Points to Note

1. Math 1 & 2 are “squeezed” within one trimester, so the pace will be hectic. Mentally prepare for a struggle and there is no honeymoon period.
2. Learn the logic behind the math to understand it. Visualize by using graphs. Try to incorporate common sense and intuition. Using rote memory is the sure way to NOT learn anything and become miserable.
3. Each tutorial typically have more questions than what we can finish during class. So it is best for you to attempt the tutorial beforehand and request that I go through the questions you find difficulty in.

Points to Note

4. Much, if not all, of the math you learn will be applied in the engineering & data science modules. Eg, linear algebra & multivariable calculus are used in machine learning, vector calculus is used in electricity & magnetism and Laplace transform is extensively applied in control engineering etc.

   Hence, you will find it beneficial throughout your degree course if you build up strong fundamentals in math.

Assessments for EDE1011 Math 1

The assessment weightages are:

1) 7% Class Participation
2) 28% Quizzes (2 MCQ quizzes, 1 hr each) - Weeks 2 & 6
3) 30% Midterm Test - Week 4 (structured questions, 2.5 hrs)
4) 35% Final Exam - Week 7 (Yes, during term-break.)

Unless otherwise specified, all assessments are **openbook** (no internet access). So you have to **bring a laptop or tablet to do the tests**. Handphones are not allowed.

Topic 1
Functions

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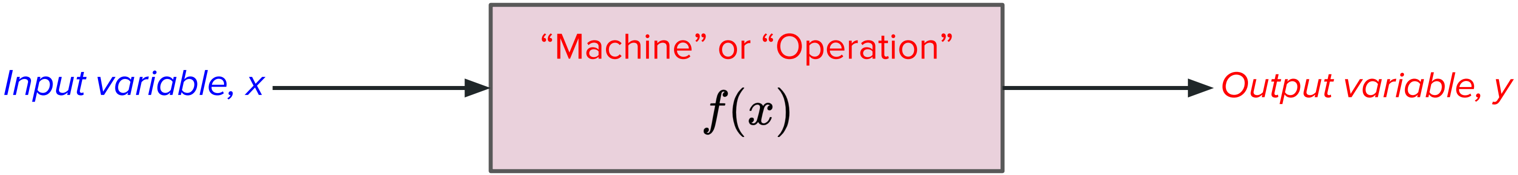
Outline

- Concept of a Function
- Domain & Range
- Elementary Functions
- Symmetry
- Composition of Functions
- Inverse Functions
- Transformations of Graphs

Concept of a Function

The basic concept of a function is that it performs some "operation" to an input and produces an output. One can also view it as a "machine" depicted below.



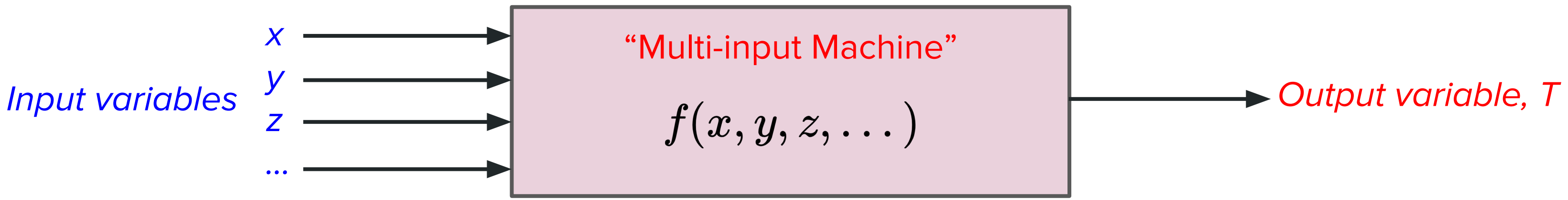
That's why it is common to write

$$y = f(x)$$

where input x is called an independent variable and output y is called a dependent variable (since y is dependent on x). In this case, the function is known as a single-variable function since it only takes in one input x.

Concept of a Multivariable Function

Clearly, a function can also take in multiple inputs, which represents a multivariable function. The "machine" is:



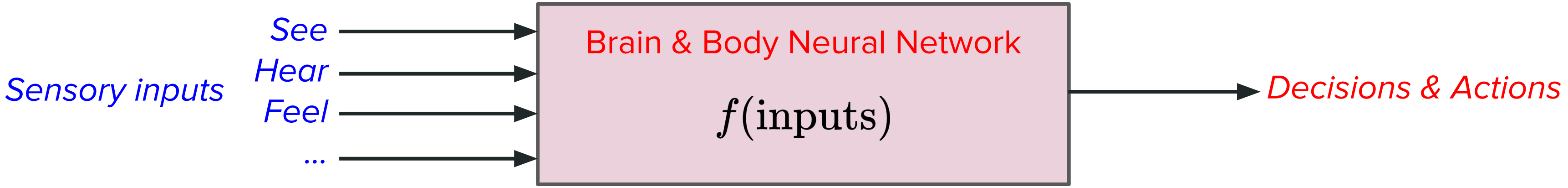
For example, the temperature (T) in a room can be a function of space (x, y, z) and time (t), i.e.

$$T = f(x, y, z, t)$$

We shall deal with multivariable functions and their calculus in Math 2.

Uses of Functions

Perhaps unknowingly, we use functions everyday. For example, your daily decisions & actions can be represented by:



And when you get a salary and check your bank account, you might use the function

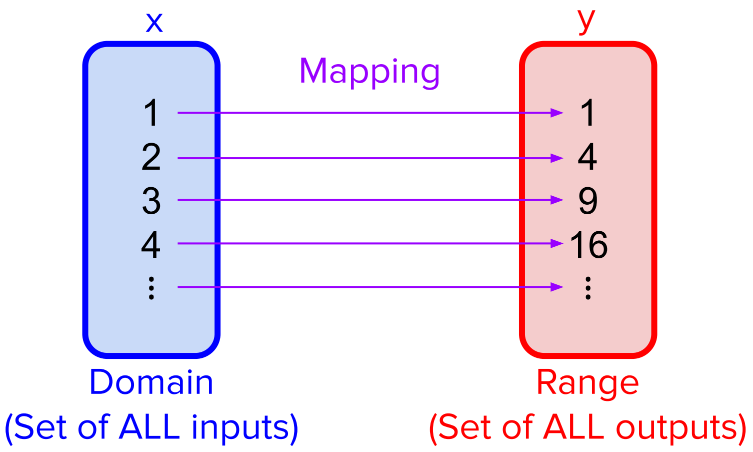
$$\begin{aligned}\text{Salary deposit}, S(t, r) &= rt - \text{CPF} \\ &= rt - 0.2rt = 0.8rt\end{aligned}$$

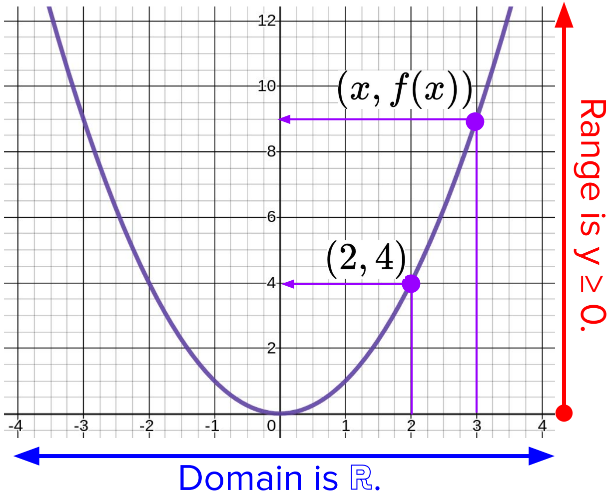
where t and r represent the hours worked and hourly pay rate respectively.

Function As Mapping

A function is also known as a mapping, where each element in the input space called "domain" is mapped to exactly one element in the output space called "range".

For example, the function $y = f(x) = x^2$ has a domain of $\mathbb{R}$ (all real numbers) and a range of $y \geq 0$. This can be seen from the graph.





Domain of a Function

The domain of a function can either be prescribed or evaluated if not. For example, the function for calculating the area of a circle has a prescribed domain of $r \geq 0$, since it does not make sense for the radius to be negative.

$$A(r) = \pi r^2, \quad r \geq 0$$

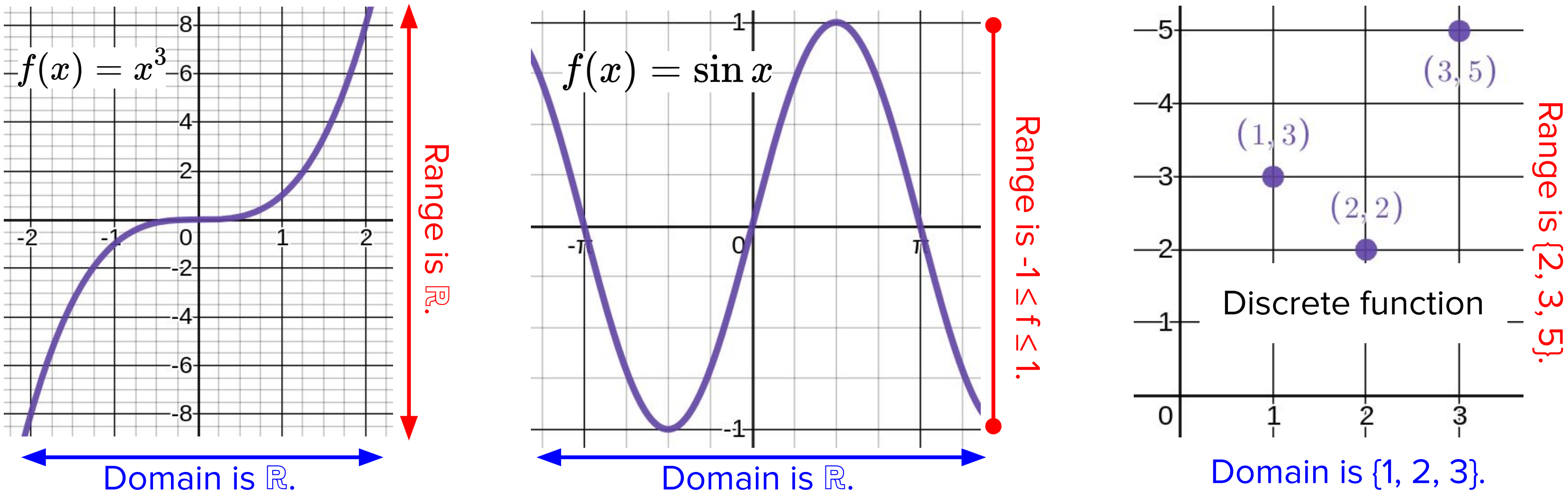
However, if a function is given as

$$f(x) = \pi x^2$$

without any prescribed domain, then the (natural) domain can be evaluated to be $\mathbb{R}$ since one can input any real number x into the function and calculate an output number. Take note that both A(r) and f(x) represent the exact same function, which is an operation that squares the input and multiply the result by π.

Range of a Function

The range of a function is simply ALL output values that the function produces when it takes in ALL elements in the domain. Some examples are shown below.



(Use https://www.desmos.com/calculator to investigate the graphs of functions. Hold shift and drag on the axes to scale.)

Notation of Sets

When defining the domain or range or some interval, there are two notations you can use.

"an element of"

1. Set-builder notation $\{x \mid x \,\text{has some property.}\}$

   Eg. $\{x \in \mathbb{R} \mid -1 \leq x \leq 3\}$, $\{y \in \mathbb{R} \mid y \geq 0\}$, $\{z \in \mathbb{R} \mid 1 < z \leq 7\}$

   "given that"

2. Interval notation

   [ ] = including

   never include ∞ as ∞ is a concept number ∴ round bracket

   ( ) ← "excluding"

   Eg. $[-1, 3]$, $[0, \infty)$, $(1, 7]$

In interval notation, when the endpoint value is included, use a square bracket. If not, use a round bracket. Note that infinity is never included.

Domain & Range

Exercise: Evaluate the domain and range of each function below.

