

Linear Algebra Class on 26 January

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2.1 Linear Transformation

Definition. \mathbf{V}, \mathbf{W} are vector spaces over \mathbb{R} . $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ is \mathbb{R} -linear map if $\mathbf{L}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{L}\mathbf{v}_1 + \mathbf{L}\mathbf{v}_2$ and $\mathbf{L}(c\mathbf{v}) = c\mathbf{L}\mathbf{v}$. In other words \mathbf{L} is called vector space homomorphism.

Properties

1. $\mathbf{L}\mathbf{0} = \mathbf{0}$

2. $\mathbf{L}(-\mathbf{v}) = -\mathbf{L}\mathbf{v}$

3. $\mathbf{L}(a\mathbf{v}_1 + b\mathbf{v}_2) = a\mathbf{L}\mathbf{v}_1 + b\mathbf{L}\mathbf{v}_2$

1. *Proof.* $\mathbf{L}\mathbf{0} = \mathbf{L}(\mathbf{0} + \mathbf{0}) = \mathbf{L}\mathbf{0} + \mathbf{L}\mathbf{0}$ □

2. *Proof.* $\mathbf{L}(-\mathbf{v}) = \mathbf{L}(-1 \cdot \mathbf{v}) = -\mathbf{L}\mathbf{v}$ □

3. $\mathbf{L}(a\mathbf{v}_1 + b\mathbf{v}_2) = \mathbf{L}(a\mathbf{v}_1) + \mathbf{L}(b\mathbf{v}_2) = a\mathbf{L}\mathbf{v}_1 + b\mathbf{L}\mathbf{v}_2$

Remark. $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for \mathbf{V} and $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ is linear. Then $\mathbf{L}(\sum_{i=1}^n a_i \mathbf{v}_i) = \sum_{i=1}^n a_i \mathbf{L}\mathbf{v}_i$
 $\therefore \mathbf{L}$ is completely determined by $\mathbf{L}\mathbf{v}_i$ for $i = 1, \dots, n$

Theorem (Linear Extension Theorem). \mathbf{V}, \mathbf{W} is vector space. Let $\mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for \mathbf{V} and $f : \mathfrak{B} \longrightarrow \mathbf{W}$ and be a function. Then There is unique $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ linear map such that $\mathbf{L}|_{\mathfrak{B}} = f(\iff \mathbf{L}\mathbf{v}_i = f(\mathbf{v}_i)$ for $i = 1, \dots, n$)

Proof. 1. existence

Define $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ by $\mathbf{L}(\sum_{i=1}^n a_i \mathbf{v}_i) := \sum_{i=1}^n a_i f(\mathbf{v}_i)$. For each $i = 1, \dots, n$

$$\begin{aligned}\mathbf{L}\mathbf{v}_i &= \mathbf{L}(0\mathbf{v}_1 + \dots + 1\mathbf{v}_i + \dots + 0\mathbf{v}_n) \\ &= 0f(\mathbf{v}_1) + \dots + f(\mathbf{v}_i) + \dots + 0f(\mathbf{v}_n) \\ &= f(\mathbf{v}_i) \\ \therefore \mathbf{L}|_{\mathfrak{B}} &= f\end{aligned}$$

2. linearity of \mathbf{L}

For $\sum_{i=1}^n a_i \mathbf{v}_i, \sum_{i=1}^n b_i \mathbf{v}_i \in \mathbf{V}$

$$\begin{aligned}
 \mathbf{L}\left(\sum_{i=1}^n a_i \mathbf{v}_i + \sum_{i=1}^n b_i \mathbf{v}_i\right) &= \mathbf{L}\left(\sum_{i=1}^n (a_i + b_i) \mathbf{v}_i\right) \\
 &= \sum_{i=1}^n (a_i + b_i) \mathbf{L} \mathbf{v}_i \\
 &= \sum_{i=1}^n a_i \mathbf{L} \mathbf{v}_i + \sum_{i=1}^n b_i \mathbf{L} \mathbf{v}_i \\
 &= \sum_{i=1}^n a_i f(\mathbf{v}_i) + \sum_{i=1}^n b_i f(\mathbf{v}_i) \\
 &= \mathbf{L}\left(\sum_{i=1}^n a_i \mathbf{v}_i\right) + \mathbf{L}\left(\sum_{i=1}^n b_i \mathbf{v}_i\right)
 \end{aligned}$$

3. uniqueness

Let $M : \mathbf{V} \longrightarrow \mathbf{W}$ be linear map such that $M|_{\mathfrak{B}} = f$. For any $\sum_{i=1}^n a_i \mathbf{v}_i \in \mathbf{V}$,

$$\begin{aligned}
 \mathbf{L}\left(\sum_{i=1}^n a_i \mathbf{v}_i\right) &= \sum_{i=1}^n a_i f(\mathbf{v}_i) \\
 &= \sum_{i=1}^n a_i M \mathbf{v}_i \\
 &= M\left(\sum_{i=1}^n a_i \mathbf{v}_i\right) \\
 \therefore \mathbf{L} &= M
 \end{aligned}$$

□

Definition.

