2. Explain why the limit below is an indeterminate form. Give examples showing why the limit can be different values dependent on f(x) and g(x).

$$\lim_{x \to a} f(x)^{g(x)} = 0^0$$

$$\lim_{x \to a} f(x)^{g(x)} = 0^0$$

$$(0.00\ldots1)^0 = 1$$

$$0^{0.00\ldots1} = 0$$

$\therefore$ indeterminate

3. Determine the horizontal asymptotes of

$$f(x) = \begin{cases} \frac{1-4x^2}{x^2+3x-1}, & x < 0 \\ \frac{x(x^2+x+1)}{5x^3-7}, & x \geq 0 \end{cases}$$

ANS: y = -4, y = ⅕.

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1-4x^2}{x^2+3x-1} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x \to -\infty} \frac{1/x^2 - 4}{1 + 3/x - 1/x^2}$$

$$= \lim_{x \to -\infty} \frac{1/\infty - 4}{1 + 3/\infty - 1/\infty} = \frac{0-4}{1+0-0}$$

$$= -4$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x(x^2+x+1)}{5x^3-7} = \lim_{x \to \infty} \frac{x^3+x^2+x}{5x^3-7} \cdot \frac{1/x^3}{1/x^3}$$

$$= \lim_{x \to \infty} \frac{1 + 1/x + 1/x^2}{5 - 7/x^3}$$

$$= \frac{1}{5}$$

4

a) $\lim_{x\to 5} \frac{x^2-25}{x+5}$ $= \lim_{x\to 5} \frac{\cancel{(x+5)}(x-5)}{\cancel{(x+5)}}$

$= 5-5 = 0$

b) $\lim_{x\to -5} \frac{x^2-25}{x+5}$ $= \lim_{x\to -5} \frac{\cancel{(x+5)}(x-5)}{\cancel{(x+5)}}$

$= -5-5 = -10$

c) $\lim_{x\to 1}(2-e^x)\cos(\pi x)$ $= (2-e^x)(-1)$

$= e^x - 2$

d) $\lim_{x\to\infty}\{e^{-x}-7\}$ $= \frac{e}{\infty} - 7$

$= 0 - 7 = -7$

e) $\lim_{x\to 1} x^{\ln x}$ $= 1^0 = 1$

f) $\lim_{x\to\infty} \frac{2x^3}{1-x^3}$ $\cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{2}{1/x^3 - 1}$

$= \frac{2}{-1} = -2$

i) $\lim_{x\to\infty} \frac{\sqrt{x}-3}{9-x}$ $= \lim_{x\to\infty} \frac{\sqrt{x}-3}{(3-\sqrt{x})(3+\sqrt{x})}$

$= \lim_{x\to\infty} \frac{\cancel{\sqrt{x}-3}}{-\cancel{(\sqrt{x}-3)}(3+\sqrt{x})}$

$= \frac{1}{\infty} = 0$

4. Evaluate the following limits.

a) $\lim_{x\to 5} \frac{x^2-25}{x+5}$

b) $\lim_{x\to -5} \frac{x^2-25}{x+5}$

c) $\lim_{x\to 1} (2-e^x)\cos(\pi x)$

d) $\lim_{x\to\infty} \{e^{-x}-7\}$

e) $\lim_{x\to 1} x^{\ln x}$

f) $\lim_{x\to\infty} \frac{2x^3}{1-x^3}$

g) $\lim_{x\to 9} \frac{\sqrt{x}-3}{9-x}$

h) $\lim_{x\to -1} \frac{\sqrt{x+2}-1}{3-\sqrt{5-4x}}$

i) $\lim_{x\to\infty} \frac{\sqrt{x}-3}{9-x}$

j) $\lim_{x\to -\infty} x^3 e^{-x}$

k) $\lim_{x\to\infty} \left\{\sqrt{x^2+10}-x\right\}$

ANS: **a)** 0. **b)** -10. **c)** e - 2. **d)** -7. **e)** 1. **f)** -2. **g)** -⅙. **h)** ¾. **i)** 0. **j)** -∞. **k)** 0.

g) $\lim_{x\to 9} \frac{\sqrt{x}-3}{9-x}$

$$= \lim_{x\to 9} \frac{\sqrt{x}-3}{9-x} = \lim_{x\to 9} \frac{\sqrt{x}-3}{(3-\sqrt{x})(3+\sqrt{x})}$$

$$= \lim_{x\to 9} \frac{\cancel{\sqrt{x}-3}}{-(\cancel{\sqrt{x}-3})(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{1}{-(\sqrt{x}+3)} = -\frac{1}{3+3} = -\frac{1}{6}$$

h) $\lim_{x\to -1} \frac{\sqrt{x+2}-1}{3-\sqrt{5-4x}}$

$$= \lim_{x\to -1} \frac{\sqrt{x+2}-1}{3-\sqrt{5-4x}} \left(\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}\right)\left(\frac{3+\sqrt{5-4x}}{3+\sqrt{5-4x}}\right)$$

