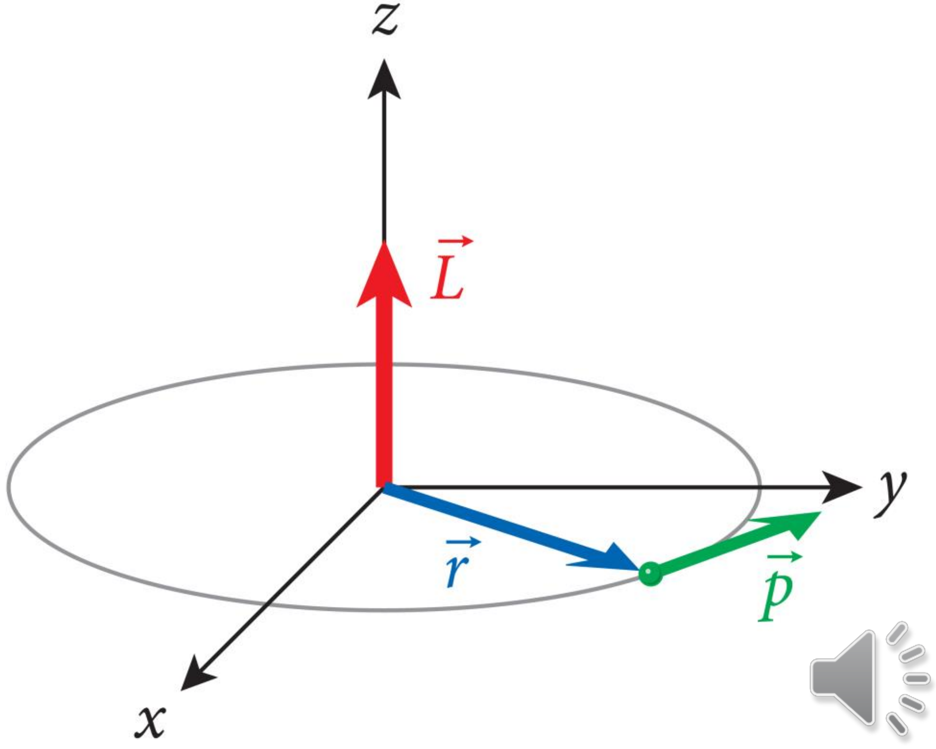
Angular Momentum

- We have discussed linear momentum: $\vec{p} = m\vec{v}$

- Now we introduce the rotational equivalent, angular momentum:
  - We will use the symbol *L* (sometimes **H** or **h**) to denote angular momentum.

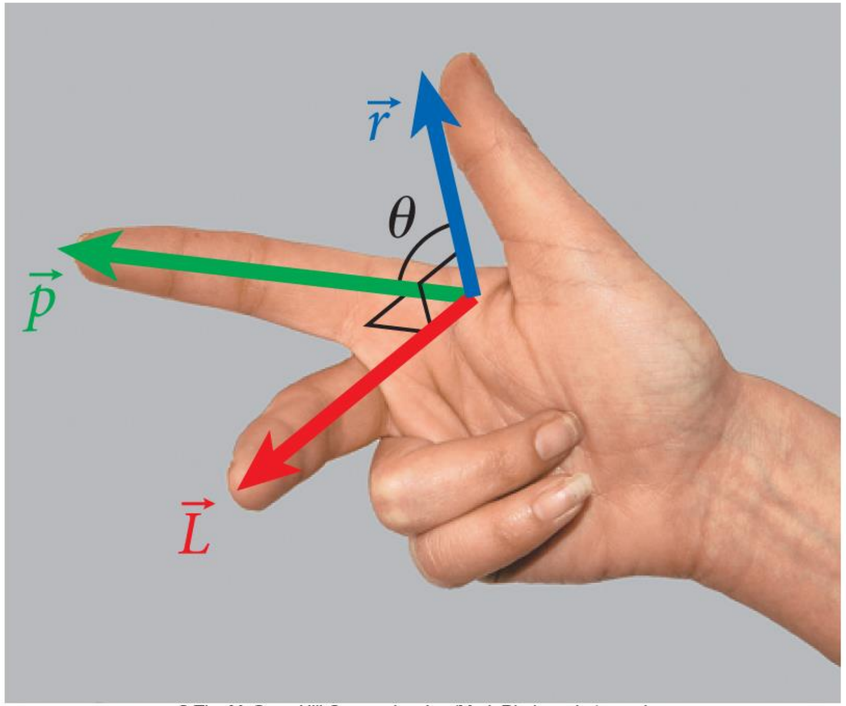


- We start by defining the angular momentum of a point particle: $\vec{L} = \vec{r} \times \vec{p}$

- The magnitude of the angular momentum is given by: $L = rp\sin\theta$

Angular Momentum

- We can define the direction of the angular momentum using the right hand rule:



- Angular momentum is *always perpendicular* to the momentum vector and to the coordinate vector.

