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1 Dual Space

$\mathfrak{B} := \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ a basis for $\mathbf{V}$, $f : \mathbf{V} \longrightarrow \mathbb{R}$ linear
f is called "linear functionals on $\mathbf{V}$"
$\mathbf{V}^* := \{\text{all linear functionals}\}$
$\mathbf{V}^* := \mathfrak{L}(\mathbf{V}, \mathbb{R})$ $\mathfrak{B}^* := \{\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*\}$, where $\mathbf{v}_j^*(\mathbf{v}_i) := \delta_{ij}$

Theorem 1.1. *$\mathfrak{B}^*$ is a basis for $\mathbf{V}^*$ and called dual basis for $\mathbf{V}^*$*

Proof. linear independent
Suppose $a_1\mathbf{v}_1^* + \cdots + a_n\mathbf{v}_n^* = \mathbf{0}$
For all $i = 1, \ldots, n$ $(a_1\mathbf{v}^*{}_1 + \cdots + a_n\mathbf{v}_n^*)\mathbf{v}_i = a_1\mathbf{v}_1^*(\mathbf{v}_i) + \cdots + a_n\mathbf{v}_n^*(\mathbf{v}_i) = 0$
Since $\mathbf{v}_j^*(\mathbf{v}_i) = \delta_{ij}$, $a_j = 0$ for all $j = 1, \ldots, n$
$\therefore \mathfrak{B}^*$ is linearly independent ◻

Proof. $\mathfrak{B}^*$ span $\mathbf{V}^*$
Let $f \in \mathbf{V}^*$ and $\mathbf{v} \in \mathbf{V}$ be given. Put $\mathbf{v} := \sum_{i=1}^n a_i\mathbf{v}_i$
$f(\mathbf{v}) = f(\sum_{i=1}^n a_i\mathbf{v}_i) = \sum_{i=1}^n a_i f(\mathbf{v}_i)$

$$\text{Since } a_i = \mathbf{v}_i^*(\sum_{i=1}^n a_i\mathbf{v}_i)$$
$$\sum_{i=1}^n a_i f(\mathbf{v}_i) = \sum_{i=1}^n f(\mathbf{v}_i)\mathbf{v}_i^*(\mathbf{v})$$
$$f(\mathbf{v}) = \sum_{i=1}^n f(\mathbf{v}_i)\mathbf{v}_i^*(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}$$
$$\therefore f = \sum_{i=1}^n f(\mathbf{v}_i)\mathbf{v}_i^* \in \operatorname{span}\mathfrak{B}^*$$

◻

Corollary 1.1.1. $\mathbf{V} \cong \mathbf{V}^*$ *but not naturally isomorphic*

Theorem 1.2. $L : \mathbf{V} \longrightarrow \mathbf{W}$ *linear,* $\mathfrak{B}, \mathfrak{C}$ *are bases for* $\mathbf{V}, \mathbf{W}$ *and* $\mathfrak{B}^*, \mathfrak{C}^*$ *are dual bases for* $\mathbf{V}^*, \mathbf{W}^*$ *respectively . Let* $L^* : \mathbf{W}^* \longrightarrow \mathbf{V}^*$ *defined by* $f \mapsto f \circ L$*. Then* L^* *is linear and* $[L^*]_{\mathfrak{B}^*}^{\mathfrak{C}^*} = ([L]_{\mathfrak{C}}^{\mathfrak{B}})^t$

Proof. L^* is linear
Suppose $f, g \in \mathbf{W}^*$, and $a \in \mathbb{R}$ be given.

$$\begin{aligned} L^*(af + g) &= (af + g) \circ L \\ &= af \circ L + g \circ L \\ &= a(f \circ L) + g \circ L \\ &= aL^* f + L^* g \\ &\therefore L^* \text{ is linear} \end{aligned}$$

□

Note. $(f + g) \circ L = f \circ L + g \circ L$
$\because ((f + g) \circ L)\mathbf{v} = (f + g)(L\mathbf{v}) = f(L\mathbf{v}) + g(L\mathbf{v})$

