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ABSTRACT Although studies of scientific practice are now common, relatively little attention has been given to the technical detail of the social objects of the sciences. That detail, and the ordinariness of that detail, establish the legitimacy of practitioners' work and make that work recognizable as the work of a discovering science. In particular, the close examination of mathematical proofs exhibits the embeddedness of mathematics within a surrounding culture of proving, and leads to an appreciation of what it means to be a member of such a culture and to be engaged in the work of doing mathematics.

Cultures of Proving

Eric Livingston

Whether mathematics concerns a domain of ideal, immutable objects, whether it is based on empirical observation, whether mathematics is reducible to formal logic, or whether mathematical truth depends on conventions of definition and reasoning are, for the anthropologist, propositions that require neither assent nor denial. They are accounts of nature and stories of origins, possibly true, possibly false, but always belonging to the communities from within which they arise. For the anthropologist, 'mathematics' refers to the products of a culture, and the anthropologist seeks to inquire into the practices of communities that produce 'mathematics'.[1]

Support for an anthropology of mathematics comes from many sources: the history of mathematics,[2] philosophy,[3] the sociology of science,[4] and recent studies in ethnomathematics.[5] Yet central to the practice of mathematics is the activity of proving: proving is what mathematicians do, and it is the ability of mathematicians to engage in proving that makes mathematicians recognizable to each other as mathematicians. Although much has been written on proofs and proving – in the philosophy and history of mathematics, in mathematics education, and in technical studies of mathematical foundations – there appear to be no ethnographically descriptive studies of proving as a cultural activity.[6]

This paper gathers together observations on five aspects of proving: (1) the manner in which mathematical reasoning transcends its written description; (2) the form of mathematical argumentation; (3) the materiality of mathematical culture; (4) the rôle of definitions in proving; and (5) the phenomenology of mathematical discovery. Taken together, a picture of mathematical culture emerges that is quite different from more traditional accounts.

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Mathematical Reasoning

Figure 1 illustrates a central theme of Gestalt psychology: immediate perception involves the perception of organized 'wholes' or 'gestalts', not the perception of sense data from which a figure – the chalice or the two heads – is inferred. When Figure 1 is seen as a chalice, the details of Figure 1 are related to each other as mutually elaborating features – the base, the knops, the flattened bowl. Similarly, when Figure 1 is seen as two silhouetted heads, its details are seen in terms of, and articulate, a different totality. A second theme, however, is implicit in the first. In that Figure 1 can be seen in two incompatible ways – as a depiction of a chalice or as two silhouetted heads – the physical details of the figure do not determine its perception. The seen figure is not literally 'in' the physical object; the perception of either chalice or silhouetted heads goes beyond – or transcends – what retrospectively appears to be the material grounds of that perception.[7]

Mathematical reasoning, both in its concrete detail and in its transcendence of that concreteness as an organization of those details, is analogous to these features of gestalt perception. The details of a written mathematical argument are seen in terms of the reasoning that argument describes, yet that reasoning goes beyond the literal details of the written argument. Figure 2, an example of an early Greek method of proof that used 'pebbles' to represent numbers, provides an illustration.[8]

Figure 2 is intended as a proof that the sum of any number of even numbers is an even number. One interpretation of the figure is that it demonstrates this proposition for only one particular sum of even numbers. Yet Figure 2 is inspected by provers to find the organization of reasoning and practice that it describes – in this case, a methodic procedure – which, like a perceptual gestalt, is not literally present in its individual details. The numbers in parentheses indicate that the demonstration is organized as a sequence of steps. Step 1 shows[9] that *every* even number can be counted out as an equal number of black and white dots.

FIGURE 1

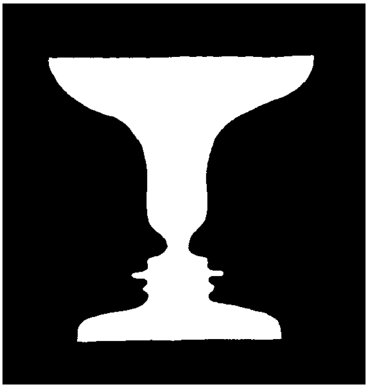
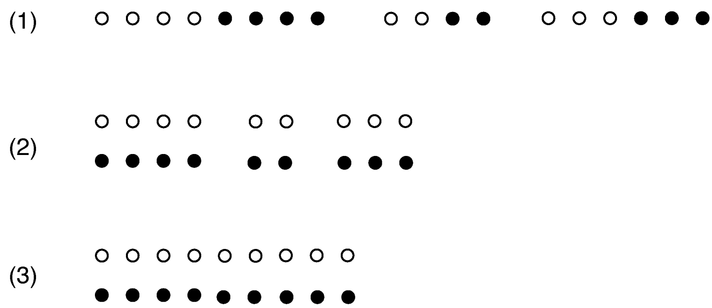


FIGURE 2

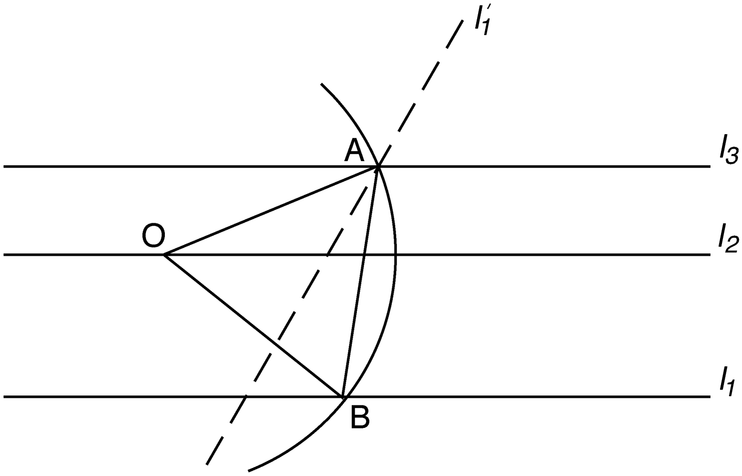


The transition to step 2 shows that, for each even number individually, the black and white dots can be placed in a one-to-one correspondence with each other. In step 3, all the dots have been brought together; hence, the dots now represent the sum of the original numbers. In that the one-to-one correspondence in the separate groups of dots can be maintained when the groups are joined, the resulting sum is also an even number. Given any number of even numbers, the same procedure can be followed and, therefore, the general proposition – that the sum of any number of even numbers is an even number – holds.

In analogy with perceptual gestalts, mathematical proofs are discovered, and rediscovered on subsequent occasions, as organized totalities of reasoning and practice: the material detail of an argument articulates a coherent ‘whole’ of reasoning that is not present in any of the argument’s individual details. First, once the pebble proof is understood – that is, once the visual argument is realized as a description of the proof that the visual argument describes – Figure 2 is seen in terms of the reasoning it has come to reveal; the details of Figure 2 fit together as a description of a proof that the sum of even numbers is even. Second, when someone first looks at Figure 2, it is unclear how Figure 2 is a ‘proof’ of anything. The proof of Figure 2 does not literally reside in the visual description of the proof; the reasoning involved in the demonstration is neither ‘in’ nor not ‘in’ the proof figure. Provers must ‘find’ the proof in the figure. Provers inspect materially definite writings (in this case, arrangements of dots), see through the notational particulars (the dots) to what they represent, and organize, rearrange and rework such displays to find gestalts of reasoning and practice adequate to a stated theorem. The success of that work is the revelation of how such material details compose a description of the discovered gestalt.

