| $f(x)$ | $f'(x)$ | $f(x)$ | $f'(x)$ |
|---|---|---|---|
| $x^n$ | $nx^{n-1}$ | $e^x$ | $e^x$ |
| $\ln(x)$ | $1/x$ | $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ | $\tan(x)$ | $\sec^2(x)$ |
| $\cot(x)$ | $-\mathrm{cosec}^2(x)$ | $\sec(x)$ | $\sec(x)\tan(x)$ |
| $\mathrm{cosec}(x)$ | $-\mathrm{cosec}(x)\cot(x)$ | $\tan^{-1}(x)$ | $1/(1+x^2)$ |
| $\sin^{-1}(x)$ | $1/\sqrt{1-x^2}$ for $\lvert x\rvert<1$ | $\cos^{-1}(x)$ | $-1/\sqrt{1-x^2}$ for $\lvert x\rvert<1$ |
| $\sinh(x)$ | $\cosh(x)$ | $\cosh(x)$ | $\sinh(x)$ |
| $\tanh(x)$ | $\mathrm{sech}^2(x)$ | $\coth(x)$ | $-\mathrm{cosech}^2(x)$ |
| $\mathrm{sech}(x)$ | $-\mathrm{sech}(x)\tanh(x)$ | $\mathrm{cosech}(x)$ | $-\mathrm{cosech}(x)\coth(x)$ |
| $\sinh^{-1}(x)$ | $1/\sqrt{x^2+1}$ | $\cosh^{-1}(x)$ | $1/\sqrt{x^2-1}$ for $x>1$ |
| $\tanh^{-1}(x)$ | $1/(1-x^2)$ for $\lvert x\rvert<1$ | $\coth^{-1}(x)$ | $-1/(x^2-1)$ for $\lvert x\rvert>1$ |

# Trigonometry Function Identities

## Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

## Reciprocal Indentities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

## Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\csc^2\theta - \cot^2\theta = 1$$

## Even/Odd Indentities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta$$

## Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\frac{\pi}{2} \text{ radians} = 90°$$

## Sum/Difference Indentities

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

## Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

## Half Angle Indentities

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

## Sum to Product of Two Angles

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

## Product to Sum of Two Angles

$$\sin\theta\sin\phi = \frac{[\cos(\theta - \phi) - \cos(\theta + \phi)]}{2}$$

$$\cos\theta\cos\phi = \frac{[\cos(\theta - \phi) + \cos(\theta + \phi)]}{2}$$

$$\sin\theta\cos\phi = \frac{[\sin(\theta + \phi) + \sin(\theta - \phi)]}{2}$$

$$\cos\theta\sin\phi = \frac{[\sin(\theta + \phi) - \sin(\theta - \phi)]}{2}$$

# Laws of Logarithms

Similar to the laws of exponents, we have the following laws of logarithms.

a) $\log_a(xy) = \log_a x + \log_a y$ → $a^{LHS} = a^{\log_a(xy)} = xy$  $a^{RHS} = a^{\log_a x + \log_a y}$

$= a^{\log_a x} \cdot a^{\log_a y}$

$= x \cdot y$

b) $\log_a(x/y) = \log_a x - \log_a y$

c) $\log_a x^y = y \log_a x$

d) $\log_a x = \log_b x / \log_b a$ (change of base)

e) $\log_a 1 = 0$ ~ since $\log_a\left(\frac{x}{x}\right) = \log_a x - \log_a x = 0$

Exercise: Prove each law above using the laws of exponents.

# Table of Derivatives

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

| $f(x)$ | $f'(x)$ | $f(x)$ | $f'(x)$ |
|---|---|---|---|
| $c$ | $0$ | $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $x^r$ | $rx^{r-1}$ | $\cos^{-1} x$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| $e^x$ | $e^x$ | $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |
| $a^x$ | $a^x \ln a$ | $kg(x)$ | $kg'(x)$ |
| $\ln x$ | $\frac{1}{x}$ | $g(x) \pm h(x)$ | $g'(x) \pm h'(x)$ |
| $\log_a x$ | $\frac{1}{x \ln a}$ | $g(x)h(x)$ | $g'h + h'g$ |
| $\sin x$ | $\cos x$ | $\frac{g(x)}{h(x)}$ | $\frac{hg' - gh'}{h^2}$ |
| $\cos x$ | $-\sin x$ | $g(h(x))$ | $g'(h)h'(x)$ |
| $\tan x$ | $\sec^2 x$ | | |

# Derivative of $a^x$ & $x^r$

We can also use logarithmic differentiation to derive the derivative of the exponential function with any base a > 1.

$$f(x) = a^x \to \ln f(x) = x \ln a$$
$$\frac{d}{dx} \to \frac{f'(x)}{f(x)} = \ln a$$
$$\to f'(x) = f(x) \ln a = a^x \ln a$$

Exercise: Using logarithmic differentiation, show that the derivative of **f(x) = x^r** where **r is a real number** is

$$f'(x) = rx^{r-1}$$

$$\ln f = r \ln x$$
$$\frac{d}{dx} \to \frac{f'}{f} = \frac{r}{x}$$
$$f'(x) = f(x) \cdot \frac{r}{x} = x^r\left(\frac{r}{x}\right) = rx^{r-1}$$

Since r is a constant

# Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

# Squeeze Theorem

For example, try to evaluate the limit

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

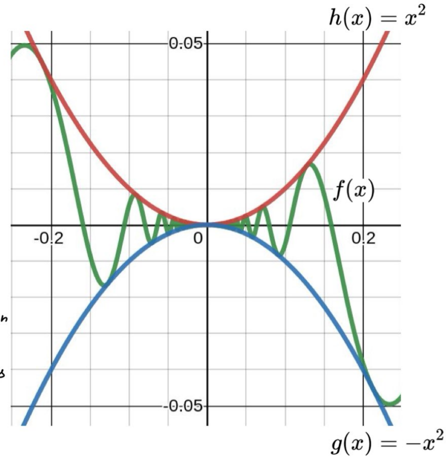
Notice that although the limit of sin(1/x) DNE, we know that the range of the sine function is [-1, 1]. So

$$-1 \le \sin(1/x) \le 1$$

While on inequality from the function that is bounded and work towards f(x)

$$\times x^2 \to -x^2 \le x^2 \sin(1/x) \le x^2$$

$$\text{And } \underline{\lim_{x\to 0} -x^2 = 0}, \quad \underline{\lim_{x\to 0} x^2 = 0}$$




Hence, by squeeze theorem,

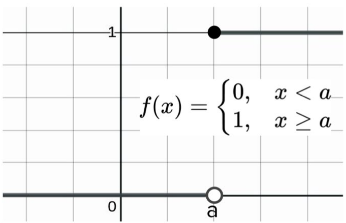
$$\underline{\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0}$$

# Types of Discontinuity

We have seen the various types of discontinuity in earlier slides. They are described below. The removable discontinuity is called 'removable' because one can define f(a) = L to 'remove' the discontinuity. This cannot be done for other types.

