Matriculation Number: ______________________

Circuit Theory

Exam:	EDE1201 / Examination	**Date:**	Monday 12th April, 2021
Examiner:	Dr.-Ing. Michael Joham	**Time:**	16:30 – 18:10

Working instructions

- This exam consists of **20 pages** with a total of **8 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 90 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: **maximally 5 two-sided DIN A4 cheat sheets**
- Subproblems marked by * can be solved without results of previous subproblems.
- **All** answers to be written on the spaces provided. Additional sheets are included at the end of the booklet. Other additional answer sheets will not be accepted.
- Please use only **blue** or **black** pens for writing.
- Laptops, calculators, mobile phones, smart watches, or any wireless devices are **not** allowed in the examination hall. Please **switch off** all electronic devices, put them into your bag, and close the bag.

At the end of the examination,

Please ensure that you have written your **Student ID** on each page on top right corner.

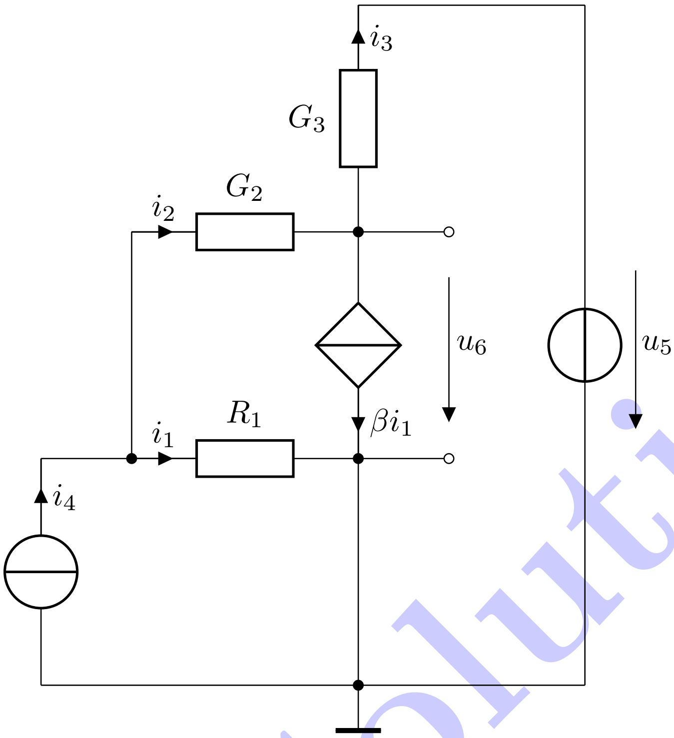
Failure to do so will mean that your work will not be identified.

The university reserves the right not to mark your script if you fail to follow these instructions.

Left room from ________ to ________ / Early submission at ________

Problem 1 Kirchhoff's Laws (6 credits)

Consider the following circuit.



0
1

a)* What is the number of nodes of the circuit?

$n = 4$ √

0
1

b)* Give the number of branches of the circuit.

$b = 6$

0
1
2

c) Find the node incidence matrix.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$ √√

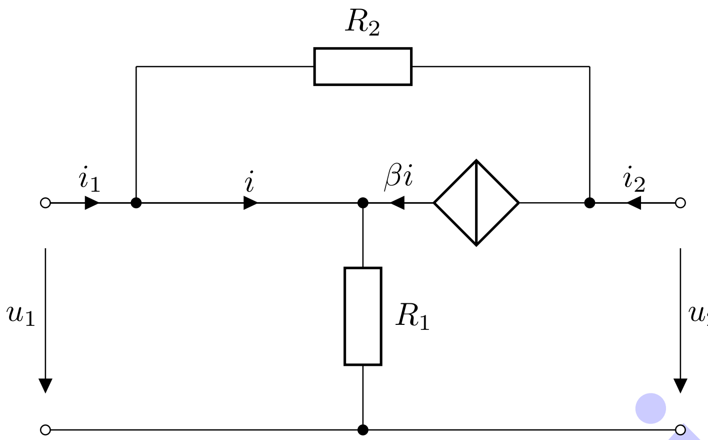
d) Determine the loop incidence matrix.