D R

a) $f(x) = 7x^4$ D is $\mathbb{R}$ $\therefore$ R is $[0,\infty)$

b) $y(x) = \sqrt{x-3}$ since $\underbrace{x-3 \geq 0}_{x \geq 3}$, D is $[3,\infty)$ R is $[0,\infty)$

c) $X(r) = \dfrac{1}{r-2}$

$r-2 \neq 0$, D is $(-\infty,-2) \cup (2,\infty)$ or $\{r \in \mathbb{R} / r \neq 2\}$

R is $(-\infty,0) \cup (0,\infty)$ or $\{x \in \mathbb{R} / x \neq 0\}$

d) $h(t) = R\cos t$ D is $\mathbb{R}$ Range is $[-R,R]$

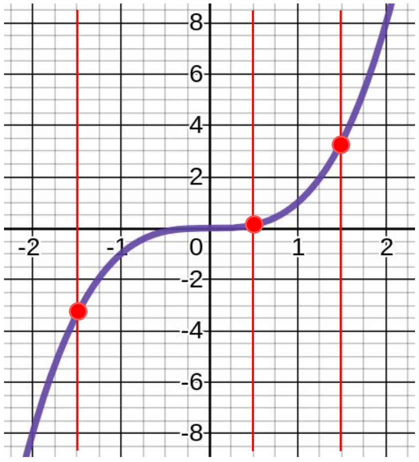
In (d), R is known as a constant parameter in the function h(t).

Vertical Line Test

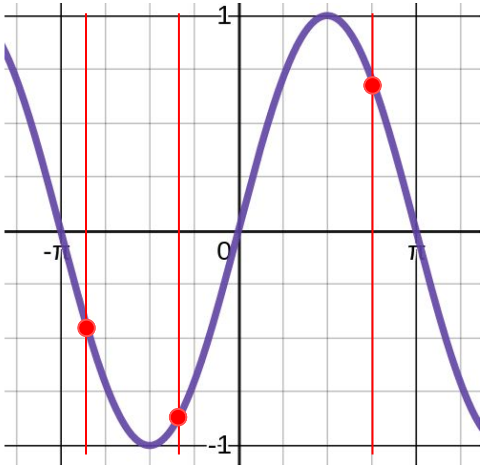
$y(x) = \pm\sqrt{1-x^2} \begin{cases} +\sqrt{1-x^2} \\ -\sqrt{1-x^2} \end{cases}$

A function must map an input to exactly one output, defined such that the function output is deterministic (not ambiguous). This means any vertical line intersects the graph of a function in its domain exactly once, illustrated below.

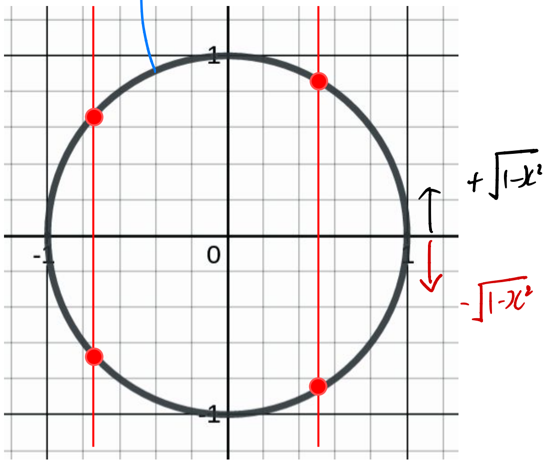
$x^2 + y^2 = 1$



$y = f(x) = x^3$



$y = f(x) = \sin x$

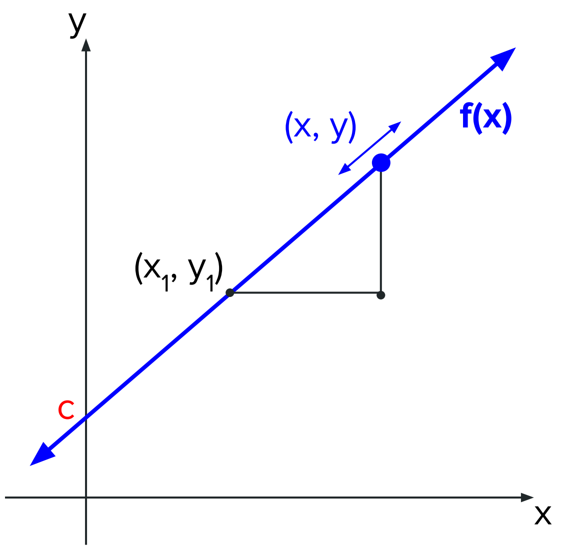


$y \neq f(x)$

Linear Functions

A **line**ar function is one where the graph of the function is a line. For a line, it has a constant slope defined by

$$m = \frac{y - y_1}{x - x_1} = \frac{f(x) - y_1}{x - x_1}, \quad (x, y) \neq (x_1, y_1)$$



Rearranging, we get the point-slope form of the equation of a line as

$$f(x) - y_1 = m(x - x_1)$$

Further rearranging allows us to get the slope-intercept form

$$f(x) = mx + (y_1 - mx_1) = mx + c$$

where $f(0) = c$ is the y-intercept.

Linear Function

Yet another form of the equation of a line known as standard form is

$$ay + bx = k$$

where a, b and k are constants. One can convert between the various forms of a line easily.

Example: Determine the equation of a line passing through points (2, -4) and (-4, 11) in all forms. Verify that all forms give the same line in Desmos.

$$m = \frac{y - y_1}{x - x_1} = \frac{-4 - 11}{2 - (-4)} = -\frac{5}{2}$$

$$f(x) - y_1 = m(x - x_1)$$

$$f(x) - (-4) = -\frac{5}{2}(x - 2)$$

$$f(x) + 4 = -\frac{5}{2}(x - 2)$$

$$f(x) = -\frac{5}{2}x + 5 - 4$$

$$f(x) = -\frac{5}{2}x + 1$$

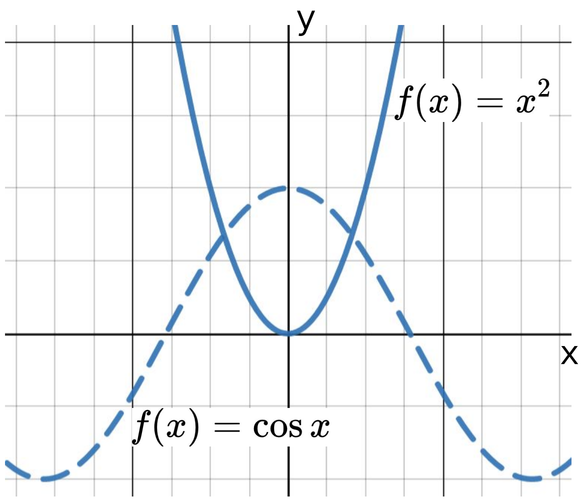
$$2y = -5x + 2$$

$$2y + 5x = 2$$

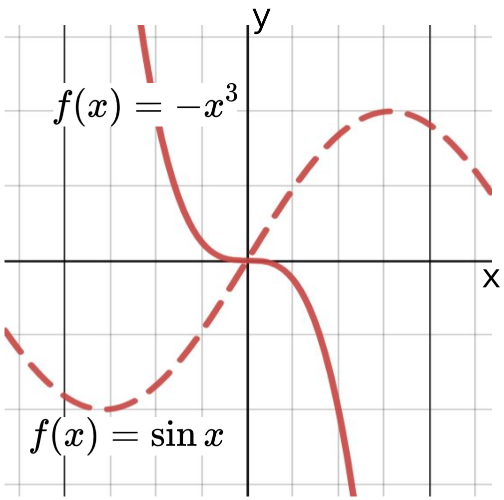
ANS: $f(x) + 4 = -\frac{5}{2}(x - 2), \quad f(x) = -\frac{5}{2}x + 1, \quad 2y + 5x = 2$

Symmetry of Functions

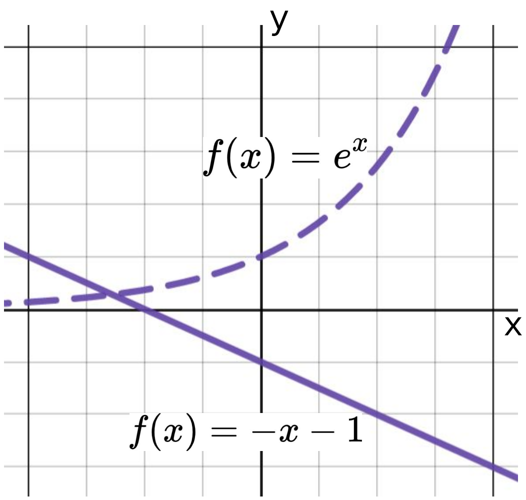
A function can be defined as even or odd, illustrated below. An even function looks like half of it is reflected about the y-axis (symmetry about the y-axis). An odd function looks like half of it is rotated 180° about the origin (symmetry about the origin). A function can also be neither even nor odd.



Even functions



Odd functions



Neither even nor odd

Symmetry of Functions

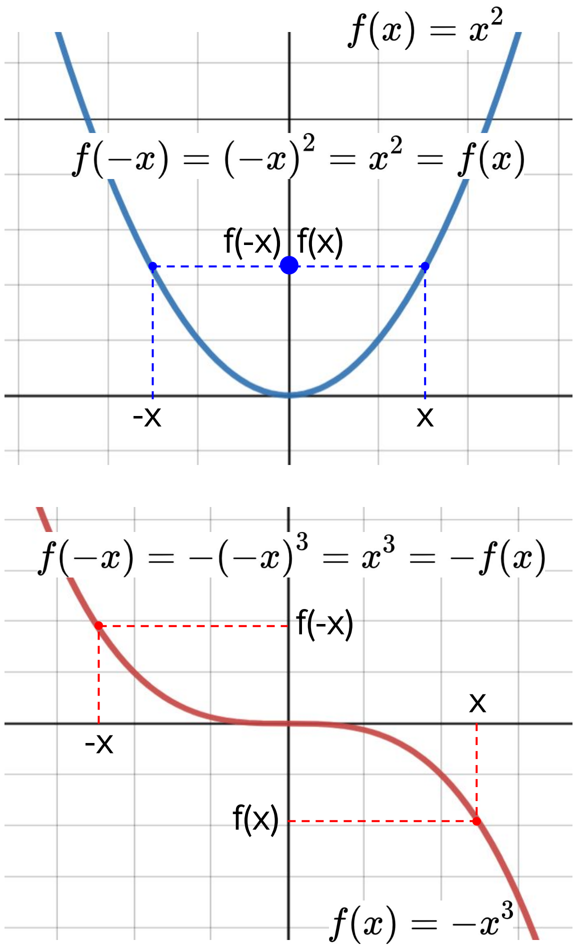
From the graph, we observe that an even function satisfies

$$f(-x) = f(x)$$

for all x in its domain. Correspondingly, an odd function satisfies

$$f(-x) = -f(x)$$

If a function does not satisfy either of the above relations, then it's neither even nor odd.



Symmetry of Functions

Exercise: Evaluate the symmetry of each function below, if any.

a) $f(x) = |x| + 3$

$f(-x) = |-x| + 3 = |x| + 3 = f(x)$, $\therefore f(x)$ is even

b) $g(x) = (x-1)^2$

$g(-x) = (-x-1)^2 = [(-1)^2(x+1)^2] = (x+1)^2 \left\{ \begin{array}{l} \neq -g(x) \\ \neq g(x) \end{array} \right\}$ neither odd or even

c) $h(x) = 1/x^{2n+1}$, $n \in \mathbb{Z}$(all integers)

$h(-x) = \dfrac{1}{(-x)^{2n+1}} = \dfrac{1}{(-1)^{2n+1}x^{2n+1}} = \dfrac{1}{(-1)x^{2n+1}} = -\dfrac{1}{x^{2n+1}} = -h(x)$

$\therefore h(x)$ is odd

since $\underset{n=1}{2n+1} = 3,\ \underset{n=2}{5},\ \underset{n=3}{7} \ldots$

is always odd

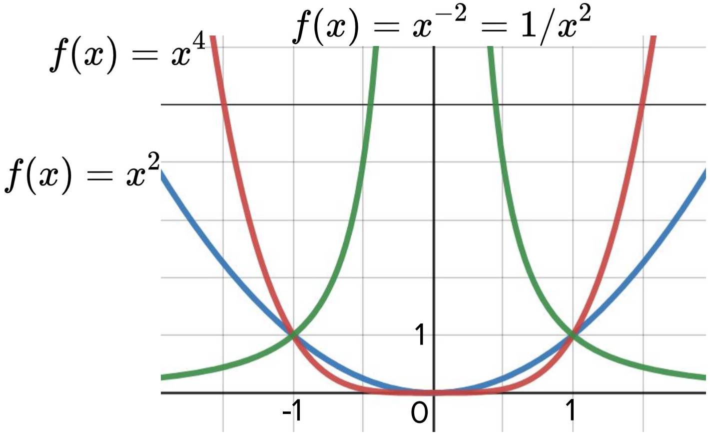
ANS: a) Even, b) Neither, c) Odd

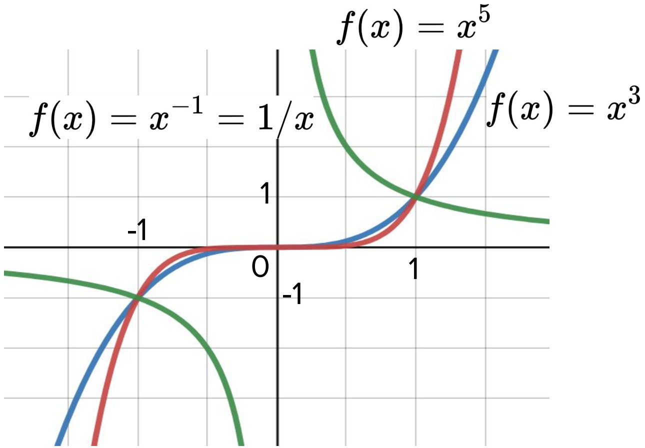
Power Functions

A power function is one of the general form

$$f(x) = ax^b$$

where a and b are real constants. Some power functions are shown below. For an even (odd) integer b, the function is even (odd).



