B) \leftrightarrow C] \subseteq E[A]$

$E[\neg(A \to B) \not\leftrightarrow C] \subseteq E[A]$

$E[A \wedge \neg B \not\leftrightarrow C] \subseteq E[A]$

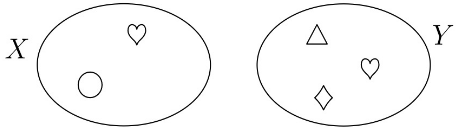
$(E[A] \cap E[B]) \,\Delta\, E[C] \subseteq E[A]$

$(E[A] \setminus E[B]) \,\Delta\, E[C] \subseteq E[A]$

$\{(1,0,0),(1,1,0)\} \,\Delta\, \{(0,1,0),(1,1,0)\} \subseteq E[A]$

$\{(1,0,0),(0,1,0)\} \subseteq \{(0,1,0),(1,0,0),(1,1,0)\} \Leftrightarrow t$

3.6 Given are the sets X und Y:



To describe these sets in propositional logic, the elements are coded using the following table:

Element	○	◊	△	♡
Coding (a,b)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$

Give for both sets X and Y a valid characteristic function A and B, respectively (see formulary p. 49).

	a b	set X A	set Y B
○	(0,0)	1	0
◊	(0,1)	0	1
△	(1,0)	0	1
♡	(1,1)	1	1

All propositional forms with on-set $E[A] = \{(0,0), (1,1)\}$ are a valid characteristic function of set X

$A(a,b) \Longleftrightarrow (\neg a \wedge \neg b) \vee (a \wedge b)$

either CCNF or CDNF

$A(a,b) \Longleftrightarrow a \leftrightarrow b$

both are valid
there are limitless possibilities

$E[B] = \{(0,1), (1,0), (1,1)\}$

$B(a,b) \Longleftrightarrow (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$

3.7 Prove the following relations between sets:

3.7.c $(A \triangle B) \subseteq A \iff B \subseteq A$

Name all laws you use.

$$(A \triangle B) \subseteq A \overset{J29}{\iff} \overline{(A \triangle B)} \cup A = G$$

$$\overset{J9}{\iff} \overline{(A \cup B) \cap \overline{(A \cap B)}} \cup A = G$$

$$\overset{H10}{\underset{H3}{\iff}} (\bar{A} \cap \bar{B}) \cup \underbrace{(A \cap B) \cup A}_{A} = G$$

$$\overset{H5}{\iff} (\bar{A} \cap \bar{B}) \cup A = G$$

$$\overset{H3}{\iff} \underbrace{(\bar{A} \cup A)}_{G} \cap (\bar{B} \cup A) = G$$

$$\overset{H8,\, H5}{\iff} \bar{B} \cup A = G \overset{J29}{\iff} B \subseteq A$$

3.8 Prove the following implications in set notation using the deductive proof scheme. Make sure you solely use sets of laws. Are all implications valid? The universal set is G.

3.8.b $[(\mathcal{A} \triangle \mathcal{G}) \underset{t}{\cup} (\mathcal{B} \cap \mathcal{C}) = \mathcal{G}] \wedge [(\mathcal{A} \cup \mathcal{B}) \underset{t}{\triangle} \mathcal{G} \neq \mathcal{C}] \implies \mathcal{B} \cap \underset{f}{\mathcal{C}} \neq \emptyset$

this implication is not valid show with Counterexample : $t \Rightarrow f$

1. Assumption : $B \cap C = \emptyset$

$[(A \triangle G) = G] \wedge [(A \cup B) \triangle G \neq C] \Rightarrow f$

$\underbrace{\bar{A} = G}_{t} \wedge \underbrace{[(A \cup B) \triangle G \neq C]}_{t} \Rightarrow f$ JIS

2. Assumption $t \Leftrightarrow \bar{A} = G \Leftrightarrow A = \emptyset$

$t \wedge [(\emptyset \cup B) \triangle G \neq C] \Rightarrow f$

$\bar{B} \neq C \Rightarrow f$

3. Assumption $B = \emptyset \wedge C = \emptyset$

Counter exampl : $A = B = C = \emptyset$

3.8.e $\left(\overline{\mathcal{A}} \cup \overline{\mathcal{B}} = \mathcal{G}\right) \wedge \left(\overline{\mathcal{A}} \subseteq \overline{\mathcal{C}}\right) \implies \mathcal{C} \setminus \overline{\mathcal{B}} = \emptyset$

1)	$\overline{A} \cup \overline{B} = G$	
2)	$\overline{A} \subseteq \overline{C}$	$\Rightarrow C \setminus \overline{B} = \emptyset$
3)	$B \subseteq \overline{A}$	1, J29
4)	$B \subseteq \overline{C}$	2, 3, J34
5)	$B \cap C = \emptyset$	4, J29
6)	$\overline{\overline{B}} \cap C = \emptyset$	5, H9
7)	$C \setminus \overline{B} = \emptyset$	6, J3, H9

3.9 Transform the implications given in set form in the previous exercise into implications in predicate logic.
The predicates are defined as follows:

$$x \in \mathcal{A} \iff Px \qquad x \in \mathcal{C} \iff Sx$$
$$x \in \mathcal{B} \iff Qx \qquad x \in \mathcal{D} \iff Tx\,,$$

The universal set is $\mathcal{G}$.

