Topic 3
Vectors & Geometry

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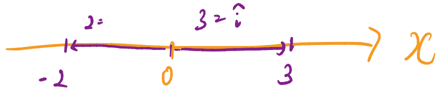
Outline

- Definition of a Vector
- Algebraic Operations of Vectors
- Unit and Basis Vectors
- Dot and Cross Products
- Complex Numbers
- Vector Equations of Lines & Planes

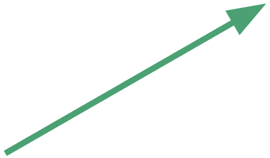
A Vector

A scalar is also a "vector"



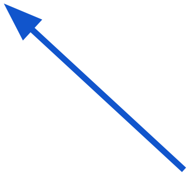
A vector is defined as an entity with a **magnitude** and a **direction**.

Examples:



Pushing force of 220 Newtons at an angle of 30° from the horizontal

Gravitational acceleration of 9.81 m/s^2 towards the center of the earth



Wind velocity of 3 m/s at a bearing of N45°W

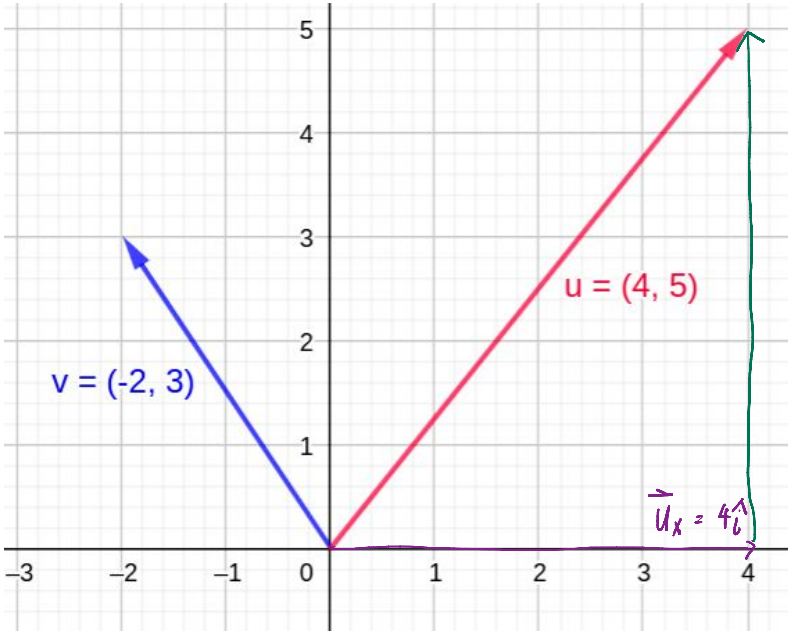
Vector Notation

A vector when defined in a coordinate system can have the following notations:

$$\vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -2\hat{i} + 3\hat{j} = (-2,\ 3)$$

$$\vec{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4\hat{i} + 5\hat{j} = (4,\ 5)$$

Matrix notation (column vector) | Unit vector notation | Ordered set notation



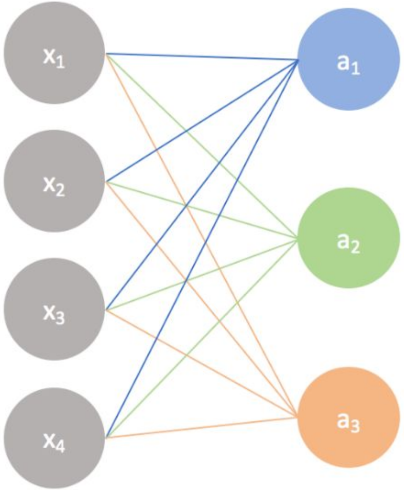
Note that a vector is usually indicated by:

1. An arrow above a letter, or
2. A bold letter

Representations of a Vector

A vector need not only represent a quantity with a ‘physical’ magnitude and direction, like a force or velocity.

In computer science, a vector is commonly treated as a list (of elements).



Neural Network in Machine Learning
(https://www.jeremyjordan.me/intro-to-neural-networks/)

$$\begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} = \begin{bmatrix} w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + b \\ w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + b \\ w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + b \end{bmatrix} \xrightarrow[\textit{activation}]{} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Vector of inputs Vector of biases Vector of outputs

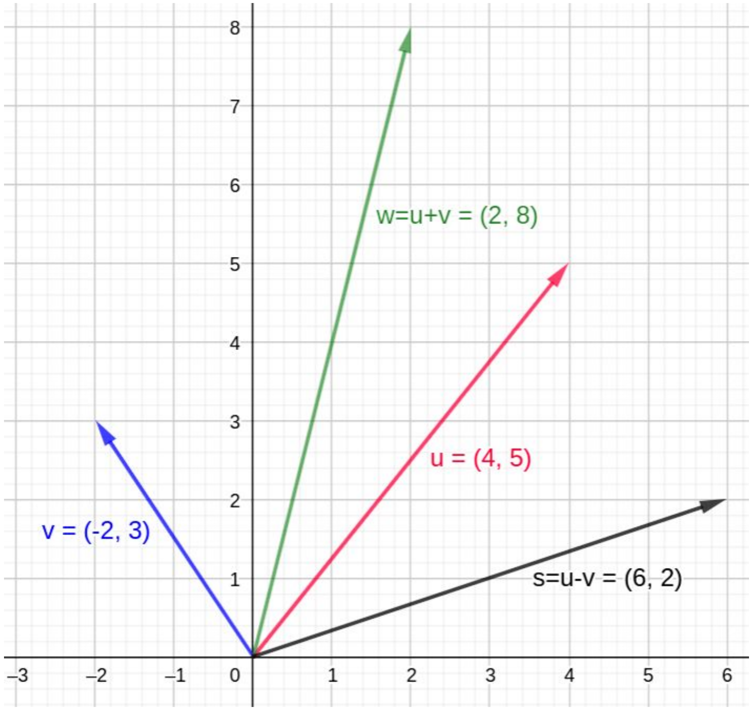
Vector Addition / Subtraction

Since a vector is a matrix, addition and subtraction follows the **same rules for matrices.**

$$\vec{w} = \vec{u} + \vec{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\vec{s} = \vec{u} - \vec{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

The negative of a vector reverses its direction but preserves its magnitude.

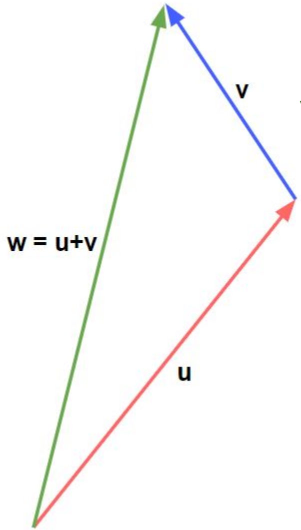


Vector Addition / Subtraction

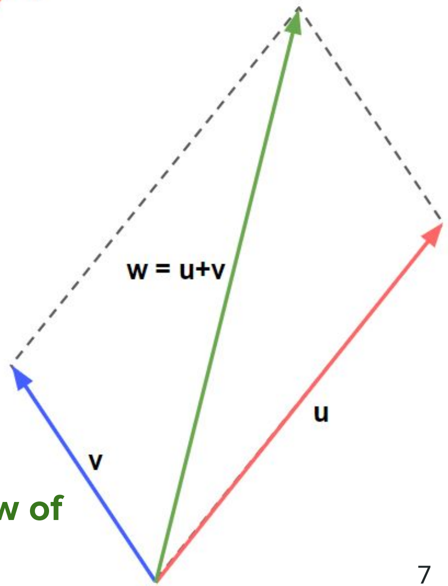
Graphically, vectors are added by one of the following ways illustrated.

The top diagram illustrates the **triangle law of vector addition**, which states that if two vectors are represented by the two adjacent sides of a triangle taken in order, then their resultant is the closing side of the triangle taken in the reverse order.

Alternatively, as illustrated by the bottom diagram, given two vectors that are drawn from a point and hence represents the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the parallelogram drawn from the same point. This is called the **parallelogram law of vector addition**.