Figure 3 – the construction of an equilateral triangle subject to specific conditions – provides another example.[10] The description of the construction leaves out many details – for example, how a point can be rotated around another point by 60° (Figure 4),[11] which, in turn, requires a number of other constructions and proofs, among them the proof that 3 times the constructed angle is, in Euclidean terms, a straight angle

FIGURE 3



Theorem. Given three parallel lines, an equilateral triangle can be constructed with each of its vertices on a different line.

Proof. Let l_1, l_2 and l_3 be three parallel lines, and let O be an arbitrary point on one of the lines, say l_2. Let l_1' be the rotation of l_1 through an angle of 60 degrees around O, and let A be the point of intersection of l_1' with l_3. A circle with centre O through A will intersect l_1 at a point B such that angle AOB is 60 degrees. Then triangle OBA is an equilateral triangle: by construction, line segments OA and OB have equal length; therefore triangle OBA is an isosceles triangle, and angle OBA equals angle OAB. Since the sum of the angles of a triangle equals 180 degrees, these two angles must each be 60 degrees. Since all the angles of triangle OBA are equal, it follows that triangle OBA is an equilateral triangle.

(Figure 5).[12] Since two points determine a line and a rotation preserves lines, the rotation of two distinct points of l_1 determines the rotation of l_1 through 60°.

Whether all such details can be supplied is not at issue: the description in Figure 3 gives those details needed to see how the construction – that gestalt of reasoning and practice – can be made; what is left out is left out to make that gestalt clear. Although intended to amplify that description, Figures 4 and 5, in fact, have this same character, each providing sufficient detail to give definiteness and clarity to their respective arguments. Similarly, the claim in the proof in Figure 3 that 'a circle with centre O through A will intersect l_1 at a point B such that angle AOB is 60 degrees', by specifying the point of intersection, invites provers to see that A is the image of B under a rotation of l_1 through 60°. The written statement points to the reasoning that must be found in order to find the appropriateness of the claim. Each of these elaborating constructions and justifications illustrate how deeply even elementary proofs are embedded in mathematical practice: the description of the construction in Figure 3 is, in fact, designed for an audience – the practitioners of a mathematical culture – for whom that description is adequate.

FIGURE 4

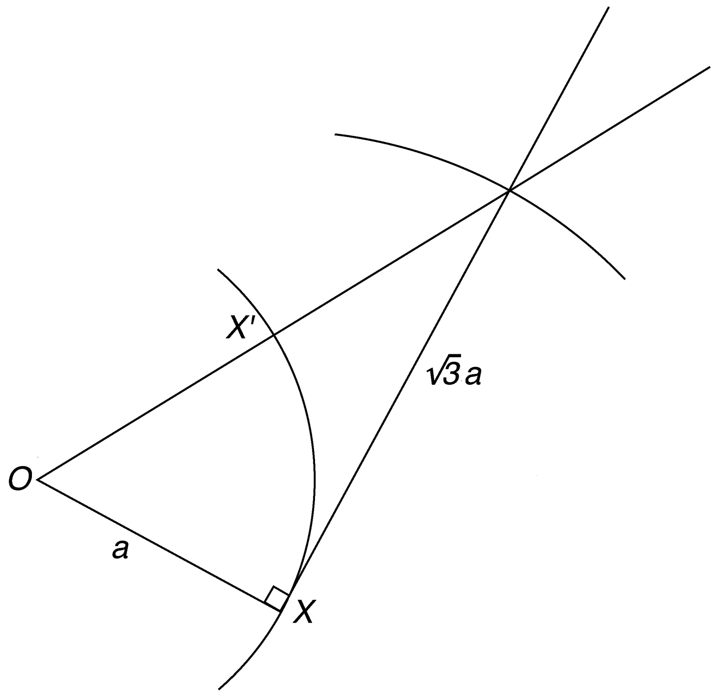
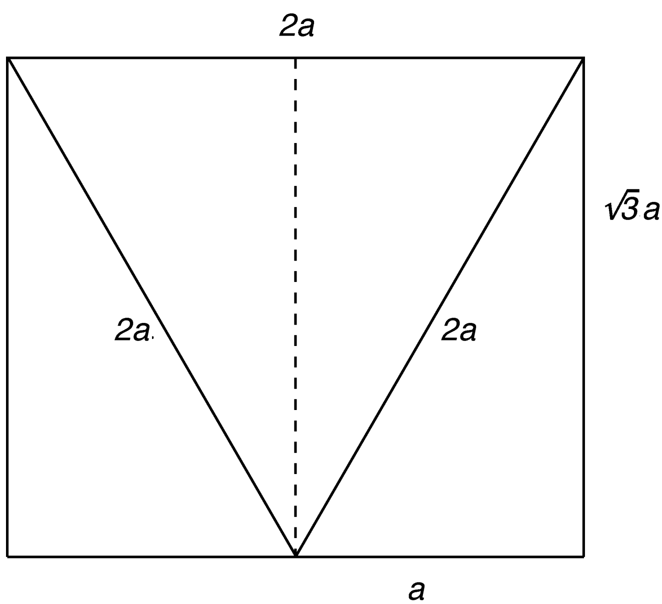


FIGURE 5



The subsequent proof in Figure 3 that the constructed triangle is equilateral illustrates another typical feature of such gestalts, almost invisible because of its naturalness to provers. Relevant features of the constructed triangle are described in an order that makes it appear as if the perception of how those details fit together is a matter of logical deduction: line segments OA and OB have equal length, therefore OBA is an isosceles triangle; since angle AOB is 60°, and the sum of the angles of a triangle equal 180°, angles OBA and OAB must each be 60°. Prior to, or co-temporaneous with, the development of this line of reasoning, a prover must see the relevance of such observations and how they do, in fact, 'fit together'. Again, the issue is not whether a proof must – in an absolute sense – be capable of description as a sequence of steps of a formal deduction, but that provers often compose proofs so as to give the sense that a particular proof could be so described. The proof in Figure 3 describes an organization of reasoning about Euclidean constructions and their properties, and it describes an organization of practice – how to put together such reasoning – so as to show the theorem that the proof claims to reveal.

Although more evident in advanced mathematics, the gestalt character of proofs is present in the simplest proofs as well. Consider, for example, the proof that the identity element of a group is unique. The identity of a group is an element e such that, when multiplied by any other element a, it gives a: that is, $a = e \cdot a = a \cdot e$. The proof that the identity element is unique consists of a single line:

Let e and e' be two identity elements. Then $e = e' \cdot e = e'$.