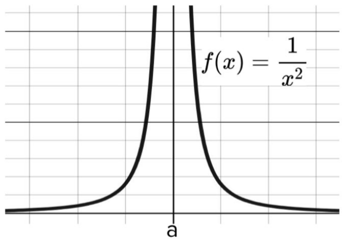
**Jump discontinuity**

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$
$$\lim_{x\to a} f(x) \text{ D.N.E.}$$



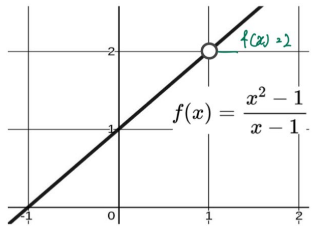

**Infinite discontinuity**

$$\lim_{x\to a^\pm} f(x) = \pm\infty$$
$$\lim_{x\to a} f(x) \text{ D.N.E.}$$




**Removable discontinuity**

$$\lim_{x\to a} f(x) = L$$
$$f(a) \neq L$$



# Rationalizing

Rationalizing the numerator or denominator (or both) is another technique to evaluate a limit. Consider

$$\lim_{x\to 1}\frac{\sqrt{2-x}-1}{x-1}=\frac{0}{0}$$

which gives an indeterminate form after direct substitution. And factoring does not work. Notice that the numerator is an irrational expression, which can be rationalized by multiplying by its conjugate, i.e.

$$\lim_{x\to 1}\frac{\sqrt{2-x}-1}{x-1}\cdot\frac{\sqrt{2-x}+1}{\sqrt{2-x}+1}=\lim_{x\to 1}\frac{2-x-1}{(x-1)\left(\sqrt{2-x}+1\right)}$$

$$=\lim_{x\to 1}\frac{-(x-1)}{(x-1)\left(\sqrt{2-x}+1\right)}=\lim_{x\to 1}\frac{-1}{\sqrt{2-x}+1}=-\frac{1}{2}$$

(Verify in Desmos.)

# Evaluating Limits

We have covered quite a few techniques of limit evaluation. The table below provides a general guide of when to use which (may not necessarily work).

| Technique | Generally works for: |
|---|---|
| Direct substitution | Continuous functions. **Always try first.** |
| Factoring and divide off | Rational functions giving indeterminate form of 0/0. |
| Compare growth rates | Ratio of elementary functions as x approaches $\pm\infty$. |
| Divide by x to the largest power | Rational functions giving indeterminate form of $\pm\infty/\pm\infty$. |
| Rationalizing | Functions with irrational (root) expressions. |
| Squeeze theorem | Functions with oscillatory behaviour. |
| L'Hopital's Rule | Functions giving giving indeterminate form of 0/0 or $\pm\infty/\pm\infty$. (Covered in a later topic.) |

# Derivative of $a^x$ & $x^r$

We can also use logarithmic differentiation to derive the derivative of the exponential function with any base a > 1.

$$f(x) = a^x \to \ln f(x) = x \ln a$$
$$\frac{d}{dx} \to \frac{f'(x)}{f(x)} = \ln a$$
$$\to f'(x) = f(x) \ln a = a^x \ln a$$

Exercise: Using logarithmic differentiation, show that the derivative of **f(x) = x$^{\mathbf{r}}$** where **r is a real number** is

$$f'(x) = rx^{r-1}$$

$$\ln f = r \ln x$$
$$\frac{d}{dx} \to \frac{f'}{f} = \frac{r}{x}$$
$$f'(x) = f(x) \cdot \frac{r}{x} = x^r\left(\frac{r}{x}\right) = rx^{r-1}$$

Since r is a constant

# Integrals of Elementary Functions

Since now we know that integration and differentiation are inverse operations, the integrals of elementary functions can be deduced from their derivatives by 'reverse engineering'. Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}x}c = 0 \to \int 0\,dx = c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}kx = k \to \int k\,dx = kx + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1} \to \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \to \int e^x\,dx = e^x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a \to \int a^x\,dx = \frac{a^x}{\ln a} + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x} \to \int \frac{1}{x}\,dx = \ln x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \to \int \cos x\,dx = \sin x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \to \int \sin x\,dx = -\cos x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \to \int \sec^2 x\,dx = \tan x + c$$

Integrals of other elementary functions not shown here will be derived in the next topic.



# Some Other Integrals

And we also have

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \to \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} \to \int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2} \to \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

# Derivative of ln(x) & $\log_a x$

The derivative of **y(x) = ln(x)** can be derived using implicit differentiation.

$$y = \ln x \to x = e^y$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \to 1 = y' e^y$$
$$\to y' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

And for **f(x) = $\log_a$x**, its derivative can be worked out to be

$$f(x) = \log_a x = \frac{\ln x}{\ln a}$$
$$\to f'(x) = \frac{1}{\ln a}\frac{d}{dx}\ln x = \frac{1}{x \ln a}$$

# Symmetry of Functions

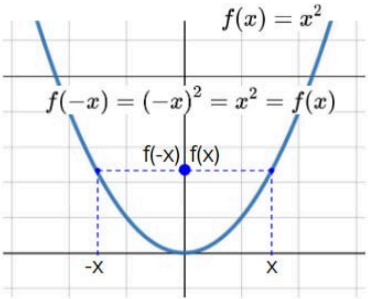
From the graph, we observe that an even function satisfies

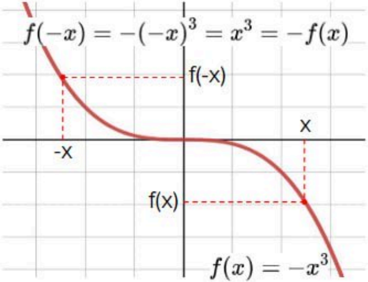
$$f(-x) = f(x)$$

for all x in its domain. Correspondingly, an odd function satisfies

$$f(-x) = -f(x)$$

If a function does not satisfy either of the above relations, then it's neither even nor odd.



# Integration by Substitution of Trigonometric Relations

Notice that in the earlier exercise, the substitution of u = Rcos x is used to **rewrite the root expression for integration**. There are other root expressions where appropriate trigonometric relations can help, as tabulated below (k is a constant).

| Root Expression | Substitution | Result |
|---|---|---|
| $\sqrt{k^2 - x^2}$ | $x = k\sin\theta$ or $k\cos\theta$ | $k\cos\theta$ or $k\sin\theta$ |
| $\sqrt{k^2 + x^2}$ | $x = k\tan\theta$ | $k\sec\theta$ |
| $\sqrt{x^2 - k^2}$ | $x = k\sec\theta$ | $k\tan\theta$ |

$$\sqrt{k^2(\underbrace{1+\tan^2\theta}_{\sec^2\theta})}$$

$$= k\sec\theta$$

Sub

As one can observe, the selection of the substitution is such that the expression being rooted can be **reduced to a single trigonometric expression which removes the root** in the result.

