EDE1011 ENGINEERING MATHEMATICS 1

Tutorial 4

Linear Transformations & Eigendecomposition

1. By intuition, define a matrix that does each of the following linear transformations in $\mathbb{R}^2$:

   a) Matrix A: Scale up by *k* times.
   b) Matrix B: Rotate CCW by angle *θ*.
   c) Matrix C: Reflect about the line y = x

   Hence, compose a single matrix D that performs all of the above linear transformations in the same order.

ANS: $D = \begin{bmatrix} k\sin\theta & k\cos\theta \\ k\cos\theta & -k\sin\theta \end{bmatrix}$

2. Solve the following SLEs using the matrix inverse. Which one is inconsistent or not linearly independent? Compare with your answers in tutorial 2.

a) $\begin{matrix} 3x + 4y = 1 \\ 2x + 3y = 12 \end{matrix}$

Does not exist

b) $\begin{matrix} 3x - 2y = 4 \\ -6x + 4y = 7 \end{matrix}$

Does not exist

c) $\begin{matrix} u + v + w - 6 = 0 \\ 4v + w + u = 5 \\ 6 = w + 3v + u \end{matrix}$

ANS: **a)** x = -45, y = 34. **b)** Inconsistent / not L.I. **c)** Inconsistent / not L.I.

3. Given the matrix B below, for what values of p is B not invertible?

$$B = \begin{bmatrix} 2 & 1 & p \\ 3 & 4 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

set $\det(B) = 0$
Solve for B

ANS: p = 3

4. Determine the inverse of the following matrices, if it exists.

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 3 & 1 \\ -1 & 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

ANS:

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 12 & -9 \\ 1 & -1 & 2 \\ -3 & 8 & -6 \end{bmatrix}, \quad \underbrace{B^{-1}\,\text{D.N.E.}}_{\det(B)=0}, \quad C = \frac{1}{2}\begin{bmatrix} -2 & 2 & 2 \\ 2 & -1 & -1 \\ -2 & 3 & 1 \end{bmatrix}$$

5. **Modelling Problem:** In a MMORPG, there are three items that you can equip to increase your attack, defence and dexterity points. The number of each item you can equip are x, y and z respectively. The contributions of each item are shown below:

ITEM	ATTACK	DEFENCE	DEXTERITY
Dragon Scale (x)	-20	40	10
Griffin Claw (y)	50	10	-10
Elven Crystal (z)	10	10	60

In order to clear a level boss, you need to increase your attack by 320 points and defence by 280 points. If the total number of items you can equip is 22, determine the number of each item to equip for the boss fight. (Hint: Form a SLE and solve using the matrix inverse.)

ANS: x = 2, y = 4, z = 16.

6. Find the eigenvalues and eigenvectors for the following matrices. Explain the eigenvectors for matrix E.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 7 & -3 \\ -2 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \; D = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}, \; E = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

ANS:

$$A: \; \lambda_1 = 1, \; \lambda_2 = 6, \; \vec{v_1} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \; \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B: \; \lambda_1 = 1, \; \lambda_2 = 8, \; \vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; \vec{v_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$C: \; \lambda_1 = 1, \; \lambda_2 = 2, \; \vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \; \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; D: \; \lambda_1 = -4, \; \lambda_2 = 3, \; \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \vec{v_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$E: \; \lambda_{1,2} = 4, \; \vec{v}_{1,2} = \begin{bmatrix} a \\ b \end{bmatrix} \; \forall \, a, \, b \mid \vec{v_1} \text{ not along the same span of } \vec{v_2}$$

7. Diagonalize:

$$A = \begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & -2 \\ -1 & 8 \end{bmatrix}$$

That is, find an invertible matrix P and diagonal matrix D such that each matrix is expressed as PDP^{-1}. Verify your answer.

ANS: $A = \begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\frac{1}{-4}\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}\frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$

8. Given the following matrix,

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix}$$

$\lambda_{1,2} = -2, 6$

$\vec{v}_{1,2} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

compute $A^5 + 7A^4 + I$. Also compute $\sqrt{5I + A}$ if it exists.

ANS: $A^5 + 7A^4 + I = \begin{bmatrix} 6369 & 10480 \\ 6288 & 10561 \end{bmatrix}, \sqrt{5I + A} = \frac{1}{8}\begin{bmatrix} 5\sqrt{3} + 3\sqrt{11} & 5\sqrt{11} - 5\sqrt{3} \\ 3\sqrt{11} - 3\sqrt{3} & 3\sqrt{3} + 5\sqrt{11} \end{bmatrix}$

$A^5 + 7A^4 + I = PD^5P^{-1} + 7PD^4P^{-1} + PIP^{-1}$

$= P(D^5 + 7D^4 + I)P^{-1}$

9. For a diagonalisable matrix A, write down the formula for sin(A), cos(A) and ln(A). Are there any restrictions on the eigenvalues for the matrix functions to be defined?

no restriction for sin(A) & cos(A)

For ln(A), $\lambda_n > 0$ for $\ln \lambda_n$ to be defined

$\sin(A) = \begin{bmatrix} \sin\lambda_1 & 0 & 0 \\ 0 & \sin\lambda_2 & \vdots \\ \vdots & \vdots & \sin\lambda_n \end{bmatrix} P^{-1}$

10. Determine the eigendecomposition of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$$

ANS: $A = \begin{bmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}\frac{1}{-56}\begin{bmatrix} 0 & 8 & 0 \\ -21 & -14 & -14 \\ -7 & -2 & 14 \end{bmatrix}$

11. Show that $\mathbf{x_1}$ is an eigenvector of matrix A.

$$A = \begin{bmatrix} -1 & 1 & 2 \\ -6 & 2 & 6 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the other eigenvectors and hence determine the eigendecomposition of A.

ANS: $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

12. Show that $\mathbf{v_1}$ is an eigenvector of matrix A.

$$A = \begin{bmatrix} -5 & 8 & 32 \\ 2 & 1 & -8 \\ -2 & 2 & 11 \end{bmatrix}, \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Find the other eigenvectors and describe the eigenspace geometrically. Diagonalize A.

ANS: $$A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 5 \\ -1 & 2 & 6 \\ 1 & -1 & -4 \end{bmatrix}$$

For more practice problems (& explanations), check out:

1) https://openstax.org/books/precalculus-2e/pages/9-7-solving-systems-with-inverses

End of Tutorial 4
(Email to youliangzheng@gmail.com for assistance.)