Trinity Western University
Department of Mathematical Sciences
MATH250 (Linear Algebra)
Final Examination Solution

1. a) If A and B are nonzero $n \times n$ matrices and $AB = 0$. Show that both A and B are singular.

Solution:

We prove the result by contradiction. Let one of the two matrices A or B be nonsingular. To be specific we take A to be nonsingular. Then A^{-1} exists. Multiply both sides of $AB = 0$ on the left by A^{-1}. We obtain $A^{-1}AB = A^{-1}0 \Rightarrow IB = 0 \Rightarrow B = 0$, which contradicts the assumption that B is nonzero. Similarly it can be shown than if B is nonsingular then $A = 0$, which again results into a contradiction. Hence both A and B are singular.

(b) Consider the system

$$2x - 3y + 5z = 0$$
$$-x + 7y - z = 0$$
$$4x - 11y + kz = 0$$

For what value(s) of k will the system have non-trivial solution. Also find these solutions.(Indicate in each case how many solutions will you get, also giving the solutions if they exist).

Solution:

The coefficient matrix is

$$A = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 7 & -1 \\ 4 & -11 & k \end{pmatrix}$$

For non-trivial solutions $\det(A) = 0$

$$\left|\begin{array}{ccc|cc} 2 & -3 & 5 & 2 & -3 \\ -1 & 7 & -1 & -1 & 7 \\ 4 & -11 & k & 4 & -11 \end{array}\right. = 0$$

$\Rightarrow 14k + 12 + 55 - (140 + 22 + 3k) = 0$

$\Rightarrow 11k - 95 = 0 \Rightarrow k = \frac{95}{11}$.

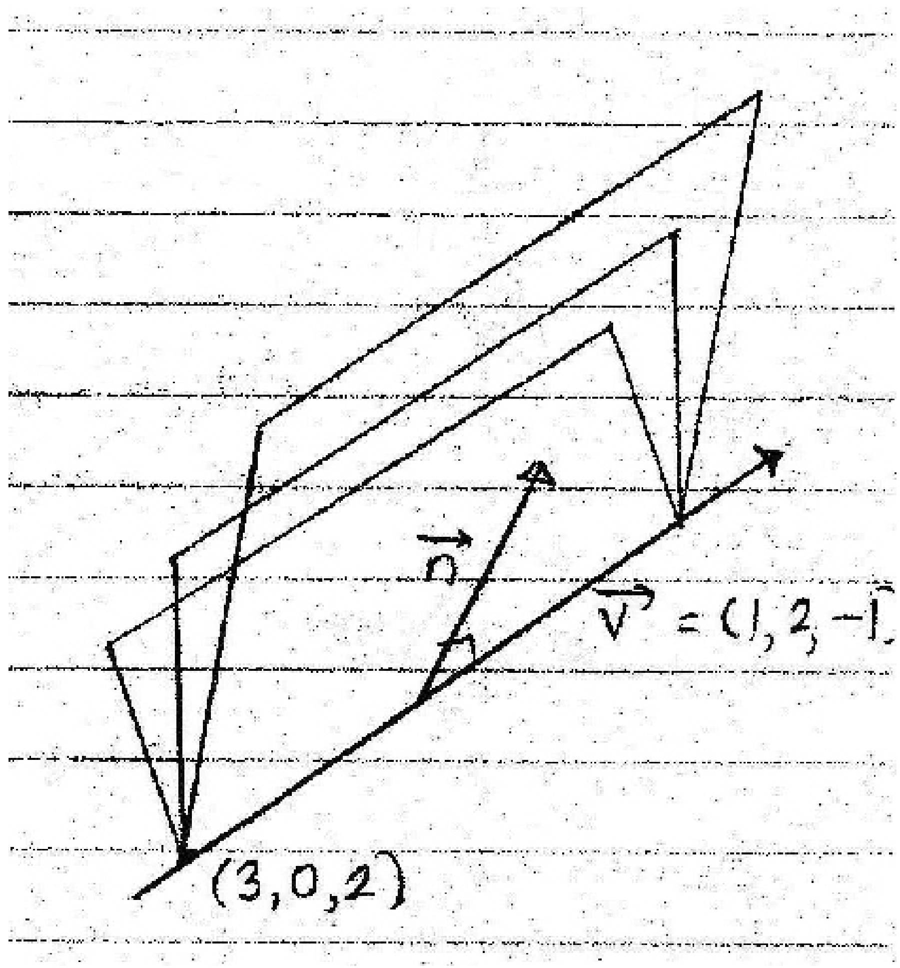
The solution space is simply the nullspace of A, which is orthogonal to the row space. Taking the cross product of the first two rows, we get

(3 - 35, -5 + 2, 14 - 3) or (-32, -3, 11).

