Topic 6
Vector Calculus II

Dr Teh Yong Liang
Email: youliangzheng@gmail.com

Outline

- Parametric Surfaces
- Tangent Vectors on a Surface
- Area of a Surface
- Surface Integrals
- Flux Across a Surface
- Divergence Theorem
- Circulation & Stokes Theorem

Vector Equation of a Plane

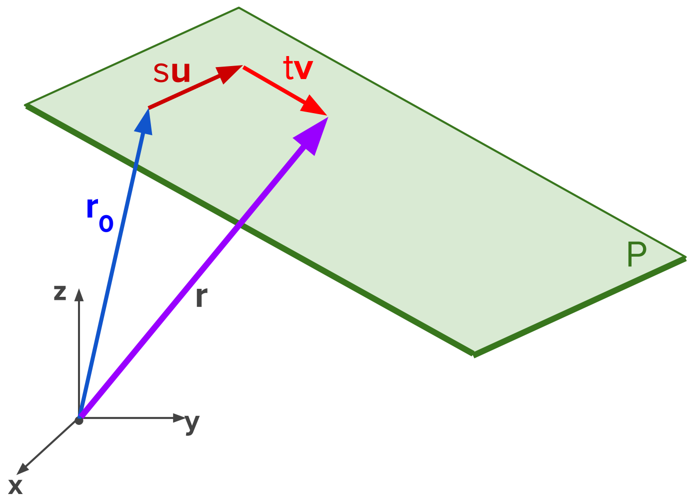
Recall from Math 1 (Topic 3) that a plane P defined by the Cartesian equation

$$ax + by + cz = k$$

can be parameterized by letting

$$x = s, y = t \rightarrow z = \frac{k - ax - by}{c}$$

such that the vector equation of the plane is



$$\mathbf{r}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} s \\ t \\ \frac{k}{c} - \frac{a}{c}s - \frac{b}{c}t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{k}{c} \end{bmatrix}}_{\mathbf{r}_0} + s\underbrace{\begin{bmatrix} 1 \\ 0 \\ -\frac{a}{c} \end{bmatrix}}_{\mathbf{u}} + t\underbrace{\begin{bmatrix} 0 \\ 1 \\ -\frac{b}{c} \end{bmatrix}}_{\mathbf{v}}$$

Parametric Surfaces

Such a vector function of a plane is also called a parametric representation because the vector function is a **function of parameters**. Parametric equations extend readily to general surfaces as well. For a surface defined by z = f(x, y), parameterizations are

$$x(s,t), y(s,t) \to z(s,t) = f(x(s,t), y(s,t))$$

such that its vector function is

$$\mathbf{r}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} x(s,t) \\ y(s,t) \\ f(x(s,t), y(s,t)) \end{bmatrix}$$

For example, the parabolic surface $z = x^2 + y^2$ can also be defined by the vector function

$$\mathbf{r}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} s \\ t \\ s^2 + t^2 \end{bmatrix}$$

Parametric Surfaces

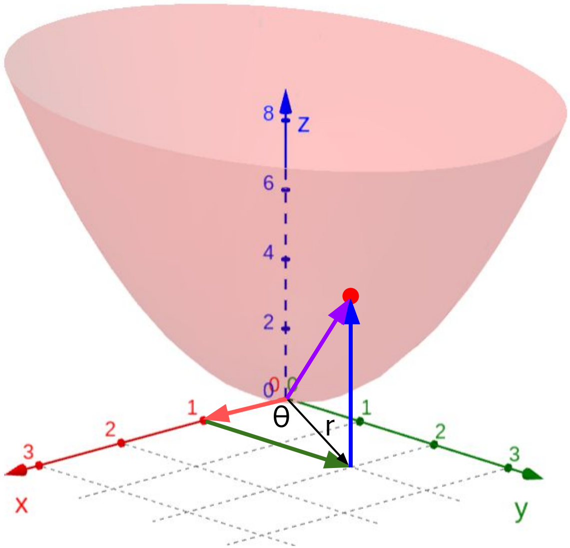
Parametric equations are not unique. Eg, for the parabolic surface $z = x^2 + y^2$, another parameterization using polar coordinates is

$$x = r\cos\theta,\ y = r\sin\theta \to z = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

