# Linear Algebra Class on 5 January

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## 1.4 Linear Combination, Linear Independence, Dimension

**Definition.** *$\boldsymbol{Sym}_n(\mathbb{R}) := \{A \in \mathfrak{M}_n(\mathbb{R}) \mid A^t = A\}$ and $\boldsymbol{Alt}_n(\mathbb{R}) := \{A \in \mathfrak{M}_n(\mathbb{R}) \mid A^t = -A\}$*

**Note.** $A \in \mathfrak{M}_n(\mathbb{R})$ $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$
*A square matrix can be written the sum of symmetric and skew-symmetric matrix.*

**Definition.** *Let $\mathbf{V}$ be a vector space and $S$ a nonempty subset of $\mathbf{V}$. A vector $\mathbf{v} \in \mathbf{V}$ is called a **linear combination** of vectors $S$ if there exist a finite number of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in \mathbf{F}$ such that $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$.*

**Definition.** *Let $S$ be a nonempty subset of a vector space $\mathbf{V}$. Then **span** of $S$, denoted $spanS$, is the set consisting of all linear combinations of the vectors in $S$. For convenience, we define $span\emptyset := \{\boldsymbol{0}\}$.*

**Remark.** *$spanS$ is a subspace of $\mathbf{V}$. In particular, if $\mathbf{V} = spanS$, then we say that $S$ generates(spans) $\mathbf{V}$ or $S$ is a generating subset of $\mathbf{V}$.*

*Proof.* Suppose $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n, b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n \in S$ be given. We want to show that $c(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) + b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n \in S$.

$$c(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) + b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = (ca_1 + b_1)\mathbf{v}_1 + \cdots + (ca_n + b_n)\mathbf{v}_n \in \mathrm{span}S$$

$\square$

**Definition.** *$S$ is linearly independent if $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \boldsymbol{0} \Longrightarrow a_1 = \cdots = a_n = 0$*

**Example 1.** *$\mathbf{V} := \mathbb{R}^2$, $S_1 = \{(1,0),(0,1),(1,1)\}$ is linearly dependent subset because $1(1,0)+1(0,1)-1(1,1) = (0,0)$. But $S_2 = \{(1,0),(0,1)\}$ is linearly independent subset because $a_1(1,0) + a_2(0,1) = (0,0) = (a_1, a_2)$. Note that $\mathbb{R}^2 = spanS_1$, but $(3,2) = 3(1,0) + 2(0,1) + 0(1,1) = 1(1,0) = 0(0,1) + 2(1,1)$. The expression is not unique.*

**Definition.** *$S$ is called a basis for $\mathbf{V}$ if $\mathbf{V} = spanS$ and $S$ is linearly independent.*

**Note.** *Every vector space has a basis.*

**Definition.** *$dim\mathbf{V} := |S|$ where $S$ is is a basis for $\mathbf{V}$ and called "dimension of $\mathbf{V}$".*

**Note.** *All bases for $\mathbf{V}$ have the same cardinality.*

**Theorem.** *Let $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ subset of $\mathbf{V}$. Then $\beta$ is a basis if and only if $\forall \mathbf{v} \in \mathbf{V}$ can be uniquely expressed as linear combinations of vectors of $\beta$.*