Let $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ be linear map. $\ker \mathbf{L} := \{\mathbf{v} \in \mathbf{V} \mid \mathbf{L} \mathbf{v} = \mathbf{0}\}$ $\text{im} \mathbf{L} := \{\mathbf{L} \mathbf{v} \in \mathbf{W} \mid \mathbf{v} \in \mathbf{V}\}$

Theorem. Let $\varphi : \mathbf{V} \longrightarrow \mathbf{W}$ be linear map. Then $\ker \varphi = \mathbf{0} \iff \varphi$ is one to one.

Proof. \implies Suppose that $\varphi \mathbf{v}_1 = \varphi \mathbf{v}_2$, $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$. By linearity $\varphi(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$, which implies that $\mathbf{v}_1 - \mathbf{v}_2 \in \ker \varphi$. Since $\ker \varphi = \mathbf{0}$, $\mathbf{v}_1 = \mathbf{v}_2$

\impliedby Let $\mathbf{v} \in \ker \varphi$ be given. Since φ is linear map, $\varphi \mathbf{0} = \mathbf{0}$. Also $\varphi \mathbf{v} = \mathbf{0}$ for $\mathbf{v} \in \ker \varphi$. Since φ is one to one, $\varphi \mathbf{v} = \varphi \mathbf{0} \implies \mathbf{v} = \mathbf{0} \therefore \varphi$ is one to one. □

Theorem. Let $\varphi : \mathbf{V} \longrightarrow \mathbf{W}$ be linear map. φ is one-to-one $\iff [\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \text{ is linearly independent subset of } \mathbf{V} \implies \{\varphi \mathbf{v}_1, \dots, \varphi \mathbf{v}_k\} \text{ is linearly independent subset of } \mathbf{W}]$

Proof. \implies Suppose that $\sum_{i=1}^k a_i \varphi(\mathbf{v}_i) = \mathbf{0}$. Then left-hand-side is $\varphi\left(\sum_{i=1}^k a_i \mathbf{v}_i\right) = \mathbf{0}$. Since φ is one-to-one $\sum_{i=1}^k a_i \mathbf{v}_i = \mathbf{0}$. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent, all a_i is zero. $\therefore \{\varphi \mathbf{v}_1, \dots, \varphi \mathbf{v}_k\}$ is linearly independent.

independent.

\Leftarrow Suppose $\varphi \mathbf{v} = \varphi \mathbf{v}'$

$$\varphi \mathbf{v} - \varphi \mathbf{v}' = \mathbf{0}$$

$$\varphi(\mathbf{v} - \mathbf{v}') = \mathbf{0}$$

$$\mathbf{v} - \mathbf{v}' = \mathbf{0}$$

Suppose $\mathbf{v} - \mathbf{v}' \neq \mathbf{0}$. Then $\{\mathbf{v} - \mathbf{v}'\}$ is linearly independent subset of \mathbf{V} , but $\{\varphi(\mathbf{v} - \mathbf{v}')\} = \{\mathbf{0}\}$ is linearly dependent subset of \mathbf{W} . But it contradicts to the assumption. $\therefore \mathbf{v} = \mathbf{v}'$

φ is one-to-one. □

Theorem. Let $\varphi : \mathbf{V} \rightarrow \mathbf{W}$ be linear map.

φ is onto $\iff [\text{span} S = \mathbf{V} \implies \text{span} \varphi S = \mathbf{W}]$

Proof. \implies Put $S := \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ then $\varphi S := \{\varphi \mathbf{v}_1, \dots, \varphi \mathbf{v}_k\}$

Suppose $\mathbf{w} \in \mathbf{W}$ be given. Since φ is onto, there are some $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{w} = \varphi \mathbf{v}$. Since S is

spanning set, so $\mathbf{v} = \sum_{i=1}^k a_i \mathbf{v}_i$ and $\mathbf{w} = \varphi(\sum_{i=1}^k a_i \mathbf{v}_i)$. Since φ is linear map, $\mathbf{w} = \sum_{i=1}^k a_i \varphi \mathbf{v}_i$

$\therefore \mathbf{w} \in \text{span} \varphi S$

\Leftarrow Suppose $S := \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $\text{span} S = \mathbf{V}$. Then $\text{span} \varphi S = \mathbf{W}$ by the assumption. Since φS is

spanning set of \mathbf{W} , $\mathbf{w} = \sum_{i=1}^k a_i \varphi \mathbf{v}_i = \varphi(\sum_{i=1}^k a_i \mathbf{v}_i)$

$\therefore \varphi$ is onto. □

Corollary. Let $\varphi : \mathbf{V} \rightarrow \mathbf{W}$ be linear map.

φ is bijective $\iff [\mathfrak{B} \text{ is a basis for } \mathbf{V} \implies \varphi \mathfrak{B} \text{ is a basis for } \mathbf{W}]$

Proof. By previous two theorems, it's trivial to show that φ is bijective $\iff [\mathfrak{B} \text{ is a basis for } \mathbf{V} \implies \varphi \mathfrak{B} \text{ is a basis for } \mathbf{W}]$ □