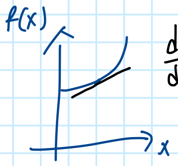
Some remarks on differential operators:
gradient, divergence, curl (rot): The Del Operator (Nabla Operator)

Del / Nabla operator : $\vec{\nabla}$

(i) gradient : acts on a scalar function $f(x,y,z)$

result is a vector

Similar to slop of a curve in one-dimensional case



$\frac{df}{dx} \to \frac{\partial f}{\partial x}$

$df = \underbrace{\frac{\partial f}{\partial x}}\, dx$ small change in X

tells how much the function f changes with respect to X

$$\text{grad } f(\underbrace{x,y,z}_{\vec{r}}) = \vec{\nabla} \cdot f(\vec{r}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot f(x,y,z)$$

$$= \begin{pmatrix} \partial/\partial x\, f \\ \partial/\partial y\, f \\ \partial/\partial z\, f \end{pmatrix} = \frac{\partial}{\partial x} f(x,y,z) \cdot \vec{e}_x + \frac{\partial}{\partial y} f(x,y,z) \cdot \vec{e}_y + \frac{\partial}{\partial z} f(x,y,z) \cdot \vec{e}_z$$

gradient points always in the direction of the steepest increase

(ii) Divergence (div) : acts on a vector field $\vec{f}(\vec{r}) = \vec{f}(x,y,z)$

$$\vec{\nabla} \cdot \vec{f}(x,y,z) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z = \text{div}(\vec{f}(x,y,z))$$

result is a scalar

div is correlated with the sources of a field and to the flux of a quantity

div corresponds to closed surface integrals $\Rightarrow$ see (1.5) Gauss' Law

(iii) Curl operator

Del-operator applied to a vector field $\vec{f}(x,y,z)$ as a "vector product"

$$\vec{\nabla} \times \vec{f}(x,y,z) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \left(\frac{\partial}{\partial y} f_z - \frac{\partial}{\partial z} f_y\right) + \left(\frac{\partial}{\partial z} f_x - \frac{\partial}{\partial x} f_z\right) \cdot \vec{e}_y + \left(\frac{\partial}{\partial x} f_y - \frac{\partial}{\partial y} f_x\right) \cdot \vec{e}_z$$

result is a vector

Tells us something about vorticity of a vector field

Related to closed path integrals $\oint$

$\operatorname{div}(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2) = \vec{\mathbf{V}}_2 \cdot \operatorname{rot} \vec{\mathbf{V}}_1 - \vec{\mathbf{V}}_1 \cdot \operatorname{rot} \vec{\mathbf{V}}_2.$ (13.98)

13.2.7.3 Expressions of Vector Analysis in Cartesian, Cylindrical, and Spherical Coordinates (see Table 13.3)

Table 13.3 Expressions of vector analysis in Cartesian, cylindrical, and spherical coordinates

	x, y, z **Cartesian coordinates**	r, φ, θ **Cylindrical coordinates**	**Spherical coordinates**
$d\vec{s} = d\vec{\mathbf{r}}$	$\vec{\mathbf{e}}_x dx + \vec{\mathbf{e}}_y dy + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_\rho d\rho + \vec{\mathbf{e}}_\varphi \rho d\varphi + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_r dr + \vec{\mathbf{e}}_\vartheta r d\vartheta + \vec{\mathbf{e}}_\varphi r \sin\vartheta d\varphi$
$\operatorname{grad} U$	$\vec{\mathbf{e}}_x \frac{\partial U}{\partial x} + \vec{\mathbf{e}}_y \frac{\partial U}{\partial y} + \vec{\mathbf{e}}_z \frac{\partial U}{\partial z}$	$\vec{\mathbf{e}}_\rho \frac{\partial U}{\partial \rho} + \vec{\mathbf{e}}_\varphi \frac{1}{\rho}\frac{\partial U}{\partial \varphi} + \vec{\mathbf{e}}_z \frac{\partial U}{\partial z}$	$\vec{\mathbf{e}}_r \frac{\partial U}{\partial r} + \vec{\mathbf{e}}_\vartheta \frac{1}{r}\frac{\partial U}{\partial \vartheta} + \vec{\mathbf{e}}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial U}{\partial \varphi}$
$\operatorname{div} \vec{\mathbf{V}}$	$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(V_\vartheta \sin\vartheta)$ $+ \frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial \varphi}$
$\operatorname{rot} \vec{\mathbf{V}}$ Curl = rot	$\vec{\mathbf{e}}_x \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)$ $+\vec{\mathbf{e}}_y \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$ $+\vec{\mathbf{e}}_z \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$	$\vec{\mathbf{e}}_\rho \left(\frac{1}{\rho}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right)$ $+\vec{\mathbf{e}}_\varphi \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)$ $+\vec{\mathbf{e}}_z \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\varphi) - \frac{1}{\rho}\frac{\partial V_\rho}{\partial \varphi}\right)$	$\vec{\mathbf{e}}_r \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(V_\varphi \sin\vartheta) - \frac{\partial V_\vartheta}{\partial \varphi}\right]$ $+\vec{\mathbf{e}}_\vartheta \frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial V_r}{\partial \varphi} - \frac{\partial}{\partial r}(r V_\varphi)\right]$ $+\vec{\mathbf{e}}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}(r V_\vartheta) - \frac{\partial V_r}{\partial \vartheta}\right]$
ΔU	$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2}$ $+ \frac{\partial^2 U}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right)$ $+ \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial U}{\partial \vartheta}\right)$ $+ \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2 U}{\partial \varphi^2}$