Hölder Spaces

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Motivation via Arzelà-Ascoli

Suppose we have a sequence of functions u_n and wish to prove that this sequence converges uniformly (we often encounter this, for example, when solving PDEs). If we are working on a compact subset $X \subset \mathbb{R}^n$, the Arzelà-Ascoli theorem provides a convenient criteria for uniform convergence: pointwise boundedness and equicontinuity. Recall these are defined as follows, letting $C(X)$ denote the space of continuous functions $X \to \mathbb{R}$.

Def (pointwise boundedness). A subset $\mathcal{F} \subset C(X)$ is *pointwise bounded* if for all $x \in X$, the set $\{u(x) : u \in \mathcal{F}\} \subset \mathbb{R}$ is bounded.

Def (equicontinuity). A subset $\mathcal{F} \subset C(X)$ is *equicontinuous* if for all $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies we have the bound $|u(x) - u(y)| < \varepsilon$ for all $u \in \mathcal{F}$.

Importantly, given such a x and ε as above, the same δ must work for all $u \in \mathcal{F}$. Keep in mind the subset $\mathcal{F}$ above is often simply a sequence u_n of continuous functions. The Arzelà-Ascoli theorem then promises

Theorem (Arzelà-Ascoli). If $\mathcal{F} \subset C(X)$ is equicontinuous and pointwise bounded, then there exists a subsequence $u_k \subset \mathcal{F}$ such that $u_k \to u \in C(X)$ uniformly.

Equicontinuity is generally harder than pointwise boundedness to verify when using Arzelà-Ascoli. A convenient sufficient condition for equicontinuity is there exists a fixed $C > 0$ so that

$$|u(x) - u(y)| \leq C\|x - y\| \tag{1}$$

for all $u \in \mathcal{F}$. This immediately implies equicontinuity because for any $\varepsilon > 0$ we can simply take $|x-y| < \varepsilon/C$. However, an even weaker sufficient condition which is often easier to check is that for some fixed $\alpha \in (0, 1]$ there exists $C > 0$ such that

$$|u(x) - u(y)| \leq C\|x - y\|^\alpha \tag{2}$$

for all $u \in \mathcal{F}$. In this case, we get equicontinuity by taking $\|x - y\| < (\varepsilon/C)^{1/\alpha}$ for any given $\varepsilon > 0$. If u satisfies (1), it is called *Lipshitz continuous* and if u satisfies (2), it is called *α-Hölder continuous*. This terminology is due to the following.

Proposition. If $u : X \to \mathbb{R}$ satisfies either (1) or (2), it is a continuous function.

Proof. Suppose u is α-Hölder continuous with Hölder constant $C > 0$. Then or any $\varepsilon > 0$, taking $\|x - y\| < (\varepsilon/C)^{1/\alpha}$ guarantees

$$|u(x) - u(y)| \leq C|x - y|^\alpha < \varepsilon.$$

□

Given an α-Hölder continuous function $u : X \to \mathbb{R}$, the minimum constant C is given by the *α-Hölder semi-norm*

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha}.$$

This semi-norm is convenient because $\{[u]_\alpha : u \in \mathcal{F}\}$ bounded implies $\mathcal{F}$ is equicontinuous. Additionally, note that if $\{\|u\|_{\sup} : u \in \mathcal{F}\}$ is bounded, then $\mathcal{F}$ is pointwise bounded where $\|\cdot\|_{\sup}$ denotes the uniform norm.

This motivates defining the *α-Hölder norm*

$$\|u\|_\alpha := \|u\|_{\sup} + [u]_\alpha$$

because if $\{\|u\|_\alpha : u \in \mathcal{F}\}$ is bounded, then $\mathcal{F}$ is equicontinuous and pointwise bounded, so Arzelà-Ascoli applies and there exists $u_k \in \mathcal{F}$ such that $u_k \to u \in C(X)$ uniformly.

The point is the α-Hölder norm $\|\cdot\|_\alpha$ is useful because it allows us to conclude uniform convergence due to Arzelà-Ascoli. We can rephrase this oberservation using the language of functional analysis, which begins by considering the function space of α-Hölder continuous functions.

Def (Hölder space). For a compact subset $X \subset \mathbb{R}^n$ and $\alpha \in (0, 1]$, the *Hölder space* $C^\alpha(X)$ is

$$C^\alpha(X) := \{u \in C(X) : \|u\|_\alpha < \infty\}.$$