Topic 7
Introduction to Laplace Transform

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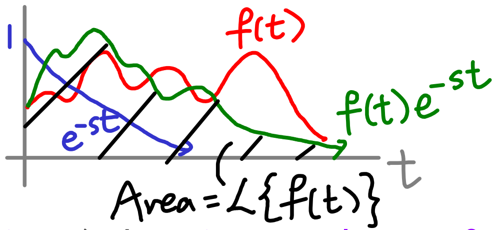
Outline

- Definition of the Laplace Transform
- Laplace Transform of Elementary Functions
- Properties of the Laplace Transform

The Laplace Transform



Functions w/o L.T.

e^{t^2}, $\ln t$, $\frac{1}{t}$, $\tan t$...

Since area is infinite
(integral doesn't converge).

The Laplace transform is simply an integration (a.k.a. integral transform) defined by:

L.T.

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

where the input is a function of a real variable (usually time), and the output is another function of a complex variable, s = σ+iω.

Area is finite.

For the Laplace transform of a function f(t) to **exist**, the **improper integral must converge**.

Example: The Laplace transform of f(t) = k (constant) is,

$$L\{k\} = \int_0^\infty ke^{-st}\,dt = k\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty = k\left[-\frac{1}{s}(0-1)\right] = \overbrace{\frac{k}{s}}^{F(s)}, \quad s > 0$$

Laplace Transform of Elementary Functions

Easy to verify by integration by parts.

The Laplace transform of $f(t) = t$, t^2, t^3 and t^n respectively are:

$$L\{t\} = \int_0^\infty te^{-st}\,dt = \left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_0^\infty = \frac{1}{s^2}, \quad s > 0$$

$$L\{t^2\} = \int_0^\infty t^2e^{-st}\,dt = \left(-\frac{t^2}{s}e^{-st} - \frac{2t}{s}e^{-st} - \frac{2}{s^3}e^{-st}\right)\Big|_0^\infty = \frac{2}{s^3}, \quad s > 0$$

$$L\{t^3\} = \int_0^\infty t^3e^{-st}\,dt = \left(-\frac{t^3}{s}e^{-st} - \frac{3t^2}{s^2}e^{-st} - \frac{6t}{s^3}e^{-st} - \frac{6}{s^4}e^{-st}\right)\Big|_0^\infty = \frac{6}{s^4}, \quad s > 0$$

$$\vdots$$

$$L\{t^n\} = \int_0^\infty t^ne^{-st}\,dt = \frac{n!}{s^{n+1}}, \quad s > 0$$

Laplace Transform of Elementary Functions

The Laplace transform of f(t) = sin(ωt) is (using integration by parts twice):

$$L\left\{\sin(\omega t)\right\} = F(s) = \int_0^\infty \sin(\omega t) e^{-st}\, dt = -\sin(\omega t)\frac{e^{-st}}{s}\Big|_0^\infty + \frac{\omega}{s}\int_0^\infty \cos(\omega t) e^{-st}\, dt$$

$$= 0 + \frac{\omega}{s}\left[-\cos(\omega t)\frac{e^{-st}}{s}\Big|_0^\infty - \frac{\omega}{s}\underbrace{\int_0^\infty \sin(\omega t) e^{-st}\, dt}_{F(s)}\right]$$

$$F(s) = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}F(s)$$

$$\rightarrow \left(1 + \frac{\omega^2}{s^2}\right)F(s) = \frac{\omega}{s^2} \rightarrow F(s) = \frac{\omega}{s^2+\omega^2}, \quad s > 0$$

Using the same approach, the Laplace transform of f(t) = cos(ωt) can be evaluated as:

* Not ω!

$$L\left\{\cos(\omega t)\right\} = \frac{s}{s^2+\omega^2}, \quad s > 0$$

Laplace Transform of Elementary Functions

The Laplace transform of f(t) = e^{at} is:

$$L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[-\frac{1}{s-a}e^{-(s-a)t}\right]\Big|_0^\infty = \frac{1}{s-a}, \quad s > a$$

Example: State the Laplace transforms of the following functions.

