# Advanced Algebra II HW1

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January 20, 2022

**Problem 1.** Show $p(x) = x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$. Let $\theta$ be a root of this polynomial. Compute

$$(1+\theta)(1+\theta+\theta^2) \qquad \text{and} \qquad \frac{1+\theta}{1+\theta+\theta^2}$$

in $\mathbb{Q}(\theta)$, as $\mathbb{Q}$-linear combinations of $1, \theta$, and $\theta^2$.

*Solution.* Since 2 divides the non-leading coefficients of $p(x)$, but 4 does not divide the "1" coefficient, Eisenstein's criterion shows that $p(x)$ is irreducible in $\mathbb{Q}[x]$. Let $\theta \in \mathbb{C}$ be a root of $p(x)$. Since $p(x)$ is irreducible and has degree 3, we know that

$$\mathbb{Q}(\theta) = \left\{a + b\theta + c\theta^2 : \ a, b, c, \in \mathbb{Q}\right\},$$

and in this field that $\theta^3 = 2\theta + 2$. Thus

$$(1+\theta)(1+\theta+\theta^2) = \theta^3 + 2\theta^2 + 2\theta + 1 = 2\theta^2 + 4\theta + 3.$$

Next, since $p(x)$ is irreducible of degree 3, it is relatively prime to all quadratic polynomials in $\mathbb{Q}[x]$. Thus, from the division algorithm, there exist polynomials $a(x), b(x) \in \mathbb{Q}[x]$ with $a(x)(1+x+x^2) + b(x)p(x) = 1$. Using long division twice, we find

$$\begin{aligned} x^3 - 2x - 2 &= (x^2+x+1)(x-1) - (2x+1), \\ 4(x^2+x+1) &= -(2x+1)\,(-2x-1) + 3. \end{aligned}$$

Solving the second equation for 3 and eliminating a $2x+1$ term using the first equation gives

$$3 = (2x+1)p(x) + (-2x^2+x+5)(x^2+x+1).$$

Since $\mathbb{Q}(\theta) \cong \mathbb{Q}[x]/\langle p(x)\rangle$, the above equation shows that $(1+\theta+\theta^2)^{-1} = \frac{1}{3}(-2\theta^2+\theta+5)$ in $\mathbb{Q}(\theta)$. Thus

$$\frac{1+\theta}{1+\theta+\theta^2} = \frac{(1+\theta)(-2\theta^2+\theta+5)}{3} = \frac{-2\theta^3-\theta^2+6\theta+5}{3} = \frac{-\theta^2+2\theta+1}{3}.$$

■

**Problem 2.** Let $K/F$ be a field extension of degree $n$. Show that for any $\alpha \in K$, the "left multiplication by $\alpha$" map $m_\alpha : K \to K$ given by $x \mapsto \alpha \cdot x$ is an $F$-linear transformation. Deduce that $K$ is isomorphic to a subfield of the ring $\mathrm{Mat}_n(F)$ of $n \times n$ matrices over $F$.

*Proof.* Let $\alpha \in K$: additivity of $m_\alpha$ follows from the distributive law in $K$, and $F$-scalar multiplication follows from commutative multiplication in $K$ and $F$. Thus $m_\alpha : K \to K$ is $F$-linear. By fixing a basis $B = \{b_1, ..., b_n\}$ for $K$ over $F$, we can represent each $m_\alpha$ as an $n \times n$ diagonal matrix $[m_\alpha]_B$ with entries in $F$. The function $\phi : K \to \mathrm{Mat}_n(F)$ given by $\alpha \mapsto [m_\alpha]_B$ is a ring homomorphism (since by definition of $m_a$, we have $m_a \circ m_b = m_b \circ m_a = m_{ab}$ and $m_a + m_b = m_{a+b}$ for any $a, b \in K$.) Injectivity of $\phi$ follows since if $m_a$ is the zero matrix, then $ab_1 + ... + ab_n = 0$, and we have $a = 0$. ■