*Proof.* $\Longrightarrow$
Suppose that $\beta = \{b\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbf{V}$. Then $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + n\mathbf{v}_n = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n \forall \mathbf{v} \in \mathbf{V}$. $(a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n = \mathbf{0}$. Since $\beta$ is a linearly independent subset of $\mathbf{V}$, $(a_i - b_i) = 0$ for all $i$. I.e., $a_i = b_i$ for all $i$.
$\therefore$ $\mathbf{v}$ can be uniquely expressed as linear combinations of vectors of $\beta$.
$\Longleftarrow$
Since every $\mathbf{v}$ can be uniquely expressed as linear combinations of $\beta$, span$\beta = \mathbf{V}$. Suppose $a_1\mathbf{v}_1 + a_n\mathbf{v}_n = 0$ and let's assume that there is at least non-zero coefficeint $a_i$. With out the loss of generality, suppose that $a_1 \neq 0$. Then $\mathbf{v}_1 = -a_1^{-1}(a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n)$. But it contradicts to the assumption. Because $\mathbf{v}_1$ can be expressed as two different linear combinations. $\mathbf{v}_1 = 1 \cdot \mathbf{v}_1 = -a_1^{-1}(a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n)$. Thus there cannot be non-zero coefficient $a_i$, i.e., $a_i = 0$ for $i = 1, \ldots, n$.
$\therefore$ $\beta$ is a basis for $\mathbf{V}$. $\square$

**Example 2.**

1. *A basis of* $\mathbb{R}^n$ *is* $\mathcal{E} := \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ *and is called standard basis for* $\mathbb{R}^n$. $dim\mathbb{R}^n = |\mathcal{E}| = n$
2. *A basis of* $\mathfrak{M}_{m\times n}(\mathbb{R})$ *is* $E^{ij}$ *where* $(E^{ij})_{ij} = 1$ *otherwise 0. Thus* $A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E^{ij}$ *and* $dim\mathfrak{M}_{m\times n}(\mathbb{R}) = mn$
3. $\mathbf{P}_n(\mathbb{R}) = \{f(x) \in \mathbf{P}(R) \mid \deg f(x) \leq n\} = \{a_0 + a_1X + \cdots + a_nX^n \mid a_i\mathbb{R}\} = span\{1, X, \ldots, X^n\}$
$dim\mathbf{P}_n(\mathbb{R}) = n + 1$
4. *A basis of* $\mathbf{U}_n(\mathbb{R}) := \{A \in \mathfrak{M}_n(\mathbb{R}) \mid a_{ij} = 0\, \forall i > j\}$ *is* $\{E^{ij} \mid 1 \leq i \leq j \leq n\ \}$ *and dimension is* $\frac{n(n+1)}{2}$.
5. *A basis of* $\mathbf{L}_n(\mathbb{R}) := \{A \in \mathfrak{M}_n(\mathbb{R}) \mid a_{ij} = 0\, \forall i < j\}$ *is* $\{E^{ij} \mid 1 \leq j \leq i \leq n\}$ *and dimension is* $\frac{n(n+1)}{2}$.
6. *A basis of* $\mathbf{D}_n(\mathbb{R}) := \{A \in \mathfrak{M}_n(\mathbb{R}) \mid a_{ij} = 0\, \forall i \neq j\}$ *is* $\{E^{ij} \mid i = j\}$ *and dimension is* $n$.
7. *A basis of* $\boldsymbol{Sl}_n(\mathbb{R}) := \{A \in \mathfrak{M}_n(\mathbb{R}) \mid trA = 0\}$ *is* $\{E^{ij} \mid i \neq j\} \cup \{E^{ii} - E^{i+1,i+1} \mid 1 \leq i < n\}$ *and dimension is* $n^2 - 1$
8. *A basis of* $\boldsymbol{Alt}_n(\mathbb{R}) := \{A \in \mathfrak{M}_n(\mathbb{R}) \mid A^t = -A\}$ *is* $\{E^{ii} \mid i = 1, \ldots, n\} \cup \{E^{ij} - E^{ji} \mid i < j\}$ *and dimension is* $\frac{n(n+1)}{2}$.
9. *A basis of* $\boldsymbol{Sym}_n(\mathbb{R}) := \{A \in \mathfrak{M}_n(\mathbb{R}) \mid A^t = A\}$ *is* $\{E^{ii} \mid i = 1, \ldots, n\} \cup \{E^{ij} + E^{ji} \mid i < j\}$ *and dimension is* $\frac{n(n+1)}{2}$