**Trinity Western University**
**Department of Mathematical Sciences**
**MATH250 (Linear Algebra)**
**Sample Final Examination Solution**

1. a) Let $A$ and $B$ be $n \times n$ matrices, show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Use this identity to prove that $AB - BA = I$ is impossible.
**Solution:**

We first prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, and $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & a_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$

$\Rightarrow [AB]_{ii} = A_i \cdot B_i = a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni}$

Therefore

$$\begin{aligned}\operatorname{tr}(AB) &= [AB]_{11} + [AB]_{22} + \cdots + [AB]_{nn} \\ &= (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}) + \\ &\quad \cdots + (a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}) \\ &= (b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1}) + (b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2n}a_{n2}) + \\ &\quad \cdots + (b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nn}a_{nn}) \text{ on rearranging terms} \\ &= \operatorname{tr}(BA)\end{aligned}$$

If $AB - BA = I$, then $\operatorname{tr}(AB - BA) = \operatorname{tr}(I) \Rightarrow \operatorname{tr}(AB) - \operatorname{tr}(BA) = n \Rightarrow 0 = n$, which is clearly impossible.

b) Discuss the solution of the system of equations

$$\begin{aligned} x + ay - z &= 1 \\ -x + (a-2)y + z &= -1 \\ 2x + 2y + (a-2)z &= 1 \end{aligned}$$

for various values of a (Indicate in each case how many solutions will you get, also giving the solutions if they exist).

**Solution:**

The augmented matrix is

$$\begin{pmatrix} 1 & a & -1 & 1 \\ -1 & a-2 & 1 & -1 \\ 2 & 2 & a-2 & 1 \end{pmatrix} \qquad R_{12}(1),\ R_{13}(-2)$$

$$\sim \begin{pmatrix} 1 & a & -1 & 1 \\ 0 & 2a-2 & 0 & 0 \\ 0 & 2-2a & a & -1 \end{pmatrix}$$

The only non-zero entry in the second row could be $2a-2$, though it could also become zero. Therefore we must distinguish between two cases (A) $2a - 2 \neq 0$, and (B) $2a - 2 = 0$.

Case (A) $2a-2 \neq 0 \Rightarrow a \neq 1$
Performing $R_2(\frac{1}{2a-2})$, we get

$$\sim \begin{pmatrix} 1 & a & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2-2a & a & -1 \end{pmatrix} \qquad R_{23}(2a-2)$$

$$\sim \begin{pmatrix} 1 & a & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & -1 \end{pmatrix}$$

Again looking for the first non-zero entry in the third row we find $a$, but it could also be zero. Thus we must again distinguish between cases (A1) $a \neq 0$, and (A2) $a = 0$

Case (A1) $a \neq 0$
Performing $R_3(\frac{1}{a})$, we get

$$\sim \begin{pmatrix} 1 & a & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{a} \end{pmatrix}$$

The equivalnet system of equations is

$$x + ay - z = 1,\ y = 0,\ z = -\frac{1}{a}$$

Solving backward we get

$$x = 1 - \frac{1}{a},\ y = 0,\ z = -\frac{1}{a}$$

Hence if $a \neq 1$ and $a \neq 0$, we get a unique solution.
Case (A2) $a = 0$
The augmented matrix becomes

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

the third row of which leads to $0x + 0y + 0z = -1$. There are no values of $x,\ y,$ and $z$ which can satisfy it. Hence when $a = 0$, the system is inconsistent.

Case (B) $2a - 2 = 0 \Rightarrow a = 1$
In this case the augmented matrix becomes

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad R_{23}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_{21}(1)$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The equivalent system of equations is

$$x + y = 0,\ z = -1,\ 0 = 0$$

Solving for leading variables, we get

$$x = -y,\ z = -1$$

Since we cannot solve for $y$, we assign it an arbitrary value $t$, then the solution becomes

$x = -t,\ y = t,\ z = -1,\ t \in \mathbb{R}$

So when $a = 1$, we get infinitely many solutions.