$$= \lim_{x\to -1} \frac{[(x+2)-1][3+\sqrt{5-4x}]}{[9-(5-4x)][\sqrt{x+2}+1]} = \lim_{x\to -1} \frac{[x+1][3+\sqrt{5-4x}]}{[4+4x][\sqrt{x+2}+1]}$$

$$= \lim_{x\to -1} \frac{[\cancel{x+1}][3+\sqrt{5-4x}]}{4[\cancel{x+1}][\sqrt{x+2}+1]}$$

$$= \lim_{x\to -1} \frac{3+\sqrt{9}}{4(\sqrt{1}+1)}$$

$$= \frac{6}{8} = \frac{3}{4}$$

j) $\lim_{x \to -\infty} x^3 e^{-x} = \lim_{x \to -\infty} \frac{x^3}{e^x}$

$= \frac{-\infty}{e^{-\infty}}$

$= \frac{-\infty}{0^+} = -\infty$

k) $\lim_{x \to \infty} \left\{ \sqrt{x^2+10} - x \right\} \frac{\sqrt{x^2+10} + x}{\sqrt{x^2+10} + x}$

$= \infty - \infty$

$= \lim_{x \to \infty} \frac{x^2+10-x^2}{\sqrt{x^2+10} + x}$

$= \frac{10}{\infty}$

$= 0$

5. Using the squeeze theorem, evaluate the limits below.

a) $\lim_{x\to\infty} e^{-x}(7\sin x + 4)$

b) $\lim_{x\to -3}\left\{|f(x)|\sqrt{x+3}-1\right\}$ where $-5 \leq f(x) \leq 5,\ x \neq -3$

$0 \leq |f(x)| \leq 5$

ANS: **a)** 0. **b)** -1.

5a

$-1 \leq \sin x \leq 1$

$-7 \leq 7\sin x \leq 7$

$-3 \leq 7\sin x + 4 \leq 11$

$\underbrace{e^{-x}(-3)} \leq e^{-x}(7\sin x+4) \leq \underbrace{e^{-x}(11)}$

$\lim_{x\to\infty} e^{-x}(-3) = 0$ $\qquad$ $\lim_{x\to\infty} e^{-x}(11) = 0$

by squeeze theorem,

$e^{-x}(7\sin x+4) = 0$

5b

$0 \leq |f(x)| \leq 5$

$0 \leq |f(x)|\sqrt{x+3} \leq 5\sqrt{x+3}$

$\underbrace{0-1} \leq |f(x)|\sqrt{x+3} - 1 \leq \underbrace{5\sqrt{x+3} - 1}$

$\lim_{x\to -3} -1 = -1$ $\qquad$ $\lim_{x\to -3} 5\sqrt{x+3} - 1 = -1$

$\therefore$ by squeeze theorem,

$|f(x)|\sqrt{x+3} - 1 = -1$

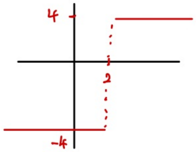
Evaluate the limits at the discontinuities

6. Determine any discontinuity and its type for each of the functions below.

For rational $f(x)$, just look for points where the denominator $= 0$

a) $f(x) = \dfrac{1}{|x|}$

b) $f(x) = \dfrac{4|x-2|}{x-2} = \begin{cases} \dfrac{4(x-2)}{x-2} = 4, & x-2>0 \to x>2 \\ \dfrac{4[-(x-2)]}{x-2} = -4, & x-2<0 \to x<2 \end{cases}$



c) $f(x) = \dfrac{\sqrt{x}-3}{9-x}$

ANS: **a)** Infinite discontinuity at x = 0. **b)** Jump discontinuity at x = 2.

c) Removable discontinuity at x = 9.

1) There are no discontinuities each piecewise function since they are constants

2) Check for discontinuities at interval transitions

$\lim_{x\to 2^-} f(x) = -4$, $\lim_{x\to 2^+} f(x) = 4 \neq \lim_{x\to 2^-} f(x) = -4$

so $f(x)$ is discontinuous at $x=2$ with a jump discontinuity

a) $f(x) = \dfrac{1}{x}$ infinity discontinuity at $x=0$

c) $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{9-x} = \dfrac{\cancel{\sqrt{x}-3}}{-(\cancel{\sqrt{x}-3})(3+\sqrt{x})}$

$= -\dfrac{1}{3+3} = -\dfrac{1}{6} \neq f(9)$

$\therefore$ removable discontinuity at $x=9$

7. Evaluate the interval where each function below is continuous.

a) $f(x) = \begin{cases} \overbrace{x^2 - 3}^{f_1(x)}, & -3 \le x < 2 \\ \underbrace{\frac{5}{3+x}}_{f_2(x)}, & x \ge 2 \end{cases}$

b) $f(x) = \begin{cases} \frac{x^2-4}{x+2}, & x < 0 \\ 2, & x = 0 \\ \frac{3}{4-x}, & x > 0 \end{cases}$

c) $f(x) = \dfrac{x^2 + 3x + 2}{x^3 - x^2 - 2x}$

ANS: **a)** [-3, $\cancel{2}$ ∞). **b)** (-∞, -2) U (-2, 0) U (0, 4) U (4, ∞).
c) (-∞, -1) U (-1, 0) U (0, 2) U (2, ∞).