Here, a prover must see that the middle term of the equation, $e' \cdot e$, can be read in two ways (as an identity element e' acting on the left, and as an identity element e acting on the right). The equation also shows what it means to say that the identity of a group is unique (if there are two, they are the same). The argument is not literally 'in' the written equation; the written equation provides the material grounds for finding the gestalt of reasoning which, when found, the equation is seen to describe.

Such proof-specific gestalts are, for provers, in their diversity throughout mathematical practice, the observable, discoverable logic and rationality of mathematical reasoning. The rationality of such reasoning is not that of formal logic, but seems to be this: what a mathematical argument describes is, in fact, observably so and, hence, pointless to dispute. This character of mathematical reasoning, most obvious in works such as Jean-Pierre Serre's *A Course in Arithmetic*,[13] is present at all levels of proving, and makes more understandable the difficulties nonprovers have when a proof is explained by further proofs. A proof's gestalt – idiosyncratic to the particular demonstration, formed through the details of that demonstration – is an organization of the practices of proving that exhibit the reasoning of that gestalt. Mathematical reasoning and practice appear to be nothing other – but, also, nothing less – than the arts of finding and describing such gestalts.

The Form of Mathematical Argumentation

Although mathematical argumentation is often discussed in terms of the rules of deductive inference, the actual presentation of proofs has a particular and peculiar form, pervasive throughout mathematical practice. Nowhere in the literature does this form seem to have found direct comment, reflecting its obviousness and transparency to provers. When a prover stands before other provers and proves a theorem, the prover does not literally present a mathematical proof; the prover engages in the arts of description. The prover describes a proof of a theorem, as if that achievement were already in hand, and other provers attend to the prover's work at the blackboard as a description of that proof. When a question is raised, for example, about a particular course of writing or a particular verbal statement, the question is addressed by reference to the objects and to the proof that are being described; as part of the witnessed production of the mathematical demonstration, the blackboard writings and the verbal statements of the prover are reviewed, elaborated, clarified, modified and corrected in terms of their projected achievement. In effect, a prover describes the achievement of the description that he or she is producing – the proof – and, therein, recommends that description to other provers as such an achievement.

The written mathematical argument has this same form: e is not the identity of a group, and $\cdot$ is certainly not the binary operator defined on a group. The lines and angles of Figure 3 are not really Euclidean lines and angles, although the proof concerns Euclidean lines and angles. In Figure 2, the dots are certainly not numbers (nor are symbolic representations such as 'i' or, for that matter, '1'); the dots are used to represent numbers and, through that representation, to reflect and show forth proof-specific properties of them. Provers see through the representations of mathematical objects to the objects that are represented, and use those representations to inspect and discover properties of mathematical objects.

No claim is being made here about the actual existence of mathematical objects or the underlying nature of mathematical proofs.[14] Instead, an observation is offered about the general way that proofs are presented: mathematical arguments, in actual practice, are presented as descriptions of proofs that existed prior to their presentation. In this way, the presentation of mathematical argumentation always maintains a relationship between, on the one hand, concrete detail and reasoning ('dots' and reasoning about 'dots') and, on the other, transcendental subject matter (numbers). It does so by continually describing that which the description reveals, in this case, the properties of numbers. The statement of the theorem for which Figure 2 is offered as a 'proof' – that the sum of any even number of numbers is an even number – provides the transcendental perspective from within which the demonstration is to be seen: in step 1, the larger spaces between the groupings of dots indicate that each group represents one number; the equal number of black and white dots within each group show that the separate groupings represent even numbers.

Given the statement of the theorem, the presentation of the dots in step 1 is seen to make these facts evident. Offered as descriptions, the statements of the argument (in this case, the lines of dots of the Figure 2) are also claims on, and recommendations to, provers to find the grounds and adequacy of those statements as descriptions of a transcendental state of affairs.

Oddly, Figure 2, like all mathematical argumentation, is a description: it is a description of an organization of practical action and reasoning – the organization of 'dots' as numbers and the configuration of reasoning about those 'dots' to show that the sum of even numbers is an even number. Figure 2 is not the proof but a *description* of a proof, the description of a gestalt of reasoning and practice adequate to the statement of the theorem. The statement of the theorem provides the organizational theme of the argument, as well as a summary of its accomplishment. The achievement of mathematical argumentation is that it makes the proof witnessable, therein exhibiting the practical adequacy of its description as a description of that proof and, therein, justifying the use of the word 'proof'.

For those who witness a mathematical demonstration, the demonstration confers on the arts of its production the appreciation that those arts belong to no one in particular. They are seen – that is, they *appear* to provers – as universal arts and, therein, as reason itself – an indication that the underlying practices that make such a perception transparent belong to a particular culture.[15] Such achievements, regularly discovered, produced and witnessed by practising provers, sustain the practices of proving, for members of a mathematical culture, as the practices of producing such gestalts and, in consequence, the practices of provers as the practices of a discovering science – mathematics.

The Materiality of Mathematical Culture

Although the demonstration in Figure 2 concerns abstract entities – a Platonic domain of numbers[16] – and involves reasoning about those entities, at the same time, the reasoning of the demonstration is both extremely concrete and specific to the particular demonstration, involving 'these' particular dots arranged in 'this' particular way. Thus, in Greek, the activity of proving, ἀποδείκνυμι, is a pointing away from other objects at an object and, therein, a pointing out, a showing forth, a making known.[17] This materiality of mathematical reasoning is evident in any mathematical presentation or article. The materiality of a pebble-proof is no different, in this respect, from the use of proof-figures in geometry, the use of symbols for mathematical operators or, in fact, the use of any of the notational specifics of mathematical exposition.

The pebble-proofs of Pythagorean arithmetic provide a clear example. A Pythagorean triple is a triple of natural numbers (a, b, c) such that $a^2 + b^2 = c^2$. Interpreted geometrically, the numbers in a Pythagorean triple are the sides of a right triangle with all the sides having integral lengths: (3, 4, 5) is a Pythagorean triple, as is (5, 12, 13). Two ways of generating Pythagorean triples are given below, the first attributed to Pythagoras, the second to Archytas and to Plato.[18]

Theorem 1 For any odd number M, $(M, [M^2 - 1]/2, [M^2 + 1]/2)$ is a Pythagorean triple.

Theorem 2 For any even number N, $(N, [N^2/4] - 1, [N^2/4] + 1)$ is a Pythagorean triple.

A gnomon (Figure 6) is the L-shaped set of dots which, when added to a square of dots, makes the next square. The name apparently comes from a carpenter's tool for making right angles. A proof of Theorem 1 is based on the observation that the sequence of gnomons represents the sequence of odd numbers 1, 3, 5, 7 . . . . Figure 7 allows us to see that the square of an odd number is odd (since that square is the sum of the square of an even number, two times that even number, and an additional unit). Thus, given any odd number M, M^2 will be the gnomon of some square. Since a gnomon and the preceding square make another square (Figure 6), and the square of an odd number is a gnomon, the remaining step of the proof of Theorem 1 is to find how to count the sides of the square given the size of the gnomon. Figure 7 shows that the side of the square preceding a gnomon of size K will be $(K - 1)/2$; Figure 8 shows that the side of the square containing a gnomon of size K will be $(K + 1)/2$. The proof of Theorem 1 results by applying this method of counting to a gnomon of size M^2.