We can summarize the situation as under

When $a \neq 0,\ a \neq 1$, we get a unique solution $x = 1 - \frac{1}{a},\ y = 0,\ z = -\frac{1}{a}$

When $a = 0$, there is no solution

When $a = 1$, there are infinitely many solutions $x = -t,\ y = t,\ z = -1,\ t \in \mathbb{R}$

2. a) Find the equation of the plane containing the lines $(x, y, z) = (1, -1, 2) + t(1, 0, 1)$, and $(x, y, z) = (0, 0, 2) + t(1, -1, 0)$.

**Solution:**

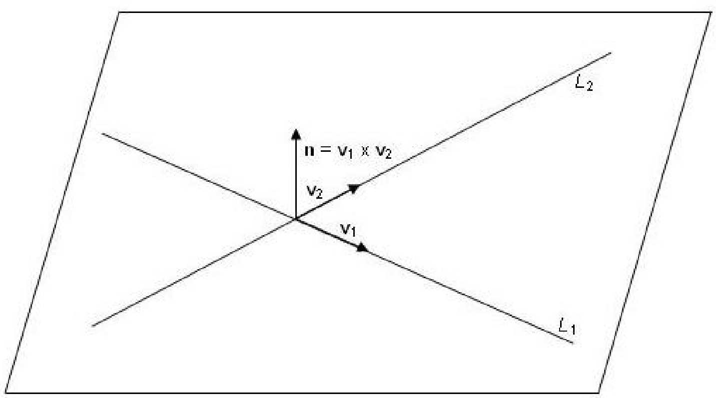
The two given lines are

$L_1 : (x, y, z) = (1, -1, 2) + t(1, 0, 1)$

and

$L_2 : (x, y, z) = (0, 0, 2) + t(1, -1, 0)$

with direction vectors $\mathbf{v}_1 = (1, 0, 1)$ and $\mathbf{v}_2 = (1, -1, 0)$ respectively.




The normal $\mathbf{n}$ to the plane is given by

$$\begin{aligned}\mathbf{n} &= \mathbf{v}_1 \times \mathbf{v}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} \\ &= \mathbf{i} + \mathbf{j} - \mathbf{k} \\ &= (1,\ 1,\ -1)\end{aligned}$$

Since the plane contains line $L_1$, it also contains all points on line $L_1$, including the point (1, -1, 2).

Hence the equation of the plane is

$1(x-1)+1(y-(-1))+(-1)(z-2)=0$
$\Rightarrow x-1+y+1-z+2=0$
$\Rightarrow x+y-z+2=0$

b) Find if $W=\{A|A \text{ in } M_{22},\ AX=0\}$, $X$ a fixed 2 x 2 matrix, is a subspace of $M_{22}$. If not indicate why it is not.

**Solution:**

1. Clearly the zero matrix 0 belongs to $W$, since it satisfies $0X=0$.
2. Let $A$ and $B$ belong to $W$, then
$AX=0$ and $BX=0$
Adding we get $(A+B)X=0$, showing that $A+B$ belongs to $W$.
3. If $A$ belongs to $W$ then $AX=0$
Multiplying by $k$, we get $kAX=k0 \Rightarrow (kA)X=0$, showing that $kA \in W$
Since all the three properties of the subspace are satisfied, $W$ is a subspace of $M_{22}$.

3. a) Prove that a vector in a vector space has exactly one negative.

**Solution:**

If possible let the vector $\mathbf{u}$ have two negatives $\mathbf{v}$ and $\mathbf{w}$.
Then we must have
$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}=\mathbf{0}$ and $\mathbf{u}+\mathbf{w}=\mathbf{w}+\mathbf{u}=\mathbf{0}$
Adding $\mathbf{w}$ to the first equation we get
$\mathbf{w}+(\mathbf{u}+\mathbf{v})=\mathbf{w}+\mathbf{0}$
$\Rightarrow (\mathbf{w}+\mathbf{u})+\mathbf{v}=\mathbf{w}$
$\Rightarrow \mathbf{0}+\mathbf{v}=\mathbf{w} \Rightarrow \mathbf{v}=\mathbf{w}$
Thus the two negatives are equal, and the vector $\mathbf{u}$ has only one negative.

