MA 565: Homework #12

Sean Grate

December 4, 2019

1. *Question*: Let $A \neq 0$ be a nilpotent matrix in $F^{n \times n}$, where F is a field. Prove:

   (a) $\mathrm{tr}\,(A) = 0$.

   (b) $\det\,(I_n + A) = \det\,(I_n - A) = 1$.

   (c) $I_n + A$ is not diagonalizable.

Answer. (a) Recall from Homework #10 that 0 is the only eigenvalue of A. So the characteristic polynomial is $\chi_A = x^n$ and from Remark 16.5, we know that $a_{n-1} = -\mathrm{tr}\,(A)$. But $a_{n-1} = 0$ which implies that $\mathrm{tr}\,(A) = 0$.

(b) Note that if $\alpha \in F$, then $\lambda \in \sigma(A) \implies \lambda + \alpha \in \sigma(\alpha I_n + A)$ since

$$(\alpha I_n + A)v = \alpha v + Av = \alpha v + \lambda v = (\lambda + \alpha)v.$$

and

$$(\alpha I_n - A)v = \alpha v - Av = \alpha v - \lambda v = (-\lambda + \alpha)v.$$

So this implies that 1 is the only eigenvalue of both $I_n + A$ and $I_n - A$. Hence $\det\,(I_n + A) = \det\,(I_n - A) = 1$.

(c) Suppose $I_n + A$ is diagonalizable. Then by Proposition 17.8(c), this implies that $\exists\ S \in GL_n(F)$ such that $S^{-1}(I_n + A)S = D$, where D is a diagonal matrix. Then this implies that

$$\begin{aligned} D &= S^{-1}(I_n + A)S \\ &= (S^{-1} + A)S \\ &= I_n + S^{-1}AS \\ D' \coloneqq D - I_n &= S^{-1}AS \\ SD'S^{-1} &= A. \end{aligned}$$

Note that D' is also a diagonal matrix. So by Homework #10, since A is nilpotent and similar to a diagonal matrix, this implies that $A = 0$ which is a contradiction. Therefore $I_n + A$ is not diagonalizable.

□

2. *Question*: Let A, B, C be square matrices over a field F such that $A \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$. Argue that the polynomials satisfy

$$\mu_A = \operatorname{lcm}(\mu_B, \mu_C),$$

where lcm denotes the least common multiple.

Answer. First note that

$$xI_n - A = \begin{bmatrix} xI - B & 0 \\ 0 & xI - C \end{bmatrix}$$

and by Proposition 14.11, $\det(xI_n - A) = \det(xI - B)\det(xI - C)$. Furthermore, observe that $\mu_A(A) = 0 \iff \mu_A(B) = \mu_A(C) = 0$ which implies that $\mu_B \mid \mu_A$ and $\mu_C \mid \mu_A$. But since μ_A has minimal degree, this implies that $\mu_A = \operatorname{lcm}(\mu_B, \mu_C)$. $\square$

3. *Question*: Let $A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$. Show that $\langle v \mid w \rangle := v^\top A w$ defines an inner product on $\mathbb{R}^3$.

Answer. Let $v_1, v_2, v, w \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. There are four properties to show:

(a) (*Linearity I*)

$$\begin{aligned} \langle v_1 + v_2 \mid w \rangle &= (v_1 + v_2)^\top A w \\ &= (v_1^\top + v_2^\top) A w \\ &= (v_1^\top A + v_2^\top A) w \\ &= v_1^\top A w + v_2^\top A w \\ &= \langle v_1 \mid w \rangle + \langle v_2 \mid w \rangle . \end{aligned}$$

(b) (*Linearity II*)

$$\langle \lambda v \mid w \rangle = (\lambda v)^\top A w = \lambda v^\top A w = \lambda \langle v \mid w \rangle .$$