such that its vector function is

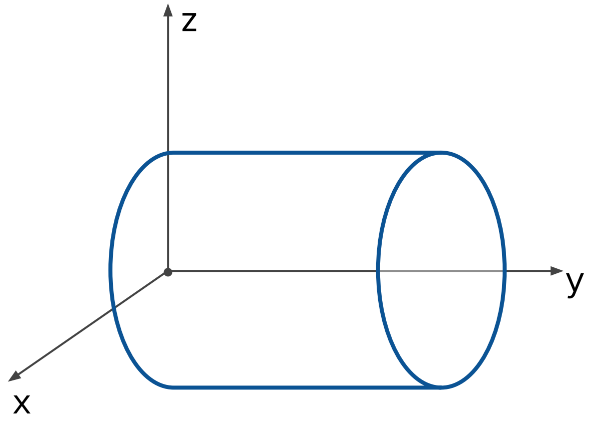
$$\underline{\mathbf{r}(r,\theta)} = \begin{bmatrix} \underline{x(r,\theta)} \\ \underline{y(r,\theta)} \\ \underline{z(r,\theta)} \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{bmatrix}$$

The choice of the parameters generally depends on the type of surface as well as the operations to be performed on the vector functions. This is analogous to the choice of coordinate system in multiple integration.



Parametric Surfaces

Example: Parameterize the infinite cylindrical surface below and state its vector function. The diameter of the cylinder is 4 cm.



ANS: $\mathbf{r}(s, t) = [2\cos(t), s, 2\sin(t)]^T$.

Tangent Vectors

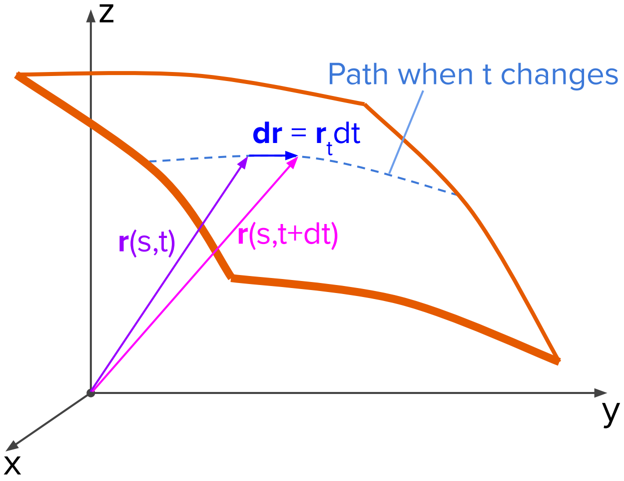
From the vector function of a surface, we can obtain the **tangent vectors** at any point on the surface by **differentiating the vector function w.r.t. the parameters**. From the graph, notice that when parameter t changes from t to t + dt, the vector **r** changes such that the change vector **dr** is

$$\mathbf{dr} = \mathbf{r}(s, t + dt) - \mathbf{r}(s, t) = \mathbf{r}_t \, dt$$

Since dt is a scalar change, $\mathbf{r}_t$ must be the tangent velocity vector in the direction of the t-path. The above **dr** is similar to the tangent vector for a path **r**(t) where

$$\mathbf{dr} = \mathbf{r}'(t) \, dt$$

as illustrated in the last topic on line integrals.



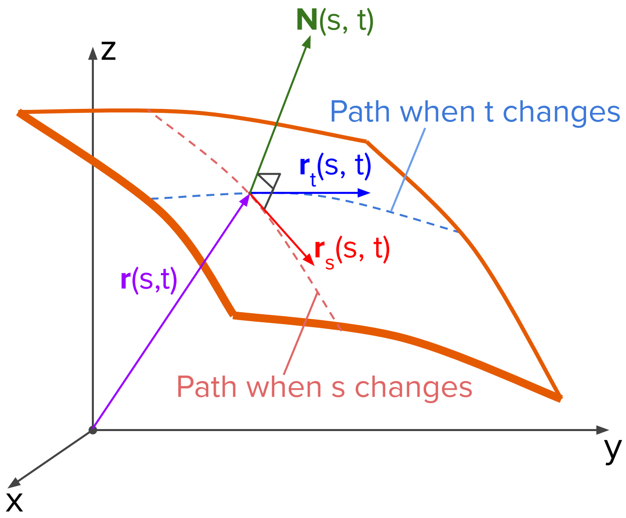
Tangent & Normal Vectors

Clearly, the same analysis can be applied when parameter s changes. So the tangent change vector along the s-path is