**Problem 3.** Show that $p(x) = x^4 + 3x + 3$ is irreducible over $\mathbb{Q}(\sqrt[3]{2})$.

*Proof.* Since 3 divides the non-leading coefficients of $p(x)$, but 9 does not divide the constant coefficient, Eisenstein's criterion shows that $p(x)$ is irreducible in $\mathbb{Q}[x]$. Then if $\theta$ is a root of $p(x)$, $[\mathbb{Q}(\theta) : \mathbb{Q}] = 4$. Since the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $x^3 - 2$, it follows that $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$. Since $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\theta)$ are both subfields of $\mathbb{Q}(\theta, \sqrt[3]{2})$, by the tower law, both 3 and 4 divide $[\mathbb{Q}(\theta, \sqrt[3]{2}) : \mathbb{Q}]$. Therefore 12 divides $[\mathbb{Q}(\theta, \sqrt[3]{2}) : \mathbb{Q}]$, and since

$$[\mathbb{Q}(\theta, \sqrt[3]{2}) : \mathbb{Q}] = [\mathbb{Q}(\theta, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})] \cdot [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = [\mathbb{Q}(\theta, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})] \cdot 3,$$

it follows that 4 divides $[\mathbb{Q}(\theta, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})]$. However this is at most 4 since $p(x)$ is degree 4, therefore $[\mathbb{Q}(\theta, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})] = 4$. It follows that $p(x)$ is the minimal polynomial for $\theta$ over $\mathbb{Q}(\sqrt[3]{2})$, which is irreducible. ■

**Problem 4.** Let $K/F$ be a field extension, and let $\alpha \in K$. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

*Proof.* Notice $F \subset F(\alpha^2) \subset F(\alpha)$. By the tower law,

$$[F(\alpha) : F] = [F(\alpha) : F(\alpha^2)] \cdot [F(\alpha^2) : F],$$

therefore $[F(\alpha) : F(\alpha^2)]$ must be odd. Since $F(\alpha^2) \subset F(\alpha) = F(\alpha^2, \alpha)$, and $\alpha$ is a root of $f(x) = x^2 - (\alpha^2) \in F(\alpha^2)[x]$, we conclude $[F(\alpha) : F(\alpha^2)]$ is at most two. Since it is odd, it must be 1, therefore $F(\alpha) = F(\alpha^2)$. ■

**Problem 5.** Let $K_1$ and $K_2$ be finite extensions of a field $F$, contained in a field $K$. Prove that the $F$-algebra $K_1 \otimes_F K_2$ is a field iff $[K_1K_2 : F] = [K_1 : F]\,[K_2 : F]$.

*Proof.* We may write $K_1 = F(a_1, ..., a_n)$ and let $K_2 = F(b_1, ..., b_m)$ where $\{a_i\}$ and $\{b_j\}$ are vector space bases. Then the elements $\{a_ib_j : i = 1, ..., n, j = 1, ..., m\}$ span $K_1K_2$.

The $F$-bilinear map $K_1 \times K_2 \to K_1K_2$ given by $(x, y) \mapsto xy$ induces an $F$-linear map $\phi : K_1 \otimes_F K_2 \to K_1K_2$, by the universal property of the tensor product of modules. The map $\phi$ is surjective, since $a_i \otimes b_j \mapsto a_ib_j$, and therefore the (linear span of the) spanning set $\{a_ib_j\}$ is in the image. Moreover, $\phi$ is a ring homomorphism, since for generic tensors $\sum_{i=1}^{\ell} x_i \otimes y_i$ and $\sum_{j=1}^{k} z_j \otimes w_j$,

$$
\begin{aligned}
\phi\left(\sum_{i=1}^{\ell} x_i \otimes y_i \ \cdot\ \sum_{j=1}^{k} z_j \otimes w_j\right) = \phi\left(\sum_{i,j} x_iz_j \otimes y_iw_j\right) &= \sum_{i,j} \phi\left(x_iz_j \otimes y_iw_j\right) \\
&= \sum_{i,j} x_iy_iz_jw_j \\
&= \sum_{i=1}^{\ell} x_iy_i \ \cdot\ \sum_{j=1}^{k} z_jw_j \\
&= \phi\left(\sum_{i=1}^{\ell} x_i \otimes y_i\right) \cdot\ \phi\left(\sum_{j=1}^{k} z_j \otimes w_j\right)
\end{aligned}
$$

Recall the dimension of $K_1 \otimes_F K_2$ over $F$ is $mn$. Since $\phi$ is surjective, then $[K_1K_2 : F] = mn$ iff $\phi$ is a linear isomorphism, i.e. injective. Since $\phi$ is a ring homomorphism, it follows that $[K_1K_2 : F] = mn$ iff $K_1 \otimes_F K_2 \cong K_1K_2$ as rings. This proves the claim since ker $\phi \neq 0$ if and only if $K_1 \otimes_F K_2$ has a nontrivial ideal, and is not a field. ∎