Topic 7
Applications of Derivatives I

Dr Teh Yong Liang
Email: youliangzheng@gmail.com

Outline

- L'Hopital's Rule
- Linear Approximation
- Taylor Series
- Minima & Maxima
- Concavity
- Absolute (Global) Extrema in a Closed Interval
- Optimization

Recap: Indeterminate Forms of Limits

In many cases, an indeterminate form arises when evaluating a limit by direct substitution, such as

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{1^2 - 1}{1 - 1} = \frac{0}{0}$$

And there are other indeterminate forms as shown below.

$$\lim_{x \to a} f(x) = \frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 \times \infty, \infty - \infty, 1^{\infty}, 0^{0}, \infty^{0}$$

As shown, an indeterminate form can arise when a function is undefined at an input value. However, a limit **might** still exist even when this happens. It is **important not to confuse an indeterminate form with a limit that D.N.E., like infinities**.

L'Hopital's Rule

L'Hopital's rule enables one to evaluate some indeterminate forms easily, subject to some conditions. Consider functions f(x) and g(x) that are differentiable at x = a and suppose we want to evaluate the limit

$$L = \lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{where} \quad \underline{\lim_{x \to a} f(x) = 0}, \ \underline{\lim_{x \to a} g(x) = 0}$$

So direct substitution results in the indeterminate form 0/0. We can rewrite the limit as

$$L = \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} \left[\frac{1/(x-a)}{1/(x-a)} \right]$$

$$= \cancel{\lim_{x \to a} \frac{\frac{f(x)-f(a)}{h}}{\frac{g(x)-g(a)}{h}}} = \frac{\lim_{x \to a} \frac{f(x)-\underline{f(a)}}{x-a}}{\lim_{x \to a} \frac{g(x)-\underline{g(a)}}{x-a}} = \frac{f'(a)}{g'(a)}$$

L'Hopital's Rule

Since f'(x) and g'(x) are continuous at x = a, we have

$$\underline{\lim_{x \to a} \frac{f(x)}{g(x)}} = \frac{f'(a)}{g'(a)} = \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} = \underline{\lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

For example, to evaluate the limit

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

which has the indeterminate form 0/0, we can use L'Hopital's rule to obtain

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{2x}{1} = 2$$

This is the same result if one uses factoring and dividing off to evaluate the limit.

L'Hopital's Rule

L'Hopital's rule can also be applied to the indeterminate form of ±∞/±∞. Consider

$$L = \lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{where} \quad \lim_{x \to a} f(x) = \pm\infty, \ \lim_{x \to a} g(x) = \pm\infty$$

We can rewrite

$$L = \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{1/g(x)}{1/f(x)} = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{0}{0}$$

So applying L'Hopital's rule on the limit of p(x)/q(x), using chain rule we get

$$L = \lim_{x \to a} \frac{p'(x)}{q'(x)} = \lim_{x \to a} \frac{\frac{-g'(x)}{g^2(x)}}{\frac{-f'(x)}{f^2(x)}} = \lim_{x \to a} \left\{ \frac{g'(x)}{f'(x)} \frac{f^2(x)}{g^2(x)} \right\}$$

$$= \lim_{x \to a} \frac{g'(x)}{f'(x)} \left[\lim_{x \to a} \frac{f(x)}{g(x)} \right]^2 = L^2 \lim_{x \to a} \frac{g'(x)}{f'(x)}$$

L'Hopital's Rule

So we can rearrange to obtain

$$L = 1/\lim_{x \to a} \frac{g'(x)}{f'(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Therefore, in a nutshell, to evaluate limits with indeterminate forms of 0/0 and ±∞/±∞, we can use the L'Hopital's rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Even for other indeterminate forms such as 1^{∞}, 0×∞ or ∞ - ∞, L'Hopital's rule can still work if one can manipulate them into 0/0 and ±∞/±∞, as demonstrated in the following exercise.

L'Hopital's Rule

$\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Exercise: Evaluate the limits below.

a) $\displaystyle\lim_{x\to\infty} \frac{2x^2-3x+1}{x^2+5}$

LH

$= \displaystyle\lim_{x\to\infty} \frac{4x-3}{2x+5} = \frac{\infty}{\infty}$

LH

$= \displaystyle\lim_{x\to\infty} \frac{4}{2} = 2$

b) $\displaystyle\lim_{x\to\infty} \underbrace{\left(1+\frac{1}{x}\right)^x}_{} = 1^\infty$

$e^{\ln\{\ \}}$

$= e^{\lim_{x\to\infty}\{\ln(1+\frac{1}{x})^x\}}$

$= e^{\lim_{x\to\infty}\{\underbrace{x\ln(1+\frac{1}{x})}_{\infty(0)}\}}$

$= e^{\lim_{x\to\infty}\left\{\frac{\ln(1+\frac{1}{x})}{1/x}\right\}}$ $\frac{0}{0}$

L'H

$= e^{\lim_{x\to\infty}\left\{\frac{\frac{1}{1+\frac{1}{x}}\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{-1/x^2}}\right\}}$

$= e^{\lim_{x\to\infty}\left\{\frac{1}{1+\frac{1}{x}}\right\}}$

$= e^1 = e$

c) $\displaystyle\lim_{x\to 1}\left\{\underbrace{\frac{x}{x-1}-\frac{1}{\ln x}}_{=\infty-\infty}\right\}$

$= \displaystyle\lim_{x\to 1}\left\{\frac{x\ln x-(x-1)}{(x-1)\ln x}\right\} = \frac{0-0}{0(0)} = \frac{0}{0}$

L'H

$= \displaystyle\lim_{x\to 1}\left\{\frac{\ln x+x(\frac{1}{x})-1}{\frac{(x-1)}{x}+\ln x}\right\}$

$= \displaystyle\lim_{x\to 1}\frac{\ln x}{1-\frac{1}{x}+\ln x} = \frac{0}{0}$

L'H

$\displaystyle\lim_{x\to 1}\frac{1/x}{\frac{1}{x^2}+\frac{1}{x}} = \lim_{x\to 1}\frac{\cancel{1/x}}{\cancel{\frac{1}{x}}(\frac{1}{x}+1)} = \frac{1}{2}$

ANS: a) 2. b) e. c) 1/2.

