0.1 Softmax [20 points]

I) [10 point] Prove that softmax is invariant to constant sifts in the input, i.e., for any input vector x and a constant scalar c, the following holds:

softmax(x) = softmax(x -+-c) ,

where softmax(xh —4— and x -kc means adding c to every dimension of x.

2) [10 point] Let z = W x -F c, where W and c are some matrix and vector, respectively. $\frac{\partial J}{\partial c}$. Let

J = log .

① Given $\text{softmax}(x)_i = \frac{e^{x_i}}{\sum_{i'} e^{x_{i'}}}$

$$\text{softmax}(x+c)_i = \frac{e^{x_i + c}}{\sum_{i'} e^{x_{i'}} e^{c}} = \frac{e^{x_i} e^{c}}{\sum_{i'} e^{x_{i'}} e^{c}} = \frac{e^{x_i} \cancel{e^{c}}}{\cancel{e^{c}} \sum_{i'} e^{x_{i'}}}$$

cancel

cancel

Calculate the derivatives of J w.r.t. W and c, respectively, i. e. , calculate _uu- and

$$= \frac{e^{x_i}}{\sum_{i'} e^{x_{i'}}} \equiv \text{softmax}(x)$$

② $J = \sum_i \text{Log softmax}(z)_i$

if $\text{softmax}(x)_i \triangleq \frac{e^{x_i}}{\sum_{i'} e^{x_{i'}}}$ then,

$\text{softmax}(z)_i \triangleq \frac{e^{z_i}}{\sum_{i'} e^{z_{i'}}}$ and,

$\boxed{j = i'}$

$$J = \cancel{\sum_i} \log \frac{e^{z_i}}{\cancel{\sum_{i'}} e^{z_{i'}}} = \log \frac{e^{z_i}}{i' e^{z_{i'}}} = \log \frac{e^{z_i}}{j^2 e^{z}}$$

$$\frac{dJ}{dw} = \frac{d}{dw}\left(\log \frac{e^{z_i}}{j^2 e^{z}}\right) = 0$$ since all term are constants

then $j = i'$

$$\frac{dJ}{dc} = \frac{d}{dc}\left(\log \frac{e^{z_i}}{j^2 e^{z}}\right) = 0$$ since all term are constants

0.2 Logistic Regression with Regularization [20 points]

1) [10 point] Let the data be (Xi, where Xi G IR^d and Yi G {O, 1}. Logistic regression is a binary classification model, with the probability of Yi being I as:

$$p(Yi = 1; Xi\ 0) = a\ 0^T Xi \triangleq \frac{1}{1+e^{-\boldsymbol{\theta}^T \mathbf{x}_i}},$$

where is the model parameter. Assume we impose an 102 regularization term on the parameter, defined as:

$$\mathcal{R}(\boldsymbol{\theta}) = \frac{\lambda}{2}\boldsymbol{\theta}^T\boldsymbol{\theta}$$

① $p(y_i = 1, X_i, \theta) = \sigma(\theta^T x_i) \triangleq \frac{1}{1+e^{-\theta^T x_i}}$

$R(\theta) = \frac{\lambda}{2}\theta^T\theta$

$J(\theta) = \frac{1}{m}\left[y^T \log(p(x,\theta)) + (1-y)^T \log(1-p(x,\theta))\right.$

$\underbrace{+\frac{\lambda}{2}\theta^T\theta}_{L_2 \text{ Regularization}}$

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with a positive constant A. Write out the final objective function for this logistic regression with regularization model.

② $\frac{dJ(\theta)}{d\theta} = \frac{d}{d\theta}\frac{1}{m}\sum_{i=1}^{m}\left[y_i \log(\sigma(x_i)) + (1-y_i)\log(1-\sigma(x_i))\right)$

2) [10 point] If we use gradient descent to solve the model parameter. Derive the updating rule for O. Your answer should contain the derivation, not just the final answer.

$= \frac{1}{m}\sum_{i=1}^{m}\left[y_i \frac{1}{\sigma(y_i)}\sigma(\theta^T x_i)(1-\sigma(\theta^T x_i)) \cdot \frac{d}{d\theta}(\theta^T x_i) +\right.$

$- \frac{1-y_i}{1-\sigma(x_i)}(1-\sigma(\theta^T x_i))\,\sigma(\theta^T x)\frac{d}{d\theta}(\theta^T x_i)$

$$= \frac{1}{m} \sum x_i \left[y_i - y_i \sigma(\theta^T x_i) - \sigma(\theta^T x_i) + y_i \sigma(\theta^T x_i) \right]$$

$$= \frac{1}{m} \sum x_i \left[y_i - \sigma(\theta^T x_i) \right]$$

0.3 Derivative of the Softmax Function [30 points]

I) [10 point] Define the loss function as

$$J(\mathbf{z}) = -\sum_{k=1}^{K} y_k \log \tilde{y}_k \ ,$$

where , and , • • • , !/K) is a known probability vector. Derive the DJ(z)

Note z — ZK) is a vector so DJ(z) is in the form of a vector. Your answer should contain the derivation, not just the final answer.