7a

1. Check for discontinuity :

$f_1(x)$ has no discontinuity

$f_2(x)$ has discontinuity at $x = -3$ but outside of range so ignore

so $f(x)$ is cont in $[-3, \infty)$

c) $f(x) = \dfrac{x^2 + 3x + 2}{x^3 - x^2 - 2x}$ $= \dfrac{(x+2)(x+1)}{x(x^2-x-2)} = \dfrac{(x+2)(x+1)}{x[(x-2)(x+1)]}$

so $f(x)$ is cont in $\{\mathbb{R} \mid x \neq -1, 0, 2\}$

7. Evaluate the interval where each function below is continuous.

a) $$f(x) = \begin{cases} x^2 - 3, & -3 \le x < 2 \\ \frac{5}{3+x}, & x \ge 2 \end{cases}$$

b) $$f(x) = \begin{cases} \frac{x^2-4}{x+2}, & x < 0 \quad f_1(x) \\ 2, & x = 0 \\ \frac{3}{4-x}, & x > 0 \quad f_2(x) \end{cases}$$

c) $$f(x) = \frac{x^2+3x+2}{x^3-x^2-2x}$$

ANS: **a)** [-3, $\cancel{2}$ ∞). **b)** (-∞, -2) U (-2, 0) U (0, 4) U (4, ∞).
c) (-∞, -1) U (-1, 0) U (0, 2) U (2, ∞).

b) $\frac{x^2-4}{x+2} = \frac{(x-2)(x+2)}{x+2}$, discontinuity at $x=2$ or $x=-2$ (ignore)

$\frac{3}{4-x}$, discontinuity at $x=4$

1) Check for discontinuity in each function

In $f_1(x)$, there is a discont. at $x=-2$ which is in $x<0$, so consider it

In $f_2(x)$, there is a discont. at $x=4$ which is in $x>0$, so consider it

2) Check for discontinuity at interval functions

At $x=0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} f_1(x) = \frac{0-4}{0+2} = \underbrace{-2}_{\neq f(0)} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f_2(x) = \frac{3}{4} \neq \lim_{x\to 0^-} f(x)$$

$\lim_{x\to 0} f(x)$ DNE, so discont. at $\underline{x=0}$

so $f(x)$ is conti. in $\{\mathbb{R} \mid x \neq -2, 0, 4\}$

8. Determine the value of c if f(x) is to be continuous in ℝ.

$$f(x) = \begin{cases} \overbrace{cx^2 + 2x}^{f_1(x)}, & x < 2 \\ \underbrace{x^3 - cx}_{f_2(x)}, & x \geq 2 \end{cases}$$

ANS: c = ⅔.

$$\lim_{x \to 2^-} f_1(x) = \lim_{x \to 2^+} f_2(x)$$

$$\lim_{x \to 2^-} cx^2 + 2x = \lim_{x \to 2^+} x^3 - cx$$

$$4c + 4 = 8 - 2c$$

$$6c = 4$$

$$c = \frac{4}{6} = \frac{2}{3}$$

9. Determine the value of a & b if f(x) is to be continuous in ℝ.

$$f(x) = \begin{cases} \overbrace{ax + b}^{f_1(x)}, & x < 1 \\ 4, & x = 1 \\ \underbrace{2ax - b}_{f_2(x)}, & x > 1 \end{cases}$$

ANS: a = 8/3, b = 4/3.

$$\lim_{x \to 1^-} f_1(x) = \lim_{x \to 1^+} f_2(x) = f(1) = 4$$

$$\lim_{x \to 1^-} ax + b = \lim_{x \to 1^+} 2ax - b = 4$$

$$a + b = 4 \quad -①$$

$$2a - b = 4 \quad -②$$

$$① + ② : \quad 3a + 0b = 8$$

$$a = \frac{8}{3}$$

$$\therefore b = 4 - \frac{8}{3}$$

$$= \frac{4}{3}$$

10. Show that a solution exists for each equation below and determine the interval where the solution lies.

a) $x^7 + x^5 + x^3 + x = 7$

b) $x^3 e^x - 99 = 0 = f(x)$

c) $\ln x = \dfrac{1}{\sqrt{x}}$

a) $x^7 + x^5 + x^3 + x - 7 = 0$

b) Trial: $f(0) = -99 < 0$

$f(1) = e - 99 < 0$

$f(2) = 8e^2 - 99 < 0$

$f(3) = 27e^3 - 99 > 0$

$\Big\}$ since $f(x)$ is conť. in $[2,3]$, by IVT, at least one sol$^{\underline{n}}$ exists in $[2,3]$