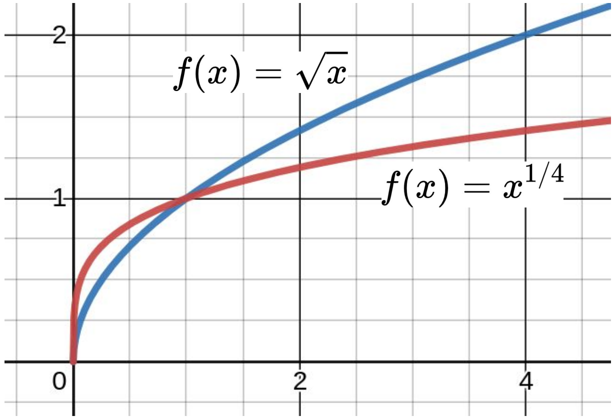


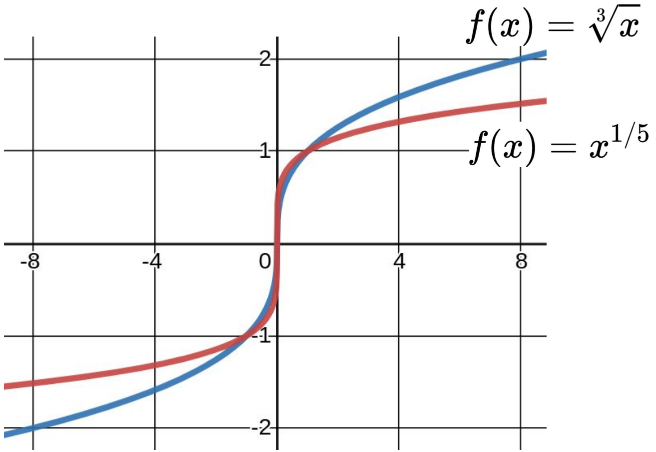
Root Functions

A root function is a power function of the form

$$f(x) = x^{\frac{1}{n}}$$

where n is an integer greater than one. For even integers of n, the function is only defined for $x \geq 0$ if the output is a real number. Clearly, a larger n leads to a function increasing more slowly for $x \geq 1$.





Polynomial Functions

A polynomial function is one of the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where constants a_0 to a_n are called coefficients and n is a positive integer called the degree of the polynomial. For example, the quadratic function below is a polynomial of degree 2.

$$f(x) = 2x^2 - 3x - 7$$

A polynomial of degree 1 is a linear function and one of degree 0 is a constant function. Since a polynomial is defined for any real input x, its (natural) domain is always $\mathbb{R}$, unless prescribed otherwise.

End Behaviour of a Function

When the input of a function approaches positive or negative infinity, the behaviour of the function output is called its end behaviour. For example, the quadratic function

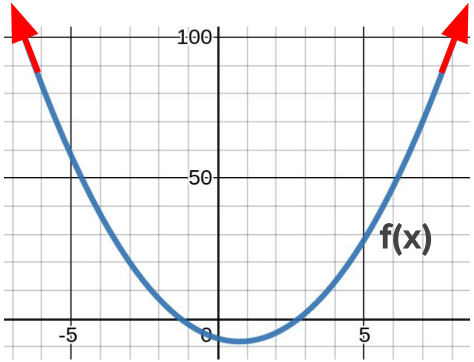
$$f(x) = 2x^2 - 3x - 7$$

approaches +∞ as x approaches +∞. This is because when x is large, the x^2 term become much larger (dominant) over other terms and so f(x) behaves more like $2x^2$. Using limit notation, we can write

$$\lim_{x \to +\infty} f(x) = +\infty$$

It should be easy to also see that

$$\lim_{x \to -\infty} f(x) = +\infty$$

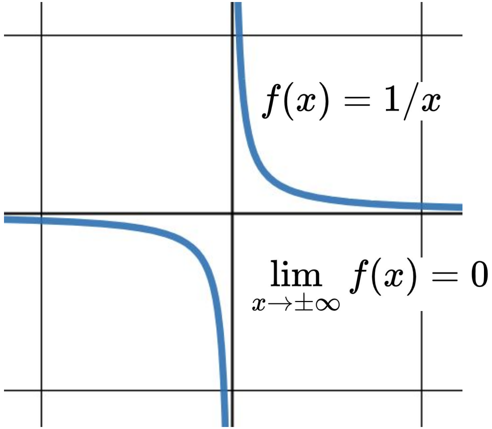


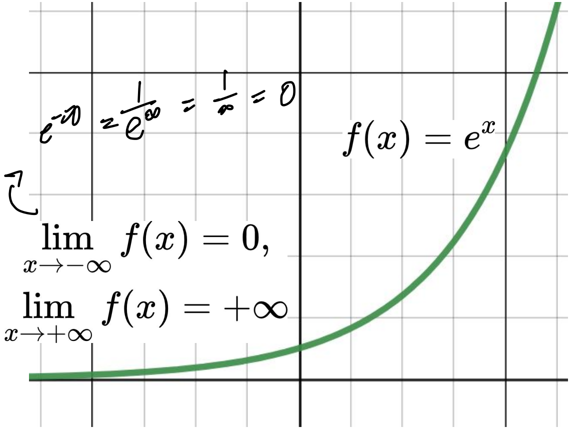
End Behaviour of a Function

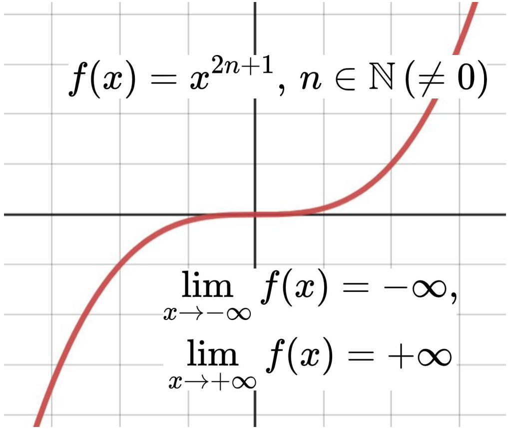
Hence, for a quadratic function $f(x) = ax^2 + bx + c$, we have

$$a > 0 : \lim_{x \to \pm\infty} f(x) = +\infty \quad , \qquad a < 0 : \lim_{x \to \pm\infty} f(x) = -\infty$$

Below are more examples on the end behaviour of some functions.







Zeros (Roots) of a Function

The zeros or roots of a function f(x) are values of the input x that satisfies f(x) = 0, where the graph intersects the x-axis. For example, the roots of

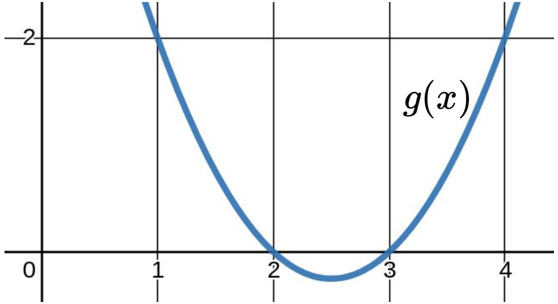
$$f(x) = 3x - 1$$

can be found by

$$f(x) = 3x - 1 = 0 \rightarrow x = 1/3$$

Another example in finding the roots of a quadratic function (by factorization) is shown below.

$$g(x) = x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$
$$x = 2, 3$$



Quadratic Formula

A quadratic function might not be easy to factorize in order to find its roots. In any case, the quadratic formula can be used, derived (by completing the square) as follows.

$$f(x) = ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

(Completed-square form)

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(Quadratic formula)

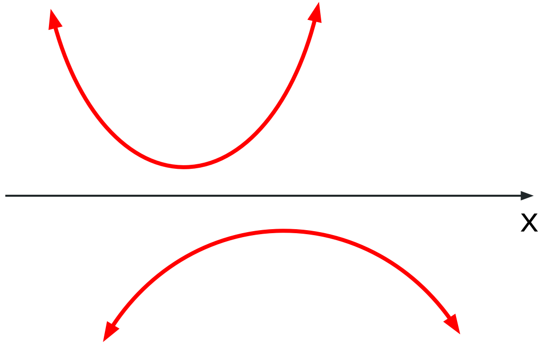
Roots of a Quadratic Function

Depending on the discriminant $b^2 - 4ac$, there might be zero, one or two real roots, illustrated below.

$$b^2 - 4ac < 0$$

$$x_{1,2} = \alpha \pm i\beta$$

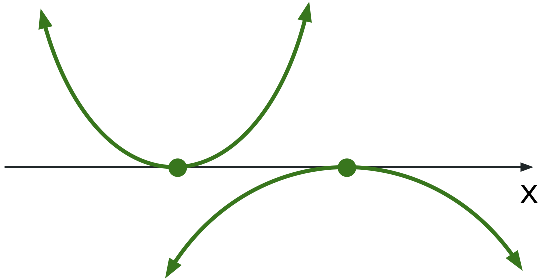
No real roots
(complex roots)



$$b^2 - 4ac = 0$$

$$x_{1,2} = -\frac{b}{2a}$$

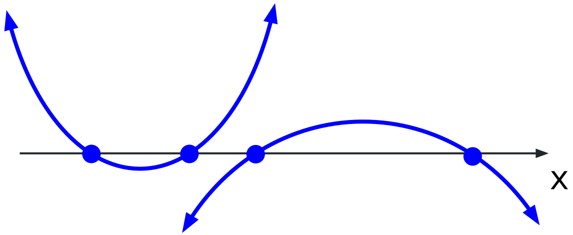
Real & equal roots



$$b^2 - 4ac > 0$$

$$x_1 \neq x_2$$

Real & distinct roots



Zeros (Roots) of a Function

Exercise: Evaluate the roots of each function below.

a) $f(x) = \sqrt{2x-5} - 1$ a) $\sqrt{2x-5} - 1 = 0$ $\left(\sqrt{2x-5}\right)^2 = 1^2$ $2x - 5 = 1$, $2x = 6$, $x = 3$

b) $g(x) = x^2 - x - 12$ b) $(x+3)(x-4) = 0$ $x = -3$ or $x = 4$

c) $h(x) = x^2 + 4x + 5 = 0$ $x = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} = -2 \pm \frac{2i}{2} = -2 \pm i$

d) $p(x) = 1/x^2 = 0$

has no solution

roots does not exist

ANS: a) x = 3, b) x = -3, 4, c) x = -2 ± i, d) DNE

Rational Functions

A rational function is a fraction of polynomials, generally of the form

$$f(x) = \frac{p(x)}{q(x)}$$

Some examples are

$$f(x) = \frac{3}{x-4} \qquad g(x) = \frac{2x+1}{x-5} \qquad h(x) = \frac{x^2+2x+1}{2x-6}$$

In f(x), where p(x) is a constant and q(x) is a linear function, it is also called a reciprocal function. Rational functions are undefined at the roots of the denominator polynomials. At these points, the function could possibly approach infinities and have vertical asymptotes. (Take note that not all denominator roots give rise to asymptotes, to be explained in the topic on limits.)