2 [10 point] Assume the above softmax is the output layer of an FNN. Briefly explain how $\frac{\partial \dot{}}{\partial \mathbf{W}}$ the derivative is used in the backpropagation algorithm.

(1) $J(z) = -\sum_{k=1}^{K} y_k \log \tilde{y}_k$ & $\tilde{y}_k = \dfrac{e^{z_k}}{\sum_{k'} e^{z_{k'}}}$ & $\dfrac{dJ(z)}{dz} = \begin{bmatrix} dJ/dz_1 \\ \vdots \\ dJ/dz_K \end{bmatrix}$

so

$$\frac{dJ(z)}{dz} = -\frac{d}{dz} \sum_{k=1}^{K} y_k \log \tilde{y}_k$$

$$= -\frac{d}{dz} \sum_{k=1}^{K} y_k \log \frac{e^{z_k}}{\sum_{k'} e^{z_{k'}}}$$

3) [10 points] Let z = h + b), where a(•) is the sigmoid function, W is a matrix, b and h are

$$= -\frac{d}{dz} \sum_{k=1}^{K} y_k z_k + \frac{d}{dz} \sum_{k=1}^{K} y_k \log \sum_{k'} e^{z_{k'}}$$

vectors. Use the chain rule to calculate the gradient of W and b, i.e., and Db, respectively (it is

$$= -y_k + \frac{d}{dz} \sum_{k=1}^{K} y_k \log \sum_{k'} e^{z_{k'}}$$

enough to derive the gradients for one element of the matrix/vector parameter).

$$= -y_k + \sum_{k=1}^{K} y_k \cdot \frac{d}{dz} \log \sum_{k'} e^{z_{k'}}$$

$$= -y_k + \sum_{k=1}^{K} y_k \cdot \frac{1}{\sum_{k'} e^{z_{k'}}} \cdot e^{z_{k'}}$$

$$= -y_k + \frac{e^{z_k}}{\sum e^{z_{k'}}} \Big(\underbrace{\sum_{k=1}^{K} y_k}_{1} \Big)$$

$= -y_k + \tilde{y}_k$ this means $\frac{dJ(z)}{dz} = \begin{bmatrix} -y_1 + \tilde{y}_1 \\ -y_2 + \tilde{y}_2 \\ \vdots \\ -y_k + \tilde{y}_k \end{bmatrix}$

(2) The derivative is used in backpropigation to update the weights of a FNN during training to help minimize loss.

(3) $z = \sigma(W^T h + b)$

$\frac{dJ}{dW} = ?$ $\frac{dJ}{db} = ?$

$\frac{dJ}{dW} = \frac{dJ}{dz} \cdot \frac{dz}{dW}$ $\frac{dJ}{dz} = \begin{bmatrix} -y_1 + \tilde{y}_1 \\ -y_2 + \tilde{y}_2 \\ \vdots \\ -y_k + \tilde{y}_k \end{bmatrix}$

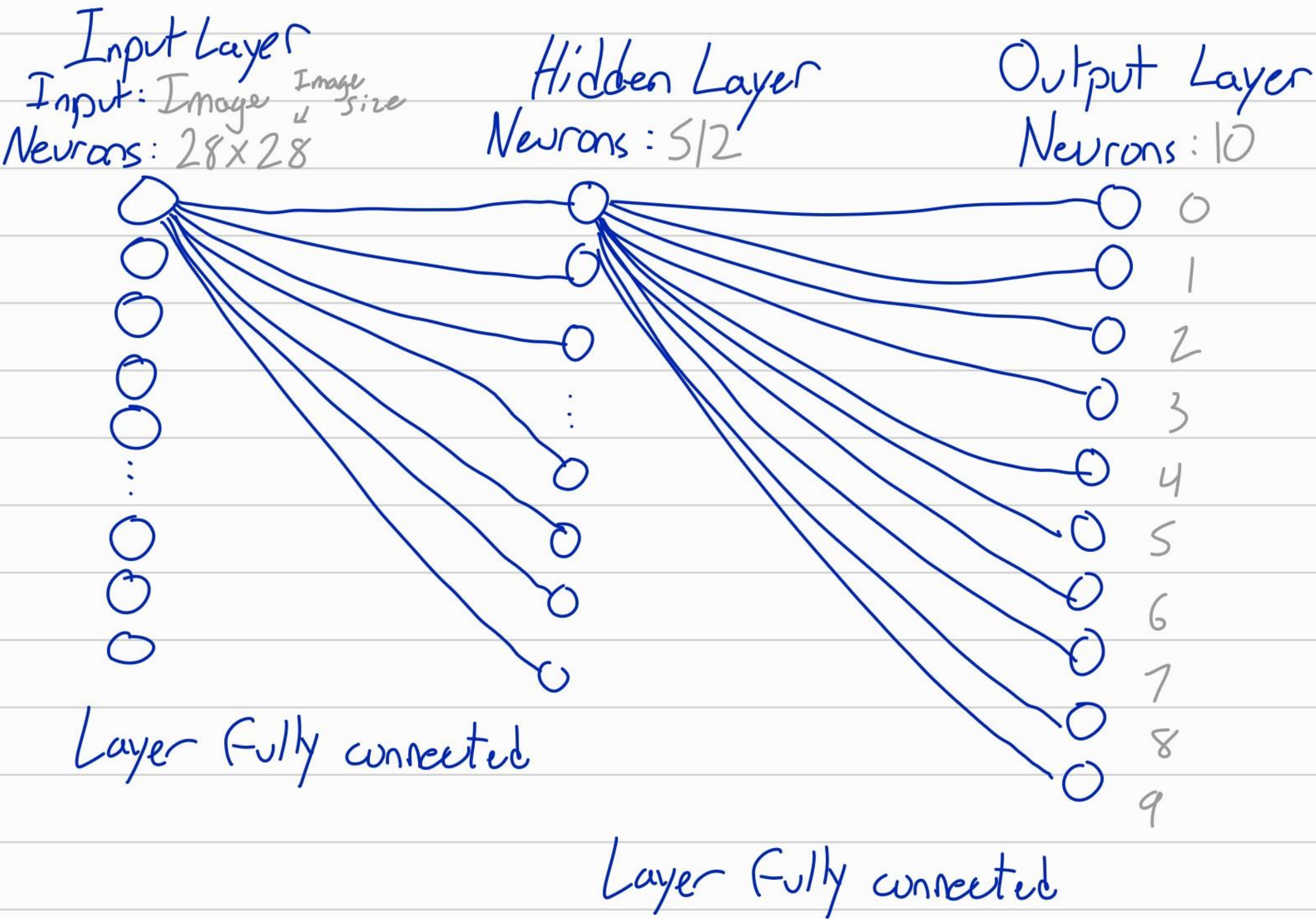
0.4 MNIST with F NN [30 points]

