1. By intuition, define a matrix that does each of the following linear transformations in $\mathbb{R}^2$:

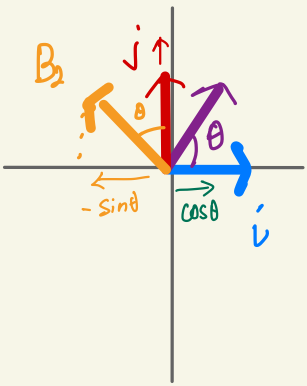
   a) Matrix A: Scale up by k times.
   b) Matrix B: Rotate CCW by angle θ.
   c) Matrix C: Reflect about the line y = x

   Hence, compose a single matrix D that performs all of the above linear transformations in the same order.

ANS: $D = \begin{bmatrix} k\sin\theta & k\cos\theta \\ k\cos\theta & -k\sin\theta \end{bmatrix}$

a) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \rightarrow T_A(\vec{v}) = A\vec{v}$

b)



$B = \begin{bmatrix} \vec{B_1} & \vec{B_2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\downarrow$

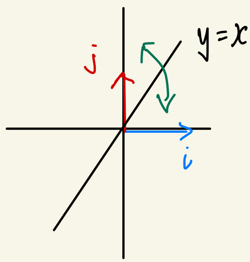
$T_B(\vec{v}) = B\vec{v}$

c)

$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\downarrow$

$T_C(\vec{v}) = C\vec{v}$



$D = C(B(A))$

$T_D(\vec{v}) = T_C(T_B(T_A(\vec{v}))) = C(B(A\vec{v}))$

$D = CBA$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

2. Solve the following SLEs using the matrix inverse. Which one is inconsistent or not linearly independent? Compare with your answers in tutorial 2.

a) $$\begin{aligned} 3x + 4y &= 1 \\ 2x + 3y &= 12 \end{aligned}$$

Does not exist

b) $$\begin{aligned} 3x - 2y &= 4 \\ -6x + 4y &= 7 \end{aligned}$$

Does not exist

c) $$\begin{aligned} u + v + w - 6 &= 0 \\ 4v + w + u &= 5 \\ 6 &= w + 3v + u \end{aligned}$$

ANS: **a)** x = -45, y = 34. **b)** Inconsistent / not L.I. **c)** Inconsistent / not L.I.

a) $$\underbrace{\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1} \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

$$= \begin{pmatrix} -45 \\ 34 \end{pmatrix}$$

$\therefore\ x = -45,\ y = 34$

b) $3x - 2y = 4 \rightarrow R_1$

$-6x + 4y = 7 \rightarrow R_2$

$R_2 = -2R_1$ $\therefore$ not L. I.

Check

$$\underbrace{\begin{pmatrix} 3 & -2 \\ -6 & 4 \end{pmatrix}}_{B} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$$

$\det(B) = 0$

c) $u + v + w = 6$

$u + 4v + w = 5$

$u + 3v + w = 6$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix}}_{C} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix}$$

$\det(C) = 1(4-3) - 1(1-1) + 1(3-4)$

$= 1 - 0 - 1$

$= 0$

since $\det(C) = 0$

$C^{-1} = 0$

$\therefore$ Inconsistent and not L. I.

3. Given the matrix B below, for what values of p is B not invertible?

$$B = \begin{bmatrix} 2 & 1 & p \\ 3 & 4 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

set $\det(B) = 0$
Solve for B

ANS: p = 3

Let $\det(B) = 0$

$$2(28-2) - 1(21+1) + p(-6-4) = 0$$

$$52 - 22 - 10p = 0$$

$$10p = 30$$

$$p = 3$$

4. Determine the inverse of the following matrices, if it exists.

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 2 & 0 & -3 \\ 0 & 3 & 1 \\ -1 & 4 & 2 \end{bmatrix} \\ + \end{array}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