Linear Approximation

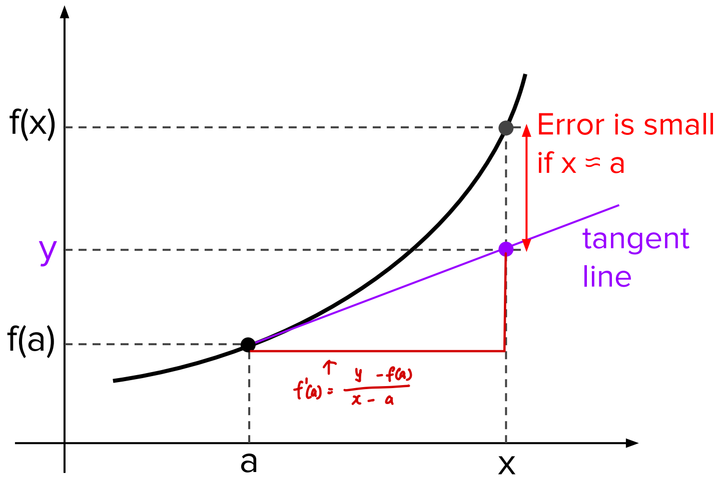
A linear approximation is simply using the tangent line to approximate the value of a function, as illustrated in the figure. The equation of the tangent line at x = a is

$$y - f(a) = f'(a)(x - a)$$

Letting y = L(x), called the linearization of a function, we can approximate f(x) by

$$f(x) \approx \underline{L(x) = f(a) + f'(a)(x - a)}$$

From the graph, we can observe that the linear approximation would be good if x is close to the point a (called the expansion point).



Linear Approximation

For example, we can approximate the value of √27 by defining

$$f(x) = \sqrt{x}$$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a) \end{aligned}$$

And choosing the expansion point a = 25 such that f(a) is easily calculated and a is close to x = 27, the linear approximation is

$$L(27) = \sqrt{25} + \frac{1}{2\sqrt{25}}(27-25) = 5\frac{1}{5} = 5.2$$

A check reveals that √27 = 5.196... which is very close to the approximation.

Linear Approximation

Exercise: Estimate the value of 1000/21 using linear approximation.

Let $f(x) = \frac{1000}{x}$

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

$$= \frac{1000}{a} - \frac{1000}{a^2}(x-a)$$

Choose $a = 20$ (close to $x = 21$)

$$f(21) \approx \frac{1000}{20} - \frac{1000}{20^2}(21-20)$$

$$= 50 - \frac{1}{4}$$

$$= 47.5$$

ANS: 47.5

Linear Approximation

Exercise: A cubic box has a $\underbrace{\text{volume}}_{V = x^3}$ of 24 cm^3. Estimate the $\underbrace{\text{length of each side}}_{\text{let } x(V) = V^{1/3}}$ using linear approximation.

$$x(V) \approx x(a) + x'(a)(V-a)$$

$$\approx a^{1/3} + \frac{1}{3}a^{-2/3}(V-a)$$

choose $a = 27$,

$$x(24) \approx 27^{\frac{1}{3}} + \frac{1}{3}(27^{1/3})^{-2}(V-a)$$

$$\approx 3 + \frac{1}{3}(3)^{-2}(24-27)$$

$$\approx 3 + \frac{1}{27}(-3)$$

$$\approx 3 - \frac{1}{9} \approx 2\frac{8}{9} \text{ cm}$$

ANS: $2\frac{8}{9}$ cm

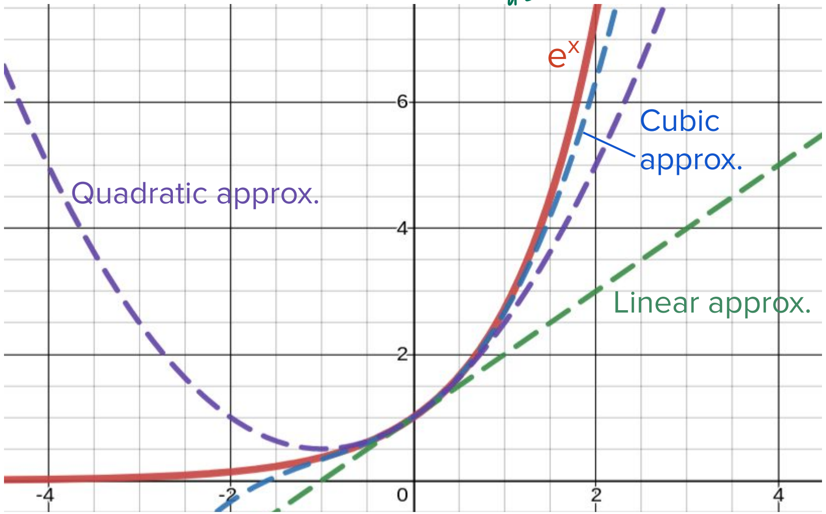
Higher-Degree Polynomial Approximation

While the linear approximation uses a linear function, a more accurate estimation of a function value can be given by higher-degree polynomial approximations. For example, the Nth-degree polynomial approximation for $f(x) = e^x$ expanded about $\underbrace{x = 0}_{a=0}$ is

$$T_N(x) = \overset{n=0}{1} + \overset{n=1}{x} + \overset{n=2}{\frac{x^2}{2!}} + \overset{n=3}{\frac{x^3}{3!}} + \dots + \overset{n=N}{\frac{x^N}{N!}} = \sum_{n=0}^{N} \frac{x^n}{n!}$$

From the graph, notice that as **more higher-degree terms** are included, the approximation curves **get closer** to the curve of e^x.

So how do we work out $T_N(x)$?



Verify at https://www.desmos.com/calculator/vbxkj7g7sh.

Derivation of the Taylor Series

The polynomial approximation can be derived by letting a power series represent the function f(x). Ideally, we want the infinite power series to **converge** to f(x), i.e.

$$f(x) = T(x) = a_0 + a_1(\cancel{x -} a) + a_2(\cancel{x -} a)^2 + \ldots + a_n(\cancel{x -} a)^n + \ldots = \sum_{n=0}^{\infty} a_n (x-a)^n$$

Now, the problem is to get the coefficients a_n. To get a_0, simply substitute $\underline{x = a}$ into the above to obtain

$$\underline{a_0 = f(a)}$$

To get a_1, differentiate and substitute x = a:

$$f'(x) = a_1 + 2a_2(x-a) + \ldots + na_n(x-a)^{n-1} + \ldots$$
$$\rightarrow a_1 = f'(a)$$

Derivation of the Taylor Series

Repeat differentiating and substituting x = a, we get the other coefficients as

$$f''(x) = 2a_2 + 3(2)a_3(x-a) + \ldots + n(n-1)a_n(x-a)^{n-2} + \ldots$$

$$\rightarrow a_2 = \frac{f''(a)}{2}$$

$$f'''(x) = 3(2)a_3 + 4(3)(2)a_4(x-a)^4 + \ldots + n(n-1)(n-2)a_n(x-a)^{n-3} + \ldots$$

$$\rightarrow a_3 = \frac{f'''(a)}{3!}$$

$$\vdots$$

$$\rightarrow a_n = \frac{f^{(n)}(a)}{n!}$$

Derivation of the Taylor Series

Substituting the coefficients into the infinite power series gives

$$f(x) = T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The expression on the RHS is called the Taylor series. The Nth-degree Taylor approximation of f(x) is therefore

$$\boxed{f(x) \approx T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Eg. For f(x) = e^x expanded about a = 0, since all derivatives of e^x are itself, so we have

When expanded about a=0, the Taylor series is also called the the Maclaurin series

$$f^{(n)}(0) = e^0 = 1$$

$$\rightarrow e^x \approx \sum_{n=0}^{N} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^N}{N!}$$

Taylor Approximation

Exercise: Evaluate the Taylor approximation of f(x) = e^x expanded about x = a. Plot it to show the convergence to f(x) as more terms are added in the Taylor approximation.

$$f(x) \approx T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\Rightarrow \quad f^{(n)}(x) = e^x$$

$$f^n(a) = e^a$$

$$T_N(x) = \sum_{n=0}^{N} \frac{e^a}{n!}(x-a)^n = e^a \sum_{n=0}^{N} \frac{(x-a)^n}{n!}$$

ANS: $e^x \approx T_N(x) = e^a \sum_{n=1}^{N} \frac{(x-a)^n}{n!}$

Taylor Series

Exercise: Evaluate the Taylor series of f(x) = sin(x) expanded about $\overbrace{x = 0}^{a=0}$. Plot it to show the convergence to f(x) as more terms are added. (The Taylor series expanded about x = 0 is also called the Maclaurin series.)