# Table of Integrals

## BASIC FORMS

(1) $\int x^n dx = \frac{1}{n+1} x^{n+1}$

(2) $\int \frac{1}{x} dx = \ln x$

(3) $\int u dv = uv - \int v du$

(4) $\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$

## RATIONAL FUNCTIONS

(5) $\int \frac{1}{ax+b} dx = \frac{1}{a}\ln(ax+b)$

(6) $\int \frac{1}{(x+a)^2} dx = \frac{-1}{x+a}$

(7) $\int (x+a)^n dx = (x+a)^n \left( \frac{a}{1+n} + \frac{x}{1+n} \right), \ n \neq -1$

(8) $\int x(x+a)^n dx = \frac{(x+a)^{1+n}(nx+x-a)}{(n+2)(n+1)}$

(9) $\int \frac{dx}{1+x^2} = \tan^{-1} x$

(10) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a)$

(11) $\int \frac{xdx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2)$

(12) $\int \frac{x^2 dx}{a^2+x^2} = x - a\tan^{-1}(x/a)$

(13) $\int \frac{x^3 dx}{a^2+x^2} = \frac{1}{2}x^2 - \frac{1}{2}a^2 \ln(a^2+x^2)$

(14) $\int (ax^2+bx+c)^{-1} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1}\left( \frac{2ax+b}{\sqrt{4ac-b^2}} \right)$

(15) $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a}\left[ \ln(a+x) - \ln(b+x) \right], \ a \neq b$

(16) $\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln(a+x)$

(17) $$\int \frac{x}{ax^2+bx+c} dx = \frac{\ln(ax^2+bx+c)}{2a} - \frac{b}{a\sqrt{4ac-b^2}} \tan^{-1}\left( \frac{2ax+b}{\sqrt{4ac-b^2}} \right)$$

## INTEGRALS WITH ROOTS

(18) $\int \sqrt{x-a}\, dx = \frac{2}{3}(x-a)^{3/2}$

(19) $\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a}$

(20) $\int \frac{1}{\sqrt{a-x}} dx = 2\sqrt{a-x}$

(21) $\int x\sqrt{x-a}\, dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2}$

(22) $\int \sqrt{ax+b}\, dx = \left( \frac{2b}{3a} + \frac{2x}{3} \right)\sqrt{b+ax}$

(23) $\int (ax+b)^{3/2} dx = \sqrt{b+ax}\left( \frac{2b^2}{5a} + \frac{4bx}{5} + \frac{2ax^2}{5} \right)$

(24) $\int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3}(x \pm 2a)\sqrt{x \pm a}$

(25) $\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x}\sqrt{a-x} - a\tan^{-1}\left( \frac{\sqrt{x}\sqrt{a-x}}{x-a} \right)$

(26) $\int \sqrt{\frac{x}{x+a}} dx = \sqrt{x}\sqrt{x+a} - a\ln\left[ \sqrt{x} + \sqrt{x+a} \right]$

(27) $\int x\sqrt{ax+b}\, dx = \left( -\frac{4b^2}{15a^2} + \frac{2bx}{15a} + \frac{2x^2}{5} \right)\sqrt{b+ax}$

(28) $$\int \sqrt{x}\sqrt{ax+b}\, dx = \left( \frac{b\sqrt{x}}{4a} + \frac{x^{3/2}}{2} \right)\sqrt{b+ax} - \frac{b^2 \ln\left( 2\sqrt{a}\sqrt{x} + 2\sqrt{b+ax} \right)}{4a^{3/2}}$$

(29) $$\int x^{3/2}\sqrt{ax+b}\, dx = \left( -\frac{b^2\sqrt{x}}{8a^2} + \frac{bx^{3/2}}{12a} + \frac{x^{5/2}}{3} \right)\sqrt{b+ax} - \frac{b^3 \ln\left( 2\sqrt{a}\sqrt{x} + 2\sqrt{b+ax} \right)}{8a^{5/2}}$$

(30) $\int \sqrt{x^2 \pm a^2}\, dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln\left( x + \sqrt{x^2 \pm a^2} \right)$

(31) $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2} - \frac{1}{2}a^2 \tan^{-1}\left( \frac{x\sqrt{a^2-x^2}}{x^2-a^2} \right)$

(32) $\int x\sqrt{x^2 \pm a^2} = \frac{1}{3}(x^2 \pm a^2)^{3/2}$

(33) $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left( x + \sqrt{x^2 \pm a^2} \right)$

(34) $\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

(35) $\int \frac{x}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$

(36) $\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2}$

(37) $\int \frac{x^2}{\sqrt{x^2 \pm a^2}}dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}\ln\left(x+\sqrt{x^2 \pm a^2}\right)$

(38) $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\frac{1}{2}x\sqrt{a-x^2} - \frac{1}{2}a^2\tan^{-1}\left(\frac{x\sqrt{a^2-x^2}}{x^2-a^2}\right)$

(39) $$\int \sqrt{ax^2+bx+c}\ dx = \left(\frac{b}{4a}+\frac{x}{2}\right)\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\left(\frac{2ax+b}{\sqrt{a}}+2\sqrt{ax^2+bc+c}\right)$$

(40) $$\int x\sqrt{ax^2+bx+c}\ dx = \left(\frac{x^3}{3}+\frac{bx}{12a}+\frac{8ac-3b^2}{24a^2}\right)\sqrt{ax^2+bx+c} - \frac{b(4ac-b^2)}{16a^{5/2}}\ln\left(\frac{2ax+b}{\sqrt{a}}+2\sqrt{ax^2+bc+c}\right)$$

(41) $\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}\ln\left[\frac{2ax+b}{\sqrt{a}}+2\sqrt{ax^2+bx+c}\right]$

(42) $$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}}\ln\left[\frac{2ax+b}{\sqrt{a}}+2\sqrt{ax^2+bx+c}\right]$$

**LOGARITHMS**

(43) $\int \ln x dx = x\ln x - x$

(44) $\int \frac{\ln(ax)}{x}dx = \frac{1}{2}(\ln(ax))^2$

(45) $\int \ln(ax+b)dx = \frac{ax+b}{a}\ln(ax+b) - x$

(46) $\int \ln(a^2x^2 \pm b^2)dx = x\ln(a^2x^2 \pm b^2) + \frac{2b}{a}\tan^{-1}\left(\frac{ax}{b}\right) - 2x$

(47) $\int \ln(a^2-b^2x^2)dx = x\ln(a^2-b^2x^2) + \frac{2a}{b}\tan^{-1}\left(\frac{bx}{a}\right) - 2x$

(48) $$\int \ln(ax^2+bx+c)dx = \frac{1}{a}\sqrt{4ac-b^2}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) - 2x + \left(\frac{b}{2a}+x\right)\ln\left(ax^2+bx+c\right)$$