Asymptotes

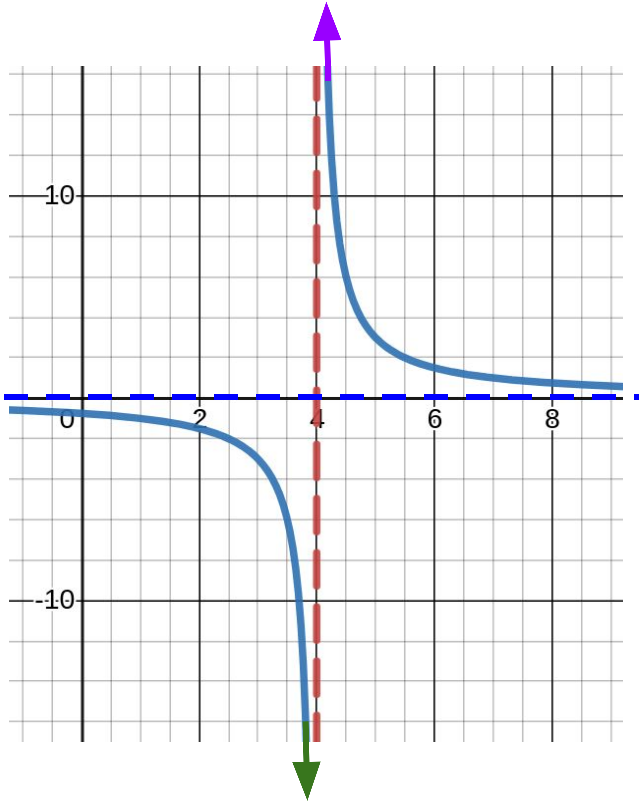
For example, the vertical asymptote of $f(x) = \dfrac{3}{x-4}$

can be clearly evaluated to be at x = 4. When x approaches 4 from the left (3.99...), f(x) approaches -∞. When x approaches 4 from the right (4.00...1), f(x) approaches +∞. Using limit notation, we can write

$$\lim_{x \to 4^-} f(x) = -\infty, \qquad \lim_{x \to 4^+} f(x) = +\infty$$

And we can also find the horizontal asymptote at y = 0 by

$$\lim_{x \to \pm\infty} f(x) = 0$$



What is the domain and range of f(x)?

{x ∈ ℝ | x ≠ 4}   {f ∈ ℝ | f ≠ 0}

Asymptotes

Exercise: Determine the roots and asymptotes of the function below and state its domain and range. Sketch its graph.

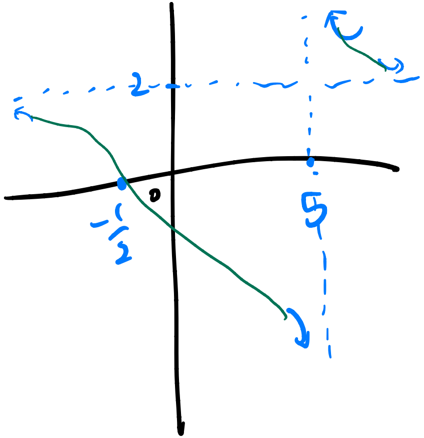
Domain

$\{x \in \mathbb{R} \mid x \neq 5\}$

Range

$\{y \in \mathbb{R} \mid y \neq 2\}$

$$g(x) = \frac{2x+1}{x-5}$$



$$\frac{2x+1}{x-5} = 0$$

$$x = -\frac{1}{2}$$

horizontal asymptote : $\lim_{x \to \pm\infty} g(x) = \frac{2(\infty)+1}{(\infty)-5}$

$$= \frac{2x}{x}$$

$$= 2$$

vertical asymptote : Let $x - 5 = 0$

$$x = 5$$

$$\lim_{x \to 5^-} g(x) = \frac{2(4.9999)+1}{4.9999-5} = \frac{+}{-0.00\ldots1}$$

$$= -\infty$$

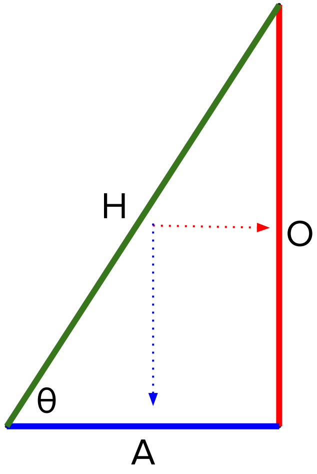
ANS: Root at x = -½, vertical asymptote at x = 5, horizontal asymptote at y = 2.
Domain is {ℝ | x ≠ 5} and range is {ℝ | y ≠ 2}.

$\lim_{x \to 5^+} g(x) = +\infty$

Trigonometric Functions

In a right-angled triangle, the cosine and sine functions of an angle output the projected length of the hypotenuse onto the sides adjacent and opposite to the angle respectively.

$$A = H\cos\theta, \quad O = H\sin\theta$$



Hence we can define the (familiar) relations

$$\cos\theta = \frac{A}{H}, \quad \sin\theta = \frac{O}{H}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{O}{A}$$

And the reciprocal of the above functions are defined as

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

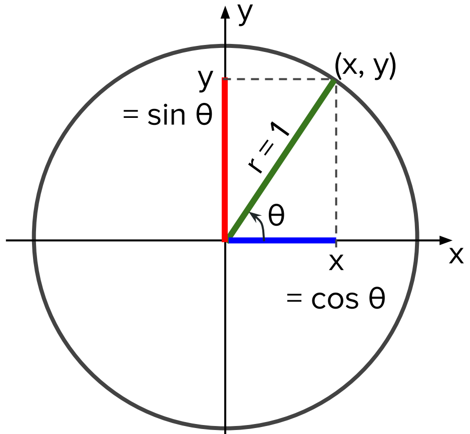
Trigonometric Functions

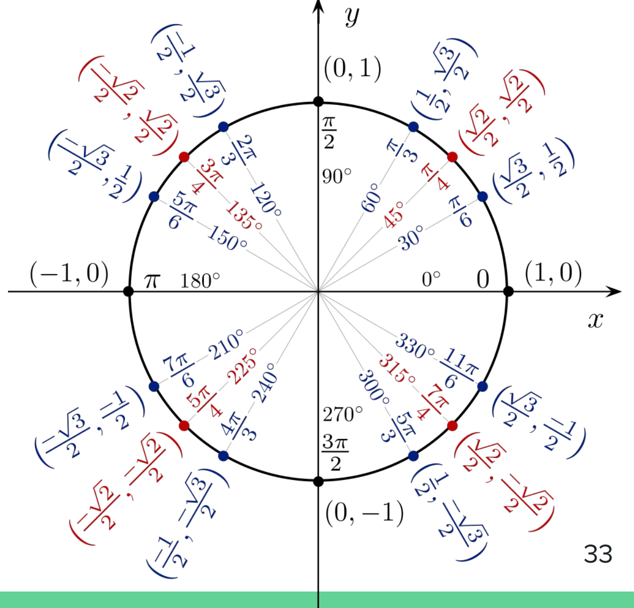
More formally, the cosine and sine functions take in an angle as input and outputs a coordinate on a unit circle. So

$$x(\theta) = \cos\theta, \quad y(\theta) = \sin\theta$$

outputs the projected x and y coordinates respectively as shown. It is easy to see the coordinates at some angles. Eg.

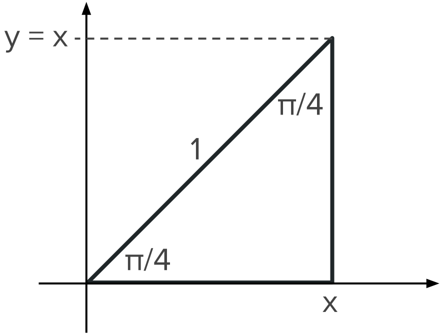
$$\begin{gathered}
x(0) = \cos 0 = 1, \quad y(0) = \sin 0 = 0 \\
x(\pi/2) = \cos(\pi/2) = 0, \quad y(\pi/2) = \sin(\pi/2) = 1 \\
x(\pi) = \cos(\pi) = -1, \quad y(\pi) = \sin(\pi) = 0 \\
x(3\pi/2) = \cos(3\pi/2) = 0, \quad y(3\pi/2) = \sin(3\pi/2) = -1 \\
x(2\pi) = x(0) = 1, \quad y(2\pi) = y(0) = 0
\end{gathered}$$





Trigonometric Functions

The coordinates at θ = π/4 (45°) & π/6 (30°) can be derived using Pythagoras theorem.

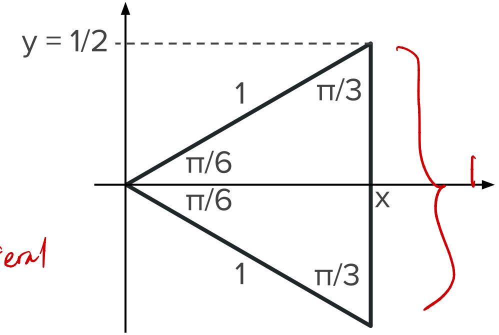


$$x^2 + y^2 = 1 \rightarrow x^2 + x^2 = 2x^2 = 1$$

$$x = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

$$\rightarrow y = x = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

Since △ is equilateral



$$2y = 1 \rightarrow y = 1/2 = \sin\frac{\pi}{6} = \cos\frac{\pi}{3}$$

$$x^2 + y^2 = 1 \rightarrow x^2 + \frac{1}{4} = 1$$

$$\rightarrow x = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} = \sin\frac{\pi}{3}$$

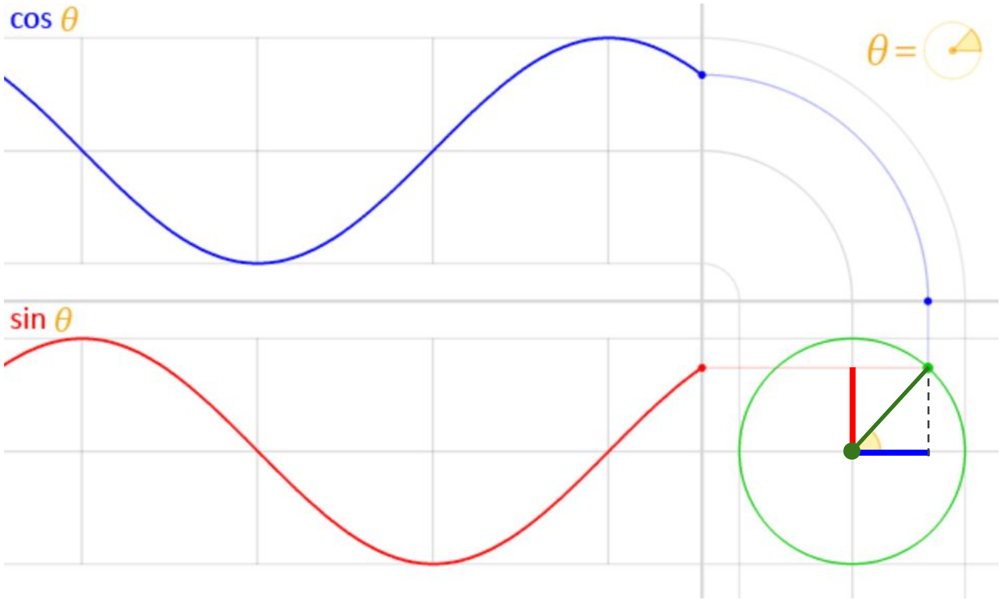
Trigonometric Functions

One can imagine that as the radius "hand" rotates CCW, the x & y coordinates change accordingly in a periodic manner, giving rise to the graphs as shown. We have

$$\cos\theta = \cos(\theta \pm 2\pi),$$
$$\sin\theta = \sin(\theta \pm 2\pi)$$

A periodic function with period p is one that satisfies

$$f(x) = f(x \pm p)$$

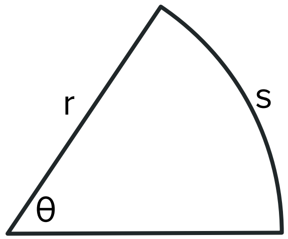


See animation of the cosine & sine graph at https://en.wikipedia.org/wiki/Sine_and_cosine#/media/File:Circle_cos_sin.gif.

Radian!

It is important to use the radian measure when computing trigonometric functions unless otherwise specified. One reason is that it is defined by the arc length s, where

$$s = r\theta$$
$$\rightarrow \theta = \frac{s}{r} \text{ (radian)}$$



So, for a circle with circumference 2πr, the subtended angle is 2π which is in radian.
Another reason is because of the derivatives below are only true if θ is in radian.

$$\frac{d}{d\theta}\sin\theta = \cos\theta, \quad \frac{d}{d\theta}\cos\theta = -\sin\theta$$

We will come back to derivatives again, so do not worry about it for now.

Radian!