To find!

$$f(x) \approx T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$n=0$, $f(0) = \sin(0) = 0$

$n=1$, $f'(0) = \cos(0) = 1$

$n=2$, $f''(0) = -\sin(0) = 0$

$n=3$, $f'''(0) = -\cos(0) = -1$

$n=4$, $f''''(0) = \sin(0) = 0$

Sub into $T_N(x)$:

$$T_N(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$= 0 + (1)x + 0 + \frac{(-1)x^3}{3!} + 0 + \frac{(1)x^5}{5!} + 0 + \frac{(-1)x^7}{7!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$= \sum (-1)^k \frac{x^n}{n!}, \quad \underbrace{n = 1, 3, 5, 7, \ldots}_{\text{let } n = 2k+1}$$

k=0 k=1 k=2 k=3

$k = 0, 1, 2, 3$

$$\sin(x) \approx T_N(x) = \sum_{k=0}^{N} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

hence $\Rightarrow \sin(x) = T_N(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

ANS: $\sin x = T(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

Taylor Series

The Taylor series for some functions can be deduced from the known Taylor series for another function. For example, for f(x) = e^{-2x}, its Taylor series can be obtained by **replacing x with -2x** in the Taylor series of e^{x}, i.e.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\rightarrow e^{-2x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-2x)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} x^n = \underset{n=0}{1} - \underset{n=1}{2x} + \underset{n=2}{2x^2} - \frac{4x^3}{3} + \ldots$$

This works even when the function is not elementary, such as

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{3n} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{3} + \ldots$$

Plot the above in Desmos to verify.

Taylor Series

Exercise: From the Maclaurin series of sin(x), determine the Maclaurin series of f(x) = 2sin(-πx). Verify in Desmos.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$2\sin(-\pi x) = 2\sum_{k=0}^{\infty} \frac{(-1)^k(-\pi x)^{2k+1}}{(2k+1)!} = 2\sum_{k=0}^{\infty} \frac{(-1)^k(-1)^{2k+1}\pi^{2k+1}x^{2k+1}}{(2k+1)!}$$

$$= 2\sum_{k=0}^{\infty} \frac{(-1)^{3k+1}\pi^{2k+1}x^{2k+1}}{(2k+1)!}$$

same result as $(-1)^{3k+1}$

$$= 2\sum_{k=0}^{\infty} \frac{(-1)^{k+1}\pi^{2k+1}x^{2k+1}}{(2k+1)!}$$

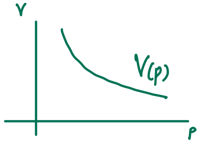
ANS: $2\sin(-\pi x) = 2\sum_{k=0}^{\infty} \frac{(-1)^{k+1}\pi^{2k+1}x^{2k+1}}{(2k+1)!}$

Motivation for Optimization of a Function

In many applications, one seeks to find the ‘best’ performance of a process or system. For example, in doing business, a typical objective is to maximize profit, which could be described by

price of item sold

$$\text{Profit, } Q(p) = \underbrace{pV(p)}_{\text{revenue from sales}} - \underbrace{C(p)}_{\text{cost of production}}$$



where p is the price of an item sold, V(p) is the volume sold and C(p) is the cost of production. From the basic principles of supply and demand, we know volume sold depends on the price of the item, hence V(p). And the cost of production varies with volume, so C(V(p)) = C(p).

Intuitively, we know that a business owner cannot set the price too low or too high or else there would be zero or negative profit. This implies there must be an ‘ideal’ price such that the profit is at a maximum. The **motivation** is to find this ‘ideal’ price.

Motivation for Optimization of a Function

Depending on the context, optimization can also be finding the minimum of a function. For example, a production engineer might aim to minimize the material usage in manufacturing, or an air-conditioning consultant might want to minimize the energy consumption in the cooling of a building.

Hence, optimization of a function usually means minimizing or maximizing some performance metric (the goal of many tasks).

A vital process in machine learning is to minimize the ‘error’ function.

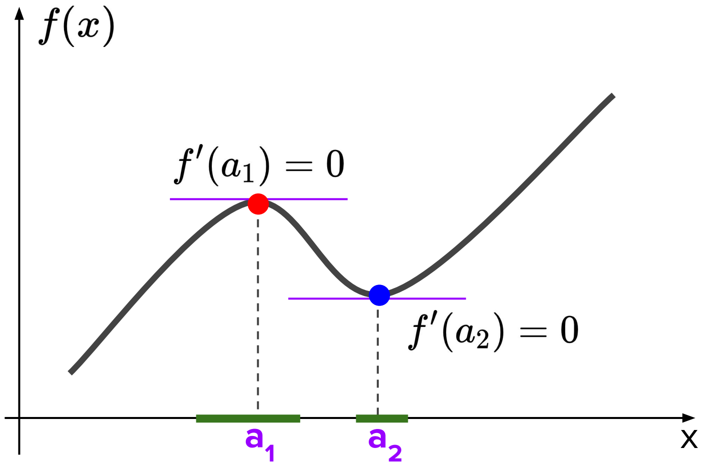
In reality, the performance functions are usually multivariable functions and subjected to constraints, so we will learn how to handle them in Math 3. We shall firstly learn the basics of optimization with single-variable functions in this topic.

Local Maximum & Minimum (Extremum)

Formally, a point (a, f(a)) is called a local max / min point of a function f(x) if there is an open interval (c, d) such that:

$$f(a) \geq f(x) \quad \text{(local max)}$$
$$f(a) \leq f(x) \quad \text{(local min)}$$

for all x in (c, d).



If f(x) is smooth at x = a, then we observe that the tangent to the curve is horizontal, i.e.

$$f'(a) = 0$$

Then (a, f(a)) is also called a stationary point (f(x) is 'stationary' when x goes from a^- to a^+).