(c) (*Conjugate-symmetric*) Suppose $v = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$ and $w = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$. Then

$$\begin{aligned}
\langle v \mid w \rangle &= v^\top A w \\
&= \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \\
&= \begin{bmatrix} 4r_1 + 2r_2 & 2r_1 + 3r_2 & r_3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \\
&= 4r_1t_1 + 2r_2t_1 + 2r_1t_2 + 3r_2t_2 + r_3t_3 \\
&= 4t_1r_1 + 2t_1r_2 + 2t_2r_1 + 3t_2r_2 + t_3r_3 \\
&= \overline{4t_1r_1 + 2t_1r_2 + 2t_2r_1 + 3t_2r_2 + t_3r_3} \\
&= \begin{bmatrix} \overline{4t_1 + 2t_2} & \overline{2t_2 + 3t_2} & \overline{t_3} \end{bmatrix} \begin{bmatrix} \overline{r_1} \\ \overline{r_2} \\ \overline{r_3} \end{bmatrix} \\
&= \overline{w^\top A v} \\
&= \overline{\langle w \mid v \rangle}.
\end{aligned}$$

(d) (*Positive-definite*) Suppose $v = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$ and $v \neq 0$. Then this implies that at least one of r_1, r_2, r_3 is non-zero. So

$$\begin{aligned}
\langle z \mid z \rangle &= v^\top A v \\
&= \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \\
&= \begin{bmatrix} 4r_1 + 2r_2 & 2r_1 + 3r_2 & r_3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \\
&= 4r_1^2 + 2r_2r_1 + 2r_1r_2 + 3r_2^2 + r_3^2 \\
&= 4r_1^2 + 3r_2^2 + r_3^2 + 4r_1r_2.
\end{aligned}$$

Since at least of r_1, r_2, r_3 is non-zero, then this implies $\langle v \mid v \rangle$ is nonzero. If $v = 0$, then $\langle v \mid v \rangle = 0$.

So $\langle v \mid w \rangle$ defines an inner product on $\mathbb{R}^3$. □

4. *Question*: Let $(V, \langle \mid \rangle)$ be an inner product space over $\mathbb{R}$ of finite dimension $n \geq 2$. Fix a vector $w \in V$ with $||w|| = 1$, and consider the map $\varphi \colon V \to V$, $v \mapsto v - 2\langle v \mid w \rangle \cdot w$.

(a) Show φ is a linear map and $||\varphi(v)|| = ||v||$ for all $v \in V$.

(b) Prove φ has eigenvalues 1 and -1 and the corresponding eigenspaces satisfy

$$\operatorname{eig}(\varphi; 1)^{\perp} = \operatorname{eig}(\varphi; -1) \quad \text{and} \quad V = \operatorname{eig}(\varphi; 1) \oplus \operatorname{eig}(\varphi; -1).$$

(c) Describe the map φ geometrically.

Answer. (a) Note that φ is linear since if $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned}
\varphi(v_1 + \lambda v_2) &= (v_1 + \lambda v_2) - 2\langle v_1 + \lambda v_2 \mid w\rangle \cdot w \\
&= v_1 + \lambda v_2 - 2(\langle v_1 \mid w\rangle + \langle \lambda v_2 \mid w\rangle) w \\
&= v_1 + \lambda v_2 - 2(\langle v_1 \mid w\rangle + \lambda \langle v_2 \mid w\rangle) w \\
&= v_1 + \lambda v_2 - 2\langle v_1 \mid w\rangle \cdot w - 2\lambda \langle v_2 \mid w\rangle \cdot w \\
&= (v_1 - 2\langle v_1 \mid w\rangle \cdot w) + \lambda (v_2 - 2\langle v_2 \mid w\rangle \cdot w) \\
&= \varphi(v_1) + \lambda \varphi(v_2).
\end{aligned}$$