$f(t) = t^6$

$F(s) = \frac{6!}{s^{6+1}}$

$= \frac{6!}{s^7}$ //

$g(t) = \sin(3t)$ $\quad \omega = 3$

$G(s) = \frac{3}{s^2+3^2}$

$= \frac{3}{s^2+9}$ //

$h(t) = \cos(7t)$ $\quad \omega = 7$

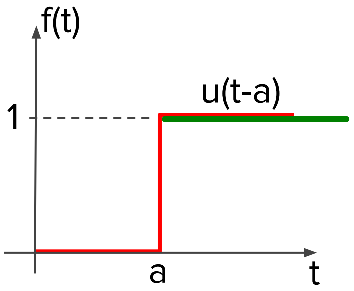
$H(s) = \frac{s}{s^2+49}$ //

$p(t) = e^{-5t}$ $\quad a = -5$

$P(s) = \frac{1}{s+5}$ //

Laplace Transform of Unit-Step Function

Exercise: Evaluate the Laplace transform of the unit-step function, u(t-a).



$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)\cdot e^{-st}\,dt$$

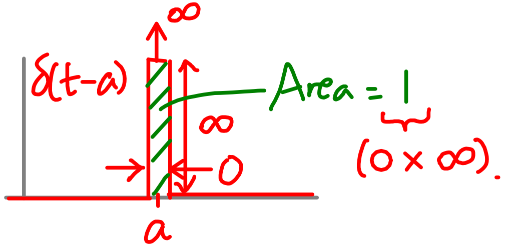
$$= \int_0^a 0 \cdot e^{-st}\,dt + \int_a^\infty (1)\cdot e^{-st}\,dt$$

$$= 0 + \left(-\frac{1}{s}e^{-st}\right)\Bigg|_a^\infty = -\frac{1}{s}\left[0 - e^{-as}\right]$$

$$= \frac{e^{-as}}{s} \quad //$$

ANS: $L\{u(t-a)\} = \dfrac{e^{-as}}{s}$

Laplace Transform of Delta Function



The Dirac delta function δ(t-a) is defined as one that is **zero everywhere except at t = a**, where it is infinitely large. Aka as the unit-impulse, the delta function has an area = 1. The delta function is used to **model impact forces** and **voltage spikes** etc.

The Laplace transform of f(t) = δ(t-a) and f(t) = g(t)δ(t-a) are:

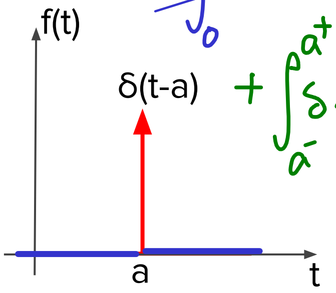
$\cancel{\int_0^{a^-} 0 \cdot e^{-st}\,dt} \to 0$

$t=a$ in (a^-, a^+).

$$L\{\delta(t-a)\} = \int_0^\infty \delta(t-a)e^{-st}\,dt = \int_0^\infty \delta(t-a)e^{-sa}\,dt$$

$+\int_{a^-}^{a^+} \delta \cdot e^{-sa}\,dt + \cancel{\int_{a^+}^{\infty} 0 \cdot e^{-st}\,dt} \to 0$

$$= e^{-as}\int_0^\infty \delta(t-a)dt = e^{-as}, \quad s > 0$$



$t=a$ in (a^-, a^+).

$$L\{g(t)\delta(t-a)\} = \int_0^\infty g(t)\delta(t-a)e^{-st}\,dt = \int_0^\infty \delta(t-a)g(a)e^{-sa}\,dt$$

$$= g(a)e^{-as}\int_0^\infty \delta(t-a)dt = g(a)e^{-as}, \quad s > 0$$

Properties of Laplace Transform

Since the Laplace transform is an integration, it is therefore linear, i.e.

$$L\left\{f(t)\right\} = \int_0^\infty kf(t)e^{-st}\,dt = k\int_0^\infty f(t)e^{-st}\,dt = kF(s)$$

$$\rightarrow L\left\{kf(t)\right\} = kL\left\{f(t)\right\}$$

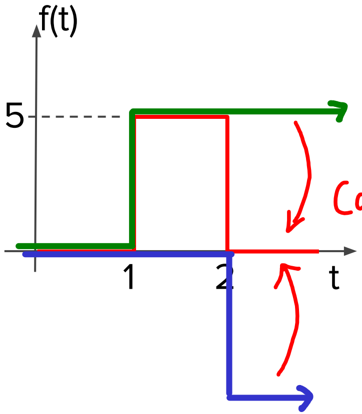
$$L\left\{f(t)+g(t)\right\} = \int_0^\infty [f(t)+g(t)]e^{-st}\,dt = \int_0^\infty f(t)e^{-st}\,dt + \int_0^\infty g(t)e^{-st}\,dt$$
$$= F(s) + G(s)$$

$$\rightarrow L\left\{f(t)+g(t)\right\} = L\left\{f(t)\right\} + L\left\{g(t)\right\}$$

