Lutzer's Rotating Hammer

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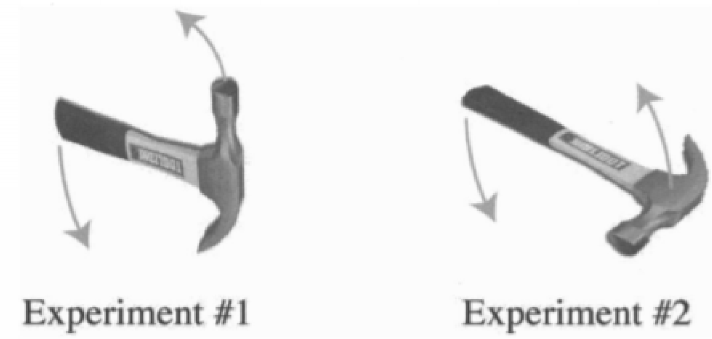
1 Introduction

The paper "Hammer Juggling, Rotational Inertia, and Eigenvalues" by Carl Lutzer begins with a simple system: throwing a rotating hammer. If thrown vertically from the hilt, with the head down, (if you're coordinated), you'll catch in with the head down again. However, thrown with the head to the left side, it'll flip horizontal mid-air and you'll catch it with the head on the right side. The reason for this phenomena can be traced to angular velocity, and with a bit of math magic, we can derive information about the Moment of inertia of the system and explain this phenomena in mathematical terms.

NOTE: We will be using dot notation for derivates, where $\dot{x}=\frac{dx}{dt}$



2 The Rotating Hammer System (in physics terms)

Before we delve into the math, we must first refresh our physics terminology so that our notation is consistent and can create some intuition for our system. Here we will define some key physics concepts that we will be manipulating throughout our work.

Center of Mass: When an object spins freely, its mass is pushing and pulling on itself, resulting in the rotation. However, when two masses exert a force on one another, they accelerate in equal but exactly opposite directions. So, the average location of the mass of the object must not be effected by rotation. This is possible only if the object rotates about the average location of its mass. This average is called the "center of mass" and it is the point about an object will freely rotate.

Angular Velocity: In basic mechanics, there are two types of motion, rotational and translational. Translational is our normal concept of motion: a car moves down a road. Rotational is similar, if we imagine our car driving on a circle. The car moves radially, so it's more useful to think of movement as that with respect to the circle. We define angular velocity as velocity times $2\pi \cdot r$, where r is the radius. This measures how quickly an object completes a cycle. Now, rotating is just like transversing a circle, but the center is at the center of mass. Rotation about the center of mass is fixed, so it will be useful for us when analyzing our system.

Euler's Equation Of Motion: In physics, one of the basic equations of motion is Newton's equation, F = ma. However, this is for force and motion in a two dimensional setting. When appling equations of motion for a rotational system, we need to use a different form, with new rotational variables. That is Euler's equation, governed by

$$\tau = M\dot{\omega} + \omega \times M\omega = \frac{dL}{dt} \tag{1}$$

where τ is Torque, or rotational force, L is angular momentum, ω is Angular Velocity, and M is the Moment of Inertia Tensor.

Moment of Inertia Tensor: Practically, M, the Moment of Inertia Tensor can be thought of as the rotational mass. M is a symmetric 3×3 matrix, which holds moments of inertia (the willingness to rotate) in each direction in 3D space. These moments of inertia affect how much force is required to rotate the object in one direction. For example, a dumbbell is easy to spin along the handle, which means it has a low moment of inertia in that direction. However, it's harder to rotate pivoting from the center and rotating the heads. Thus a large moment of inertia. The Moment of Inertia Tensor holds these moments of inertia as eigenvalues, one of which for each direction of which the object can rotate.