Also, $||\varphi(v)|| = ||v||$ for all $v \in V$ since

$$\begin{aligned}
||\varphi(v)||^2 &= ||v - 2\langle v \mid w\rangle \cdot w||^2 \\
&= \langle v - 2\langle v \mid w\rangle \cdot w \mid v - 2\langle v \mid w\rangle \cdot w\rangle \\
&= \langle v \mid v - 2\langle v \mid w\rangle \cdot w\rangle - 2\langle v \mid w\rangle \langle w \mid v - 2\langle v \mid w\rangle \cdot w\rangle \\
&= \langle v \mid v\rangle - 2\langle v \mid w\rangle \langle v \mid w\rangle - 2\langle v \mid w\rangle \langle w \mid v\rangle + 4\langle v \mid w\rangle \langle v \mid w\rangle \langle w \mid w\rangle \\
&= \langle v \mid v\rangle - 2\langle v \mid w\rangle^2 - 2\langle v \mid w\rangle^2 + 4\langle v \mid w\rangle^2 \langle w \mid w\rangle \\
&= \langle v \mid v\rangle - 4\langle v \mid w\rangle^2 + 4\langle v \mid w\rangle^2 \langle w \mid w\rangle \\
&= ||v||^2 - 4\langle v \mid w\rangle^2 + 4\langle v \mid w\rangle^2 ||w||^2 \\
&= ||v||^2 - 4\langle v \mid w\rangle^2 + 4\langle v \mid w\rangle^2 \\
&= ||v||^2.
\end{aligned}$$

Hence $||\varphi(v)|| = ||v||$ as desired.

(b) To find the eigenvalue of φ, note that $\lambda \in \mathbb{R}$ is an eigenvalue of φ if and only if $\varphi(v) = \lambda v$. This implies $||\varphi(v)|| = ||\lambda v|| = |\lambda|\,||v||$ by Proposition 18.8(a). But since $||\varphi(v)|| = ||v||$, this implies $|\lambda| = 1$. The only such possible values for λ are 1 and -1.

Observe that if $v \in \operatorname{eig}(\varphi; -1)$, then $\varphi(v) = -v$ and so we get

$$\begin{aligned}
-v &= v - 2\langle v \mid w\rangle \cdot w \\
-2v &= -2\langle v \mid w\rangle \cdot w \\
v &= \langle v \mid w\rangle \cdot w.
\end{aligned}$$

This implies that $w \in \operatorname{eig}(\varphi; -1)$. So for any vector $v \in \operatorname{eig}(\varphi; -1)$, we have $w = \lambda v$. Now suppose $u \in \operatorname{eig}(\varphi; 1)$. Then since $\varphi(u) = u$, we have that

$$\begin{aligned} u &= u - 2 \langle u \mid w \rangle \cdot w \\ 0 &= -2 \langle u \mid w \rangle \cdot w \\ 0 &= \langle u \mid w \rangle \cdot w \end{aligned}$$

and since $||w|| = 1$, this implies $w \neq 0$ by Proposition 18.8(b). Hence $\langle u \mid w \rangle = \langle u \mid \lambda v \rangle = \overline{\lambda} \langle u \mid v \rangle = 0$. Thus we have shown that for any pair of vectors $u \in \operatorname{eig}(\varphi; 1), v \in \operatorname{eig}(\varphi; -1)$, we have $\langle v \mid u \rangle = 0$. Therefore $\operatorname{eig}(\varphi; 1)^{\perp} = \operatorname{eig}(\varphi; -1)$. As a consequence of Theorem 17.10(c), we thus also have that $V = \operatorname{eig}(\varphi; 1) \oplus \operatorname{eig}(\varphi; -1)$.

(c) Geometrically, φ reflects a vector $v \in V$ across the hyperplane determined by the vector w.

$\square$

Supplemental Material

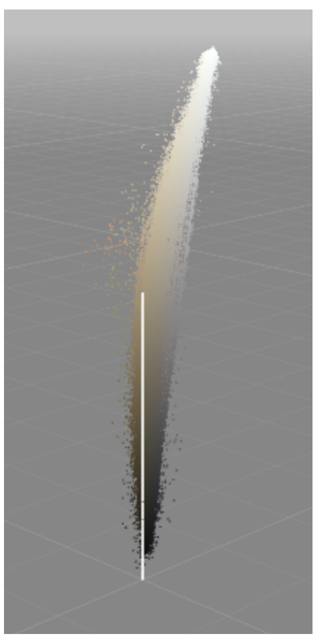
There are some nice visualizations you can get using the inner product defined in Question #4. My printer is only black and white, so the results look a lot better in color. After applying φ to the RGB color values for each pixel, some simple transformations had to be done in order to rescale the values into $[0, 255] \times [0, 255] \times [0, 255]$ and get a comprehensible image. However, the points in RGB space are the raw RGB values before and after applying φ. Each point is a position in RGB space (colored accordingly). After applying φ to the points in RGB space, the coloring remained the same for consistency, but the position of the points was not altered (unlike in the image).

