FREE-ENERGY PRINCIPLE (FEP) TUTORIAL

Perception as an inference process - predictive processing

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Background reading

Papers:

- Bogacz, R. (2017). A tutorial on the free-energy framework for modelling perception and learning. Journal of mathematical psychology, 76, 198-211.
- Lanillos, P., & Cheng, G. (2018, October). Adaptive robot body learning and estimation through predictive coding. In 2018 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS) (pp. 4083-4090). IEEE.
- Sancaktar, C., van Gerven, M., & Lanillos, P. (2019). End-to-end pixel-based deep active inference for body perception and action. arXiv preprint arXiv:2001.05847.
- Friston, K. J., Daunizeau, J., Kilner, J., & Kiebel, S. J. (2010). Action and behavior: a free-energy formulation. Biological cybernetics, 102(3), 227-260.
- Buckley, C. L., Kim, C. S., McGregor, S., & Seth, A. K. (2017). The free energy principle for action and perception: A mathematical review. Journal of Mathematical Psychology, 81, 55-79.

Blog:

- https://msrmblog.github.io/is-this-my-body-1/
- https://msrmblog.github.io/is-this-my-body-2/

Problem definition

We have an agent that lives in a 1-dimensional space environment. The agent goal is to infer its position in the space x given the sensory input s. It has a sensor that it is able to sense its rotation over the environment. After exploring the 1-dimensional space, it has learnt a generative model of the world depending on its internal state x, which is x^2 - See image below. *This generative model defines the prediction of the sensory input depending on its internal state.*

toy_problem_def.jpg

The agent starts with a wrong belief of his internal state, which yields into a bad estimation of it position. As it can be seen in the image below his actual position is misaligned with his estimated position. We will use a simplified model of predictive processing minimizing the Variational Free Energy (VFE) to solve this task.

Problem1.png

Overview

In this toy example you will see a basic FEP model at work. The following will be done:

a. Look at the formulas for the variational free energy
b. Implement the generative model
c. Run the generative model
d. Inspect the results

Tasks:

1. Implement the updating step for the generative model

1. Test its robustness in regard to different parameter intializations
   - What effects does it have if the initial values are further away from the agents actual position?
   - What values will make the system to be stable?
   - What if we initialize the belief with a negative position?
   - What happens if we add noise to the sensory measurement?

1. Describe according to the FEP how the agent would act in this environment

Part a: Variational Free Energy (VFE) - with the Laplace approximation

$$F = KL[q(x)||p(x|s)] - lnp(s) \rightarrow_{\text{laplace}} F = -\ln p(s,\mu) - \frac{1}{2}(\ln|\Sigma| + \ln 2\pi)$$

Perception is defined as the minimization of F:

$$\mu = \arg\min_s F(\mu, s)$$

We compute it using the following differential equation (gradient optimization):

$$\dot{\mu} = -\frac{\partial F}{\partial \mu} = -\frac{\partial \ln p(s|\mu)p(\mu)}{\partial \mu}$$

(Eq. 1)

Assuming that the sensors measurements s are defined by a generative model with Gaussian noise z: $s = g(\mu) + z$

We have:

$$\dot{\mu} = \frac{\partial g(\mu)}{\partial \mu}^T \Sigma_s^{-1}(s - g(\mu))$$

(Eq. 2)

Question a.1: Write down the derivation from (Eq. 1) to (Eq. 2)

1. The likelihood $p(s|\mu)$ that defines the generative model follows a Normal distribution with mean μ and σ standard deviation: $p(s|mu) \sim \mathcal{N}(s;\mu,\sigma^2)$
2. The prior $p(\mu)$ is assumed to be uniform