3 The Math

3.1 The Approach

There is a natural way to coordinatize a spinning object into an orthornormal basis along three "principle axes". All calculations we perform are in reference to these three natural axes. Using Euler's equation creates a relationship between the angular acceleration and the angular velocity about these axes. This paired with the assumption of 0 torque being applied to the system gives rise to a first order system of differential equations. We find the equilibrium solutions, which correspond to rotations around each of the principle axes. We linearize the system near the equlibrium solutions to analyze the local behavior. In doing so, we find that equilibrium solutions about the shortest and longest axis are semi-stable centers. So, when a hammer is thrown about one of these axes, it's rotation pattern will stay fairly constant. On the other hand, the equilibrium solutions about the axis of intermidiate length are unstable saddles. So, when an object is rotated about this intermidiate axis will have have a rotation with a more chaotic appearance.

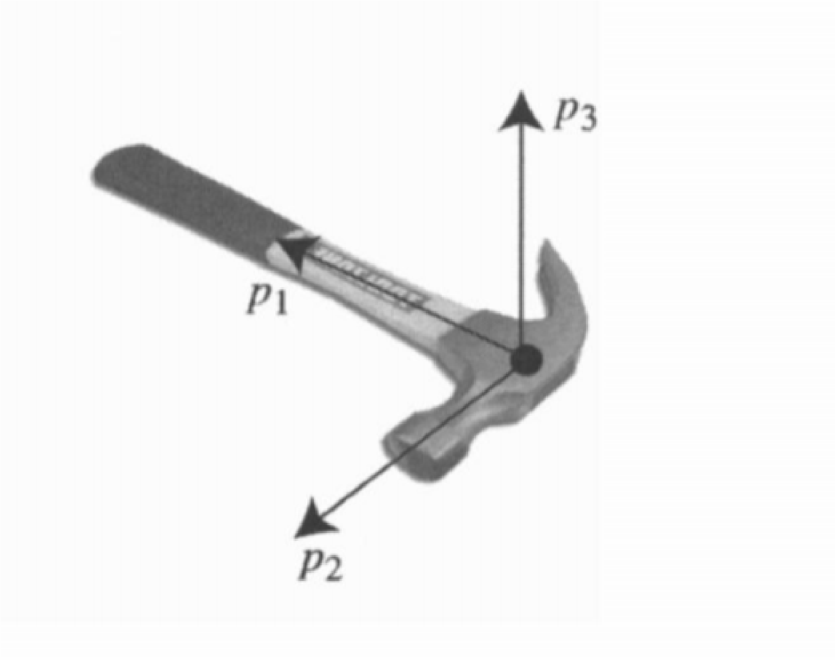


Figure 1: Three Principle Axes

3.2 Principle axes and Euler's equation

First off, we must consider the angular velocity of our object. Since we live in three dimensions, our object will have three principle axis to rotate around, which we denote p_1, p_2, p_3. These will be centered at the center of mass, since objects rotate about the center of mass. Each p refers to an arbitrary dimension, say p_1 is length, p_2 is width, and p_3 is height. Since this labeling is arbitrary, we're going to make an assumption that length > width > height. With this assumption we'll see the unstable rotation about p_2 as we will show.

Now with now our orthonormal basis at the center of mass, we can express rotational velocity as a linear combination, so that,

$$\omega(t) = \alpha_1(t)p_1 + \alpha_2(t)p_2 + \alpha_3(t)p_3 \tag{2}$$

Where $\omega(t)$ is the rotational velocity of the object as any instant in time. Because the object is rigid, every point on the object shares the same angular velocity. Next, we have to consider the Moment of Inertia Tensor, which is a 3x3 matrix that holds our eigenvalues in each p direction

$$M = \begin{pmatrix} \lambda_1 \; 0 \; 0 \\ 0 \; \lambda_2 \; 0 \\ 0 \; 0 \; \lambda_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{3}$$