The white line is the fixed (randomly generated) vector w. A rotation was applied to all points in order to ensure that w was oriented along the conventional z-axis for ease of interpreting the results. This naturally allows the gray grid to act as the plane defined by w (since w now acts as a normal vector to it). From this, we can see the geometric interpretation of φ on $\mathbb{R}^3$. The images are each respectively from

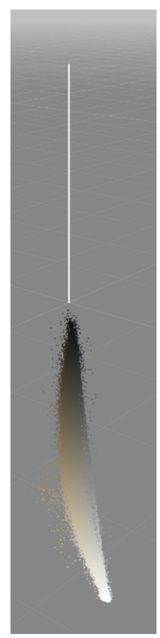
- https://provectapet.com/provecta-for-cats/
- https://en.wikipedia.org/wiki/Mona_Lisa
- https://weheartit.com/articles/337024026-rainbow-tag
- https://www.scenic-safaris.com/blog/2017/06/5-places-for-spectacular-sunsets-in-grand-teton
- https://www.oprahmag.com/entertainment/a29739536/cat-meme-taylor-armstrong-explained/

(a) A cat.

(b) Image after applying φ to the RGB colors.



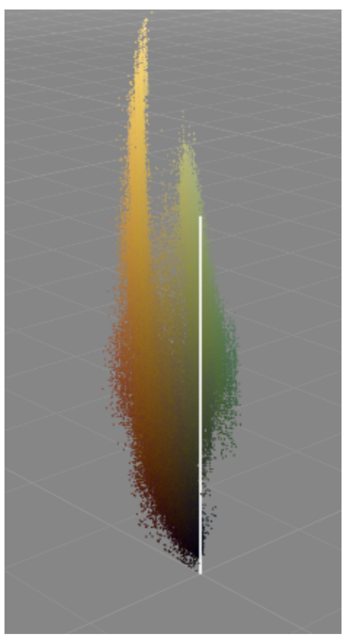
(c) Original RGB colors (plotted in $\mathbb{R}^3$).



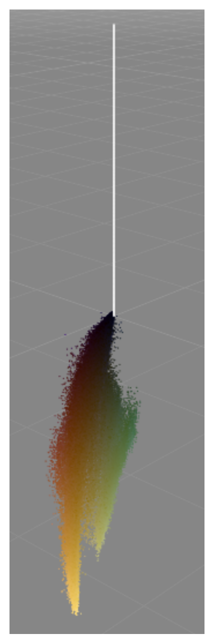
(d) RGB colors after applying φ (plotted in $\mathbb{R}^3$).

(a) The Mona Lisa.

(b) Image after applying φ to the RGB colors.

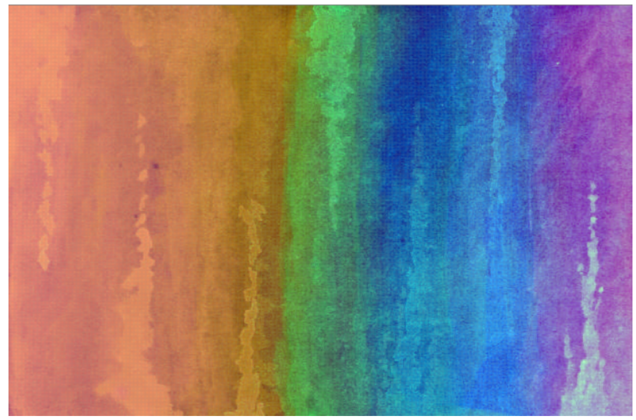


(c) Original RGB colors (plotted in $\mathbb{R}^3$).