Tip: use the function of the Normal density function (pdf). wiki

Your derivation here:

$$\dot{\mu} = -\frac{\partial F}{\partial \mu} = -\frac{\partial \ln p(s|\mu)p(\mu)}{\partial \mu}$$

$$= -\frac{\partial}{\partial \mu}\ln(\frac{1}{\sqrt{2\pi\Sigma}}\exp(\frac{1}{2}\frac{(s-g(\mu))^2}{\Sigma}))$$

$$= -\frac{\partial}{\partial \mu}\ln(\frac{1}{\sqrt{2\pi\Sigma}}) + \ln(\exp(\frac{1}{2}\frac{(s-g(\mu))^2}{\Sigma}))$$

The $\ln(\frac{1}{\sqrt{2\pi\Sigma}})$ factor equals to 0 as it doesn't contain a μ term. And the $\ln$ and $\exp$ also cancel each other out.

$$= -\frac{\partial}{\partial \mu}\frac{1}{2}\frac{(s-g(\mu))^2}{\Sigma}$$

$$= -\frac{\partial}{\partial \mu}\frac{1}{2}\Sigma^{-1}(s-g(\mu))^2$$

$$= -\frac{1}{2}\Sigma^{-1}\frac{\partial}{\partial \mu}(s-g(\mu))^2$$

Apply the derivative rule

$$= -\frac{1}{2}\Sigma^{-1}2(s-g(\mu))\frac{\partial}{\partial \mu}g(\mu)$$

2's cancell out and rewriting gives...

$$\dot{\mu} = \frac{\partial g(\mu)^T}{\partial \mu}\Sigma_s^{-1}(s-g(\mu))$$

Part b: Generative model definition

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt

# Generative model definition
def g(x):      # forward model
    return x**2

def dg(x):     # inverse model
    return 2*x
```

In [2]:

```
# Plotting actual and inferred position of the agent
```

```
def plot_results(real_x, mu_0):

    x = np.linspace(-2, 2, 1000)
    y = np.linspace(2, 2, 1000)
    plt.plot(x, y, 'k')
    plt.plot(real_x, 2, 'r8', markersize=15)
    plt.plot(mu_0, 2, 'b8', markersize=15)
    plt.plot(0, 2, 'k|', markersize=8)

    plt.xlim([-1,1])  # adjust plot when choosing values outside of range -1 and 1
    plt.legend(['Space', 'actual position', 'estimated position', 'x=0'])
    plt.xticks([])
    plt.yticks([])
    plt.title('Estimated vs. actual position')

    plt.show()
```

Initialization

In [3]:

```
dt = 0.001    # integration step
ftime = 1   # maximum time considered

time = np.arange(0, ftime, dt) # Time array
N = time.shape[0] # Steps

real_x = 0.1    # real position of the ball
real_x_dot = 0  # change on the position

mu_0 = 0.5      # internal belief
mu_dot = 0.0     # change on the internal belief

sigma_s = 0.01    # variance of the sensory process
z = 0                 # for now
```

Question b.1: Compute the sensor measurement

In [4]:

```
# your code here


s_measure = g(real_x) + z
s_measure
```

Out[4]: 0.010000000000000002

Question.b.2: Compute the predicted sensor value depending on the brain variable μ

In [5]:

```
# your code here

mu_dot = dg(mu_0) * sigma_s ** -1 * ( s_measure - g(mu_0))
mu_0 += dt * mu_dot

mu_0
```

Out[5]:
```
0.476
```

Question b.3: Compute the prediction error

In [6]:
```
# your code here