Triangle law of vector addition



Parallelogram law of vector addition

Magnitude & Direction

Using Pythagoras theorem and simple trigonometry, we can define for:

$$\vec{V} = \begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

Magnitude (Length):

$$\left|\vec{V}\right| = \sqrt{{V_x}^2 + {V_y}^2}$$

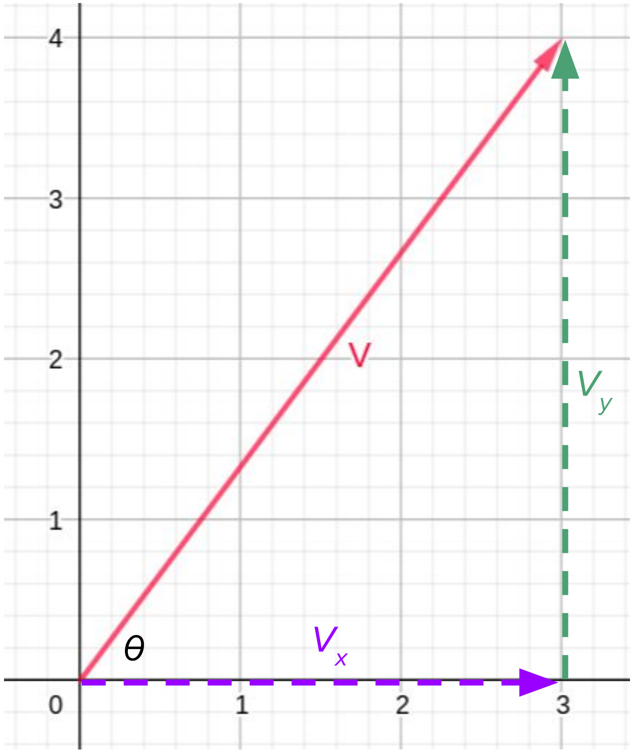
Direction (Angle):

$$\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)$$

x & y components:

$$V_x = \left|\vec{V}\right| \cos\theta$$

$$V_y = \left|\vec{V}\right| \sin\theta$$



Magnitude & Direction

Example: Determine the components, magnitude and direction of vectors **u** and **v**.

$u_x = 4, \quad u_y = 5 \qquad \vec{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$|\vec{u}| = \sqrt{4^2 + 5^2} = \sqrt{41}$

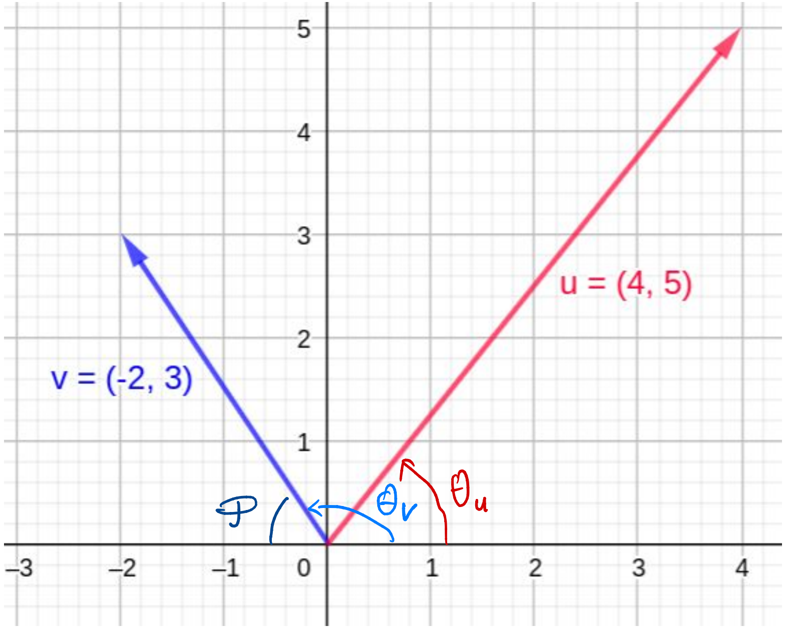
$\theta = \tan^{-1}\left(\frac{5}{4}\right)$

$= 0.896 \text{ rad}$

$v_x = -2, \quad v_y = 3 \qquad \vec{V} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$|\vec{V}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$

$\theta_v = \pi - \varphi = \pi - \tan^{-1}\left(\frac{3}{2}\right) = 2.16 \text{ rad}$



ANS: |**u**| = √(41), |**v**| = √(13), θ_u = 0.896 rad, θ_v = 2.16 rad

Vectors in Higher Dimensions

Similar rules apply to vectors in 3D and above. In 3D, two angular coordinates are required to specify the direction of the vector.

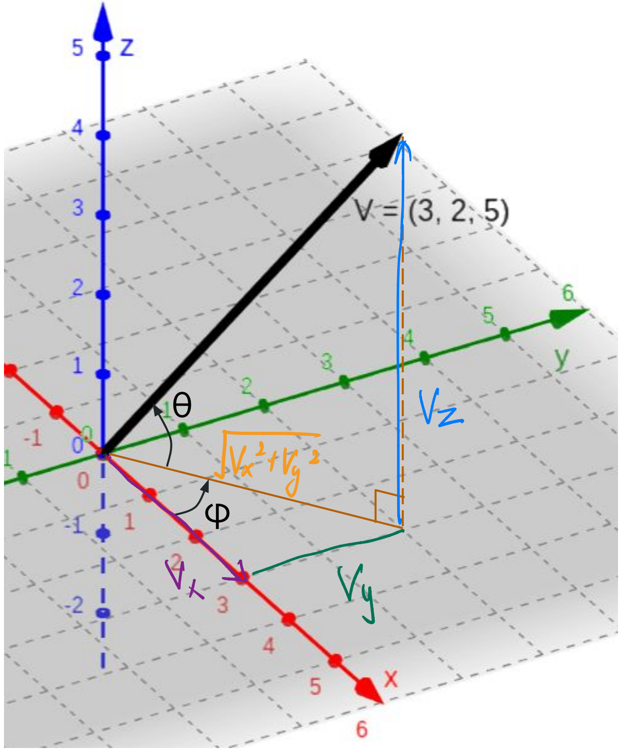
Exercise: Determine the components, magnitude and direction of vector **V**. $V_x = 3$, $V_y = 2$, $V_z = 5$

$$|\vec{V}| = \sqrt{\left(\sqrt{V_x^2 + V_y^2}\right)^2 + V_z^2} = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$= \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38}$$

$$\varphi = \tan^{-1}\left(\frac{V_y}{V_x}\right) = \tan^{-1}\left(\frac{2}{3}\right) = 0.588 \text{ rad}$$

$$\theta = \tan^{-1}\left(\frac{V_z}{\sqrt{V_x^2 + V_y^2}}\right) = \tan^{-1}\left(\frac{5}{\sqrt{13}}\right) = 0.946 \text{ rad}$$



ANS: |**V**| = √(38), φ = 0.588 rad, θ = 0.946 rad

Scaling a Vector

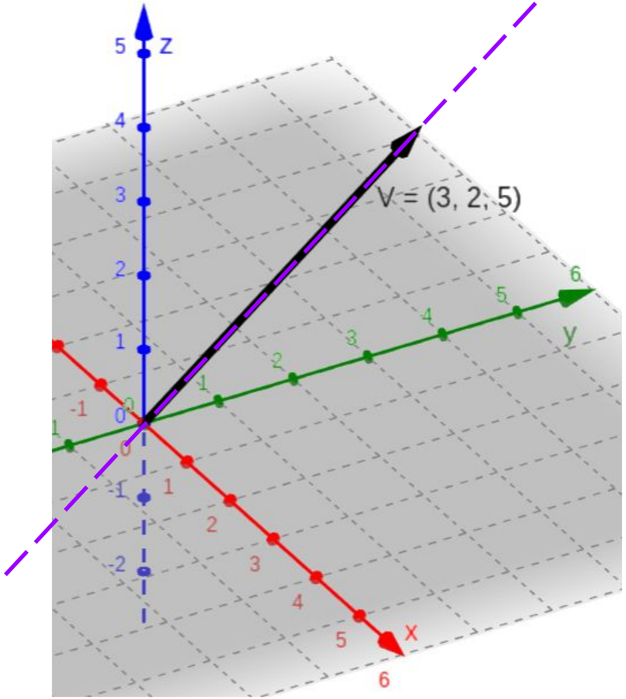
Multiplying a vector by a scalar changes the length and/or direction along a line collinear with the vector (along its linear span).

$$\lambda\vec{V} = \lambda\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \lambda V_x \\ \lambda V_y \\ \lambda V_z \end{bmatrix}$$

The scalar, λ, scales the vector V according to:

$|\lambda| > 1$ lengthens V,
$|\lambda| < 1$ shortens V and
$\lambda < 0$ flips the direction of V.

Notice that the same rule of scalar multiplication for matrices applies, since a vector is also a matrix.