$$\mathbf{dr} = \mathbf{r}(s,t) - \mathbf{r}(s+ds,t) = \mathbf{r}_s\,ds$$

So, the tangent velocity vector in the direction of the s-path is $\mathbf{r}_s$, as shown on the graph. With the 2 tangent vectors $\mathbf{r}_s$ and $\mathbf{r}_t$ at any point on the surface, the normal vector can be evaluated by the cross product, i.e.

$$\mathbf{N}(s,t) = \mathbf{r}_s(s,t) \times \mathbf{r}_t(s,t)$$



The other normal vector can be obtained by swapping the tangent vectors above in the cross product.

Tangent Vectors on a Plane

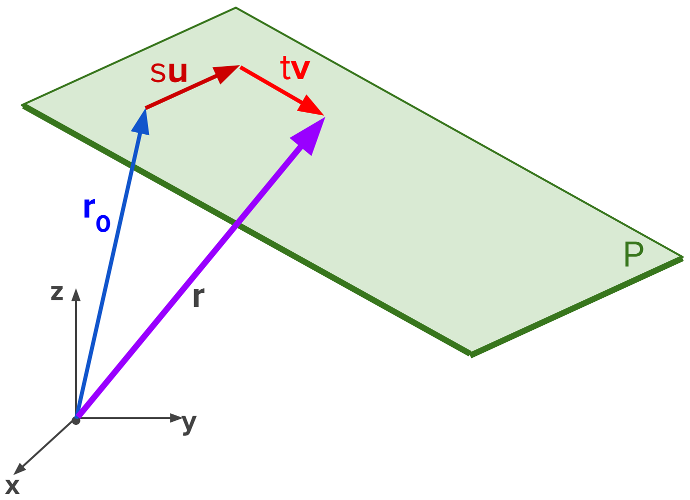
Notice that a planar surface defined by the vector function

$$\mathbf{r}(s,t) = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$$

has the tangent vectors

$$\mathbf{r}_s(s,t) = \mathbf{u}, \qquad \mathbf{r}_t(s,t) = \mathbf{v}$$

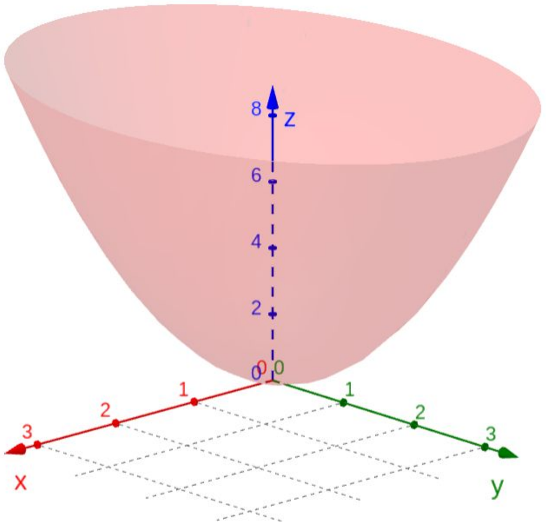
as logically expected. Since the plane has **no curvature**, the **tangent vectors are constant vectors**.



Tangent Vectors

Example: For the parabolic surface defined below, evaluate the tangent vectors and the normal vector that is pointing outward (away from the z-axis).

$$\mathbf{r}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{bmatrix}$$



ANS: $\mathbf{r}_r = [\cos\theta \quad \sin\theta \quad 2r]^T$, $\mathbf{r}_\theta = [-r\sin\theta \quad r\cos\theta \quad 0]^T$, $\mathbf{N} = [2r^2\cos\theta \quad 2r^2\sin\theta \quad -r]^T$.