Hence the non-trivial solutions (when $k = 95/11$) are $(-32, -3, 11)t$.

2. a) Find the equation of all planes containing the line $(x, y, z) = (3, 0, 2) + t(1, 2, -1)$.

Solution:



Note that there are more than one plane containing the given line. In fact, there are infinitely many. So we expect a parameter to occur in the equation of the planes.

Clearly the normal $\mathbf{n} = (a, b, c)$ will be perpendicular to the direction vector $\mathbf{v} = (1, 2, -1)$ of the line.

Therefore $\mathbf{n} \cdot \mathbf{v} = 0$.

$\Rightarrow (a, b, c) \cdot (1, 2, -1) = 0$

$\Rightarrow a + 2b - c = 0$

$\Rightarrow c = a + 2b$

Also the plane passes through the point (3, 0, 2). Therefore its equation is

$a(x-3) + b(y-0) + c(z-2) = 0$

$\Rightarrow a(x-3) + by + (a+2b)(z-2) = 0$

$\Rightarrow a(x-3+z-2) + b(y+2z-4) = 0$

$\Rightarrow a(x+z-5) + b(y+2z-4) = 0$

It appears that there are two parameters a and b in the equation. In reality there is only one parameter, for we can divide the equation by either a or b (whichever is non-zero), in which case there will be only one parameter; either b/a or a/b in the equation.

b) Find if $W = \{(a,\ a-1,\ c)|a,\ c \text{ in } \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. If not indicate why it is not.

Solution:

For W to be a subspace we need to prove the following:
1) W is not empty.
Clearly W is not empty, since (0, -1, 0) $\in W$.
2) W is closed with respect to addition.
Let $\mathbf{u} = (a,\ a-1,\ c)$ and $\mathbf{v} = (b,\ b-1,\ d)$
$\mathbf{u} + \mathbf{v} = (a+b,\ a+b-2,\ c+d) = (p,\ p-2,\ q)$
where $p = a+b$ and $q = c+d$.
Obviously $\mathbf{u} + \mathbf{v} \notin W$, since the x- and y- components differ by 2, and not by one.
3) W is closed with respect to addition.
Consider $0\mathbf{u} = 0(a, a-1, c) = (0, 0, 0)$.
$0u \notin W$, since the x- and y- components are equal, whereas they must differ by one.
Hence W is not a subspace, since it is neither closed with respect to addition, nor with respect to scalar multiplication.

3. a) Prove that a vector in a vector space has exactly additive identity (zero).

Solution:

If possible let the vector space have two zeros $\mathbf{0}_1$ and $\mathbf{0}_2$.
Then we must have
$\mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2 + \mathbf{0}_1 = \mathbf{0}_1$ (since $\mathbf{0}_2$ is a zero)
and $\mathbf{0}_2 + \mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$ (since $\mathbf{0}_1$ is a zero)
From the above two equations we get $\mathbf{0}_1 = \mathbf{0}_2$. Thus the two zeros are identical, or in other words, there is exactly one zero.

b) Find the basis for the null space of

$$A = \begin{pmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{pmatrix}$$

Then compute rank (A) and verify the theorem stating that

rank (A) + nullity $(A) = n$,

n, being the number of columns in A.