This matrix may seem arbitrary, but is necessary for Euler's equation of motion. Essentially it splits our object into having three axis to rotate around and the eignevalues hold the inertia (ie willingness to rotate) in each direction. Note: $\lambda_1 > \lambda_2 > \lambda_3 > 0$ due to our assumption of length > width > height. Now we can apply this to our system. Now, since we only throw our object and let it spin, we determine there must be no outside torque on our system. Thus we set the torque value the zero vector and Euler's equation in each direction becomes,

$$0 = M\dot{\omega} + \omega \times M\omega \tag{4}$$

3.3 Translating Euler's equation into Differential Equation System

We now expand equation 4 into matrix form and simplify into a system of differential equations.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \dot{\alpha_1} \\ \dot{\alpha_2} \\ \dot{\alpha_3} \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \times \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1\dot{\alpha_1} \\ \lambda_2\dot{\alpha_2} \\ \lambda_3\dot{\alpha_3} \end{pmatrix} + \begin{pmatrix} (\lambda_3 - \lambda_2)\alpha_2\alpha_3 \\ (\lambda_1 - \lambda_3)\alpha_1\alpha_3 \\ (\lambda_2 - \lambda_1)\alpha_1\alpha_2 \end{pmatrix} \tag{6}$$

Now we have our equations that we will derive our eigenvalues and equations of motion from.

$$\begin{aligned} \lambda_1\dot{\alpha_1} &= (\lambda_2 - \lambda_3)\alpha_2\alpha_3 \\ \lambda_2\dot{\alpha_2} &= (\lambda_3 - \lambda_1)\alpha_1\alpha_3 \\ \lambda_3\dot{\alpha_3} &= (\lambda_1 - \lambda_2)\alpha_1\alpha_2 \end{aligned}$$

3.4 Analysis of System of Differential Equations

We begin to analyze the system of differential equations by finding the equilibrium values. The system is at equilibrium when the angular vecolcity in each of the three directions is 0. So, $\dot{\alpha_1} = \dot{\alpha_2} = \dot{\alpha_3} = 0$ or $\vec{w} = \vec{0}$. Equilibrium occurs in three different situations:

- $\alpha_2 = \alpha_3 = 0$ for any α_1
- $\alpha_1 = \alpha_3 = 0$ for any α_2
- $\alpha_1 = \alpha_2 = 0$ for any α_3

So, we find that we do not simply have equilibrium points. Rather, the freedom of one alpha to vary in each case implies that the system has three equilibrium *lines*. Furthermore, because two of the α_i values are 0 in each case, it is the $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ axes that hold all equlibrium values. To determine the local behavior of the solution about each equlibrium point, we linearize the equation into something we can deal with at each point.

The Jacobian matrix takes the following form,

$$DT = \begin{pmatrix} \partial_{\alpha_1}\omega_1 & \partial_{\alpha_2}\omega_1 & \partial_{\alpha_3}\omega_1 \\ \partial_{\alpha_1}\omega_2 & \partial_{\alpha_2}\omega_2 & \partial_{\alpha_3}\omega_2 \\ \partial_{\alpha_1}\omega_3 & \partial_{\alpha_2}\omega_3 & \partial_{\alpha_3}\omega_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\lambda_1}(\lambda_2 - \lambda_3)\alpha_3 & \frac{1}{\lambda_1}(\lambda_2 - \lambda_3)\alpha_2 \\ \frac{1}{\lambda_2}(\lambda_3 - \lambda_1)\alpha_3 & 0 & \frac{1}{\lambda_2}(\lambda_3 - \lambda_1)\alpha_1 \\ \frac{1}{\lambda_3}(\lambda_3 - \lambda_1)\alpha_2 & \frac{1}{\lambda_3}(\lambda_3 - \lambda_1)\alpha_1 & 0 \end{pmatrix}$$

So,

$$\begin{pmatrix} \dot{\omega_1} \\ \dot{\omega_2} \\ \dot{\omega_3} \end{pmatrix} \approx \begin{pmatrix} 0 & \frac{1}{\lambda_1}(\lambda_2 - \lambda_3)\alpha_3 & \frac{1}{\lambda_1}(\lambda_2 - \lambda_3)\alpha_2 \\ \frac{1}{\lambda_2}(\lambda_3 - \lambda_1)\alpha_3 & 0 & \frac{1}{\lambda_2}(\lambda_3 - \lambda_1)\alpha_1 \\ \frac{1}{\lambda_3}(\lambda_3 - \lambda_1)\alpha_2 & \frac{1}{\lambda_3}(\lambda_3 - \lambda_1)\alpha_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 - \mathrm{eq}_1 \\ \alpha_2 - \mathrm{eq}_2 \\ \alpha_3 - \mathrm{eq}_3 \end{pmatrix}$$

