Hölder Spaces

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Motivation via Arzelà-Ascoli

Suppose we have a sequence of functions u_n and wish to prove that this sequence converges uniformly (we often encounter this, for example, when solving PDEs). If we are working on a compact subset $X \subset \mathbb{R}^n$, the Arzelà-Ascoli theorem provides a convenient criteria for uniform convergence: pointwise boundedness and equicontinuity. Recall these are defined as follows, letting $C(X)$ denote the space of continuous functions $X \to \mathbb{R}$.

Def (pointwise boundedness). A subset $\mathcal{F} \subset C(X)$ is *pointwise bounded* if for all $x \in X$, the set $\{u(x) : u \in \mathcal{F}\} \subset \mathbb{R}$ is bounded.

Def (equicontinuity). A subset $\mathcal{F} \subset C(X)$ is *equicontinuous* if for all $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies we have the bound $\|u(x) - u(y)\| < \varepsilon$ for all $u \in \mathcal{F}$.

Importantly, given such a x and ε as above, the same δ must work for all $u \in \mathcal{F}$. Keep in mind the subset $\mathcal{F}$ above is often simply a sequence u_n of continuous functions. The Arzelà-Ascoli theorem then promises

Theorem (Arzelà-Ascoli). If $\mathcal{F} \subset C(X)$ is equicontinuous and pointwise bounded, then there exists a subsequence $u_k \subset \mathcal{F}$ such that $u_k \to u \in C(X)$ uniformly.

Equicontinuity is generally harder than pointwise boundedness to verify when using Arzelà-Ascoli. A convenient sufficient condition for equicontinuity is there exists a fixed $C > 0$ so that

$$\|u(x) - u(y)\| \leq C\|x - y\| \tag{1}$$

for all $u \in \mathcal{F}$. This immediately implies equicontinuity because for any $\varepsilon > 0$ we can simply take $\|x-y\| < \varepsilon/C$. However, an even weaker sufficient condition which is often easier to check is that for some fixed $\alpha \in (0, 1]$ there exists $C > 0$ such that

$$\|u(x) - u(y)\| \leq C\|x - y\|^\alpha \tag{2}$$

for all $u \in \mathcal{F}$. In this case, we get equicontinuity by taking $\|x - y\| < (\varepsilon/C)^{1/\alpha}$ for any given $\varepsilon > 0$. If u satisfies (1), it is called *Lipshitz continuous* and if u satisfies (2), it is called *α-Hölder continuous*. This terminology is due to the following exercise.

Exercise. Show that if $u : \mathbb{R}^n \to \mathbb{R}$ satisfies either (1) or (2), it is a continuous function.

Given an α-Hölder continuous function $u : X \to \mathbb{R}$, the minimum constant C is given by the *α-Hölder semi-norm*

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha}.$$

This semi-norm is convenient because $\{[u]_\alpha : u \in \mathcal{F}\}$ bounded implies $\mathcal{F}$ is equicontinuous. Additionally, note that if $\{\|u\|_{\sup} : u \in \mathcal{F}\}$ is bounded, then $\mathcal{F}$ is pointwise bounded where $\|\cdot\|_{\sup}$ denotes the uniform norm. This motivates defining the *α-Hölder norm*

$$\|u\|_\alpha := \|u\|_{\sup} + [u]_\alpha$$

because if $\{\|u\|_\alpha : u \in \mathcal{F}\}$ is bounded, then $\mathcal{F}$ is equicontinuous and pointwise bounded, so Arzelà-Ascoli applies and there exists $u_k \in \mathcal{F}$ such that $u_k \to u \in C(X)$ uniformly.

Hölder spaces

The point is the α-Hölder norm $\|\cdot\|_\alpha$ is useful because it allows us to conclude uniform convergence due to Arzelà-Ascoli. We can rephrase this oberservation using the language of functional analysis, which begins by considering the function space of α-Hölder continuous functions.

Def (Hölder space). Given a subset $\Omega \subset \mathbb{R}^n$, the *Hölder space* $C^\alpha(\Omega)$ for some $\alpha \in (0,1]$ is

$$C^\alpha(\Omega) := \{u \in C(\Omega) : \|u\|_\alpha < \infty\}.$$

Prop. The *Hölder space* $C^\alpha(\Omega)$ is a Banach space

Compact embedding theorems

/recall motivation for Holder spaces... rephrase in terms of compact embeddings/

$C^{0,\alpha} \to C^0$

$C^{0,\alpha} \to C^{0,\beta}$

$C^{k,\alpha} \to C^k$ */hopefully using earlier embedding/*

$C^{k,\alpha} \to C^{k,\beta}$ */hopefully using earlier embedding/*