e_s = s_measure - mu_0
e_s
```

Out[6]:
```
-0.46599999999999997
```

Part c: Implementing the FEP model

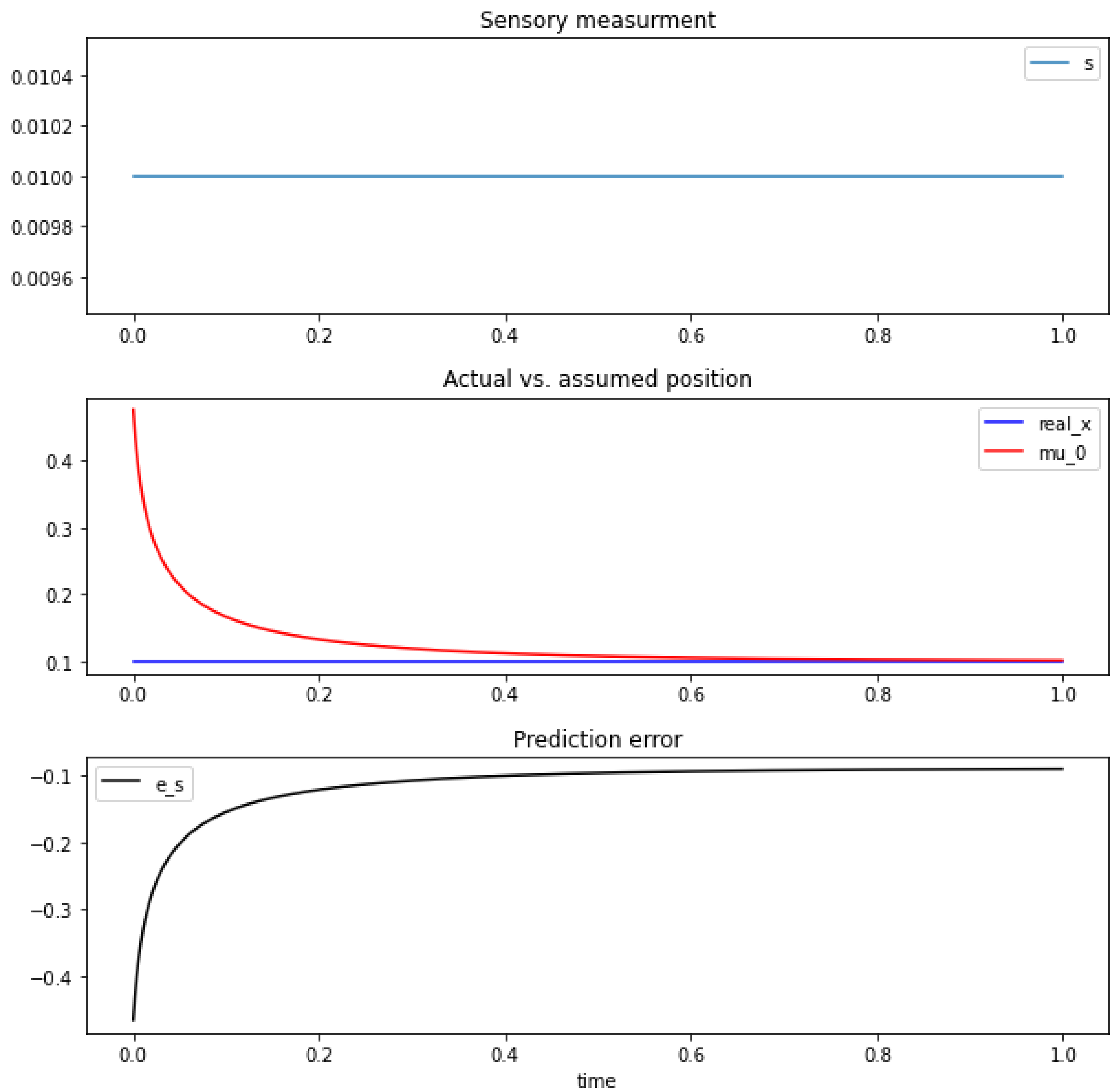
Question c.1: Write the function fep(parameters) that executes the FEP model

In [7]:
```
def fep(real_x, mu_0):
    """
    Takes real position real_x and belief in position mu_0 as input and runs FEP for

    Returns data which is a 4 x N array that collects s, real_x, mu_0 and e_s for N

    """
    data = np.zeros((4, N)) # (4, 1000)


    for i in range(0, N):
        s      = g(real_x)       # sensor reading
        mu_0   +=  dt * (dg(mu_0) * sigma_s ** -1 * ( s - g(mu_0))) # change in mu_0
        e_s    = s - mu_0        # error between sensor and belief

        data[0,i] = s
        data[1,i] = real_x
        data[2,i] = mu_0
        data[3,i] = e_s

    return data

# Initialization
real_x = 0.1    # real position of the ball
mu_0 = 0.5      # internal belief

data = fep(real_x, mu_0)
```

Part d: Inspect the results

In [8]:
```
def plot_variables(data):