Unit Vectors

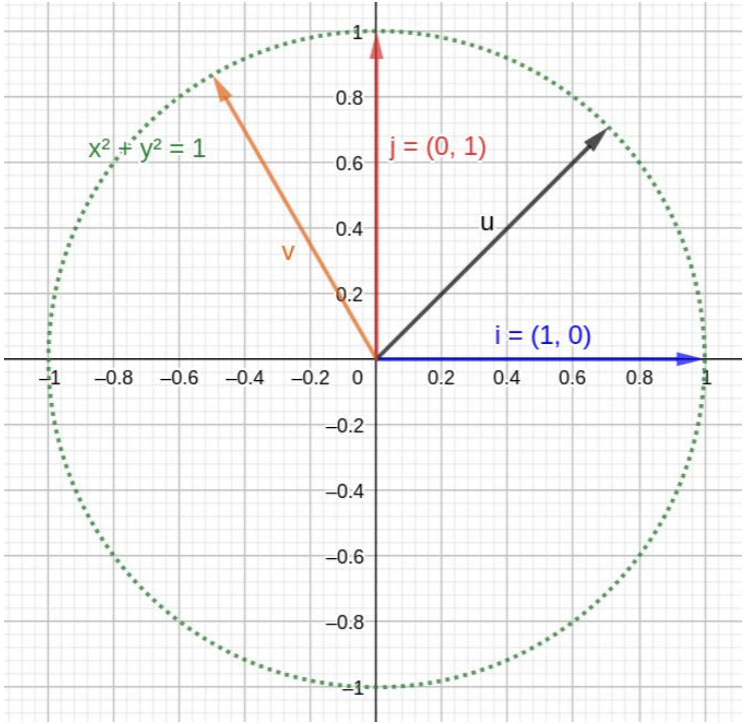
A unit vector is a vector with a magnitude / length of 1.

Examples are:

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \rightarrow |\vec{V}| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1 \;//$$



Exercise: Show that the above vectors are all unit vectors, using the Pythagoras theorem.

Exercise

Determine if the vectors below are unit vectors.

$$\vec{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Yes, since

$|\vec{p}| = 1$

$$\vec{q} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

no, since

$|\vec{q}| = \sqrt{3} > 1$

$$\vec{r} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Yes, since

$|\vec{r}| = 1$

As long as one of the component > 1

$$\vec{s} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 2 \\ 0 \end{bmatrix} \quad 2 \to s_z$$

no, since

$s_z > 1$

Unit Vector Along a Vector's Direction

Given any vector **V**, a unit vector in the same direction can be found by:

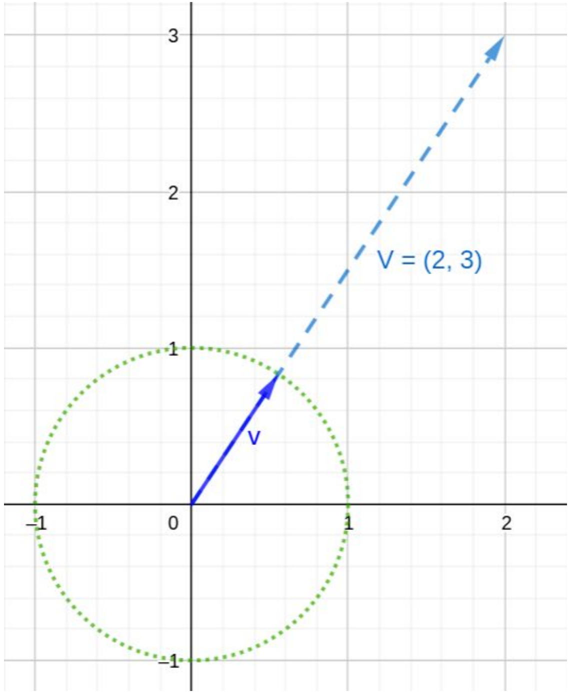
$$\hat{v} = \frac{\vec{V}}{|\vec{V}|}$$

Example:

For $\vec{V} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\quad |\vec{V}| = \sqrt{2^2 + 3^2} = \sqrt{13}$

So, $\hat{v} = \frac{1}{|\vec{V}|}\vec{V} = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Verify for yourself that |**v**| is 1.



Basis Vectors

Span $\mathbb{R}^3 = \vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$

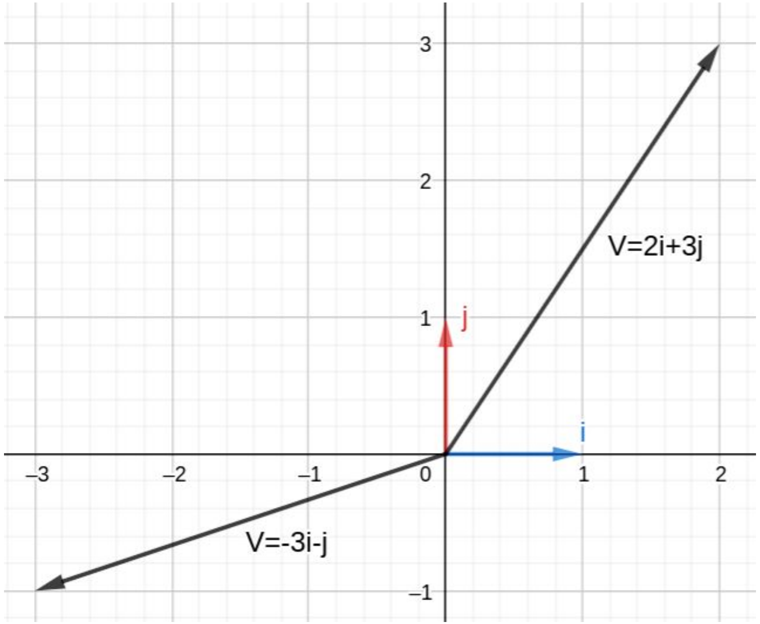
Unit vectors, **i** and **j**, are also known as basis vectors for the plane ($\mathbb{R}^2$). This is because any point in the plane can be reached by a linear combination of **i** and **j**.

$$\vec{V} = a\hat{i} + b\hat{j}$$

By varying scalars a and b, **V** can point to any coordinate in the plane (or span $\mathbb{R}^2$). Notice that **i** and **j** must not be parallel in order for this to be true.

In fact, any set of two **non-parallel** vectors that can span $\mathbb{R}^2$ can be used as basis vectors.

The set **{i, j}** is just one of many. But since they are so commonly used, they are also known as the *standard basis*.



Basis Vectors

$$\left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} : \vec{V}_1 = -2\vec{u}_1 - 3\vec{u}_2$$

$$\underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{\vec{u}_1}, \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\vec{u}_2}$$

$$= -2\begin{pmatrix} -1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} //$$

Exercise: Draw another possible set of basis vectors below and update the vectors indicated by **V** using the new basis.

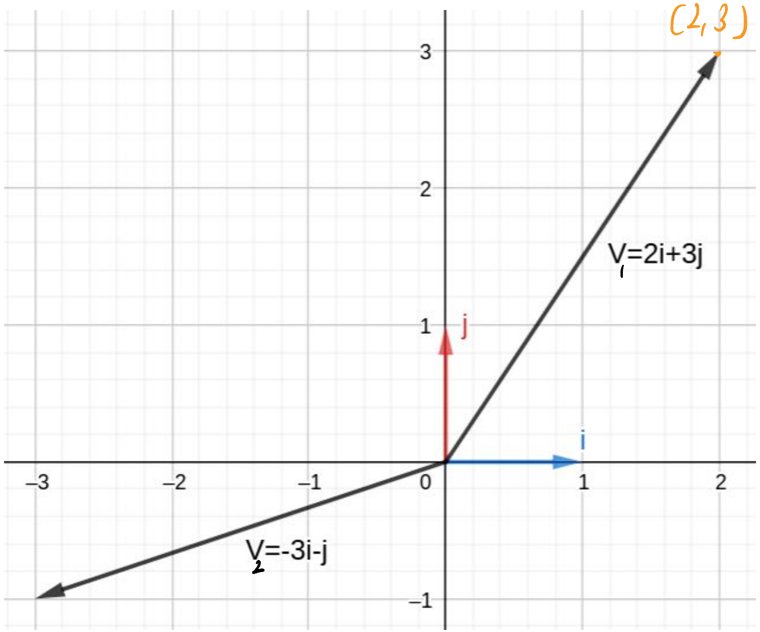
Basis set: $\{\vec{V}_1, \vec{V}_2\}$

$$\vec{V}_1 = k_1\vec{V} + k_2\vec{V}_2 = (1)\vec{V}_1 + 0\vec{V}_2$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} //$$