☐ 0 ☐ 1 ☐ 2

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \checkmark\checkmark$$

Problem 2 Two-Ports (16 credits)

Given is the following two-port $\mathcal{N}$.



Note that $R_1 > 0$ and $R_2 > 0$.

0 1 2 a)* Determine u_1 depending on i.

$u_1 = (1+\beta)iR_1$ ✓

0 1 2 b)* Find u_2 depending on i and i_2.

$u_2 = R_2(i_2 - \beta i) + u_1$ ✓
$u_2 = R_2(i_2 - \beta i) + (1+\beta)iR_1$ ✓

By appropriate analysis, it can be shown that $i = \frac{i_1+i_2}{1+\beta}$.

c) Give the resistance matrix $\boldsymbol{R}_{\mathcal{N}}$ of the two-port $\mathcal{N}$.

$$\boldsymbol{R}_{\mathcal{N}} = \begin{bmatrix} R_1 & R_1 \\ R_1 - \frac{\beta R_2}{1+\beta} & R_1 + \frac{R_2}{1+\beta} \end{bmatrix} \surd\surd\surd$$

d) Find β such that the two-port $\mathcal{N}$ is reciprocal.

For $\mathcal{N}$ to be reciprocal, $\boldsymbol{R}_{\mathcal{N}} = \boldsymbol{R}_{\mathcal{N}}^{\mathrm{T}}$ must hold.
Therefore, $R_1 = R_1 - \frac{\beta R_2}{1+\beta}$
$\beta = 0$ √

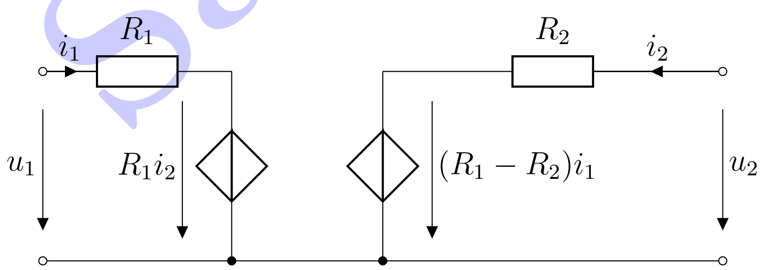
The alternative two-port $\mathcal{M}$ has the following resistance matrix

$$\boldsymbol{R}_{\mathcal{M}} = \begin{bmatrix} R_1(1+\frac{1}{5+\beta}) & R_1 + \frac{R_2}{1+\beta} \\ R_1 - \frac{\beta R_2}{1+\beta} & R_2 \end{bmatrix}.$$

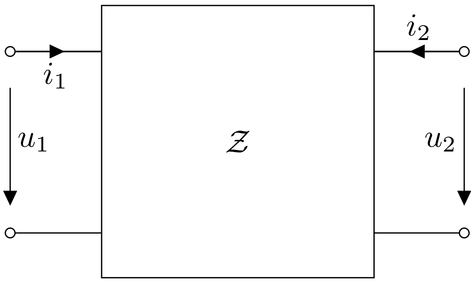
e)* Give the resistance matrix $\boldsymbol{R}_{\infty}$ of $\mathcal{M}$ for the limit $\beta \to \infty$.

$$\boldsymbol{R}_{\infty} = \begin{bmatrix} R_1 & R_1 \\ R_1 - R_2 & R_2 \end{bmatrix} \surd$$

f) Draw the equivalent circuit diagram for the resistance matrix $\boldsymbol{R}_{\infty}$.



Now, consider the two-port $\mathcal{Z}$.