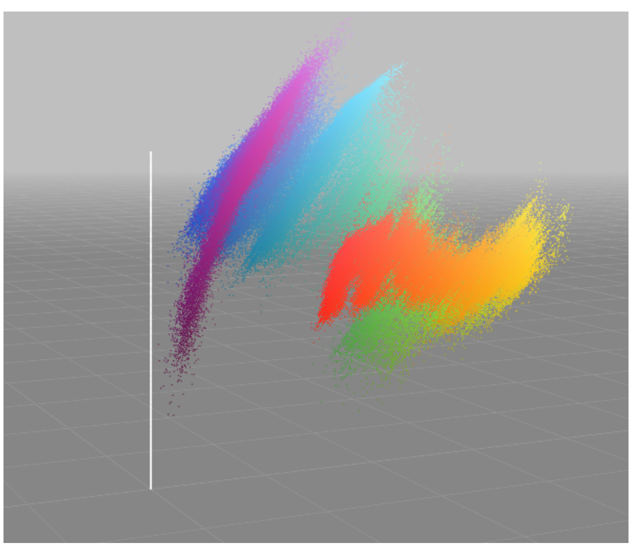


(d) RGB colors after applying φ (plotted in $\mathbb{R}^3$).

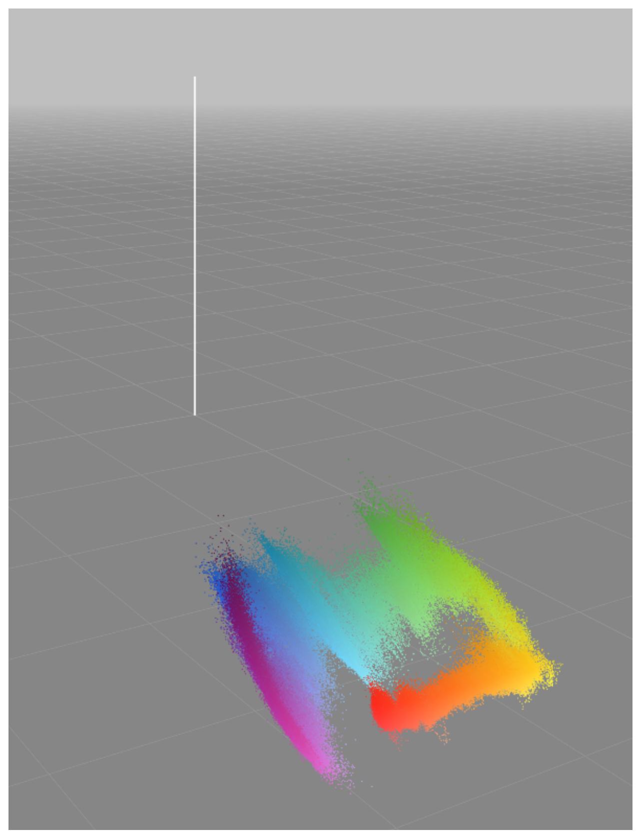
(a) A colorful image.



(b) Image after applying φ to the RGB colors.



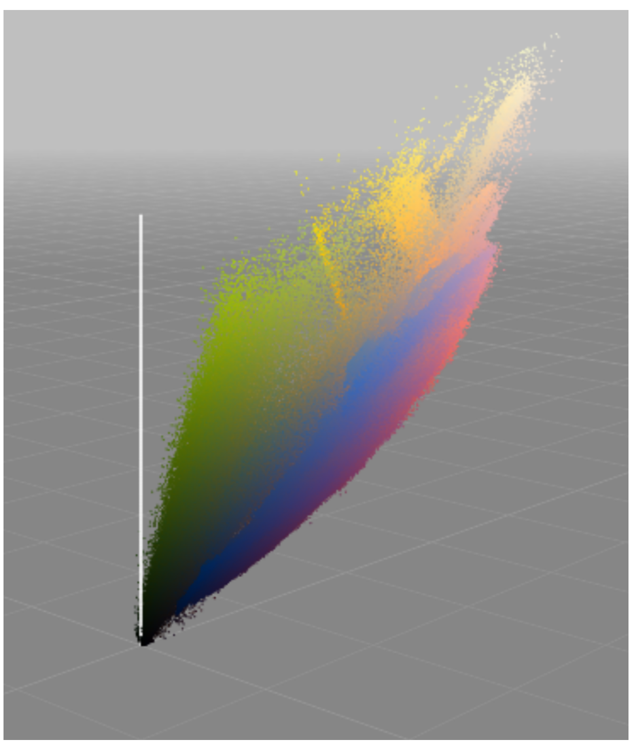
(c) Original RGB colors (plotted in $\mathbb{R}^3$).