To find the formula in Theorem 2, a prover might look for a similar relationship between gnomons and the squares that precede them. In that

FIGURE 6

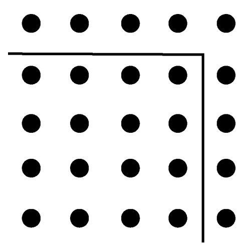


FIGURE 7

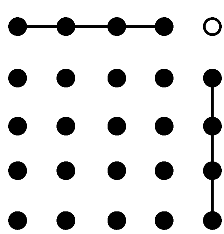


FIGURE 8

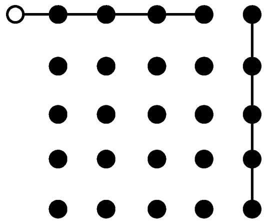


FIGURE 9

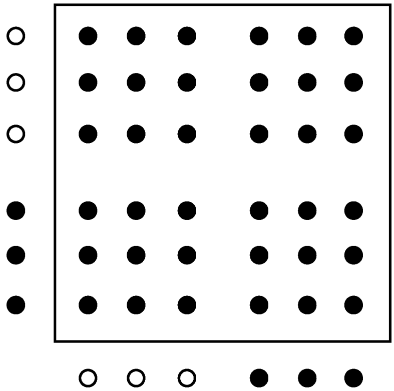
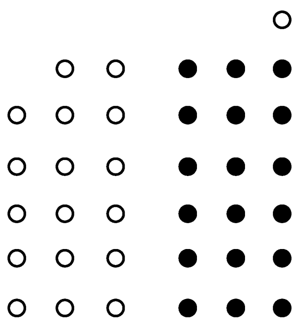


FIGURE 10



the square of an even number N is 4 times some number (Figure 9), $N^2/2$ will be even, and $[(N^2/2) - 1]$ and $[(N^2/2) + 1]$ will be two adjacent gnomons (Figure 10). Using the counting techniques already developed, Theorem 2 follows.

These examples illustrate how proving is steeped in a material culture, analogous to the manner in which the play of chess, for chess players, is tied to the physical board, the chess pieces and the position of those pieces on the board. Although geometric shapes and numbers represented by dots have been used in the examples above, there seems to be no essential difference, in regard to the materiality of proving, to Euclid's use of geometric constructions, Diophantus' use of algebraic equations such as $x^2 + y^2 = z^2$ and $y = mx + z$,[19] or, for that matter, algebraic topologists' use of commutative diagrams. While the materiality of such cultures may differ – for example, the use of dots or the use of symbolic notation such as i – and the consequences of that materiality may be substantial (for example, the discovery and extraction of category theory from algebraic topology) – the immediate 'physicality' of proving remains. Mathematics is frequently presented as the most abstract of disciplines; at the same time, cultures of proving seem to be extremely concrete, material cultures.

The Rôle of Definitions in Theorem Proving

For the logician, definitions are notational abbreviations for longer strings of symbols; for the Platonist, definitions identify properties of ideal objects. Whatever truth is contained in each, both are stories told about the practices of a culture by members of that culture. The practical prover views mathematical definitions in terms of how they function within the practices of mathematical culture.

The problem of identifying Pythagorean triples provides an example. A descriptive formula will clarify the discussion. From the standpoint of a description of cultural practice, a proof – as an ongoing activity – is the pairing of, on the one hand, a description of that proof and, on the other, the reasoning of the proof and the organization of the practices of proving that the proof-description describes.[20] This can be written as follows:

(1) a-proof-description / the-reasoning-and-the-organization-of-the-practices-of-proving-that-the-proof-description-describes

The left-hand side of (1) is the description of a proof, such as those given at the blackboard or in texts, and the right-hand side of (1) is that which is being described, the reasoning and organization of the practices of proving which, when paired with the left-hand side, exhibit the gestalt of reasoning and practice of which the proof consists. The written mathematical arguments in this paper – involving dot representations of numbers, Euclidean constructions and the uniqueness of the identity element of a group – are all examples of proof descriptions; the formula (1) is used to indicate the relationship between such descriptions and the reasoning and practices that such descriptions describe and, at the same time, configure.

Consider, for example, the problem of generating all Pythagorean triples, viewed in terms of the representation of numbers as dots configuring a square. The solution involves the characterization of all pairs of numbers a and b such that they have the relationships shown in Figure 11: the square of each can be counted out around the physical representation of the square of the other to form a larger square. Another way of visualizing the problem is to begin with the encompassing square and ask the conditions under which the dots in that square can be divided, as in Figure

FIGURE 11

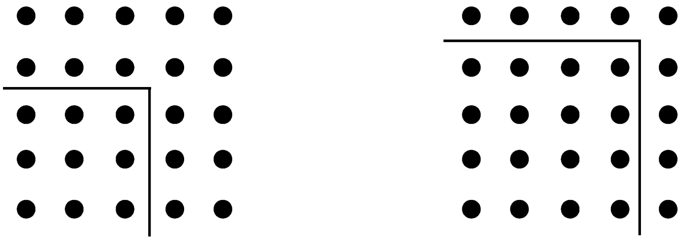
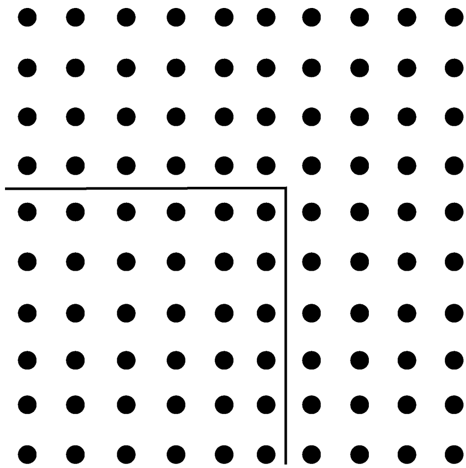


FIGURE 12



12, such that the dots in the L-shaped portion of the figure are the square of some number.

The immediate difficulty of this approach is that, at present, the prover has very little to work with regarding the analysis of such configurations of dots: the right-hand side of (1) – the practices of proving in which the proof is to be embedded – is not particularly rich. If, however, a gnomon is defined as the dots added to a square to form the next larger square, Figures 11 and 12 can begin to be seen in terms of the relationship between a square and its gnomon, or between a square and a series of adjacent gnomons. Once gnomons are identified, it is also possible to see that gnomons represent only some of the integers – 1, 3, 5, 7 . . . . The other integers – 2, 4, 6, 8 . . . – have the property that each of them can be divided into two equal groups of dots; in contrast, each of the numbers represented by gnomons, when divided into two equal groups of dots, have an excess of one dot. One set of numbers can be defined as *even* numbers; the second set, the set of numbers represented by gnomons, as *odd* numbers.