$$\vec{V}_2 = k_1\vec{V}_1 + k_2\vec{V}_2 = 0\vec{V}_1 + (1)\vec{V}_2$$

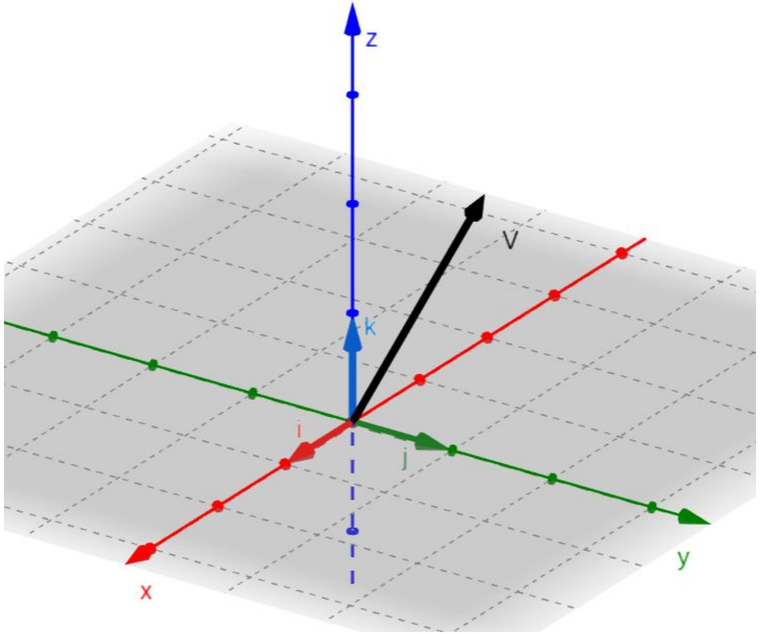
$$= \begin{pmatrix} -3 \\ -1 \end{pmatrix} //$$



Basis Vectors

In a similar context, {**i**, **j**, **k**} represents the set of *standard basis* vectors for a three-dimensional space ($\mathbb{R}^3$).

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Exercise: Determine the *standard basis* for $\mathbb{R}^n$.

$$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \hat{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Complex Numbers

$i = \sqrt{-1}$

A complex number, $z = x + iy$, can be represented on the complex plane as a vector, where the basis is {1, i}. x is called the real part of z, Re{z}, and y is called the imaginary part, Im{z}. From the figure, a complex number can also be written in polar form as

$$z = r(\cos\varphi + i\sin\varphi)$$

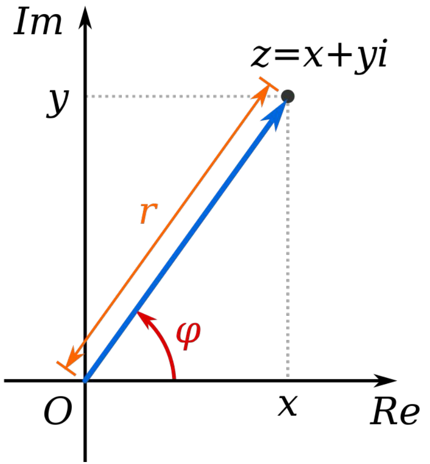
where r and φ are called the absolute value and argument of z respectively.

$r = |z|$

$\varphi = \arg(z)$

Using Taylor series (Topic 7), an exponential form of z can be derived from the polar form to be

$$z = re^{i\varphi}$$



https://en.wikipedia.org/wiki/Complex_number#/media/File:Complex_number_illustration_modarg.svg

Dot Product

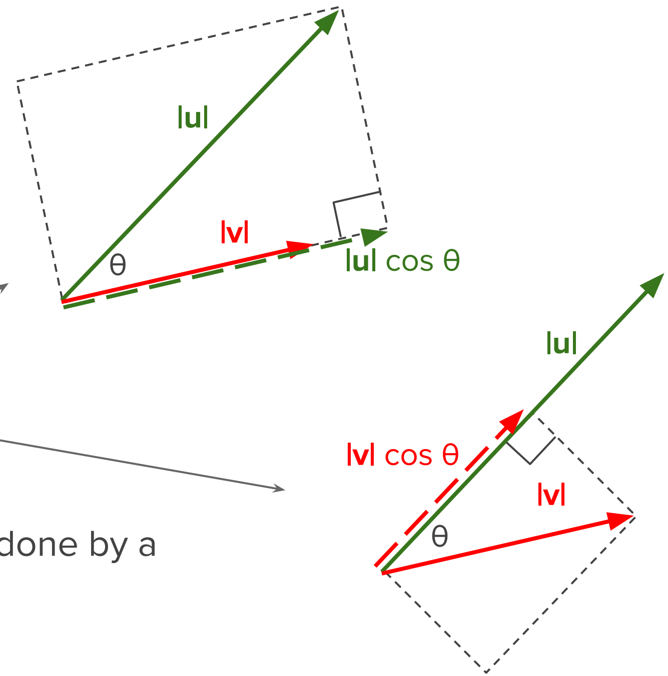
The dot product of two vectors, **u** and **v**, is defined by:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\, |\mathbf{v}| \cos\theta, \qquad 0 \le \theta \le \pi$$

where θ is the angle between the vectors.

Graphically, it represents either of the following:

1. Length of projection of u onto v x Length of v
2. Length of projection of v onto u x Length of u



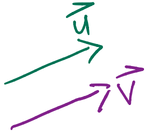
The dot product is commonly applied to compute the work done by a force acting on an object.

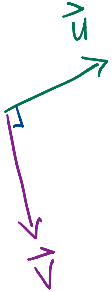
Dot Product

Exercise: What values does the dot product give for the angles below? Give a layman description of the main characteristic of the dot product.

θ	0	π/2	π	(0, π/2)	(π/2, π)
u.v	$\lvert\vec{u}\rvert\lvert\vec{v}\rvert$	0	$-\lvert\vec{u}\rvert\lvert\vec{v}\rvert$	>0	<0

since $\cos 0 = 1$ | $\cos\frac{\pi}{2} = 0$ | $\cos\pi = -1$ | $\cos\theta > 0$ | $\cos\theta < 0$





The dot product accounts for the similarity in the directions of the two vectors, giving a positive value when in a similar direction, negative when in opposite direction

Dot Product

The dot product of two vectors, **u** and **v**, is more directly computed from:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_m v_m = \sum_{i=1}^{m} u_i v_i$$

Notice that the dot product of two vectors outputs a scalar value. Hence, it is also known as a scalar product (aka inner product).

Combining the two definitions of the dot product, the angle between two vectors, **u** and **v**, can be found by the following relation (to be proven in tutorial 3).

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{m} u_i v_i = |\mathbf{u}||\mathbf{v}| \cos\theta \rightarrow \theta = \cos^{-1}\left(\frac{\sum_{i=1}^{m} u_i v_i}{|\mathbf{u}||\mathbf{v}|}\right)$$

Work Done by a Force

Example: A force vector (4, -1, 3) acts on an object as it moves through a displacement of (-2, -5, 1). Determine the work done by the force and the angle between the two vectors. Are the values logical?

$$\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix} = -8 + 5 + 3$$
$$= 0$$

$$\theta = \cos^{-1}\left(\frac{W}{|\vec{F}||\vec{d}|}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

since F is perpendicular to the displacement d, this implies that the displacement is not caused by force F, hence the work done by F must be 0. Logical

ANS: Work done = 0. Angle = π/2

Properties of the Dot Product

The following hold for vectors **u**, **v** and **w**, and real number k.

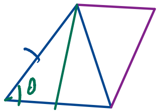
1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$
4. $\mathbf{0} \cdot \mathbf{u} = 0$
5. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
6. $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u}$ (for **u** and **v** as column vectors)

$\vec{u} \cdot \vec{u} = \Sigma u_i^2 = \left(\sqrt{u_1^2 + u_2^2 + \ldots}\right)^2 = |\vec{u}|^2$

$u_1 v_1 + u_2 v_2 + \ldots = (u_1 \; u_2 \; \ldots) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} = \vec{u}^T v = (v_1 \; v_2 \; \ldots) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} = v^T u$

The student should be able to verify each property above.