About some equlibrium point $\vec{\mathrm{eq}} = \langle \mathrm{eq}_1,\ \mathrm{eq}_2,\ \mathrm{eq}_3 \rangle$ with respect to the basis p_1, p_2, p_3.

If we do rotate around an axis, perfectly, say α_1, we have zeros in the other two directions, which means no other rotational motion. However, in the real world it is impossible to perfectly flip an object. We will say that we will rotate the object very close to a principle axis, such that the other axes will be approximately zero. So, we use the approximation of the system through the Jacobian Matrix to analyze this approximation.

3.5 Eigenvalues of Stable Rotation

For the three axes of the system, stable rotation occurs when rotating around either the smallest or largest axis (p_1 *or* p_3). We will do the math for p_1 since p_3 is a similar derivation. This means we are considering equilibrium case where $\alpha_2 = \alpha_3 = 0$ for any α_1

Let's begin the math then. We consider our approximately perfect throw, so our initial velocities are $\alpha_1(0) \neq 0$ and $\alpha_2(0) \approx \alpha_2(0) \approx 0$. Thus our equation for p_1 motion becomes,

$$\begin{aligned} 0 &= \lambda_1 \dot{\alpha_1} + (\lambda_3 - \lambda_2)\alpha_2\alpha_3 \\ 0 &= \lambda_1 \dot{\alpha_1} + \approx 0 \\ 0 &\approx \lambda_1 \dot{\alpha_1} \end{aligned}$$

This implies that α_1 is constant with time(or nearly so), so when applying to the Jacobian, we can eliminate one dimension, yielding,

$$\begin{pmatrix} \dot{\alpha_2} \\ \dot{\alpha_3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\lambda_2}(\lambda_3 - \lambda_1)\alpha_1 \\ \frac{1}{\lambda_3}(\lambda_1 - \lambda_2)\alpha_1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{7}$$

Thus we have a new matrix with new eigenvalues to consider. We'll name this new equation $\dot{x} = Ax$ for simplicity. We now compute the eigenvalues of A to be,

$$\lambda_A = \pm i \sqrt{\frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)\alpha_1^2}{\lambda_2\lambda_3}} \tag{8}$$

Note: We pop out an i because of our assumption of $\lambda_1 > \lambda_2 > \lambda_3$. Thus $(\lambda_1 - \lambda_2)$ is a negative quantity, so we rewrite as $-(\lambda_2 - \lambda_1)$ to keep the inside positive, which in turn, pops out an i. For simplicity, we will denote $\lambda_A = \pm i\Theta$. Now, from differential equations, we know that a solution to x(t) is

$$x(t) = c_1 e^{i\Theta t} + c_2 e^{-i\Theta t} \tag{9}$$

With eigenvectors of $\frac{1}{2}(c_1 e^{i\Theta t} + c_2 e^{-i\Theta t})$ and $\frac{1}{2i}(c_1 e^{i\Theta t} - c_2 e^{-i\Theta t})$ we will observe real solutions that form a center about a point. So, over time, the system will not evolve, but rather revolve around the center axis. Thus, if α_2 and α_3 are initially small, they will remain small, thus we have a mathematical justification for stable rotation.