Properties of Laplace Transform

$$\int_1^2 5e^{-st}\,dt = \frac{5}{s}\left(e^{-s} - e^{-2s}\right)$$

Exercise: Using the linearity property, evaluate the Laplace transform of the following rectangular pulse.



Cancel to 0 after t=2.

$$f(t) = \underbrace{5u(t-1)}_{\text{'turn on'}} \underbrace{-\,5u(t-2)}_{\text{'turn off'}}$$

$$\mathcal{L}\{f(t)\} = 5\mathcal{L}\{u(t-1)\} - 5\mathcal{L}\{u(t-2)\}$$

$$= \frac{5e^{-s}}{s} - \frac{5e^{-2s}}{s} = \frac{5}{s}\left(e^{-s} - e^{-2s}\right) //$$

ANS: $L\{f(t)\} = \frac{5}{s}\left(e^{-s} - e^{-2s}\right)$

Shifting Properties of Laplace Transform

$\int_0^{\infty} f(t)e^{-st}\,dt = F(s)$

When a function f(t) is multiplied by e^{at}, its Laplace transform can be evaluated as:

$$L\left\{f(t)e^{at}\right\} = \int_0^{\infty} f(t)e^{at}e^{-st}\,dt = \int_0^{\infty} f(t)e^{-(s-a)t}\,dt = F(s-a), \quad s > a$$

shifted by a.

This property of Laplace transform is called **shifting in the s-domain**.

"s-shifting"

Example: Evaluate the LT of g(t) = te^{3t} and verify the above property.

$$\mathcal{L}\{te^{3t}\} = \int_0^{\infty} te^{3t}\cdot e^{-st}\,dt = \int_0^{\infty} te^{-(s-3)t}\,dt$$

① f(t)

	u	dv
+	t	$e^{-(s-3)t}$
−	1	$\frac{-1}{s-3}e^{\cdots}$
	0	$\frac{1}{(s-3)^2}e^{\cdots}$

$$= \left\{e^{-(s-3)t}\left[\frac{-t}{s-3} - \frac{1}{(s-3)^2}\right]\right\}\Bigg|_0^{\infty} = 0 - \left(-\frac{1}{(s-3)^2}\right)$$

$$= \frac{1}{(s-3)^2}$$

ANS: $L\left\{te^{3t}\right\} = \dfrac{1}{(s-3)^2}$

By s-shifting, $a = 3$,

$$\mathcal{L}\{\underset{\substack{\uparrow \\ 1\ f(t)}}{t}\} = \underbrace{\frac{1}{s^2} = F(s)}_{2} \longrightarrow \overbrace{F(s-3) = \frac{1}{\underbrace{(s-3)^2}_{\text{same as by integration.}}}}^{3} = \mathcal{L}\{te^{3t}\}.$$

Process of applying s-shifting:
1) Identify f(t).
2) Get F(s) = L{f(t)}.
3) Write down F(s-a) by replacing s with s-a in F(s).

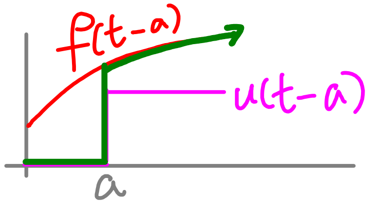
Shifting Properties of Laplace Transform

When a function f(t-a) is multiplied by u(t-a), its Laplace transform can be evaluated as:

$$L\{f(t-a)u(t-a)\} = \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt = \int_a^\infty f(t-a)\cancelto{1}{u(t-a)}e^{-st}\,dt$$

$$= \int_a^\infty f(\underbrace{t-a}_{\tau})e^{-st}\,dt$$

$t = \tau + a$



Let τ = t-a, so dτ = dt, the above integral becomes:

$$L\{f(t-a)u(t-a)\} = \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau = \int_0^\infty f(\tau)e^{-s\tau}e^{-as}\,d\tau$$

$$= e^{-as}\underbrace{\int_0^\infty f(\tau)e^{-s\tau}\,d\tau}_{\mathcal{L}\{f(t)\}} = e^{-as}F(s), \quad s > 0$$

This property of Laplace transform is called shifting in the time (t)-domain.