These definitions identify and extract patterns of reasoning about 'dots', and identify and extract practices of arranging dots to find such patterns of reasoning. In this way, the context in which provers try to find the right-hand side of (1) becomes a much denser domain of practice. Distinguishing gnomons as odd numbers, and knowing that the square of an odd number is an odd number, provides a *basis* for finding in Figure 6 the proof of Theorem 1. The utility of the definitions is intimately connected to what can be proved through their use; through that use they become part of the same domain of practice that those definitions were originally intended to clarify. Theorems 1 and 2 and their proofs come to clarify the meaning, exhibit the relevance and substantiate the use of the same definitions that make their proofs possible. Such proofs and theorems 'immortalize' those definitions.

It would be wrong – and historically inaccurate – to suggest that these particular proofs are the reason for the introduction of the definitions of gnomons and of odd and even numbers. But neither is it correct to suppose, as the form of mathematical exposition often suggests, that definitions precede the proofs that are discovered through those definitions, or that definitions precede the theorems that are formulated in their terms. Instead, Pythagorean number theorists were engaged in proving many things about the field of proving that they were simultaneously creating and discovering – that the sum of even numbers is even, that the square of even numbers is even, that the sum of a sequence of odd numbers (gnomons) 1, 3, 5 . . . (2n–1) is n^2, and so on. It is within such a culture of proving that mathematical definitions are introduced and gain their value. When mathematicians are viewed as members of a culture, and proving is examined as an activity of that culture, rather than having definitions first and proofs later, provers are seen to begin and remain amid the practices of proving; definitions – and the practical utility and consequentiality of definitions – are part of the culture of proving in which those definitions are embedded. The claim, however, is not simply that this is so, as if agreement were solicited to an abstract proposition. Instead, attention to the ethnography of proving begins to allow examination of how this relationship develops within and as mathematical practice.

A Phenomenology of Mathematical Discovery

Throughout the literature of mathematics and on mathematics, descriptive analyses of what it looks like to be engaged in discovering a proof are extremely rare.[21] Legends of provers, shrouded in a cult of genius, are sometimes offered in place of what most provers know for themselves to be hard work. Imre Lakatos' articles are often cited,[22] yet these involve a 'rational reconstruction' of an historical process,[23] used by Lakatos to argue various points in the philosophy of science and mathematics. Honoured as George Polya's text *How to Solve It* may be,[24] the heuristic advice it offers seems applicable only after the value of that advice, in a particular situation, is realized; the text offers little as a description of how provers, in actual situations of discovery work, go about finding proofs.

Mathematical discovery work appears to be a fragile phenomenon: until the gestalt of reasoning of which the 'proof' consists is found, that work, though definite in its details, is only promissory; once that gestalt is found, its written exposition is presented as a description of a transcendental proof, available prior to the material argument. In terms of the previous formula that a proof consists of a pairing

> a-proof-description / the-reasoning-and-the-organization-of-the-practices-of-proving-that-the-proof-description-describes

the two sides of the pairing arise together and are inseparably entwined. The problem of mathematical discovery work – and, therein, the problem of the description of discovery work – is that of finding the details that make such a gestalt visible while, at the same time, the proof-specific relevance of those details is not assured until that gestalt is found.

When provers engage in discovery work, they seem to produce partial material arguments. Such partial arguments can be described as pairings of description and reasoning as well:

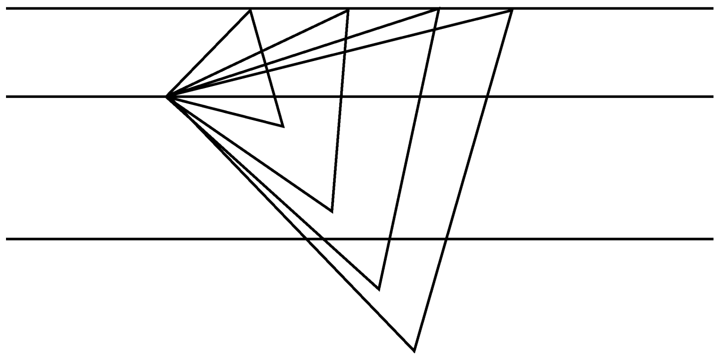
> a-*partial*-proof-description / the-associated-reasoning-and-organization-of-the-practices-of-proving-that-the-*partial*-proof-description-describes.

The idea is that, as a prover 'writes'[25] a partial or 'fragmented' description of a projected proof, the prover simultaneously configures a course of reasoning. The reasoning so embedded in the partial material argument is reasoning that, for provers, is reasoning any competent prover could see in the partial argument. As if the profession perched on provers' shoulders, provers are constrained in their writing and thinking. As provers engage in the 'work' of proving, they also, at the same time, as part of that activity, analyze that work. They cannot write anything, having their own way with words and written expression, but write, and examine what they write, as that which other mathematicians will recognize as mathematical argumentation. The activity of proving builds within itself and maintains its own analyzability as mathematical argumentation.

As such partial arguments are produced, they are subject to any number of operations: they are inspected, integrated, discarded, revised, compared, combined and reworked, thereby embedding within the material detail of the written argument a dense texture of reasoning. In such layered arguments, provers look for the coherence of reasoning – the gestalt – of which the projected proof consists, at the same time that they are stabilizing within their work the communally recognized practices of proving. When provers arrange, rearrange and rework the material details of a prospective and developing proof, they are, in fact, orienting to and composing the cultural substance of their work.

The construction of the equilateral triangle in Figure 3 provides a first example. A prover might begin by sketching a number of equilateral

FIGURE 13



triangles. The idea of drawing them with one vertex fixed, as in Figure 13, might then arise. Figure 13 suggests that, for some length of the side of the triangle, one such triangle will fit exactly – that is, it will have each of its vertices on one of the parallel lines. Since, as in Figure 13, two of the vertices can be positioned on the parallel lines, one approach might be to consider the locus of points of the third vertex. Showing that the distance of the third vertex from the upper line increases monotonically gives a proof of the existence of such an equilateral triangle, but not a Euclidean construction. It is possible that such considerations could lead to the construction.

Given such initial considerations, an alternative approach might develop and seem reasonable. Figure 14 begins with that which is to be proved – that an equilateral triangle (the one in the figure with two of its sides given as dotted lines) can be constructed with each of its vertices on a different line. If that triangle is rotated 60° about any vertex (as indicated in the figure by the arrow), one of the rotated sides will be coincident with one of the sides of the original triangle. As shown in Figure 14, in that it is possible to rotate the line as well, the point of intersection of that line with the non-rotated line will determine a second vertex of the triangle. The construction of the triangle then follows.[26] The proof given in Figure 3 had its origins in this reasoning. Although that proof might have included such reasoning as heuristic motivation, it does not exclude that reasoning in order to hide it. Figure 3 describes the reasoning and organization of practice required to prove the theorem; from the perspective of the actual proof, the above reasoning is, in fact, only heuristic motivation.

A second example is more involved; it illustrates how provers 'fiddle' with the details of material arguments and, therein, with what is shown through such arguments. It also builds on what, at first, seems wayward reasoning.

