1.4.3 Electric potential

From fundamental law of electrostatics: electrostatic field is a gradient field

gradient field: the field can be expressed as the gradient of a potential function ϕ

ϕ is a scalar field $\phi(\vec{r}) = \phi(x,y,z)$

Electric potential $\phi(\vec{r}) \Rightarrow \boxed{\vec{E} = -\operatorname{grad}\phi(\vec{r})}$ (1.12)

Why does this reflect the conservative character of the vector field?

Since $\operatorname{Curl}(\operatorname{grad}\phi) = 0$; always!

$\Rightarrow \operatorname{Curl}(\vec{E}) = \operatorname{Curl}(-\operatorname{grad}\phi) = 0 \Rightarrow \oint \vec{E}d\vec{r} = \oint(-\operatorname{grad}\phi) = 0$

(i) Reference potential:

$\vec{E} = -\operatorname{grad}\phi$; Calculate ϕ from $E(\vec{r})$ by Integration

$$\phi(\vec{r}) = \int_{P_0}^{P} \vec{E}d\vec{r} + C$$

$\phi = \phi_1$

$\phi = \phi_1 + C \quad C = \text{Const}$ $\Big\}$ $\operatorname{grad}\phi$ is the same, since $\operatorname{grad}C = 0$

$\Downarrow$

Potential function is only defined up to a Constant Potential ϕ_0

This is called reference potential ϕ_0, which can be chosen freely; in general it is chosen accordingly to the application

(technical application $\Rightarrow$ ground potential; electric mask potential)

(zero potential, reference potential for voltage measurements)

(1.13) $\boxed{\phi(\vec{r}) = \phi_0 - \int_{P_0}^{P(\vec{r})} \vec{E}d\vec{r}}$ electric potential function

P_0 = reference potential at reference point $\vec{r}_0$

$P(\vec{r})$ = potential of an arbitrary point in space

$P(\vec{r}) = P(x,y,z)$ is a variable point in space

$$\phi(\vec{r}) = \phi_0 - \int_{\vec{r}=\vec{r}_0}^{\vec{r}=\vec{r}(x,y,z)} \underline{\vec{E}(x',y',z')\,d\vec{r}} \qquad \vec{E} = \vec{E}(x,y,z)$$

$\Downarrow$

$\phi(x,y,z)$; result of this calculation is a scalar function

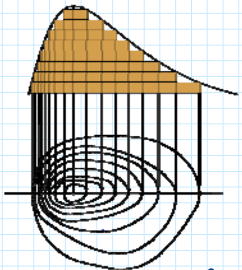
$$U_{12} = \int_{P_1}^{P_2} \underline{\vec{E}(x,y,z)\, d\vec{r}} = \int_{\vec{r}_1(x_1,y_1,z_1)}^{\vec{r}_2(x_2,y_2,z_2)} \vec{E}(x,y,z)\, d\vec{r}$$ = result is a value for U (not a function of x, y, z)

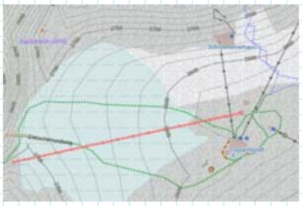
U_{12} is a fixed potential difference $U_{12} = \phi_2 - \phi_1$

$$= \cancel{\phi_0} - \int_{P_0}^{P_2} \vec{E} d\vec{r} - \cancel{\phi_0} + \int_{P_0}^{P_1} \vec{E} d\vec{r}$$

$$= U_{12}$$

(iii) Equipotential surfaces/equipotential lines (cf. Lines of constant height in a map for hiking)

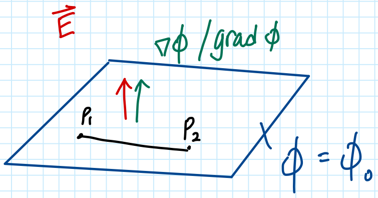




Iso · surfaces of a potential function : $\underline{\phi(\vec r) = \text{Const.} = \phi_0}$

$\downarrow$

describe a surface is 3D space



If we move from P_1 to P_2 ($P_1, P_2 \in \phi = \phi_0$)

=> no work is performed

It can be derived mathematically : $\nabla\phi(\vec r) \perp$ Iso-surface $\phi = \phi_0$

We know that $\vec E$ is a gradient field (since fulfils curl$(\vec E) = 0$)

$\vec E = -\text{grad}\ \phi \parallel \text{grad}\ \phi \perp \phi = \phi_0$

$\vec E$ - field lines are always perpendicular to equipotential surfaces / iso-surfaces.

field lines have a direction (vector field)

potential lines don't have a direction (scalar field)

