Linear Algebra Exam

April 20, 2019

Cayley-Hamilton Theorem

Theorem 1. $T : V \to V$ *linear operator* $\to \phi_T(T) = \mathbf{O}$

Proof. Let $\mathbf{v}$ *in* V be given. If $\mathbf{v} = \mathbf{O},$ *then* $\phi_T(T)\mathbf{v} = \mathbf{O}$

$\because \phi_T(T) : linear$

Suppose that $\mathbf{v} \neq \mathbf{O}$, Let W be the T-cyclic subspace of V (dim W = k)

Enough To Show) $\phi_{T|_W}(T) = \mathbf{O}$

($\because \phi_{T|_W} \mid \phi_T(t)$)

$\phi_T(T)\mathbf{v} \ = \ q(T)\phi_{T|_W}(T)\mathbf{v} \ = \ \mathbf{O}$

Since $\{\mathbf{v}, T\mathbf{v}, \cdots, T^{k-1}\mathbf{v}\}$: basis for$W$ there are $a_0, \ldots, a_{k-1}$ *in* $\mathbb{R}$ s.t.

$$a_0\mathbf{v} + a_1T\mathbf{v} + \cdots + a_{k-1}T^{k-1}\mathbf{v} + T^k\mathbf{v} = O$$

Also, we know $\phi_{T|W}(t) = t^k + a_{k-1}T^{k-1} + \cdots + a_1t + a_0$

Thus $\phi_{T|W}(T)\mathbf{v} = (T^k + \cdots + a_0I)\mathbf{v} = 0$

□

Chapter 6. Inner Product Space

V : $\mathbb{C}$ - vector space

$\langle , \rangle \ : \ \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ inner product if

$(\mathbf{v}, \mathbf{w}) \ \mapsto \langle , \rangle(\mathbf{v}, \mathbf{w}) =: \langle \mathbf{v}, \mathbf{w} \rangle$

(1)(positive-definite) $\langle \mathbf{v}, \mathbf{v} \rangle \geqslant 0$, $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \leftrightarrow \mathbf{v} = \mathbf{0}$

(2)(conjugate symmetry) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \overline{\mathbf{w}, \mathbf{v}} \rangle$

(3)(linearity on the first component)

$\langle \mathbf{v_1} + \mathbf{v_2}, \mathbf{w} \rangle = \langle \mathbf{v_1}, \mathbf{w} \rangle + \langle \mathbf{v_2}, \mathbf{w} \rangle$

c$\cdot\langle \mathbf{v}, \mathbf{w} \rangle$ = c $\langle \mathbf{v}, \mathbf{w} \rangle$, c $\in \mathbb{C}$

e.g.

1. $\mathbb{R}^{\mathrm{n}}$: $\mathbb{R}$ - vector space

   $\mathbf{a} := (a_1, \ldots, a_n) \in \mathbb{R}^{\mathrm{n}}$, $\mathbf{b} := (b_1, \ldots, b_n) \in \mathbb{R}^{\mathrm{n}}$

   $\mathbf{a} \cdot \mathbf{b} := \sum_{i=1}^{n} a_ib_i$ -> inner product?

   (1) $<\mathbf{a}, \mathbf{a}> = \sum_{i=1}^{n} a_i \geqslant 0$, $<\mathbf{a}, \mathbf{a}> = 0 \leftrightarrow \mathbf{a} = \mathbf{0}$

   (2) $<\mathbf{a}, \mathbf{b}> = \sum_{i=1}^{n} a_ib_i = \sum_{i=1}^{n} b_ia_i = <\mathbf{b}, \mathbf{a}>$

   (3) linear -> O.K

2. $C([0,1]) :=$ { all complex valued continuoun functions defined on [0,1] }
$= \{f : [0,1] \to \mathbb{C} \mid f : continuous\}$

$\langle f,\ g\rangle := \int_0^1 f(x)\overline{g(x)}dx \in \mathbb{C}$

(1)

(2) $\overline{\langle g, f\rangle} = \overline{\int_0^1 g(x)\overline{f(x)}\ dx} = \int_0^1 \overline{g(x)\overline{f(x)}}\ dx = \int_0^1 f(x)\overline{g(x)}\ dx\ \ =\ \ \langle f, g\rangle$

(3) $\langle f_1 + f_2, g\rangle = \int_0^1 (f_1(x) + f_2(x))\overline{g(x)}dx = \int_0^1 f_1(x)\overline{g(x)} + \int_0^1 f_2(x)\overline{g(x)}\ dx = \langle f_1, g\rangle + \langle f_2, g\rangle$

$\langle c \cdot f, g\rangle = \int_0^1 c \cdot f(x) \cdot \overline{g(x)}\ dx = c \cdot \int_0^1 f(x)\overline{g(x)}\ dx = c\langle f, g\rangle$

3. $A \in M_{n\times n}(\mathbb{C})$

$\langle A, B\rangle := tr(B^*A) \in \mathbb{C}$

(1) A $= (a_{ij}),\ A^* = (\overline{a_{ji}})$
$(A^*A)_{kk} = \sum_{l=1}^n [A^*]_{kl}[A]_{lk}$
$\because [A^*]_{kl} = \overline{a_{lk}} = \sum_{l=1}^n \overline{a_{lk}}a_{lk} = \sum_{l=1}^n \mid a_{kl} \mid^2 \geqslant 0$

$tr(A^*A) = \sum_{k=1}^n(\sum_{l=1}^n \mid a_{lk} \mid^2) \geqslant 0$

(2) $\overline{\langle B, A\rangle} = \overline{tr(A^*B)} = tr\overline{(A * B)^t} = tr((A^*B)^*) = tr(B^*A) = \langle A, B\rangle$