1) CIO points] Design an FNN for MNIST classification. Draw the computational graph of your model.

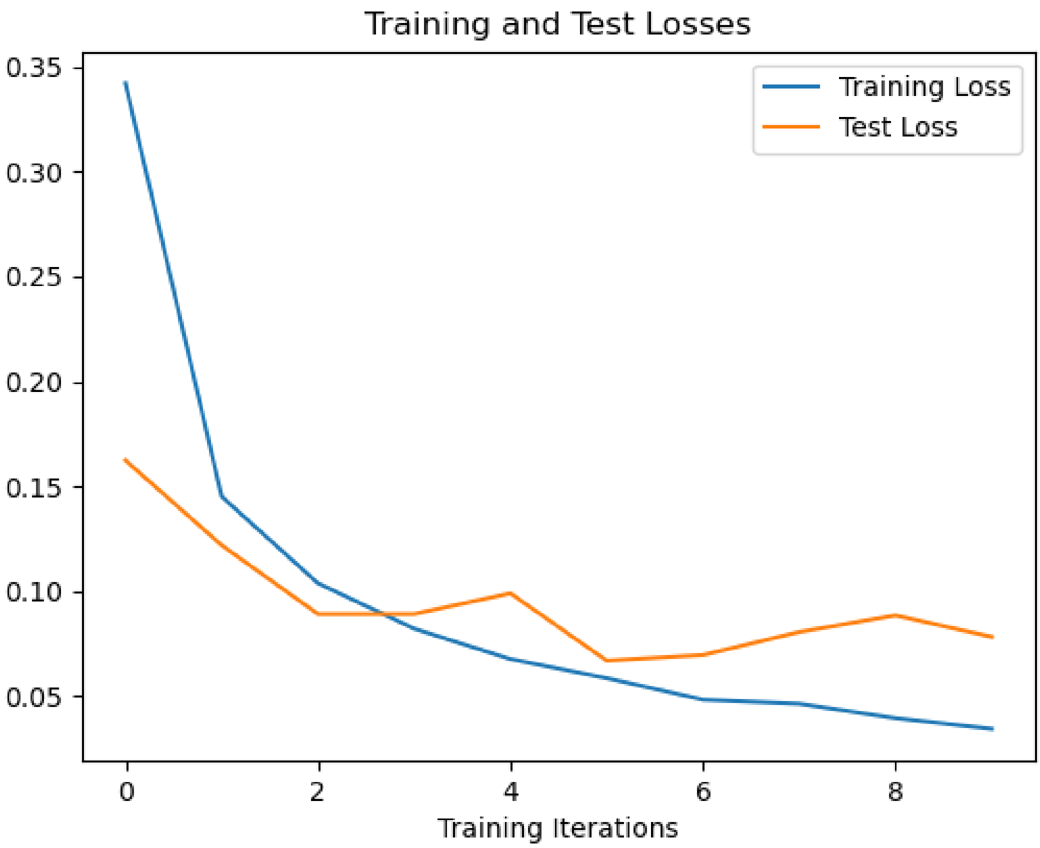
2) [20 points] Implement the model and plot two curves in one figure: i) training loss vs. training iterations; ii) test loss vs. training iterations.

   – You can use any packages, but Pytorch is recommended.

1



References: Code adapted from https://towardsdatascience.com/handwritten-digit-mnist-pytorch-977b5338e627



Training and Test Losses
Training Loss
Test Loss
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
2
4
6
8
Training Iterations