3.6 Eigenvalues of Unstable Rotation

Now we look at the special case when we rotate about p_2. Proceeding from before, we now have,

$$0 \approx \lambda_2 \dot{\alpha_2}$$

Once again, this implies

$$\begin{aligned} 0 &= \lambda_1 \dot{\alpha_1} + (\lambda_3 - \lambda_2)\alpha_2\alpha_3 \\ 0 &\approx \lambda_2 \dot{\alpha_2} \\ 0 &= \lambda_3 \dot{\alpha_3} + (\lambda_2 - \lambda_1)\alpha_2\alpha_1 \end{aligned}$$

This implies α_2 is nearly constant, so when applying to the Jacobian we obtain a 2x2 matrix, denoted by

$$\begin{pmatrix} \dot{\alpha_1} \\ \dot{\alpha_3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\lambda_1}(\lambda_2 - \lambda_3)\alpha_2 \\ \frac{1}{\lambda_3}(\lambda_1 - \lambda_2)\alpha_2 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} \tag{10}$$

The matrix, which we'll call B, has eigenvalues

$$\lambda_B = \pm \sqrt{\frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_2)\alpha_2^2}{\lambda_1\lambda_3}} \tag{11}$$

Now we observe that no i pops out since both internal quantities are negative. Thus we can rename this matrix $\lambda_B = \pm\Theta$. Since this comes from a differential equation, we now know that a solution is

$$x(t) = c_1 e^{\Theta t} + c_3 e^{-\Theta t} \tag{12}$$

As t evolves, we see the negative theta term decay, but the positive term grows exponentially. Thus the rotational functions of α_1 and α_3 grow with time. As opposed to the other axis, where we saw close rotation by the axis, this exponential growth explains our unstable rotation mathematically.

4 The Details

4.1 Deriving Euler's Equation of Motion

Euler's equation is an expression for torque, so we begin our derivation with the definition of torque. Torque, $\vec{\tau}$, is the time derivative of angular momentum, $\vec{L}$.

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

However, this result assumes an interial reference frame. We however, are using our coordinates $\vec{p_1}, \vec{p_2}, \vec{p_3}$. Our coordinates are defined by the "three principle axes" of the object, so they rotate with the object itself. In other words, the coordinates themselves are functions of time. If an object has angular momentum $\vec{\omega}$, the rate of change of a principle axis, p_i, to a stationary observer is given by:

$$\frac{d\vec{p_i}}{dt} = \vec{\omega} \times \vec{p_i} \tag{13}$$

Now we are prepared to solve for torque in terms of our principle axes.

$$\vec{\tau} = \frac{d}{dt}\left(\vec{L}\right)$$

We now expand $\vec{L}$ out to be $L_1\vec{p_1} + L_2\vec{p_2} + L_3\vec{p_3}$ where L_i is the component of $\vec{L}$ in direction p_i.

$$\vec{\tau} = \frac{d}{dt}\left(L_1\vec{p_1} + L_2\vec{p_2} + L_3\vec{p_3}\right)$$

Both the coordinates and the angular momentum with respect to the coordinates change with time, so Both L_i and p_i are functions of time. Therefore, we apply product rule.

$$\begin{aligned}
\vec{\tau} &= \left(\frac{dL_1}{dt}\vec{p_1} + L_1\frac{d\vec{p_1}}{dt}\right) + \left(\frac{dL_2}{dt}\vec{p_2} + L_2\frac{d\vec{p_2}}{dt}\right) + \left(\frac{dL_3}{dt}\vec{p_3} + L_3\frac{d\vec{p_3}}{dt}\right) \\
\vec{\tau} &= \left(\frac{dL_1}{dt}\vec{p_1} + \frac{dL_2}{dt}\vec{p_2} + \frac{dL_3}{dt}\vec{p_3}\right) + \left(L_1\frac{d\vec{p_1}}{dt} + L_2\frac{d\vec{p_2}}{dt} + L_3\frac{d\vec{p_3}}{dt}\right)
\end{aligned}$$

Note the left hand side is the derivative of $\vec{L}$ with respect to the $\vec{p_i}$ basis vectors. We denote properties with respect to rotating coordinates with a subscripted "r". And we apply the substitution given by equation 13 to the right hand side.

$$\vec{\tau} = \frac{d\vec{L_r}}{dt} + L_1\omega \times \vec{p_1} + L_2\omega \times \vec{p_2} + L_2\omega \times \vec{p_2}$$