Shifting Properties of Laplace Transform

Example: Evaluate the LT of $g(t) = t^2u(t-2)$ by using the time-shifting property.

Rewrite in $f(t-2)u(t-2)$.

a

Process:

1. Get $f(t)$.
2. Get $F(s)$.
3. Write $e^{-as}F(s)$.

$$g(t) = (t-2+2)^2u(t-2) = [(t-2)^2 + 4(t-2) + 4]u(t-2)$$

$$G(s) = \mathcal{L}\{\underbrace{[(t-2)^2 + 4(t-2) + 4]}_{f(t-2)}u(t-2)\}$$

$$f(t-2) \xrightarrow{1} f(t) = t^2 + 4t + 4 \xrightarrow{2} F(s) = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$$

3

$$= \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)e^{-2s}$$

ANS: $G(s) = 2e^{-2s}\left(\frac{1}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right)$

OR: Do term by term.

$$G(s) = \mathcal{L}\left\{\underbrace{(t-2)^2}_{f_1(t-2)} u(t-2) + 4(t-2)u(t-2) + 4u(t-2)\right\}$$

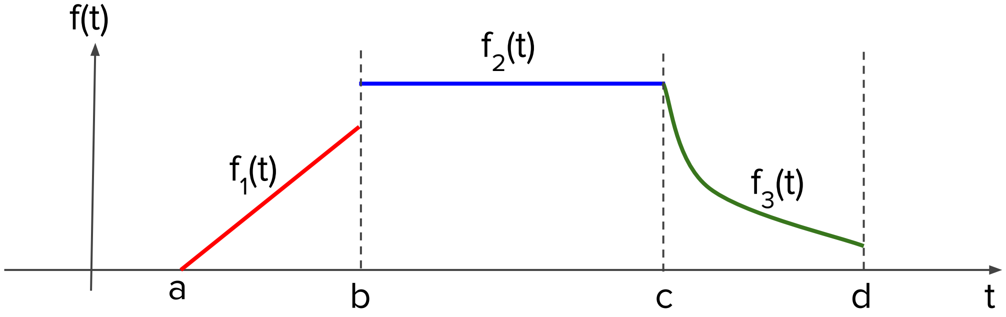
$$f_1(t-2) \xrightarrow{1} f_1(t) = t^2 \xrightarrow{2} F_1(s) = \frac{2}{s^3}$$

3

$$= \left(\frac{2}{s^3}e^{-2s} + \cdots\right)$$

Rewriting Piecewise Functions Using u(t-a)

A piecewise function can be rewritten into a **single function** using the **unit-step function**. Generally, we can deduce:



$$f(t) = \begin{cases} f_1(t), & a \leq t < b \\ f_2(t), & b \leq t < c \\ f_3(t), & c \leq t < d \\ & \vdots \end{cases} = \underbrace{f_1(t)u(t-a)}_{\text{On } f_1 \text{ at } t = a} + \underbrace{[f_2(t) - f_1(t)]u(t-b)}_{\text{On } f_2 \text{ + Off } f_1 \text{ at } t = b} + \underbrace{[f_3(t) - f_2(t)]u(t-c)}_{\text{On } f_3 \text{ + Off } f_2 \text{ at } t = c} + \ldots$$

Rewriting Piecewise Functions Using u(t-a)

Example: Rewrite f(t) using the unit-step function and evaluate its Laplace transform.

$$f(t) = \begin{cases} 2, & 0 \le t < 1 \\ t, & 1 \le t < 3 \\ e^{-5t}, & 3 \le t \end{cases}$$

$$= 2u(t) + (t-2)u(t-1) + (e^{-5t} - t)u(t-3)$$

on 2. on t off 2. on e^{-5t} off t.