Cross Product

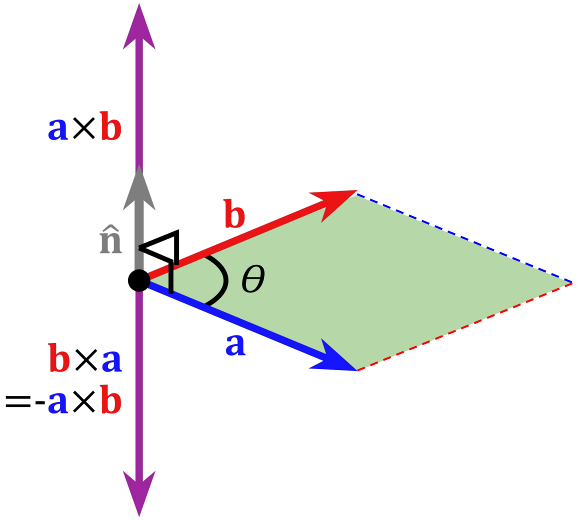


The cross product of two vectors, **a** and **b**, produces a third vector, **axb**, that is orthogonal to both **a** and **b**. It is defined by:

$$\mathbf{a} \times \mathbf{b} = \underline{|\mathbf{a}|\,|\mathbf{b}|\sin(\theta)}\ \hat{n}, \qquad 0 \le \theta \le \pi$$

where:

θ is the angle between the vectors and
n is the unit vector normal to a plane containing **a** & **b**
with direction given by the right-hand rule.



The magnitude of the cross product is equal to the area of the parallelogram formed by vectors a & b.

Note that the cross product is only defined for vectors in $\mathbb{R}^3$.

Source:
https://en.wikipedia.org/wiki/Cross_product

Cross Product

Exercise: What values does the magnitude of the cross product give for the angles below? Give a layman description of the main characteristic of the cross product.

θ	0	π/2	π	(0, π)
|axb|	0	$\|\vec{a}\|\|\vec{b}\|$	0	>0

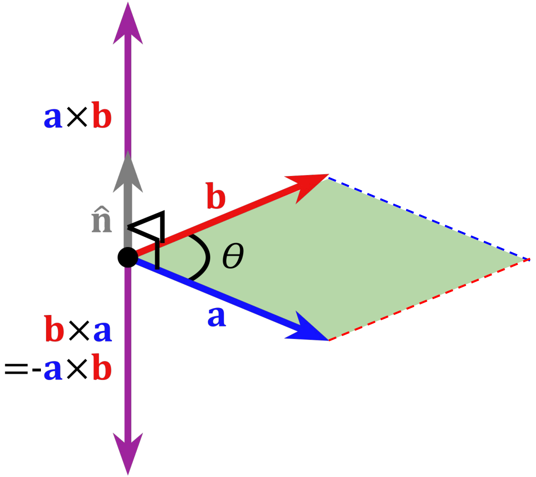
Since $\sin 0 = 0$ $\quad \sin\frac{\pi}{2} = 1$ $\quad \sin\pi = 0$

The cross product accounts for the degree of perpendicularity between 2 vectors, giving a positive value if any for the length of the resultant vector

Cross Product

The cross product of two vectors, **a** and **b**, is more directly be computed from:

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$



Tip: To obtain the expression in any row, cover that row and perform a 'cross multiplication' of the other two rows.

Notice that the cross product of two vectors produces a vector, hence it is also known as a vector product.

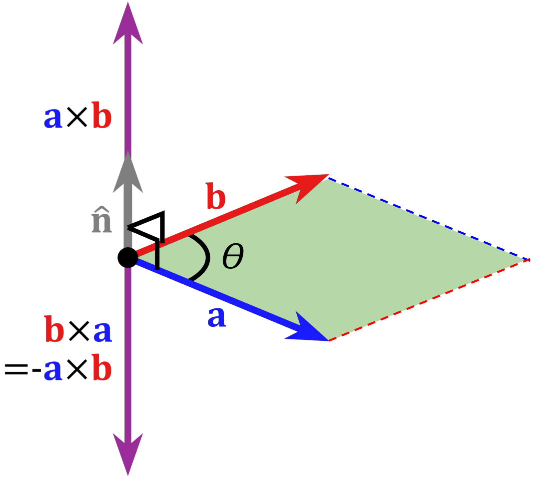
Derivation of the Cross Product

The cross product formula can be derived from Gauss-Jordan elimination (row ops).

Since **a** x **b** is perpendicular to both **a** and **b**, we have:

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = a_1 x + a_2 y + a_3 z = 0$$

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = b_1 x + b_2 y + b_3 z = 0$$



which gives the SLE in matrix form: $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Notice that the SLE has 2 equations but 3 unknowns, hence giving a line solution. The vector **a** x **b** will span this line.

Performing row ops:

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \end{array}\right] \xrightarrow{a_1 R_2 - b_1 R_1} \left[\begin{array}{ccc|c} a_1 b_1 & a_2 b_1 & a_3 b_1 & 0 \\ 0 & a_1 b_2 - a_2 b_1 & a_1 b_3 - a_3 b_1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{R_1}{a_1 b_1},\ \frac{R_2}{a_1 b_2 - a_2 b_1}} \left[\begin{array}{ccc|c} 1 & \frac{a_2 b_1}{a_1 b_1} & \frac{a_3 b_1}{a_1 b_1} & 0 \\ 0 & 1 & \frac{a_1 b_3 - a_3 b_1}{a_1 b_2 - a_2 b_1} & 0 \end{array}\right]$$

From the above row-echelon form, we solve:

$$R_2 \longrightarrow y + \frac{a_1 b_3 - a_3 b_1}{a_1 b_2 - a_2 b_1} z = 0 \longrightarrow z = -\frac{a_1 b_2 - a_2 b_1}{a_1 b_3 - a_3 b_1} y$$

$$\text{Let } y = -(a_1 b_3 - a_3 b_1) \longrightarrow z = a_1 b_2 - a_2 b_1$$

$$R_1 \longrightarrow x + \frac{a_2 b_1}{a_1 b_1} y + \frac{a_3 b_1}{a_1 b_1} z = 0$$

$$\longrightarrow x = \frac{-a_2 b_1 y - a_3 b_1 z}{a_1 b_1} = \frac{a_2 b_1 (a_1 b_3 - a_3 b_1) - a_3 b_1 (a_1 b_2 - a_2 b_1)}{a_1 b_1}$$

$$= a_2 b_3 - a_3 b_2$$

$$\longrightarrow \quad \vec{a} \times \vec{b} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Cross Product

Exercise: Compute the cross products of $\vec{a} = (3, 1, 0)$ and $\vec{b} = (0, 2, 0)$ in both directions and sketch the resultant vectors. What did you notice about their directions?

↳ $\vec{a} \times \vec{b}$ is opposite to $\vec{b} \times \vec{a}$, vice versa

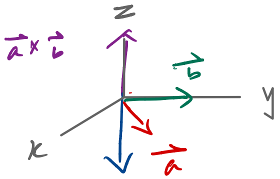
$$\vec{a} \times \vec{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1(0) - 0(2) \\ -[(3(0) - 0(0)] \\ 3(2) - 1(0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$$\vec{b} \times \vec{a} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2(0) - 0(1) \\ -[0(0) - 0(3)] \\ 0(1) - 2(3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$\vec{a}$ & $\vec{b}$



ANS: (0, 0, 6) & (0, 0, -6).

Properties of the Cross Product

The following hold for vectors **a**, **b** and **c**, and real number k.

1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
2. $k(\mathbf{a} \times \mathbf{b}) = k\mathbf{a} \times \mathbf{b} = \mathbf{a} \times k\mathbf{b}$
3. $\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}$
4. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ (zero vector)

The student should be able to verify each property above.

Vector Equation of a Line in $\mathbb{R}^2$

For a line L in the plane defined by the Cartesian equation y(x) = mx + c, a vector equation in the form below can be used to define the same line.