b) Find the basis for the null space of

$$A=\begin{pmatrix} 3 & 5 & 5 & 2 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & -2 & -2 \\ -2 & 0 & -4 & -4 & -2 \end{pmatrix}$$

Then compute rank $(A)$ and verify the theorem stating that
rank $(A)$ + nullity $(A)=n$,
$n$, being the number of columns in $A$.

**Solution:**

The augmented matrix is

$$\begin{pmatrix} 3 & 5 & 5 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & -2 & -2 & 0 \\ -2 & 0 & -4 & -4 & 2 & 0 \end{pmatrix} \quad R_{12}$$

$$\tilde{} \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 0 \\ 3 & 5 & 5 & 2 & 0 & 0 \\ 1 & 1 & 1 & -2 & -2 & 0 \\ -2 & 0 & -4 & -4 & 2 & 0 \end{pmatrix} \quad R_{12}(-3),\ R_{13}(-1),\ R_{14}(2)$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 5 & -1 & -4 & -3 & 0 \\ 0 & 1 & -1 & -4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_{23}$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & -4 & -3 & 0 \\ 0 & 5 & -1 & -4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_{23}(-5)$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & -4 & -3 & 0 \\ 0 & 0 & 4 & 16 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3(\tfrac{1}{4})$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & -4 & -3 & 0 \\ 0 & 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_{31}(-2),\ R_{32}(1)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -6 & -5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The equivalent system of equations is

$x_1 - 6x_4 - 5x_5 = 0$
$x_2 = 0$
$x_3 + 4x_4 + 3x_5 = 0$
$0 = 0$

Solving for leading variables, we obtain

$x_1 = 6x_4 + 5x_5$
$x_2 = 0$
$x_3 = -4x_4 - 3x_5$

Setting the free variables $x_4 = s,\ x_5 = t$, we get the following solution

$x_1 = 6s + 5t$
$x_2 = 0$
$x_3 = -4s - 3t$
$x_4 = s$
$x_5 = t,$

which can be put in the vector form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6s+5t \\ 0 \\ -4s-3t \\ s \\ t \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s \begin{pmatrix} 6 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The basis for the nullspace of $A$ is therefore $\{\mathbf{v}_1,\ \mathbf{v}_2\}$, where
$\mathbf{v}_1$ = (6, 0, -4, 1, 0), and $\mathbf{v}_2$ = (5, 0, -3, 0, 1).

Since $A$ contains three leading ones, rank$(A) = 3$. Also nullity$(A) = 2$
rank$(A)+$ nullity$(A) = 3 + 2 = 5$, the number of columns in $A$.

4. Find a subset of the vectors that forms a basis for the space spanned by the vectors; then express each vector that is not in the basis as a linear combination of the basis vectors.
$\mathbf{v}_1$ = (1, 0, -1, 3), $\mathbf{v}_2$ = (2, 1, 0, -2), $\mathbf{v}_3$ = (-1, 1, 2, 1), $\mathbf{v}_4$ = (-3, 2, 4, 11)

**Solution:**

We form a matrix $A$ consisting of the given vectors as its rows.

$$A = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 2 & 1 & 0 & -2 \\ -1 & 1 & 2 & 1 \\ -3 & 2 & 4 & 11 \end{pmatrix}$$

$$\Rightarrow A^T = \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ -1 & 0 & 2 & 4 \\ 3 & -2 & 1 & 11 \end{pmatrix} \qquad R_{13}(1),\ R_{14}(-3)$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & -8 & 4 & 20 \end{pmatrix} \qquad R_{21}(-2),\ R_{23}(-2),\ R_{24}(8)$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 12 & 36 \end{pmatrix} \qquad R_3(-1)$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 12 & 36 \end{pmatrix} \qquad R_{31}(3),\ R_{32}(-1),\ R_{34}(-12)$$

$$\sim \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3 \quad \mathbf{w}_4 \end{array}$$

The vectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ form the basis of the column space of $A^T$. Hence the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis of the rowspace of $A$. Further it can be seen that

$$\mathbf{w}_4 = 2\mathbf{w}_1 - \mathbf{w}_2 + 3\mathbf{w}_3$$

Therefore

$$\mathbf{v}_4 = 2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_2$$

5. Let $U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ and $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$

Define $\langle U,\ V\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$

Use the Gram-Schmidt algorithm to transform the basis $B$ given below into an orthonormal basis

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \right\}$$

**Solution:**

Let $\mathbf{u}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{u}_4 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Note that we have changed the order of the vectors, so that we can have as many orthogonal vectors to begin with as possible.

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2,\ \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - \frac{(0)(1) + (1)(0) + (0)(1) + (1)(0)}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3,\ \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3,\ \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$= \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} - \frac{(0)(1) + (2)(0) + (0)(1) + (1)(0)}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} -$$

$$- \frac{(0)(0) + (2)(1) + (0)(0) + (1)(1)}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

So $\mathbf{v}_3$ can be taken as $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$

$$\begin{aligned}
\mathbf{v}_4 &= \mathbf{u}_4 - \frac{\langle \mathbf{u}_4,\ \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4,\ \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4,\ \mathbf{v}_3\rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \frac{(1)(1)+(1)(0)+(0)(1)+(0)(0)}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \\
&\quad - \frac{(1)(0)+(1)(1)+(0)(0)+(0)(1)}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - \\
&\quad - \frac{(1)(0)+(1)(1)+(0)(0)+(0)(-1)}{2}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}
\end{aligned}$$

So $\mathbf{v}_4$ can be taken as $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

Hence the new orthogonal basis is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \right\}$$

It can be normalized by dividing each vector by its norm. We get the orthonormal basis as

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \right\}$$

6. For the matrix $A$, obtain $A^n$, where $n$ is a positive integer.

$$A = \begin{pmatrix} 8 & 7 & 7 \\ -5 & -6 & -9 \\ 5 & 7 & 10 \end{pmatrix}$$

**Solution:**

The characteristic equation of the matrix is

$$\det(A) - \lambda(M_{11}+M_{22}+M_{33}) + \lambda^2(a_{11}+a_{22}+a_{33}) - \lambda^3 = 0$$

$\Rightarrow 24 - 35\lambda + 12\lambda^2 - \lambda^3 = 0$

$\Rightarrow (1-\lambda)(24 - 11\lambda + \lambda^2) = 0$

$\Rightarrow (1-\lambda)(3-\lambda)(8-\lambda) = 0$

$\Rightarrow \lambda = 1,\ 3,\ 8$

$$\lambda = 1,\ A - \lambda I = \begin{pmatrix} 7 & 7 & 7 \\ -5 & -7 & -9 \\ 5 & 7 & 9 \end{pmatrix},\ \mathbf{p} = (-14, 28, -14) = -14(1, -2, 1)$$

$$\lambda = 3,\ A - \lambda I = \begin{pmatrix} 5 & 7 & 7 \\ -5 & -9 & -9 \\ 5 & 7 & 7 \end{pmatrix},\ \mathbf{p} = (0, 10, -10) = 10(0, 1, -1)$$

$$\lambda = 8,\ A - \lambda I = \begin{pmatrix} 0 & 7 & 7 \\ -5 & -14 & -9 \\ 5 & 7 & 2 \end{pmatrix},\ \mathbf{p} = (35, -35, 35) = 35(1, -1, 1)$$