Critical Points

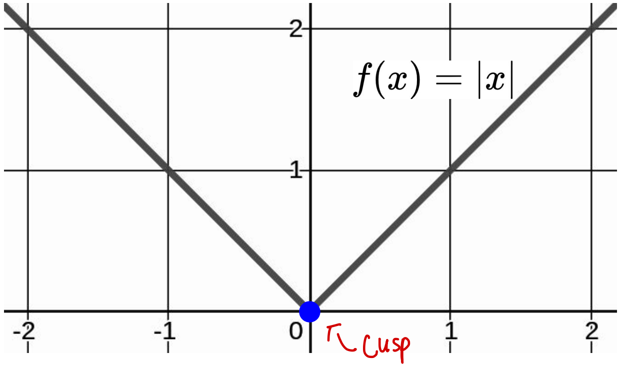
① Stationary pt : $f'(x) = 0$
② Cusp pt : $f'(x)$ DNE
Extrema (min/max)

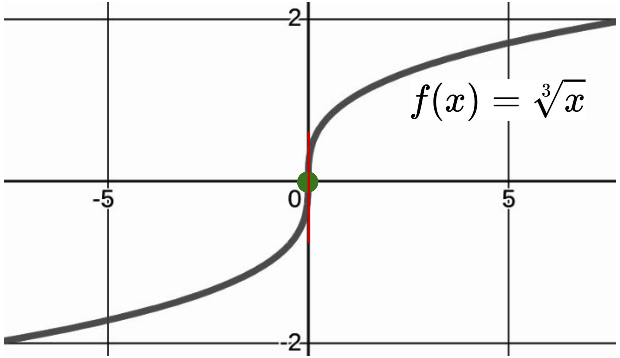
Note that **not all extrema are stationary points**. For example, f(x) = |x| has a minimum at (0, 0), yet f'(0) DNE. Hence, to find **potential** extrema, we solve

$$f'(x) = 0, \;\; f'(x) \text{ D.N.E.}$$

Any point that satisfies either of the two conditions above is called a critical point. However, **not all critical points are extrema**. For example, f(x) = $x^{1/3}$ has a critical point at (0, 0) since f'(0) DNE, but it is clearly not an extrema.

So further verification is required to identify an extremum after the critical points are evaluated.

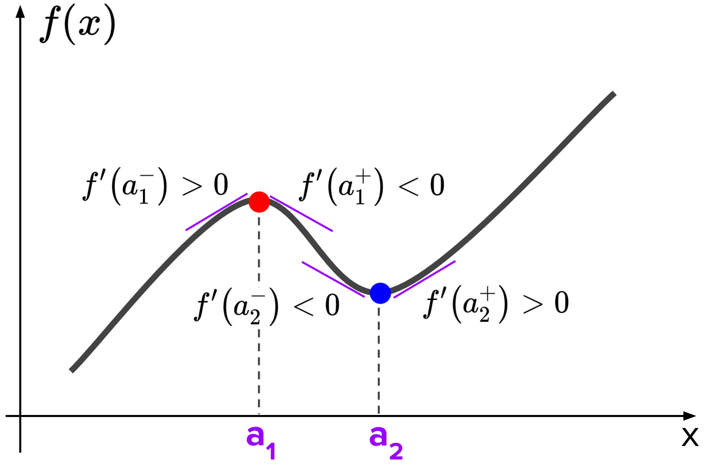




First-Derivative Test

To identify if a critical point is a local extremum, we can make use of the derivative. From the graph, we observe that

$$f'\left(a^-\right) > 0,\ f'\left(a^+\right) < 0 \quad \text{(local max)}$$
$$f'\left(a^-\right) < 0,\ f'\left(a^+\right) > 0 \quad \text{(local min)}$$



This is called the first-derivative test. If a critical point does not satisfy any of the two conditions above, then it is not a local extremum. For example, the critical point (0, 0) of $f(x) = x^{1/3}$ has

$$f'\left(0^-\right) > 0,\ f'\left(0^+\right) > 0$$

which makes it neither a local max or min according to the MVT.

First-Derivative Test

Example: Find the critical points of the function below and identify the extrema. Verify your answers in Desmos.

$$f(x) = \sqrt[3]{x}(4 - x)$$

Step 1: solve $f'(x) = 0$ & $f'(x)$ DNE

$$f'(x) = \frac{1}{3x^{2/3}}(4-x) + (-1)(x^{1/3})\frac{3x^{2/3}}{3x^{2/3}} = \frac{4 - x - 3x}{3x^{2/3}}$$

$$= \frac{4-4x}{3x^{2/3}}$$

$f'(x) = 0$

$\frac{4-4x}{3x^{2/3}} = 0$

$4 - 4x = 0$

$x = 1$

(stationary pt)

$y = f(1) = 3$

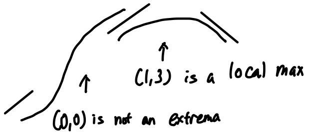
$f'(x)$ DNE $\rightarrow 3x^{2/3} = 0$

$x = 0 \rightarrow f(0) = 0$

Critical pts at $(1, 3)$ & $(0, 0)$

Step 2: Classify critical points

	$x=0$	$x=1$	
$4-4x$	+	+	−
$3x^{2/3}$	+	+	+
$f'(x)$	+	+	−



$$f'(x) = \frac{4-4x}{3x^{2/3}} \rightarrow f''(x) = \frac{3x^{2/3}(-4) - 2x^{-1/3}(4-4x)}{9x^{4/3}}$$

$$= \frac{x^{2/3}(-12+8) - 8x^{-1/3}}{9x^{4/3}} = \frac{-4x^{2/3} - 8x^{1/3}}{9x^{4/3}}$$

$$= \frac{-4\cancel{x^{-1/3}}(x+2)}{9x^{\cancel{4/3}\ 5/3}}$$

At $x=1$ (stationary pt),

$f''(1) = \frac{-4(1+2)}{9(1)} = -\frac{4}{3} < 0 \rightarrow$ local max

ANS: Critical points at (0, 0), (1, 3). Local max at (1, 3).

First-Derivative Test

Exercise: Find the critical points of the function below and identify the extrema. Verify your answers in Desmos.

$$f(x) = \frac{x^3}{3} - 4|x+1|$$

$$\begin{cases} \frac{x^3}{3} - 4(x+1), & x+1 \geq 0 \to x \geq -1 \\ \frac{x^3}{3} + 4(x+1), & x+1 < 0 \to x < -1 \end{cases}$$

$$f'(x) = \begin{cases} x^2 - 4, & x > -1 \\ x^2 + 4, & x < -1 \end{cases}$$

Step 1: solve $f'(x) = 0$

$x^2 - 4 = 0$

$x^2 = 4$

$x = -2, 2$

$\uparrow$ (NA since $x > -1$)

$x^2 + 4 = 0$

$\Rightarrow$ no sol^n

$f(2) = \frac{2^3}{3} - 4(3)$

$= -\frac{28}{3}$

$f'(x)$ DNE

Check at $x = -1$

$\left.\begin{array}{l} \lim_{x \to -1^-} x^2 + 4 = 5 \\ \lim_{x \to -1^+} x^2 - 4 = -3 \end{array}\right\} \neq \ f'(-1)$ DNE