ANS: $A^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 12 & -9 \\ 1 & -1 & 2 \\ -3 & 8 & -6 \end{bmatrix}$, $\underbrace{B^{-1}\text{ D.N.E.}}_{\det(B)=0}$, $C = \frac{1}{2}\begin{bmatrix} -2 & 2 & 2 \\ 2 & -1 & -1 \\ -2 & 3 & 1 \end{bmatrix}$

$$\det(A) = 2(6-4) - 0 + (-3)(3)$$

$$= 4 - 9$$

$$= -5$$

$$A_{adj} = \begin{bmatrix} 2 & -1 & 3 \\ -12 & 1 & -8 \\ 9 & -2 & 6 \end{bmatrix}^T = \begin{bmatrix} 2 & -12 & 9 \\ -1 & 1 & -2 \\ 3 & -8 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-5}\begin{bmatrix} 2 & -12 & 9 \\ -1 & 1 & -2 \\ 3 & -8 & 6 \end{bmatrix}$$

$$d(B) = 1(1+3) - 0 + 2(-2)$$

$$= 4 - 4 = 0$$

$\therefore B^{-1}$ does not exist

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\det(C) = 1(-1-1) - 2(0-2) + 0$$

$$= -2 + 4$$

$$= 2$$

$$C_{adj} = \begin{bmatrix} -2 & 2 & -2 \\ 2 & -1 & 3 \\ 2 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -1 & -1 \\ -2 & 3 & 1 \end{bmatrix}$$

$$C^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 2 & 2 \\ 2 & -1 & -1 \\ -2 & 3 & 1 \end{bmatrix}$$

5. **Modelling Problem:** In a MMORPG, there are three items that you can equip to increase your attack, defence and dexterity points. The number of each item you can equip are x, y and z respectively. The contributions of each item are shown below:

ITEM	ATTACK	DEFENCE	DEXTERITY
Dragon Scale (x)	-20	40	10
Griffin Claw (y)	50	10	-10
Elven Crystal (z)	10	10	60

In order to clear a level boss, you need to increase your attack by 320 points and defence by 280 points. If the total number of items you can equip is 22, determine the number of each item to equip for the boss fight. (Hint: Form a SLE and solve using the matrix inverse.)

ANS: x = 2, y = 4, z = 16.

$$-20x + 50y + 10z = 320$$

$$\downarrow$$

$$-2x + 5y + z = 32 \quad - R_1$$

$$4x + y + z = 28 \quad - R_2$$

$$x + y + z = 22 - R_3$$

$$\underbrace{\begin{bmatrix} -2 & 5 & 1 \\ 4 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{v}} = \underbrace{\begin{pmatrix} 32 \\ 28 \\ 22 \end{pmatrix}}_{\vec{b}}$$

$$\vec{v} = A^{-1} \vec{b}$$

$$\det(A) = (5-1) - (-2-4) + (-2-20)$$

$$= 4 + 6 - 22$$

$$= -12$$

$$A_{adj} = \begin{bmatrix} 0 & -3 & 3 \\ -4 & -3 & 7 \\ 4 & 6 & -22 \end{bmatrix}^T = \begin{bmatrix} 0 & -4 & 4 \\ -3 & -3 & 6 \\ 3 & 7 & -22 \end{bmatrix}$$

$$-\frac{1}{12}\begin{pmatrix} 0 & -4 & 4 \\ -3 & -3 & 6 \\ 3 & 7 & -22 \end{pmatrix}\begin{pmatrix} 32 \\ 28 \\ 22 \end{pmatrix} = -\frac{1}{12}\begin{pmatrix} -24 \\ -48 \\ -192 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4 \\ 16 \end{pmatrix}$$

$x = 2$, $y = 4$, $z = 16$

6. Find the eigenvalues and eigenvectors for the following matrices. Explain the eigenvectors for matrix E.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 7 & -3 \\ -2 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \; D = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}, \; E = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

