1. A surface is defined by the vector function

$$\mathbf{r}(s,t) = [s^2\cos t \quad s^2\sin t \quad s]^T$$

a) Evaluate the normal vectors to the surface at (1, 0, -1).

b) Determine the Cartesian equation of the tangent plane at (1, 0, -1).

c) Determine the Cartesian equation of the surface in the form F(x, y, z) = 0.

ANS: **a)** $\mathbf{N} = \pm[1 \quad 0 \quad 2]^T$. **b)** $x + 2z = -1$. **c)** $x^2 + y^2 - z^4 = 0$.

a) $\vec{r}_s = \begin{pmatrix} 2s\cos t \\ 2s\sin t \\ 1 \end{pmatrix} \qquad \vec{r}_t = \begin{pmatrix} -s^2\sin t \\ s^2\cos t \\ 0 \end{pmatrix}$

$$\vec{N} = \vec{r}_s \times \vec{r}_t = \begin{pmatrix} -s^2\cos t \\ -s^2\sin t \\ 2s^3\cos t + 2s^3\sin^2 t \end{pmatrix}$$

$$= \begin{pmatrix} -s^2\cos t \\ -s^2\sin t \\ 2s^3 \end{pmatrix}$$

At $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad 1 \rightarrow s^2\cos t = 1$, $\quad -1 \rightarrow s = -1$, $\quad t = 0$

$\vec{N}$ at $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$

2. Evaluate the flux of the vector field below across the triangular surface S that is the plane 2x - 2y + z = 2 cut out by the coordinate planes. The surface is orientated with an upward-pointing normal.

$$z = 2 - 2x + 2y \qquad \mathbf{F}(x, y, z) = [x \quad y \quad z]^T$$

ANS: Flux = 1.

parameterize surface S

$$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ 2-2x+2y \end{pmatrix}$$

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

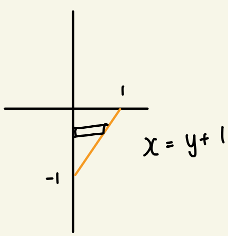
$$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \leftarrow \text{upward}$$

$$\vec{F} \cdot \vec{N} = \begin{pmatrix} x \\ y \\ 2-2x+2y \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$= 2x - 2y + 2 - 2x + 2y$$

$$= 2$$

$$\text{Flux} = \iint \vec{F} \cdot \vec{N} \, dx\,dy = \int_{-1}^{0} \int_{0}^{y+1} 2 \, dx\,dy$$



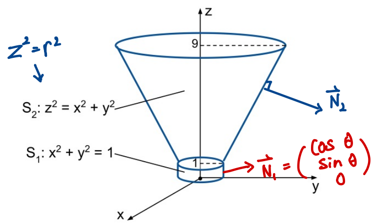
$$= \int_{-1}^{0} [2y+2] \, dy$$

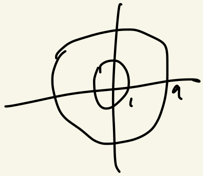
$$= [y^2 + 2y] \Big|_{-1}^{0}$$

$$= 0 - (1-2)$$

$$= 1$$

4. A surface $S = S_1 + S_2$ that looks like a funnel is shown below.





a) Determine the outward-pointing normals of surfaces S_1 and S_2.

b) Evaluate the flux of **F** below through S, which is orientated by outward-pointing normals.

$$\mathbf{F}(x, y, z) = [-y \quad x \quad z]^T$$

ANS: **a)** $S_1 : \mathbf{N} = [x \quad y \quad 0]^T$. $S_2 : \mathbf{N} = [x \quad y \quad -z]^T$. **b)** -1456π/3.

a) S_1: $N_1 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

S_2 = $N_2 = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$ ← outward

b) Parameterise S_2 : $\vec{r}(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix}$

$$N = \vec{r}_r \times \vec{r}_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r\cos^2\theta + r\sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix}$$ ← Inward

$$\vec{N}_0 = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ -r \end{pmatrix}$$

$$\text{Flux} = \iint \vec{F} \cdot \vec{N}_0 \, dr\,d\theta = \iint \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ r \end{pmatrix} \cdot \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ -r \end{pmatrix} dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^9 -r^2 \, dr\,d\theta$$

$$= -\int_0^{2\pi} d\theta \cdot \int_1^9 r^2\, dr$$

$$= -2\pi \cdot \left[\frac{r^3}{3}\right]_1^9$$

$$= \frac{-1456\pi}{3}$$

Parameterise s_1 : $\vec{N}_0 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

$$\text{Flux} : \iint \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} dr\,d\theta = \iint 0\, dr\,d\theta$$

$$= 0$$

$$\therefore \text{Flux}_S = \text{Flux}_{s_1} + \text{Flux}_{s_2}$$

$$= \frac{-1456\pi}{3}$$

5. Use the divergence theorem to evaluate the flux of the vector field below through surface S of the unit cube in the domain [0, 1] × [0, 1] × [0, 1],

integration used in

$$\mathbf{V}(x, y, z) = \left[ze^{x^2} \quad 3y \quad 2 - yz\right]^T$$

ANS: Flux = e/2 + 2.