Consider the work of finding a proof of the theorem that an angle inscribed in a semicircle is a right angle. Figure 15 shows an initial proof-figure for a prospective proof, clarifying the descriptive sense of the statement of the theorem. The inscribed angle is placed in a visibly general

FIGURE 14

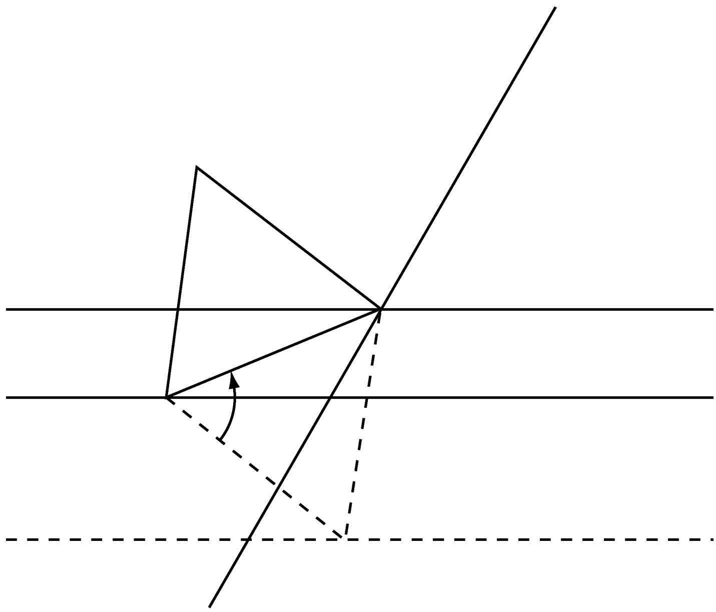
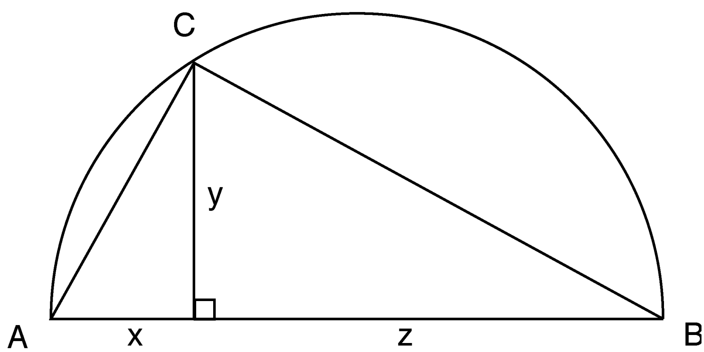


FIGURE 15



position – not, for example, at the centre of the semicircle – so that any reasoning about the angle will not implicitly include any special property of the drawn figure.

In Figure 15, an auxiliary line – a perpendicular from the vertex of the inscribed angle to the diameter – has also been drawn, and notation has been introduced. The motivation for introducing the perpendicular in Figure 15 is provisional: the line is used to 'embed' prospective reasoning into the figure that, potentially, will become definite in that line's subsequent use in the developing proof.

The auxiliary line in Figure 15 exhibits a potentially relevant fact: if the inscribed angle is a right angle, a particular relationship holds between the marked features of the triangle, interpreted as lengths:

(2) $$x/y = y/z$$

The direct bearing of this course of reasoning for the proof of the theorem has not yet been shown.

Does this work have any relevance to the proof of the theorem? The redrawing of Figure 15 as Figure 16 is an attempt to establish that relevance. Assuming that angle ACB is a right angle, the midpoint of the hypotenuse of triangle ACB is marked by a large dot. New notation is used, where R indicates the possibility that the marked segment is the radius of a circle; x is the distance from the centre of the circle to the base of the perpendicular, and (R – x) measures the distance from the perpendicular to the vertex of angle BAC. The notation is arranged as part of the following argument, where equation (2) is expressed in the notation of Figure 16:

$$(R - x)/y = y/(R + x).$$

Cross multiplication leads to the equation

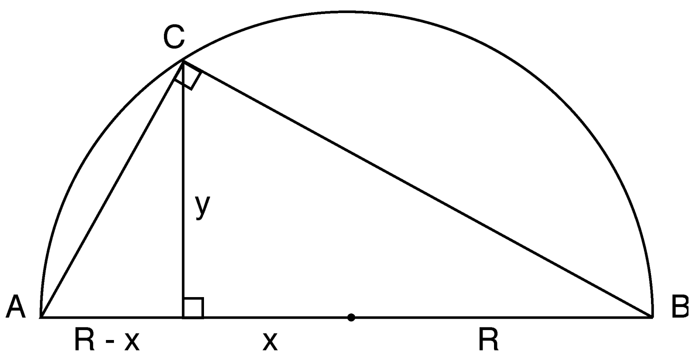
$$R^2 - x^2 = y^2$$

or, rearranged, to

$$R^2 = x^2 + y^2,$$

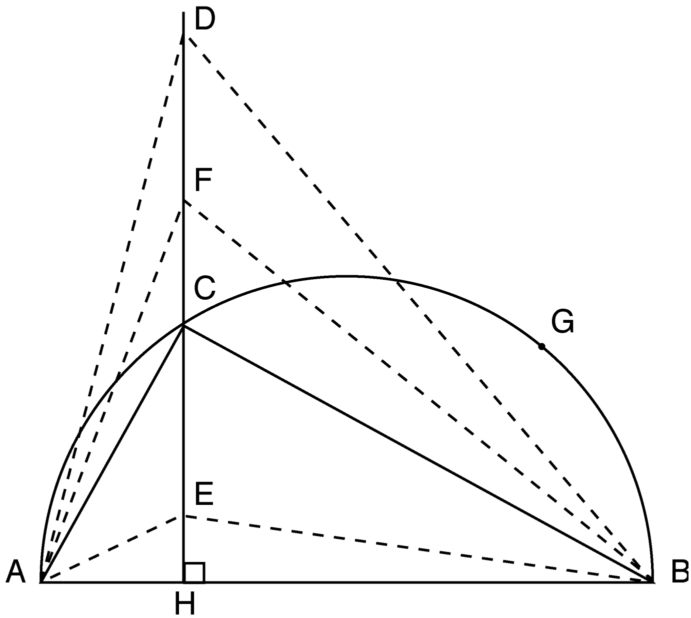
the equation of a circle.

FIGURE 16



What has actually been shown is that, given a right triangle, the vertex of its right angle lies on a circle with the hypotenuse of the triangle as a diameter. This is not the desired theorem. A prover might, however, reason about Figure 16 in the manner illustrated in Figure 17. If one begins with the assumption that angle ACB is a right angle (that which is to be proved), the semicircle ACGB can be drawn (that which has already been shown). If, in Figure 17, the angle on the semicircle is stretched far enough upward, say to D, it will certainly become an acute angle. If the angle is pushed far enough downward, say to E, the angle will become obtuse.