The transmission matrix of $\mathcal{Z}$ is given by

$$\boldsymbol{A}_{\mathcal{Z}} = \begin{bmatrix} 1 & R_2 \\ -\frac{1}{R_1} & 1 \end{bmatrix}.$$

0
1
2

g)* Is the two-port $\mathcal{Z}$ current-controlled, i.e., does its resistance matrix exist? Justify your answer.

Yes, since $a_{21} \neq 0$, the resistance matrix exists. √√

0
1
2
3

h)* Find a two-port matrix for the two-port $\mathcal{Z}^{\mathrm{d}}$ which is dual to $\mathcal{Z}$. The duality constant is denoted by R_{d}.

From the transmission representation of $\mathcal{Z}$:

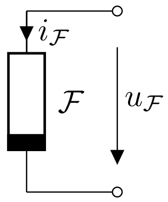
$R_{\mathrm{d}} i_1^{\mathrm{d}} = R_{\mathrm{d}} i_2^{\mathrm{d}} - R_2 \frac{u_2^{\mathrm{d}}}{R_{\mathrm{d}}}$ √

$\frac{u_1^{\mathrm{d}}}{R_{\mathrm{d}}} = -\frac{1}{R_1} R_{\mathrm{d}} i_2^{\mathrm{d}} - \frac{u_2^{\mathrm{d}}}{R_{\mathrm{d}}}$ √

$\Rightarrow \boldsymbol{A}_{\mathcal{Z}^{\mathrm{d}}} = \begin{bmatrix} -1 & \frac{R_{\mathrm{d}}^2}{R_1} \\ -\frac{R_2}{R_{\mathrm{d}}^2} & -1 \end{bmatrix}$ √

Problem 3 Non-Linear One-Port (6 credits)

Given is the non-linear one-port $\mathcal{F}$.



The characteristic of $\mathcal{F}$ can be written as

$$i_{\mathcal{F}} = g(u_{\mathcal{F}}) = \frac{2}{3u_0 R}\left(u_{\mathcal{F}} - \frac{u_0}{2}\right)^2 - \frac{u_0}{R}$$

where $u_0 > 0$ and $R > 0$ are constants.
Due to the connection of $\mathcal{F}$, its operating point voltage is known as $U_{\mathcal{F}} = 2u_0$.

a)* Find the operating point current $I_{\mathcal{F}}$ of $\mathcal{F}$. 0 1

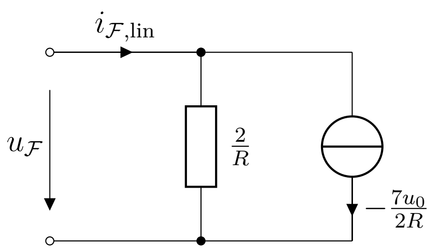
$I_{\mathcal{F}} = g(U_{\mathcal{F}}) = \frac{2}{3u_0R}(2u_0 - \frac{u_0}{2})^2 - \frac{u_0}{R} = \frac{2}{3u_0R}\frac{9u_0^2}{4} - \frac{u_0}{R} = \frac{u_0}{2R}$ $\checkmark$

b) Linearize $\mathcal{F}$ in the given operating point with $U_{\mathcal{F}} = 2u_0$, that is, find $i_{\mathcal{F},\text{lin}}$. 0 1 2 3

slope: $G_{\mathcal{F}} = \frac{\mathrm{d}i_{\mathcal{F}}}{\mathrm{d}u_{\mathcal{F}}}\Big|_{u_{\mathcal{F}}=U_{\mathcal{F}}} = \frac{4}{3u_0R}\left(u_{\mathcal{F}} - \frac{u_0}{2}\right)\Big|_{u_{\mathcal{F}}=2u_0} = \frac{2}{R}$ $\checkmark\checkmark$

