T-invariant and T-cyclic subspace

Seanie Lee

04 May 2019

Theorem 1. *Let* $\mathbf{T}$ *be a linear operator on a finite-dimensional vector space* $\mathbf{V}$ *such that the characteristic polynomial of* $\mathbf{T}$ *splits. Let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the distinct eigenvalues of* $\mathbf{T}$ *Then,*

(1). $\mathbf{T}$ *is diagonalizable if and only if the multiplicty of* λ_i *is equal to dim(*$\mathbf{E}_{\lambda_i}$*) for all i*

(2). If $\mathbf{T}$ *is diagonalizable and* β_i *is an ordered basis for* $\mathbf{E}_{\lambda_i}$*, for each i, then* $\beta = \beta_1 \cup \cdots \cup \beta_k$ *is an ordered basis for* $\mathbf{V}$ *consisting of eignevectors of* $\mathbf{T}$

Note. $[\mathbf{T}]_\beta = \begin{pmatrix} \lambda_1 I_{n_1} & & & \text{\Large 0} \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ \text{\Large 0} & & & \lambda_k I_{n_k} \end{pmatrix}$: block diagonal matrix

Definition 2. $\boldsymbol{T} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ *is linear operator and* $\boldsymbol{W} \leq \boldsymbol{V}$*. If* $\boldsymbol{TW} \subseteq W$*, then* $\boldsymbol{W}$ *is called T-invariant subspace.* $\mathbf{0}, \boldsymbol{V}$*, and* $\boldsymbol{E}_\lambda$ *are examples of it.*

Lemma 3. $M := \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, $\det M = \det A \det C$

Proof. 1. if $\det A = 0$

Then there are some linearly dependent column vectors in A, so $\det M = \det A \det C = 0$

2. if $\det A \neq 0$

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix} \det \begin{pmatrix} I_n & A^{-1}B \\ 0 & C \end{pmatrix} = \det A \det C$$

□

Note. $\boldsymbol{v} = \boldsymbol{W}_1 \bigoplus \cdots \bigoplus \boldsymbol{W}_k$, $\boldsymbol{T} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$, and β_i is a basis for $\boldsymbol{W}_i$. Then $\beta = \beta_1 \biguplus \cdots \biguplus \beta_k$ is a basis for $\boldsymbol{V}$. But $[\boldsymbol{T}]_\beta$ is not a block matrix because $\boldsymbol{T}(\beta_i) \not\subset \boldsymbol{W}_i$. But if $\boldsymbol{W}_1$ is T-invariant,
then $[\boldsymbol{T}]_\beta = ¢\boldsymbol{W}_1 c_{\beta_1} B0C$

Moreover if $\boldsymbol{W}_i$ is T-invariant for all i the, $[T]_\beta = ¢\boldsymbol{W}_1 c_{\beta_1}$ $\text{\Large 0}$ $[\boldsymbol{T}|_{\boldsymbol{W}_2}]_{\beta_2}$ $\ddots$ $\text{\Large 0}$ $[\boldsymbol{T}|_{\boldsymbol{W}_k}]_{\beta_k}$

and by Lemma 3. $\phi_{\boldsymbol{T}}(t) = \phi_{\boldsymbol{T}_1}(t) \times \cdots \times \phi_{\boldsymbol{T}_k}(t)$; where $\boldsymbol{T}_i := \boldsymbol{T}|_{\boldsymbol{W}_1}$

Definition 4. $\boldsymbol{T} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ *linear. span*$\{\boldsymbol{v}, \boldsymbol{Tv}, \boldsymbol{T^2 v}, \ldots, \}$ *is* $\boldsymbol{T}$*-cyclic subspace of* $\boldsymbol{v} \in \boldsymbol{V}$

Note. $\boldsymbol{T}$-cyclic subspace of $\boldsymbol{v} \in \boldsymbol{V}$ is the smallest $\boldsymbol{T}$-invariant of $\boldsymbol{V}$ containing $\boldsymbol{v}$.

Proof. 1. $\boldsymbol{T}$-invariant.

$\boldsymbol{W} :=$ span$\{\boldsymbol{v}, \boldsymbol{T}\boldsymbol{v}, \boldsymbol{T}^2\boldsymbol{v}, \ldots,\}$ and $\boldsymbol{w} = \sum_{i=0}^{\infty} a_i \boldsymbol{T}^i \boldsymbol{v} \in \boldsymbol{W}$

$\boldsymbol{T}\boldsymbol{w} = \sum_{i=0}^{\infty} a_i \boldsymbol{T}^{i+1} \boldsymbol{v} \in \boldsymbol{W}$

$\therefore \boldsymbol{W}$ is $\boldsymbol{T}$-invariant.