Rewrite in $f(t-a)u(t-a)$.

$$= 2u(t) + (t-1-1)u(t-1) + \left[e^{-5(t-3+3)} - (t-3+3)\right]u(t-3)$$

$$= 2u(t) + (t-1)u(t-1) - u(t-1) + \underline{e^{-15} \cdot e^{-5(t-3)} u(t-3)} - (t-3)u(t-3) - 3u(t-3)$$

ANS: $f(t) = 2u(t) + (t-2)u(t-1) + (e^{-5t} - t)u(t-3)$, $F(s) = \dfrac{2}{s} + e^{-s}\left(\dfrac{1}{s^2} - \dfrac{1}{s}\right) + e^{-3s}\left(\dfrac{e^{-15}}{s+5} - \dfrac{1}{s^2} - \dfrac{3}{s}\right)$

$$\mathcal{L}\{f(t)\} = \frac{2e^{0s}}{s} + \frac{1}{s^2}e^{-s} - \frac{e^{-s}}{s} + e^{-15}\cdot\frac{1}{s+5}e^{-3s} - \frac{1}{s^2}e^{-3s} - \frac{3}{s}e^{-3s}$$

$$= \frac{2}{s} + e^{-s}\left(\frac{1}{s^2} - \frac{1}{s}\right) + e^{-3s}\left(\frac{e^{-15}}{s+5} - \frac{1}{s^2} - \frac{3}{s}\right) //$$

$$\underbrace{e^{-5(t-3)}}_{f(t-3)} u(t-3)$$

$$\downarrow$$

$$f(t) = e^{-5t}$$

$$\downarrow$$

$$F(s) = \frac{1}{s+5}$$

Derivative of Laplace Transform

When the transformed function F(s) is differentiated, we notice that:

$$\frac{\mathrm{d}F(s)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}\,dt = \int_0^\infty f(t)(-t)e^{-st}\,dt$$

$$= -\int_0^\infty tf(t)e^{-st}\,dt = -L\{tf(t)\}$$

$\frac{d}{ds}$ $- \int_0^\infty t(-t)f(t)e^{-st}\,dt$

$= \int_0^\infty t^2 f(t)e^{-st}\,dt$

Therefore, when a function f(t) is multiplied by t, its Laplace transform is:

$$L\{tf(t)\} = -F'(s)$$

Further differentiating F(s) reveals that:

$$L\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

$n=2$

$= L\{t^2 f(t)\}$

$= F''(s)$

Derivative of Laplace Transform

Example: Using the derivative of Laplace transform, evaluate the Laplace transform of the following functions. What did you notice in (a)?

a) $h(t) = \underbrace{t^{2}e^{-t}}_{f(t)}$ $n=2$

$\downarrow$

$F(s) = \dfrac{1}{s+1}$

$H(s) = (-1)^2 F''(s)$

$= \dfrac{d}{ds}\underbrace{\left[\dfrac{-1}{(s+1)^2}\right]}_{F'(s)} = \dfrac{2}{(s+1)^3}$ //

For s-shifting:

$h(t) = \underbrace{t^2}_{f(t)} e^{-t}$

$\downarrow$

$F(s) = \dfrac{2}{s^3}$

$\downarrow$

$H(s) = F(s+1)$

$= \dfrac{2}{(s+1)^3}$

same ✓

b) $g(t) = \underbrace{te^{-t}\sin(3t)}_{f(t)}$ $n=1$

$\downarrow$ use s-shifting on $\dfrac{3}{s^2+9}$

$F(s) = \dfrac{3}{(s+1)^2+9}$

$G(s) = -F'(s) = -\dfrac{3(-1)[2(s+1)]}{[(s+1)^2+9]^2}$

ANS: a) $H(s) = \dfrac{2}{(s+1)^3}$. b) $G(s) = \dfrac{6(s+1)}{\left[(s+1)^2+9\right]^2}$

Table of Laplace Transforms

Consolidating previous results, we create a table for easy reference (not exhaustive).

$f(t)$	$F(s)$
k	$\frac{k}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$

$f(t)$	$F(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
$g(t)\delta(t-a)$	$g(a)e^{-as}$
$g(t)e^{at}$	$G(s-a)$
$g(t-a)u(t-a)$	$e^{-as}G(s)$
$t^n g(t)$	$(-1)^n G^{(n)}(s)$
$g(t)+h(t)$	$G(s)+H(s)$
$kg(t)$	$kG(s)$

$$\boxed{F(s) = \int_0^\infty f(t)e^{-st}\,dt}$$

End of Topic 7

We shall continue our struggle in Math 3.
All the very best till then.

The End?

You will find much of the math being employed in the engineering & data science modules, so it's more of a new beginning!