$f(-1) = -\frac{1}{3} - 4|0| = -\frac{1}{3}$

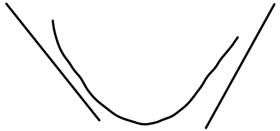
Critical points at $(2, -\frac{28}{3})$ and $(-1, -\frac{1}{3})$

Step ②: Classify

$$f'(x) = \begin{cases} x^2 - 4, & x > -1 \\ x^2 + 4, & x < -1 \end{cases}$$

f_1'

$f'(2^-) = 4^- - 4 < 0, \quad f'(2^+) = 4^+ - 4 > 0$



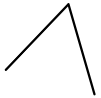
$f''(x) = f_1''(x) = 2x$

At $x = 2$,

$f''(2) = 4 > 0 \to$ local min

$(2, -\frac{28}{3})$ is a local min

$\lim_{x \to -1^-} x^2 + 4 = 5, \quad \lim_{x \to -1^+} x^2 - 4 = -3$



$(-1, -\frac{1}{3})$ is a local max

ANS: Critical points at (-1, -1/3) (local max) and (2, -28/3) (local min).

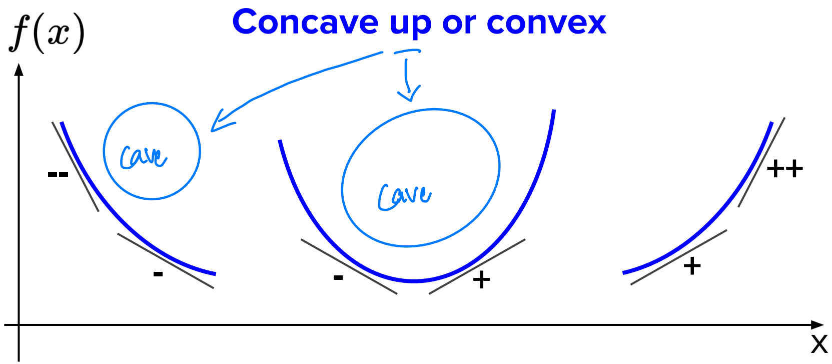
Concavity

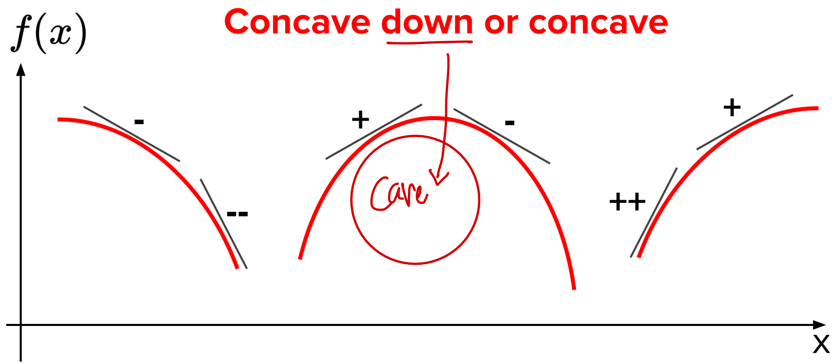
Besides using the first-derivative test to identify if a critical point is a local max or min, there is another method which looks at how the function curves, called concavity. Concavity of a function is defined by

$f\uparrow \rightarrow f'>0$

$$f'(x)\uparrow \;:\; f''(x) > 0 \quad (\overbrace{\text{concave up or convex}}^{\text{curve up}})$$

$$f'(x)\downarrow \;:\; f''(x) < 0 \quad (\underbrace{\text{concave down or concave}}_{\text{curve down}})$$





Second-Derivative Test

So from the second-derivative, we can see that a critical point at x = a can be classified according to

$$f''(a) < 0 \quad (\text{local max})$$
$$f''(a) > 0 \quad (\text{local min})$$

which is consistent with the first-derivative test. However, the second-derivative test can only be applied to **stationary points where f(x) is smooth**, since a function is not differentiable at cusps.

Exercise: For the last two exercises, use the second-derivative test to classify the stationary points. In what situation is it the preferred test?

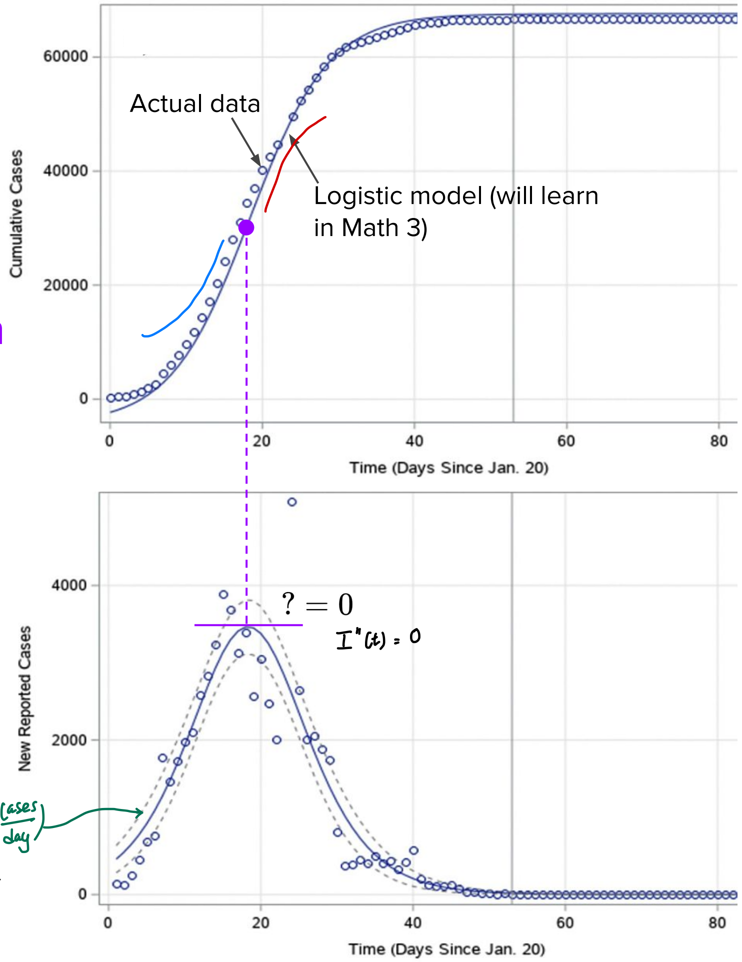
It is the preferred test if $f''(x)$ is easy to evaluate. And also if there are only stationary points to classify. If there are non stationary critical points, then just use the 1st derivative test for all critical points.

Points of Inflection

Besides extrema, inflection points are also of interest in many cases. For example, during a pandemic, the cumulative infections can be modelled by the logistic equation and the inflection point signifies that daily new infections have reached a peak. This would imply that the infection wave has roughly passed its halfway mark with an end in sight. The authorities can make use of this information for policy measures.

If cumulative infections is I(t), what does the daily new cases represent?