2. Let $\boldsymbol{U}$ be any $\boldsymbol{T}$-invariant subspace of $\boldsymbol{V}$ containing $\boldsymbol{v}$. Since $\boldsymbol{U}$ is $\boldsymbol{T}$-invariant and $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{T}\boldsymbol{v} \in \boldsymbol{V}$.Repeatedly $\boldsymbol{T}^k\boldsymbol{v} \in \boldsymbol{V}$ for all k. Thus $\{\boldsymbol{v}, \boldsymbol{T}\boldsymbol{v}, \boldsymbol{T}^2\boldsymbol{v}, \ldots\} \subset \boldsymbol{U}$ $\therefore$ span$\{\boldsymbol{v}, \boldsymbol{T}\boldsymbol{v}, \boldsymbol{T}^2\boldsymbol{v}, \ldots\} \leq \boldsymbol{U}$

$\square$

Theorem 5. *$\boldsymbol{W}$ is $\boldsymbol{T}$-cyclic subspace of $\boldsymbol{V}$ generated by a nonzero vector $\boldsymbol{v} \in \boldsymbol{V}$ and dim$\boldsymbol{W} = k$*

1. *$\{\boldsymbol{v}, \boldsymbol{T}\boldsymbol{v}, \ldots, \boldsymbol{T}^{k-1}\boldsymbol{v}\}$ is a basis for $\boldsymbol{W}$*

2. *$a_0\boldsymbol{v} + a_1\boldsymbol{T}\boldsymbol{v} + \cdots + a_{k-1}\boldsymbol{T}^{k-1}\boldsymbol{V} + a_k\boldsymbol{T}^k\boldsymbol{v} = 0 \Longrightarrow \phi_{\boldsymbol{T}|\boldsymbol{W}}(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0$*

Proof. 1. Let l be the largest integer such that $\beta := \{\boldsymbol{v}, \boldsymbol{T}\boldsymbol{v}, \ldots, \boldsymbol{T}^{l-1}\boldsymbol{v}\}$ is linearly independent. Let $\boldsymbol{Z} :=$ spanβ. Then β is a basis for $\boldsymbol{Z}$. Since β is linearly independent set and $\beta \cup \{\boldsymbol{T}^i\boldsymbol{v}\}$ is linearly dependent, $\boldsymbol{T}^i\boldsymbol{v} \in$ spanβ for $i = l, l+1, \ldots$. So, $\boldsymbol{T}^l\boldsymbol{v} \in \boldsymbol{Z}$. Note that $\boldsymbol{Z}$ is $\boldsymbol{T}$-invariant because of the following reason.

$$\boldsymbol{w} = b_0\boldsymbol{v} + b_1\boldsymbol{T}\boldsymbol{v} + \cdots + b_{l-1}\boldsymbol{T}^{l-1} \in \boldsymbol{Z}$$
$$\boldsymbol{T}\boldsymbol{w} = b_0\boldsymbol{T}\boldsymbol{v} + b_1\boldsymbol{T}^2\boldsymbol{v} + \cdots + b_{l-2}\boldsymbol{T}^{l-1}\boldsymbol{v} + b_{l-1}\boldsymbol{T}^l\boldsymbol{v} \in \boldsymbol{Z}$$

Moreover $\boldsymbol{Z}$ is $\boldsymbol{T}$-invariant subspace containing $\boldsymbol{v}$ and $\boldsymbol{W}$ is the smallest $\boldsymbol{T}$-invariant subspace containing $\boldsymbol{v}$, $\boldsymbol{W} \leq \boldsymbol{Z}$ which implies that $k \leq l$. But dim$\boldsymbol{W} = k$ and $l \leq k$, thus $k = l$

$\therefore \beta$ is linearly independent subset of $\boldsymbol{W}$.

Since β is linearly independent and $\beta \cup \{\boldsymbol{T}^i\boldsymbol{v}\}$ is linearly dependent $\Longrightarrow \boldsymbol{T}^i\boldsymbol{v} \in$ spanβ for $i \geq k$.