    # Plot variables

    plt.figure(figsize=(10, 10))
    plt.subplots_adjust(hspace=0.3)

    ax1 = plt.subplot(311)
```

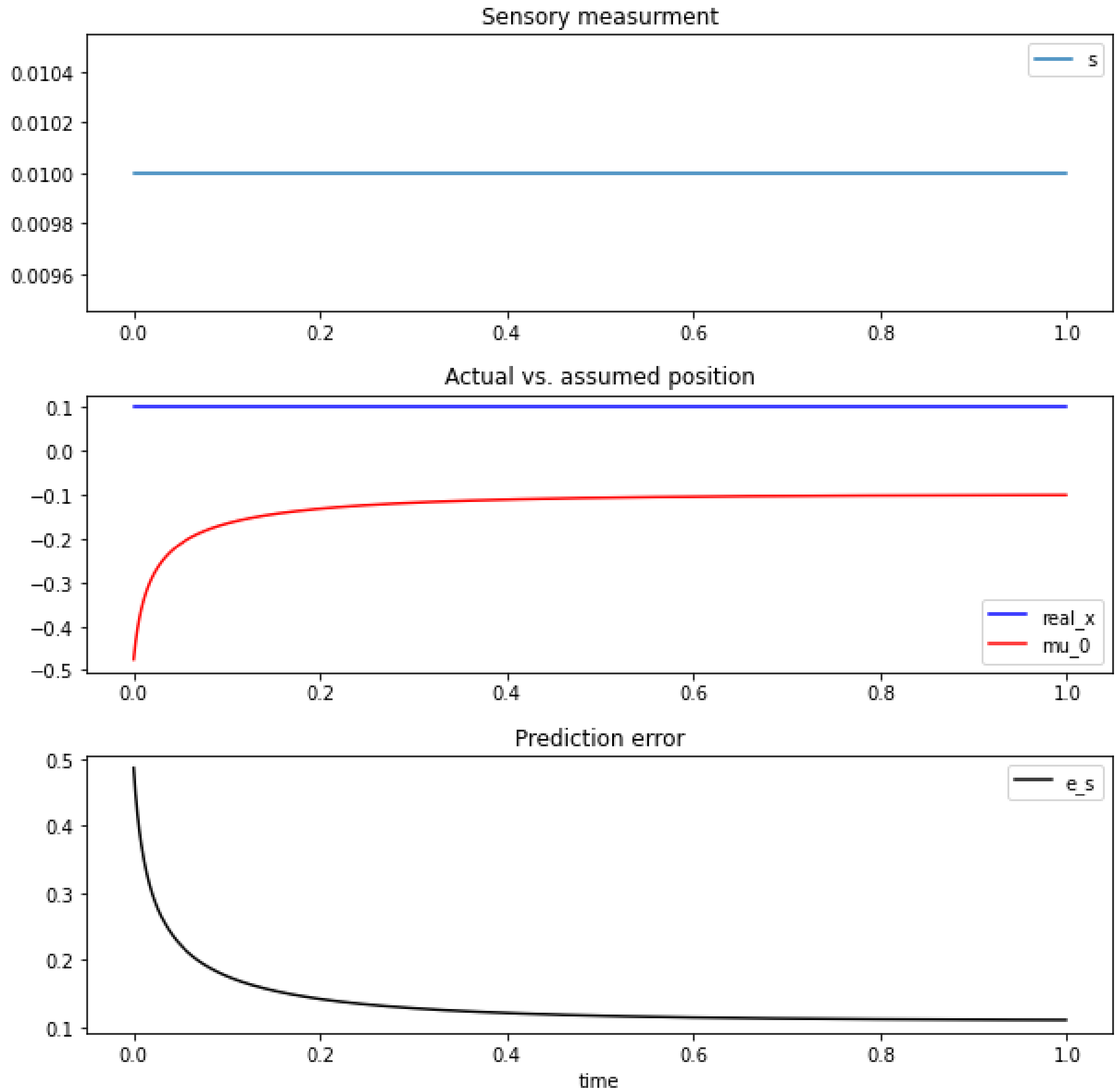
```
    ax1.plot( time, data[0, :] )
    ax1.legend(['s'])
    ax1.title.set_text('Sensory measurment')