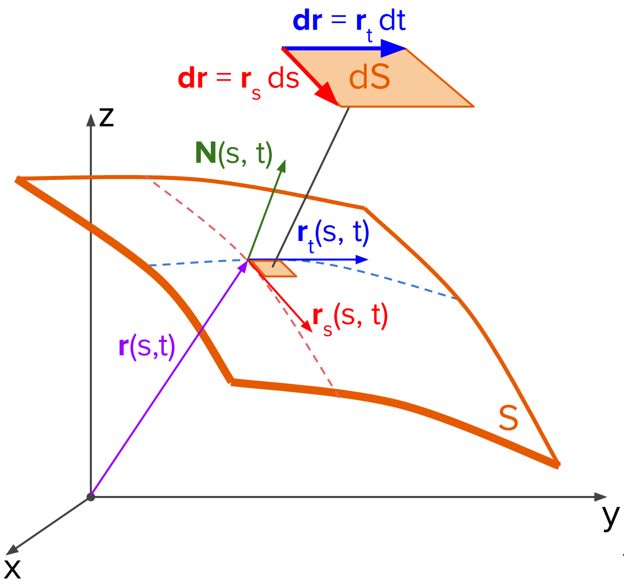
Area of a Surface

Recall from Math 1 that the **magnitude of the cross product gives the area of the parallelogram** spanned by the two vectors. So, the elemental area dS on a surface S is

$$dS = |\mathbf{r}_s ds \times \mathbf{r}_t dt| = |\mathbf{r}_s \times \mathbf{r}_t|\, dsdt = |\mathbf{N}(s,t)|\, dsdt$$

noting that ds & dt are scalars so they can factored out. The total surface area of S can then be evaluated by **summing up all the elemental areas**, i.e.

$$Area = \iint_S |\mathbf{r}_s \times \mathbf{r}_t| dsdt = \iint_S |\mathbf{N}(s,t)|\, dsdt$$



Scalar Surface Integrals

Analogous to scalar line integrals, we can also define the scalar surface integral of a scalar function f(x, y, z), which is simply **summing up the function multiplied by an elemental area** over a surface S. Hence we have

$$\iint_S f(x,y,z)\,dS = \iint_S f(\mathbf{r}(s,t))\,|\mathbf{r}_s \times \mathbf{r}_t|\,dsdt$$

We shall focus on vector surface integrals in this course.

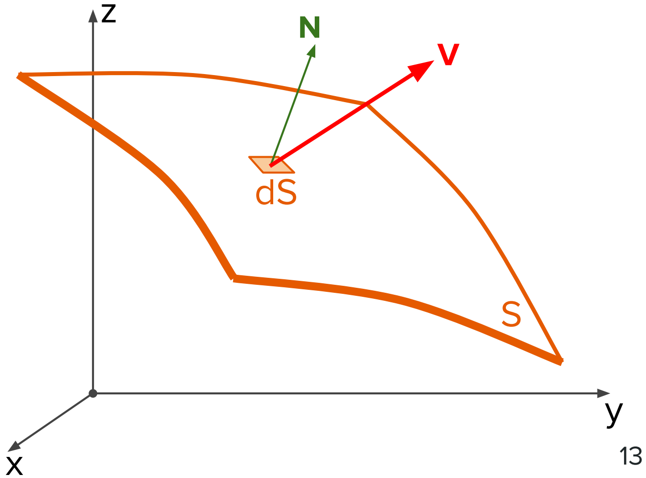
Vector Surface Integrals - Flux

The surface integral of a vector function **F**(x, y, z) represents the ‘flux’ of the vector field across a surface. To make the concept more intuitive, consider a velocity field **V**(x, y, z) flowing across a surface S as shown. At the elemental area dS, the velocity component in the direction of the normal vector is

$$V_n = \mathbf{V} \cdot \frac{\mathbf{N}}{|\mathbf{N}|}$$

Hence, the volumetric flowrate across the elemental area dS parameterized by s & t is

$$dQ = V_n\, dS = \mathbf{V} \cdot \frac{\mathbf{N}}{|\mathbf{N}|} |\mathbf{N}|\, dsdt = \mathbf{V} \cdot \mathbf{N}\, dsdt$$