ANS:

$$A: \; \lambda_1 = 1, \; \lambda_2 = 6, \; \vec{v_1} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \; \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B: \; \lambda_1 = 1, \; \lambda_2 = 8, \; \vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; \vec{v_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$C: \; \lambda_1 = 1, \; \lambda_2 = 2, \; \vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \; \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; D: \; \lambda_1 = -4, \; \lambda_2 = 3, \; \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \vec{v_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$E: \; \lambda_{1,2} = 4, \; \vec{v}_{1,2} = \begin{bmatrix} a \\ b \end{bmatrix} \; \forall \, a, \, b \; | \; \vec{v_1} \text{ not along the same span of } \vec{v_2}$$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix}$$

$$\lambda^2 - \underbrace{(a+d)}_{Tr(A)}\lambda + \underbrace{(ad-bc)}_{det(A)} = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = 6$$

For $\lambda_1 = 1$:

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}$$

$$3x + 2y = 0 \qquad y = -\frac{3}{2}x$$

$$2y = -3x$$

$$\therefore \vec{v_1} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

For $\lambda_2 = 6$:

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$-2x + 2y = 0 \qquad \therefore \vec{v_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y - x = 0$$

$$y = x$$

$$B - \lambda I = \begin{bmatrix} 7-\lambda & -3 \\ -2 & 2-\lambda \end{bmatrix}$$

$$\det(B - \lambda I) = \lambda^2 - 9\lambda + 8 = 0$$

$$(\lambda - 8)(\lambda - 1) = 0$$

$$\lambda_1 = 8 \text{ or } \lambda = 1$$

For $\lambda_1 = 8$:

$$\left[\begin{array}{cc|c} -1 & -3 & 0 \\ -2 & -6 & 0 \end{array}\right]$$

$R_2 \to 2R_1$

$$-x - 3y = 0$$

$$x = -3y$$

$$\therefore \vec{v}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 1$

$$\left[\begin{array}{cc|c} 6 & -3 & 0 \\ -2 & 1 & 0 \end{array}\right]$$

$R_1 \to -3R_2$

$$-2x + y = 0$$

$$2x = y$$

$$\therefore \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

6. Find the eigenvalues and eigenvectors for the following matrices. Explain the eigenvectors for matrix E.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix},\ B = \begin{bmatrix} 7 & -3 \\ -2 & 2 \end{bmatrix},\ C = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix},\ D = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix},\ E = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

ANS:

$$A:\ \lambda_1 = 1,\ \lambda_2 = 6,\ \vec{v_1} = \begin{bmatrix} 2 \\ -3 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B:\ \lambda_1 = 1,\ \lambda_2 = 8,\ \vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$C:\ \lambda_1 = 1,\ \lambda_2 = 2,\ \vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ D:\ \lambda_1 = -4,\ \lambda_2 = 3,\ \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$E:\ \lambda_{1,2} = 4,\ \vec{v}_{1,2} = \begin{bmatrix} a \\ b \end{bmatrix}\ \forall\, a,\, b \mid \vec{v_1} \text{ not along the same span of } \vec{v_2}$$

c) $\lambda^2 - 3\lambda - 2 = 0$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9 - 4(1)(-2)}}{2}$$

$$= \frac{3}{2} \pm \frac{\sqrt{17}}{2}$$

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{17}}{2}$$

$$(C - \lambda_1 I)\vec{v_1} = \vec{0}$$

$$\begin{bmatrix} \frac{3}{2} - \frac{\sqrt{17}}{2} & 1 \\ 2 & -\frac{3}{2} - \frac{\sqrt{17}}{2} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$y = -\frac{3 - \sqrt{17}}{2} x \longrightarrow \vec{v_1} = \begin{pmatrix} 1 \\ \frac{\sqrt{17} - 3}{2} \end{pmatrix}$$

Check: $2x - \left[\frac{3 + \sqrt{17}}{2} \left(-\frac{3 - \sqrt{17}}{2} x \right) \right] = 2x + \frac{9-17}{4} x$