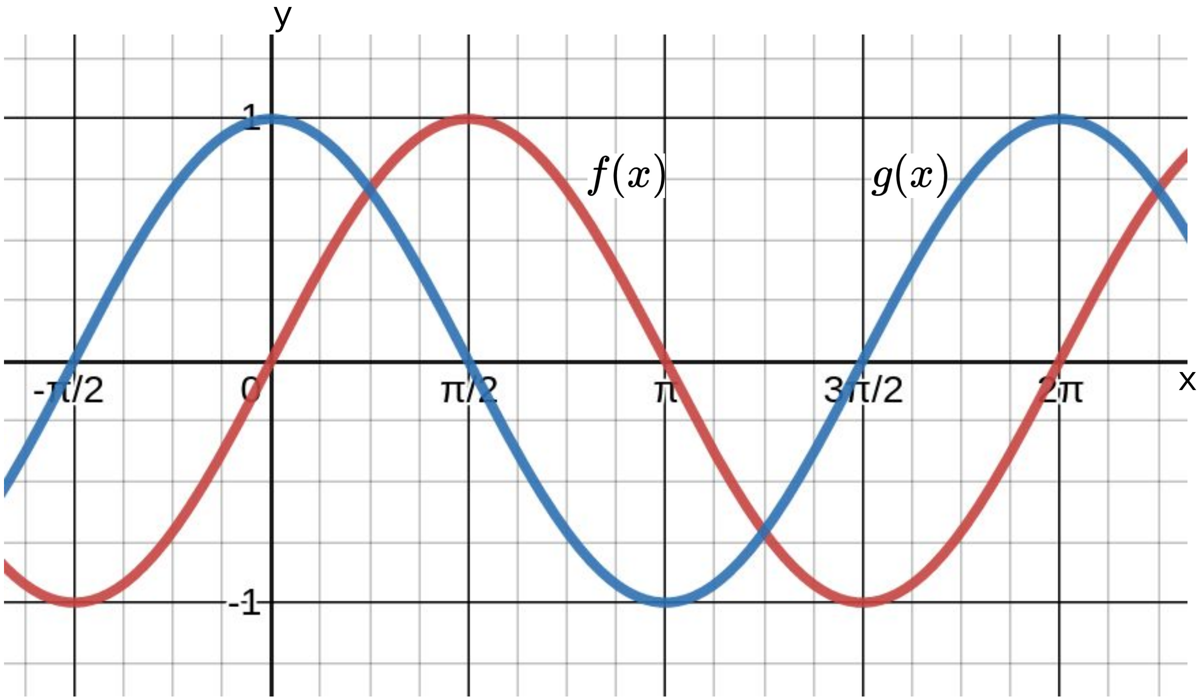
Notice that the graphs of

$$f(x) = \sin x,$$
$$g(x) = \cos x$$

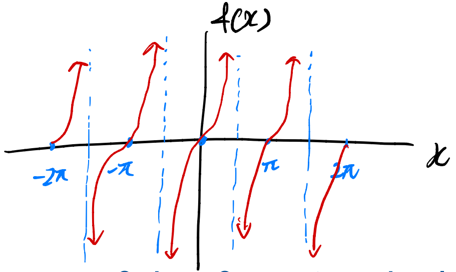
are always given such that the input variable x is in radian.

What is the $\overbrace{\text{domain}}^{\mathbb{R}}$ and range of the cosine & sine functions?

$[-1, 1]$



Trigonometric Functions



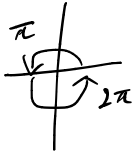
↙ By inspection

period = π

Exercise: Determine the roots and asymptotes of the function below and sketch it in [-2π, 2π]. What is the period?

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$\frac{\sin x}{\cos x} = 0$$



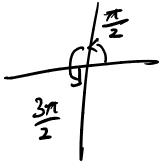
$\sin x = 0$

$x = 0, \pi, 2\pi, \ldots$

$\therefore x = n\pi \ \forall n \in \mathbb{Z}$

"for all"

$\cos x = 0$



$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$= \frac{2n+1}{2}\pi \ \forall n \in \mathbb{Z}$

ANS: Roots at x = nπ, vertical asymptotes at x = π(2n+1)/2, period = π.

Trigonometric Identities

Pythagorean Identities

$\div \cos^2 x$

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \cot^2 x = \csc^2 x \qquad \tan^2 x + 1 = \sec^2 x$$

$\div \sin^2 x$

Compound Angle Identities

(Proof: https://en.wikipedia.org/wiki/Proofs_of_trigonometric_identities#Angle_sum_identities)

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Let $y = x$

Let $y = x$

Double Angle Identities

$$\sin(2x) = 2 \sin x \cos x$$

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

Use Pythagorean identity.

Exponential Functions

An exponential function is one where the input variable is an exponent, giving the form

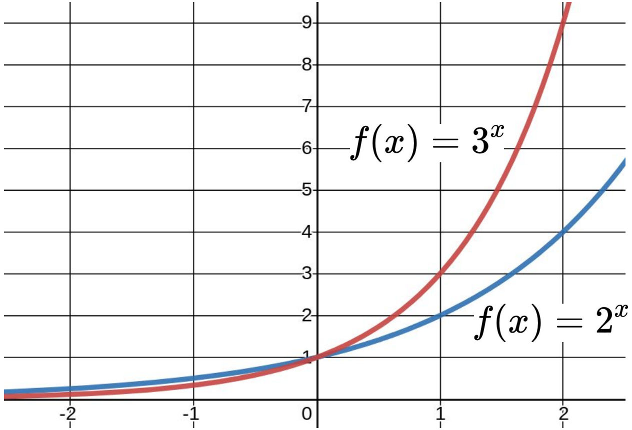
$$f(x) = a^x$$

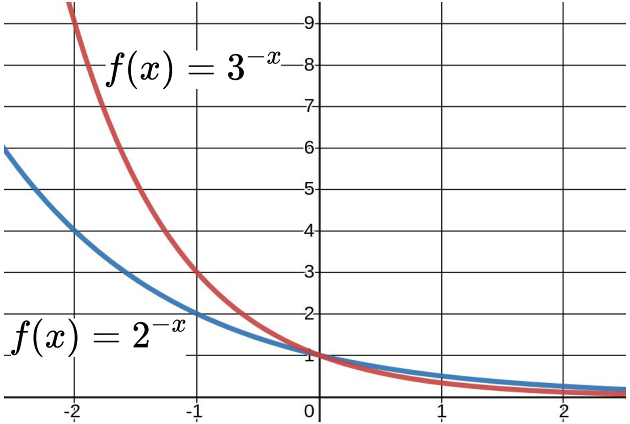
where a is a positive real number ($a \neq 1$) called the base. When x increases, the base a is multiplied by itself more times, leading to rapid growth in a^x (called exponential growth).

Correspondingly, the exponential function

$$f(x) = a^{-x} = 1/a^x$$

decreases as x increases, leading to exponential decay.





Laws of Exponents

a) $a^x a^y = a^{x+y}$

b) $a^x / a^y = a^{x-y}$

c) $(a^x)^y = a^{xy}$

d) $(ab)^x = a^x b^x$

e) $a^0 = 1$

$$a^x a^y = \overbrace{(a \times a \times a \times \ldots)}^{\text{x times}}\overbrace{(a \times a \times \ldots)}^{\text{y times}}$$
$$= \underbrace{a \times a \times a \times a \times a \times \cdots}_{\text{x+y times}} = a^{x+y}$$

$$a^x / a^y = \frac{\overbrace{\cancel{a} \times \cancel{a} \times a \times \ldots}^{\text{x times}}}{\underbrace{\cancel{a} \times \cancel{a} \times \ldots}_{\text{y times}}} = \overbrace{a \times \cdots}^{\text{x-y times}} = a^{x-y}$$

$$(a^x)^y = \overbrace{\overbrace{(a \times a \times \ldots)}^{\text{x times}} \times \overbrace{(a \times a \times \ldots)}^{\text{x times}} \times \ldots}^{\text{y times}}$$
$$= \underbrace{a \times a \times a \times a \times \cdots}_{\text{xy times}} = a^{xy}$$

$$(ab)^x = \overbrace{(ab \times ab \times \ldots)}^{\text{x times}}$$
$$= \underbrace{(a \times a \times \ldots)}_{\text{x times}}\underbrace{(b \times b \times \ldots)}_{\text{x times}} = a^x b^x$$

Exercise: Prove (e) using (b).

$\frac{a^x}{a^x} = 1 = a^{x-x} = a^0$

Combination of Functions

Fundamentally, functions can be combined in the following ways.

a) Sum / Difference: $y(x) = f(x) \pm g(x)$

b) Product: $y(x) = f(x) \cdot g(x)$

c) Quotient: $y(x) = f(x)/g(x), \qquad g(x) \neq 0$

Since y(x) is only defined where f(x) and g(x) are defined, the domain of y(x) is the intercept of the domain of f(x) and g(x). Note the additional requirement in the quotient to avoid division by zero.

Combination of Functions

Exercise: Given f(x) and g(x), evaluate each resultant function below and state its domain.

D_f is $[1, \infty)$

$$f(x) = \sqrt{x-1}, \qquad g(x) = \frac{1}{3-x}$$

D_g is $\{\mathbb{R} \mid x \neq 3\}$

a) $p(x) = f(x) - g(x)$ $= \sqrt{x-1} - \frac{1}{3-x}$ $\quad D_p$ is $[1,3) \cup (3, \infty)$

b) $q(x) = f(x) \cdot g(x)$ $= \sqrt{x-1} \cdot \frac{1}{3-x}$ $\quad D_q$ is $[1,3) \cup (3, \infty)$

c) $r(x) = g(x)/f(x)$ $= \frac{1}{3-x} \div \sqrt{x-1}$

$= \frac{1}{(3-x)(\sqrt{x-1})}$ $\quad \therefore D_r$ is $(1,3) \cup (3, \infty)$

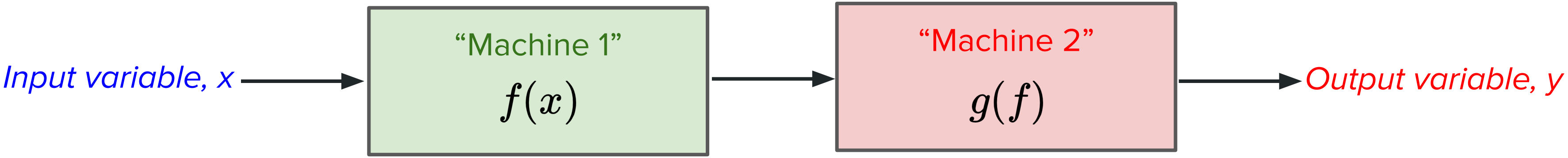
Cannot include 1 as $\sqrt{1-1} = 0$

$\frac{0}{3-1}$ = undefined

ANS: For (a) and (b), the domain is [1, 3) U (3, ∞). For (c), the domain is (1, 3) U (3, ∞).

Function Composition

A function composition (aka composite function) is simply a function of another function. The “machine” perspective is shown below.



For example,

$$f(x) = \sqrt{x}, \qquad g(x) = x^2 + 3$$

$$y_1(x) = g(f(x)) = f^2 + 3 = \left(\sqrt{x}\right)^2 + 3 = x + 3$$
$$y_2(x) = f(g(x)) = \sqrt{g} = \sqrt{x^2 + 3}$$

Domain of a Composite Function

In order for $y(x) = g(f(x))$ to be defined, the domain of $y(x)$ has to be the set in the domain of $f(x)$ for which $f(x)$ is in the domain of $g(x)$. From the earlier example, we have

$$f(x) = \sqrt{x}, \quad g(x) = x^2 + 3,$$
$$y_1(x) = g(f(x)) = x + 3$$

The range of $f(x)$ is $[0, \infty)$, which is in the domain of $g(x)$ that is $\mathbb{R}$. So the domain of $y_1(x)$ is the domain of $f(x)$, which is $[0, \infty)$.

Note that you **cannot evaluate the domain directly from the resultant function** $y_1(x)$, which is $\mathbb{R}$ (wrong). Verify this in Desmos.

Composition of Functions

Exercise: Given f(x) and g(x), evaluate each resultant function below and state its domain.