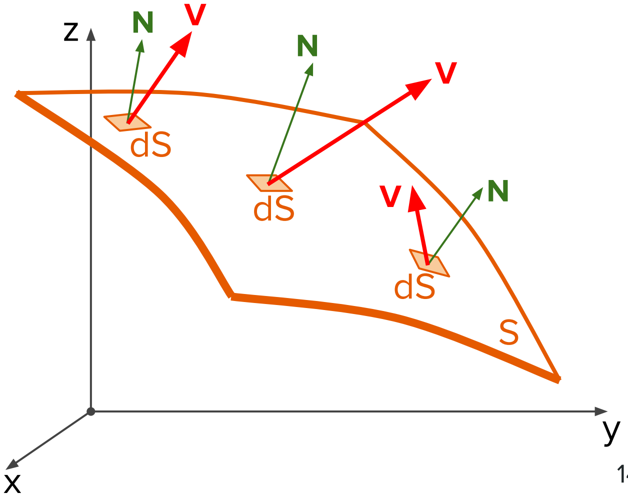
Vector Surface Integrals - Flux

So the total volumetric flowrate across surface S can be found by **summing up the flowrates across all elemental areas**, i.e.

$$Q = \iint_S dQ = \iint_S \mathbf{V} \cdot \mathbf{N}\, dsdt$$

In essence, the flux of a vector field **F** across a surface S parameterized by s & t is

$$Flux = \iint_S \mathbf{F}(\mathbf{r}(s,t)) \cdot \mathbf{N}(s,t)\, dsdt$$



So, what do you think is the flux if **F** is tangential to surface S everywhere?

Vector Surface Integrals - Flux

Example: Evaluate the flux of the vector field below across the cylindrical surface defined by $x^2 + y^2 = 1$ from $z = -2$ to $z = 2$. The surface is oriented with outward normal vectors. Is the flux in or out of the surface?

$$\mathbf{F}(x, y, z) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

ANS: Flux = 8π.

Vector Surface Integrals - Flowrate

Exercise: The velocity field of airflow in a region is modelled by the function below. If the density of the air is uniform at 1.2 kg/m^3, evaluate the mass flowrate of airflow across a dome surface defined by $z = 1 - x^2 - y^2$ above the xy-plane, oriented with outward normal vectors. Is the net airflow inwards or outwards?

$$\mathbf{V}(x, y, z) = \begin{bmatrix} -y \\ x \\ -z \end{bmatrix}$$

ANS: 0.6π kg/s (inward). Airflow is spiralling downwards into the dome.

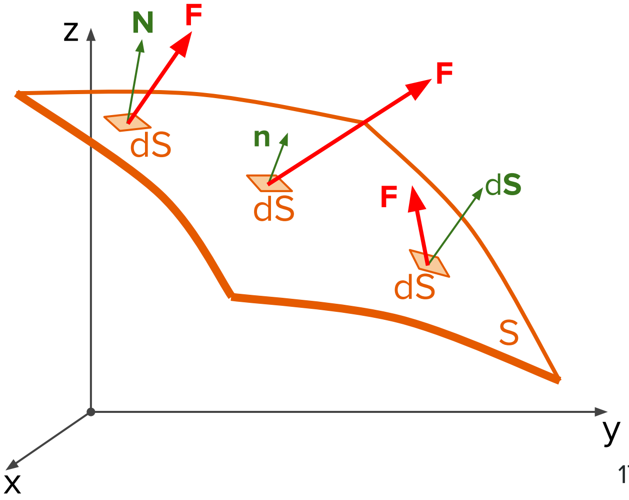
Vector Surface Integrals - Flux

There are other common notations for the flux (as you will encounter in the module of Electricity and Magnetism next trimester). The unit normal vector at any point on surface S is

$$\mathbf{n} = \frac{\mathbf{N}(s,t)}{|\mathbf{N}(s,t)|} \rightarrow \mathbf{N}(s,t) = \mathbf{n}\,|\mathbf{N}(s,t)| = \mathbf{n}\,|\mathbf{r}_s \times \mathbf{r}_t|$$

Substituting the above into the flux gives

$$Flux = \underline{\iint_S \mathbf{F}\cdot\mathbf{N}\,dsdt} = \iint_S \mathbf{F}\cdot\mathbf{n}\,|\mathbf{r}_s \times \mathbf{r}_t|dsdt$$
$$= \underline{\iint_S \mathbf{F}\cdot\mathbf{n}\,dS} = \underline{\iint_S \mathbf{F}\cdot d\mathbf{S}}$$