(49) $\int x\ln(ax+b)dx = \frac{b}{2a}x - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2-\frac{b^2}{a^2}\right)\ln(ax+b)$

(50) $\int x\ln(a^2-b^2x^2)dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2-\frac{a^2}{b^2}\right)\ln(a^2-bx^2)$

**EXPONENTIALS**

(51) $\int e^{ax}dx = \frac{1}{a}e^{ax}$

(52) $\int \sqrt{x}e^{ax}dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\mathrm{erf}\left(i\sqrt{ax}\right)$ where

$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$

(53) $\int xe^x dx = (x-1)e^x$

(54) $\int xe^{ax}dx = \left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}$

(55) $\int x^2e^x dx = e^x(x^2-2x+2)$

(56) $\int x^2e^{ax}dx = e^{ax}\left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)$

(57) $\int x^3e^x dx = e^x(x^3-3x^2+6x-6)$

(58) $\int x^ne^{ax}dx = (-1)^n\frac{1}{a}\Gamma[1+n,-ax]$ where

$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$

(59) $\int e^{ax^2}dx = -i\frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(ix\sqrt{a}\right)$

**TRIGONOMETRIC FUNCTIONS**

(60) $\int \sin x dx = -\cos x$

(61) $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x$

(62) $\int \sin^3 x dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x$

(63) $\int \cos x dx = \sin x$

(64) $\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4}\sin 2x$

(65) $\int \cos^3 x dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x$

(66) $\int \sin x\cos x dx = -\frac{1}{2}\cos^2 x$

(67) $\int \sin^2 x\cos x dx = \frac{1}{4}\sin x - \frac{1}{12}\sin 3x$

(68) $\int \sin x\cos^2 x dx = -\frac{1}{4}\cos x - \frac{1}{12}\cos 3x$

(69) $\int \sin^2 x\cos^2 x dx = \frac{x}{8} - \frac{1}{32}\sin 4x$

(70) $\int \tan x dx = -\ln\cos x$

(71) $\int \tan^2 x dx = -x + \tan x$

(72) $\int \tan^3 x dx = \ln[\cos x] + \frac{1}{2}\sec^2 x$

(73) $\int \sec x dx = \ln|\sec x + \tan x|$

(74) $\int \sec^2 x dx = \tan x$

(75) $\int \sec^3 x dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x\tan x|$

(76) $\int \sec x\tan x dx = \sec x$

(77) $\int \sec^2 x\tan x dx = \frac{1}{2}\sec^2 x$

(78) $\int \sec^n x\tan x dx = \frac{1}{n}\sec^n x\,,\quad n \neq 0$

(79) $\int \csc x dx = \ln|\csc x - \cot x|$

(80) $\int \csc^2 x dx = -\cot x$

(81) $\int \csc^3 x dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\ln|\csc x - \cot x|$

(82) $\int \csc^n x\cot x dx = -\frac{1}{n}\csc^n x\,,\quad n \neq 0$

(83) $\int \sec x\csc x dx = \ln\tan x$

**TRIGONOMETRIC FUNCTIONS WITH $x^n$**

(84) $\int x\cos x dx = \cos x + x\sin x$

(85) $\int x\cos(ax)dx = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax$

(86) $\int x^2\cos x dx = 2x\cos x + (x^2-2)\sin x$

(87) $\int x^2\cos ax dx = \frac{2}{a^2}x\cos ax + \frac{a^2x^2-2}{a^3}\sin ax$

(88) $\int x^n\cos x dx = -\frac{1}{2}(i)^{1+n}\left[\Gamma(1+n,-ix) + (-1)^n\Gamma(1+n,ix)\right]$

(89) $\int x^n\cos ax dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(1+n,-iax) - \Gamma(1+n,iax)\right]$

(90) $\int x\sin x dx = -x\cos x + \sin x$

(91) $\int x\sin(ax)dx = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax$

(92) $\int x^2\sin x dx = (2-x^2)\cos x + 2x\sin x$

(93) $\int x^3\sin ax dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2}{a^3}x\sin ax$

(94) $\int x^n\sin x dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1,-ix) - (-1)^n\Gamma(n+1,-ix)\right]$

**TRIGONOMETRIC FUNCTIONS WITH $e^{ax}$**

(95) $\int e^x\sin x dx = \frac{1}{2}e^x[\sin x - \cos x]$

(96) $\int e^{bx}\sin(ax)dx = \frac{1}{b^2+a^2}e^{bx}[b\sin ax - a\cos ax]$

(97) $\int e^x\cos x dx = \frac{1}{2}e^x[\sin x + \cos x]$

(98) $\int e^{bx}\cos(ax)dx = \frac{1}{b^2+a^2}e^{bx}[a\sin ax + b\cos ax]$

**TRIGONOMETRIC FUNCTIONS WITH $x^n$ AND $e^{ax}$**

(99) $\int xe^x\sin x dx = \frac{1}{2}e^x[\cos x - x\cos x + x\sin x]$

(100) $\int xe^x\cos x dx = \frac{1}{2}e^x[x\cos x - \sin x + x\sin x]$

**HYPERBOLIC FUNCTIONS**

(101) $\int \cosh x dx = \sinh x$

(102) $\int e^{ax}\cosh bx dx = \frac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx]$

(103) $\int \sinh x dx = \cosh x$

(104) $\int e^{ax}\sinh bx dx = \frac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx]$

(105) $\int e^x\tanh x dx = e^x - 2\tan^{-1}(e^x)$

(106) $\int \tanh ax dx = \frac{1}{a}\ln\cosh ax$

(107) $\int \cos ax\cosh bx dx = \frac{1}{a^2+b^2}[a\sin ax\cosh bx + b\cos ax\sinh bx]$

(108) $$\int \cos ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b\cos ax \cosh bx + a\sin ax \sinh bx\right]$$

(109) $$\int \sin ax \cosh bx dx = \frac{1}{a^2+b^2}\left[-a\cos ax \cosh bx + b\sin ax \sinh bx\right]$$

(110) $$\int \sin ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b\cosh bx \sin ax - a\cos ax \sinh bx\right]$$

(111) $$\int \sinh ax \cosh ax dx = \frac{1}{4a}\left[-2ax + \sinh(2ax)\right]$$

(112) $$\int \sinh ax \cosh bx dx = \frac{1}{b^2-a^2}\left[b\cosh bx \sinh ax - a\cosh ax \sinh bx\right]$$

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin \varphi \cos \theta, y = \rho \sin \varphi \sin \theta, \text{ and } z = \rho \cos \varphi.$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2, \tan \theta = \frac{y}{x}, \varphi = \arccos \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right).$$

Other relationships that are important to know for conversions are

- $r = \rho \sin \varphi$
- $\theta = \theta$
- $z = \rho \cos \varphi$

These equations are used to convert from spherical coordinates to cylindrical coordinates

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$
- $\varphi = \arccos \left( \frac{z}{\sqrt{r^2+z^2}} \right)$

These equations are used to convert from cylindrical coordinates to spherical coordinates.