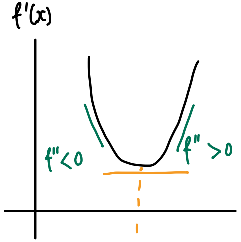
$I'(t) = \frac{dI}{dt} \left(\frac{\text{cases}}{\text{day}}\right)$



Shen, C. Y. (2020). *Logistic growth modelling of COVID-19 proliferation in China and its international implications*, International Journal of Infectious Diseases, 96, 582-589. (https://www.sciencedirect.com/science/article/pii/S1201971220303039)

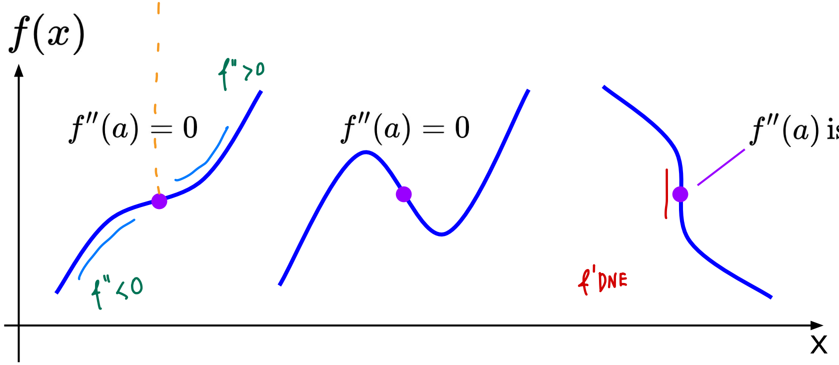
Points of Inflection

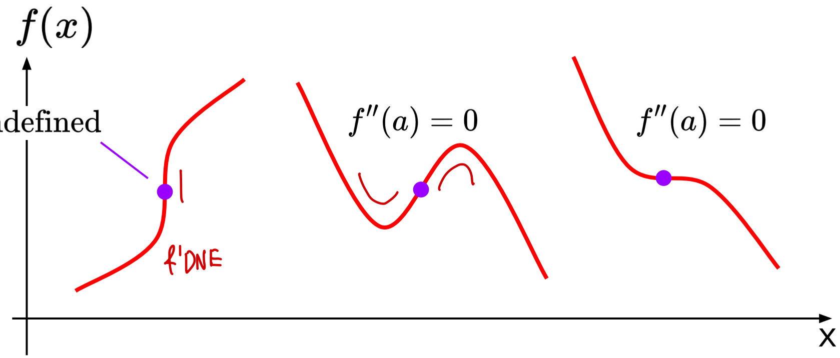
A point of inflection is characterized by a change in concavity, as illustrated by some examples below. For a **smooth** f(x), an inflection point at x = a must satisfy



$$f''(a) = 0 \quad \text{or} \quad f''(a) \text{ is undefined,}$$

$$\underline{f''(a^-) < 0,\ f''(a^+) > 0} \quad \text{or} \quad \underline{f''(a^-) > 0,\ f''(a^+) < 0}$$





Points of Inflection

Example: Determine the points of inflection for the function below. Verify your answers in Desmos.

$$f(x) = \sqrt[3]{x}(4 - x)$$

$$f''(x) = \frac{-4\cancel{x^{-1/3}}(x+2)}{9x^{\cancel{4/3}\ 5/3}} = 0$$

$-4(x+2) = 0$

$x = -4$

$f(-2) = -7.56$

f'' DNE $\rightarrow 9x^{5/3} = 0$

$x = 0$

$f(0) = 0$

Check for change of concavity.

		$x=-2$		$x=0$	
$-4(x+2)$	+		−		−
$9x^{5/3}$	−		−		+
$f''(x)$	−		+		−

Since concavity changes at $(-2, -7.56)$ & $(0,0)$, both are inflection pts.

ANS: Inflection points at (0,0) & (-2, -7.56).

Points of Inflection

Exercise: Determine the points of inflection for the function below. Verify your answers in Desmos.

$$f(x) = \frac{x^4}{12} - \frac{x^6}{30}$$

$$f'(x) = \frac{x^3}{3} - \frac{x^5}{5}$$

$$f''(x) = x^2 - x^4$$

$$= x^2(1-x^2)$$

	$x<-1$	$-1<x<0$	$0<x<1$	$x>1$
x^2	+	+	+	+
$1-x^2$	−	+	+	−
$f''(x)$	−	+	+	−

inflection pts

at $(-1, 0.05)$ & $(1, 0.05)$

$f''(x) = 0$

$x^2(1-x^2) = 0$

$x^2 = 0 \Rightarrow x = 0$, $f(0) = 0$ | $1-x^2=0 \rightarrow x^2 = 1$, $x = \pm 1$

$f''(x)$ DNE $\rightarrow$ no sol^n

$f(1) = f(-1) = 0.05$

ANS: Inflection points at (-1, 0.05) & (1, 0.05). (0, 0) is not an inflection point, but an undulation point.

$f(x) = x^2 + 4 \rightarrow f'(x) = 2x = 0 \rightarrow x = 0$ (stationary pt) $f''(x) = 2 > 0$

$\Rightarrow f''(0) = 2 \rightarrow x = 0$ is local min

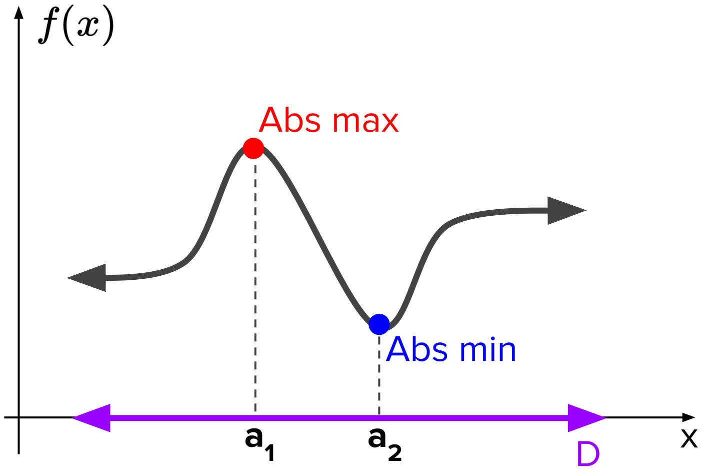
$f(0) = 4$ which is the min in the $\underbrace{\text{range of } f(x)}_{[4,\ \infty]}$, so $(0,4)$ is also a global min.