$= 2x - 2x = 0$

6. Find the eigenvalues and eigenvectors for the following matrices. Explain the eigenvectors for matrix E.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix},\ B = \begin{bmatrix} 7 & -3 \\ -2 & 2 \end{bmatrix},\ C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix},\ D = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix},\ E = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

ANS:

$$A:\ \lambda_1 = 1,\ \lambda_2 = 6,\ \vec{v_1} = \begin{bmatrix} 2 \\ -3 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B:\ \lambda_1 = 1,\ \lambda_2 = 8,\ \vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$C:\ \lambda_1 = 1,\ \lambda_2 = 2,\ \vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ D:\ \lambda_1 = -4,\ \lambda_2 = 3,\ \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$E:\ \lambda_{1,2} = 4,\ \vec{v}_{1,2} = \begin{bmatrix} a \\ b \end{bmatrix}\ \forall\, a,\ b\ |\ \vec{v_1}\ \text{not along the same span of}\ \vec{v_2}$$

$$C - \lambda I = \begin{bmatrix} 3-\lambda & -1 \\ 2 & 0-\lambda \end{bmatrix}$$

$$\det(C - \lambda I) = \lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda_1 = 2 \text{ or } \lambda_2 = 1$$

For $\lambda_1 = 2$:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$R_2 \to 2R_1$

$x - y = 0$

$x = y$

$\therefore \vec{V_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 1$:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$R_2 = R_1$

$2x = y$

$\therefore \vec{V_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$D - \lambda I = \begin{bmatrix} -2-\lambda & -2 \\ -5 & 1-\lambda \end{bmatrix}$$

$$\det(D - \lambda I) = \lambda^2 + \lambda - 12 = 0$$

$$(\lambda+4)(\lambda-3) = 0$$

$$\lambda_1 = -4 \qquad \lambda_2 = 3$$

For $\lambda_1 = -4$:

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ -5 & 5 & 0 \end{array}\right]$$

$$2x - 2y = 0$$

$$x = y$$

$$\therefore \vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -4 \end{pmatrix} = -4\vec{v_1}$$

For $\lambda_2 = 3$:

$$\left[\begin{array}{cc|c} -5 & -2 & 0 \\ -5 & -2 & 0 \end{array}\right]$$

$$-5x - 2y = 0$$

$$5x = -2y$$

$$x = -\frac{2}{5}y$$

$$\therefore \vec{v_2} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 15 \end{pmatrix}$$

$$= 3\begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

Checked

$E = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$\lambda_{1,2} = 4$

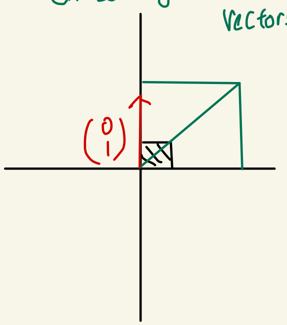
$(E - \lambda I)\vec{r} = \vec{0}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$\to 0x + 0y = 0 \to x$ & y can be anything

$\therefore$ Any vector is an vector

Choose any 2 non-parallel vectors like $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$S = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \to \lambda^2 - 2k\lambda + k^2 = 0$

$(\lambda - k)^2 = 0$

$\lambda_{1,2} = k$

For a scaling matrix, the eigenvalues are the diagonal elements.

Consider any triangular matrix : $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

$$\det \begin{bmatrix} a-\lambda & b & c \\ 0 & d-\lambda & e \\ 0 & 0 & f-\lambda \end{bmatrix} = \underbrace{(f-\lambda)(a-\lambda)(d-\lambda)}_{\lambda_{1,2,3} = a, d, f} = 0$$

For any triangular or diagonal matrix, the diagonal elements are eigenvalues

7. Diagonalize:

$$A = \begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & -2 \\ -1 & 8 \end{bmatrix}$$

That is, find an invertible matrix P and diagonal matrix D such that each matrix is expressed as PDP^{-1}. Verify your answer.