Solution:

The augmented matrix is

$$\begin{pmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{pmatrix} \qquad R_{12}(-2),\ R_{14}(1)$$

$$\sim \begin{pmatrix} 1 & -1 & 5 & -2 & 2 \\ 0 & 0 & -12 & 9 & -3 \\ 0 & 0 & -12 & 9 & -3 \\ 0 & 0 & 12 & -9 & 3 \end{pmatrix} \qquad R_{23}(-1),\ R_{24}(1)$$

$$\sim \begin{pmatrix} 1 & -1 & 5 & -2 & 2 \\ 0 & 0 & -12 & 9 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_{23}(-5)$$

The equivalent system of equations is

$x_1 - x_2 + 5x_3 - 2x_4 + 2x_5 = 0$
$-12x_3 + 9x_4 - 3x_5 = 0$
$0 = 0$
$0 = 0$

Solving for leading variables, we obtain

$x_1 = x_2 - 5x_3 + 2x_4 - 2x_5$
$x_3 = \frac{3}{4}x_4 - \frac{1}{4}x_5$

Setting the free variables $x_2 = s,\ x_4 = t,\ x_5 = u$, we get the following solution

$x_3 = \frac{3}{4}t - \frac{1}{4}u$
$x_1 = s - 5\left(\frac{3}{4}t - \frac{1}{4}u\right) + 2t - 2u = s - \frac{7}{4}t - \frac{3}{4}u$
$x_2 = s$
$x_4 = t$
$x_5 = u,$

which can be put in the vector form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} s - \frac{7}{4}t - \frac{3}{4}u \\ s \\ \frac{3}{4}t - \frac{1}{4}u \\ t \\ u \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{7}{4} \\ 0 \\ \frac{3}{4} \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} -\frac{3}{4} \\ 0 \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$$

The basis for the nullspace of A is therefore $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$, where

$\mathbf{v}_1 = (1, 1, 0, 0, 0)$, $\mathbf{v}_2 = (-\frac{7}{4}, 0, -\frac{3}{4}, 1, 0)$ and $\mathbf{v}_3 = (-\frac{3}{4}, 0, -\frac{1}{4}, 0, 1)$.

Since A contains three leading non-zero entries, $\text{rank}(A) = 2$. Also $\text{nullity}(A) =$ 3

$\text{rank}(A) + \text{nullity}(A) = 2 + 3 = 5$, the number of columns in A.

4. Find a subset of the vectors that forms a basis for the space spanned by the

vectors; then express each vector that is not in the basis as a linear combination of the basis vectors.

$\mathbf{v}_1$ = (1, 0, -1, 3), $\mathbf{v}_2$ = (2, 1, 0, -2), $\mathbf{v}_3$ = (-1, 1, 2, 1), $\mathbf{v}_4$ = (-3, 2, 4, 11)

Solution:

We form a matrix A consisting of the given vectors as its rows.

$$A = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 0 & -2 \\ -1 & 1 & 2 & 1 \\ -3 & 2 & 4 & 11 \end{pmatrix}$$

$$\Rightarrow A^T = \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ -1 & 0 & 2 & 4 \\ 3 & -2 & 1 & 11 \end{pmatrix} \qquad R_{13}(1),\ R_{14}(-3)$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & -8 & 4 & 20 \end{pmatrix} \qquad R_{21}(-2),\ R_{23}(-2),\ R_{24}(8)$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 12 & 36 \end{pmatrix} \qquad R_3(-1)$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 12 & 36 \end{pmatrix} \qquad R_{31}(3),\ R_{32}(-1),\ R_{34}(-12)$$

$$\sim \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3 \quad \mathbf{w}_4 \end{array}$$

The vectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ form the basis of the column space of A^T. Hence the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis of the rowspace of A. Further it can be seen that

$\mathbf{w}_4 = 2\mathbf{w}_1 - \mathbf{w}_2 + 3\mathbf{w}_3$

Therefore

$\mathbf{v}_4 = 2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_2$

5. Let $\mathbb{R}^3$ have the inner product $< (x, y, z), (x', y', z') >= 2xx' + yy' + 3zz'$

Use the Gram-Schmidt algorithm to transform the basis B given below into an orthonormal basis

$B = \{(1, 1, 1), (1, -1, 1), (1, 1, 0)\}$

Solution:

Let $\mathbf{u}_1 = (1,1,1),\ \mathbf{u}_2 = (1,-1,1),\ \mathbf{u}_3 = (1,1,0)$

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 = (1,1,1)\\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2,\ \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1\\
&= (1,-1,1) - \frac{2(1)(1)+(-1)(1)+3(1)(1)}{2(1)^2+(1)^2+3(1)^2}(1,1,1)\\
&= (1,-1,1) - \tfrac{2}{3}(1,1,1) = (\tfrac{1}{3},-\tfrac{5}{3},\tfrac{1}{3})\\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3,\ \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3,\ \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2\\
&= (1,1,0) - \frac{2(1)(1)+(1)(1)+3(0)(1)}{2(1)^2+(1)^2+3(1)^2}(1,1,1) -\\
&\quad - \frac{2(1)(\frac{1}{3})+(1)(-\frac{5}{3})+3(0)(\frac{1}{3})}{2(\frac{1}{3})^2+(-\frac{5}{3})^2+3(\frac{1}{3})^2}(\tfrac{1}{3},-\tfrac{5}{3},\tfrac{1}{3})\\
&= (1,1,0) - \tfrac{1}{2}(1,1,1) + \tfrac{3}{10}(\tfrac{1}{3},-\tfrac{5}{3},\tfrac{1}{3}) = (\tfrac{3}{5},0,-\tfrac{2}{5})
\end{aligned}
$$

Hence the new orthogonal basis is

$$\left\{(1,1,1),\ (\tfrac{1}{3},-\tfrac{5}{3},\tfrac{1}{3}),\ (\tfrac{3}{5},0,-\tfrac{2}{5})\right\}$$

It can be normalized by dividing each vector by its norm. We get the orthonormal basis as

$$\left\{\ \left(\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{6}}\right),\quad \left(\tfrac{1}{\sqrt{30}},-\tfrac{5}{\sqrt{30}},\tfrac{1}{\sqrt{30}}\right),\quad \left(\tfrac{3}{\sqrt{30}},0,-\tfrac{2}{\sqrt{30}}\right)\right\}$$

6. For the matrix A, obtain A^n, where n is a positive integer.

$$A = \begin{pmatrix} -1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3 \end{pmatrix}$$

Solution:

The characteristic equation of the matrix is

$$\det(A) - \lambda(M_{11}+M_{22}+M_{33}) + \lambda^2(a_{11}+a_{22}+a_{33}) - \lambda^3 = 0$$

$\Rightarrow 6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$

$\Rightarrow (1-\lambda)(6-5\lambda+\lambda^2) = 0$

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$

$\Rightarrow \lambda = 1,\ 2,\ 3$

$$\lambda = 1,\ A-\lambda I = \begin{pmatrix} -2 & 4 & -2\\ -3 & 3 & 0\\ -3 & 1 & 2 \end{pmatrix},\ \mathbf{p} = (6,6,6) = 6(1,1,1)$$

$$\lambda = 2,\ A-\lambda I = \begin{pmatrix} -3 & 4 & -2\\ -3 & 2 & 0\\ -3 & 1 & 1 \end{pmatrix},\ \mathbf{p} = (4,6,6) = 2(2,3,3)$$

$$\lambda = 3,\ A-\lambda I = \begin{pmatrix} -4 & 4 & -2\\ -3 & 1 & 0\\ -3 & 1 & 0 \end{pmatrix},\ \mathbf{p} = (2,6,8) = 2(1,3,4)$$

Thus

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

$$[P \mid I] = \left(\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right) \qquad R_{12}(-1),\ R_{13}(-1)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right) \qquad R_{21}(-2),\ R_{23}(-1)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \qquad R_{31}(3),\ R_{32}(-1)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5 & 3 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) = [I \mid P^{-1}]$$

$$\Rightarrow P^{-1} = \begin{pmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

Therefore

$$D = P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & -5 & 3 \\ -2^n & 3 \cdot 2^n & -2 \cdot 2^n \\ 0 & -3^n & 3^n \end{pmatrix}$$

$$= \begin{pmatrix} 3 - 2 \cdot 2^n & -5 + 6 \cdot 2^n - 3^n & 3 - 4 \cdot 2^n + 3^n \\ 3 - 3 \cdot 2^n & -5 + 9 \cdot 2^n - 3 \cdot 3^n & 3 - 6 \cdot 2^n + 3 \cdot 3^n \\ 3 - 3 \cdot 2^n & -5 + 9 \cdot 2^n - 4 \cdot 3^n & 3 - 6 \cdot 2^n + 4 \cdot 3^n \end{pmatrix}$$

