Problem of Quickest Ascent

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1 Introduction

In this paper, we explore a particle subject to a gravitational potential constrained on a rotating well to study the conditions required for lift to be generated. We seek to find the geometry of the well for which the particle ascends the quickest.

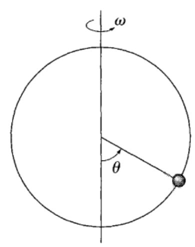

Figure 1: Proposed setup of our system

1.1 The Brachistochrone Problem

In 1696, Isaac Newton was challenged to solve a problem which would later become known as the *Brachistochrone* Problem[1] – the problem of quickest descent. It asks the following: what is the geometry of the curve such that a bead that begins sliding from rest and subject only to the influence of gravity will go from one point to another in the least possible amount of time. Of course, Sir Newton only needed an afternoon to solve the problem.

To gain some intuition of the problem, the following warm-up question is posed: what would be the fastest path from point A to B for a free particle subject to no external potentials?

Intuition says that it should be a straight line. Throughout any possible trajectory a particle can take from point A to B, there's nothing that would aid or hinder the particle from reaching its destination. Hence, there's no reason

[1]Brachistos meaning "the shortest" in Greek, and Chronos meaning "time" in Greek.

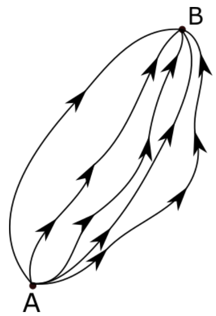

Figure 2: Many trajectories of a particle going from point A to B

for a particle to do anything except go straight from point A to B, without any detour whatsoever. It turns out that one can prove that the path the particle should take to reach point B in the minimum amount of time is the straight line between point A and point B.

However, this same line of logic does not hold when the particle is under the influence of gravity. Intuition says that the particle should take advantage of as much velocity gain from the gravitational potential to approach point B in the least amount of time. Finding the exact shape of the curve that optimizes this is the heart of the Brachistochrone problem.

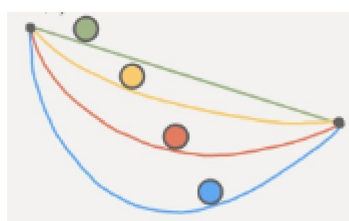

Figure 3: Trajectory of balls travelling between two points on various paths (circular path: blue, brachistochrone path: red, parabolic path: yellow, linear path: green)

Given an arbitrary starting point A for the particle at rest, the key limitation for point B is that it *must* be below point A, otherwise the particle will have no means of reaching the destination point.

1.2 Our Problem

We will constrain our attention to a particle which is again under the influence of gravity, in addition to a well which rotates at a constant angular velocity ω.

This particular configuration is interesting because the potential energy of the particle arises solely due to Earth's gravitational field. Yet due to the

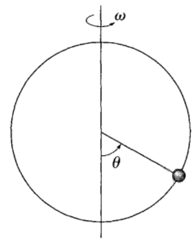

Figure 4: Proposed setup of our system

influence of the rotating well, we will see that the particle *effectively* has an extra component of potential giving it the energy required to rise.

We are interested in studying the conditions required for the lift of the particle, as well as optimizing the geometry allowing the fastest traversal between two points. In order to model this, we will place no constraint on the shape of the path of the particle[2], and we seek to find the path which minimizes the time to move between the two points. In the process, we will also seek to find a generalized sufficient condition for lift.

This model places minimal constraints on the setup of the system, and consequently will have a fighting chance of tapping into the fundamental physics of rotating reference frames. However, this model does have some drawbacks. Namely, we assumed a point-mass, and any sort of experiment which seeks to verify the claims of this paper would have to modify the equations so that a physical object such as a circular bead is being modelled. However, intuitively speaking, we do not expect the fundamental physics to change when replacing a particle with a bead, and making the change could be the subject of further study. A future paper interested in experimentally quantifying these results would also need to consider friction in the model, as it too has been neglected. These simplifying assumptions primarily function to make our upcoming mathematics easier to handle, and indeed the goal of this paper is to ensure that the a reasonable solution is reachable in the simplest case before worrying about increasing the accuracy of the model.