ANS: $A = \begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{-4} \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 5 \\ -1 & -2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda_1 = 3 \quad \lambda_2 = -1$$

For $\lambda_1 = 3$:

$$\left[\begin{array}{cc|c} 1 & 5 & 0 \\ -1 & -5 & 0 \end{array}\right]$$

$$x + 5y = 0$$

$$x = -5y$$

$$\therefore \vec{v_1} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1$

$$\left[\begin{array}{cc|c} 5 & 5 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$x + y = 0$$

$$\therefore \vec{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} -5 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{\det(P)} P_{adj}$$

$$= \frac{1}{4}\begin{bmatrix} -1 & -1 \\ -1 & -5 \end{bmatrix}$$

$$PDP^{-1} = \begin{pmatrix} -5 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{4}\begin{pmatrix} -1 & -1 \\ -1 & -5 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -15 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & -5 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 16 & 20 \\ -4 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 \\ -1 & -2 \end{pmatrix} = A$$

$$B = \begin{bmatrix} 7 & -2 \\ -1 & 8 \end{bmatrix}$$

$$B - \lambda I = \begin{bmatrix} 7-\lambda & -2 \\ -1 & 8-\lambda \end{bmatrix}$$

$$\det(B - \lambda I) = \lambda^2 - 15\lambda + 54 = 0$$

$$(\lambda - 6)(\lambda - 9) = 0$$

$$\lambda_1 = 6 \quad \lambda_2 = 9$$

For $\lambda_1 = 6$:

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$x - 2y = 0$$

$$x = 2y$$

$$\therefore \vec{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 9$:

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$-x - y = 0$$

$$x = -y$$

$$\therefore \vec{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 \\ 0 & 9 \end{pmatrix}$$

$$P^{-1} = \frac{1}{\det(P)} P_{adj}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$PDP^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 9 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 12 & -9 \\ 6 & 9 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 21 & -6 \\ -3 & 24 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -2 \\ -1 & 8 \end{pmatrix}$$

8. Given the following matrix,

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix}$$

$\lambda_{1,2} = -2, 6$

$\vec{v}_{1,2} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

compute $A^5 + 7A^4 + I$. Also compute $\sqrt{5I + A}$ if it exists.

ANS:

$$A^5 + 7A^4 + I = \begin{bmatrix} 6369 & 10480 \\ 6288 & 10561 \end{bmatrix}, \quad \sqrt{5I + A} = \frac{1}{8}\begin{bmatrix} 5\sqrt{3} + 3\sqrt{11} & 5\sqrt{11} - 5\sqrt{3} \\ 3\sqrt{11} - 3\sqrt{3} & 3\sqrt{3} + 5\sqrt{11} \end{bmatrix}$$

$$A^5 + 7A^4 + I = PD^5P^{-1} + 7PD^4P^{-1} + PIP^{-1}$$
$$= P(D^5 + 7D^4 + I)P^{-1}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda - 12 = 0$$
$$(\lambda - 6)(\lambda + 2) = 0$$
$$\lambda_1 = 6 \quad \text{or} \quad \lambda_2 = -2$$

For $\lambda_1 = 6$:

$$\begin{bmatrix} -5 & 5 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}$$
$$-5x + 5y = 0$$
$$x = y$$
$$\therefore \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -2$

$$\begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}$$
$$3x + 5y = 0$$
$$x = -\frac{5}{3}y$$
$$\therefore \vec{v}_2 = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$$P = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{8} \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$A^5 + 7A^4 + I = PD^5P^{-1} + 7PD^4P^{-1} + PIP^{-1}$$

