

# T-invariant and T-cyclic subspace

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**Theorem 1.** Let  $\mathbf{T}$  be a linear operator on a finite-dimensional vector space  $\mathbf{V}$  such that the characteristic polynomial of  $\mathbf{T}$  splits. Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be the distinct eigenvalues of  $\mathbf{T}$ . Then,

- (1).  $\mathbf{T}$  is diagonalizable if and only if the multiplicity of  $\lambda_i$  is equal to  $\dim(\mathbf{E}_{\lambda_i})$  for all  $i = 1, \dots, k$
- (2). If  $\mathbf{T}$  is diagonalizable and  $\beta_i$  is an ordered basis for  $\mathbf{E}_{\lambda_i}$ , for each  $i$ , then  $\beta = \beta_1 \cup \dots \cup \beta_k$  is an ordered basis for  $\mathbf{V}$  consisting of eigenvectors of  $\mathbf{T}$

Note.  $[\mathbf{T}]_\beta = \begin{pmatrix} \lambda_1 I_{n_1} & & 0 \\ & \lambda_2 I_{n_2} & \\ & & \ddots \\ 0 & & & \lambda_k I_{n_k} \end{pmatrix}$  : block diagonal matrix

**Definition 2.**  $\mathbf{T} : \mathbf{V} \longrightarrow \mathbf{V}$  is linear operator and  $\mathbf{W} \leq \mathbf{V}$ . If  $\mathbf{T}\mathbf{W} \subseteq \mathbf{W}$ , then  $\mathbf{W}$  is called  $\mathbf{T}$ -invariant subspace.  $\mathbf{0}, \mathbf{V}$ , and  $\mathbf{E}_\lambda$  are examples of it.

**Lemma 3.**  $M := \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ ,  $\det M = \det A \det C$

*Proof.* 1. if  $\det A = 0$

Then there are some linearly dependent column vectors in  $A$ , so  $\det M = \det A \det C = 0$

2. if  $\det A \neq 0$

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix} \det \begin{pmatrix} I_n & A^{-1}B \\ 0 & C \end{pmatrix} = \det A \det C$$

□

Note.  $\mathbf{V} = \mathbf{W}_1 \oplus \dots \oplus \mathbf{W}_k$ ,  $\mathbf{T} : \mathbf{V} \longrightarrow \mathbf{V}$ , and  $\beta_i$  is a basis for  $\mathbf{W}_i$ . Then  $\beta = \beta_1 \uplus \dots \uplus \beta_k$  is a basis for  $\mathbf{V}$ . But  $[\mathbf{T}]_\beta$  is not a block matrix because  $\mathbf{T}(\beta_i) \not\subset \mathbf{W}_i$ . But if  $\mathbf{W}_1$  is  $\mathbf{T}$ -invariant,

then  $[\mathbf{T}]_\beta = \begin{pmatrix} [\mathbf{T}|_{\mathbf{W}_1}]_{\beta_1} & B \\ 0 & C \end{pmatrix}$

Moreover if  $\mathbf{W}_i$  is  $\mathbf{T}$ -invariant for all  $i$  the,  $[\mathbf{T}]_\beta = \begin{pmatrix} [\mathbf{T}|_{\mathbf{W}_1}]_{\beta_1} & & 0 \\ & [\mathbf{T}|_{\mathbf{W}_2}]_{\beta_2} & \\ & & \ddots \\ 0 & & & [\mathbf{T}|_{\mathbf{W}_k}]_{\beta_k} \end{pmatrix}$

and by Lemma 3.  $\phi_{\mathbf{T}}(t) = \phi_{\mathbf{T}_1}(t) \times \dots \times \phi_{\mathbf{T}_k}(t)$ ; where  $\mathbf{T}_i := \mathbf{T}|_{\mathbf{W}_i}$

**Definition 4.**  $T : V \longrightarrow V$  linear.  $\text{span}\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \dots, \}$  is  $T$ -cyclic subspace of  $\mathbf{v} \in V$

Note.  $T$ -cyclic subspace of  $\mathbf{v} \in V$  is the smallest  $T$ -invariant of  $V$  containing  $\mathbf{v}$ .