Applying the distributive and scalar multiplication properties of the cross product produces the expression for angular momentum in rotating coordinates.

$$\begin{aligned}
\vec{\tau} &= \frac{d\vec{L_r}}{dt} + \omega \times \left(L_1\vec{p_1} + L_2\vec{p_2} + L_3\vec{p_3}\right) \\
\vec{\tau} &= \frac{d\vec{L_r}}{dt} + \vec{\omega} \times \vec{L}_r \qquad (14)
\end{aligned}$$

We now wish to make the substitution $L_r = M\vec{w}$ for some matrix M. To arrive at this result, we use a common formula for angular momentum of a point mass, $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r} = \langle x, y, z\rangle$ is the displacement of the mass from the origin. The equation holds for r of arbitrary coordinates, so we will leave the coordinate choice general for now. $\vec{p}$ is the linear momentum of the mass. Additionally, we know that $\vec{p} = m(\vec{w} \times \vec{r})$ for a point mass of mass m and angular velocity $\vec{w} = \langle w_x, w_y, w_z\rangle$. Now, we solve for $\vec{L}$.

$$\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} = \vec{r} \times (m(\vec{w} \times \vec{r})) \\
\vec{L} &= m\langle x, y, z\rangle \times (\langle w_x, w_y, w_z\rangle \times \langle x, y, z\rangle) \\
\vec{L} &= m\langle x, y, z\rangle \times \langle zw_y - yw_z, xw_z - zw_x, yw_x - xw_y\rangle \\
\vec{L} &= m\begin{pmatrix} y(yw_x - xw_y) - z(xw_z - zw_x) \\ z(zw_y - yw_z) - x(yw_x - xw_y) \\ x(xw_z - zw_x) - y(zw_y - yw_z) \end{pmatrix}
\end{aligned}$$

We now distinguish the w_x, w_y, and w_z terms in each entry of $\vec{L}$ to back out a matrix

$$\vec{L} = m\begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix}\begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} \qquad (15)$$

However, we are not interested in the angular momentum of a point mass. Rather, we are interested in the angular momentum of the object as a whole. A small contribution to the angular momentum, $d\vec{L}$ comes from a small portion of mass $dm = \rho \cdot dV$. So, we substitute m with $\rho \cdot dV$ in equation 15. Then, we integrate over the region Ω, which could describe a region such as a hammer. This yields,

$$\vec{L} = \begin{pmatrix} \int_\Omega (y^2+z^2)\rho dV & \int_\Omega -xy\rho dV & \int_\Omega -xz\rho dV \\ \int_\Omega -xy\rho dV & \int_\Omega (x^2+z^2)\rho dV & \int_\Omega -yz\rho dV \\ \int_\Omega -xz\rho dV & \int_\Omega -yz\rho dV & \int_\Omega (x^2+y^2)\rho dV \end{pmatrix} \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} \tag{16}$$

We give the above matrix the name M and arrive at the result $\vec{L} = M\vec{w}$. Recall that our choice of $\vec{r} = \langle x, y, z \rangle$ determines the coordinates of the equation. We can choose our $\vec{r}$ to follow our rotating coordinates given by p_i. We can now write the radius variable as $\vec{r} = x\vec{p_1} + y\vec{p_2} + z\vec{p_3}$ and we arrive at our result $\vec{L_r} = M\vec{\omega}$. Substituting the L_r values from equation 14 with $\vec{L_r} = M\vec{\omega}$ that follows from equation 16 gives Euler's equation. Note that M is constant with time.

$$\begin{aligned} \vec{\tau} &= \frac{d}{dt}(M\vec{\omega}) + \vec{\omega} \times (M\vec{\omega}) \\ \vec{\tau} &= M\dot{\omega} + \omega \times M\omega \end{aligned} \tag{Euler's Equation}$$