$$= P(D^5 + 7D^4 + I)P^{-1}$$

$$D^5 + 7D^4 + I = \begin{bmatrix} 6^5 & 0 \\ 0 & (-2)^5 \end{bmatrix} + 7\begin{bmatrix} 6^4 & 0 \\ 0 & (-2)^4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16849 & 0 \\ 0 & 81 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 16849 & 0 \\ 0 & 81 \end{bmatrix} = \begin{bmatrix} 16849 & -405 \\ 16849 & 243 \end{bmatrix}$$

$$\frac{1}{8} \begin{bmatrix} 16849 & -405 \\ 16849 & 243 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 50952 & 83840 \\ 50304 & 84488 \end{bmatrix}$$

$$= \begin{bmatrix} 6369 & 10480 \\ 6288 & 10561 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{8}\begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$\sqrt{5I + A} = P\sqrt{5I + D}\,P^{-1}$$

$$= \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \sqrt{5+6} & 0 \\ 0 & \sqrt{5-2} \end{bmatrix}\frac{1}{8}\begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} \sqrt{11} & -5\sqrt{3} \\ \sqrt{11} & 3\sqrt{3} \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 3\sqrt{11} + 5\sqrt{3} & 5\sqrt{11} - 5\sqrt{3} \\ 3\sqrt{11} - 3\sqrt{3} & 5\sqrt{11} + 3\sqrt{3} \end{bmatrix}$$

10. Determine the eigendecomposition of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$$

ANS: $$A = \begin{bmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \frac{1}{-56} \begin{bmatrix} 0 & 8 & 0 \\ -21 & -14 & -14 \\ -7 & -2 & 14 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 3 & 1 & -3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0 + (2-\lambda)\left[(1-\lambda)(-3-\lambda) - 12\right] - 0$$

$$= (2-\lambda)\left[-3 - \lambda + 3\lambda + \lambda^2 - 12\right] = 0$$

$$= (2-\lambda)\left[\lambda^2 + 2\lambda - 15\right] = 0$$

$$= (2-\lambda)\left[(\lambda+5)(\lambda-3)\right] = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = -5$$

For $\lambda_1 = 2$:

$$\begin{bmatrix} -1 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & -5 & 0 \end{array}\right] \xrightarrow{R_3 + 3R_1} \left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 7 & 0 \end{array}\right]$$

R_3: $y + 7z = 0$ $\quad$ R_1: $-x + 4z = 0$

$y = -7z$ $\quad$ $x = 4z$

$$\therefore \vec{v_1} = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3$:

$$\begin{bmatrix} -2 & 0 & 4 \\ 0 & -1 & 0 \\ 3 & 1 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$$

From R_2: $y = 0$

$-2x + 4z = 0 - R_1 \rightarrow x - 2z = 0$

$3x + \cancel{y} - 6z - R_3 \rightarrow x - 2z = 0$

same

$x = 2z$

$$\therefore \vec{v_2} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = -5$:

$$\begin{bmatrix} 6 & 0 & 4 \\ 0 & 7 & 0 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$$

From R_2 : $y = 0$

$6x + 4z = 0 - R_1 \longrightarrow 3x + 2z = 0$

same

$3x + \cancel{y} + 2z = 0 - R_3$

$3x = -2z \qquad x = -\frac{2}{3}z$

$$\therefore \vec{V_3} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$$

$$P = \begin{pmatrix} 4 & 2 & -2 \\ -7 & 0 & 0 \\ 1 & 1 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\det(P) = -(-7)(6-(-2))$$
$$= 56$$

$$P_{adj} = \begin{pmatrix} 0 & 21 & -7 \\ -8 & 14 & -2 \\ 0 & 14 & -14 \end{pmatrix}^T = \begin{pmatrix} 0 & -8 & 0 \\ 21 & 14 & 14 \\ -7 & -2 & 14 \end{pmatrix}$$

$$P^{-1} = \frac{1}{-56}\begin{pmatrix} 0 & -8 & 0 \\ 21 & 14 & 14 \\ -7 & -2 & 14 \end{pmatrix}$$

$$PDP^{-1} = \begin{pmatrix} 4 & 2 & -2 \\ -7 & 0 & 0 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}\frac{1}{56}\begin{pmatrix} 0 & -8 & 0 \\ 21 & 14 & 14 \\ -7 & -2 & 14 \end{pmatrix}$$

$$= \frac{1}{56}\begin{pmatrix} 56 & 0 & 224 \\ 0 & 112 & 0 \\ 168 & 56 & -168 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} = A$$

