Topic 7
Introduction to Laplace Transform

Dr Teh Yong Liang
Email: youliangzheng@gmail.com

Outline

- Definition of the Laplace Transform
- Laplace Transform of Elementary Functions
- Properties of the Laplace Transform

The Laplace Transform

The Laplace transform is simply an integration (a.k.a. integral transform) defined by:

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

where the input is a function of a real variable (usually time), and the output is another function of a complex variable, s = σ+iω.

For the Laplace transform of a function f(t) to **exist**, the **improper integral must converge**.

Example: The Laplace transform of f(t) = k (constant) is,

$$L\{k\} = \int_0^\infty ke^{-st}\,dt = k\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty = k\left[-\frac{1}{s}(0-1)\right] = \frac{k}{s}, \quad s > 0$$

Laplace Transform of Elementary Functions

The Laplace transform of f(t) = t, t^2, t^3 and t^n respectively are:

$$L\{t\} = \int_0^\infty te^{-st}\,dt = \left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_0^\infty = \frac{1}{s^2}, \quad s > 0$$

$$L\{t^2\} = \int_0^\infty t^2e^{-st}\,dt = \left(-\frac{t^2}{s}e^{-st} - \frac{2t}{s}e^{-st} - \frac{2}{s^3}e^{-st}\right)\Big|_0^\infty = \frac{2}{s^3}, \quad s > 0$$

$$L\{t^3\} = \int_0^\infty t^3e^{-st}\,dt = \left(-\frac{t^3}{s}e^{-st} - \frac{3t^2}{s^2}e^{-st} - \frac{6t}{s^3}e^{-st} - \frac{6}{s^4}e^{-st}\right)\Big|_0^\infty = \frac{6}{s^4}, \quad s > 0$$

$$\vdots$$

$$L\{t^n\} = \int_0^\infty t^ne^{-st}\,dt = \frac{n!}{s^{n+1}}, \quad s > 0$$

Laplace Transform of Elementary Functions

The Laplace transform of f(t) = sin(ωt) is (using integration by parts twice):

$$L\left\{\sin\left(\omega t\right)\right\} = F(s) = \int_0^{\infty} \sin\left(\omega t\right) e^{-st}\,dt = -\sin\left(\omega t\right)\frac{e^{-st}}{s}\bigg|_0^{\infty} + \frac{\omega}{s}\int_0^{\infty} \cos\left(\omega t\right) e^{-st}\,dt$$

$$= 0 + \frac{\omega}{s}\left[-\cos\left(\omega t\right)\frac{e^{-st}}{s}\bigg|_0^{\infty} - \frac{\omega}{s}\int_0^{\infty} \sin\left(\omega t\right) e^{-st}\,dt\right]$$

$$= \frac{\omega}{s^2} - \frac{\omega^2}{s^2}F(s)$$

$$\rightarrow \left(1 + \frac{\omega^2}{s^2}\right)F(s) = \frac{\omega}{s^2} \rightarrow F(s) = \frac{\omega}{s^2+\omega^2}, \quad s > 0$$

Using the same approach, the Laplace transform of f(t) = cos(ωt) can be evaluated as:

$$L\left\{\cos\left(\omega t\right)\right\} = \frac{s}{s^2+\omega^2}, \quad s > 0$$

Laplace Transform of Elementary Functions

The Laplace transform of $f(t) = e^{at}$ is:

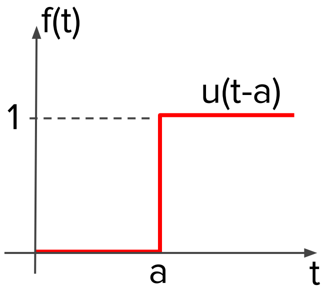
$$L\left\{e^{at}\right\} = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[-\frac{1}{s-a} e^{-(s-a)t}\right]\Big|_0^\infty = \frac{1}{s-a}, \quad s > a$$

Example: State the Laplace transforms of the following functions.

$f(t) = t^6$ $\qquad$ $g(t) = \sin(3t)$ $\qquad$ $h(t) = \cos(7t)$ $\qquad$ $p(t) = e^{-5t}$

Laplace Transform of Unit-Step Function

Exercise: Evaluate the Laplace transform of the unit-step function, u(t-a).