FIGURE 17



This reasoning, while not adequate to prove the theorem, gives the underlying idea on which a proof can be based: start with the semicircle AGB in Figure 17. Pick an arbitrary point C on the semicircle and draw a line CH through C perpendicular to the diameter of the semicircle. At some point on this perpendicular, say D, the angle ADB will be less than 90°. At some point near H, say E, the angle will be greater than 90°. By continuity, at some point F on line CH, angle AFB will be a right angle. Then, by the previous reasoning about right triangles, F lies on the semicircle with its hypotenuse AB as a diameter, and F must be the point C.

Given the preceding examples, a distinction might be offered between the discovery of proofs of theorems which are known to have proofs and the discovery of proofs of theorems whose proofs are hitherto unknown. This distinction is often confused with another distinction, that between proofs which are recognized and celebrated within the discipline as exceptional achievements and proofs which are not so recognized. As the circumstances of their daily work as professional theorem-provers, mathematicians are continually engaged in the work of proving mathematical theorems. The preceding material gives, at least, a first descriptive sense to the activity of mathematical discovery as the actual work of doing mathematics.

Conclusion

That mathematicians come together and, in each other's presence, prove theorems, not simply to their own satisfaction, but for all provers for all time, provides the sustaining grounds of mathematical activity. Aristotle distinguished the mathematical demonstration in terms of this phenomenon, as argumentation that expresses necessary truth or absolute certainty. Writers on the foundations of mathematics have, in turn, sought to explain such knowledge in many ways: that mathematics concerns abstract, Platonic objects; that the theorems of mathematics are propositions of symbolic logic; that mathematical thought reflects fundamental intuitions of the human mind; that proofs are matters of conventional reasoning and agreed or negotiated definitions. Yet the observable phenomenon of mathematical proving is not apodictic knowledge, but the appearance of apodictic knowledge to those capable of witnessing the mathematical demonstration. This paper suggests that the *appearance* of necessary truth or absolute certainty in the mathematical demonstration – independently of the ultimate truth of mathematical reasoning or the actual existence of mathematical objects – can be examined as a cultural phenomenon and as a phenomenon generated by the same practices that sustain those practices – that is, as a phenomenon belonging to a particular tribe, the tribe of mathematical theorem-provers.

The observations gathered in this paper have not been presented as the basis of an encompassing theory of mathematics, a 'master narrative' that re-interprets and, therein, attempts to 'explain' mathematical practice. If there are underlying aims to the studies on which the paper is based, they are the opposite – to show the feasibility of descriptive studies of mathematical practice and to begin to discover the cultural character of mathematics *as* mathematics. Taken together, the observations of this paper present a picture of a culture of proving quite different from more conventional presentations. They illustrate as well one of the problems of teaching the arts of proving: at all levels, proving retains the features of proving that make it such for provers, and novices must somehow be brought within such a culture for that culture to 'make sense'.

Notes

The material in this paper was first developed as the Inaugural Lecture of the Meaning and Computation Laboratory at the University of California, San Diego, 20 June 1997. I thank the members of the lab, the Department of Computer Science and Engineering, and particularly Joseph Goguen, the director of the laboratory, for stimulating my return to studies reported in *The Ethnomethodological Foundations of Mathematics* (London: Routledge & Kegan Paul, 1986), and to work undertaken in collaboration with Harold Garfinkel in the late 1970s and early 1980s. The current work also has its grounds in ethnomethodology – see, for example, Harold Garfinkel, *Studies in Ethnomethodology* (Englewood Cliffs, NJ: Prentice-Hall, 1967) and Harold Garfinkel and Harvey Sacks, 'On Formal Structures of Practical Action', in John C. McKinney and Edward A. Tiryakian (eds), *Theoretical Sociology: Perspectives and Developments* (New York: Appleton-Century-Crofts, 1970), 337–66 – and, particularly, in ethnomethodological studies of science, among them Harold Garfinkel, Michael Lynch and Eric Livingston, 'The Work of a Discovering Science Construed with Materials from the Optically-Discovered Pulsar', *Philosophy of the Social Sciences*, Vol. 11 (1981), 131–58; Michael Lynch, *Art and Artifact in Laboratory Science: A Study of Shop Work and Shop Talk in a Research Laboratory* (London: Routledge & Kegan Paul, 1985); and Eric Livingston, 'The Idiosyncratic Specificity of the Methods of Physical

Experimentation', *The Australian New Zealand Journal of Sociology*, Vol. 31 (1995), 1–22, and *The Ethnomethodological Foundations of Mathematics*, op. cit. I am indebted to Michelle Arens for her constant assistance, to Charles Livingston for our discussions on mathematics and for his help with the mathematical details of the argument, and to Michael Lynch for his close reading of the text and trenchant suggestions. I have received help from many others as well, for which I thank Randall Albury, Alan Bundy, Fred D'Agostino, David Edge, Joseph Goguen, Martin Krieger, Patrick Jobes, Peter Lucich, James Lynch, Chris Radford, Peter Toohey, and two anonymous reviewers. The illustrations for this paper were prepared using Adobe Illustrator 7.0.

1. This paper is based on the study of mathematics over a number of years and on extended observational research involving a range of techniques – for example, the material on the form of mathematical argumentation was first occasioned, in part, by the analysis of a videotaped session of theorem proving. The reference to anthropology is used here to stress the examination of lived practice as well as to highlight the distinction between cultural explanations and matters of nature or objective fact; it does not imply a commitment to conventionally understood ethnographic description, nor to any explicit research methodology as a means of substantiating real worldly claims otherwise not in evidence. Instead, the discovered praxeological character of cultural objects recommends those methods adequate to their further articulation and specification, a claim in many ways distinctive to ethnomethodological studies. Hopefully, the examples used in this paper are of sufficient clarity to direct attention to the phenomena that they are intended to help explicate.
2. See, for example, Abraham Seidenberg, 'Did Euclid's Elements, Book I, Develop Geometry Axiomatically?', *Archive for History of Exact Sciences*, Vol. 14 (1975), 263–95.
3. For example, Ludwig Wittgenstein, *Philosophical Investigations* (Oxford: Blackwell, 1953) and Wittgenstein, *Remarks on the Foundations of Mathematics* (Oxford: Blackwell, 1956).
4. For example, Imre Lakatos, 'Proofs and Refutations', *The British Journal for the Philosophy of Science*, Vol. 14 (1963/4), 1–25, 120–39, 221–43, 296–342; David Bloor, *Knowledge and Social Imagery* (London: Routledge & Kegan Paul, 1976), esp. 74–140; and Donald MacKenzie, 'Slaying the Kraken: The Sociohistory of a Mathematical Proof', *Social Studies of Science*, Vol. 29, No. 1 (February 1999), 7–60.
5. See Marcia Ascher, *Ethnomathematics: A Multicultural View of Mathematical Ideas* (Pacific Grove, CA: Brooks/Cole, 1991) and Arthur B. Powell and Marilyn Frankenstein (eds), *Ethnomathematics: Challenging Eurocentrism in Mathematics Education* (Albany: State University of New York, 1997).
6. Other than my *The Ethnomethodological Foundations of Mathematics*, I am aware of only one article attempting such a descriptive analysis: Bartel L. van der Waerden, 'How the Proof of Baudet's Conjecture was Found', in Leonid Minsky (ed.), *Studies in Pure Mathematics* (London & New York: Academic Press, 1971), 251–60. This article is written from a perspective, and with aims, different from the present paper, yet much of van der Waerden's discussion is compatible with the material presented here – the materiality and concreteness of mathematical reasoning, the substantiation of various categorizations and of the general argument through subsequent work, the 'fiddling' with proof-specific detail, and the piecing together of partial arguments. I thank Alan Bundy for referring me to this article.
7. An instructive review of the theory of Gestalt perception is given in Aron Gurwitsch, *The Field of Consciousness* (Pittsburgh, PA: Duquesne University, 1964), 87–153. The theme of mutually elaborating detail – 'Gestalt contextures' – is described there, a theme subsequently developed by Garfinkel in undergraduate lectures at UCLA as part of the analysis of social objects. The analogy between Gestalt figures such as Figure 1 and 'Gestalt switches' involving incommensurable perspectives has wide currency in social studies of science. However, the central point in the text is different: in that Figure 1 can be seen in two incompatible ways, the material figure itself does not determine its perception. The analogy for proofs is that a written mathematical