*Proof.* 1.  $T$ -invariant.

$$\mathbf{W} := \text{span}\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \dots, \} \text{ and } \mathbf{w} = \sum_{i=0}^{\infty} a_i T^i \mathbf{v} \in \mathbf{W}$$

$$T\mathbf{w} = \sum_{i=0}^{\infty} a_i T^{i+1} \mathbf{v} \in \mathbf{W}$$

$\therefore \mathbf{W}$  is  $T$ -invariant.

2. Let  $\mathbf{U}$  be any  $T$ -invariant subspace of  $V$  containing  $\mathbf{v}$ . Since  $\mathbf{U}$  is  $T$ -invariant and  $\mathbf{v} \in V$ ,  $T\mathbf{v} \in \mathbf{U}$ . Repeatedly  $T^k \mathbf{v} \in \mathbf{U}$  for all  $k$ . Thus  $\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \dots\} \subset \mathbf{U} \therefore \text{span}\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \dots\} \leq \mathbf{U}$

□

**Theorem 5.**  $\mathbf{W}$  is  $T$ -cyclic subspace of  $V$  generated by a nonzero vector  $\mathbf{v} \in V$  and  $\dim \mathbf{W} = k$

1.  $\{\mathbf{v}, T\mathbf{v}, \dots, T^{k-1}\mathbf{v}\}$  is a basis for  $\mathbf{W}$

$$2. a_0 \mathbf{v} + a_1 T\mathbf{v} + \dots + a_{k-1} T^{k-1} \mathbf{v} + T^k \mathbf{v} = 0 \implies \phi_{T|_{\mathbf{W}}}(t) = t^k + a_{k-1} t^{k-1} + \dots + a_1 t + a_0$$

*Proof.* 1. Let  $l$  be the largest integer such that  $\beta := \{\mathbf{v}, T\mathbf{v}, \dots, T^{l-1}\mathbf{v}\}$  is linearly independent. Let  $\mathbf{Z} := \text{span}\beta$ . Then  $\beta$  is a basis for  $\mathbf{Z}$ . Since  $\beta$  is linearly independent set and  $\beta \cup \{T^i \mathbf{v}\}$  is linearly dependent,  $T^i \mathbf{v} \in \text{span}\beta$  for  $i = l, l+1, \dots$ . So,  $T^l \mathbf{v} \in \mathbf{Z}$ . Note that  $\mathbf{Z}$  is  $T$ -invariant because of the following reason.

$$\mathbf{w} = b_0 \mathbf{v} + b_1 T\mathbf{v} + \dots + b_{l-1} T^{l-1} \mathbf{v} \in \mathbf{Z}$$

$$T\mathbf{w} = b_0 T\mathbf{v} + b_1 T^2 \mathbf{v} + \dots + b_{l-2} T^{l-1} \mathbf{v} + b_{l-1} T^l \mathbf{v} \in \mathbf{Z}$$

Moreover  $\mathbf{Z}$  is  $T$ -invariant subspace containing  $\mathbf{v}$  and  $\mathbf{W}$  is the smallest  $T$ -invariant subspace containing  $\mathbf{v}$ ,  $\mathbf{W} \leq \mathbf{Z}$  which implies that  $k \leq l$ . But  $\dim \mathbf{W} = k$  and  $l \leq k$ , thus  $k = l$

$\therefore \beta$  is linearly independent subset of  $\mathbf{W}$ .

Since  $\beta$  is linearly independent and  $\beta \cup \{T^i \mathbf{v}\}$  is linearly dependent  $\implies T^i \mathbf{v} \in \text{span}\beta$  for  $i \geq k$ .

$\therefore \beta$  is a basis for  $\mathbf{W}$

$$2. [T|_{\mathbf{W}}]_{\beta} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix} \text{ and } \phi_{[T|_{\mathbf{W}}]_{\beta}}(t) = \det \begin{pmatrix} t & 0 & \cdots & 0 & a_0 \\ -1 & t & \cdots & 0 & a_1 \\ 0 & -1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t + a_{k-1} \end{pmatrix}$$