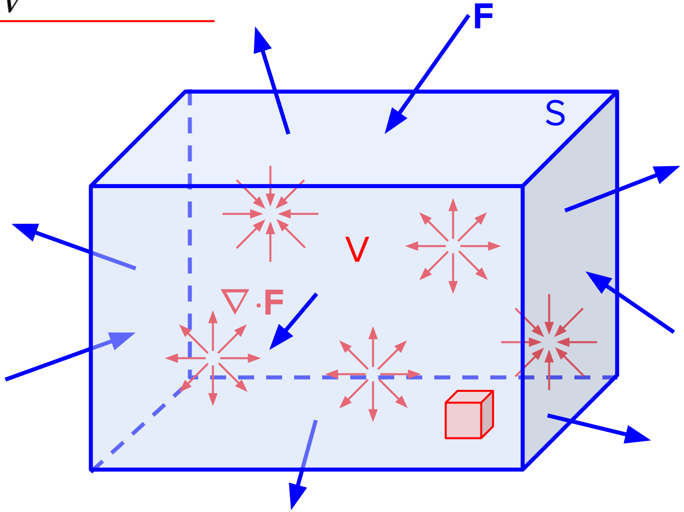
where d**S** = **n**·dS. The three forms of flux are **equivalent**.

Divergence Theorem

The divergence theorem (aka Gauss's theorem) relates the flux of a vector field across a **closed** surface S to its divergence in the volumetric region V **enclosed by S**, given by

$$Flux = \underline{\iint_S \mathbf{F} \cdot \mathbf{N}\, dsdt} = \underline{\iiint_V \nabla \cdot \mathbf{F}\, dV}$$

Intuitively, by treating **F** as a velocity field, one can imagine that net flowrate out of a closed surface S must be equal to the total 'outflow-ness' inside S. From this perspective, the divergence theorem can also be understood as the conservation of volume flowrate.

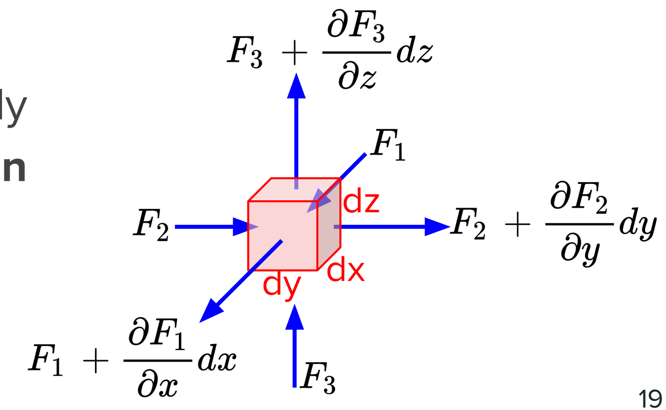


Proof of the Divergence Theorem

Considering a cube element in the volume V enclosed by surface S as shown, the infinitesimal flux across the element is

$$dFlux = \frac{\partial F_1}{\partial x} dx\, dydz + \frac{\partial F_2}{\partial y} dy\, dxdz + \frac{\partial F_3}{\partial z} dz\, dydz$$

$$= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV = \nabla \cdot \mathbf{F}\, dV$$



To get the total flux across the entire volume V, one simply **sums up all the infinitesimal fluxes across all elements in V**, hence giving

$$Flux = \int_V dFlux = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

Proof of the Divergence Theorem

Since the total flux across volume V **must be equal** to the flux across the surface S enclosing the volume, we have the divergence theorem.

$$Flux = \iint_S \mathbf{F} \cdot \mathbf{N} \, dsdt = \iiint_V \nabla \cdot \mathbf{F} \, dV$$

Divergence Theorem

Example: Continuing from an earlier example, verify the divergence theorem for the vector field below across the closed cylindrical surface defined by $x^2 + y^2 = 1$ from $z = -2$ to $z = 2$. The surface is oriented with outward normal vectors.