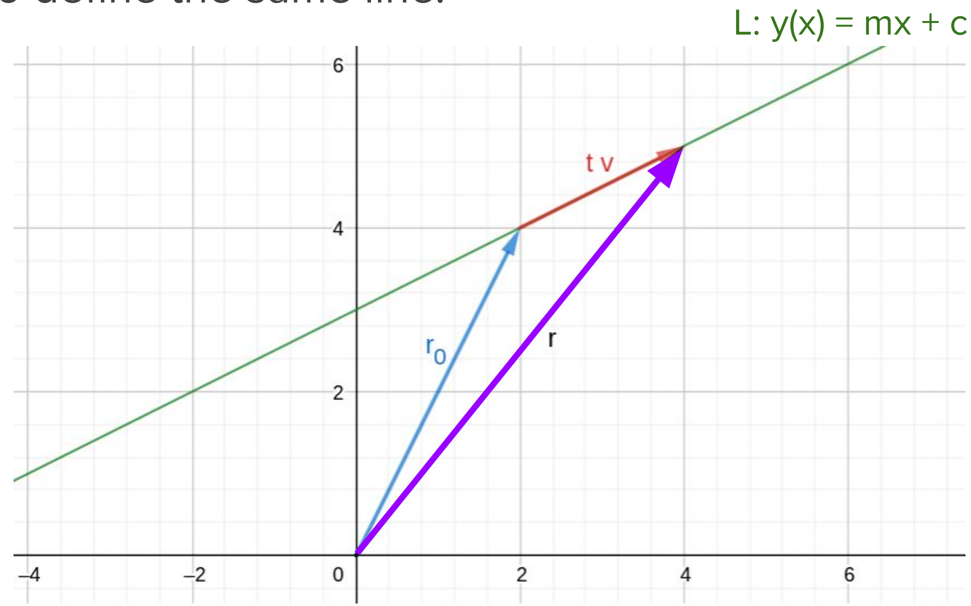
$$\mathbf{r} = \begin{bmatrix} x \\ y(x) \end{bmatrix} = \mathbf{r_0} + t\mathbf{v}$$

where:

$\mathbf{r_0}$ is a position vector to any point on L,
v is any vector in the direction of L and
t is a scalar.

When t is varied, vector **r** is able to define all points on line L, hence called a vector equation.

Note that $\mathbf{r_0}$ and **v** are not unique.



Vector Equation of a Line in $\mathbb{R}^2$

One method to obtain the vector equation is described below.

1. Define the position vector, $\mathbf{r_0}$, by finding a point (x_0, y_0) on the line, i.e.

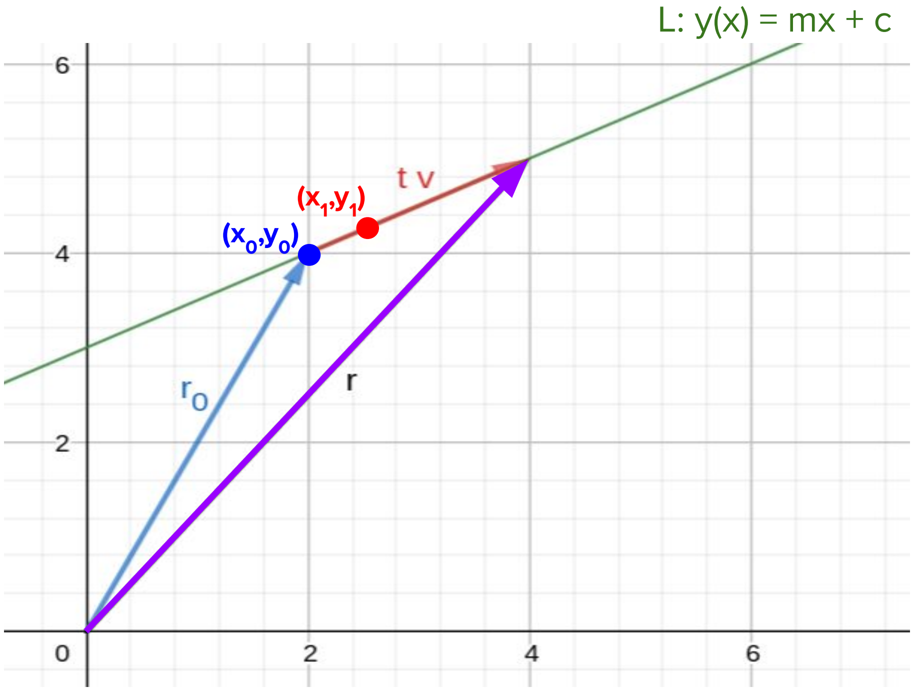
$$\mathbf{r_0} = \begin{bmatrix} x_0 \\ y_0 = y(x_0) \end{bmatrix}$$

2. Finding another point (x_1, y_1) on the line. Then, define the direction vector, $\mathbf{v}$, by

$$\mathbf{v} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}$$

3. Finally, write:

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$



Procedure is similar for a line in $\mathbb{R}^3$.

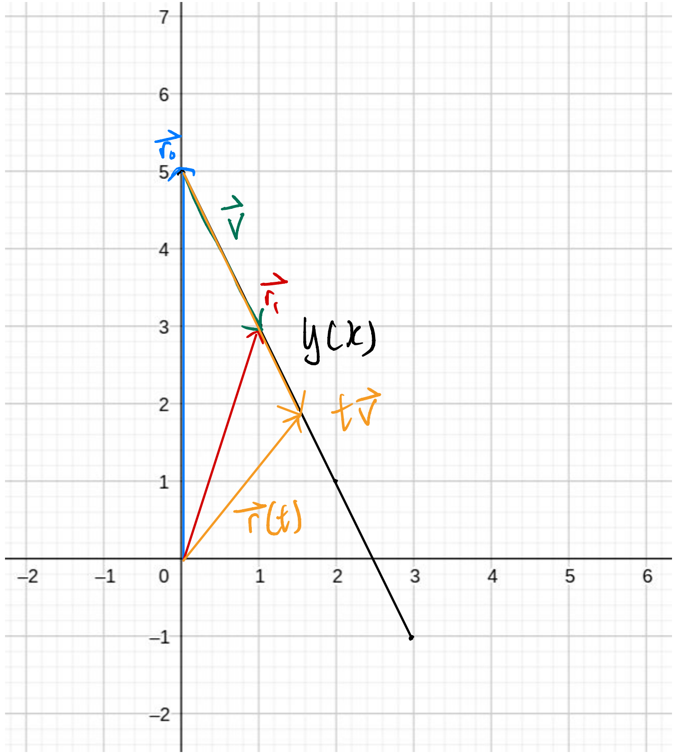
Vector Equation of a Line in $\mathbb{R}^2$

Example: Determine the vector equation of a line L given by y(x) = -2x + 5. Draw it on the graph.

$$\vec{r}_0 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1-0 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

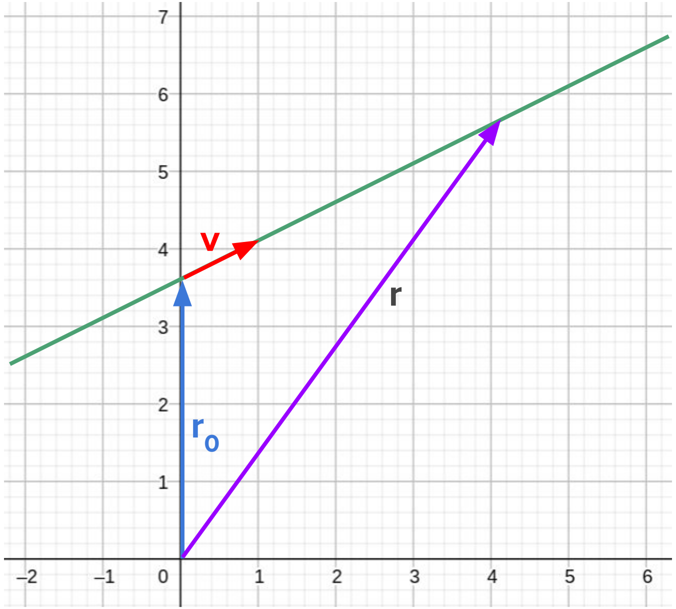


ANS: (Not unique) **r** = (0, 5) + t(1, -2). Check: https://www.geogebra.org/3d/xyddmhkq.

Vector Equation by Parameterization

Parameterization is a faster way to obtain the vector representation of lines and planes etc. Eg, given a line with the cartesian equation, $y(x) = mx + c$, letting $x = t$, we get $y = mt + c$. In vector notation, we get:

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ mt + c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ c \end{bmatrix}}_{\mathbf{r_0}} + t \underbrace{\begin{bmatrix} 1 \\ m \end{bmatrix}}_{\mathbf{v}}$$



In the above, t is called a parameter. Note that parameterization is not unique (other functions of t can be defined).

Vector Equation by Parameterization

Exercise: Determine the vector equation of a line L given by y(x) = -2x + 5 again, this time using parameterization. Use another parametrization to represent the same line.