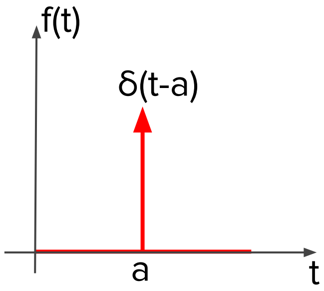


ANS: $L\left\{u(t-a)\right\} = \dfrac{e^{-as}}{s}$

Laplace Transform of Delta Function

The Dirac delta function δ(t-a) is defined as one that is **zero everywhere except at t = a**, where it is infinitely large. Aka as the unit-impulse, the delta function has an area = 1. The delta function is used to **model impact forces** and **voltage spikes** etc.

The Laplace transform of f(t) = δ(t-a) and f(t) = g(t)δ(t-a) are:



$$L\{\delta(t-a)\} = \int_0^\infty \delta(t-a)e^{-st}\,dt = \int_0^\infty \delta(t-a)e^{-sa}\,dt$$
$$= e^{-as}\int_0^\infty \delta(t-a)dt = e^{-as}, \quad s > 0$$
$$L\{g(t)\delta(t-a)\} = \int_0^\infty g(t)\delta(t-a)e^{-st}\,dt = \int_0^\infty \delta(t-a)g(a)e^{-sa}\,dt$$
$$= g(a)e^{-as}\int_0^\infty \delta(t-a)dt = g(a)e^{-as}, \quad s > 0$$

Properties of Laplace Transform

Since the Laplace transform is an integration, it is therefore linear, i.e.

$$L\left\{f(t)\right\} = \int_0^\infty kf(t)e^{-st}\,dt = k\int_0^\infty f(t)e^{-st}\,dt = kF(s)$$

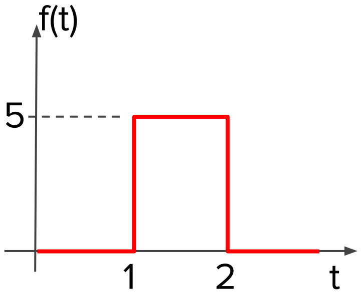
$$\rightarrow L\left\{kf(t)\right\} = kL\left\{f(t)\right\}$$

$$L\left\{f(t)+g(t)\right\} = \int_0^\infty [f(t)+g(t)]e^{-st}\,dt = \int_0^\infty f(t)e^{-st}\,dt + \int_0^\infty g(t)e^{-st}\,dt$$
$$= F(s) + G(s)$$

$$\rightarrow L\left\{f(t)+g(t)\right\} = L\left\{f(t)\right\} + L\left\{g(t)\right\}$$

Properties of Laplace Transform

Exercise: Using the linearity property, evaluate the Laplace transform of the following rectangular pulse.



ANS: $L\{f(t)\} = \frac{5}{s}\left(e^{-s} - e^{-2s}\right)$

Shifting Properties of Laplace Transform

When a function f(t) is multiplied by e^{at}, its Laplace transform can be evaluated as:

$$L\left\{f(t)e^{at}\right\} = \int_0^\infty f(t)e^{at}e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a), \quad s > a$$

This property of Laplace transform is called **shifting in the s-domain**.

Example: Evaluate the LT of $g(t) = te^{3t}$ and verify the above property.

ANS: $L\left\{te^{3t}\right\} = \dfrac{1}{(s-3)^2}$

Shifting Properties of Laplace Transform

When a function f(t-a) is multiplied by u(t-a), its Laplace transform can be evaluated as:

$$L\{f(t-a)u(t-a)\} = \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt = \int_a^\infty f(t-a)u(t-a)e^{-st}\,dt$$
$$= \int_a^\infty f(t-a)e^{-st}dt$$

Let τ = t-a, so dτ = dt, the above integral becomes:

$$L\{f(t-a)u(t-a)\} = \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau = \int_0^\infty f(\tau)e^{-s\tau}e^{-as}\,d\tau$$
$$= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-as}F(s), \quad s > 0$$

This property of Laplace transform is called shifting in the time (t)-domain.