7. A trucking company has 100 trucks, which are dispatched from three locations A, B and C. Each truck at A, B and C uses 40, 30 and 30 units of fuel daily, and 2,500 units per day are available. The costs of labor to operate and maintain each truck are \$70, \$80 and \$70 per truck per day at the three locations, and \$8,000 per day is the maximum that the company can pay for labor. How many trucks should be allocated to each location if the daily profits per truck are \$300, \$250, and \$200 at locations A, B, and C?

Solution:

Let the trucks dispatched from locations A, B and C be x, y and z respectively. The profit f is given by

$f = 300x + 250y + 200z.$

The constraints are

$x + y + z \leq 100 \qquad \text{(number of trucks)}$

$40x + 30y + 30z \leq 2500 \qquad \text{(fuel)}$

$70x + 80y + 30z \leq 8000 \qquad \text{(labor)}$

$x,\ y,\ z\ \geq 0$

Introducing the slack variables u, v, and w, the constraints change into the following equations

$x + y + z + u = 100$

$40x + 30y + 30z + v = 2500$

$70x + 80y + 30z + w = 8000$

and the equation for the objective function is written as

$-300x - 250y - 200z + f = 0$

The initial tableau is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ \boxed{40} & 30 & 30 & 0 & 1 & 0 & 0 & 2500 \\ 70 & 80 & 30 & 0 & 0 & 1 & 0 & 8000 \\ \boxed{-300} & -250 & -200 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \left| \begin{array}{l} 100 \\ 62\frac{1}{2} \\ 114\frac{2}{7} \\ \end{array} \right. \qquad R_2(\tfrac{1}{40})$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ 1 & \frac{3}{4} & \frac{3}{4} & 0 & \frac{1}{40} & 0 & 0 & \frac{125}{2} \\ 70 & 80 & 30 & 0 & 0 & 1 & 0 & 1800 \\ -300 & -250 & -200 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad R_{21}(-1),\ R_{23}(-70),\ R_{24}(300)$$

$$\sim \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 1 & -\frac{1}{40} & 0 & 0 & \frac{75}{2} \\ 1 & \boxed{\frac{3}{4}} & \frac{3}{4} & 0 & \frac{1}{40} & 0 & 0 & \frac{125}{2} \\ 0 & \frac{55}{2} & \frac{35}{2} & 0 & -\frac{7}{4} & 1 & 0 & 3625 \\ 0 & \boxed{-25} & 25 & 0 & \frac{15}{2} & 0 & 1 & 18750 \end{pmatrix} \left| \begin{array}{l} 150 \\ 83\frac{1}{3} \\ 131\frac{9}{11} \\ \end{array} \right. \qquad R_2(\tfrac{4}{3})$$

$$\sim \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 1 & -\frac{1}{40} & 0 & 0 & \frac{75}{2} \\ \frac{4}{3} & 1 & 1 & 0 & \frac{1}{30} & 0 & 0 & \frac{250}{3} \\ 0 & \frac{55}{2} & \frac{35}{2} & 0 & -\frac{7}{4} & 1 & 0 & 3625 \\ 0 & -25 & 25 & 0 & \frac{15}{2} & 0 & 1 & 18750 \end{pmatrix} \qquad R_{21}(-\tfrac{1}{4}),\ R_{23}(-\tfrac{55}{2}),\ R_{24}(-25)$$

$$\sim \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{30} & 0 & 0 & \frac{50}{3} \\ \frac{4}{3} & 1 & 1 & 0 & \frac{1}{30} & 0 & 0 & \frac{250}{3} \\ -\frac{110}{3} & 0 & -10 & 0 & -\frac{8}{3} & 1 & 0 & \frac{4000}{3} \\ \frac{100}{3} & 0 & 50 & 0 & \frac{25}{3} & 0 & 1 & \frac{62500}{3} \end{pmatrix}$$

Since none of the entries in the last row is negative, it is the final tableau.