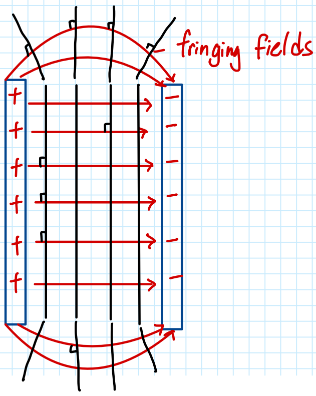
Exercise: Draw the field lines and equipotental lines for the following charge configurations:

a) Dipole field of two opposite point charges

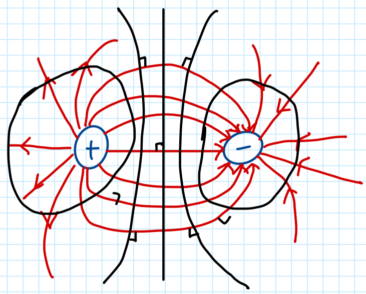
b) Two oppositely charged parallel electrically conducting plates

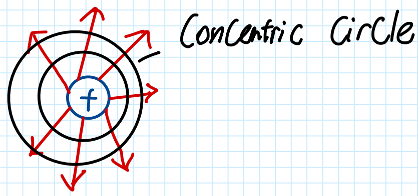
monopole field

b)



a)





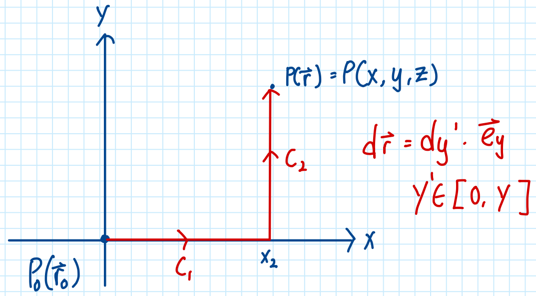
Example: Given is the following electrical field:

$$\vec{E}\text{-field} \quad E_x = k\cdot y\,, \quad E_y = k\cdot x\,, \quad E_z = 0 \qquad \vec{E} = \begin{pmatrix} ky \\ kx \\ 0 \end{pmatrix}$$

Calculate the potential function. The reference potential is 0 at the reference point, which is located at the origin of the coordinate system.

After having calculated the potential function, please draw a qualitative sketch of the equipotential lines and the electric field lines in the xy-plane.

Reference potential $\phi_0 = \phi(\vec{r}_0) = 0 \qquad \vec{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$



To solve : $\displaystyle \phi(\vec{r}) = \phi_0 - \int_{P_0}^{P(\vec{r})} \vec{E}(\vec{r}\,')\,d\vec{r}\,'$

$C_1 : (0,0) \to (x,0)$

$C_2 : (x,0) \to (x,y)$

$d\vec{r} = \vec{e}_x \cdot dx'$

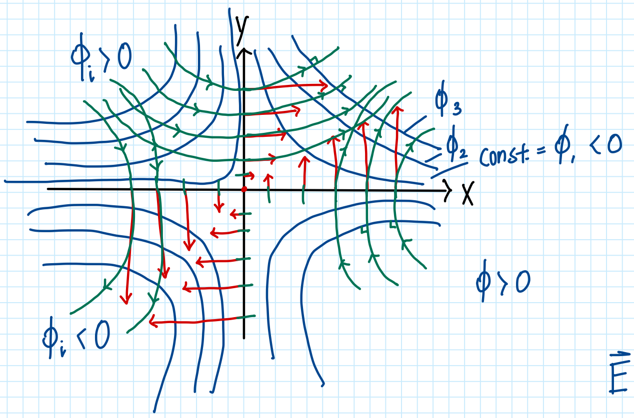
$x' = [0, x]$

$$\phi(x,y) = -\int_{P_0}^{P(\vec{r})} \vec{E}(\vec{r}\,')\,d\vec{r}\,' = -\left(\underbrace{\int_{(0,0)}^{(x,0)} (ky'\cdot\vec{e}_x + kx'\cdot\vec{e}_y)\,\vec{e}_x\cdot dx'}_{C_1} + \underbrace{\int_{(x,0)}^{(x,y)} (ky'\cdot\vec{e}_x + kx'\cdot\vec{e}_y)\,\vec{e}_x\cdot dy'}_{C_2}\right)$$

$$= -\int_{(0,0)}^{(x,0)} \underbrace{\begin{pmatrix} ky' \\ kx' \\ 0 \end{pmatrix}}_{\vec{E}(\vec{r})} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cdot dx'}_{d\vec{r}\,'} - \int_{(x,0)}^{(x,y)} \begin{pmatrix} ky' \\ kx' \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dy'$$

$$\phi(x,y) = -\int_{(0,0)}^{(x,0)} ky'\,dx' - \int_{(x,0)}^{(x,y)} kx'\,dy' = -\left[ky'x'\right]_{(0,0)}^{(x,0)} - \left[kx'y'\right]_{(x,0)}^{(x,y)}$$

$$\boxed{\phi(x,y) = -kxy}$$



Equipotential lines in X-Y plane:

$\phi(x,y) = \phi_j = -kxy$

$y = -\dfrac{\phi_j}{kx}$

$y \sim -\dfrac{1}{x}$

$\phi_j \quad j = 1, 2, 3$

$\vec{E} = \begin{pmatrix} ky \\ kx \\ 0 \end{pmatrix}$

$y = 0$ = x-axis

$x = 0$ = y-axis