Linear Algebra class on $9^{\textsf{th}}$ March

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1 Elementary operations Elementary matrices

(1) row multiplying transformations

$$\begin{pmatrix} \boldsymbol{r_1} \\ \vdots \\ \boldsymbol{r_i} \\ \vdots \\ \boldsymbol{r_m} \end{pmatrix} \longrightarrow \begin{pmatrix} \boldsymbol{r_1} \\ \vdots \\ \mathrm{c}\boldsymbol{r_i} \\ \vdots \\ \boldsymbol{r_m} \end{pmatrix} \cdots \begin{pmatrix} \boldsymbol{e_1} \\ \vdots \\ \mathrm{c}\boldsymbol{e_i} \\ \vdots \\ \boldsymbol{e_m} \end{pmatrix}$$

(2) row switching transformations

$$\begin{pmatrix} \boldsymbol{r_1} \\ \vdots \\ \boldsymbol{r_i} \\ \vdots \\ \boldsymbol{r_j} \\ \vdots \\ \boldsymbol{r_m} \end{pmatrix} \longrightarrow \begin{pmatrix} \boldsymbol{r_1} \\ \vdots \\ \boldsymbol{r_j} \\ \vdots \\ \boldsymbol{r_i} \\ \vdots \\ \boldsymbol{r_m} \end{pmatrix} \cdots \begin{pmatrix} \boldsymbol{e_1} \\ \vdots \\ \boldsymbol{e_j} \\ \vdots \\ \boldsymbol{e_i} \\ \vdots \\ \boldsymbol{e_m} \end{pmatrix}$$

(3) row addition transformations

$$\begin{pmatrix} \boldsymbol{r_1} \\ \vdots \\ \boldsymbol{r_i} \\ \vdots \\ \boldsymbol{r_j} \\ \vdots \\ \boldsymbol{r_m} \end{pmatrix} \longrightarrow \begin{pmatrix} \boldsymbol{r_1} \\ \vdots \\ \boldsymbol{r_i} + c\boldsymbol{r_j} \\ \vdots \\ \boldsymbol{r_j} \\ \vdots \\ \boldsymbol{r_m} \end{pmatrix} \cdots \begin{pmatrix} \boldsymbol{e_1} \\ \vdots \\ \boldsymbol{e_i} + c\boldsymbol{e_j} \\ \vdots \\ \boldsymbol{e_j} \\ \vdots \\ \boldsymbol{e_m} \end{pmatrix}$$

2 Row space, column space, rank

$A \in \boldsymbol{M}_{m\times n}(\mathbb{R})$
R(A)$:=$ span$\{\boldsymbol{r_1,\dots,r_m}\}$ row space
C(A)$:=$ $\{$span$\boldsymbol{C_1,\dots,C_n}\}$ column space
N(a) $:= \{\boldsymbol{x} \in \mathbb{R} | A\boldsymbol{x} = \boldsymbol{0}\}$ null space dimR(A) $:=$ row rank, dimN(A) $:=$ nullity of A, dimC(A) $:=$ column rank

Theorem 2.1 (Rank Theorem)**.** *row rank = column rank*

By the theorem of 2.1, define rankA $:=$ column rank and write rankA.
Note that rankA $\le min$(m,n) and A has full rank if rankA = m or n

Note.
$L : \boldsymbol{V} \longrightarrow \boldsymbol{W}$ linear
$\mathfrak{B}, \mathfrak{C}$ are bases for $\boldsymbol{V, W}$ respectively. rankL = rank$[L]_{\mathfrak{C}}^{\mathfrak{B}}$
Note.
C(A) = span$\{\boldsymbol{C_1},\dots,\boldsymbol{C_n}\} = \{x_1\boldsymbol{C_1} + \dots + x_n\boldsymbol{C_n} | x_1,\dots,x_n \in \mathbb{R}\} = \{A\boldsymbol{X} | \boldsymbol{x} \in \mathbb{R}^n\} = \mathrm{im}L_A$

Note.
N(A) = $\{\boldsymbol{x} \in \mathbb{R}^n | A\boldsymbol{x} = \boldsymbol{0}\} = \mathrm{ker}L_A$

Theorem 2.2. $A \in M_{m\times n}(\mathbb{R}), B \in M_{n\times k}(\mathbb{R})$

$$rank(AB) \le rankA \tag{1}$$

$$rank(AB) \le rankB \tag{2}$$

Proof. (1)
C(AB) = AB$(\mathbb{R}^k) \subseteq A(\mathbb{R}^n) = C(A)$
$\therefore$rank$AB \le$ rank$A$ □

Proof. (2)
$\boldsymbol{C}(B^tA^t) = B^tA^t(\mathbb{R}^m) \subseteq B^t(\mathbb{R}^n) = \boldsymbol{C}(B^t)$
$\therefore$rank$A^tB^t \le$ rankB^t
$\therefore$ By the rank theorem rankAB = rank$B^tA^t \le$ rankB^t = rankB □

Theorem 2.3. $A \sim B \Longrightarrow rankA = rankB$

Proof. $B = Q^{-1}AQ$. Since Q has full rank, rankB = rank$(Q^{-1}AQ)$ = rank(AQ) = rankA □