$\vec{r}(t) = \begin{pmatrix} t \\ -2t+5 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Let $x = -2t$, $y = 4t+5$

$\vec{r_2}(t) = \begin{pmatrix} -2t \\ 4t+5 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Let $x = t+1$, $y = -2(t+1) + 5 = -2t+3$

$\vec{r_3}(t) = \begin{pmatrix} t+1 \\ -2t+3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

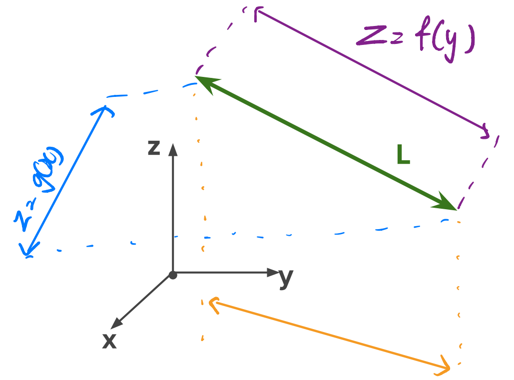
ANS: (Not unique) **r** = (0, 5) + t(1, -2). Check: https://www.geogebra.org/3d/xyddmhkq.

Vector Equation of a Line in $\mathbb{R}^3$

The same procedure can be applied to obtain the vector equation of a line in $\mathbb{R}^3$, given by the Cartesian equations:

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

Example: Draw the projections of the line L onto the coordinate planes and state their respective equations. (Hence, understand the Cartesian equation of a line in $\mathbb{R}^3$.)



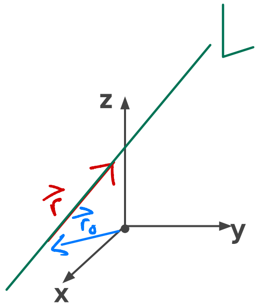
Vector Equation of a Line in $\mathbb{R}^3$

Exercise: Using the same procedure for a line in $\mathbb{R}^2$, define the vector equation of a line L defined below. Represent it on the graph.

$$\frac{x-1}{2} = y+1 = z$$

Let $z = t$, $y = t-1$, $x = 2t+1$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 2t+1 \\ t-1 \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\vec{r}_0} + t\underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{\vec{v}}$$

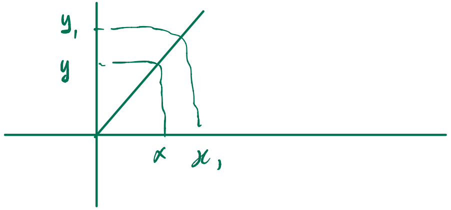


Let $y = t$

$$\vec{r}_2(t) = \begin{pmatrix} 2t+3 \\ t \\ t+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Ans: (Not unique) Let z = t, **r**(t) = (1, -1, 0) + t(2, 1, 1).
Check https://www.geogebra.org/3d/fry9rad6.

Vector Equation of a Plane in $\mathbb{R}^3$



A plane in $\mathbb{R}^3$ is defined by the Cartesian equation below:

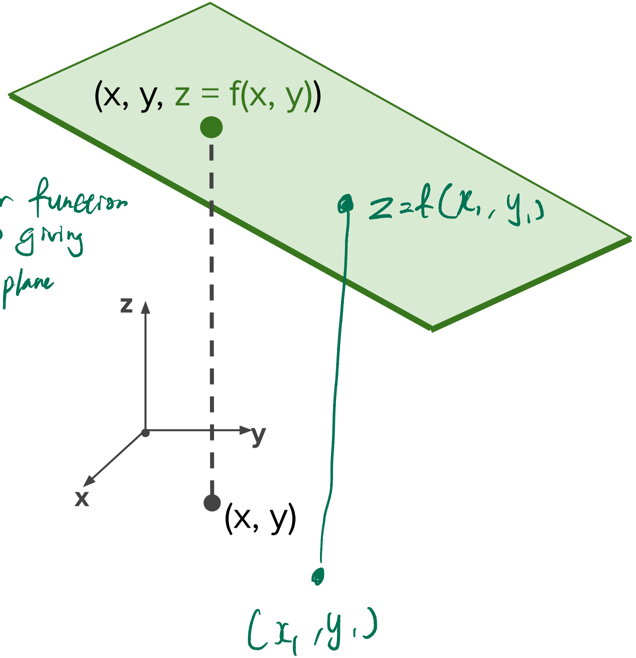
$$ax + by + cz = k \longrightarrow z = f(x,y)$$

$$z = -\frac{a}{c}x - \frac{b}{c}y + k$$

where a, b, c and k are constants.

$$= px + qy + k \sim \text{linear function so giving a plane}$$

Exercise: From the above equation, define z = f(x,y) and explain the graphical meaning. (Hence, understand the Cartesian equation of a plane in $\mathbb{R}^3$.)



Vector Equation of a Plane in $\mathbb{R}^3$

For a plane P in $\mathbb{R}^3$ defined by $ax + by + c = k$, a vector equation in the form below can be used to define the same plane.

$$\mathbf{r}(s,t) = \mathbf{r_0} + s\mathbf{u} + t\mathbf{v}$$

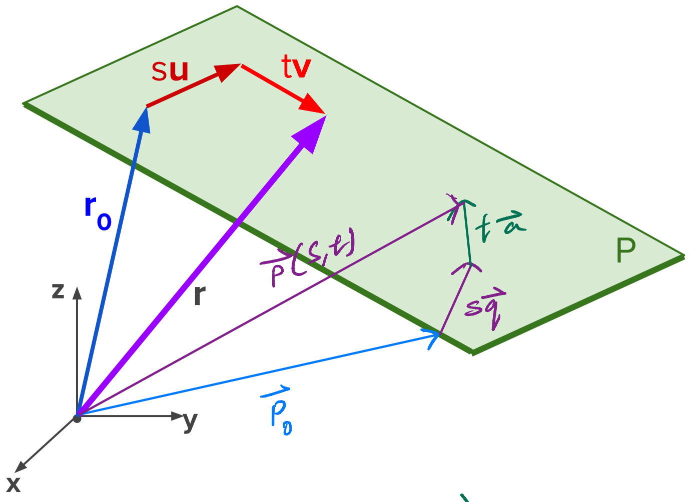
where:

- $\mathbf{r_0}$ is a position vector to any point on P,
- **u** and **v** are any 2 non-parallel vectors on P and
- **t** and **s** are scalars.

When t and s are varied, vector **r** is able to define all points on plane P, hence a vector equation.

Note that $\mathbf{r_0}$, **v** and **u** are not unique.

Exercise: Define and draw another vector equation of plane P.



$\vec{p}(s,t) = \vec{p}_0 + s\vec{q} + t\vec{a}$

Vector Equation of a Plane in $\mathbb{R}^3$

Example: Determine the vector equation of a plane P given by $x - 4y + 2z = 7$. (Hint: Procedure is analogous to that of a line. Sketch a graph to label the vectors.)

Let $y = t$, $z = s$, $x = 7 - 2s + 4t$

$$\vec{r}(s,t) = \begin{pmatrix} 7 - 2s + 4t \\ t \\ s \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} -2s \\ 0 \\ s \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad //$$

must be linearly independent

Let $x = 2t + 3$, $y = s - 2$,

$$z = \frac{7 - 2t - 3 + 4s - 8}{2} = \frac{4s - 2t - 4}{2} = 2s - t - 4$$

$$\vec{r}_2(s,t) = \begin{pmatrix} 2t + 3 \\ s - 2 \\ 2s - t - 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 2t \\ 0 \\ -t \end{pmatrix} + \begin{pmatrix} 0 \\ s \\ 2s \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Ans: (Not unique) Let y = t & z = s, **r**(t) = (7, 0, 0) + t(4, 1, 0) + s(-2, 0, 1).
Check https://www.geogebra.org/3d/besjxasn.