We have

$-\frac{1}{3}x + u - \frac{1}{30}v = \frac{50}{3}$

$\frac{4}{3}x + y + z + \frac{1}{30}v = \frac{250}{3}$

$-\frac{110}{3}x - 10z - \frac{8}{3}v + w = \frac{4000}{3}$

$\frac{100}{3}x + 50z + \frac{25}{3}v + f = \frac{62500}{3}$

From the last equation

$$f = \frac{62500}{3} - \frac{100}{3}x - 50z - \frac{25}{3}v$$

For maximum f, $x = 0$, $z = 0$ and $v = 0$, which when substituted in the first three equations above gives

$$u = \frac{50}{3},\ y = \frac{250}{3},\ w = \frac{4000}{3}$$

Hence for maximum profit trucks should be dispatched from location B only. The number of trucks is $83\frac{1}{3}$ per day, and then the profit is $\$20,833\frac{1}{3}$. (Strictly speaking, one can't take the number of trucks as a fraction - that means then one must deal with various cases e.g., $x = 0,\ y = 83,\ z = 0$ etc.)

8. A car survey has found that 80% of those who were driving a car five years ago are now driving a car, 10% are now driving a minivan, and 10% are now driving a sport utility vehicle. Of those who were driving a minivan five years ago, 20% are now driving a car, 70% are now driving a minivan, and 10% are now driving a sport utility vehicle. Finally, of those who were driving a sport utility vehicle five years ago, 10% are now driving a car, 30% are now driving a minivan, and 60% are now driving a sport utility vehicle.

a) Determine the transition matrix for this Markov chain.

b) Suppose that 70% of those questioned were driving cars five years ago, 20% were driving minivans, and 10% were driving sport utility vehicles. Estimate the percentage of these persons driving each type of vehicle now.

c) Under the conditions in (b), estimate the percentage of these persons who will be driving each type of vehicle five years from now? Ten years from now?

d) Determine the percentage of these persons driving each type of vehicle in the long run, assuming that the present trend continues indefinitely.

Solution:

(a) The transition matrix is

$$P = \begin{pmatrix} 0.8 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{pmatrix}$$

(b) The initial state vector is $\mathbf{x}^{(0)} = \begin{pmatrix} 0.7 \\ 0.2 \\ 0.1 \end{pmatrix}$

The current state vector will be $\mathbf{x}^{(1)} = P\mathbf{x}^{(0)}$.

$$\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{pmatrix} 0.8 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} 0.7 \\ 0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.61 \\ 0.24 \\ 0.15 \end{pmatrix}$$

Thus currently 61% will be driving a car, 24% will be driving a minivan, and 15% will be driving a SUV.

(c) Five years from now the state vector will be $\mathbf{x}^{(2)} = P\mathbf{x}^{(1)}$

$$\mathbf{x}^{(2)} = P\mathbf{x}^{(1)} = \begin{pmatrix} 0.8 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} 0.61 \\ 0.24 \\ 0.15 \end{pmatrix} = \begin{pmatrix} 0.551 \\ 0.274 \\ 0.175 \end{pmatrix}$$

So after five years 55.1% will be driving a car, 27.4% will be driving a minivan, and 17.5% will be driving a SUV.

Ten years from now the state vector will be $\mathbf{x}^{(3)} = P\mathbf{x}^{(2)}$

$$\mathbf{x}^{(3)} = P\mathbf{x}^{(2)} = \begin{pmatrix} 0.8 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} 0.551 \\ 0.274 \\ 0.175 \end{pmatrix} = \begin{pmatrix} 0.5131 \\ 0.2994 \\ 0.1875 \end{pmatrix}$$

Hence after ten years 51.31% will be driving a car, 29.94% will be driving a minivan, and 18.75% will be driving a SUV.