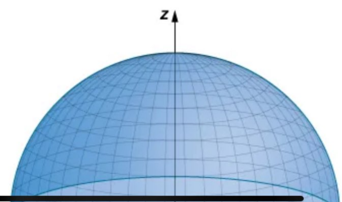
We have discussed parameterizations of various surfaces, but two important types of surfaces need a separate discussion: spheres and graphs of two-variable functions. To parameterize a sphere, it is easiest to use spherical coordinates. The sphere of radius $\rho$ centered at the origin is given by the parameterization

$$\mathbf{r}\left(\phi,\theta\right)=\left\langle\rho\cos\theta\sin\phi,\rho\sin\theta\sin\phi,\rho\cos\phi\right\rangle,0\leq\theta\leq2\pi,0\leq\phi\leq\pi.$$

The idea of this parameterization is that as $\phi$ sweeps downward from the positive *z*-axis, a circle of radius $\rho\sin\phi$ is traced out by letting $\theta$ run from 0 to $2\pi$. To see this, let $\phi$ be fixed. Then

$$\begin{aligned}x^2+y^2&=\left(\rho\cos\theta\sin\phi\right)^2+\left(\rho\sin\theta\sin\phi\right)^2\\&=\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)\\&=\rho^2\sin^2\phi\\&=\left(\rho\sin\phi\right)^2.\end{aligned}$$

This results in the desired circle (Figure 6.61).



# Jacobian of a Vector Field

Analogous to the gradient of a scalar field (function), the Jacobian of a vector field represents the rate of change (ROC) of the vector function w.r.t. each independent variable. Eg, for a vector field

$$\mathbf{F}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

its Jacobian (matrix) is

$$\mathbf{J}_{\mathbf{F}}(x,y) = \begin{bmatrix} \dfrac{\partial \mathbf{F}}{\partial x} & \dfrac{\partial \mathbf{F}}{\partial y} \end{bmatrix} = \begin{array}{cl} \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\[2ex] \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} & \begin{array}{l} \longleftarrow \text{ROC of x-component of } \mathbf{F} \\[4ex] \longleftarrow \text{ROC of y-component of } \mathbf{F} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{ROC w.r.t. x} & \text{ROC w.r.t. y} \end{array} & \end{array}$$

# Divergence of a Vector Field

The divergence of a vector field is a **scalar** quantity that measures the degree of 'outflow-ness' the vector field is **at a point**. For a 2D & 3D vector field, the divergence are respectively defined as

$\frac{\partial}{\partial x}$

$$\nabla \cdot \mathbf{F} = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}, \qquad \nabla \cdot \mathbf{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$\frac{\partial}{\partial y}$

Notice that the divergence is a sum of the **rate of change of each vector component in its own direction**.

But how does this scalar quantity measure 'outflow-ness'?

# Curl of a Vector Field

The curl of a **3D** vector field is a **vector** quantity that measures the circulation (rotation effect) of the vector field **at a point**. It is defined by

$$\nabla \times \mathbf{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{bmatrix}$$

To understand the curl more intuitively, firstly consider the flow velocity field **V**(x, y, z) = [y, 0, 0]$^T$ as shown on the next slide. The curl is

$$\nabla \times \mathbf{V} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \leftarrow$$

# Scalar Line Integral

Hence, the scalar line integral of f(x, y) along path C parameterized by **r**(t) = [x(t), y(t)] is

$$\int_C f(x,y)\,ds = \int_C f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt = \int_C f(\mathbf{r}(t))\,\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

Similarly , for a scalar line integral of f(x, y, z) along a path C in 3D space parameterized by **r**(t) = [x(t), y(t), z(t)], we have

$$\int_C f(x,y,z)\,ds = \int_C f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt = \int_C f(\mathbf{r}(t))\,\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

One can deduce that the total arc length of path C can be computed by the scalar line integral

$$\int_C ds = \int_C |\mathbf{r}'(t)|\ dt$$

# Vector Line Integral

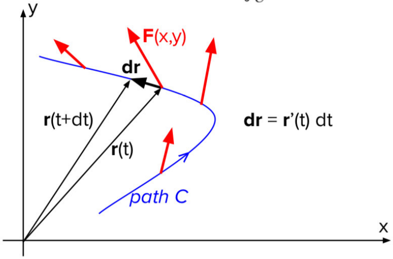
Analogous to a scalar line integral, the vector line integral sums up the dot-product of a vector function with an infinitesimal displacement **dr** along a path C. Using parameterization **r**(t) of the path, the vector line integral of **F**(x, y) is

$$\int_C \mathbf{F}(x,y)\cdot\mathbf{dr} = \int_C \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

$$= \int_C \begin{bmatrix} f_1(\mathbf{r}(t)) \\ f_2(\mathbf{r}(t)) \end{bmatrix}\cdot\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} dt$$

$$= \int_C f_1(\mathbf{r}(t))x'(t) + f_2(\mathbf{r}(t))y'(t)\,dt$$

$$\text{Vector line integral} = \int_C \mathbf{F}(x,y)\cdot\mathbf{dr}$$




**dr**: infinitesimal displacement vector.
**r'**(t): tangential velocity vector to path.

# A Conservative Vector Field

Hence, **all conservative vector fields are gradient fields (& vice-versa)**.

But, how do we know if a vector field is conservative in the first place before the scalar potential is used to compute the line integral? It turns out that all conservative vector fields (gradient fields) have **zero curl**, because

$$\nabla \times \nabla E = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_{zy} - E_{yz} \\ -(E_{zx} - E_{xz}) \\ E_{yx} - E_{xy} \end{bmatrix} = \mathbf{0}$$

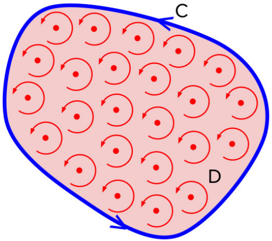
by symmetry of mixed partials. So one can **check for zero curl of a vector field before using the scalar potential** to evaluate a line integral.

# Green's Theorem

The Green's theorem relates the curl of a vector field **F**(x, y) **inside** a closed curve to the line integral **along** the curve, defined by

closed curve C.

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dA = \oint_C \mathbf{F} \cdot \mathbf{dr}$$

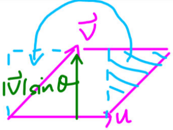



Imagine curve C behaves like a 'conveyor belt' being moved by circulative flow inside it.

Intuitively, one can imagine that the sum of circulation (rotation effect) of a vector field **within a region (D)** has a net circulative effect on the **boundary (C) of the region**, that is the line integral.

