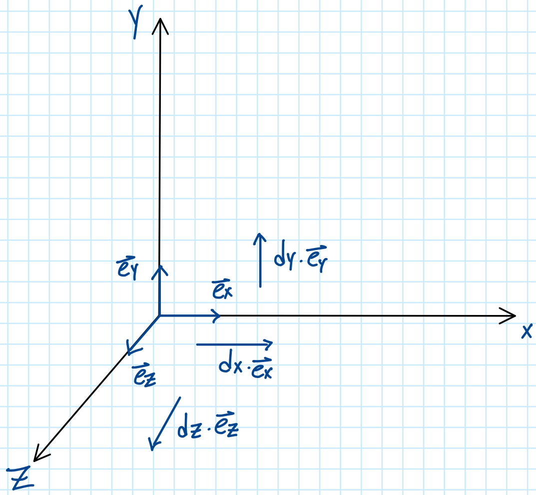
Math5: Different Coordinate Systems (relevant for this lecture)

1) Cartesian Coordinates:



orthonormal basis: $\vec{e}_X$, $\vec{e}_Y$, $\vec{e}_Z$ Unit Vector

representation of position vector:

$$\vec{r} = V_X \cdot \vec{e}_X + V_y \cdot \vec{e}_y + V_Z \cdot \vec{e}_Z = \begin{pmatrix} V_X \\ V_Y \\ V_Z \end{pmatrix}$$

Integration in cartesian coordinates:

differential volume element for volume integral: $dV = dx\,dy\,dz$

path/curve integral: Integration along a Curve in a vector

- in x-/y-/z-direction: $dx \cdot \vec{e}_X$; $dy \cdot \vec{e}_Y$; $dz \cdot \vec{e}_Z$

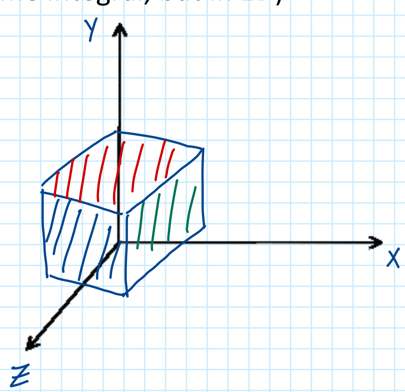
surface integral:

a) To calculate surface area (scalar, content of area; cf. volume integral, but in 2D)

Side ~~Top~~ surface (parallel to xy-plane): $dx\,dy$

Top ~~Side~~ surface (parallel to xz-plane): $dx\,dz$

Side surface (parallel to yz-plane): $dy\,dz$



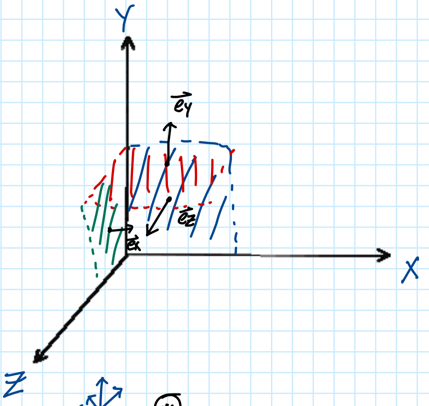
b) To calculate flux of vector field through respective surface (flux, which is penetrating this surface)
Vectorial Surface element $d\vec{a}$

b) To calculate flux of vector field through respective surface (flux, which is penetrating this surface)

Side
~~Top~~ surface (parallel to xy-plane): $dx \cdot dy \cdot \vec{e}_z$

Top
~~Side~~ surface (parallel to xz-plane): $dx\, dz \cdot \vec{e}_y$

Side surface (parallel to yz-plane): $dy dz \cdot \vec{e}_x$



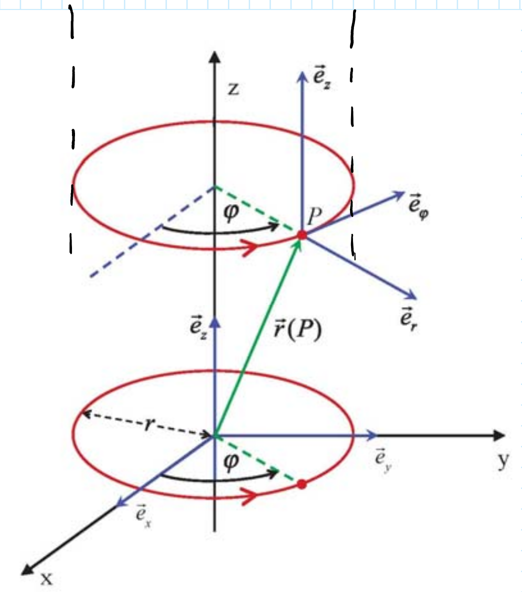
In EaM: We often spherical problems $\Rightarrow$ $\vec{E}$-field of point charge, e.g. ⊛

or cylindric problems $\Rightarrow$ wires with electric current I

⊛ $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{e}_r$ $\quad |\vec{r}-\vec{r}_Q|^2 = |\vec{r}|^2 = (x^2+y^2+z^2) \Rightarrow |\vec{r}|^2 = r$