Angular Momentum

- Now let's take the time derivative of the angular momentum
$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \left(\left(\frac{d}{dt}\vec{r}\right)\times\vec{p}\right) + \left(\vec{r}\times\frac{d}{dt}\vec{p}\right) = \left(\vec{v}\times\vec{p}\right) + \left(\vec{r}\times\vec{F}\right)$$

- We can see that: $\vec{v}\times\vec{p} = 0 \quad \left(\vec{v} \parallel \vec{p}\right)$

- And we remember that: $\vec{r}\times\vec{F} = \vec{\tau}$

- So we get: $$\frac{d}{dt}\vec{L} = \vec{\tau}$$
$$\left(\text{reminds you of } \frac{d\vec{p}}{dt} = \vec{F}\right)$$

Angular Momentum

- Let's revisit the relationship between the linear velocity, the coordinate vector, and the angular velocity.
- For circular motion we had:

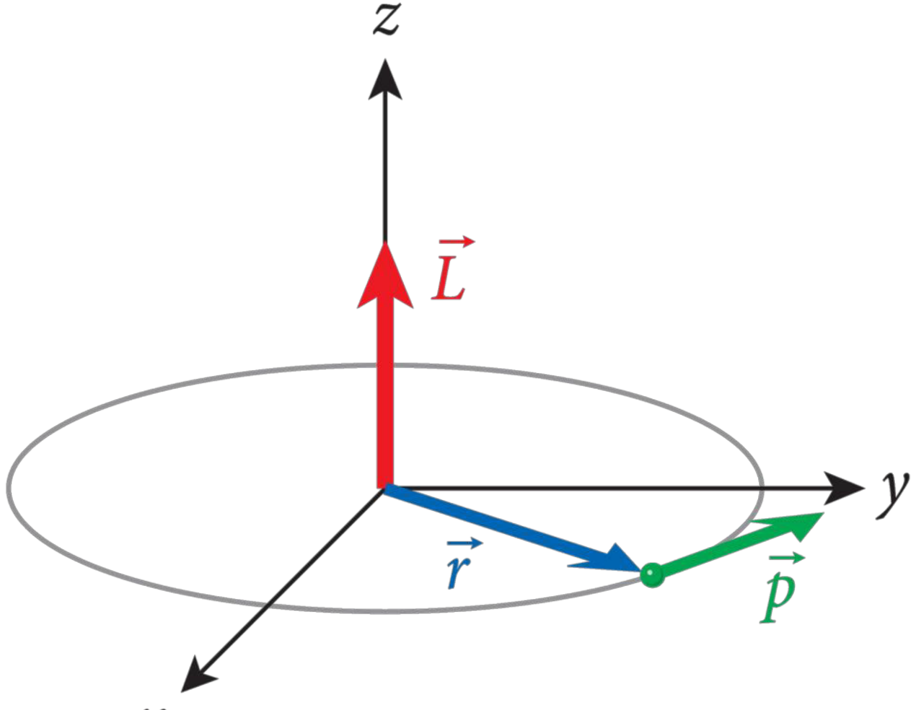
$$\omega = v / r$$

- We can write the angular velocity vectors as:

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}$$

- Which gives us:

$$\vec{L} = \vec{\omega}\left(mr^2\right)$$



Angular Momentum

- We can generalize our results for a single particle to a system of *n* particles:

$$\vec{L} = \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i = \sum_{i=1}^{n} m_i \vec{r}_i \times \vec{v}_i$$

- Take the time derivative to get the relationship to torque:

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\left(\sum_{i=1}^{n} \vec{L}\right) = \frac{d}{dt}\left(\sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i\right) = \sum_{i=1}^{n} \frac{d}{dt}\left(\vec{r}_i \times \vec{p}_i\right)$$

$$= \sum_{i=1}^{n} \underbrace{\underbrace{\left(\frac{d}{dt}\vec{r}_i\right)}_{\text{Equals } \vec{v}_i} \times \vec{p}_i}_{\text{Equals 0, } \vec{v}_i \| \vec{p}_i} + \sum_{i=1}^{n} \frac{d}{dt}\vec{r}_i \times \left(\frac{d}{dt}\vec{p}_i\right) = \sum_{i=1}^{n}\left(\vec{r}_i \times \vec{F}_i\right) = \sum_{i=1}^{n} \vec{\tau}_i = \vec{\tau}_{net}$$

- As expected, the time derivative of the total angular momentum for a system of particles is total net external torque acting on the system.

Angular Momentum

- Representing a rigid body as a collection of point particles which maintain their relative distance constant implies that all the particles will rotate with the same angular velocity.

$$\mathbf{L} = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} m_i r_i \times v_i = \sum_{i=1}^{n} m_i r_{i\perp}^2 \boldsymbol{\omega}$$

- Since the angular velocity is constant:

$$\mathbf{L} = \sum_{i=1}^{n} m_i r_{i\perp}^2 \boldsymbol{\omega} = \boldsymbol{\omega} \sum_{i=1}^{n} m_i r_{i\perp}^2 = I\boldsymbol{\omega}$$

Angular Momentum

- If the net torque is zero, then:

$$\text{if } \vec{\tau}_{net} = 0 \Rightarrow \vec{L} = \text{constant} \Rightarrow \vec{L}(t) = \vec{L}(t_0) \equiv \vec{L}_0$$

- Angular momentum is conserved:

$$I\vec{\omega} = I_0\vec{\omega}_0$$

- The conservation of angular momentum has many interesting consequences:
  - Gyroscopes
  - Divers
  - Dancers
  - Ice-skaters…

(a) (b) (c)