3.8.b $[(\mathcal{A} \triangle \mathcal{G}) \cup (\mathcal{B} \cap \mathcal{C}) = \mathcal{G}] \wedge [(\mathcal{A} \cup \mathcal{B}) \triangle \mathcal{G} \neq \mathcal{C}] \implies \mathcal{B} \cap \mathcal{C} \neq \emptyset$

1) Resolve In-land Equality Relations

- Use relations in formulary pg 49/50

- here relevant

$\cdot\ A = G \iff \forall_{x \in G}\ x \in A \qquad \cdot\ A \neq B \iff \exists_{x \in G} (x \in A \not\leftrightarrow x \in B) \qquad \cdot\ A \neq \emptyset \iff \exists_{x \in G}\ x \in A$

$$\forall_{x \in G}\ x \in [(A \triangle G) \cup (B \cap C)] \wedge \exists_{x \in G}\ x \in [(A \cup B) \triangle G] \not\leftrightarrow x \in C \Rightarrow \exists_{x \in G}\ x \in (B \cap C)$$

2) Convert set-operations to propositional logic operators

- formulary p 47

$$\forall_{x \in G} [(x \in A \not\leftrightarrow x \in G) \vee (x \in B \wedge x \in C)] \wedge \exists_{x \in G} [(x \in A \vee x \in B) \not\leftrightarrow x \in G] \not\leftrightarrow x \in C \Rightarrow \exists_{x \in G}\ x \in B \wedge x \in C$$

3) Substitute the predicates

$x \in A \iff Px\ ;\ x \in B \iff Qx\ ;\ x \in C \iff Sx\ ,\ x \in D \iff Tx$

$$\forall_{x \in G} [(Px \not\leftrightarrow t) \vee (Qx \wedge Sx)] \wedge \exists_{x \in G} [(Px \vee Qx) \not\leftrightarrow t] \not\leftrightarrow Sx \Rightarrow \exists_{x \in G} Qx \wedge Sx$$

4. Simplify

$$\forall_{x \in G} [\neg Px \vee (Qx \wedge Sx)] \wedge \exists_{x \in G} \neg (Px \vee Qx) \not\leftrightarrow Sx \Rightarrow \exists_{x \in G} Qx \wedge Sx$$

3.10 Transform the implications in predicate logic given in exercise 2.4 into set notation. The predicates are defined as follows:

$$x \in \mathcal{A} \iff Px \qquad x \in \mathcal{C} \iff Sx$$
$$x \in \mathcal{B} \iff Qx \qquad x \in \mathcal{D} \iff Tx\,, \qquad \text{The universal set is } \mathcal{G}.$$

2.4.d $$\neg \underset{x\,\in\,\mathcal{G}}{\exists} (Px \longrightarrow \neg Qx) \wedge \left(\neg \underset{x\,\in\,\mathcal{G}}{\exists} Qx \vee \underset{x\,\in\,\mathcal{G}}{\forall} Sx\right) \implies \underset{x\,\in\,\mathcal{G}}{\exists} Sx$$

$$\neg \underset{x \in G}{\exists} (\neg Px \vee \neg Qx) \wedge (\neg \underset{x \in G}{\exists} Qx \vee \underset{x \in G}{\forall} Sx) \Rightarrow \underset{x \in G}{\exists} Sx$$

$$\neg \underset{x \in G}{\exists} (x \in \bar{A} \vee x \in \bar{B}) \wedge (\neg \underset{x \in G}{\exists} (x \in B) \vee \underset{x \in G}{\forall} (x \in C)) \Rightarrow \underset{x \in G}{\exists} (x \in C)$$

$$(\bar{A} \vee \bar{B} = \emptyset) \wedge (B = \emptyset \vee C = G) \Rightarrow C \neq \emptyset$$

Side Calculation :

From formulary pg 51 :

$$A = \emptyset \iff \underset{x \in G}{\forall} x \notin A \iff \underset{x \in G}{\forall} \neg x \in A$$

$$\iff \neg \underset{x \in A}{\exists} x \in A$$

3.12 Given is the following implication in set notation:

$$A = B \Longrightarrow A \cup C = B \cup C$$

$$x \in A \iff Px;\ x \in B \iff Qx;\ x \in C \iff Sx; x \in D \iff Tx$$
$$A, B, C, D \subseteq G$$

Transform the expressions into predicate logic.
Perform the deductive proof once in predicate logic notation and once in set notation.