D_f is $[1, \infty)$

R_f is $[0, \infty)$

$$f(x) = \sqrt{x - 1}, \qquad g(x) = -x^2$$

D_g is $\mathbb{R}$

R_g is $(-\infty, 0]$

a) $p(x) = f(g(x)) = \sqrt{g-1} = \sqrt{-x^2-1}$ since R_g is entirely not in D_f, so D_p is undefined $\{\}$

b) $q(x) = g(f(x)) = -f^2 = -(x-1)$ since R_f is entirely in the D_g, so $D_q = D_f$ is $[1, \infty)$

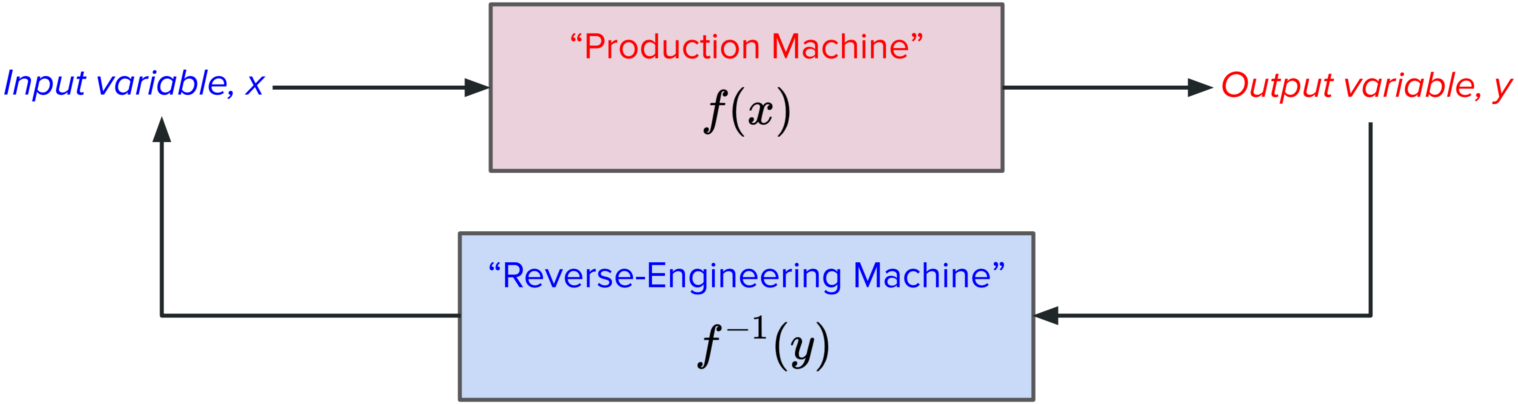
c) $r(x) = f(f(x)) = \sqrt{f-1} = \sqrt{\sqrt{x-1}-1}$

For $r(x)$ to be defined, D_r is the set in D_f such that $R_f = D_f = [1, \infty)$. Hence min x at $f = \sqrt{x-1}$, so $x_{min} = 2$. So D_r is $[2, \infty)$.

ANS: a) Undefined since domain is the empty set. b) Domain is [1, ∞). c) Domain is [2, ∞).

Inverse of a Function

For a function *f*, an inverse function f^{-1} might exist such that it “reverses” the operation executed by *f*. The “machine” perspective is shown below.



Formally we have

$$f^{-1}(f(x)) = x, \qquad f\left(f^{-1}(y)\right) = y$$

So, what is the inverse function of $f^{-1}(x)$? $\left(f^{-1}_{(x)}\right)^{-1} = f(x)$

Inverse of a Function

Some inverse functions are intuitive to evaluate. For example, the inverse of the squaring function is the square root function.

$$f(x) = x^2 \to f^{-1}(x) = \sqrt{x}$$

The above functions satisfy

$$f^{-1}(f(x)) = \sqrt{x^2} = x, \qquad f\left(f^{-1}(x)\right) = \left(\sqrt{x}\right)^2 = x$$

Check

Exercise: Intuitively, state the inverse function of

$g^{-1}(g(x)) = \frac{1}{\frac{1}{x}}$

$= x$

Checks

$f^{-1}(f) = \frac{kx}{k} = x$

$$f(x) = kx, \qquad g(x) = 1/x$$

$f^{-1}(x) = \frac{x}{k}$

$g^{-1}(x) = \frac{1}{x}$

Inverse of a Function

Generally, to obtain the inverse of a function, firstly recognize from the “machine” perspective that we want

$$x = f^{-1}(y)$$

which means we simply have to isolate x from y = f(x). And since by convention, usually we want $f^{-1}(x)$, simply change the variable y to x. For example,

Can be + or - depending on the domain of f(x). See next slide.

$$f(x) = 2x^2 + 1 \longrightarrow \text{Let } y = 2x^2 + 1$$

$$\text{Isolate } x \rightarrow 2x^2 = y - 1 \rightarrow x = \sqrt{\frac{y-1}{2}} = f^{-1}(y)$$

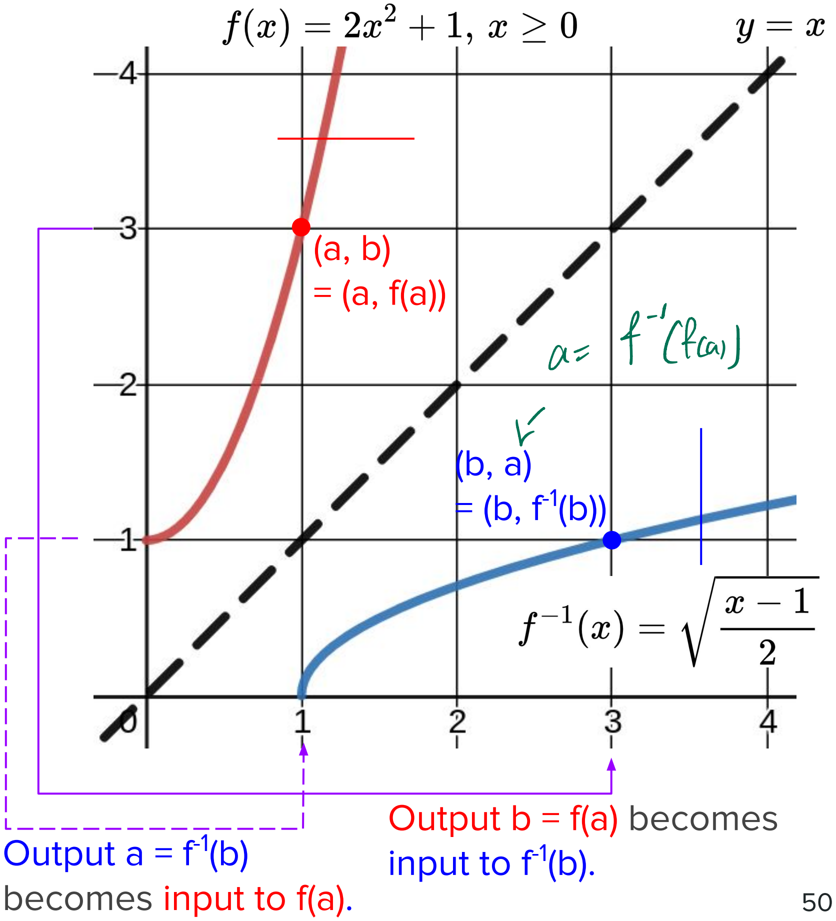
$$\text{Replace } y \text{ by } x \rightarrow f^{-1}(x) = \sqrt{\frac{x-1}{2}}$$

One-to-one Functions

The graphs of f(x) and $f^{-1}(x)$ are mirror reflections of each other about the line y = x, because any coordinate (a, b) on f(x) is swapped to become (b, a) on $f^{-1}(x)$. Refer to the "machine" flowchart.

Since $f^{-1}(x)$ must pass the vertical line test to qualify as a function, this means f(x) must pass both the horizontal and vertical line tests, because the mirror reflection of a horizontal line about y = x is a vertical line. f(x) is then called a one-to-one function, which means

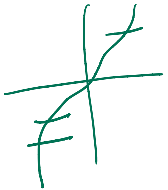
$$f(x_1) \neq f(x_2) \quad \forall\, x_1 \neq x_2$$



One-to-one Functions

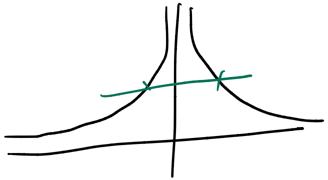
Exercise: Which of the following functions is one-to-one? For those that isn't, how would you constraint the domain so that it has an inverse?

$$f(x) = x^3$$



1 to 1

$$g(x) = 1/x^2$$

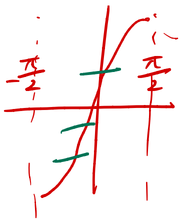


not 1 to 1

Domain can be reduced

$(-\infty, 0)$ or $(0, \infty)$

$$h(x) = \sin x$$



not 1 to 1

Domain can be reduced to

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

ANS: a) Yes. b) No. x < 0 or x > 0. c) No. -π/2 ≤ x ≤ π/2.

Domain & Range of Inverse Functions

Because of the reflection symmetry about y = x, the domain and range of $f^{-1}(x)$ are the range and domain of f(x) respectively. This means the domain and range of $f^{-1}(x)$ **cannot be evaluated independently**, since $f^{-1}(x)$ is "dependent" on f(x).

Let $y = \sqrt{1-x}$ D_f is $(-\infty, 1]$

R_f is $[0, \infty)$

Example: Determine the inverse of $y^2 = 1 - x$

$$f(x) = \sqrt{1-x}$$

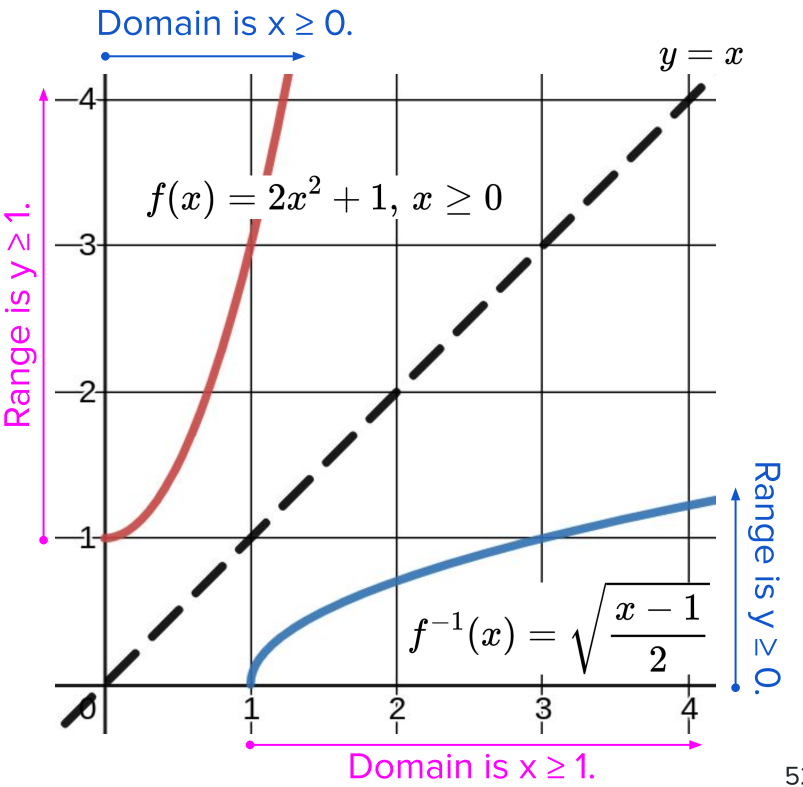
$x = 1 - y^2 = f^{-1}(y)$

$D_{f^{-1}}$ is $[0,\infty)$. $R_{f^{-1}}$ is $(-\infty, 1]$ ← $f^{-1}(x) = 1 - x^2$

and its domain and range. Are they the same as as that evaluated independently?

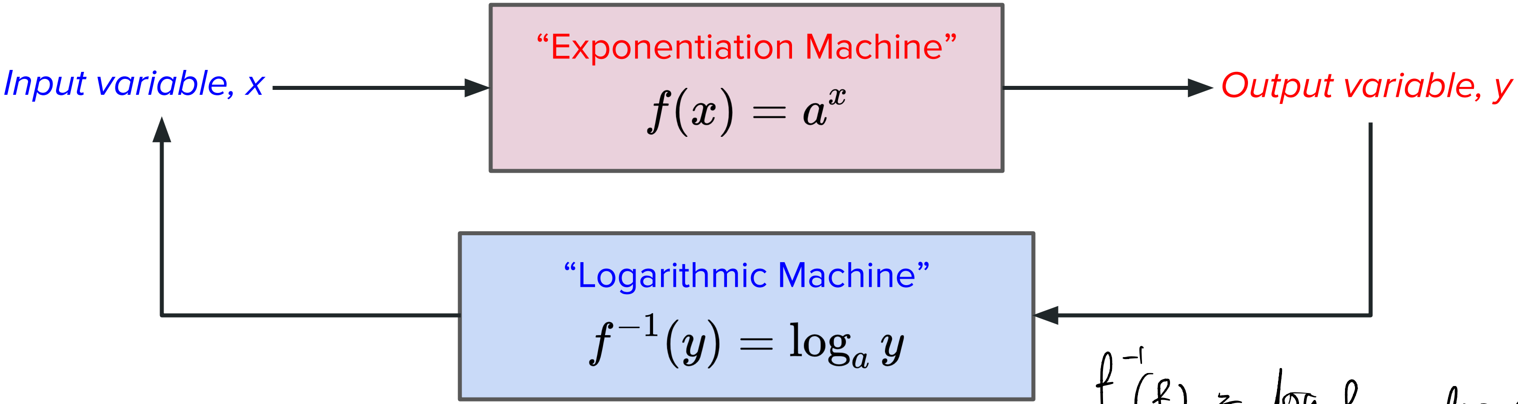
D is R ✗ wrong

R is $(-\infty, 1]$ ✗ wrong



Logarithmic Functions

The logarithmic function is the inverse of an exponential function. The “machine” perspective is shown below. So, given an input y and a base a, the function $x = \log_a(y)$ returns the exponent x.



