EDE1012 MATHEMATICS 2

Tutorial 2

Techniques of Integration

1. Evaluate the following integrals using integration by substitution.

a) $\displaystyle\int x\sqrt{1-x^2}\,dx$

b) $\displaystyle\int x^2\sin\left(x^3\right)dx$

c) $\displaystyle\int x^5\sqrt{1+x^2}\,dx$

d) $\displaystyle\int_{-5/2}^{-2} x(2x+5)^8\,dx$

e) $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^3 x}\,dx$

f) $\displaystyle\int_1^{e^{\pi/4}} \frac{\sec^2(\ln x)}{x}\,dx$

g) $\displaystyle\int_1^2 \frac{2\ln x}{x}\,dx$

ANS: **a)** $\dfrac{-\left(1-x^2\right)^{3/2}}{3}+c$. **b)** $\dfrac{-\cos\left(x^3\right)}{3}+c$.
c) $\dfrac{\left(1+x^2\right)^{7/2}}{7}-\dfrac{2\left(1+x^2\right)^{5/2}}{5}+\dfrac{\left(1+x^2\right)^{3/2}}{3}+c$. **d)** $\dfrac{-41}{360}$. **e)** $\dfrac{1}{2}$. **f)** 1. g) $(\ln 2)^2$.

2. Evaluate each integral below in terms of f(x). k is a constant.

a) $\displaystyle\int f'(x)\,dx$

b) $\displaystyle\int f'(kx)\,dx$

c) $\displaystyle\int x f'\left(kx^2\right)dx$

ANS: **a)** $f(x)+c$. **b)** $\dfrac{f(kx)}{k}+c$. **c)** $\dfrac{f\left(kx^2\right)}{2k}+c$

3. Set up a definite integral that represents the area of a quarter of the ellipse below. Evaluate it to obtain the area formula of the full ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ANS: $b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx=\frac{\pi ab}{4}$

4. Using substitution of trigonometric relations, evaluate the following integrals.

a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

b) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$

c) $\displaystyle\int \frac{x^3}{\sqrt{x^2-4}}\,dx$

ANS: **a)** $\cos^{-1}\left(\frac{x}{3}\right)-\frac{\sqrt{9-x^2}}{x}+c$. **b)** $-\frac{\sqrt{1+x^2}}{x}+c$. **c)** $4\sqrt{x^2-4}+\frac{\left(x^2-4\right)^{3/2}}{3}+c$.

5. Using partial fractions or otherwise, evaluate the integrals below.

a) $\displaystyle\int \frac{3x}{x^2+2x-8}\,dx$

b) $\displaystyle\int \frac{x^2-1}{x^2+x-6}\,dx$

c) $\displaystyle\int \frac{x^5}{x^3-9x}\,dx$

d) $\displaystyle\int \frac{x^2-1}{x^2+5x-6}\,dx$

e) $\displaystyle\int \frac{x}{x^2+2x+1}\,dx$

ANS: **a)** $2\ln|x+4|+\ln|x-2|+c$. **b)** $\frac{3}{5}\ln|x-2|-\frac{8}{5}\ln|x+3|+x+c$.
c) $\frac{27}{2}\ln\left|\frac{x-3}{x+3}\right|+\frac{x^3}{3}+9x+c$. **d)** $x-5\ln|x+6|+c$. **e)** $\ln|x+1|+\frac{1}{x+1}+c$.

6. Using integration by parts, evaluate the following.

a) $\int x^4 \ln(2x)\,dx$

b) $\int x^2 \cos x\,dx$

c) $\int e^{2x}\cos(4x)\,dx$

d) $\int (1-x^3)e^{x/2}\,dx$

ANS: **a)** $\dfrac{x^5\ln(2x)}{5}-\dfrac{x^5}{25}+c$. **b)** $(x^2-2)\sin x+2x\cos x+c$.
c) $\dfrac{\mathrm{e}^{2x}[2\sin(4x)+\cos(4x)]}{10}+c$. **d)** $-2(x^3-6x^2+24x-49)e^{x/2}+c$.