Deduction in set - notation

1) $A = B \quad \Rightarrow A \cup C = B \cup C$

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2) $A \subseteq B$ 1, J28, E4

3) $B \subseteq A$ 1, J28, E4

4) $\underbrace{C \subseteq C}_{t}$ J20

5) $A \cup C \subseteq B \cup C$ 2, 4, J35

6) $B \cup C \subseteq A \cup C$ 3, 4, J35

7) $A \cup C = B \cup C$ 5, 6, J28

Deduction in Predicate Logic

$$\forall_{x\in G}(P_x \longleftrightarrow Q_x) \Rightarrow \forall_{x\in G}[(P_x \vee S_x) \longleftrightarrow (Q_x \vee S_x)]$$

$$1)\ \forall_{x\in G}(P_x \longleftrightarrow Q_x) \Rightarrow \forall_{x\in G}[(P_x \vee S_x) \longleftrightarrow (Q_x \vee S_x)]$$

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2) $\forall_{x\in G} P_x \to Q_x$	1, C11, E4, F5
3) $\forall_{x\in G} Q_x \to P_x$	1, C11, E4, F5
4) $\forall_{x\in G} S_x \to S_x$	D6
5) $\forall_{x\in G} [(P_x \vee Q_x) \to (Q_x \vee S_x)]$	2, 4, E15, F5
6) $\forall_{x\in G} [(Q_x \vee S_x) \to (P_x \vee S_x)]$	3, 4, E15, F5
7) $\forall_{x\in G} [(P_x \vee S_x) \longleftrightarrow (Q_x \vee S_x)]$	5, 6, C11, F5

$\boxed{\textbf{3.13}}$ Given is the following implication:

$$V_1 \wedge V_2 \wedge V_3 \Longrightarrow S$$

with:

$$\begin{aligned}
V_1 &\iff \neg \underset{x \in G}{\exists} (Qx \wedge \neg Sx) \\
V_2 &\iff \underset{x \in G}{\forall} (\neg Sx \longleftrightarrow Tx) \vee Tx \\
V_3 &\iff \neg \underset{x \in G}{\exists} (Px \wedge \neg Qx \wedge \neg Sx) \\
S &\iff \underset{x \in G}{\forall} (Px \vee Qx \longrightarrow Tx)
\end{aligned}$$

$$x \in A \iff Px;\; x \in B \iff Qx;\; x \in C \iff Sx; x \in D \iff Tx$$
$$A, B, C, D \subseteq G$$

Transform the implication into set notation and perform the deduction.

$V_1 \Leftrightarrow \underset{x \in G}{\forall} \neg (Q_x \wedge \neg S_x) \Leftrightarrow \overline{(B \cap \overline{C})} = G$

$\Leftrightarrow B \cap \overline{C} = \emptyset$

$V_2 \Leftrightarrow (\overline{C} \triangle D) \cup D = G$

$V_3 \Leftrightarrow \underset{x \in G}{\forall} \neg (P_x \wedge \neg Q_x \wedge \neg S_x) \Leftrightarrow A \cap \overline{B} \cap \overline{C} = \emptyset$

$V_4 \Leftrightarrow A \cup B \subseteq D$

1) $B \cap \bar{C} = \emptyset$

2) $(\bar{C} \Delta D) \cup D = G$

3) $A \cap \bar{B} \cap \bar{C} = \emptyset$ $\qquad \Rightarrow A \vee B \subseteq D$

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4) $(\bar{C} \cap \bar{D}) \cup \underbrace{(C \cap D) \cup D}_{D} = G$ $\qquad$ 2, J11

5) $(\bar{C} \cap \bar{D}) \cup D = G$ $\qquad$ 4, H5

6) $(\bar{C} \cup D) \cap \underbrace{(\bar{D} \cup D)}_{G} = G$ $\qquad$ 5, H3

7) $\bar{C} \cup D = G$ $\qquad$ 6, H8, H6

8) $C \subseteq D$ $\qquad$ 7, J29

9) $A \cap \bar{B} \subseteq C$ $\qquad$ 3, J29

10) $B \subseteq C$ $\qquad$ 1, J29

11) $(A \cap \bar{B}) \wedge B \subseteq C \vee C$ $\qquad$ 9, 10, J35

12) $(A \cup B) \wedge (\bar{B} \cup B) \subseteq C$ $\qquad$ 11, H3, H4

13) $A \cup B \subseteq C$ $\qquad$ 12, H8, H6

14) $A \cup B \subseteq D$ $\qquad$ 8, 13, J34