$\therefore \beta$ is a basis for $\boldsymbol{W}$

2. $[\boldsymbol{T}|_{\boldsymbol{W}}]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$ and $\phi_{[\boldsymbol{T}|_{\boldsymbol{W}}]_\beta}(t) = \det \begin{pmatrix} t & 0 & \cdots & 0 & a_0 \\ -1 & t & \cdots & 0 & a_1 \\ 0 & -1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t + a_{k-1} \end{pmatrix}$

$$\text{Put } b_i \coloneqq \det \begin{pmatrix} t & 0 & \cdots & 0 & a_i \\ -1 & t & \cdots & 0 & a_{i+1} \\ 0 & -1 & \cdots & 0 & a_{i+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t+a_{k-1} \end{pmatrix}_{(k-i)\times(k-i)}$$

$$b_0 = \begin{pmatrix} t & 0 & \cdots & 0 & a_0 \\ -1 & t & \cdots & 0 & a_1 \\ 0 & -1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t+a_{k-1} \end{pmatrix}$$

$$b_0 = t\det \begin{pmatrix} t & 0 & \cdots & 0 & a_1 \\ -1 & t & \cdots & 0 & a_2 \\ 0 & -1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t+a_{k-1} \end{pmatrix} + (-1)^{k+1}a_0 \det \begin{pmatrix} -1 & t & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

$$\begin{aligned} b_0 &= tb_1 + (-1)^{k+1+k-1}a_0 \\ b_0 &= tb_1 + a_0 \\ b_1 &= tb_2 + a_1 \\ &\vdots \\ b_{k-2} &= tb_{k-1} + a_{k-2} \\ b_{k-1} &= t + a_{k-1} \\ b_0 &= t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0 \end{aligned}$$

$$\therefore \phi_{[\boldsymbol{T}|\boldsymbol{W}}(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0$$

□

Theorem 6. *If $\boldsymbol{F} = \mathbb{C}$ and $\boldsymbol{T} \in \mathfrak{L}(\boldsymbol{V}, \boldsymbol{V})$, then $\exists \mathfrak{B}$ such that $\mathfrak{B}$ is a basis for $\boldsymbol{V}$ and $[\boldsymbol{T}]_{\mathfrak{B}}$ is upper-triangular matrix.*

Proof. Induction on matrix size n. Since (1x1) matrix is upper-triangular matrix, let's assume $n \geq 2$. Since $\boldsymbol{F} = \mathbb{C}$, $\exists \boldsymbol{v}_1$ such that $\boldsymbol{T}\boldsymbol{v}_1 = \lambda \boldsymbol{v}_1$. Construct a basis $\mathfrak{C} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ by Basis Extension Theorem. Then $[\boldsymbol{T}]_{\mathfrak{C}} = \left(\begin{array}{c|c} \lambda & * \\ \hline 0 & B \end{array}\right)$, where $B \in \mathfrak{M}_{n-1,n-1}(\mathbb{C})$. By Induction hypothesis, $\exists P \in \mathfrak{M}_{n-1,n-1}(\mathbb{C})$ such that $p^{-1}BP$ is upper-triangular matrix. Put $U \coloneqq \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right)$.

Since, $U^{-1} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^{-1} \end{array}\right)$, $[\boldsymbol{T}]_{\mathfrak{C}} \sim U^{-1}\left(\begin{array}{c|c} \lambda & * \\ \hline 0 & B \end{array}\right) U = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^{-1} \end{array}\right) \left(\begin{array}{c|c} \lambda & * \\ \hline 0 & B \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right) = \left(\begin{array}{c|c} \lambda & * \\ \hline 0 & P^{-1}BP \end{array}\right)$

Since $P^{-1}BP$ is upper-triangular matrix, $[\boldsymbol{T}]_{\mathfrak{C}}$ is similar to upper-triangular matrix. □

Example 1. *Let* $A \in \mathbf{M}_{n\times n}(\mathbb{R})$ *be the matrix defined by* $A_{ij} = 1$ *for all* i *and* j. *Find the characteristic polynomial of* A. $\phi_A(t) = t^{n-1}(t-n)$

Proof. Since, rank$A = 1$, dim$(kerA) =$ n-1, which implies that $\exists \boldsymbol{x}$ such that $A\boldsymbol{x} = 0\boldsymbol{x}$ and $\boldsymbol{x} \neq 0$.Thus dim$E_0 =$ n-1. Since multiplicty of eignevalue 0 is greater than or equal to n-1, $\phi_A(t) = t^{n-1}(t-\lambda)$. Since there exists at least eigenvalue, which is 0, of A, A is similar to upper triangular matrix U by Theorem 6. Since U is upper-triangular matrix and trace $A =$ trace $U = n, -(\lambda + 0) = -n$.

$$U = \begin{pmatrix} 0 & & & & \\ & 0 & & * & \\ & & \ddots & & \\ & \mathbf{0} & & \ddots & \\ & & & & \lambda \end{pmatrix}$$

$\therefore \phi_A(t) = \phi_U(t) = t^{n-1}(t-n)$ □