$f^{-1}(f) = \log_a f = \log_a(a^x) = x$

Logically, for the same input to a log function, a bigger base leads to a smaller exponent. Eg. $\log_2(16) = 4$ and $\log_4(16) = 2$.

Natural Exponential & Logarithmic Functions

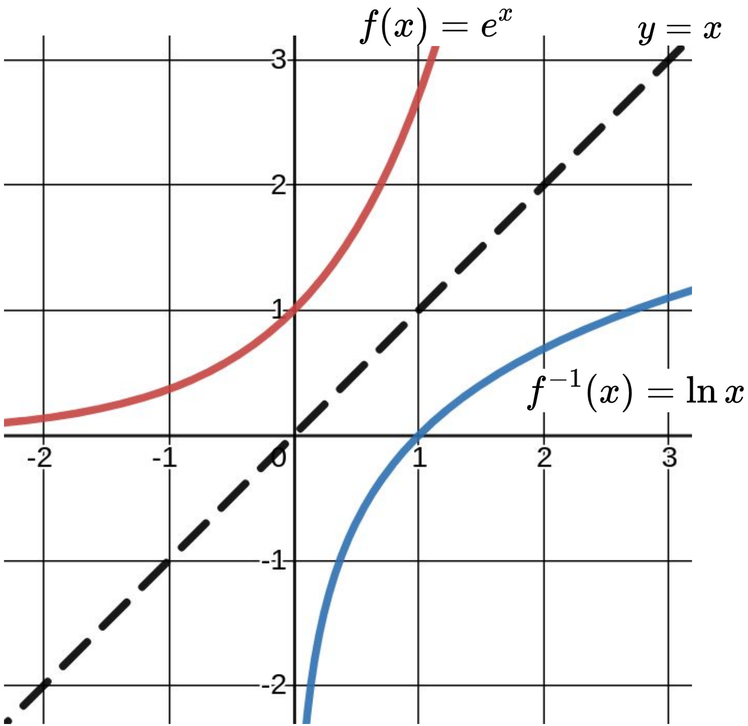
Due to the inverse relationship, the graph of $\log_a(x)$ is the reflection of a^x over the mirror line y = x.

When the base a is the natural (Euler's) number e = 2.718..., we have

$$f(x) = e^x \to f^{-1}(x) = \log_e x = \ln x$$

The number e is named "natural" for a few reasons, the most famous being the base of continuous compounding (to be elaborated in Limits), i.e.

$$\lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = Pe^{rt}$$



Laws of Logarithms

Similar to the laws of exponents, we have the following laws of logarithms.

a) $\log_a(xy) = \log_a x + \log_a y$ $\rightarrow$ $a^{LHS} = a^{\log_a(xy)} = xy$ $\quad a^{RHS} = a^{\log_a x + \log_a y}$

$= a^{\log_a x} \cdot a^{\log_a y}$

$= x \cdot y$

b) $\log_a(x/y) = \log_a x - \log_a y$

c) $\log_a x^y = y \log_a x$

d) $\log_a x = \log_b x / \log_b a$ (change of base)

e) $\log_a 1 = 0$ ~ since $\log_a\left(\frac{x}{x}\right) = \log_a x - \log_a x = 0$

Exercise: Prove each law above using the laws of exponents.

Logarithmic Functions

Exercise: Without using a calculator, solve the following.

a) $4^{\log_4 x^2} = \log_2 2^x \longrightarrow$

b) $\log_5 50 - \log_5 2 = 2^x$

c) $\ln(x-1) + 2\ln\sqrt{x} = 0$

d) $e^{2x+1}e^{-4x} = 3e$

a)
$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } x = 1$$

b)
$$\log_5 50 - \log_5 2 = 2^x$$
$$\log_5\left(\frac{50}{2}\right) = 2^x$$
$$\log_5 25 = 2^x$$
$$\log_5 5^2 = 2^x$$
$$2^x = 2$$
$$x = 1$$

ANS: a) x = 0, 1. b) x = 1. c) x = (1+√5)/2. d) -½ ln(3).

c) $\ln(x-1) + 2\ln\sqrt{x} = 0$

$\ln[(x-1)x] = 0$

$e^{\ln[(x-1)x]} = e^{0} = 1$

$x^2 - x - 1 = 0$

$x = \dfrac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{5}}{2}$

d) $e^{2x+1} e^{-4x} = 3e$

$e^{-2x+1} = 3e$

$-2x + 1 = \ln 3 + \ln e$

$-2x = \ln 3$

$x = -\dfrac{\ln 3}{2}$

Rate of Growth of Elementary Functions

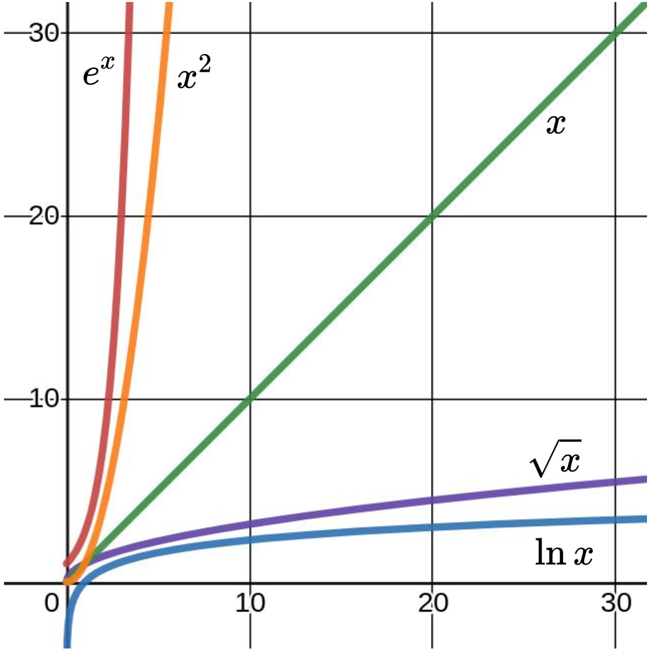
The rate of growth of increasing elementary functions can be ranked as follows.

$$\log_a x < x^{1/n} < mx < x^n < a^x$$

Faster growth rate →

Notice that the functions at opposing sides are inverses of each other, so when one increasing quickly, the other is increasing slowly. Eg, e^x increases the fastest, so its inverse ln(x) increases the slowest.

Note that the **ranking is always true after some big x**. Eg, while $x^{10} > e^x$ when x is small, $e^x > x^{10}$ after some bigger x. Verify in Desmos.



Inverse Trigonometric Functions

Since a trigonometric function takes in an angle and outputs a coordinate, its inverse takes in a coordinate and outputs the angle. The “machine” perspective is

Input angle, θ →

“Projection Machine”

$$x(\theta) = \cos\theta, \quad y(\theta) = \sin\theta$$

→ *Output coordinate, x or, y*

“Angle Machine”

$$\theta(x) = \cos^{-1} x, \quad \theta(y) = \sin^{-1} y$$

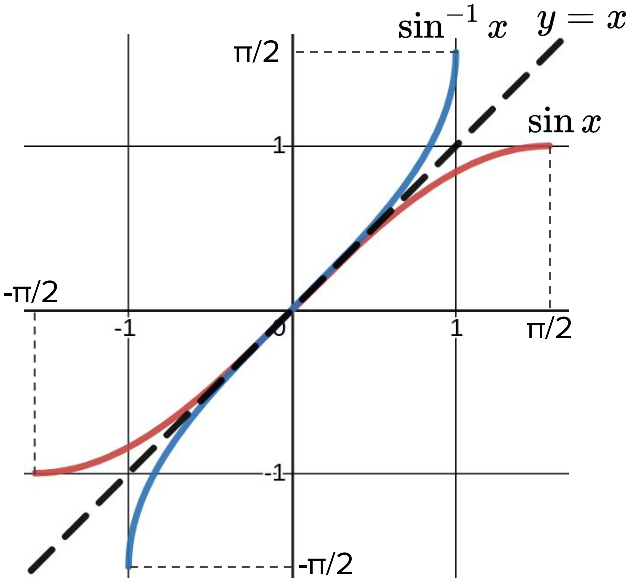
For example, since $y = \sin(\pi/2) = 1$, we get $\theta = \sin^{-1}(1) = \pi/2$. In layman, the inverse sine function returns the angle required for the projection on the y-axis to be at 1.

Inverse Trigonometric Functions

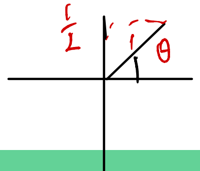
Due to the periodic nature of a trigonometric function, the domain needs to be restricted in order for it to be one-to-one and thus have an inverse. Hence we have

$$f(x) = \sin x$$
$$\textbf{Domain} : [-\pi/2,\ \pi/2],\ \textbf{Range} : [-1, 1]$$

$$f^{-1}(x) = \sin^{-1} x$$
$$\textbf{Domain} : [-1, 1],\ \textbf{Range} : [-\pi/2,\ \pi/2]$$



Exercise: Without using a calculator, state the value of $\sin^{-1}(½)$. $= 30° = \frac{\pi}{6}$



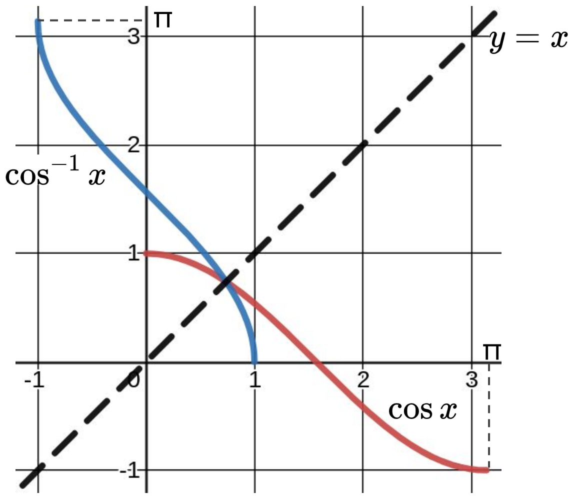
Similarly for inverse cosine and inverse tangent functions, we have

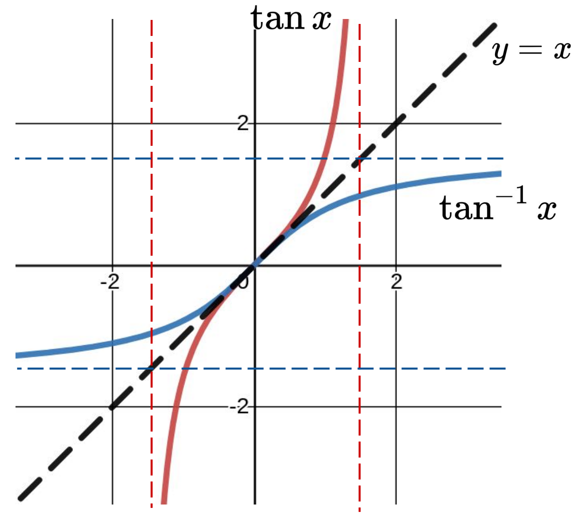
$$f(x) = \cos x$$
$$\text{Domain} : [0,\ \pi],\ \text{Range} : [-1, 1]$$

$$f^{-1}(x) = \cos^{-1} x$$
$$\text{Domain} : [-1, 1],\ \text{Range} : [0,\ \pi]$$

$$f(x) = \tan x$$
$$\text{Domain} : (-\pi/2,\ \pi/2),\ \text{Range} : \mathbb{R}$$

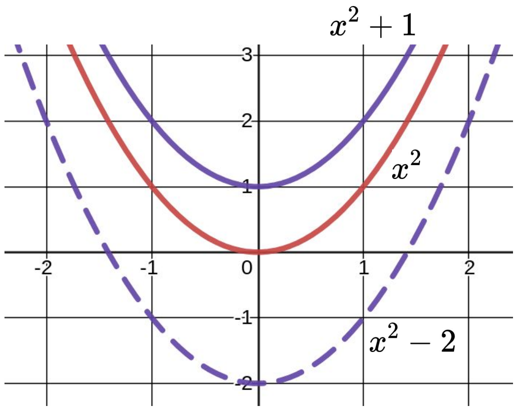
$$f^{-1}(x) = \tan^{-1} x$$
$$\text{Domain} : \mathbb{R},\ \text{Range} : (-\pi/2,\ \pi/2)$$





Transformations of Functions

The graph of some functions can be obtained by "transformations" from the graph of the "basic" function, illustrated as follows.