Note that the Green's theorem applies **only to a closed curve C** (counter-clockwise).

$$|\vec{u}\times\vec{v}| = \underbrace{\underbrace{|\vec{u}|}_{\text{base}}\,\underbrace{|\vec{v}|\sin\theta}_{\text{height}}}_{\text{Area of ▱}}$$



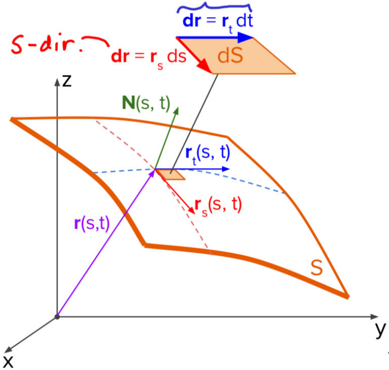

# Area of a Surface

Recall from Math 1 that the **magnitude of the cross product gives the area of the parallelogram** spanned by the two vectors. So, the elemental area dS on a surface S is

$$dS = |\mathbf{r}_s ds \times \mathbf{r}_t dt| = |\mathbf{r}_s \times \mathbf{r}_t|\, dsdt = |\mathbf{N}(s,t)|\, dsdt$$

displacement in t-dir.

displacement in s-dir.




noting that ds & dt are scalars so they can factored out. The total surface area of S can then be evaluated by **summing up all the elemental areas**, i.e.

$$Area = \iint_S |\mathbf{r}_s \times \mathbf{r}_t| dsdt = \iint_S |\mathbf{N}(s,t)|\, dsdt$$

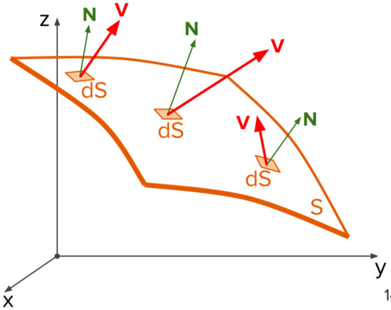
# Vector Surface Integrals - Flux

So the total volumetric flowrate across surface S can be found by **summing up the flowrates across all elemental areas**, i.e.

$$Q = \iint_S dQ = \iint_S \mathbf{V} \cdot \mathbf{N}\, dsdt$$

In essence, the flux of a vector field **F** across a surface S parameterized by s & t is

$$\boxed{Flux = \iint_S \mathbf{F}(\mathbf{r}(s,t)) \cdot \mathbf{N}(s,t)\, dsdt}$$




So, what do you think is the flux if **F** is tangential to surface S everywhere?

0, since $\vec{F}$ has no normal component to S.

# Vector Surface Integrals - Flux

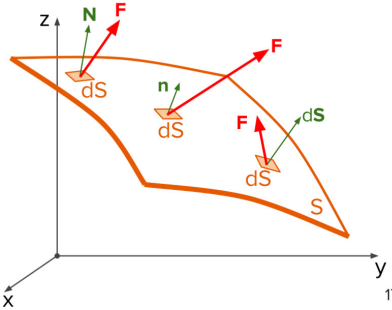
There are other common notations for the flux (as you will encounter in the module of Electricity and Magnetism next trimester). The unit normal vector at any point on surface S is

$$\mathbf{n} = \frac{\mathbf{N}(s,t)}{|\mathbf{N}(s,t)|} \rightarrow \mathbf{N}(s,t) = \mathbf{n}\,|\mathbf{N}(s,t)| = \mathbf{n}\,|\mathbf{r}_s \times \mathbf{r}_t|$$

Substituting the above into the flux gives

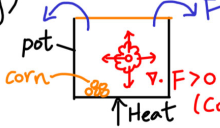
$$Flux = \underline{\iint_S \mathbf{F}\cdot\mathbf{N}\,dsdt} = \iint_S \mathbf{F}\cdot\mathbf{n}\overbrace{|\mathbf{r}_s \times \mathbf{r}_t|dsdt}^{dS}$$

$$= \underline{\iint_S \mathbf{F}\cdot\mathbf{n}\,dS} = \underline{\iint_S \mathbf{F}\cdot d\mathbf{S}}$$




where d**S** = **n**·dS. The three forms of flux are **equivalent**.

Eg)




Flux of popcorn. = Total corn pops.

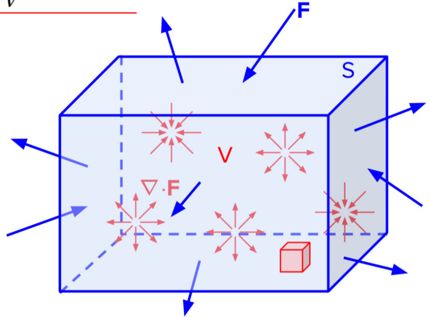
$\iint.$ $\iiint.$

# Divergence Theorem

The divergence theorem (aka Gauss's theorem) relates the flux of a vector field across a **closed** surface S to its divergence in the volumetric region V **enclosed by S**, given by

$$Flux = \underline{\iint_S \mathbf{F} \cdot \mathbf{N}\, dsdt} = \underline{\iiint_V \nabla \cdot \mathbf{F}\, dV}$$

Intuitively, by treating **F** as a velocity field, one can imagine that net flowrate out of a closed surface S must be equal to the total 'outflow-ness' inside S. From this perspective, the divergence theorem can also be understood as the conservation of volume flowrate.

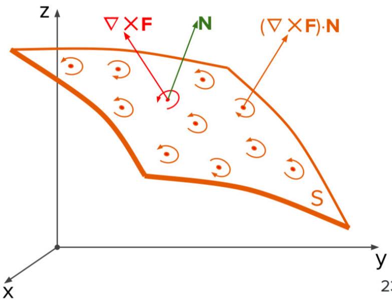


# Vector Surface Integral - Circulation

Besides the vector surface integral of flux, the surface integral of the curl of a vector field **F** can also be similarly defined, called circulation, i.e.

$$Circulation = \iint_S \underline{(\nabla \times \mathbf{F})} \cdot \mathbf{N}(s,t)\, dsdt$$

Notice that in comparison with flux (below), the vector **F** in flux is **replaced by** the curl of **F** in circulation, which means that the (total) circulation over a surface S simply the sum of all infinitesimal circulations on S.




$$Flux = \iint_S \underline{\mathbf{F}} \cdot \mathbf{N}(s,t)\, dsdt$$

# Stokes' Theorem

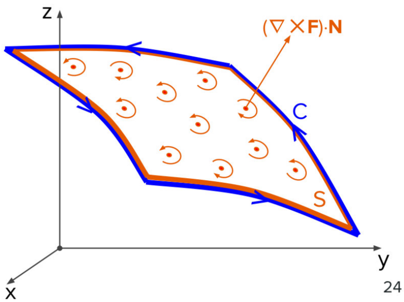
The Stokes' theorem relates the curl of a vector field **F**(x, y, z) over a surface S **inside** a closed curve C to the line integral **along** the curve, i.e.

$$Circulation = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}(s,t)\, dsdt = \oint_C \mathbf{F} \cdot \mathbf{dr}$$

$$\vec{F}(x,y,z) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

Intuitively, one can imagine that the sum of circulation (rotation effect) of a vector field **within a surface S** has a net circulative effect on the **boundary (C) of the surface**, that is the line integral.