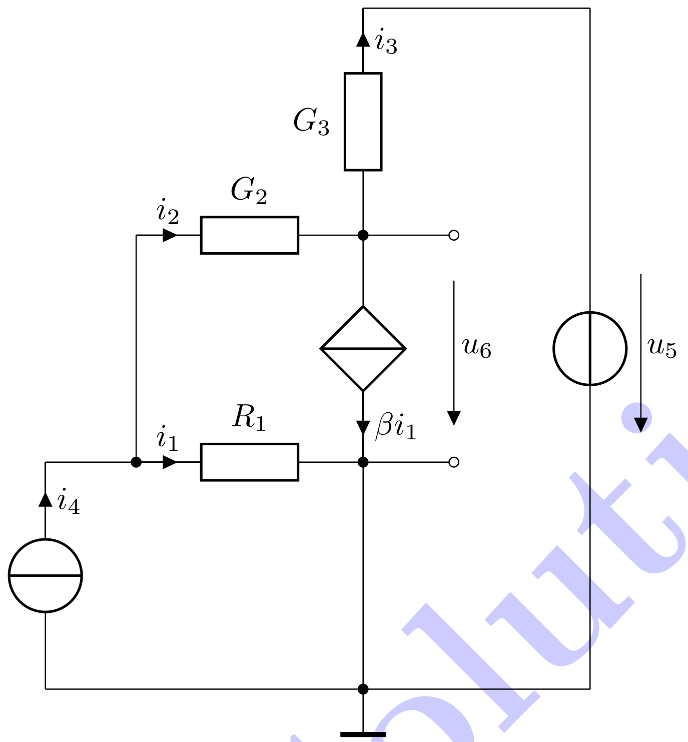
$i_{\mathcal{F},\text{lin}} = I_{\mathcal{F}} + G_{\mathcal{F}}(u_{\mathcal{F}} - U_{\mathcal{F}}) = \frac{u_0}{2R} + \frac{2}{R}(u_{\mathcal{F}} - 2u_0) = -\frac{7u_0}{2R} + \frac{2}{R}\,u_{\mathcal{F}}$ $\checkmark$

c) Draw an equivalent circuit diagram for the linearized one-port with $i_{\mathcal{F},\text{lin}} = g_{\text{lin}}(u_{\mathcal{F}})$. Give the element values depending on u_0 and R. 0 1 2



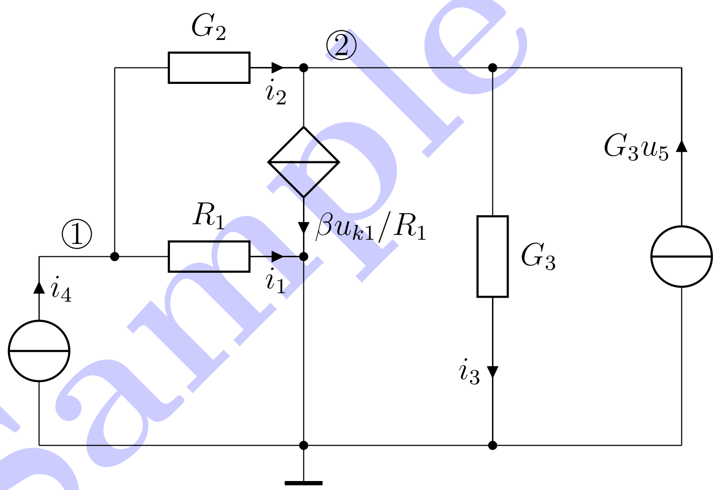
Problem 4 Nodal Analysis (10 credits)

The circuit of Problem 1 is investigated with the help of the nodal analysis.



0 1 2 3

a)* Prepare the circuit for the nodal analysis. To this end, express i_1 by the node voltages and perform a source transform. Label the nodes.



0 1

b)* Express u_6 depending on the node voltages.

