Linear Algebra Class on 27 April

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Definition 1. *x is eigen-vector such that $Ax = \lambda x$ and $x \neq 0$. λ is called eigen-value.*

Definition 2. *$\phi_A(t) \coloneqq \det(tI_n - A) \in P_n(\mathbb{R})$ is called characteristic polynomial of A*

Definition 3. *$(\lambda I_n - A)x = 0$ where $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$*
Null space of $(\lambda I_n - A)$ is called eigen space of A with respect to λ; $E_\lambda \coloneqq N(\lambda I_n - A)$

Definition 4. *A is diagonalizable if and only if $D \sim A$ for some diagonal matrix D*

Theorem 5. *$A \in \mathfrak{M}_{n\times n}(\mathbb{R})$ is diagonalizable if and only if [$\mathbb{R}^n$ has n-linearly independent eigen-vectors. ($\iff$ $\mathbb{R}^n$ has a basis consisting of eigen vectors of A)]*

Proof.

$$\begin{aligned}
A \text{ is diagonalizable} &\iff D = Q^{-1}AQ \text{ where } Q \coloneqq [x_1, \ldots, x_n] \text{ and } D \coloneqq \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \\
&\iff QD = AQ \\
&\iff \lambda_x x_j = Ax_j \text{ for } j = 1, \ldots, n \text{ where } \{x_1, \ldots, x_n\} \text{ is linearly independent} \\
&\qquad (\because Q \text{ is invertible})
\end{aligned}$$

$\therefore D \sim A \iff \mathbb{R}^n$ has n-linearly independent eigen vectors. □

Theorem 6. *Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigen-values of T. If $v_1, v_2, \ldots, v_n$ are eigenvectors of T such that λ_i corresponds to v_i ($1 \leq i \leq k$), then $\{v_1, v_2, \ldots, v_k\}$ is linearly independet.*

Proof. The proof is by mathmatical induction on k. Suppose that $k = 1$. Then $v_1 \neq 0$ since v_1 is an eigenvector, and hence $\{v_1\}$ is linearly independent. Now assume that the theorem holds for $k-1$ distinct eigenvalues, where $k - 1 \geq 1$, and that we have k eigenvectors $v_1, v_2, \ldots, v_k$ corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. We wish to show that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. Suppose that $a_1, a_2, \ldots, a_k$ are scalars such that

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0 \tag{1}$$

Applying $\mathrm{T} - \lambda_k I$ to both sides of (1), we obtain

$$\begin{aligned}
(\mathrm{T} - \lambda_k I)(a_1v_1 + \cdots + a_kv_k) = a_1(\lambda_1 - \lambda_k)v_1 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0 \\
(\because (\mathrm{T} - \lambda_k I)(a_kv_k) = a_k(\lambda_kv_k - \lambda k v_k) = 0)
\end{aligned}$$

Since distinct $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$

$$a_i(\lambda_i - \lambda_k) = 0$$

Therefore $a_i = 0$ for all $i = 1, 2, \ldots, k-1$. We have $a_k v_k = 0$. So that $a_k = 0$ ($\because v_k$ is eigenvector.) It follows that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. $\square$

Corollary 7. *If all eigenvalues are distinct, then A is diagonalizable.*

Definition 8. *A polynomial $f(t)$ in $P(F)$ splits over F if there are scalars $c, a_1, \ldots, a_n$ (not necessarily distinct) in F such that*

$$f(t) = c(t - a_1)(t - a_2) \ldots (t - a_n).$$

Definition 9. *Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The (algebraic) multiplicity of λ is the largest positive integer k for which $(t - \lambda)^k$ is a factor of $f(t)$.*

Theorem 10. *Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \leq \dim(E_\lambda) \leq m$.*

Proof. Take a basis $\{v_1, \ldots, v_k\}$ of E_λ. Extend $\{v_1, \ldots, v_k\}$ to $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ by basis extension theorem. Put $A \coloneqq [T]_\beta$. Observe that v_i $(1 \leq i \leq k)$ is an eigenvector of T corresponding to λ, and therefore

$$A = \begin{pmatrix} \lambda I_p & B \\ O & C \end{pmatrix}$$

The characteristic polynomial of T is

$$\begin{aligned} f(t) = \det(A - tI_n) &= \det \begin{pmatrix} (\lambda - t)I_k & B \\ O & C - tI_{n-k} \end{pmatrix} \\ &= \det((\lambda - t)I_k) \det(C - tI_{n-k}) \\ &= (\lambda - t)^k g(t) \end{aligned}$$

where g(t) is a polynomial. Thus $(\lambda - t)^k$ is a factor of f(t), and hence the multiplicity of is at least k. But $\dim(E_\lambda) = k$, and so $\dim(E_\lambda) \leq m$. $\square$

Theorem 11. *Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $i = 1, 2, \ldots, k$, let S_i be a finite linearly independent subset of the eigenspace E_{λ_i}. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of V.*

Proof. Suppose that for each i

$$S_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}.$$

Then $S = \{v_{ij} : 1 \leq j \leq n_i, and 1 \leq i \leq k\}$. Consider any scalars a_{ij} such that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = 0.$$

For each i, let

$$w_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}.$$

Then $w_i \in E_{\lambda_i}$ for each i, and $w_1 + \cdots + w_k = 0$. Therefore, $w_i = 0$ for all i. But each S_i is linearly independent, and hence $a_{ij} = 0$ for all j. We conclude that S is linearly independent. □

Theorem 12. *Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then*

1. *T is diagonalizable if and only if the multiplicity of λ is equal to $\dim(E_{\lambda_i)}$ for all i*
2. *If T is diagonalizable and β_i is an ordered basis for E_{λ_i}, for each i, then $\beta = \beta_1 \uplus \beta_2 \uplus \cdots \uplus \beta_k$ is an ordered basis for V consisting of eigenvectors of T.*

Proof. For each i, let m_i, denote the multiplicity of λ_i, $di = \dim(E_{\lambda_i})$, and $n = \dim(V)$.
$(\implies)$ $\exists \beta :$ basis for V consisting of eigenvectors of T.
Let $\beta_i := \beta \cap E_{\lambda_i}$, $n_i = |\beta_i|$. Then

$$n_i \leq d_i \leq m_i$$
$$\sum_i n_i = n, \sum_i m_i = n$$
$$n = \sum_i n_i \leq \sum_i d_i \leq \sum_i m_i = n$$

It follows that

$$\sum_{i=1}^{k} (m_i - d_i) = 0.$$

Since $(m_i - d_i) \geq 0$ for all i, we conclude that $m_i = d_i$ for all i.
$(\impliedby) Suppose that \mathrm{d}_i = d_i$ for all i. We simultaneously show that T is diagonalizable and prove (2). For each i, let β_i be an ordered basis for E_{λ_i}, and let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. By Theorem 11, β is linearly independet. Futhermore, since $d_i = m_i$ for all i, β contains

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i = n$$

vectors. Therefore β is an ordered basis for V consisting of eigenvectors of V, and we conclude that T is diagonalizable.

□