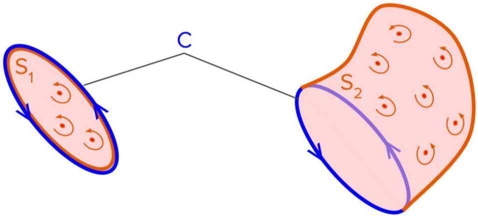
Hmm, sounds familiar?



# Stokes' Theorem

One interesting consequence of the Stokes' theorem is that the circulation over different surfaces with the **same boundary curve C** will be **equal**, since the line integral over curve C is the same. So we have

$$Circulation = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{N}(s,t)\, dsdt = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{N}(s,t)\, dsdt = \oint_C \mathbf{F} \cdot \mathbf{dr}$$



# Table of Laplace Transforms

Consolidating previous results, we create a table for easy reference.

| $f(t)$ | $F(s)$ |
|---|---|
| $k$ | $\dfrac{k}{s}$ |
| $t$ | $\dfrac{1}{s^2}$ |
| $t^n$ | $\dfrac{n!}{s^{n+1}}$ |
| $\sin(\omega t)$ | $\dfrac{\omega}{s^2+\omega^2}$ |
| $\cos(\omega t)$ | $\dfrac{s}{s^2+\omega^2}$ |
| $e^{at}$ | $\dfrac{1}{s-a}$ |

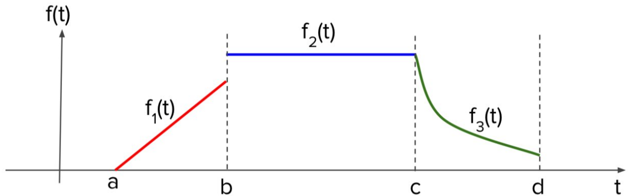
| $f(t)$ | $F(s)$ |
|---|---|
| $u(t-a)$ | $\dfrac{e^{-as}}{s}$ |
| $\delta(t-a)$ | $e^{-as}$ |
| $g(t)\delta(t-a)$ | $g(a)e^{-as}$ |
| $g(t)e^{at}$ | $G(s-a)$ |
| $g(t-a)u(t-a)$ | $e^{-as}G(s)$ |
| $t^n g(t)$ | $(-1)^n G^{(n)}(s)$ |
| $(g*h)(t)$ | $G(s)H(s)$ |
| $g(t)+h(t)$ | $G(s)+H(s)$ |
| $kg(t)$ | $kG(s)$ |

| $f(t)$ | $F(s)$ |
|---|---|
| $g'(t)$ | $sG(s)-g(0)$ |
| $g''(t)$ | $s^2G(s)-sg(0)-g'(0)$ |
| $g^{(n)}(t)$ | $s^nG(s)-\sum_{i=1}^{n} s^{n-i}g^{(i-1)}(0)$ |

$$\boxed{F(s)=\int_0^\infty f(t)e^{-st}\,dt}$$

# Rewriting Piecewise Functions Using u(t-a)

A piecewise function can be rewritten into a **single function** using the **unit-step function**. Generally, we can deduce:

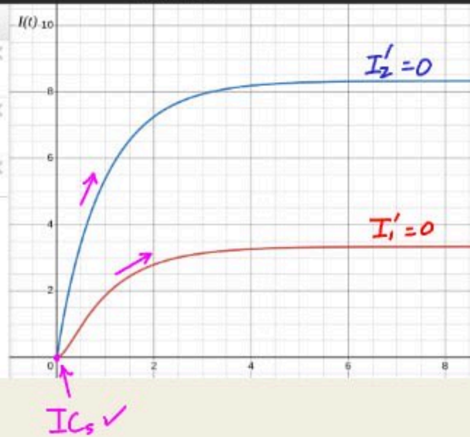



$$f(t) = \begin{cases} f_1(t), & a \le t < b \\ f_2(t), & b \le t < c \\ f_3(t), & c \le t < d \\ \vdots & \end{cases} = \underbrace{f_1(t)u(t-a)}_{\text{On } f_1 \text{ at } t = a} + \underbrace{[f_2(t) - f_1(t)]u(t-b)}_{\text{On } f_2 \text{ + Off } f_1 \text{ at } t = b} + \underbrace{[f_3(t) - f_2(t)]u(t-c)}_{\text{On } f_3 \text{ + Off } f_2 \text{ at } t = c} + \dots$$

Tut 3 Q5 RL Circuit

$$I_1(t) = \frac{10}{3} - 4e^{-t} + \frac{2}{3}e^{-6t}\{t>0\}$$

$$I_2(t) = \frac{25}{3} - 8e^{-t} - \frac{1}{3}e^{-6t}\{t>0\}$$

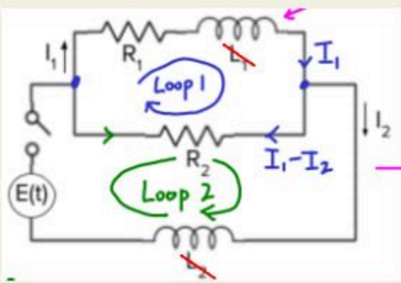


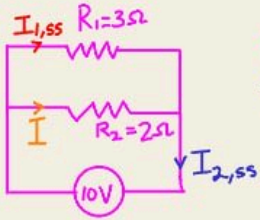

Observations:
1) The initial conditions of zero current is obeyed by the solutions as expected.

2) The currents rise to a steady value due to the voltage source being a constant 10 VDC.

3) The steady-state currents are 10/3 A and 25/3 A respectively since when I' = 0, the inductors behave like a length of wire with no impedance and hence the analysis below shows the steady-state currents as logically revealed by the current functions.

At steady-state, $\Delta V$ across inductors approach 0.



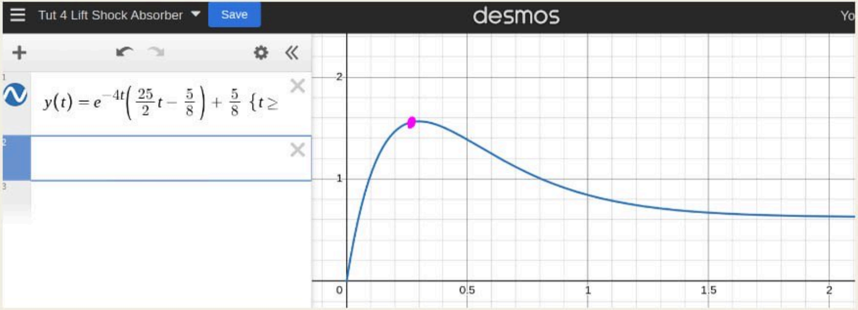




$$I_{1,ss} = \frac{10}{3} = 3\tfrac{1}{3}\,A$$

$$I_{2,ss} = I_{1,ss} + \frac{10}{2}$$

$$= 3\tfrac{1}{3} + 5 = 8\tfrac{1}{3}\,A$$

c) Plot the response y(t) and analyze. Do you think the spring and damper components are selected appropriately for the lift shock absorber system? Explain.