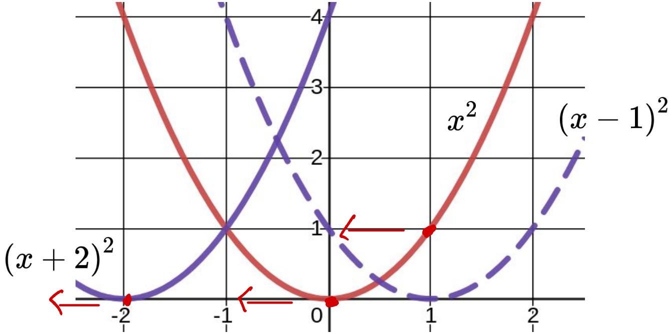
Vertical Shifting

$$y(x) = f(x) + c \rightarrow \text{shift } f(x) \text{ up by } c \text{ units}$$
$$y(x) = f(x) - c \rightarrow \text{shift } f(x) \text{ down by } c \text{ units}$$

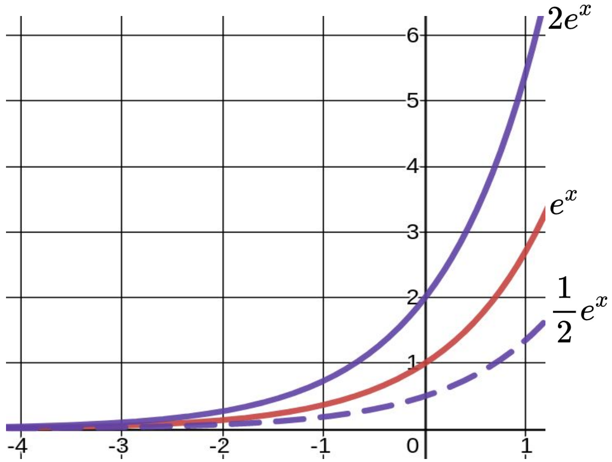
Horizontal Shifting

$$y(x) = f(x+c) \rightarrow \text{shift } f(x) \text{ left by } c \text{ units}$$
$$y(x) = f(x-c) \rightarrow \text{shift } f(x) \text{ right by } c \text{ units}$$

When the input to the function is changed to x-c, x has to increase such that function output is the same, so the graph shifts right.



Transformations of Functions



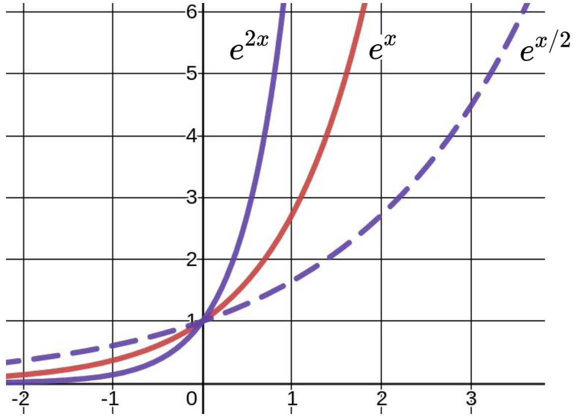
Vertical Scaling

$$y(x) = cf(x) \text{ where}$$
$$\begin{cases} c > 1 & \text{stretch } f(x) \text{ vertically by } c \text{ times} \\ 0 < c < 1 & \text{compress } f(x) \text{ vertically by } c \text{ times} \end{cases}$$

Horizontal Scaling

$2x$

$$y(x) = f(cx) \text{ where}$$
$$\begin{cases} c > 1 & \text{compress } f(x) \text{ horizontally by } 1/c \text{ times} \\ 0 < c < 1 & \text{stretch } f(x) \text{ horizontally by } 1/c \text{ times} \end{cases}$$



Notice that transformations in x occur in an opposite way to that in y.

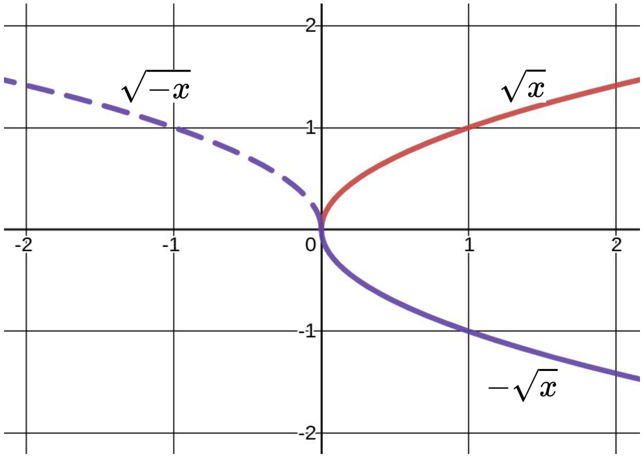
Transformations of Functions

Reflection about x-axis

$$y(x) = -f(x)$$

Reflection about y-axis

$$y(x) = f(-x)$$



The various transformations can be combined in a function. When this happens, follow (generally) the order of transformations according to a ➜ b ➜ c ➜ d as shown below.

$$y(x) = cf(a(x-b)) + d$$

Chronologically, **transformations to the input x should be done first** before those done to the function f(x) in order to get the correct graph.

Transformations of Functions

Example: From the graph of f(x), obtain the graph of y(x) in a step-by-step manner.

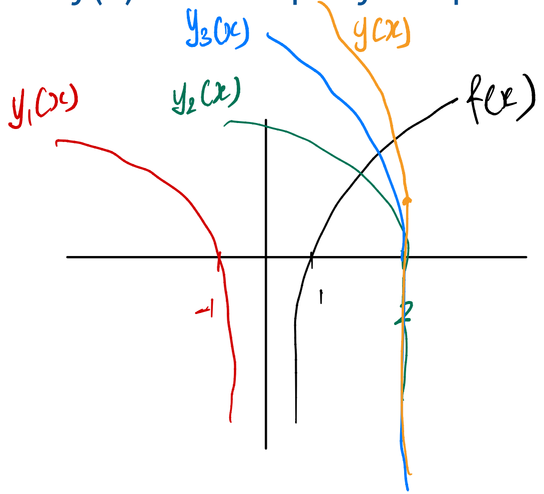
$$f(x) = \ln x\,, \quad y(x) = 2\ln(3-x) + 1$$

same

$$y_1(x) = \ln(-x)$$

$$y_2(x) = \ln(-x+3) = \ln[-(x-3)]$$

$$y_3(x) = 2\ln[-(-x-3)]$$

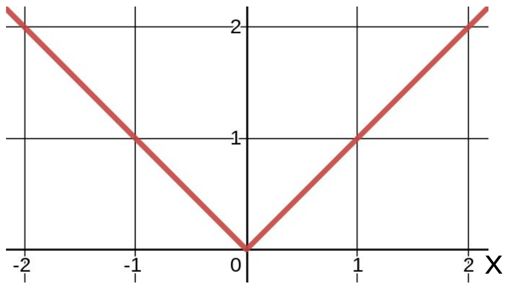


Some Other Functions

Besides the elementary functions described earlier, below are some other functions that we will encounter and explore more in later topics.

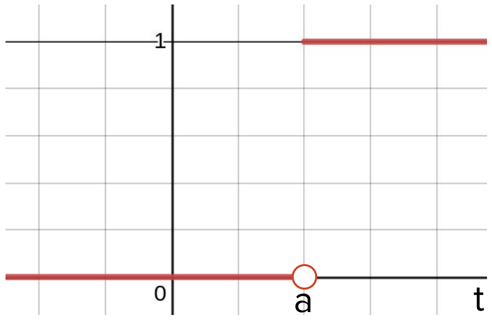
Absolute Value Function

$$f(x) = |x| = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases}$$



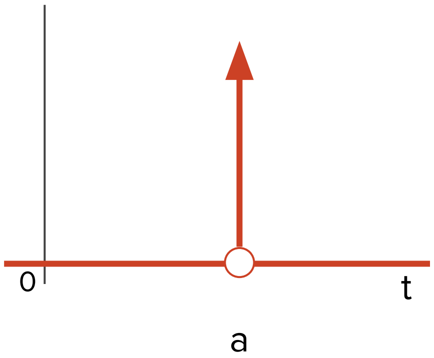
Unit-Step (Heaviside) Function

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$



Dirac delta Function

$$\delta(t-a) = \begin{cases} 0, & t \neq a \\ +\infty, & t = a \end{cases}$$



Why You & Me are Learning Math

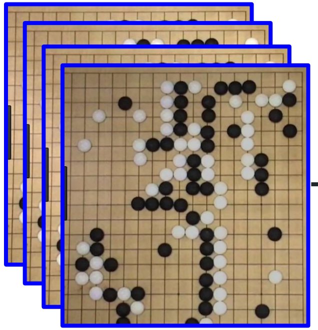
People learn math for various reasons. Some learn it to understand and analyze phenomena. Some learn it to invent and engineer products. A few are naturally attracted to its mutually-reinforcing logic and the beauty of equations. *Many are forced to do it*.

I am trained as an engineer, during which I applied math modelling to the design of mechanical systems and compared the model solutions to experimentation. I was astonished at how accurately mathematics describes the physics of the systems. From then on, I became **fascinated** and was hooked to learning math ever since.

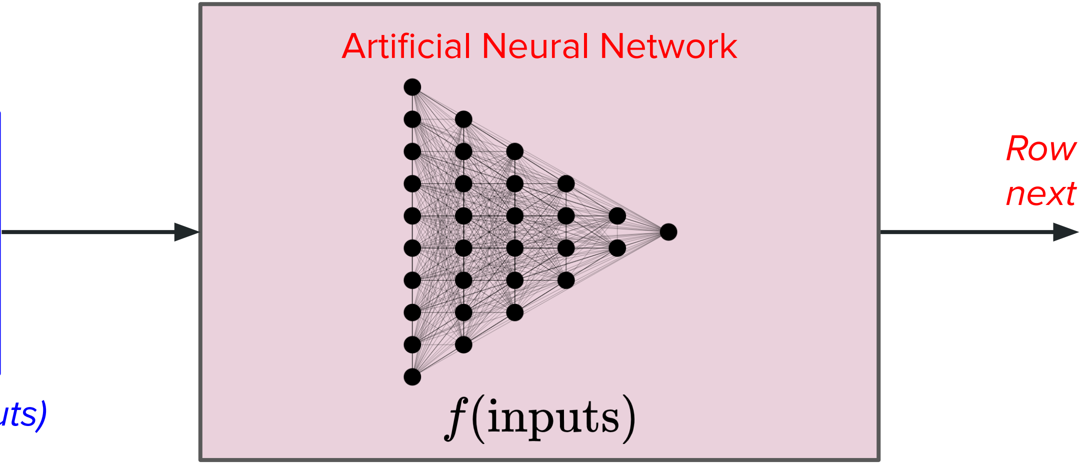
Along our journey in Maths 1, 2 & 3, I hope you find your own reasons for learning math. Most of you will find it a difficult journey, but I assure you the rewards are worth it if you struggle it through with me at the end. Because I struggled through it as well.

A Function of Artificial Intelligence (AI)

In 2016, an AI program named AlphaGo developed by Google Deepmind beat the world champion, Lee Sedol, in Go chess. As depicted below, the "trained" function is an artificial neural network that takes in the last 4 image frames of the chessboard as inputs and produces the coordinates of its next move as an output.



4 image frames (inputs)



Row i & column j for the next move (output)

End of Topic 1

*The world doesn't **function** without **Functions**. Literally.*

Full movie at https://youtu.be/WXuK6gekU1Y

You should really watch this movie. Be touched by the beauty of human tenacity & machine intelligence.