$u_6 = u_{k2}$ √

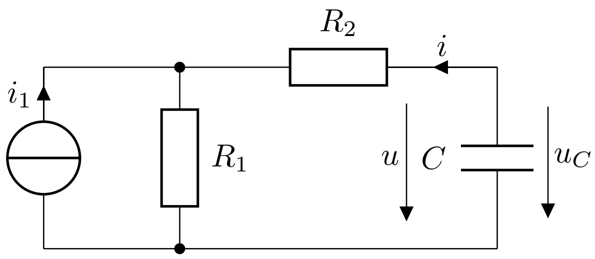
c) Formulate the equation system of the nodal analysis in the form $\boldsymbol{Y}_k \boldsymbol{u}_k = \boldsymbol{i}_q$.

0
1
2
3
4
5
6

$$\begin{bmatrix} \frac{1}{R_1} + G_2 & -G_2 \\ -G_2 + \frac{\beta}{R_1} & G_2 + G_3 \end{bmatrix} \begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix} = \begin{bmatrix} i_4 \\ G_3 u_5 \end{bmatrix}$$

Problem 5 First-Order Circuit (7 credits)

Consider the following first-order circuit with R_1,R_2,$C > 0$.



0 1 2 3

a)* Find the representation of the resistive part as a linear source.

open circuit measurment: $u_0 = R_1 i_1$ $\checkmark$

short circuit measurement (current divider rule): $i_0 = \frac{\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}} i_1 = \frac{R_1}{R_1+R_2} i_0$ $\checkmark$

$u = u_0 + \frac{u_0}{i_0} i = R_1 i_1 + (R_1 + R_2) i$ $\checkmark$

0 1

b) What is the time constant τ of the circuit?

$\tau = (R_1 + R_2)C$ $\checkmark$

0 1

c) Justify why the circuit is stable.

Since R_1,R_2,$C > 0$, also $\tau > 0$. $\checkmark$

Suppose that the excitation is constant, i.e., $i_1(t) = I_1$. Additionally, note that $u_C(0) = (R_1+R_2)I_1$. Use simply τ for the time constant.

0 1 2

d) Find $u_C(t)$ for $t \geq 0$.

$u_C(0) = (R_1 + R_2) I_1$

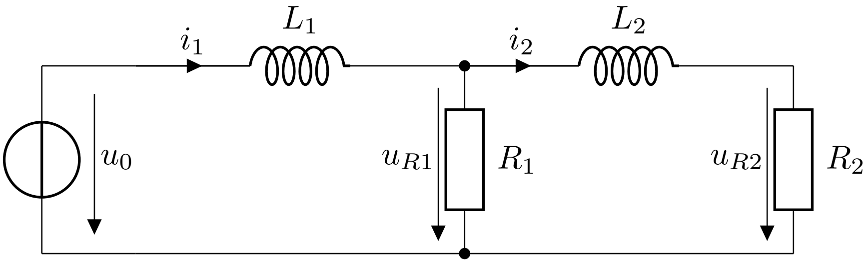
$u_{C,\infty} = R_1 I_1$ (open circuit voltage) $\checkmark$

$u_C(t) = u_{C,\infty} + (u_C(0) - u_{C,\infty})\, \mathrm{e}^{-\frac{t}{\tau}}$

$u_C(t) = R_1 I_1 + R_2 I_1\, \mathrm{e}^{-\frac{t}{\tau}}$ $\checkmark$

Problem 6 Linear Second-Order Circuit (27 credits)

In this problem, the following linear circuit with the independent constant voltage source u_0 and the two inductors L_1 and L_2 is investigated.



Note that u_{R1} is the output of the circuit.

a)* Give the state variables of the circuit.

0 1

i_1, i_2

b)* Determine the state equations of the circuit.