Thus

$$P = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$[P \mid I] = \left(\begin{array}{rrr|r|r|r} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \qquad R_{12}(2),\ R_{13}(-1)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right) \qquad R_{23}(1)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \qquad R_{31}(-1),\ R_{32}(-1)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) = [I \mid P^{-1}]$$

$$\Rightarrow P^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Therefore

$$D = P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Hence

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 8^n \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 3^n & 0 & -3^n \\ 8^n & 8^n & 8^n \end{pmatrix}$$

$$= \begin{pmatrix} 8^n & -1+8^n & -1+8^n \\ 3^n - 8^n & 2-8^n & 2-3^n-8^n \\ -3^n+8^n & -1+8^n & -1+3^n+8^n \end{pmatrix}$$

7. For the transition matrix $P = \begin{pmatrix} a & 0 \\ 1-a & 1 \end{pmatrix}$, $0 < a < 1$, show that $P$ is not regular by finding $P^n$, where $n$ is a positive integer. Further show that as $n$ increases, $P^n\mathbf{x}^{(0)}$ approaches $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for any value of $a$ and any initial vector $\mathbf{x}^{(0)}$.

**Solution:**

The characteristic equation for $P$ is

$\det(P) - (M_{11} + M_{22})\lambda + \lambda^2 = 0$

$\Rightarrow a - (a+1)\lambda + \lambda^2 = 0$

$\Rightarrow (1-\lambda)(a-\lambda) = 0$

$\Rightarrow \lambda = 1,\ a$

$\lambda = 1,\ A - \lambda I = \begin{pmatrix} a-1 & 0 \\ 1-a & 0 \end{pmatrix},\ \mathbf{q} = (0,1)$

$\lambda = a,\ A - \lambda I = \begin{pmatrix} 0 & 0 \\ 1-a & 1-a \end{pmatrix},\ \mathbf{q} = (1,-1)$

Thus

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},\ Q^{-1} = -\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence

$$P^n = QD^nQ^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$== \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ a^n & 0 \end{pmatrix} = \begin{pmatrix} a^n & 0 \\ 1-a^n & 1 \end{pmatrix}$$

Since for any positive integer $n$, the entry in the first row and second column is always zero, $P$ is not regular.

Let $\mathbf{x}^{(0)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1 + x_2 = 1$

$$P^n\mathbf{x}^{(0)} = \begin{pmatrix} a^n & 0 \\ 1-a^n & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a^n x_1 \\ (1-a^n)x_1 + x_2 \end{pmatrix}$$

Since $0 < a < 1,\ \lim_{n\to\infty} a^n = 0$

Hence

$$\lim_{n\to\infty} P^n\mathbf{x}^{(0)} = \begin{pmatrix} 0 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\text{ since } x_1 + x_2 = 1.$$

8. A wolf pack always hunts in one of the three regions $R_1$, $R_2$, and $R_3$. Its hunting habits are as follows:

1. If it hunts in one region one day, it is as likely as not to hunt there again the next day.
2. If it hunts in $R_1$, it never hunts in $R_2$ the next day.
3. If it hunts in $R_2$ or $R_3$, it is equally likely to hunt in each of the other regions the next day.

a) If the pack hunts in $R_1$ on Monday, find the probability that it hunts there on Thursday.

b) What are the long range probabilities that the pack hunts in each of the three regions?

**Solution:**

The transition matrix is

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

(a) The initial state vector is $\mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

The state vector three days later will be $\mathbf{x}^{(3)} = P^3\mathbf{x}^{(0)}$.

$$\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$\mathbf{x}^{(2)} = P\mathbf{x}^{(1)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{1}{2} \end{pmatrix}$$

$$\mathbf{x}^{(3)} = P\mathbf{x}^{(2)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{11}{32} \\ \frac{3}{16} \\ \frac{15}{32} \end{pmatrix}$$

The probability of the wolf pack hunting in $R_1$ on Thursday is $\frac{11}{32}$.