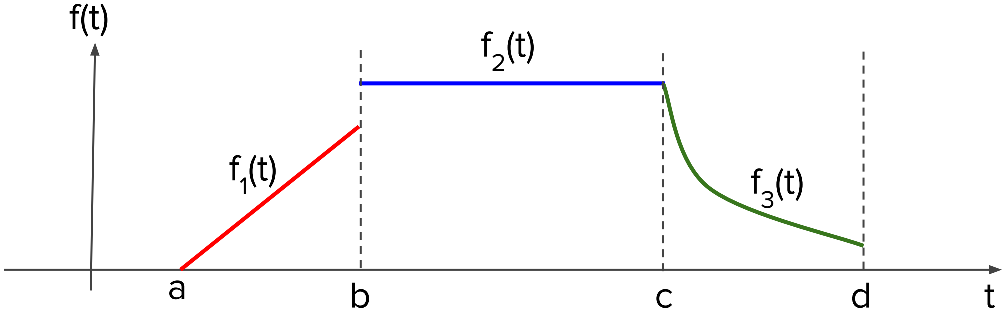
Shifting Properties of Laplace Transform

Example: Evaluate the LT of $g(t) = t^2u(t-2)$ by using the time-shifting property.

ANS: $G(s) = 2e^{-2s}\left(\frac{1}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right)$

Rewriting Piecewise Functions Using u(t-a)

A piecewise function can be rewritten into a **single function** using the **unit-step function**. Generally, we can deduce:



$$f(t) = \begin{cases} f_1(t), & a \leq t < b \\ f_2(t), & b \leq t < c \\ f_3(t), & c \leq t < d \\ & \vdots \end{cases} = \underbrace{f_1(t)u(t-a)}_{\text{On } f_1 \text{ at } t = a} + \underbrace{[f_2(t) - f_1(t)]u(t-b)}_{\text{On } f_2 + \text{Off } f_1 \text{ at } t = b} + \underbrace{[f_3(t) - f_2(t)]u(t-c)}_{\text{On } f_3 + \text{Off } f_2 \text{ at } t = c} + \ldots$$

Rewriting Piecewise Functions Using u(t-a)

Example: Rewrite f(t) using the unit-step function and evaluate its Laplace transform.

$$f(t) = \begin{cases} 2, & 0 \leq t < 1 \\ t, & 1 \leq t < 3 \\ e^{-5t}, & 3 \leq t \end{cases}$$

ANS: $f(t) = 2u(t) + (t-2)u(t-1) + \left(e^{-5t} - t\right)u(t-3),\ F(s) = \dfrac{2}{s} + e^{-s}\left(\dfrac{1}{s^2} - \dfrac{1}{s}\right) + e^{-3s}\left(\dfrac{e^{-15}}{s+5} - \dfrac{1}{s^2} - \dfrac{3}{s}\right)$

Derivative of Laplace Transform

When the transformed function F(s) is differentiated, we notice that:

$$\frac{\mathrm{d}F(s)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}\,dt = \int_0^\infty f(t)(-t)e^{-st}\,dt$$
$$= -\int_0^\infty tf(t)e^{-st}\,dt = -L\left\{tf(t)\right\}$$

Therefore, when a function f(t) is multiplied by t, its Laplace transform is:

$$L\left\{tf(t)\right\} = -F'(s)$$

Further differentiating F(s) reveals that:

$$L\left\{t^n f(t)\right\} = (-1)^n F^{(n)}(s)$$

Derivative of Laplace Transform

Example: Using the derivative of Laplace transform, evaluate the Laplace transform of the following functions. What did you notice in (a)?

a) $h(t) = t^2 e^{-t}$

b) $g(t) = te^{-t} \sin(3t)$

ANS: a) $H(s) = \dfrac{2}{(s+1)^3}$. b) $G(s) = \dfrac{6(s+1)}{\left[(s+1)^2 + 9\right]^2}$

Table of Laplace Transforms

Consolidating previous results, we create a table for easy reference (not exhaustive).

$f(t)$	$F(s)$
k	$\frac{k}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$

$f(t)$	$F(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
$g(t)\delta(t-a)$	$g(a)e^{-as}$
$g(t)e^{at}$	$G(s-a)$
$g(t-a)u(t-a)$	$e^{-as}G(s)$
$t^n g(t)$	$(-1)^n G^{(n)}(s)$
$g(t)+h(t)$	$G(s)+H(s)$
$kg(t)$	$kG(s)$

$$\boxed{F(s) = \int_0^\infty f(t)e^{-st}\,dt}$$

End of Topic 7

We shall continue our struggle in Math 3.
All the very best till then.

The End?

You will find much of the math being employed in the engineering & data science modules, so it’s more of a new beginning!