0 1 2 3 4 5

KVL: $u_0 = u_1 + u_{R1}$
KCL: $\frac{1}{R_1} u_{R1} = i_1 - i_2$
KVL: $u_{R1} = u_2 + R_2 i_2$
$u_1 = -R_1(i_1 - i_2) + u_0$
$u_2 = R_1(i_1 - i_2) - R_2 i_2$
$\dot{i}_1 = -\frac{R_1}{L_1} i_1 + \frac{R_1}{L_1} i_2 + \frac{1}{L_1} u_0$
$\dot{i}_2 = \frac{R_1}{L_2} i_1 - \frac{R_1+R_2}{L_2} i_2$

c) What are the state matrix and the input vector for the circuit?

0 1 2

$$\boldsymbol{A} = \begin{bmatrix} -\frac{R_1}{L_1} & \frac{R_1}{L_1} \\ \frac{R_1}{L_2} & -\frac{R_1+R_2}{L_2} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix}$$

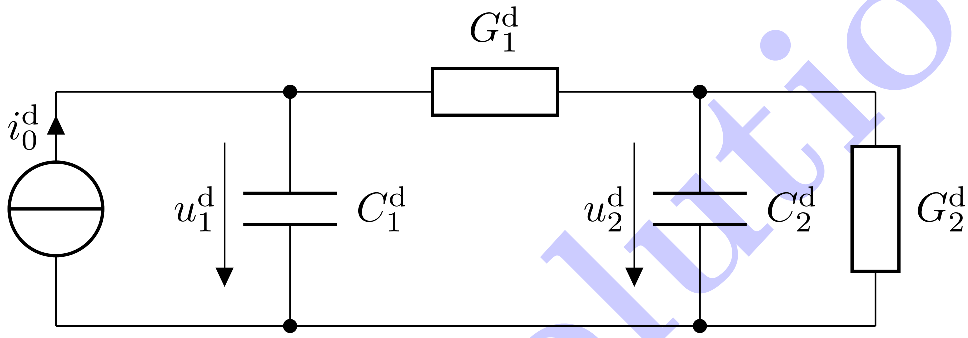
0 1 2

d) Give the output vector and the feedthrough if u_{R1} is the output.

$\boldsymbol{c}^{\mathrm{T}} = [R_1, -R_1]$ √
$d = 0$ √

0 1 2 3

e)* Draw the circuit which is dual to the given circuit.



For a particular choice of the element values, the following normalized state matrix can be obtained

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 \\ -3 & -5 \end{bmatrix}.$$

0 1 2 3

f)* Determine the eigenvalues of this state matrix.

$$\det(\boldsymbol{A} - \lambda \mathbf{1}) = \det \begin{bmatrix} -1-\lambda & 1 \\ -3 & -5-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 8 = 0 \ \checkmark$$
$$\lambda_{1/2} = -3 \pm \sqrt{9-8} = -3 \pm 1$$
$$\lambda_1 = -2 \quad \text{und} \quad \lambda_2 = -4 \ \checkmark\checkmark$$

0 1

g) Give the name of the equilibrium.

stable node √

h) Find the eigenvectors of the given state matrix. 0 1 2

$$[\boldsymbol{A} - \lambda_1 \mathbf{1}]\boldsymbol{q}_1 = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \boldsymbol{q}_1 = \mathbf{0} \quad \boldsymbol{q}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \surd$$

$$[\boldsymbol{A} - \lambda_2 \mathbf{1}]\boldsymbol{q}_2 = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \boldsymbol{q}_2 = \mathbf{0} \qquad \boldsymbol{q}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \surd$$

From now on, $u_0 = 0$. Correspondingly, the state equations read as

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -3x_1 - 5x_2.$$

i)* What is the equilibrium $\boldsymbol{x}_\infty$ of the circuit for $u_0 = 0$? 0 1 2

$\dot{x}_1 = \dot{x}_2 = 0$ $\surd$
$\boldsymbol{x}_\infty = \mathbf{0}$ $\surd$

For a different second-order circuit, the eigenvalues and eigenvectors can be written as

$$\lambda_1 = -4 \qquad \lambda_2 = 2$$

$$\boldsymbol{q}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \boldsymbol{q}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The excitation is zero. In other words, $v(t) = 0$.