argument is not, in itself, a proof; as described more fully in the text below, the proof associated with the written argument transcends the details of its written description.

8. See Árpád Szabó, trans. A.M. Ungar, *The Beginnings of Greek Mathematics* (Dordrecht, Holland: Reidel; Budapest: Akadémiai Kiadó, 1978), 192–93; and Wilbur Richard Knorr, *The Evolution of the Euclidean Elements* (Dordrecht, Holland: Reidel, 1975, 131–69). For the proof in Figure 2, see Szabó's presentation on p. 193.
9. The word 'shows' is, in fact, a reference to the gestalt of reasoning that the visual argument may be found to describe and, thus, does not refer to the self-sufficient details of a particular, isolated line of the visual display. When the visual argument is entertained as a demonstration of the general proposition, the generality of the first line of the diagram must be found as it bears on that general proposition; the word 'shows' is used in this instance as a gloss for the work of finding that generality.
10. For reasoning leading to the construction, see Howard W. Eves, *A Survey of Modern Geometry*, Vol. 1 (Boston, MA: Allyn & Bacon, 1963), 191–92, and pp. 881–82 of this paper. James Lynch has pointed out that, with minor changes in the proof in Figure 3, the assumption that lines l_1, l_2 and l_3 are parallel is not required. Here is the argument: given any three lines, at least two of which are distinct, an equilateral triangle can be constructed with each of its vertices on a different line. If the lines are not distinct or are pairwise at 60-degree angles, the argument is (relatively) simple. Hence, assume that the lines are distinct and that two of them, say l_1 and l_3, are not at 60-degree angles. Let O be a point on l_2 that is not on either l_1 or l_3. Let l_1' be a 60-degree rotation of l_1 about O, and let A be the unique point of intersection of l_1' and l_3. The circle centred at O through A and the circle centred at A through O intersect in two points, B and B'. By construction, OAB and OAB' are equilateral triangles. Since angle AOB and angle AOB' are both 60-degree angles, and line segments OB and OB' both have length equal to OA, either B or B' will lie on l_1.
11. Briefly, Figure 4 shows that, given point X, a perpendicular to OX through X can be constructed of length $\sqrt{3} \cdot$ OX, resulting in a 30–60–90-degree triangle; the angle at O is 60°, and the intersection X' of the circle with centre O and radius OX with the hypotenuse of the constructed triangle is the required rotation of X through 60°. The figure illustrates the reasoning that comes to be embedded in a written proof or proof-figure: although Figure 4 might be seen as an atemporal diagram, it is inspected by provers to find, and is then seen to reflect, the ordered, temporal sequence of reasoned actions that result in the construction of the point X'.
12. Given a right triangle with sides a, $\sqrt{3}$a, and 2a, Figure 5 shows that the angle opposite the leg of triangle of length $\sqrt{3}$a is the angle of an equilateral triangle and, hence, that the sum of three such angles is an angle of 180° or a straight angle.
13. Jean-Pierre Serre, *A Course in Arithmetic* (New York: Springer-Verlag, 1973).
14. A deeper issue involving the self-referential, reflexive or incarnate character of representations of mathematical objects is involved: 'representations' of mathematical objects are used by provers to show forth properties of the objects so represented; through the exhibition of those properties, such 'representations' are seen to be adequate representations of those objects or, more precisely, representations adequate to the practices of proving properties of such objects. The development of this theme goes beyond the aim of this paper, which is to indicate the cultural character of the details of provers' work.
15. That a group treats a way of reasoning as universal reason – that reasoning of anyone and everyone is reasoning like that – does not make that reasoning reasonable, but indicates that, among that group, how they go about reasoning in that way does not seem to be questioned.
16. Once again, a claim is not being made that the demonstration does, in fact, concern Platonic entities 'really', but that the demonstration is treated and produced, in practical circumstances of proving, as a demonstration about such Platonic entities. Although the achievement of the demonstration – the witnessed proof – sustains the properties of mathematical objects as the properties of a transcendental domain of objects (that is, as objects about which such demonstrations can be made), the

preservation of the transcendental *appearance* of mathematical objects is deeply embedded in the ways that provers go about their work. The last section of this paper, 'A Phenomenology of Mathematical Discovery', gives an indication of how this might be so.

17. Henry G. Liddell and Robert Scott, *A Greek-English Lexicon*, 9th edn revised and augmented by Henry Stuart (Oxford: Clarendon Press, 1940). Szabó discusses the early use of the word in *The Beginnings of Greek Mathematics*, op. cit. note 8, 185–96; I am indebted to Peter Toohey for his help with the Greek.
18. Knorr, op. cit. note 8, 155. In Figures 7 and 8 below, the use of line segments to join groups of dots as a device of proving is from Knorr; the reader might wish to compare his proofs of the theorems with those given here.
19. See Leonard Eugene Dickson, *History of the Theory of Number*, Vol. 2 (New York: Chelsea, 1966), 165.
20. This formulation developed in dissertation research under the direction of, and as part of collaborative studies with, Harold Garfinkel. See Eric Livingston, *The Ethnomethodological Foundations of Mathematics* (London: Routledge & Kegan Paul, 1986) and Livingston, *Making Sense of Ethnomethodology* (London: Routledge & Kegan Paul, 1987).
21. See note 6.
22. Lakatos, op. cit. note 4.
23. Imre Lakatos, 'Falsification and the Methodology of Scientific Research Programmes', in Lakatos and Alan Musgrave (eds), *Criticism and the Growth of Knowledge* (Cambridge: Cambridge University, 1970), 91–196.
24. George Polya, *How to Solve It* (Princeton, NJ: Princeton University Press, 1945).
25. In the following discussion, the word 'writing' will be used not only for physically writing or drawing, but for concrete reasoning that could be rendered in writing or drawing as well. The reference is to the materiality – the concrete physicality – of cultures of proving.
26. Eves, op. cit. note 10, formulates this argument as an illustration of the general method of homology.

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