7. Using integration by parts, prove the antiderivatives of the following inverse trigonometric functions.

$$\int \sin^{-1}x\,dx = x\sin^{-1}x+\sqrt{1-x^2}+c$$

$$\int \cos^{-1}x\,dx = x\cos^{-1}x-\sqrt{1-x^2}+c$$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x-\frac{\ln(x^2+1)}{2}+c$$

8. Using various techniques of integration, evaluate the following.

a) $\int \dfrac{\sin^{-1}(\ln x)}{x}\,dx$

b) $\int \dfrac{1+\sin x}{1+\cos x}\,dx$

c) $\int \dfrac{3e^x+4e^{-x}+2}{1-e^{2x}}\,dx$

d) $\int \dfrac{a^x}{a^x+a^{-x}}\,dx$

e) $\int \dfrac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}\,dx$

f) $\int \dfrac{x^3}{\sqrt{x^2+1}}\,dx$

g) $\int \sin^4 x\cos^2 x\,dx$

h) $\int \sin^3 x\cos^4 x\,dx$

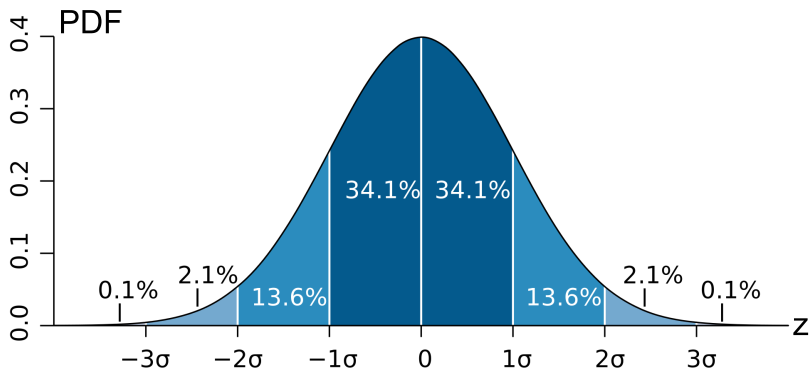
ANS: **a)** $\ln x \sin^{-1}(\ln x) + \sqrt{1-(\ln x)^2} + c$. **b)** $\tan\left(\frac{x}{2}\right) - \ln(1+\cos x) + c$.

c) $\frac{5}{2}\ln(\mathrm{e}^x+1) - \frac{9}{2}\ln|\mathrm{e}^x - 1| - 4\mathrm{e}^{-x} + 2x + c$. **d)** $\frac{\ln(a^{2x}+1)}{2\ln a} + c$.

e) $\frac{\left[\sin^{-1}(x^2)\right]^2}{4} + c$. **f)** $\frac{(x^2+1)^{3/2}}{3} - \sqrt{x^2+1} + c$. **g)** $-\frac{\sin^3(2x)}{48} - \frac{\sin(4x)}{64} + \frac{x}{16} + c$.

h) $\frac{\cos^7(x)}{7} - \frac{\cos^5(x)}{5} + c$.

9. **Use of Calculus in Statistics**
According to the probability density function (PDF) of the standard normal distribution shown below, there is a 68.27% chance that a sample's reading, z, will lie within one standard deviation from the mean, as represented by the area under the PDF.



Source: https://en.wikipedia.org/wiki/Normal_distribution#Standard_normal_distribution

Generally, the probability P that z will lie in between a and b in a normal distribution is given by the integral

$$P(a < z < b) = \int_a^b \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$

a) Using the midpoint rule, verify that $P(-1 < z < 1) \approx 0.6827$.

b) Using the trapezoidal rule, verify that $P(-2 < z < 2) \approx 0.9545$.

c) What do you think $P(-\infty < z < \infty)$ logically is? Verify it using numerical integration.

10. Using integration, evaluate the volume of each object described below. Use integration of discs and hollow cylinders.

    a) A sphere with radius R.

    b) A torus with major radius R and minor radius r.

    c) (https://openstax.org/books/calculus-volume-2/pages/2-3-volumes-of-revolution-cylindrical-shells)

    A solid of revolution with the region enclosed by $y = \sqrt{x}$ and $y = x^2$ rotated about the y-axis. Sketch the solid.

ANS: **a)** $V = \frac{4}{3}\pi R^3$. **b)** $V = 2\pi^2 R r^2$. **c)** $V = \frac{3\pi}{10}$.

For more practice problems (& explanations), check out:

1) https://openstax.org/books/calculus-volume-2/pages/3-1-integration-by-parts
2) https://openstax.org/books/calculus-volume-2/pages/3-2-trigonometric-integrals
3) https://openstax.org/books/calculus-volume-2/pages/3-3-trigonometric-substitution
4) https://openstax.org/books/calculus-volume-2/pages/3-4-partial-fractions
5) https://openstax.org/books/calculus-volume-2/pages/3-6-numerical-integration

End of Tutorial 2

(Email to youliangzheng@gmail.com for assistance.)