j)* Why is this circuit unstable? 0 1

Since $\lambda_2 = 2 > 0$. $\surd$

k)* Give the general expression of the state vector $\boldsymbol{x}(t)$ of the circuit with the given eigenvalues and eigenvectors. 0 1 2

$$\boldsymbol{x}(t) = \xi_{01}\, \mathrm{e}^{-4t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \xi_{02}\, \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \surd\surd$$

Due to $v(t) = 0$, the output equation is given by

$$y(t) = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}(t).$$

0 1 2

l)* Determine a non-trivial output vector $\boldsymbol{c}^{\mathrm{T}}$, such that $y(t)$ converges to zero.

The eigenmode with the eigenvalue $\lambda_2 = 2$ must be suppressed by $\boldsymbol{c}^{\mathrm{T}}$. Therefore, $\boldsymbol{c}$ must be orthogonal to $\boldsymbol{q}_2$, $\surd$
e.g., $\boldsymbol{c}^{\mathrm{T}} = [1, -1]$ $\surd$

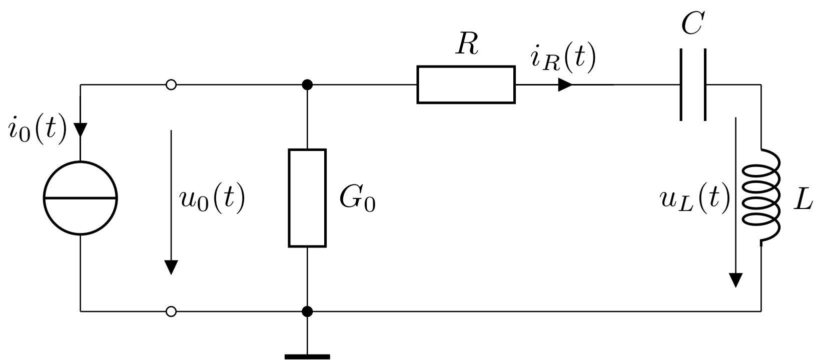
0 1

m) What is $y(t)$ for this particular choice for $\boldsymbol{c}^{\mathrm{T}}$?

$y(t) = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{q}_1 \xi_{01}\, \mathrm{e}^{\lambda_1 t} = 4\xi_{01}\, \mathrm{e}^{-4t}$ $\surd$

Problem 7 Complex Phasor Analysis (13 credits)

Given is following circuit with $G_0, R, C, L > 0$.



The current $i_0(t)$ is sinusoidal with the angular frequency ω.

a)* Give the relationship between $u_L(t)$ and $i_R(t)$.

0
1

$u_L(t) = L\dot{i}_R(t)$ ✓

Let I_R be the phasor for $i_R(t)$ and U_L the phasor for $u_L(t)$.

b)* Find U_L depending on I_R.

0
1

$U_L = \mathrm{j}\omega L I_R$ ✓

c) Determine the phasor U_0 depending on I_R.

0
1
2

$U_0 = RI_R + \frac{1}{\mathrm{j}\omega C} I_R + \mathrm{j}\omega L I_R$ ✓✓

0 1 2

d) What is, therefore, I_0 depending on U_0?

KCL: $I_0 + G_0 U_0 + I_R = 0$ $\checkmark$

$I_0 + G_0 U_0 + \frac{1}{R + \frac{1}{\mathrm{j}\omega C} + \mathrm{j}\omega L} U_0 = 0$

$I_0 = -G_0 U_0 - \frac{\mathrm{j}\omega C}{1 + \mathrm{j}\omega RC + (\mathrm{j}\omega)^2 LC} U_0$ $\checkmark$

For the voltage phasor U_0 depending on the phasor I_0 of the current source, it can be obtained that

$$U_0 = -\frac{1}{G_0} \frac{1 + \mathrm{j}\,\omega/\omega_1 - \omega^2 LC}{1 + \mathrm{j}\,\omega/\omega_2 - \omega^2 LC} I_0$$

with the constants $\omega_1, \omega_2 > 0$, where $\omega_1 \neq \omega_2$.