Obervations:

1) The initial conditions are obeyed by the solution.

2) Since the system is critically damped, there is no oscillation of the lift cabin after the impact.

3) The lift cabin reached a maximum displacement of roughly 1.55 m with an overshoot and returns to the steady-state displacement of 5/8 m.

Since the travel of the shock-absorber looks to be around 0.6 m, the system will bottom-out and be damaged by the lift cabin. Hence the components are not suitable. Increase the spring stiffness or damping coefficient to reduce the max displacement.

| S.No. | Form of the rational function | Form of the partial fraction |
|---|---|---|
| 1. | $\dfrac{px+q}{(x-a)(x-b)}, a \neq b$ | $\dfrac{A}{x-a}+\dfrac{B}{x-b}$ |
| 2. | $\dfrac{px+q}{(x-a)^2}$ | $\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}$ |
| 3. | $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$ | $\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$ |
| 4. | $\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$ | $\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$ |
| 5. | $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ | $\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$ |
| | • where $x^2 + bx + c$ cannot be factorised further | |

# Definition of Fourier Series

Now, we can summarise to give the complete definition of the Fourier series representation of a function as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where the Fourier coefficients can be evaluated from:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

# Fourier Sine Series

We can simplify the Fourier series for **odd** and **even** functions into one that contains only either **sine** or **cosine** functions. We have the following:

For an **odd function f(x)**, we can define a Fourier sine series given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

$$\text{where} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Notice that the original integral for $b_n$ is now simplified to **integrating over half of the period**. This is because f(x)sin(nπx) is **even**, so the original area under the curve can be found by **multiplying the area over a symmetric half by two**.

# Fourier Cosine Series

Similarly, for an **even function** f(x), we can define a Fourier cosine series given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

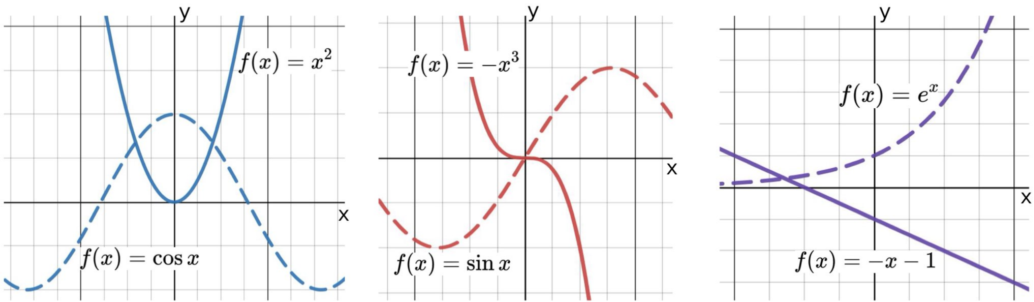
$$\text{where } a_0 = \frac{2}{L}\int_0^L f(x)dx, \;\; a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Again, since f(x) and f(x)cos(nπx) are both **even**, we only need to **integrate over half of the period** to obtain the coefficients.

Evidently, for an odd/even function, its Fourier series should only contain sine/cosine terms or else it would not be able to represent the function accurately.

# Symmetry of Functions

A function can be defined as even or odd, illustrated below. An even function looks like half of it is reflected about the y-axis (symmetry about the y-axis). An odd function looks like half of it is rotated 180° about the origin (symmetry about the origin). A function can also be neither even nor odd.




Even functions | Odd functions | Neither even nor odd

# Symmetry of Functions

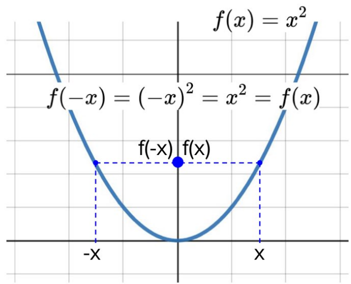
From the graph, we observe that an even function satisfies

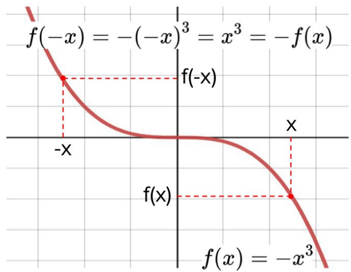
$$f(-x) = f(x)$$

for all x in its domain. Correspondingly, an odd function satisfies

$$f(-x) = -f(x)$$

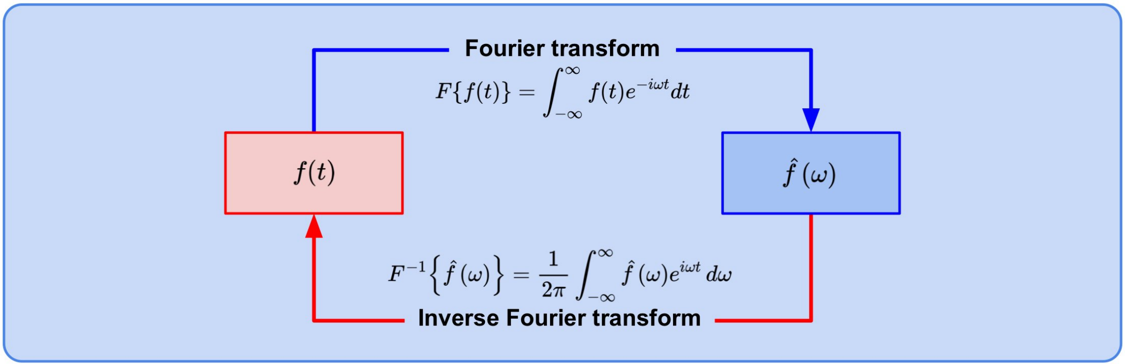
If a function does not satisfy either of the above relations, then it's neither even nor odd.



# Fourier Transform

Together, the last two transforms enable transformations between functions in the time/space domain and those in the frequency domain. Schematically, we have:



Import: `import numpy as np`

**Basic Arithmetic Functions:**

- `np.add(x, y)`, `np.subtract(x, y)`
- `np.multiply(x, y)`, `np.divide(x, y)`
- `np.power(x, y)`: Power
- `np.abs(x)`: Absolute value

**Trigonometric Functions:**

- `np.sin(x)`, `np.cos(x)`, `np.tan(x)`
- `np.arcsin(x)`, `np.arccos(x)`, `np.arctan(x)`
- `np.deg2rad(x)`, `np.rad2deg(x)`: Degree ↔ Radian conversion

**Exponential and Logarithmic Functions:**

- `np.exp(x)`: Exponential
- `np.log(x)`: Natural logarithm
- `np.log10(x)`: Logarithm base 10
- `np.log2(x)`: Logarithm base 2