$$\begin{aligned}
\text{Put } b_i &:= \det \begin{pmatrix} t & 0 & \cdots & 0 & a_i \\ -1 & t & \cdots & 0 & a_{i+1} \\ 0 & -1 & \cdots & 0 & a_{i+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t + a_{k-1} \end{pmatrix}_{(k-i) \times (k-i)} \\
b_0 &= \begin{pmatrix} t & 0 & \cdots & 0 & a_0 \\ -1 & t & \cdots & 0 & a_1 \\ 0 & -1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t + a_{k-1} \end{pmatrix} \\
b_0 &= t \det \begin{pmatrix} t & 0 & \cdots & 0 & a_1 \\ -1 & t & \cdots & 0 & a_2 \\ 0 & -1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t + a_{k-1} \end{pmatrix} + (-1)^{k+1} a_0 \det \begin{pmatrix} -1 & t & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} \\
b_0 &= tb_1 + (-1)^{k+1+k-1} a_0 \\
b_0 &= tb_1 + a_0 \\
b_1 &= tb_2 + a_1 \\
&\vdots \\
b_{k-2} &= tb_{k-1} + a_{k-2} \\
b_{k-1} &= t + a_{k-1} \\
b_0 &= t^k + a_{k-1}t^{k-1} + \cdots + a_1t + a_0
\end{aligned}$$

$$\therefore \phi_{[T]_{\mathcal{W}}}(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_1t + a_0$$

□

**Theorem 6.** If  $F = \mathbb{C}$  and  $T \in \mathcal{L}(V, V)$ , then  $\exists \mathfrak{B}$  such that  $\mathfrak{B}$  is a basis for  $V$  and  $[T]_{\mathfrak{B}}$  is upper-triangular matrix.

*Proof.* Induction on matrix size  $n$ . Since  $(1 \times 1)$  matrix is upper-triangular matrix, let's assume  $n \geq 2$ . Since  $F = \mathbb{C}$ ,  $\exists \mathbf{v}_1$  such that  $T\mathbf{v}_1 = \lambda\mathbf{v}_1$ . Construct a basis  $\mathfrak{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  by Basis Extension Theorem. Then  $[T]_{\mathfrak{C}} = \begin{pmatrix} \lambda & * \\ 0 & B \end{pmatrix}$ , where  $B \in \mathfrak{M}_{n-1, n-1}(\mathbb{C})$ . By Induction hypothesis,  $\exists P \in \mathfrak{M}_{n-1, n-1}(\mathbb{C})$

such that  $p^{-1}BP$  is upper-triangular matrix. Put  $U := \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}$ .

Since,  $U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & P^{-1} \end{pmatrix}$ ,  $[T]_{\mathfrak{C}} \sim U^{-1} \begin{pmatrix} \lambda & * \\ 0 & B \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} \lambda & * \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} \lambda & * \\ 0 & P^{-1}BP \end{pmatrix}$   
 Since  $P^{-1}BP$  is upper-triangular matrix,  $[T]_{\mathfrak{C}}$  is similar to upper-triangular matrix. □

**Example 1.** Let  $A \in \mathbf{M}_{n \times n}(\mathbb{R})$  be the matrix defined by  $A_{ij} = 1$  for all  $i$  and  $j$ . Find the characteristic polynomial of  $A$ .  $\phi_A(t) = t^{n-1}(t - n)$

*Proof.* Since,  $\text{rank} A = 1$ ,  $\dim(\ker A) = n-1$ , which implies that  $\exists \mathbf{x}$  such that  $A\mathbf{x} = 0\mathbf{x}$  and  $\mathbf{x} \neq 0$ . Thus  $\dim E_0 = n-1$ . Since multiplicity of eigenvalue 0 is greater than or equal to  $n-1$ ,  $\phi_A(t) = t^{n-1}(t - \lambda)$ . Since there exists at least eigenvalue, which is 0, of  $A$ ,  $A$  is similar to upper triangular matrix  $U$  by Theorem 6. Since  $U$  is upper-triangular matrix and  $\text{trace } A = \text{trace } U = n$ ,  $-(\lambda + 0) = -n$ .

$$U = \begin{pmatrix} 0 & & & \\ & 0 & & * \\ & & \ddots & \\ & \mathbf{0} & & \ddots \\ & & & & \lambda \end{pmatrix}$$

$$\therefore \phi_A(t) = \phi_U(t) = t^{n-1}(t - n)$$

□