Vector Equation to Cartesian Equation for a Line

We saw how the vector equation of a geometry can be obtained its Cartesian equation. Now let's see the reverse. For a line, the Cartesian equation can be obtained by reverse substitution. From

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Simply isolate t from either x(t), y(t) or z(t) (whichever is easier) and substitute t in the rest of the coordinates. Eg, from an earlier exercise:

$$\mathbf{r}(t) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2t \\ -1+t \\ t \end{bmatrix}$$

$$t = z \rightarrow \begin{cases} x = 1 + 2z \rightarrow z = \frac{x-1}{2} \\ y = -1 + z \rightarrow z = y + 1 \end{cases}$$

$$\therefore z = y + 1 = \frac{x-1}{2}$$

Vector Equation to Cartesian Equation for a Plane

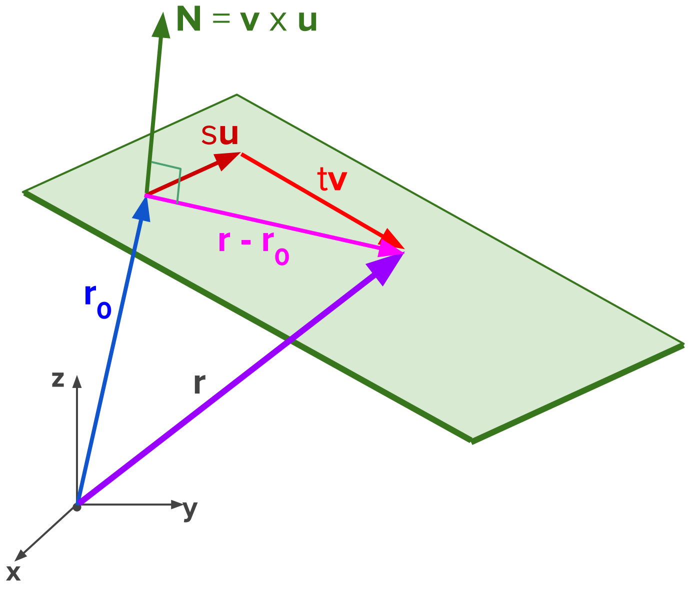
To obtain the Cartesian equation of a plane from its vector equation, the reverse substitution method is less efficient. Instead, we can make use of dot & cross products. From

$$\mathbf{r}(s,t) = \mathbf{r_0} + s\mathbf{u} + t\mathbf{v}$$

Since **u** and **v** are on the plane, a normal vector to the plane can be found by **N** = **u** x **v** (or **v** x **u**). And since any vector **r** - $\mathbf{r_0}$ on the plane is orthogonal to **N**, we have

$$\mathbf{N}\cdot(\mathbf{r}-\mathbf{r_0}) = \mathbf{N}\cdot\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \mathbf{r_0}\right) = 0$$

which gives the Cartesian equation of the plane.



Vector Equation to Cartesian Equation for a Plane

From an earlier example, the vector equation of a plane is

$$\mathbf{r}(s,t) = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

So it has the normal vector

$$\mathbf{N} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

Notice that the coefficients of the variables are equal to the components of the normal vector.

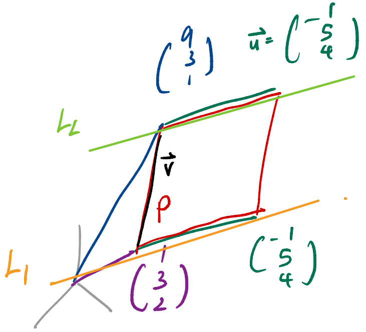
And the Cartesian equation can be evaluated to be

$$\mathbf{N} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \mathbf{r_0} \right) = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x-7 \\ y \\ z \end{bmatrix} = x - 7 - 4y + 2z = 0 \rightarrow x - 4y + 2z = 7$$

Vector & Cartesian Equations of a Plane

Exercise: Determine the vector & cartesian equation of a plane P that contains the lines L_1 and L_2 defined below. Sketch a representation.

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overset{r}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} + t \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}, \qquad L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overset{r_c}{\begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}} + s \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$$



need another vector not parallel to $\vec{u}$:

$$\vec{v} = \begin{pmatrix} 9 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{N} = \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} \times \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ -[-31] \\ -40 \end{pmatrix}$$

$$\therefore \vec{r}(s,t) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + s \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = \begin{pmatrix} -5 \\ 31 \\ -40 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-3 \\ z-2 \end{pmatrix} = 0$$

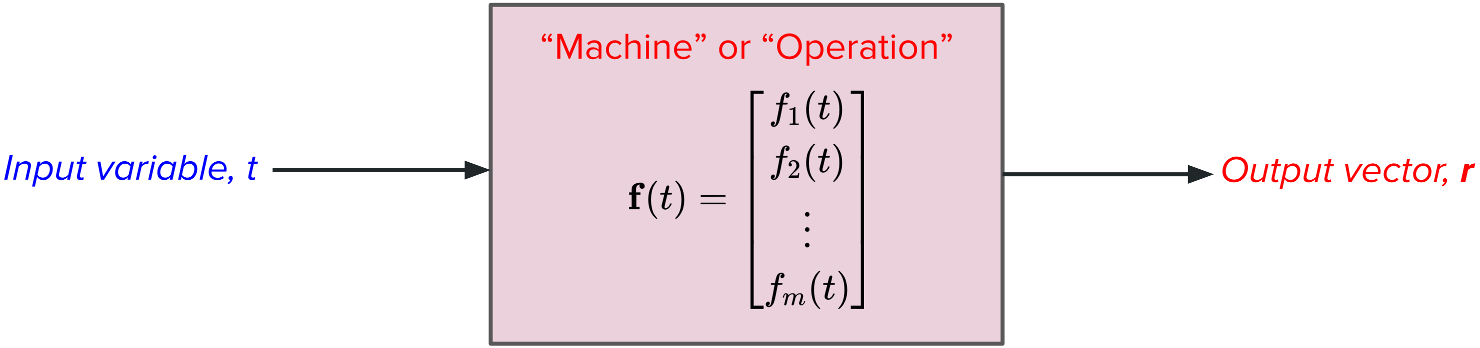
$$\Rightarrow -5(x-1) + 31(y-3) - 40(z-2) = 0$$

ANS: $\mathbf{r} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix} + s \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}$, $5x - 31y + 40z = -8$.

$$-5x + 5 + 31y - 93 - 40z + 80 = 0$$
$$-5x + 31y - 40z = 8$$

Concept of a Vector Function

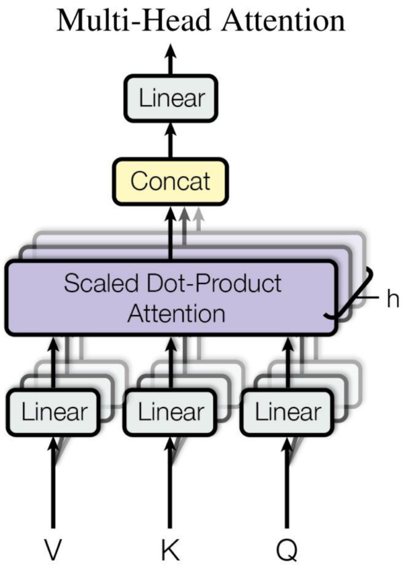
A vector equation is also a vector function. Analogous to scalar functions covered in Topic 1, a vector function is one that takes in input/s and produces an output vector. The "machine" perspective of a single-variable vector function is shown below.



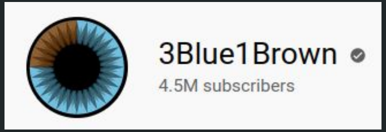
Certainly, the vector function can also take in multiple inputs (or a vector of inputs), leading to a multivariable vector function. We will explore them further in Math 2.

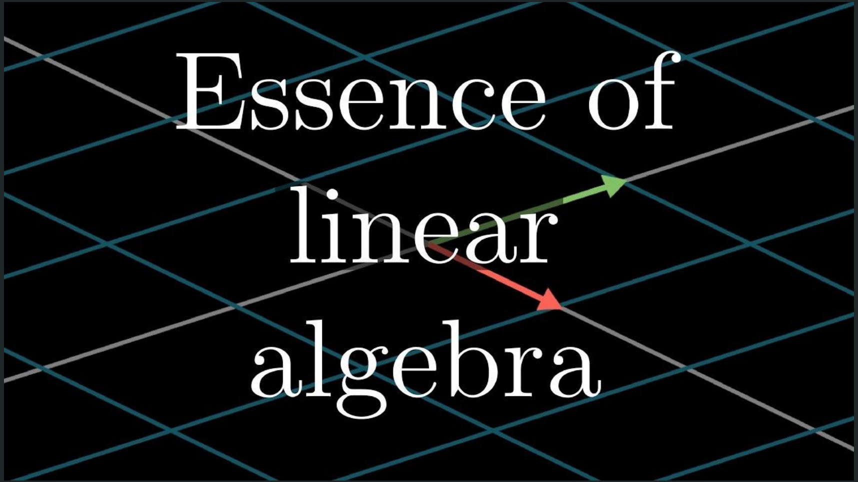
End of Topic 3

Source: https://arxiv.org/abs/1706.03762



*The **dot product** between word vectors is used by ChatGPT to focus the 'attention' on important contexts in language processing.*





https://youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

You are highly recommended to watch this series by Grant Sanderson as we cover Topics 3 & 4.