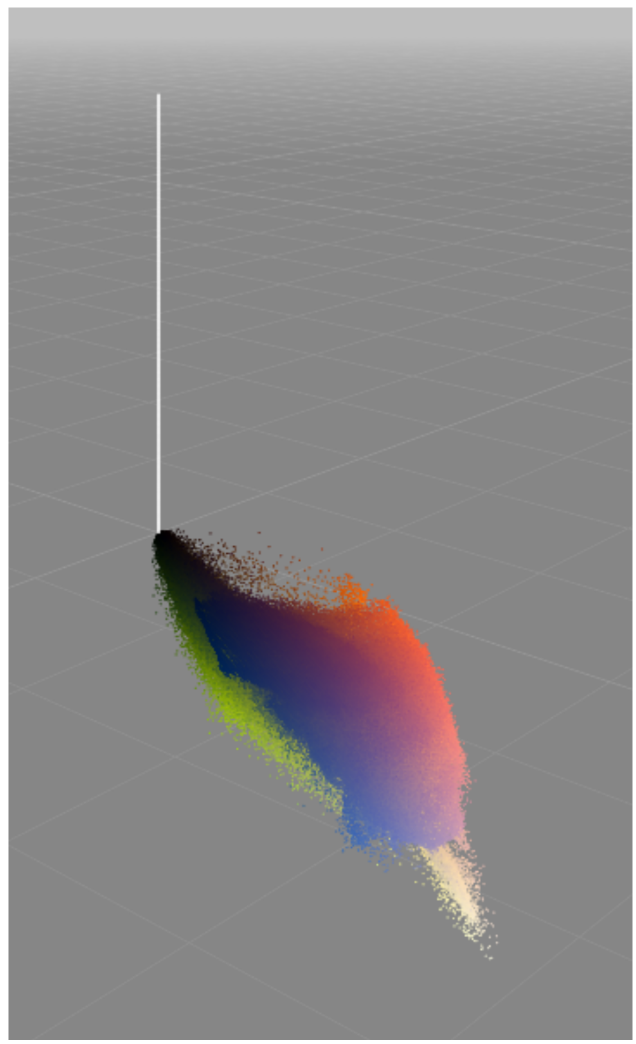
(d) RGB colors after applying φ (plotted in $\mathbb{R}^3$).

(a) The Grand Tetons.

(b) Image after applying φ to the RGB colors.