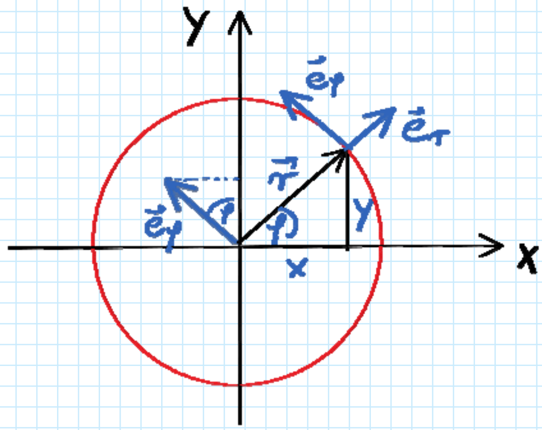
$|\vec{r}| = \sqrt{(x^2+y^2+z^2)}$ $\quad$ Spherical (r, θ, φ) $\quad |r| = r$

1) Cylindric Coordinates:



$z \to z$
$\vec{e}_z \to \vec{e}_z$
introduce r, φ

projection in xy-plane:



orthonormal basis: $\vec{e}_x, \vec{e}_y, \vec{e}_z \to \vec{e}_r, \vec{e}_\varphi, \vec{e}_z$

coordinate transformation: $x = r \cdot \cos\varphi$
$y = r \cdot \sin\varphi$
$z = z$

representation of position vector:

$$r = V_r \cdot \vec{e}_r + V_\varphi \cdot \vec{e}_\varphi \cdot V_z \cdot \vec{e}_z = \begin{pmatrix} V_r \\ V_\varphi \\ V_z \end{pmatrix}$$

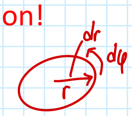
Integration in cylindrical coordinates:

Please don't forget the scale factors coming from coordinate transformation!

differential volume element for volume integral:

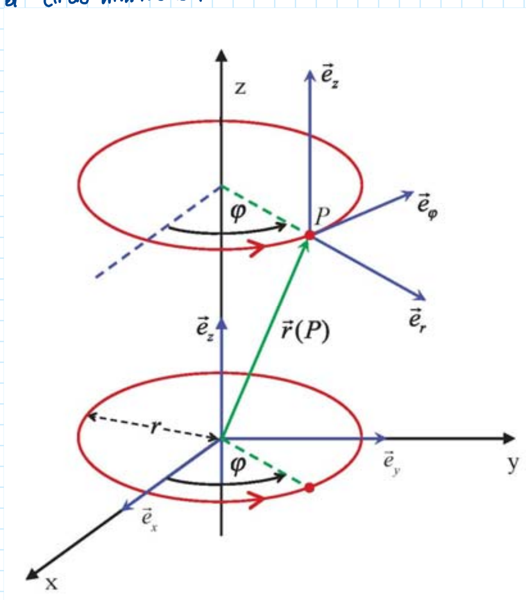
$$dx\,dy\,dz \rightarrow dV = \underbrace{r\,dr\,d\varphi}_{\text{area}} \cdot dz$$

$$\overbrace{r\,dr\,d\varphi}^{\text{area of circle}} \Rightarrow dr \cdot \underbrace{r\,d\varphi}_{\text{Circumference of circle with radius } r}$$



path/curve integral:

- in radial direction: $d\vec{r} = dr \cdot \vec{e}_r$
- in z-direction: $d\vec{r} = dz \cdot \vec{e}_z$
- In azimutal direction (along phi): $d\vec{r} = r\,d\varphi \cdot \vec{e}_\varphi$



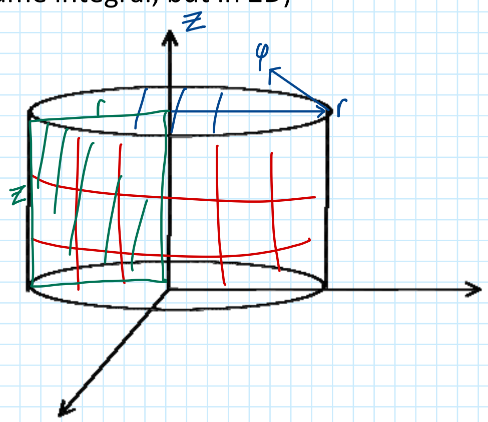
surface integral:

a) To calculate surface area (scalar, content of area; cf. volume integral, but in 2D)

Top surface: $da = r\,dr\,d\varphi$

Enveloping surface: $da = r\,d\varphi\,dz$

Longitudinal cut (rectangular area): $da = dr \cdot dz$
(mostly not needed in this lecture)



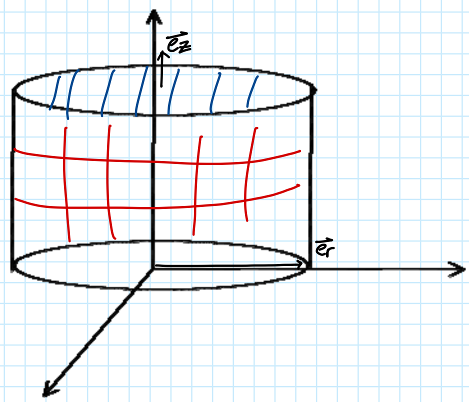
b) To calculate flux of vector field through respective surface (flux, which is penetrating this surface)

Top surface: $d\vec{a} = r\,dr\,d\varphi \;\; \vec{e}_z$

Enveloping surface: $d\vec{a} = r\,dr \cdot d\varphi \;\; \vec{e}_r$

Longitudinal cut (rectangular area):
(mostly not needed in this lecture)

$d\vec{a} = dr\,dz \cdot \vec{e}_\varphi$

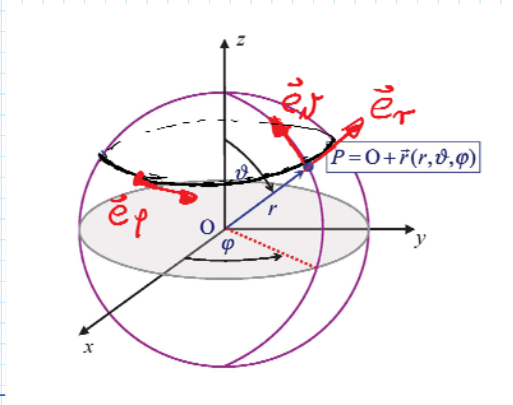


3) Spherical coordinates:

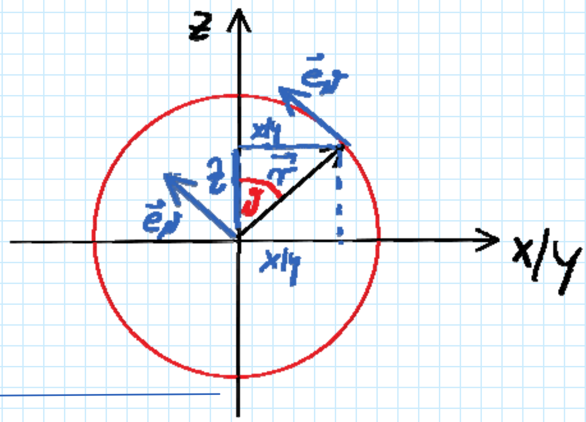
$P(x,y,z) \rightarrow P(r,\theta,\varphi)$

$\hookrightarrow \theta$ = polar angle

φ = azimuth angle



projection into xz-plane (or yz-plane)



orthonormal basis: $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\varphi$

coordinate transformation:

$x = r \cdot \cos\varphi \cdot \sin\theta$

$y = r \cdot \sin\varphi \cdot \sin\theta$

$z = r \cdot \cos\theta$

representation of position vector:

$$\vec{r} = V_r \cdot \vec{e}_r + V_\theta \cdot \vec{e}_\theta + V_\varphi \cdot \vec{e}_\varphi = \begin{pmatrix} V_r \\ V_\theta \\ V_\varphi \end{pmatrix}$$

Integration in spherical coordinates:

Please don't forget the scale factors coming from coordinate transformation!

differential volume element for volume integral:

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

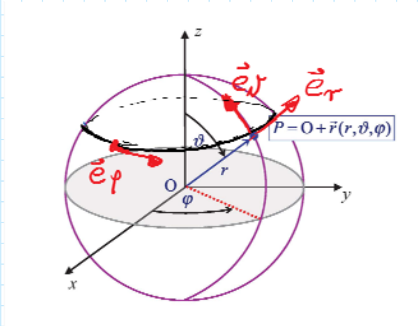
$\vec{E}$ $d\vec{r}$ work performed in a Coulomb field

Q

$W = q \int \vec{E} \cdot d\vec{r}$

path/curve integral:

- in radial direction: $d\vec{r} = dr \cdot \vec{e}_r$
- In azimutal direction (along azimut angle phi): $d\vec{r} = r\,d\varphi\, \vec{e}_\varphi$
- In direction of polar angle theta: $d\vec{r} = r\sin\theta \; \vec{e}_\theta$



surface integral:

a) To calculate surface area (scalar, content of area; cf. volume integral, but in 2D)

   We need only the surface of the sphere: $da = R^2 \sin\theta \, d\theta \, d\varphi$

   Area of a sphere with radius R

b) To calculate flux of vector field through spherical surface (flux, which is penetrating the surface of the sphere)

   We only need the flux through the closed spherical surface (electric field of a point charge is radial symmetric!)

   $d\vec{a} = R^2 \cdot \sin\theta \, d\theta \, d\varphi \, \vec{e_r}$