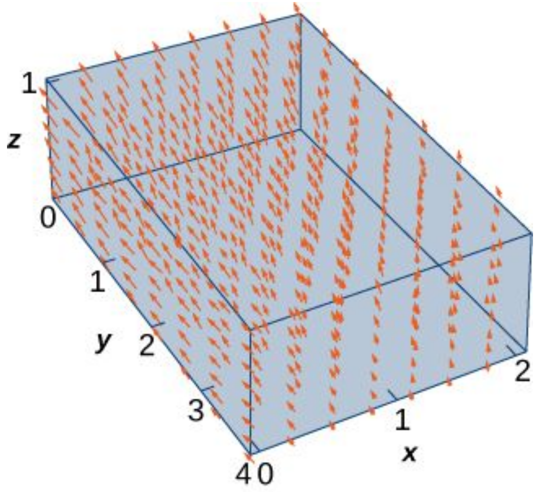
$$\mathbf{F}(x, y, z) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

ANS: Flux = 8π.

Divergence Theorem

Exercise: Using Gauss's theorem, evaluate the flux of the vector field below across the surfaces of the box shown.

$$\mathbf{F}(x,y,z) = \begin{bmatrix} x^2 + yz \\ y - z \\ 2x + 2y + 2z \end{bmatrix}$$



https://openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem

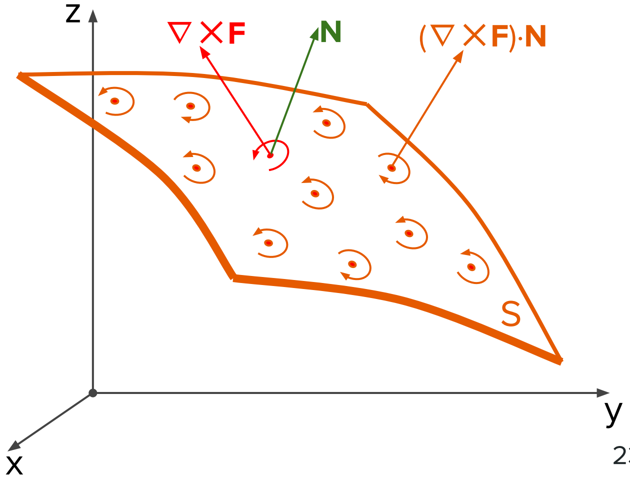
ANS: Flux = 40.

Vector Surface Integral - Circulation

Besides the vector surface integral of flux, the surface integral of the curl of a vector field **F** can also be similarly defined, called circulation, i.e.

$$Circulation = \iint_S \underline{(\nabla \times \mathbf{F})} \cdot \mathbf{N}(s,t)\, dsdt$$

Notice that in comparison with flux (below), the vector **F** in flux is **replaced by** the curl of **F** in circulation, which means that the (total) circulation over a surface S simply the sum of all infinitesimal circulations on S.



$$Flux = \iint_S \underline{\mathbf{F}} \cdot \mathbf{N}(s,t)\, dsdt$$

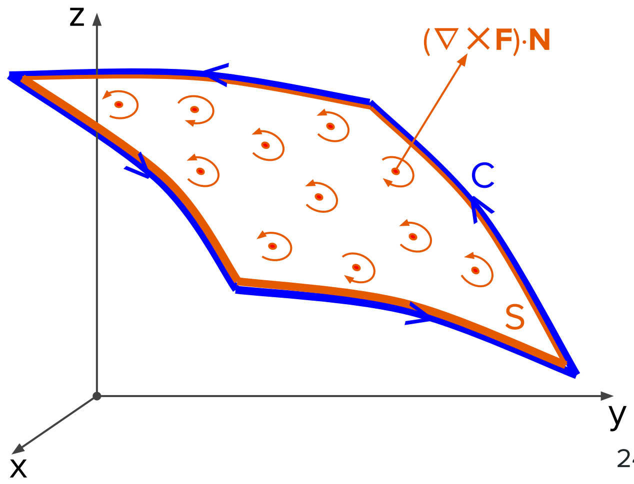
Stokes' Theorem

The Stokes' theorem relates the curl of a vector field **F**(x, y, z) over a surface S **inside** a closed curve C to the line integral **along** the curve, i.e.

$$Circulation = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}(s,t)\, dsdt = \oint_C \mathbf{F} \cdot \mathbf{dr}$$

Intuitively, one can imagine that the sum of circulation (rotation effect) of a vector field **within a surface S** has a net circulative effect on the **boundary (C) of the surface**, that is the line integral.