$$\iiint_V \vec{\nabla} \cdot \vec{v} \, dv$$

$$= \iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dv$$

$$= \iiint \left(2xze^{x^2} + 3 - y \right) dv$$

$$= \int_0^1 \int_0^1 \int_0^1 \left(2xze^{x^2} + 3 - y \right) dz \, dy \, dx$$

$$= \int_0^1 \int_0^1 \left[xz^2 e^{x^2} + 3z - yz \right]_0^1 dy \, dx$$

$$= \int_0^1 \int_0^1 \left(xe^{x^2} + 3 - y \right) dy \, dx$$

$$= \int_0^1 \left[xye^{x^2} + 3y - \frac{y^2}{2} \right] \Big|_0^1 dx$$

$$= \int_0^1 \left(xe^{x^2} + \frac{5}{2} \right) dx$$

Let $x^2 = u$
$du = 2x \, dx$

$$= \int_0^1 \cancel{x} e^u \frac{du}{2\cancel{x}} + \int_0^1 \frac{5}{2} dx$$

$$= \frac{1}{2}\left(e^1 - e^0\right) + \frac{5}{2}$$

$$= \frac{e}{2} - \frac{1}{2} + \frac{5}{2}$$

$$= \frac{e}{2} + 2$$

10. (https://openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem)
Use Stokes' theorem to evaluate the line integral below, where C is the curve given by x = cos t, y = sin t, z = sin t, 0 ≤ t ≤ 2π, traversed in the direction of increasing t.

$$\int_C [2xy^2z\,dx + 2x^2yz\,dy + (x^2y^2 - 2z)\,dz]$$

ANS: 0.

For curve C,

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ \sin t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z=y \end{pmatrix}$$

$$\vec{r}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{r}(2\pi) \rightarrow C \text{ is closed}$$

$$\oint_C \underbrace{[2xy^2z\,dx + 2x^2yz\,dy + (x^2y^2 - 2z)\,dz]}_{\begin{pmatrix} 2xy^2z \\ 2x^2yz \\ x^2y^2-2z \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \vec{F}\cdot\vec{dr}}$$

$$\text{stokes' theorem} = \iint_S \underbrace{(\vec{\nabla}\times\vec{F})}_{\vec{0}} \cdot \vec{N}\,dA$$

$$= \iint_S 0\,dA$$

$$= 0$$

$$\vec{\nabla}\times\vec{F} = \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} 2xy^2z \\ 2x^2yz \\ x^2y^2-2z \end{pmatrix}$$

$$= \begin{pmatrix} 2x^2y - 2x^2y \\ -(2xy^2 - 2xy^2) \\ 4xyz - 4xyz \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

11. Use Stokes' theorem to evaluate the line integral below, where C is the intersection curve between the plane $x+y+z=8$ and the cylinder $x^2+y^2=9$, oriented counterclockwise.

$$\int_C \mathbf{F}\cdot d\mathbf{r}, \quad \mathbf{F}(x,y,z)=\begin{bmatrix} x^2z \\ xy^2 \\ z^2 \end{bmatrix}$$

ANS: 81π/2.

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{N}\,dA$$

For curve C,

$$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ 8-x-y \end{pmatrix}$$

$$\vec{\nabla}\times\vec{F} = \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} x^2z \\ xy^2 \\ z^2 \end{pmatrix}$$

$$\vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -(0-x^2) \\ y^2-0 \end{pmatrix}$$

$$\vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

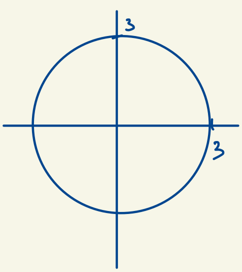
$$= \begin{pmatrix} 0 \\ x^2 \\ y^2 \end{pmatrix}$$

$$\vec{N} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(\vec{\nabla}\times\vec{F})\cdot\vec{N} = \begin{pmatrix} 0 \\ x^2 \\ y^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= x^2+y^2$$

$$\iint (x^2+y^2)\,dA = \iint (x^2+y^2)\,dx\,dy$$



$$= \int_0^{2\pi}\int_0^3 (r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4}\right]\Big|_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{81}{4}\,d\theta$$

$$= \left[\frac{81}{4} \theta \right] \Big|_0^{2\pi}$$

$$= \frac{81\pi}{2}$$

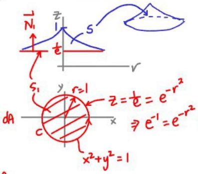
12. Using Stokes' theorem, evaluate the circulation of F over surface S defined below.

$$\iint_S (\nabla\times\vec{F})\cdot\vec{N}\,dA$$

$$\mathbf{F}(x,y,z)=\begin{bmatrix} e^{y+z}-2y \\ xe^{y+z}+y \\ e^{x+y}\end{bmatrix},\quad S:\left\{(x,y,z)\,\middle|\, z=e^{-(x^2+y^2)},\, z\geq 1/e\right\}$$

compute this! (but very tedious!)

ANS: 2π.



$$\iint_S (\nabla\times\vec{F})\cdot\vec{N}\,dA = \oint_C \vec{F}\cdot d\vec{r}$$

relate to a simpler surface.

$$= \iint_{S_1}(\nabla\times\vec{F})\cdot\vec{N}_1\,dA$$

$$= \iint_{S_1}\begin{pmatrix} \vdots \\ \vdots \\ \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}dA$$

$z=\frac{1}{e}=e^{-r^2} \Rightarrow e^{-1}=e^{-r^2} \rightarrow r^2=1$, $r=1$.

$x^2+y^2=1$

$$= \iint_{S_1} e^{y+z}-\left(e^{y+z}-2\right)dA$$

$$= \iint_{S_1} 2\,dA = 2\times \text{Area of } S_1 = 2\pi(1)^2 = 2\pi$$

parameterize S:

$$\vec{r}_s(t) = \begin{pmatrix}\cos t\\ \sin t\\ \frac{1}{e}\end{pmatrix}$$

$$\vec{r}_s\,'(t) = \begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix}$$

$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\begin{pmatrix} e^{\sin t+\frac{1}{e}}-2\sin t \\ \cos t\, e^{\sin t+\frac{1}{e}}+\sin t \\ e^{\cos t+\sin t}\end{pmatrix}\cdot\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix}dt$$