4.2 Confirming Positive Eigenvalues

In using Euler's equation, we made the assumption that the matrix M, the moment of inertia tensor, has three distinct positive eigenvalues. We will now mathematically justify this claim. In particular, if λ_1, λ_2, λ_3 are the eigenvalues of M, we will show that $0 > \lambda_1 > \lambda_2 > \lambda_3$. The matrix M is the matrix such that $\vec{L} = M\omega$ as in equation 16. The following inner product definition will allow a simplification of our matrix M.

$$\langle u, v \rangle_\Omega = \int_\Omega (u \cdot v \cdot \rho) dV$$

"Ω" represents some region of integration such as a hammer. "u" and "v" each represent some coordinate value of an orthornormal basis centered at the origin. "ρ" is some function describing the density of the object. We now rewrite M in the following way.

$$M = \begin{pmatrix} \|y\|_\Omega^2 + \|z\|_\Omega^2 & -\langle x, y \rangle_\Omega & -\langle x, z \rangle_\Omega \\ -\langle x, y \rangle_\Omega & \|x\|_\Omega^2 + \|z\|_\Omega^2 & -\langle y, z \rangle_\Omega \\ -\langle x, z \rangle_\Omega & -\langle y, z \rangle_\Omega & \|x\|_\Omega^2 + \|y\|_\Omega^2 \end{pmatrix}$$

Notice that we can now separate the matrix in the following way.

$$M = (\|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2) \cdot I - \begin{pmatrix} \langle x, x \rangle_\Omega & \langle x, y \rangle_\Omega & \langle x, z \rangle_\Omega \\ \langle x, y \rangle_\Omega & \langle y, y \rangle_\Omega & \langle y, z \rangle_\Omega \\ \langle x, z \rangle_\Omega & \langle y, z \rangle_\Omega & \langle z, z \rangle_\Omega \end{pmatrix} \tag{17}$$

Where I is the identity matrix. We will name the matrix of inner products to the right as "N".

$$M = (\|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2) \cdot I - N$$

An eigenvalue of M is a λ such that $\lambda u = Mu$ for some eigenvector u. So, we multiply the right side of our expression by some vector u. We will declare u to be of unit length.

$$Mu = \lambda u = ((\|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2) \cdot I - N)u$$

The matrix N has a lot of structure, so we will solve for Nu.

$$\begin{aligned} \lambda u &= (\|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2) \cdot u - Nu \\ Nu &= (\|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2) \cdot u - \lambda u \end{aligned}$$

Now we will take the magnitude of both sides keeping in mind u is unit length.

$$\|Nu\| = \|(\|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2 - \lambda)\| \tag{18}$$

Confirming a positive λ requires information about $\|Nu\|$. So, we will break down the left hand side of the above equation. We now declare a matrix $A = \begin{pmatrix}\mathbf{x} & \mathbf{y} & \mathbf{z}\end{pmatrix}$. Each entry of the row in A is a vector from our abstract vector space. In matrix multiplication, one vector times another will produce the dot product of the two vectors. This mirrors classic matrix multiplication when the vector has a finite amount of entries and is expaded down each column of the matrix. Notice that N, the matrix from equation 17, can be separated into the form $N = A^T A$.

$$N = \begin{pmatrix} \langle x,x\rangle_\Omega & \langle x,y\rangle_\Omega & \langle x,z\rangle_\Omega \\ \langle x,y\rangle_\Omega & \langle y,y\rangle_\Omega & \langle y,z\rangle_\Omega \\ \langle x,z\rangle_\Omega & \langle y,z\rangle_\Omega & \langle z,z\rangle_\Omega \end{pmatrix} = \begin{pmatrix}\mathbf{x} \\ \mathbf{y} \\ \mathbf{z}\end{pmatrix} \begin{pmatrix}\mathbf{x} & \mathbf{y} & \mathbf{z}\end{pmatrix} = A^T A \tag{19}$$

