Linear Algebra Class on 16 March

Dongsu Kang

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Review from last week

1. Elementary matrices:

   Elementary matrices

$$\begin{pmatrix} \mathbf{e_1} \\ \vdots \\ \vdots \\ c\mathbf{e_i} \\ \vdots \\ \vdots \\ \mathbf{e_n} \end{pmatrix} \begin{pmatrix} \mathbf{e_1} \\ \vdots \\ \mathbf{e_j} \\ \vdots \\ \mathbf{e_i} \\ \vdots \\ \mathbf{e_n} \end{pmatrix} \begin{pmatrix} \mathbf{e_1} \\ \vdots \\ \mathbf{e_i} + c\mathbf{e_j} \\ \vdots \\ \mathbf{e_j} \\ \vdots \\ \mathbf{e_n} \end{pmatrix}$$

   Inverse of Elementary matrices

$$\begin{pmatrix} \mathbf{e_1} \\ \vdots \\ \vdots \\ \left(\frac{1}{c}\right)\mathbf{e_i} \\ \vdots \\ \vdots \\ \mathbf{e_n} \end{pmatrix} \begin{pmatrix} \mathbf{e_1} \\ \vdots \\ \mathbf{e_j} \\ \vdots \\ \mathbf{e_i} \\ \vdots \\ \mathbf{e_n} \end{pmatrix} \begin{pmatrix} \mathbf{e_1} \\ \vdots \\ \mathbf{e_i} - c\mathbf{e_j} \\ \vdots \\ \mathbf{e_j} \\ \vdots \\ \mathbf{e_n} \end{pmatrix}$$

2. Rank

$$\mathbf{A} = \begin{pmatrix} \mathbf{r_1} \\ \vdots \\ \mathbf{r_m} \end{pmatrix} = (\mathbf{c_1}, \cdots, \mathbf{c_n})$$

$$\mathfrak{R}(A) := span(\{\mathbf{r_1}. \cdots, \mathbf{r_m}\}),\ dim(\mathfrak{R}(A)) : row\ rank$$

$$\mathfrak{C}(A) := span(\{\mathbf{c_1}. \cdots, \mathbf{c_m}\}),\ dim(\mathfrak{C}(A)) : column\ rank$$

$$N(A) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{O}\},\ dim(N(A)) : nullity$$

Theorem (Rank Theorem)
Let A be a $m \times n$ matrix. Then, **row rank is equal to column rank**

- by Rank Theorem, rank of A = column rank of A.
- A is "full rank" if A has $min(m, n)$

Theorem $A \in M_{m\times n}(\mathbb{R}), B \in M_{n\times k}(\mathbb{R})$

1. rank(AB) $\leq$ rank A

   $Proof)$

   $\mathbb{C}(AB) = AB(\mathbb{R}^k) \subseteq A(\mathbb{R}^n) = \mathbb{C}(A)$
2. rank(AB) $\leq$ rank B
3. rank(AB) = rank A if B is full rank

   Theorem $A \sim B \Longleftrightarrow rankA = rankB$

   $Proof)\ B = QA^{-1}Q$

   $Since\ Q\ has\ a\ full\ rank,\ rankB = rank(Q^{-1}AQ) = rankAQ = rankA$

This week :
The inverse of a matrix
$AB = I_n = BA \Rightarrow B = A^{-1}$
A : invertible
$\Rightarrow E_k \dots E_1 A = I_n,\ forsomek$
$\therefore A^{-1} = E_k \dots E_1$

A way to get an inverse matrix
Let $A \in M_{n\times n}(\mathbb{R}), C = (A \mid I_n)$

$$A^{-1}C = (A^{-1}A \mid A^{-1}I_n) = (I_n \mid A^{-1})$$

Definition let A and be $m \times n$ and $m \times p$ matrices, respectively. By the **augmented matrix**$(A \mid B)$, we mean the $m \times (n+p)$ matrix (AB), that is, the matrix whose first n columns are the columns of A, and whose last p columns are the columns of B.(from book, 161p)

Example of augmented matrix

$$\begin{cases} 3x_1 + 2x_2 + 3x_3 + 2x_4 = 1 \\ x_1 + x_2 + x_3 \qquad\quad = 3 \\ x_1 + 2x_2 + x_3 - x_4 = 2 \end{cases} \Rightarrow \left(\begin{array}{cccc|c} 3 & 2 & 3 & -2 & 1 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & 1 & -1 & 2 \end{array}\right) \tag{1}$$

Systems of Linear Equations
From above augmented matrix, we can get below matrix by making row-reduced echelon form.

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

that is eqaul to,

$$\begin{cases} x_1 + x_3 = 1 \\ x_2 \qquad\quad = 2 \\ x_4 = 3 \end{cases} = \begin{pmatrix} s \\ 2 \\ 1-s \\ 4 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

Theorem $A \in M_{r\times n}(\mathbb{R}), A\mathbf{x} = \mathbf{b}$, Suppose that rank A $=$ $rank(A \mid \mathbf{b})$

- rank A = r
  s : general solution of $A\mathbf{x} = \mathbf{b}$
  $\Rightarrow s = s_0 + t_1\mathbf{u_1} + \cdots + t_{n-r}\mathbf{u_{n-r}}$
  where $\{\mathbf{u_1}, \cdots, \mathbf{u_{n-r}}\}$: basis for N(A)
  s_0 : particular solution

- K := $\mathbf{x} \mid A\mathbf{x} = \mathbf{b}$
  $K_H := \{\mathbf{x} \mid A\mathbf{x} = \mathbf{O}\} = N(A)$

  Letting $t_1 = \cdots = t_{n-r} = 90, s = s_0 \in K$
  i.e. $K = s_0 + K_H$
  i.e. $K_H = -s_0 + K = span(\{\mathbf{u_1}, \cdots, \mathbf{u_{n-r}}\})$