11. Show that $\mathbf{x_1}$ is an eigenvector of matrix A.

$$A = \begin{bmatrix} -1 & 1 & 2 \\ -6 & 2 & 6 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the other eigenvectors and hence determine the eigendecomposition of A.

ANS: $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 \\ -6 & 2 & 6 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$= 2\vec{x_1}$$

(shown, $\lambda = 2$)

$$A - \lambda I = \begin{bmatrix} -1-\lambda & 1 & 2 \\ -6 & 2-\lambda & 6 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$\det(A - \lambda I)$

$$= (-1-\lambda)\left[(2-\lambda)(1-\lambda) - 6\right] - (-6)\left[(1-\lambda) - 2\right] + 0$$

$$= (-1-\lambda)\left[2 - 3\lambda + \lambda^2 - 6\right] + 6\left[-1-\lambda\right]$$

$$= (-1-\lambda)\left[\overbrace{\lambda^2 - 3\lambda - 4 + 6}^{\lambda^2 - 3\lambda + 2}\right]$$

$$= (-1-\lambda)(\lambda - 2)(\lambda - 1) = 0$$

$\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 1$
(found)

For $\lambda_2 = -1$:

$$\begin{bmatrix} -2 & 1 & 2 \\ -6 & 3 & 6 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ -6 & 3 & 6 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ -6 & 3 & 6 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right]$$

From R_3: $2x = 0$

From R_1: $\cancel{-2x} + y + 2z = 0$

$y = -2z$

$$\therefore \vec{v_2} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 1$:

$$\begin{bmatrix} -2 & 1 & 2 \\ -6 & 1 & 6 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$$

From R_3 : $y = 0$

From R_1 : $-2x + \cancel{y} + 2z = 0$

$z = x$

$\therefore \vec{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(P) = (-2) - 0 + (1+2)$

$= 1$

$$P_{adj} = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} = P^{-1}$$

$$PDP^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ -6 & 2 & 6 \\ 0 & 1 & 1 \end{bmatrix} = A$$

12. Show that $\mathbf{v_1}$ is an eigenvector of matrix A.

$$A = \begin{bmatrix} -5 & 8 & 32 \\ 2 & 1 & -8 \\ -2 & 2 & 11 \end{bmatrix}, \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Find the other eigenvectors and describe the eigenspace geometrically. Diagonalize A.

ANS: $$A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 5 \\ -1 & 2 & 6 \\ 1 & -1 & -4 \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} -10 & -16 & 32 \\ 4 & -2 & -8 \\ -4 & -4 & 11 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$= 3\vec{v}_1$ (so $\vec{v}_1$ is an eigenvector with $\lambda = 3$)

$$\det(A - \lambda I) = \begin{bmatrix} -5-\lambda & 8 & 32 \\ 2 & 1-\lambda & -8 \\ -2 & 2 & 11-\lambda \end{bmatrix}$$

$$= -(5+\lambda)\left[(1-\lambda)(11-\lambda) + 16\right] - 8\left[2(11-\lambda) - 16\right] + 32\left[4 + 2(1-\lambda)\right]$$

$$= -(5+\lambda)\underbrace{\left[11 - 12\lambda + \lambda^2 + 16\right]}_{\substack{\lambda^2 - 12\lambda + 27 \\ (\lambda-3)(\lambda-9)}} \quad -8(22 - 2\lambda - 16) + 32(4 + 2 - 2\lambda)$$