    ax2 = plt.subplot(312)
    ax2.plot(time, data[1, :], 'b' )
    ax2.plot(time, data[2, :], 'r' )
    ax2.legend(['real_x', 'mu_0'])
    ax2.title.set_text('Actual vs. assumed position')

    ax3 = plt.subplot(313)
    ax3.plot(time, data[3, :] , 'k' )
    ax3.legend(['e_s'])
    ax3.set_xlabel('time')
    ax3.title.set_text('Prediction error')

    plt.show();

plot_variables(data)
```



Question d.1: Now test the same with the following agent initialization and explain what happened

In [9]:

```python
real_x = 0.1     # real position of the ball
mu_0 = -0.5       # internal belief

data = fep(real_x, mu_0)
plot_variables(data)
```



Describe your findings here

We see that our sensor reading is constant, and that if our initial internal belief is positive that we converge to 0.1. However, if our initial internal belief is negative (-0.5) then we only converge to -0.1. This most likely has something to do with the models $\mathbf{g}$ and $\mathbf{dg}$.

Question d.2: Now add different levels of Gaussian noise to the sensor measurements and run the FEP model again

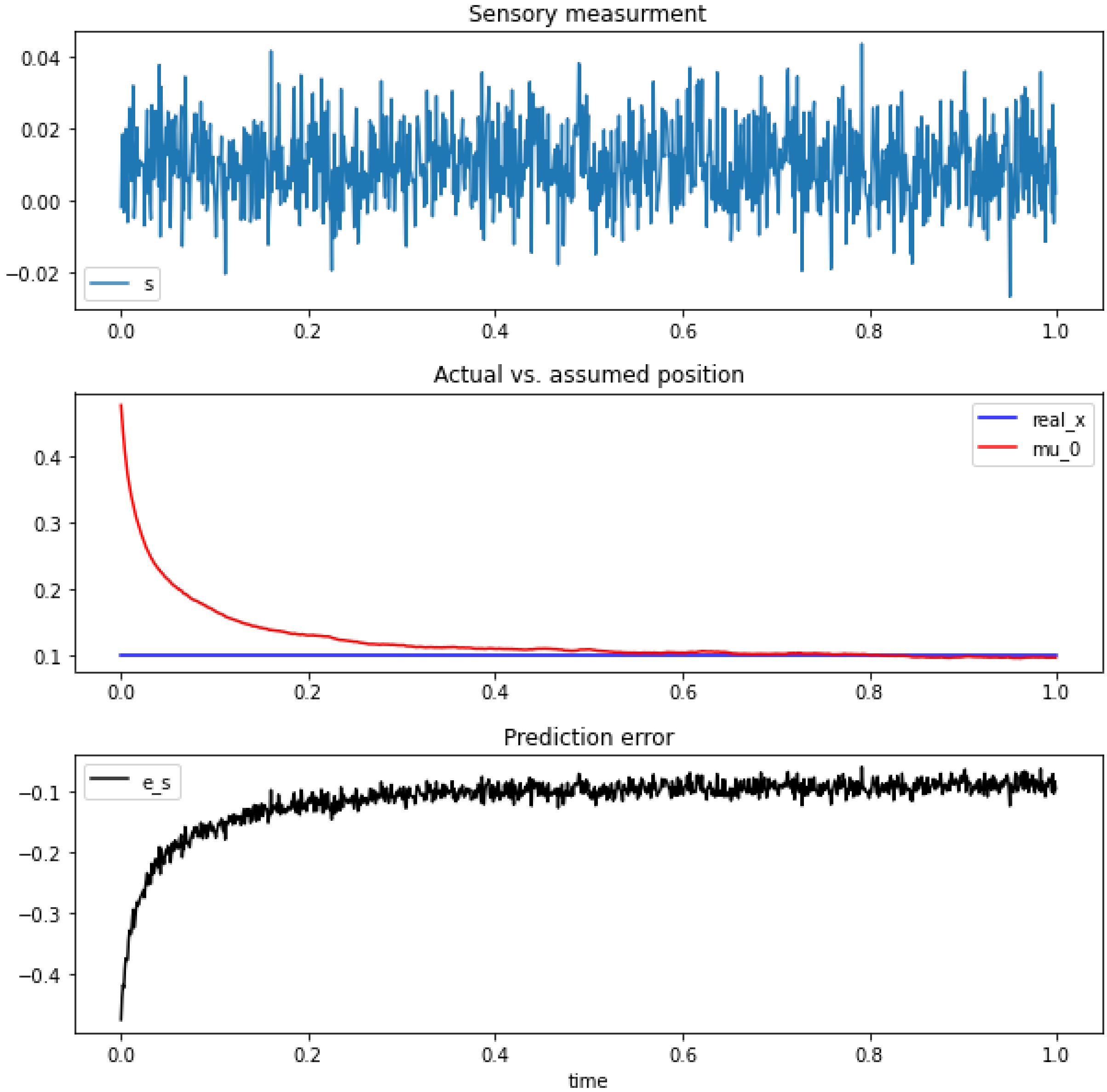
Write a function fep2 with an extra parameter that is the measurement standard deviation σ, and plot the variables

$s = g(x) + N(0, \sigma^2)$ - Increase σ to increase the input noise

In [12]:

```python
def fep2(real_x, mu_0, measurement_std, sigma_s):
    """
```

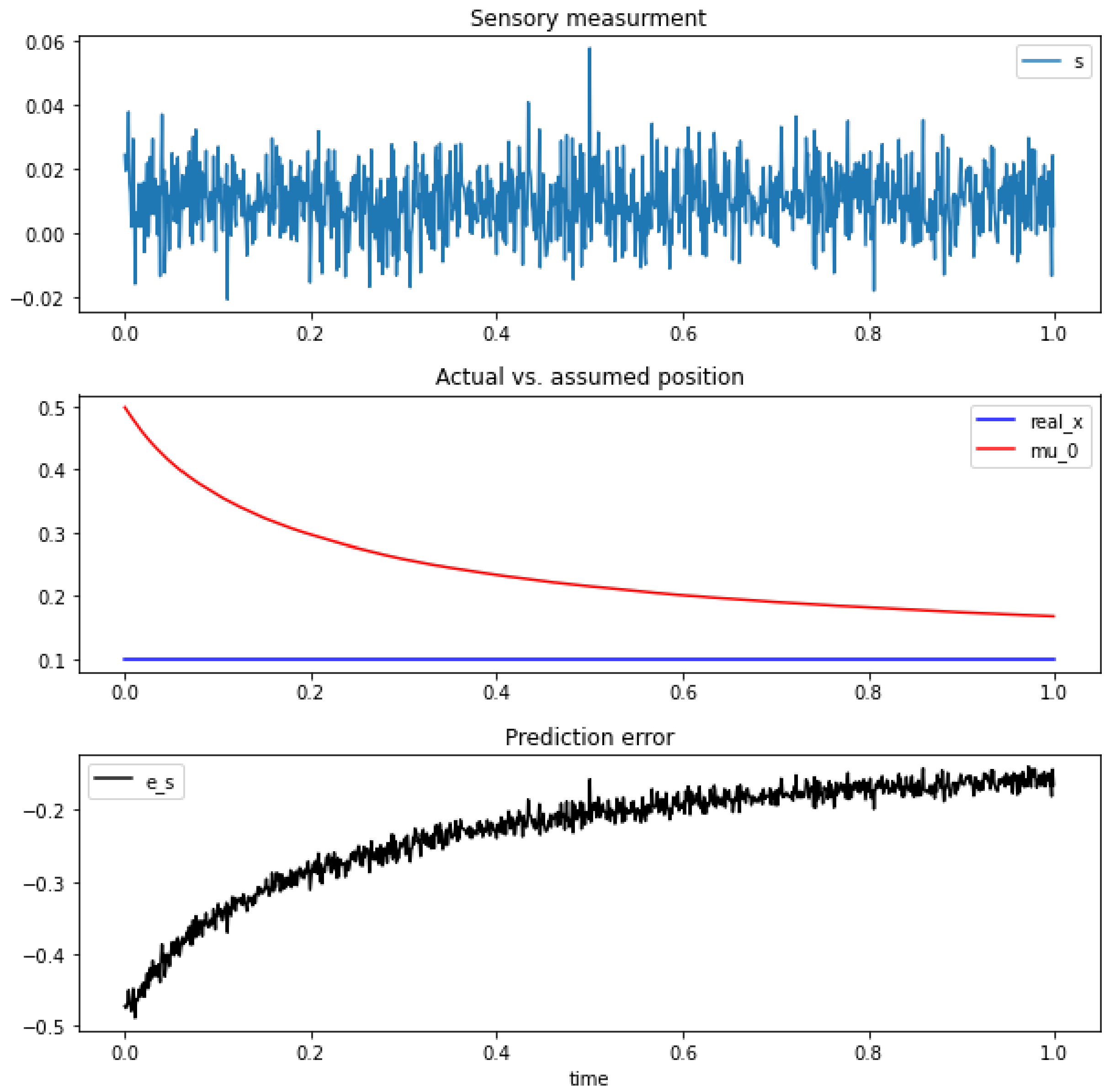
```
    Takes real position real_x, belief in position mu_0 and standard deviation sigma

