Trinity Western University
Department of Mathematical Sciences
Sample Final Examination

1. a) Let A and B n x n matrices, show that $\text{tr}(AB) = \text{tr}(BA)$. Use this identity to prove that $AB - BA = I$ is impossible.

   b) Discuss the solution of the system of equations

$$x + ay - z = 1$$
$$-x + (a - 2)y + z = -1$$
$$2x + 2y + (a - 2)z = 1$$

   for various values of a (Indicate in each case how many solutions will you get, also giving the solutions if they exist).

2.. a) Find the equation of the plane containing the lines $(x, y, z) = (1, -1, 2) + t(1, 0, 1)$, and $(x, y, z) = (0, 0, 2) + t(1, -1, 0)$

   b) Find if $W = \{A \mid A \text{ in } \mathbf{M}_{22}, AX = 0\}$, X a fixed 2 x 2 matrix, is a subspace of $\mathbf{M}_{22}$. If not indicate why it is not.

3. a) Prove that a vector in a vector space has exactly one negative.

. b) Find the basis for the null space of

$$A = \begin{pmatrix} 3 & 5 & 5 & 2 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & -2 & -2 \\ -2 & 0 & -4 & -4 & -2 \end{pmatrix}$$

   Then compute rank (A) and verify the theorem stating that

$$\text{rank}(A) + \text{nullity}(A) = n,$$

   n, being the number of columns in A.

4. Find a subset of the vectors that forms a basis for the space spanned by the vectors; then express each vector that is not in the basis as a linear combination of the basis vectors.

$$\mathbf{v}_1 = (1, 0, -1, 3),\ \mathbf{v}_2 = (2, 1, 0, -2),\ \mathbf{v}_3 = (-1, 1, 2, 1),\ \mathbf{v}_4 = (-3, 2, 4, 11)$$

5. Let $U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ and $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$

   Define $< U, V > = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$.

   Use the Gram-Schmidt algorithm to transform the basis B given below into an *orthonormal* basis

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \right\}$$

6. For the matrix A, obtain A^n, where n is a positive integer.

$$A = \begin{pmatrix} 8 & 7 & 7 \\ -5 & -6 & -9 \\ 5 & 7 & 10 \end{pmatrix}$$

7. For the transition matrix $P = \begin{pmatrix} a & 0 \\ 1-a & 1 \end{pmatrix}, 0 < a < 1$, show that P is not regular by finding P^n, where n is a positive integer. Further show that as n increases, $P^n\mathbf{x}^{(0)}$ approaches $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for any value of a and any initial vector $\mathbf{x}^{(0)}$.

8. A wolf pack always hunts in one of the three regions R_1, R_2, and R_3. Its hunting habits are as follows:
1. If it hunts in one region one day, it is as likely as not to hunt there agin the next day.
2. If it hunts in R_1, it never hunts in R_2 the next day.
3. If it hunts in R_2 or R_3, it is equally likely to hunt in each of the other regions the next day.

a) If the pack hunts in R_1 on Monday, find the probability that it hunts there on Thursday.
b) What are the long range probabilities that the pack hunts in each of the three regions?

9. A lawn mower company makes three models: standard, deluxe, and super. The construction of each mower involves three stages: motor construction, frame construction, and final assembly. The following table gives the number of hours of labor required per mower for each stage and the total number of hours of labor available per week for each stage. It also gives the profit per week. Find the weekly production schedule that maximizes the profit.

	Standard	**Deluxe**	**Super**	**Hours Available**
motor	1	1	2	2500
frame	1	2	2	2000
assembly	1	1	1	1800
PROFIT	$30	$40	$55	

10. Find the general solution of the system of equations

$$y_1' = 6y_1 + 4y_2 - 5y_3,$$
$$y_2' = 4y_1 + 6y_2 - 5y_3,$$
$$y_3' = -5y_1 - 5y_2 + 15y_3.$$