(b) Let the steady state vector be $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

The steady state is given by

$$P\mathbf{q} = \mathbf{q} \Rightarrow (P - I)\mathbf{q} = \mathbf{0}$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The nullspace (or the solution space) of $A\mathbf{x} = \mathbf{0}$ is orthogonal to the rowspace of $A$. Therefore the vector $\mathbf{q}$ must be orthogonal to the three row vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ of the matrix $P - I$. Therefore it must be along $\mathbf{r}_1 \times \mathbf{r}_2$.

$$\mathbf{r}_1 \times \mathbf{r}_2 = (\tfrac{3}{16},\ \tfrac{1}{8},\ \tfrac{1}{4}) = \tfrac{1}{16}(3, 2, 4)$$

Hence we can choose $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = t \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$

$$\Rightarrow q_1 = 3t,\ q_2 = 2t,\ q_3 = 4t$$

But $q_1 + q_2 + q_3 = 1 \Rightarrow 3t + 2t + 4t = 1 \Rightarrow 9t = 1 \Rightarrow t = \frac{1}{9}$

Thus $q_1 = \frac{1}{3},\ q_2 = \frac{2}{9},\ q_3 = \frac{4}{9}$

So in the long run out of 9 days the pack will hunt in $R_1$, 3 days, in $R_2$, 2 days, and in $R_3$, 4 days.

9. A lawn mower company makes three models: standard, deluxe, and super. The construction of each mower involves three stages: motor construction, frame construction, and final assembly. The following table gives the number of hours of labor required per mower for each stage and the total number of hours of labor available per week for each stage. It also gives the profit per week. Find the weekly production schedule that maximizes the profit.

| | Standard | Delux | Super | Hours available |
|---|---|---|---|---|
| motor | 1 | 1 | 2 | 2500 |
| frame | 1 | 2 | 2 | 2000 |
| assembly | 1 | 1 | 1 | 1800 |
| PROFIT | \$30 | \$40 | \$55 | |

**Solution:**

Let the company make $x$ units of Standard, $y$ units of Delux and $z$ units of Super models. The profit $f$ is given by

$f = 30x + 40y + 55z.$

The constraints are

$x + y + 2z \leq 2500$

$x + 2y + 2z \leq 2000$

$x + y + z \leq 1800$

$x,\ y,\ z\ \geq 0$

Introducing the slack variables $u$, $v$, and $w$, the constraints change into the following equations

$x + y + 2z + u = 2500$

$x + 2y + 2z + v = 2000$

$x + y + z + w = 1800$

and the equation for the objective function is written as

$-30x - 40y - 55z + f = 0$

The initial tableau is

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 2500 \\ 1 & 2 & \boxed{2} & 0 & 1 & 0 & 0 & 2000 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1800 \\ -30 & -40 & \boxed{-55} & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \left| \begin{array}{c} 1250 \\ 1000 \\ 1800 \end{array} \right. \qquad R_2(\tfrac{1}{2})$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 2500 \\ \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & 1000 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1800 \\ -30 & -40 & -55 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad R_{21}(-2),\ R_{23}(-1),\ R_{24}(55)$$

$$\sim \begin{pmatrix} 0 & -1 & 0 & 1 & -1 & 0 & 0 & 500 \\ \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & 1000 \\ \boxed{\frac{1}{2}} & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & 800 \\ \boxed{-\frac{5}{2}} & 15 & 0 & 0 & \frac{55}{2} & 0 & 1 & 55000 \end{pmatrix} \left| \begin{array}{c} - \\ 2000 \\ 1600 \end{array} \right. \qquad R_3(2)$$

$$\sim \begin{pmatrix} 0 & -1 & 0 & 1 & -1 & 0 & 0 & 500 \\ \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & 1000 \\ 1 & 0 & 0 & 0 & -1 & 2 & 0 & 1600 \\ \boxed{-\frac{5}{2}} & 15 & 0 & 0 & \frac{55}{2} & 0 & 1 & 55000 \end{pmatrix} \qquad R_{32}(-\tfrac{1}{2}),\ R_{34}(\tfrac{5}{2})$$

$$\sim \begin{pmatrix} 0 & -1 & 0 & 1 & -1 & 0 & 0 & 500 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 & 200 \\ 1 & 0 & 0 & 0 & -1 & 2 & 0 & 1600 \\ 0 & 15 & 0 & 0 & 25 & 5 & 1 & 59000 \end{pmatrix}$$

Since none of the entries in the last row is negative, it is the final tableau.