In order to solve our model for the problem of quickest ascent, the following background of variational calculus will be needed.

2 Background

Motion of mechanical systems obey a fundamental principle: the principle of least action. In mathematical terms, we start by defining:

[2]To be precise, the only constraints on the path of the particle will be that it is symmetric on both sides, and that it is flat at the center

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt$$

Where S is the "action" of the system, and $\mathcal{L}$ is the function (Lagrangian) by which the motion of the mechanical system is characterized. The principle of least action tells us that $\mathcal{L}$ takes on the form that minimizes the action S. Here, q is our generalized coordinate with $\dot{q}$ being the generalized velocity, and t is the independent variable with which the generalized coordinates vary (often t is just time).

It turns out that this specific form of $\mathcal{L}$ is acquired when subject to the following constraint:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

This equation is known as the Euler-Lagrange equation. The final thing we will need to know is that $\mathcal{L} = T - U$ where T is the kinetic energy of the system and U is the potential energy of the system.

3 Determining Results

3.1 Setup of Model

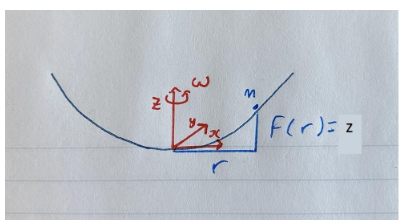

Figure 5: Model setup

This is our model of a point mass constrained to an arbitrary curve, which rotates about the z axis at a constant angular velocity ω. We will have one generalized coordinate: $s = 1$, $q = r$.

We can express the coordinates of our point mass (in cylindrical coordinates (r, θ, z)) as: $(r, \omega t, z)$.

3.2 Lagrangian

In standard cylindrical form, the kinetic energy of a particle is:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2 + (\frac{dz}{dr}\dot{r})^2)$$

The only potential energy contribution is from gravity:

$$U = mgz$$

Overall, the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2 + (\frac{dz}{dr}\dot{r})^2) - mgz$$

If we rearrange the terms to group the terms which depend on $\dot{r}$:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + (\frac{dz}{dr}\dot{r})^2) + \frac{1}{2}mr^2\omega^2 - mgz$$

Grouping the terms in this way allows us to see that the first term is the *effective* kinetic energy, and the second and third terms are the *effective* potential energy. This is effectively what gives way to the lifting phenomenon, even though the rotating reference frame doesn't explicitly add any potential energy to the system, but it *effectively* does.

This Lagrangian also doesn't have explicit time dependence, so energy is conserved:

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \implies E = \dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \mathcal{L}$$

$$E = \frac{1}{2}m(\dot{r}^2 + (\frac{dz}{dr}\dot{r})^2) - \frac{1}{2}mr^2\omega^2 + mgz$$

If we find the value of energy E at one time, then we know its value at all times since it is a conserved quantity. Let's say that the value of the energy at $t = 0$ is E_0.

So we have that:

$$E_0 = \frac{1}{2}m(\dot{r}^2 + (\frac{dz}{dr}\dot{r})^2) - \frac{1}{2}mr^2\omega^2 + mgz$$

Intuitively,

Notice that $\dot{r}^2 + (\frac{dz}{dr}\dot{r})^2$ is the square magnitude of the velocity of the bead on the curve, so we will say that:

$$E_0 = \frac{1}{2}mv^2 - \frac{1}{2}mr^2\omega^2 + mgz$$

$$\sqrt{\frac{2E_0}{m} + r^2\omega^2 - 2gz} = v$$

This equation will be an important relation in determining the optimal path of the curve for quickest ascent.

3.3 Variational Methods

The time it takes for a particle to travel from points P_1 to P_2 is:

$$t_{12} = \int_{P_1}^{P_2} \frac{ds}{v}$$

Where ds is the differential arclength. In standard cylindrical coordinates, that is:

$$ds = \sqrt{dr^2 + r^2 d\theta^2 + dz^2}$$

Since $d\theta = d\omega = 0$:

$$ds = \sqrt{dr^2 + dz^2} = \sqrt{1 + \left(\frac{dz}{dr}\right)^2} dr = \sqrt{1 + z'^2} dr$$

Where $z' = \frac{dz}{dr}$.