    Returns data which is a 4 x N array that collects s, real_x, mu_0 and e_s for N

    """
    data = np.zeros((4, N)) # (4, 1000)


    for i in range(0, N):
        s      = g(real_x) + np.random.normal(0, measurement_std**2)
        mu_0   +=  dt * (dg(mu_0) * sigma_s ** -1 * ( s - g(mu_0)))
        e_s    = s - mu_0

        data[0,i] = s
        data[1,i] = real_x
        data[2,i] = mu_0
        data[3,i] = e_s

    return data


real_x = 0.1    # real position of the ball
mu_0 = 0.5      # internal belief
measurement_std = 0.1
sigma_s = 0.01    # variance of the sensory process this is for the next exercise

data = fep2(real_x, mu_0, measurement_std, sigma_s)
plot_variables(data)
```



Question d.3: Modify the variance parameter σ that weights the sensor error accordingly and run it again

In [11]:

```
real_x = 0.1     # real position of the ball
mu_0 = 0.5       # internal belief
measurement_std = 0.1
sigma_s = 0.1     # variance of the sensory process this is for the next exercise

data = fep2(real_x, mu_0, measurement_std, sigma_s)
plot_variables(data)
```



Question d.4: How does the system deal with the noise? and what does σ_s mean to the agent in normal words?

Describe your findings here

It seems that noise in the sensor is delt with pretty succesfully, while internal belief noise has a larger effect on the prediction.

σ_s acts as a sort of normalising agent around the inverse model of the internal belief multiplied by the sensor input minus the forward model of the internal belief.

Question d.5: Now describe how you would implement the action in the FEP mathematical framework

Describe it here, you can use equations

An action is the minimisation of the free energy principle as a function of our internal belief and sensory reading. $a = arg_a min F(\mu, s(a))$

We can then use this to calculate the change in sensation with respect to our action, by applying the chain rule on $\frac{\partial F_s}{\partial a}$