Proof. $[L^*]_{\mathfrak{B}^*}^{\mathfrak{C}^*} = ([L]_{\mathfrak{C}}^{\mathfrak{B}})^t$
WTS (i,j) entry of $[L^*]_{\mathfrak{B}^*}^{\mathfrak{C}^*}$ = (j, i) entry of $[L]_{\mathfrak{C}}^{\mathfrak{B}}$
$\mathfrak{B} \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}, \mathfrak{C} \coloneqq \{w_1, \ldots, w_m\}$ are bases for $\mathbf{V}, \mathbf{W}$ respectively.
$\mathfrak{B}^* \coloneqq \{\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*\}, \mathfrak{C}^* \coloneqq \{\mathbf{w}_1^*, \ldots, \mathbf{w}_m^*\}$ are dual bases for $\mathbf{V}^*, \mathbf{W}^*$ respectively.
jth column of $[L^*]_{\mathfrak{B}^*}^{\mathfrak{C}^*} = [L^*\mathbf{w}_j^*]_{\mathfrak{B}^*} = [\mathbf{w}_j^* \circ L]_{\mathfrak{B}^*}$
Since $\mathbf{w}_j^* \circ L \in \mathbf{V}^*$, span$\mathfrak{B}^* = \mathbf{V}^*$, and $\mathbf{w}_j^* \circ L = \sum\limits_{i=1}^{n} (\mathbf{w}_j^* \circ L)(\mathbf{v}_i)\mathbf{v}_i^*$,
$[\mathbf{w}_j^* \circ L]_{\mathfrak{B}^*} = [\mathbf{w}_j^*(L\mathbf{v}_1), \ldots, \mathbf{w}_j^*(L\mathbf{v}_n)] \in \mathbb{R}^n$
Put $L\mathbf{v}_i \coloneqq \sum\limits_{k=1}^{m} a_{ik}\mathbf{w}_k$, then $\mathbf{w}_j^*(L\mathbf{v}_i) = a_{ij}$ for $i = 1, \ldots, n$. So i^{th} element is a_{ij}.,
ith column of $[L]_{\mathfrak{C}}^{\mathfrak{B}} = [L\mathbf{v}_i]_{\mathfrak{C}} = [\sum\limits_{k=1}^{m} a_{ik}\mathbf{w}_k]_{\mathfrak{C}} = (a_{i1}, \ldots, a_{im})$ and jth element is a_{ij}
$\therefore (i, j)$ of $[L^*]_{\mathfrak{B}^*}^{\mathfrak{C}^*} = (j, i)$ of $[L]_{\mathfrak{C}}^{\mathfrak{B}}$

□

2 Double Dual of V, $\mathbf{V}^{**}$

Theorem 2.1. $\phi : \mathbf{V} \longrightarrow \mathbf{V}^{**}$ *defined by* $\phi(\mathbf{v})f \coloneqq f\mathbf{v}$
ϕ *is natural isomorphism. It is not dependent on any choice of basis.*

Proof. ϕ is linear
Suppose $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}, f \in \mathbf{V}^*, a, b \in \mathbb{R}$ be given.

$$\begin{aligned} \phi(a\mathbf{v}_1 + b\mathbf{v}_2)f &= f(a\mathbf{v}_1 + b\mathbf{v}_2) \\ &= af\mathbf{v}_1 + bf\mathbf{v}_2 \\ &= a\phi(\mathbf{v}_1)f + b\phi(\mathbf{v}_2)f \\ &\therefore \phi \text{ is linear} \end{aligned}$$

□

Proof. ϕ is 1-1 ($\iff$ $\ker\phi = \mathbf{O}$)
Let $\mathbf{v} \in \ker\phi$ be given. Then $\phi(\mathbf{v}) = \mathbf{0}$. Thus, $f\mathbf{v} = 0 \quad \forall f \in \mathbf{V}^*$
Put $\mathbf{v} := \sum_{i=1}^{n} a_i\mathbf{v}_i, \mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ a basis for $\mathbf{V}$, and $\mathfrak{B}^* := \{\mathbf{v}_1^*, \dots, \mathbf{v}_n^*\}$ dual basis for $\mathbf{V}^*$.
For $j = 1, \dots, n \quad \mathbf{v}_j^*(\sum i = 1^n a_i\mathbf{v}_i) = 0$ then $a_j = 0$. Thus, $\mathbf{v} = \mathbf{0}$
$\therefore \ker\phi = \mathbf{O}$

□

Proof. ϕ is onto
Since $\dim\mathbf{V} = \dim\mathbf{V}^* = \dim\mathbf{V}^{**}$ by the Pigeon hole's principle, ϕ is onto.
$\therefore \phi$ is natural isomorphism □