(d) Let the steady state vector be $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

The steady state is given by

$P\mathbf{q} = \mathbf{q} \Rightarrow (P - I)\mathbf{q} = \mathbf{0}$

$$\Rightarrow \begin{pmatrix} -0.2 & 0.2 & 0.1 \\ 0.1 & -0.3 & 0.3 \\ 0.1 & 0.1 & -0.4 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The nullspace (or the solution space) of $A\mathbf{x} = \mathbf{0}$ is orthogonal to the rowspace of A. Therefore the vector $\mathbf{q}$ must be orthogonal to the three row vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ of the matrix $P - I$. Therefore it must be along $\mathbf{r}_1 \times \mathbf{r}_2$.

$\mathbf{r}_1 \times \mathbf{r}_2 = (0.09,\ 0.07,\ 0.04) = 0.01(9, 7, 4)$

Hence we can choose $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = t \begin{pmatrix} 9 \\ 7 \\ 4 \end{pmatrix}$

$\Rightarrow q_1 = 9t,\ q_2 = 7t,\ q_3 = 4t$

But $q_1 + q_2 + q_3 = 1 \Rightarrow 9t + 7t + 4t = 1 \Rightarrow 20t = 1 \Rightarrow t = \frac{1}{20}$

Thus $q_1 = \frac{9}{20},\ q_2 = \frac{7}{20},\ q_3 = \frac{1}{5}$

So in the long run 45% will be driving a car, 35% will be driving a minivan, and 20% will be driving a SUV.

9. Find the general solution of the system of equations

$y_1' = y_1 + y_2 + 4y_3$

$y_2' = 2y_1 - 4y_3$

$y_3' = -y_1 + y_2 + 5y_3$

Solution:

The given system of equations can be written as

$\mathbf{y}' = A\mathbf{y}$

where

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 0 & -4 \\ -1 & 1 & 5 \end{pmatrix},\ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

The characteristic equation of A is

$\det(A) - (M_{11} + M_{22} + M_{33})\lambda + (a_{11} + a_{22} + a_{33})\lambda^2 - \lambda^3 = 0$

$\Rightarrow 6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$

$\Rightarrow (1 - \lambda)(6 - 5\lambda + \lambda^2) = 0$

$\Rightarrow (1 - \lambda)(2 - \lambda)(3 - \lambda) = 0$

$\Rightarrow \lambda = 1,\ 2,\ 3$

$$\lambda = 1,\ A - \lambda I = \begin{pmatrix} 0 & 1 & 4 \\ 2 & -1 & -4 \\ -1 & 1 & 4 \end{pmatrix},\ \mathbf{p} = (0, 8, -2) = 2(0, 4, -1)$$
$$\lambda = 2,\ A - \lambda I = \begin{pmatrix} -1 & 1 & 4 \\ 2 & -2 & -4 \\ -1 & 1 & 3 \end{pmatrix},\ \mathbf{p} = (4, 4, 0) = 4(1, 1, 0)$$
$$\lambda = 3,\ A - \lambda I = \begin{pmatrix} -2 & 1 & 4 \\ 2 & -3 & -4 \\ -1 & 1 & 2 \end{pmatrix},\ \mathbf{p} = (8, 0, 4) = 4(2, 0, 1)$$

Thus

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Substituting $\mathbf{y} = P\mathbf{u}$ in the given equation, we obtain

$$P\mathbf{u}' = AP\mathbf{u} \Rightarrow \mathbf{u}' = P^{-1}AP\mathbf{u} \Rightarrow \mathbf{u}' = D\mathbf{u}$$

The last equation is equivalent to the system

$$u_1' = u_1,\ u_2' = 2u_2,\ u_3' = 3u_3$$

which has the simple solution

$$u_1 = c_1e^t,\ u_2 = c_2e^{2t},\ u_3 = c_3e^{3t}$$

Hence the solution for $\mathbf{y}$ is

$$\mathbf{y} = P\mathbf{u}$$
$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1e^t \\ c_2e^{2t} \\ c_3e^{3t} \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_2e^{2t} + 2c_3e^{3t} \\ 4c_1e^t + c_2e^{2t} \\ -c_1e^t + c_3e^{3t} \end{pmatrix}$$
$$\Rightarrow y_1 = c_2e^{2t} + 2c_3e^{3t}$$
$$y_2 = 4c_1e^t + c_2e^{2t}$$
$$y_3 = -c_1e^t + c_3e^{3t}$$