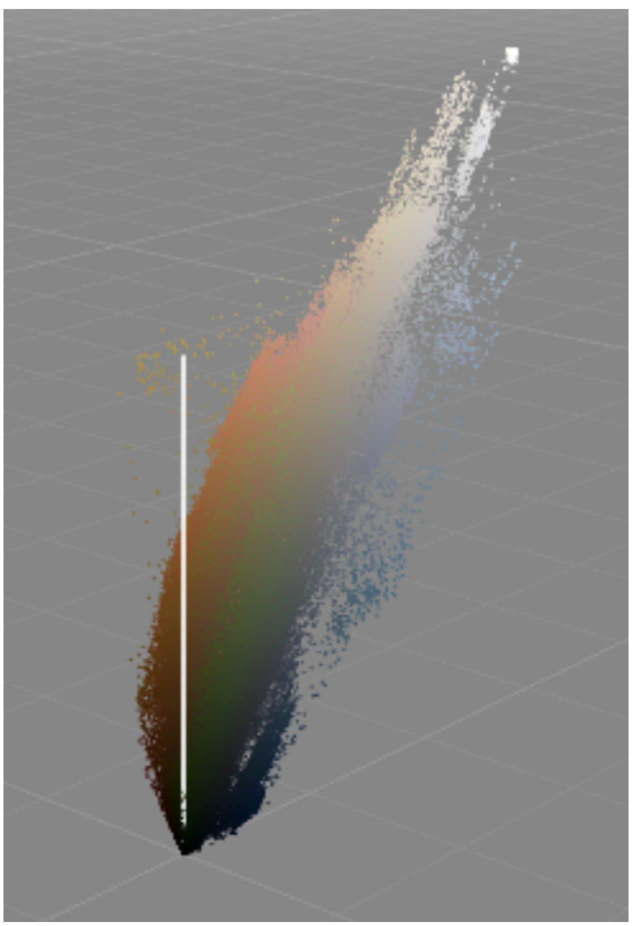
(c) Original RGB colors (plotted in $\mathbb{R}^3$).



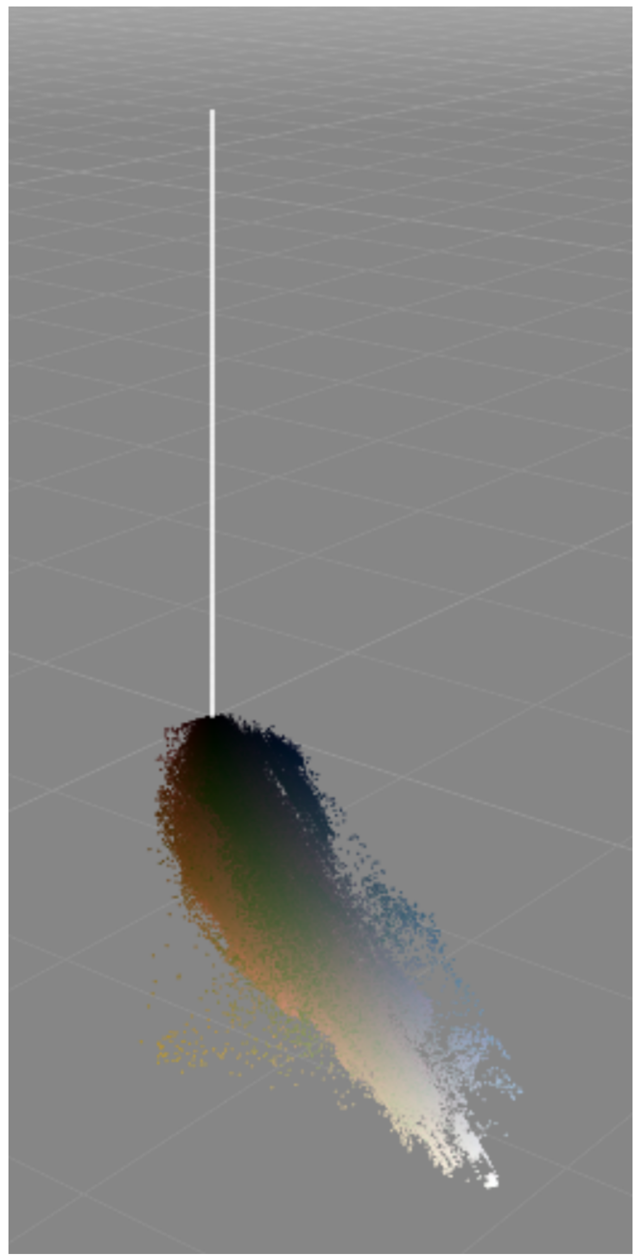
(d) RGB colors after applying φ (plotted in $\mathbb{R}^3$).

(a) A meme.

(b) Image after applying φ to the RGB colors.



(c) Original RGB colors (plotted in $\mathbb{R}^3$).



(d) RGB colors after applying φ (plotted in $\mathbb{R}^3$).