Furthermore, v in the integral for time is the velocity of the particle on its path, which we've already determined from the previous section:

$$v = \sqrt{\frac{2E_0}{m} + r^2\omega^2 - 2gz}$$

Plugging this into the integral:

$$t_{12} = \int_{P_1}^{P_2} \frac{\sqrt{1+z'^2}}{\sqrt{\frac{2E_0}{m} + r^2\omega^2 - 2gz}} dr$$

This is where our background on variational calculus is required. In order to find the exact function z which describes the curve that allows quickest ascent, then we must apply the Euler-Lagrange equations to our integrand, which becomes the function we must "vary" in order to minimize the action. So our function which we will vary will be:

$$h(r, z, z') = (1 + z'^2)^{1/2} (\frac{2E_0}{m} + r^2\omega^2 - 2gz)^{-1/2}$$

It is worth noticing that when we set $\omega = 0$, this reduces to the regular Brachistochrone problem(1). Now, we will seek to compute the Euler-Lagrange equations:

$$\frac{d}{dr}\left(\frac{\partial h}{\partial \dot{z}}\right) - \frac{\partial h}{\partial z} = 0$$

Notice that our independent variable is r instead of time. We will cut to the chase and go straight to the solution in order to skip a page of derivatives and algebra. However, the entire work has been included in the appendix for the reader to convince themselves of the mathematics:

$$z''\left(\frac{2E_0}{m} + r^2\omega^2 - 2gz\right) - (g + \omega^2 r z')(1 + z'^2) = 0$$

4 Analysis

4.1 Conservation of Energy and Lift Conditions

Firstly, let's scrutinize the equation we found from the conservation of energy which related velocity to the other parameters of the system:

$$\sqrt{\frac{2E_0}{m} + r^2\omega^2 - 2gz} = v$$

This equation alone reveals a good deal of physics to us, and we can gain intuition for the system. Firstly, the velocity will be a real number, which tells us that the contents inside the square root must be non-negative:

$$\frac{2E_0}{m} + r^2\omega^2 - 2gz \geq 0$$

This tells us that we can decrease the height of the particle without any limit, as decreasing z corresponds to increasing the value of the left hand side. However, there is a limit of increasing the height, as if z becomes too large, the whole left-hand side would become negative, and thereby no longer maintaining the inequality. This matches our intuition, as we expect the maximum height of the particle depends on how much energy we put into the system (for example as initial velocity) as well as how fast the frame is rotating!

Rearranging for ω:

$$\omega^2 \geq \frac{2gz}{r^2} - \frac{2E_0}{mr^2}$$

At a fixed E_0, in order to raise the particle to any given height z, then there exists a minimum ω at which the frame must rotate in order to achieve the corresponding height. Furthermore, this equation tells us that $-\omega$ and ω both correspond to the same equation above, which is also exactly what we expect, as there couldn't possibly be a bias towards one orientation of rotation.

4.2 Solutions for Optimal Curve

Let's numerically solve the differential equation, to see if it yield behavior consistent with what we've discussed above.

$$z''\left(\frac{2E_0}{m} + r^2\omega^2 - 2gz\right) - (g + \omega^2 r z')(1 + z'^2) = 0$$

The simplest possible scenario is when we provide no energy to the system, no rotation to the frame, and the particle rests at the center of the curve, and by boundary conditions we set the center of the curve to be flat. In this scenario, we expect the problem to have reduced to the original Brachistochrone problem: only the gravitational field acting on the particle and nothing further.

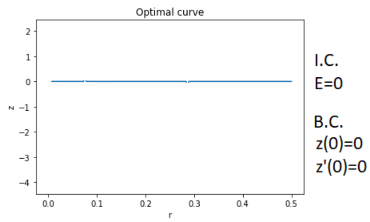

Figure 6: Particle subject to initial conditions E=0, $\omega = 0$, and boundary conditions z(0)=0, z'(0)=0. Graph of z by r.

This is an interesting result, as it doesn't match our prediction, which is that the solution should be cycloid like in a Brachistochrone problem. However, this particle was given zero external energy on a flat starting point. The particle has no energy to tip off from its stable equilibrium point to start falling downwards. This could be the reason why we don't see a cycloid.

Let's see what happens to the curve when we keep the same initial conditions as above, except provide an angular velocity of $\omega = \pi$ radians per second. In this case, we expect the particle to rise up to some critical height, which in the energy equation corresponds to the maximum height z such that the square root remains positive.

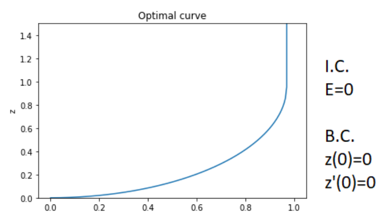

Figure 7: Particle subject to initial conditions E=0, $\omega = \pi$, and boundary conditions z(0)=0, z'(0)=0. Graph of z by r.

What we see here is that the curve slowly rises up until a cutoff point which qualitatively seems to be close to $r = 1$. This corresponds to the maximum height the particle can reach.

What about if we increase ω? We would expect that this maximum height increases with it. Not only this, we expect that the maximum height to increase with square root proportionality to ω from our previous equation for the minimum ω at a given height z.

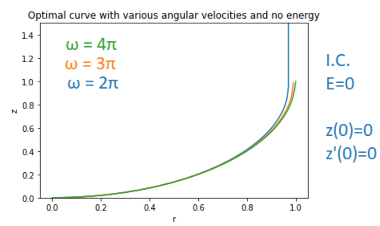

Figure 8: Particle subject to initial conditions E=0, $\omega = \pi$, and boundary conditions z(0)=0, z'(0)=0. Graph of z by r. Three separate curves

The plot above seems to contain both of these properties. As ω increases linearly, the spacing between the maximum heights increases, but at a rate slower than linear. We would have to conduct several more measurements to see if the rate of growth is a square root power law. Any valid model of our physical scenario will need to satisfy this property.

Now that we've gained a small amount of confidence of the behavior of the curve in some standard cases, let's see if the curve satisfies the quickest ascent property as well.

Using the same operating conditions as in Figure 6, we find $h(r, z, z')$ at each value of r. This is now possible because we've determined the shape of z. Using a numerical integrator, we find that the time it takes to traverse to $0.9m$ is about 1.21 seconds.

Now, let's compare how the particle would fair on a circular trajectory from $(r, z) = (0, 0)$ to $(0.9, 0.95)$ (This is the initial and final position of the particle from Figure 6). A particle in this configuration on a circular path has the following equation of motion(2):

$$\ddot{\theta} = (\omega^2 \cos\theta - g/R)\sin\theta$$

We solve this differential equation numerically, which yields θ as a function of t. The angle at which the particle finishes its trajectory is $\theta = 38$ degrees. The time which corresponds to this angle is approximately $t = 1.41$ seconds.

So our model's curve defeats the corresponding circular trajectory!

5 Discussion and Conclusion

Overall, we've considered a model of a particle in a rotating well under the influence of gravity, and have determined generalized constraints that must be met in order for the particle to generate lift:

$$\omega^2 \geq \frac{2gz}{r^2} - \frac{2E_0}{mr^2}$$

Additionally, we've also attempted to find the curve that allows the particle to rise in the fastest possible manner, and have tested to ensure that it satisfies some basic properties that we would expect to hold. For example, our curve satisfies the generalized lifting constraints we determined earlier. We found the curve to be defined by:

$$z'' \left(\frac{2E_0}{m} + r^2\omega^2 - 2gz \right) - (g + \omega^2 r z')(1 + z'^2) = 0$$

We also compared the time it takes for a particle to traverse two points along the optimized curve from our model, with a circular trajectory, and found that our curve gave better results. Overall, we've gained confidence that this curve is indeed the optimal curve needed from going from point A to B which minimizes the travel time. Future studies could characterize how friction affects our results, and also consider rigid bodies such as a circular bead instead of a point particle.

References

[1] Weisstein, Eric W. *Brachistochrone Problem.* https://mathworld.wolfram.com/BrachistochroneProblem.html

[2] John R. Taylor *Classical Mechanics.* University Science Books, 2005

6 Appendix

Work for deriving the optimized curve ODE:

$$z'' \left(\frac{2E_0}{m} + r^2\omega^2 - 2gz \right) - (g + \omega^2 r z')(1 + z'^2) = 0$$

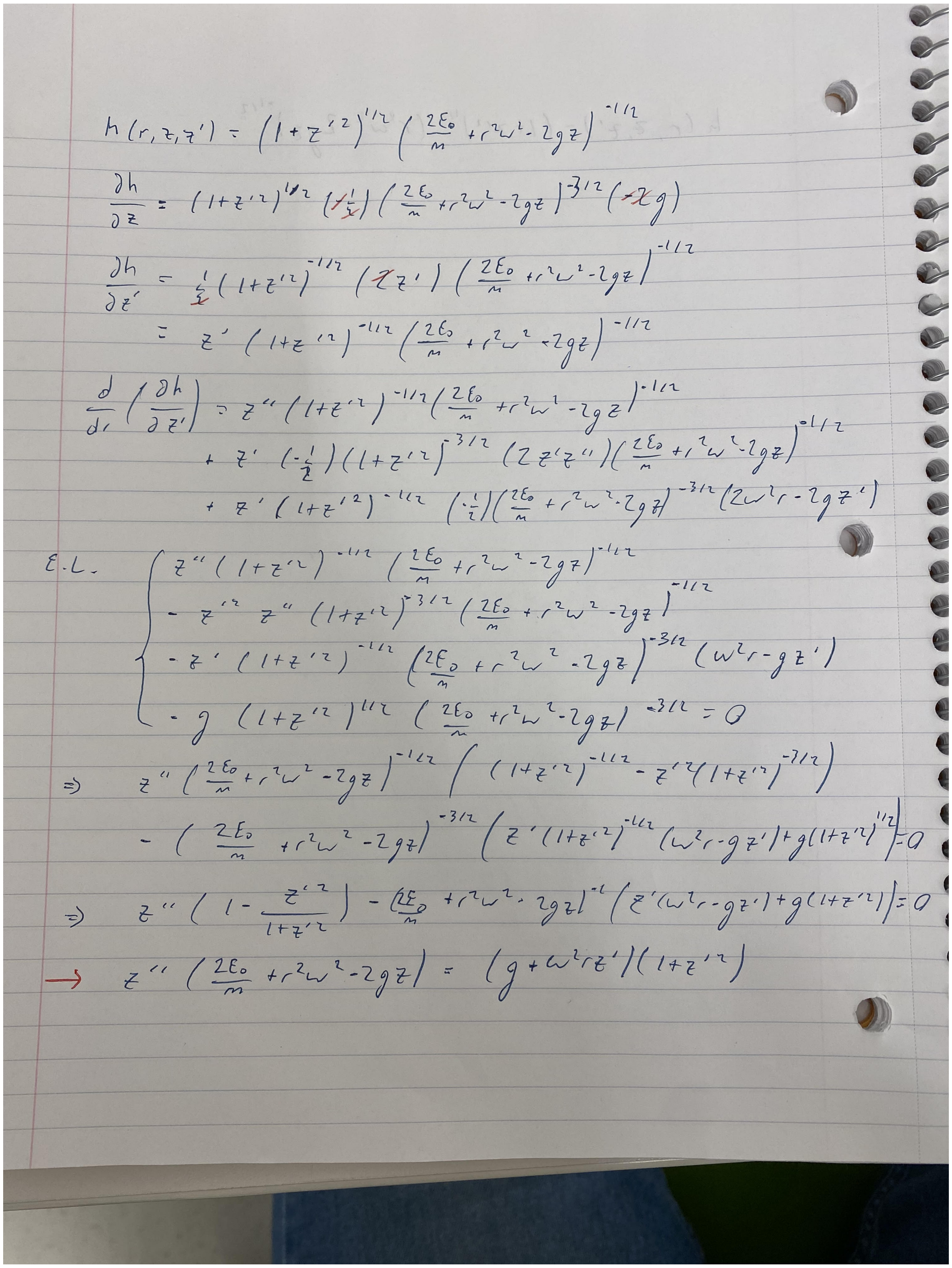

Figure 9: Work for deriving the optimized curve ODE