Absolute (Global) Extrema

Formally, a point (a, f(a)) is called an absolute (global) max / min point of a function f(x) if:

$$f(a) \geq f(x) \quad \text{(absolute max)}$$
$$f(a) \leq f(x) \quad \text{(absolute min)}$$

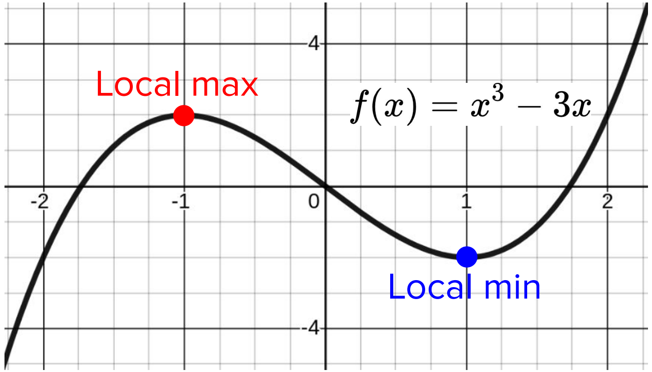
for all x in the domain D of f(x).



Some functions might not have an absolute (global) extrema. Eg:

$$f(x) = x^3 - 3x$$

does not have absolute max and min in $\mathbb{R}$., although it has a local max and a local min, since the range of $f(x)$ is $-\infty, \infty$

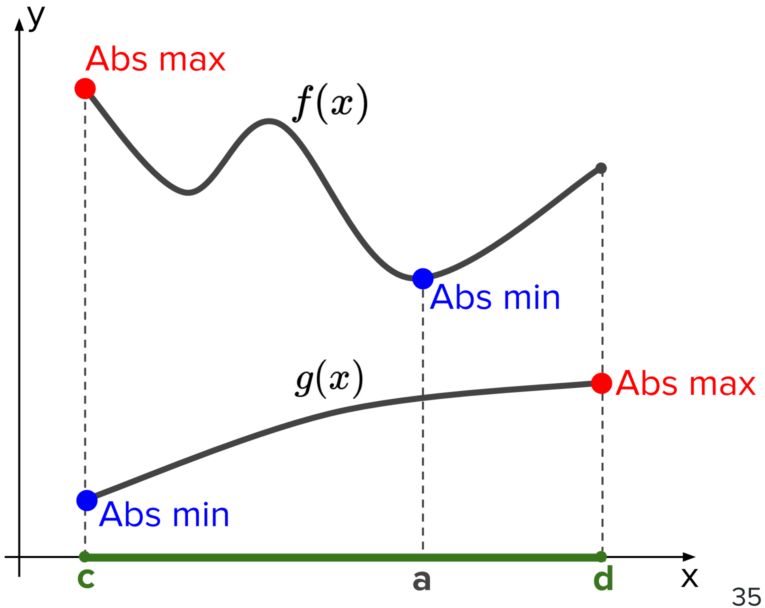


Absolute (Global) Extrema in a Closed Interval

For a function f(x) that is **continuous** on a closed interval, f(x) **must** attain an absolute max & min, either at a critical point or an endpoint. This statement is called the **extreme value theorem**.

For example, as shown in the graph, f(x) attains an absolute max at the endpoint (c, f(c)) and an absolute min at a stationary point (a, f(a)) in the closed and boundary interval [c, d].

And, g(x) attains its absolute min g(c) and absolute max g(d) at the two endpoints in the closed and boundary interval [c, d].



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Absolute (Global) Extrema in a Closed Interval

From the extreme value theorem, the following steps to find the absolute extrema on a closed and bounded interval [c, d] can be deduced.

1. Evaluate function values at all **critical points** in [c, d].
2. Evaluate function values f(c) and f(d) at **endpoints**.
3. Compare function values from steps 1 & 2 to obtain the absolute max & min.

Note that there is no need to identify if a critical point is a local max or min because a direct comparison between the function values reveals the absolute extrema easily.

Absolute (Global) Extrema in a Closed Interval

Example: Determine the absolute extrema of the function below.

$$f(x) = \sin x + \cos x,\ \ 0 \leq x \leq 2\pi$$

Since $f(x)$ is continuous in the closed interval $[0, 2\pi]$, EVT applies and the abs min and max exists in $[0, 2\pi]$

Step 1: Get critical points $f'(x) = \cos x - \sin x$

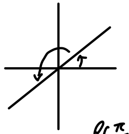
$f'(x) = 0$

$\cos x - \sin x = 0$

$\sin x = \cos x$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$



$f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f(\frac{5\pi}{4}) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

$f'(x)$ DNE → no sol^{n}

Step 2: Get $f(x)$ at endpoints

$f(0) = \sin 0 + \cos 0 = 1 = f(2\pi)$

Step 3: Compare

$(\frac{5\pi}{4}, -\sqrt{2})$ is the abs min

$(\frac{\pi}{4}, \sqrt{2})$ is the abs max

ANS: Abs max at (π/4, √2). Abs min at (5π/4, -√2).

Absolute (Global) Extrema in a Closed Interval

Exercise: Determine the absolute extrema of the function below.

$$f(x) = \begin{cases} x^2 - 1, & -3 \leq x < 0 \quad (f_1) \\ -\cos x, & 0 \leq x \leq \pi \quad (f_2) \end{cases} \quad [-3, \pi]$$

Both f_1 and f_2 are continuous in their own interval, so we just check at $x=0$ (interval transition)

$\lim_{x \to 0^-} x^2 - 1 = -1 \qquad \lim_{x \to 0^+} -\cos x = -1 = f(0)$

$\Rightarrow \lim_{x \to 0} f(x) = f(0)$, so $f(x)$ is cont. at $x=0$

so $f(x)$ is continuous in $[-3, pi]$ hence the EVT applies

Step 1: $f'(x) = \begin{cases} 2x, & -3 < x < 0 \\ \sin & 0 < x < \pi \end{cases}$

$f'(x) = 0$

$2x = 0$, $x = 0$ | $\sin x = 0$, $x = 0, \pi$

$f(0) = -1$, $f(\pi) = -\cos\pi = 1$

$f'(x)$ DNE

Only need to check at $x=0$

$\left.\begin{array}{l} \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} 2x = 0 \\ \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \sin x = 0 \end{array}\right\} =$

$\therefore f'(x)$ DNE has no sol^{n}

Step 2: $f(-3) = (-3)^2 - 1 = 8$

$f(\pi) = -\cos\pi = 1$

Step 3: Comparing

Abs max at $(-3, 8)$

Abs min at $(0, -1)$

ANS: Abs min at (0, -1). Abs max at (-3, 8).

Optimization

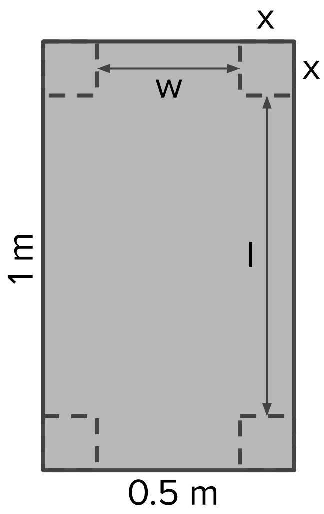
We shall now apply the techniques of finding extrema to **optimization problems**. Generally, the steps are

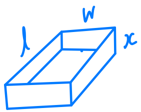
1) Interprete the problem and state the objective function to maximize or minimize.
2) State equations representing constraints if any.
3) Substitute the constraint equations into the objective function to obtain a single-variable objective function.
4) Evaluate the required extremum and answer the original problem.