Hmm, sounds familiar?



Stokes' Theorem

The Stokes' theorem is in fact similar to the Green's theorem, i.e.

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \oint_C \mathbf{F} \cdot \mathbf{dr}$$

In fact, one can observe the Green's theorem is a **special case** of the Stokes' theorem when the **surface S is flat on the xy-plane**. Hence, the Stokes' theorem is **more general** and applies to a surface in 3D space.

The proof for Stokes' theorem is similar to that for the Green's theorem, so we shall leave the student to self-explore.

Stokes' Theorem

Example: Apply Stokes' theorem to evaluate the circulation of vector field **F** below over the parabolic surface S defined by $z = 1 - x^2 - y^2$ above the xy-plane. Then, calculate the circulation over S directly to verify Stokes' theorem.

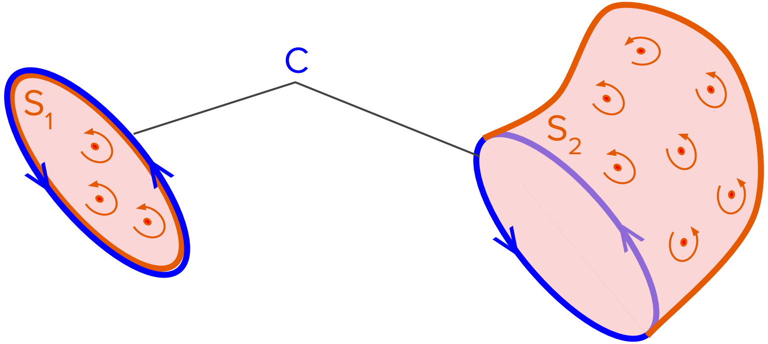
$$\mathbf{F}(x, y, z) = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$

ANS: Circulation = π.

Stokes' Theorem

One interesting consequence of the Stokes' theorem is that the circulation over different surfaces with the **same boundary curve C** will be **equal**, since the line integral over curve C is the same. So we have

$$Circulation = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{N}(s,t)\, dsdt = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{N}(s,t)\, dsdt = \oint_C \mathbf{F} \cdot \mathbf{dr}$$



Stokes' Theorem

Exercise: For the last example, calculate the circulation of **F** over the unit disc $x^2 + y^2 \leq 1$ and thus verify the consequence of Stoke's theorem stated in the last slide.

$$\mathbf{F}(x, y, z) = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$

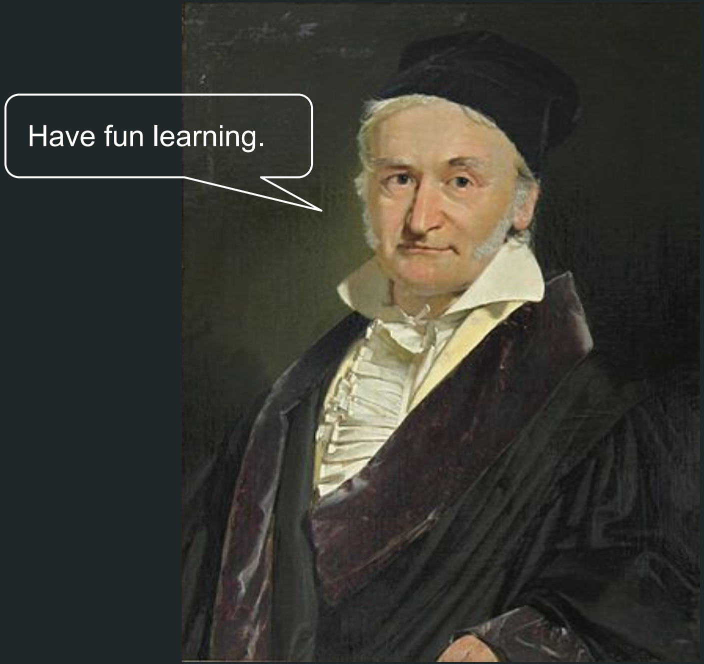
ANS: Circulation = π.

End of Topic 6

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.

Carl Friedrich Gauss
(of Gauss's Theorem)

Carl Friedrich Gauss



Source: https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss