Math2: Integration along a curve/path inside a vector field

Problem: mechanical work - calculation of work for an arbitrary path and inside a location-depending force field

- Curve/path/line integral in a vector field = Integration of a vectorial quantity along a given path/curve (e.g., mechanical work)

What are you doing, when calculating a path/curve integral?

You follow the curve and sum up the contributions of the vector field along this curve ("projection of vectors onto this curve")

Solution of problem (consists basically of 3 steps):

(i) Parametrize the curve C(P1,P2): find suitable way of parametrizing

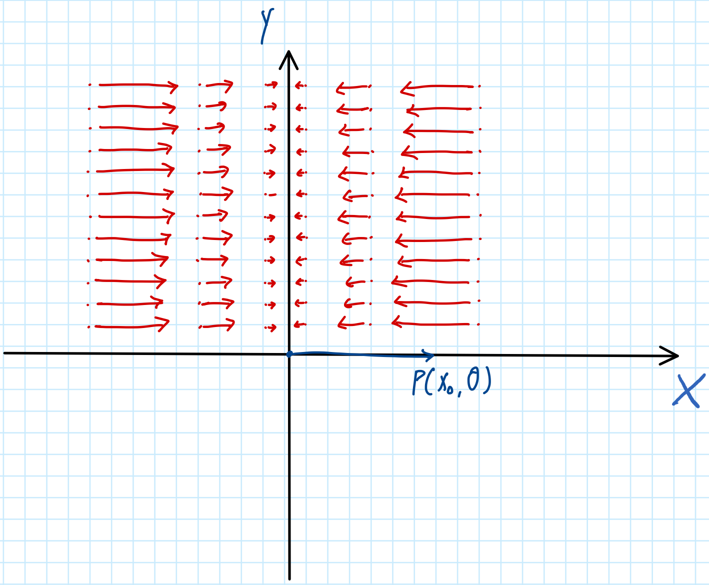
(ii) Parametrize vector field coordingly (insert the parametrization of (i)

(iii) adapt differential line element inside integral to parametrization

Example 1:

Caluculate work W along path C(P,P) in force field $\vec{f}(\vec{r}) = -f_0 \vec{e}_x = \begin{pmatrix} -f_0 x \\ 0 \end{pmatrix}$ $f_0 = \text{const.} > 0$

Kurve C is given by $C(0,P)$ where $P = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$; $x_0 = \text{const.}$



(i) Parameterize C : Curve parameter s $(s \hat{=} x)$ $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\vec{r} \to \vec{r}(s) = \begin{pmatrix} s \\ 0 \end{pmatrix} = s \cdot \vec{e}_x$$

$$s \in [0, x_0]$$

(ii) Parameterize force field : $f(\vec{r}) = -f_0 x . \vec{e}_x = \begin{pmatrix} -f_0 x \\ 0 \end{pmatrix}$

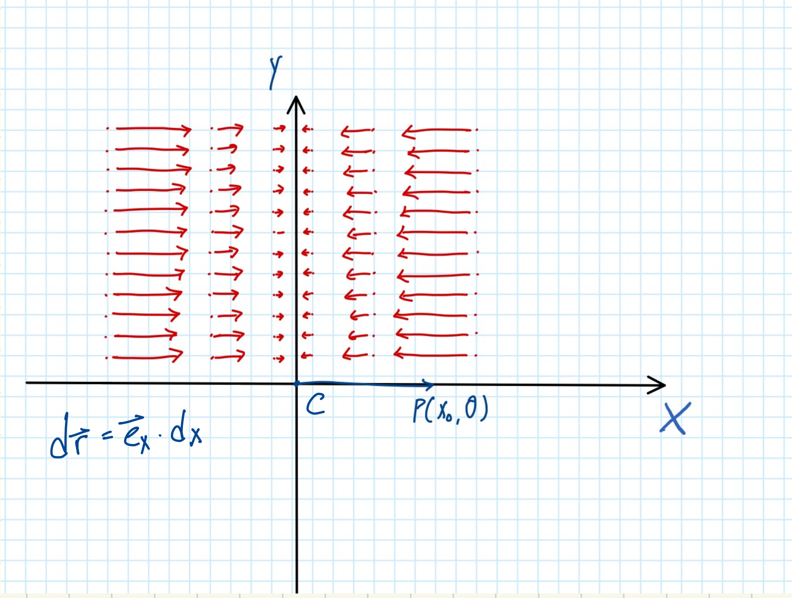
$$f(\vec{r}(s)) = -f_0 s . \vec{e}_x = \begin{pmatrix} -f_0 s \\ 0 \end{pmatrix}$$

(iii) line element : $d\vec{r}(s) = \vec{t}\, ds = \frac{d\vec{r}(s)}{ds} \cdot ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot ds = \vec{e}_x \cdot ds$

To Solve

$$W = \int_C \vec{f}(\vec{r})\, d\vec{r} = \int \underbrace{(-f_0 s . \vec{e}_x)}_{\vec{f}(\vec{r})} \cdot \underbrace{\vec{e}_x \cdot ds}_{\vec{t} \cdot ds} = \int_0^{x_0} \begin{pmatrix} -f_0 s \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds$$

$$= \int_0^{x_0} -f_0 s \cdot ds = \left[-\frac{1}{2} f_0 s^2 \right]_0^{x_0} = -\frac{1}{2} f_0 x_0^2$$



$d\vec{r} = \vec{e}_x \cdot dx$

Short way (by inspection)

$$W = \int_0^{x_0} \vec{f}(\vec{r}) \cdot \vec{e}_x \cdot dx$$

$$= \int_0^{x_0} (-f_0 x)\vec{e}_x \cdot \vec{e}_x \cdot dx$$

$$= \int_0^{x_0} (-f_0 x)\, dx$$

Example 2:

Consider vector field $\vec{F}(\vec{r}) = (x^2 + y)\vec{e}_x + (x+y)\vec{e}_y = \begin{pmatrix} x^2+y \\ x+y \end{pmatrix}$

Calculate line/path integral along curve C(P_1P_2): $y = x^2$ where: $P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $P_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

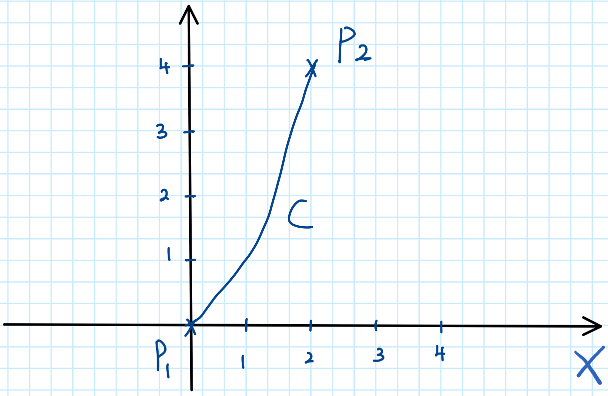
(i) Parameterize Curve C:

$$s \to \vec{r}(s) \qquad s \in [0,2]$$

$$\vec{r}(s) = s \cdot \vec{e}_x + s^2 \vec{e}_y = \begin{pmatrix} s \\ s^2 \end{pmatrix}$$

$$\vec{r}(s=0) = P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{r}(s=2) = P_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$



$$\vec{F}(\vec{r}) \to \vec{F}(\vec{r}(s)) = (s^2 + s^2)\vec{e}_x + (s + s^2)\vec{e}_y = \begin{pmatrix} s^2+s^2 \\ s+s^2 \end{pmatrix}$$

$$= 2s^2 \cdot \vec{e}_x + (s+s^2)\vec{e}_y = \begin{pmatrix} 2s^2 \\ s+s^2 \end{pmatrix}$$

Parameterize $d\vec{r}$

$$d\vec{r}(s) = \vec{t} \cdot ds = \frac{d\vec{r}(s)}{ds} \cdot ds$$

$$\vec{r}(s) = \begin{pmatrix} s \\ s^2 \end{pmatrix} = s \cdot \vec{e}_x + s^2 \vec{e}_y$$

$$\vec{t} = \frac{d\vec{r}(s)}{ds} = \vec{e}_x + 2s \cdot \vec{e}_y = \begin{pmatrix} 1 \\ 2s \end{pmatrix}$$

$$d\vec{r}(s) = \begin{pmatrix} 1 \\ 2s \end{pmatrix} \cdot ds = (\vec{e}_x + 2s \cdot \vec{e}_y) \cdot ds$$

$$W = \int_{C(P_1,P_2)} \vec{F}(\vec{r})\, d\vec{r} = \int_0^2 (2s^2 \vec{e}_x + (s+s^2)\vec{e}_y) \cdot (\vec{e}_x + 2s \cdot \vec{e}_y) \cdot ds$$

$$= \int_0^2 (2s^2 + 2s(s+s^2))\, ds = \int_0^2 (2s^2 + 2s^2 + 2s^3)\, ds$$

$$= \int_0^2 (4s^2 + 2s^3)\, ds$$

$$= \left[\frac{4}{3}s^3 + \frac{1}{2}s^4\right]_0^2 = \left[\frac{4}{3} \cdot 8 + \frac{1}{2} \cdot 16\right] = \frac{32}{3} + 8$$