We have

$-y + u - v = 500$

$y + z + v - w = 200$

$x - v + 2w = 1600$

$15y + 25v + 5w + f = 59000$

From the last equation

$f = 59000 - 15y - 25v - 5w$

For maximum $f$, $y = 0$, $v = 0$ and $w = 0$, which when substituted in the first three equations above gives

$u = 500,\ z = 200,\ x = 1600$

Hence the company must make 1600 units of Standard model and 200 units of Super model to maximize the profit. The maximum profit is \$59,000.

10. Find the general solution of the system of equations

$$y_1' = 6y_1 + 4y_2 - 5y_3$$
$$y_2' = 4y_1 + 6y_2 - 5y_3$$
$$y_3' = -5y_1 - 5y_2 + 15y_3$$

**Solution:**

The given system of equations can be written as

$$\mathbf{y}' = A\mathbf{y}$$

where

$$A = \begin{pmatrix} 6 & 4 & -5 \\ 4 & 6 & -5 \\ -5 & -5 & 15 \end{pmatrix},\ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

The characteristic equation of $A$ is

$$\det(A) - (M_{11} + M_{22} + M_{33})\lambda + (a_{11} + a_{22} + a_{33})\lambda^2 - \lambda^3 = 0$$
$$\Rightarrow 200 - 150\lambda + 27\lambda^2 - \lambda^3 = 0$$
$$\Rightarrow (2-\lambda)(100 - 25\lambda + \lambda^2) = 0$$
$$\Rightarrow (2-\lambda)(5-\lambda)(20-\lambda) = 0$$
$$\Rightarrow \lambda = 2,\ 5,\ 20$$

$$\lambda = 2,\ A - \lambda I = \begin{pmatrix} 4 & 4 & -5 \\ 4 & 4 & -5 \\ -5 & -5 & 13 \end{pmatrix},\ \mathbf{p} = (27, -27, 0) = 27(1, -1, 0)$$

$$\lambda = 5,\ A - \lambda I = \begin{pmatrix} 1 & 4 & -5 \\ 4 & 1 & -5 \\ -5 & -5 & 10 \end{pmatrix},\ \mathbf{p} = (-15, -15, -15) = -15(1, 1, 1)$$

$$\lambda = 20,\ A - \lambda I = \begin{pmatrix} -14 & 4 & -5 \\ 4 & -14 & -5 \\ -5 & -5 & -5 \end{pmatrix},\ \mathbf{p} = (-90, -90, 180) = -90(1, 1, -2)$$

Thus

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 20 \end{pmatrix}$$

Substituting $\mathbf{y} = P\mathbf{u}$ in the given equation, we obtain

$$P\mathbf{u}' = AP\mathbf{u} \Rightarrow \mathbf{u}' = P^{-1}AP\mathbf{u} \Rightarrow \mathbf{u}' = D\mathbf{u}$$

The last equation is equivalent to the system

$$u_1' = 2u_1,\ u_2' = 5u_2,\ u_3' = 20u_3$$

which has the simple solution

$$u_1 = c_1 e^{2t},\ u_2 = c_2 e^{5t},\ u_3 = c_3 e^{20t}$$

Hence the solution for $\mathbf{y}$ is

$$\mathbf{y} = P\mathbf{u}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{5t} \\ c_3 e^{20t} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 e^{5t} + c_3 e^{20t} \\ -c_1 e^{2t} + c_2 e^{5t} + c_3 e^{20t} \\ c_2 e^{5t} - 2c_3 e^{20t} \end{pmatrix}$$

$$\Rightarrow y_1 = c_1 e^{2t} + c_2 e^{5t} + c_3 e^{20t}$$

$$y_2 = -c_1 e^{2t} + c_2 e^{5t} + c_3 e^{20t}$$

$$y_3 = c_2 e^{5t} - 2c_3 e^{20t}$$