We now stop and introduce the concept of the "operator norm" of a matrix. Consider a linear transformation, B. Consider a unit sphere in the pre-image space. We apply the transformation B, which stretches and deforms the sphere, creating some shape in the image. The operator norm is simply the farthest distance a point on the surface in the image ends up from the origin. The operator norm of B would be denoted $\|B\|_*$ Formally, the operator norm is defined to be:

$$\|B\|_* = \max_{\|x\|=1} \|Bx\| \tag{Operator Norm}$$

Where u is any vector of magnitude 1. We will make use of a few operator norm properties.

$$\|Ax\| \le \|A\|_* \|x\| \tag{20}$$

This result follows from $\|Au\| \le \|A\|_*$. For unit u. So if $x = \|x\|u$, it follows that $\|Ax\| \le \|A\|_* \|x\|$ by direct substitution. The second property is as follows:

$$\|AB\|_* \le \|A\|_* \|B\|_* \tag{21}$$

To demonstrate this inequality, consider $\|ABu\|$ for unit u. The product Bu will result in a new vector v. Now we have $\|ABu\| = \|Av\| \le \|A\|_* \|v\|$ by equation 20. By the definition of v, $\|v\| \le \|B\|_*$ which we can substitute to arrive at $\|AB\|_* \le \|A\|_* \|B\|_*$. We now turn to the third property.

$$\|A\|_* = \|A^T\|_* \text{ if } AA^T = A^T A \tag{22}$$

To verify this result, we will evaluate $\|A^T x\|^2$ and $\|Ax\|^2$ and demonstrate equality. $\|A^T x\|^2 = (A^T x)^T A^T x = x^T A A^T x$. Similarly, $\|Ax\|^2 = (Ax)^T Ax = x^T A^T A x$. Under the condition that $AA^T = A^T A$, the equality clearly holds. The two matrices thus produce the same magnitude for any input vector x, including the x that produces the maximum. Therefore, they will share the same operator norm.

We now return to equation 19. We are interested in simplifying $\|Nu\| = \|A^T A u\|$. Recall that we defined u to be unit. By the definition of the operator norm, $\|A^T A u\| \leq \|A^T A\|_*$. By equation 21, $\|A^T A\|_* \leq \|A^T\|_* \|A\|_*$. And by equation 22, $\|A^T\|_* \|A\|_* = \|A^T\|_*^2$. Note that the defintion of A satisfies the condition $AA^T = A^T A$. Thus, we have

$$\|Nu\| \leq \|A^T\|_*^2 \tag{23}$$

The value $\|A^T\|_*$ is the maximum value of $\|A^T u\|$ for a unit v. So, we now examine $A^T v$.

$$A^T v = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} (\mathbf{v}) = \begin{pmatrix} \langle x, v \rangle_\Omega \\ \langle y, v \rangle_\Omega \\ \langle z, v \rangle_\Omega \end{pmatrix}$$

We find the value of $\|A^T v\|^2$ follows from the above resulting vector.

$$\begin{aligned} \|A^T v\|^2 &= \langle x, v \rangle_\Omega^2 + \langle y, v \rangle_\Omega^2 + \langle z, v \rangle_\Omega^2 \\ \|A^T v\|^2 &\leq \|x\|_\Omega^2 \|v\|_\Omega^2 + \|y\|_\Omega^2 \|v\|_\Omega^2 + \|z\|_\Omega^2 \|v\|_\Omega^2 \end{aligned}$$

We declared v unit, so $\|v\|_\Omega^2 = 1$. Furthermore, equality only holds if vectors x, y, and z are all in the same direction as v which only occurs in a one dimensional object. Therefore, we can ditch the equality case. We now have,

$$\|A^T v\|^2 < \|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2 \tag{24}$$

By equation 23 and the above,

$$\|Nu\| < \|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2 \tag{25}$$

We can now make a substitution for Nu given by equation 18 and arrive at the following:

$$\|(\|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2 - \lambda)\| < \|x\|_\Omega^2 + \|y\|_\Omega^2 + \|z\|_\Omega^2 \tag{26}$$

This equality can only hold if $\lambda > 0$. We have now verified positive eigenvalues for the matrix M that appears in Euler's equation, justifying the assumption in our previous argument.