In some cases, constraint equations are implicitly defined and therefore cannot be substituted into the objective function. We shall learn how to deal with such cases in Math 3.

Optimization

Example: A designer at Ikea is tasked to design an open box to be made from a piece of cardboard for containment of small loose parts. Each piece of cardboard is 1 m by 0.5 m, which is to be made into a box by cutting a square from each corner and folding up the sides. Determine the length x in order for the box to have the largest volume.





Step 1: Interpret the problem & state the objective function.

$$\text{Max: Volume}, V = lwx$$

Step 2: State constraint equation/s, if any.

$$l + 2x = 1 \rightarrow l = 1 - 2x$$
$$w + 2x = 0.5 \rightarrow w = 0.5 - 2x$$

$[0, 0.25]$

$x = 0$ (no box)

$2x = 0.5$ (max x you can cut)

Step 3: Substitute the constraint equations into the objective function.

$$\begin{aligned}\text{Max: Volume}, V(x) &= (1-2x)(0.5-2x)x \\ &= 4x^3 - 3x^2 + 0.5x\end{aligned}$$

And, notice that x must be in the closed interval [0, 0.25], where there is no box at the endpoints. Since V(x) is continuous, EVT can be applied.

Step 4: Evaluate required extremum & answer the problem.

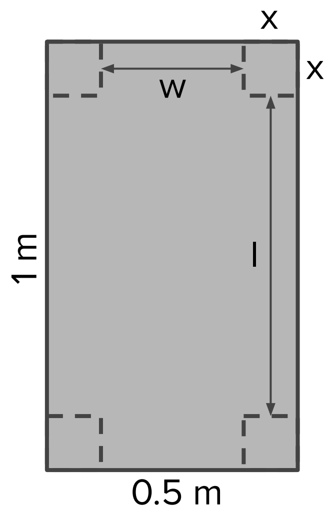
$$V'(x) = 12x^2 - 6x^{\cancel{2}} + 0.5 = 0$$

$$\rightarrow x = \frac{6 \pm \sqrt{36 - 4(12)(0.5)}}{24} = \frac{6 \pm \sqrt{12}}{24} = 0.106,\ \overbrace{0.394}^{>0.25}\ \text{(reject)}$$

$$\rightarrow V(0.106) = 0.024$$

$$\text{At endpoints, } V(0) = 0,\ V(0.25) = 0$$

Hence, the length x of 0.106 m will result in a box with the largest volume of 0.024 m^3.



Optimization

Exercise: An engineer at Nestle wants to know the dimensions of the cylindrical can for their condensed milk product which uses the least material. Given that the volume of each can must be 350 ml and the thickness of the material is uniform, determine the dimensions of the can.

Min: $\underbrace{\text{surface area}}_{\text{(material)}}$, $A = 2\pi r^2 + 2\pi r h$

Constraint: $V = \pi r^2 h = 350$

$\Rightarrow h = \dfrac{350}{\pi r^2}$

$A(r) = 2\pi r^2 + 2\cancel{\pi r}\left(\dfrac{350}{\cancel{\pi} r^{\cancel{2}}}\right)$

$= 2\pi r^2 + \dfrac{700}{r}$

$A'(r) = 4\pi r - \dfrac{700}{r^2} = 0$

$\dfrac{4\pi r^3 - 700}{\cancel{r^2}} = 0$

$\rightarrow r^3 = \dfrac{700}{4\pi}$

$r = \left(\dfrac{175}{\pi}\right)^{1/3}$

$A'(r)$ DNE $\rightarrow r^2 = 0$, $r = 0$ (na since Vol $= 0$)

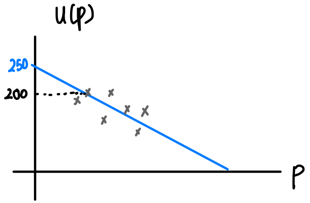
$A''(r) = 4\pi + \dfrac{1400}{r^3} > 0$ for $r > 0$

Since $A(r)$ is concave up (convex) for $r > 0$, $r = \left(\dfrac{175}{\pi}\right)^{1/3}$ is both the local and global min.

$\therefore$ min material at $d = 2r = 7.64$ cm & $h = \dfrac{350}{\pi r^2} = 7.64$ cm

ANS: Diameter = 7.64 cm, Height = 7.64 cm.

Optimization



Exercise: A business analyst in a car-rental company is tasked to maximize their revenue from a fleet of 200 cars during a peak month. From the data collected, a regression analysis reveals the relationship of rented units, U, and price/day, p, to be

→ price per day per unit

$$U(p) = 250 - p \quad \text{(units per month)}$$

Determine the price/day to set for the company to maximize their revenue during the peak month.

Step 1: Max : Revenue , $R = 30Up$

Step 2 : Constraint : $U(p) = 250 - p$ (Demand curve)

Step 3: $R(p) = 30(250-p)p = 30(250p - p^2)$

Step 4: $R'(p) = 30(250-2p)$

$R'(p) = 0$

$\cancel{30}(250-2p) = 0$

$2p = 250$

$p = 125$

$R'(p)$ DNE → no soln

$R''(p) = -60 < 0$ for all p

∴ At p = 125, there is a global max since R(p) is concave down everywhere.

$125

Hence, set the price/day to $125

U(125) = 125 units rented out

ANS: p = $125

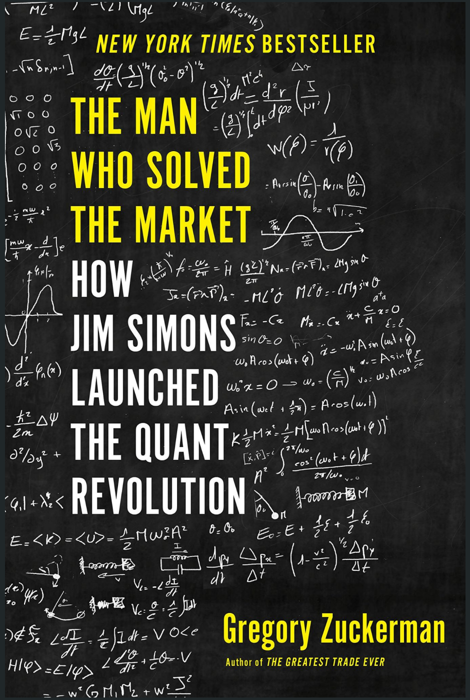
End of Topic 7

"I thought it was the coolest thing – what a life, to go out at two a.m. with friends and do math over coffee. It seemed like the world's greatest career."

Jim Simons
Founder of Renaissance Technology

Renaissance's flagship Medallion fund is famed for the best track record on Wall Street, returning more than 66% annualized before fees and 39% after fees over a 30-year span from 1988 to 2018.
https://en.wikipedia.org/wiki/Renaissance_Technologies

Application of math at the most profitable level.



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