$$-8\left[2(-\lambda+3)\right] + 32\left[2(-\lambda+3)\right]$$

$$= -(5+\lambda)(\lambda-3)(\lambda-9) + 16(\lambda-3) - 64(\lambda-3)$$

$$= (\lambda-3)\left[-(5+\lambda)(\lambda-9) + 16 - 64\right]$$

$$= (\lambda-3)\left[-\lambda^2 + 4\lambda + \underbrace{45 - 48}_{-3}\right]$$

$$\begin{matrix} -\lambda & 1 \\ \lambda & -3 \end{matrix}$$

$$= (\lambda-3)\left[(-\lambda+1)(\lambda-3)\right]$$

$$= (\lambda-3)^2(-\lambda+1) = 0$$

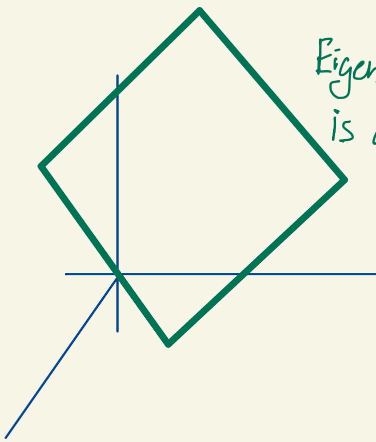
$$\lambda_{1,2} = 3 \ (AM = 2), \quad \lambda_3 = 1$$

For $\lambda_{1,2} = 3$,

$$\begin{bmatrix} -8 & 8 & 32 \\ 2 & -2 & -8 \\ -2 & 2 & 8 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$$

$$-2x + 2y + 8z = 0$$

$$x = y + 4z$$



Eigenspace for $\lambda = 3$ is a plane ($GM = 2$)

$$\vec{v_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

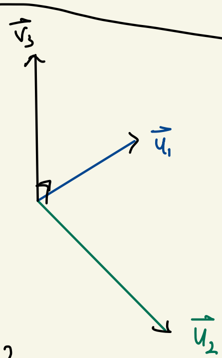
($\vec{v_1}$ not parallel to $\vec{v_2}$)

For $\lambda_3 = 1$

$$\begin{array}{c} \\ \vec{u}_1 \to \\ \vec{u}_2 \to \end{array} \begin{bmatrix} -6 & 8 & 32 \\ 2 & 0 & -8 \\ -2 & 2 & 10 \end{bmatrix} \overset{\vec{v}_3 \downarrow}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \vec{0}$$

Another way to get eigenvector

$$\left.\begin{array}{l} \vec{u}_1 \cdot \vec{v}_3 = 0 \\ \\ \vec{u}_2 \cdot \vec{v}_3 = 0 \end{array}\right\}$$



$$\vec{v}_3 = \vec{u}_1 \times \vec{u}_2 = \begin{pmatrix} 2 \\ 0 \\ -8 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 16 \\ -4 \\ 4 \end{pmatrix}$$

Choose $\vec{v}_3 = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$

$$\therefore \vec{v}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 4 & -2 & -2 \\ -7 & 0 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

$$\det(P) = 1(0) - 3(0-(-7)(-2)) + 1(0-(-7)(-2))$$

$$= 28$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$P_{adj} = \begin{pmatrix} 0 & 7 & -21 \\ -4 & 6 & -14 \\ 0 & 14 & -14 \end{pmatrix}^T = \begin{pmatrix} 0 & -4 & 0 \\ 7 & 6 & 14 \\ -21 & -14 & -14 \end{pmatrix}$$

$$P^{-1} = \frac{1}{28} \begin{pmatrix} 0 & -4 & 0 \\ 7 & 6 & 14 \\ -21 & -14 & -14 \end{pmatrix}$$

$$PDP^{-1} = \begin{pmatrix} 4 & -2 & -2 \\ -7 & 0 & 0 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \frac{1}{28}$$