0 1 2

e)* Find the complex power of the source depending on I_0.

$P = \frac{1}{2} U_0 I_0^*$ $\checkmark$

$P = -\frac{1}{2} \frac{1}{G_0} \frac{1 + \mathrm{j}\,\omega/\omega_1 - \omega^2 LC}{1 + \mathrm{j}\,\omega/\omega_2 - \omega^2 LC} |I_0|^2$ $\checkmark$

Now, $\omega, G_0, R, L > 0$ are constant.

0 1 2 3

f) Determine C depending on ω, G_0, R, and L such that the blind power of the source is zero.

$P = -\frac{1}{2} \frac{1}{G_0} \frac{(1 + \mathrm{j}\,\omega/\omega_1 - \omega^2 LC)(1 - \mathrm{j}\,\omega/\omega_2 - \omega^2 LC)}{(1 - \omega^2 LC)^2 + (\omega/\omega_2)^2} |I_0|^2$ $\checkmark$

$P_\mathrm{B} = \mathrm{Im}\{P\} = -\frac{1}{2} \frac{1}{G_0} \frac{(\omega/\omega_1 - \omega/\omega_2)(1 - \omega^2 LC)}{(1 - \omega^2 LC)^2 + (\omega/\omega_2)^2} |I_0|^2 = 0$ $\checkmark$

$1 - \omega^2 LC = 0$

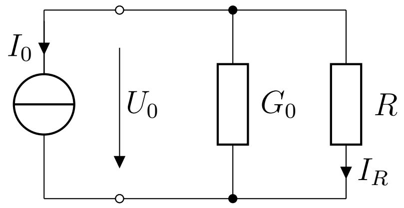
$C = \frac{1}{\omega^2 L}$ $\checkmark$

For the corresponding choice for C, we get

$$P = -\frac{1}{2}\,\frac{1}{G_0}\,\frac{R}{R + 1/G_0}\,|I_0|^2.$$

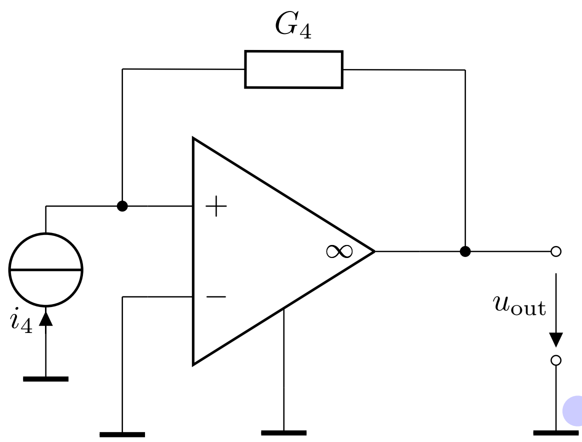
g)* Draw the corresponding equivalent circuit diagram. Mark U_0, I_0 and I_R.

0
1
2



Problem 8 **Op-Amp Circuit (5 credits)**

Consider the following Op-Amp circuit.



a)* Find the voltage u_{out} depending on i_4 and U_{sat} if the Op-Amp is operated in the linear region.

$u_{\text{out}} = -\frac{i_4}{G_4}$ √√

Now, we assume that the Op-Amp is in the positive saturation. The saturation voltage is denoted by U_{sat}.

b)* Give u_{out} depending on i_4 and U_{sat}.

$u_{\text{out}} = U_{\text{sat}}$ √

c) Determine the range for i_4 such that the Op-Omp is in positive saturation.

$u_{\text{d}} = U_{\text{sat}} + \frac{i_4}{G_4}$ √
With $u_{\text{d}} > 0$, we can infer that $i_4 > -U_{\text{sat}} G_4$. √

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.