# Hölder Spaces

Sean Richardson

## Motivation via Arzelà-Ascoli

Suppose we have a sequence of functions $u_n$ and wish to prove that this sequence converges uniformly (we often encounter this, for example, when solving PDEs). If we are working on a compact subset $X \subset \mathbb{R}^n$, the Arzelà-Ascoli theorem provides a convenient criteria for uniform convergence: pointwise boundedness and equicontinuity. Recall these are defined as follows, letting $C(X)$ denote the space of continuous functions $X \to \mathbb{R}$.

**Def (pointwise boundedness).** A subset $\mathcal{F} \subset C(X)$ is *pointwise bounded* if for all $x \in X$, the set $\{u(x) : u \in \mathcal{F}\} \subset \mathbb{R}$ is bounded.

**Def (equicontinuity).** A subset $\mathcal{F} \subset C(X)$ is *equicontinuous* if for all $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies we have the bound $|u(x) - u(y)| < \varepsilon$ for all $u \in \mathcal{F}$.

Importantly, given such a $x$ and $\varepsilon$ as above, the same $\delta$ must work for all $u \in \mathcal{F}$. Keep in mind the subset $\mathcal{F}$ above is often simply a sequence $u_n$ of continuous functions. The Arzelà-Ascoli theorem then promises

**Theorem (Arzelà-Ascoli).** If $\mathcal{F} \subset C(X)$ is equicontinuous and pointwise bounded, then there exists a subsequence $u_k \subset \mathcal{F}$ such that $u_k \to u \in C(X)$ uniformly.

Equicontinuity is generally harder than pointwise boundedness to verify when using Arzelà-Ascoli. A convenient sufficient condition for equicontinuity is there exists a fixed $C > 0$ so that

$$|u(x) - u(y)| \leq C\|x - y\| \tag{1}$$

for all $u \in \mathcal{F}$. This immediately implies equicontinuity because for any $\varepsilon > 0$ we can simply take $|x-y| < \varepsilon/C$. However, an even weaker sufficient condition which is often easier to check is that for some fixed $\alpha \in (0, 1]$ there exists $C > 0$ such that

$$|u(x) - u(y)| \leq C\|x - y\|^\alpha \tag{2}$$

for all $u \in \mathcal{F}$. In this case, we get equicontinuity by taking $\|x - y\| < (\varepsilon/C)^{1/\alpha}$ for any given $\varepsilon > 0$. If $u$ satisfies (1), it is called *Lipshitz continuous* and if $u$ satisfies (2), it is called *$\alpha$-Hölder continuous*. This terminology is due to the following.

**Proposition.** If $u : X \to \mathbb{R}$ satisfies either (1) or (2), it is a continuous function.

*Proof.* Suppose $u$ is $\alpha$-Hölder continuous with Hölder constant $C > 0$. Then or any $\varepsilon > 0$, taking $\|x - y\| < (\varepsilon/C)^{1/\alpha}$ guarantees

$$|u(x) - u(y)| \leq C|x - y|^\alpha < \varepsilon.$$

□

Given an $\alpha$-Hölder continuous function $u : X \to \mathbb{R}$, the minimum constant $C$ is given by the *$\alpha$-Hölder semi-norm*

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha}.$$

This semi-norm is convenient because $\{[u]_\alpha : u \in \mathcal{F}\}$ bounded implies $\mathcal{F}$ is equicontinuous. Additionally, note that if $\{\|u\|_{\sup} : u \in \mathcal{F}\}$ is bounded, then $\mathcal{F}$ is pointwise bounded where $\|\cdot\|_{\sup}$ denotes the uniform norm.

This motivates defining the *$\alpha$-Hölder norm*

$$\|u\|_\alpha := \|u\|_{\sup} + [u]_\alpha$$

because if $\{\|u\|_\alpha : u \in \mathcal{F}\}$ is bounded, then $\mathcal{F}$ is equicontinuous and pointwise bounded, so Arzelà-Ascoli applies and there exists $u_k \in \mathcal{F}$ such that $u_k \to u \in C(X)$ uniformly.

# Hölder spaces over a compact subset

The point is the $\alpha$-Hölder norm $\|\cdot\|_\alpha$ is useful because it allows us to conclude uniform convergence due to Arzelà-Ascoli. We can rephrase this oberservation using the language of functional analysis, which begins by considering the function space of $\alpha$-Hölder continuous functions.

**Def (Hölder space).** For a compact subset $X \subset \mathbb{R}^n$ and $\alpha \in (0,1]$, the *Hölder space* $C^\alpha(X)$ is

$$C^\alpha(X) := \{u \in C(X) : \|u\|_\alpha < \infty\}.$$

It is over $C^\alpha(X)$ that $[\cdot]_\alpha$ is a semi-norm and $\|\cdot\|_\alpha$ is a norm, which we prove below.

**Proposition.** $[\cdot]_\alpha$ is a semi-norm over $C^\alpha(X)$.

*Proof.* First we check positive homogeneity

$$[cu]_\alpha = \sup_{x \neq y} \frac{|cu(x) - cu(y)|}{\|x-y\|^\alpha} = |c| \cdot \sup_{x \neq y} \frac{|u(x) - u(y)|}{\|x-y\|^\alpha} = |c| \cdot [u]_\alpha.$$

Next we check the triangle inequality

$$[u+v]_\alpha = \sup_{x \neq y} \frac{|(u+v)(x) - (u+v)(y)|}{\|x-y\|^\alpha} \leq \sup_{x \neq y} \frac{|u(x) - u(y)|}{\|x-y\|^\alpha} + \sup_{x \neq y} \frac{|v(x) - v(y)|}{\|x-y\|^\alpha} \leq [u]_\alpha + [v]_\alpha.$$

**Proposition.** $\|\cdot\|_\alpha$ is a norm over $C^\alpha(X)$.

*Proof.* Positive homogeneity of $\|\cdot\|_\alpha$ follows immediately from that of $[\cdot]_\alpha$ and $\|\cdot\|_{\sup}$:

$$\|cu\|_\alpha = \|cu\|_{\sup} + [cu]_\alpha = |c| \cdot (\|u\|_{\sup} + [u]_\alpha) = |c| \cdot \|u\|_\alpha.$$

Similarly, the triangle inequality of $\|\cdot\|_\alpha$ follows from that of $[\cdot]_\alpha$ and $\|\cdot\|_{\sup}$:

$$\|u+v\|_\alpha = \|u+v\|_{\sup} + [u+v]_\alpha \leq \|u\|_{\sup} + [u]_\alpha + \|v\|_{\sup} + [v]_\alpha = \|u\|_\alpha + \|v\|_\alpha.$$

Finally, positive definiteness of $\|\cdot\|_\alpha$ follows from the positive definiteness of $\|\cdot\|_{\sup}$. Indeed, if $0 = \|u\|_\alpha = \|u\|_{\sup} + [u]_\alpha$ then we must have $\|u\|_{\sup} = 0$, implying $u = 0$.

**Proposition.** The *Hölder space* $C^\alpha(X)$ is a Banach space.

*Proof.* Take a Cauchy sequence $u_k \in C^\alpha(X)$. Then

## Compact embedding theorems

*/recall motivation for Holder spaces... rephrase in terms of compact embeddings/*

$C^{0,\alpha} \to C^0$

$C^{0,\alpha} \to C^{0,\beta}$

$C^{k,\alpha} \to C^k$ */hopefully using earlier embedding/*

$C^{k,\alpha} \to C^{k,\beta}$ */hopefully using earlier embedding/*