Linear Algebra Class on 8 June

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6.5 Normal and Self-adjoint Operator

Theorem (Spectral Theorem)**.** $\mathbf{T}$ *is linear operator on a complex inner product space* $\mathbf{V}$ *with finite dimension.*
$\mathbf{T}$ *is normal* $\iff \left[\exists\beta : \text{orthonormal basis such that } \beta = \{\mathbf{x}_1, \ldots \mathbf{x}_n\} \quad \mathbf{T}\mathbf{x}_i = \lambda_i \mathbf{x}_i\right]$

Proof. $\Longrightarrow$
Suppose that $\mathbf{T}$ is normal. Since $\phi_{\mathbf{T}}(t) \in \mathbb{C}[t]$, by the fundamental theorem of algebra, $\phi_{\mathbf{T}}(t)$ splits. So by the Schur's theorem, there is an orthonormal basis $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ such that $[\mathbf{T}]_\beta$ is upper-triangular matrix. Note that $\mathbf{x}_1$ is eigenvector of $\mathbf{T}$($\because$ since $[\mathbf{T}]_\beta$ is upper triangualr matrix, $[\mathbf{T}]_\beta = \lambda_1 \mathbf{e}_1 + 0\mathbf{e}_2 + \cdots + 0\mathbf{e}_n$). Assume that $\{\mathbf{x}_1, \ldots \mathbf{x}_{k-1}\}$ is the set of eigenvectors of $\mathbf{T}$. We want to show that $\mathbf{x}_k$ is eigenvector of $\mathbf{T}$. Put $A \coloneqq [\mathbf{T}]_\beta$. Then $\mathbf{T}\mathbf{x}_k = A_{1k}\mathbf{x}_1 + A_{2k}\mathbf{x}_2 + \cdots + A_{kk}\mathbf{x}_k$ ($\because$ Since A is upper-triangular matrix , $A_{ik} = 0$ for $i > k$). For $j < k$

$$\begin{aligned} A_{jk} &= \langle \mathbf{T}\mathbf{x}_k, \, \mathbf{x}_j \rangle \\ &= \langle \mathbf{x}_k, \, \mathbf{T}^*\mathbf{x}_j \rangle \\ &= \langle \mathbf{x}_k, \, \overline{\lambda}_j \mathbf{x}_j \rangle \\ &= \lambda_j \langle \mathbf{x}_k, \, \mathbf{x}_j \rangle \\ &= 0 \end{aligned}$$

Thus $\mathbf{T}\mathbf{x}_k = A_{kk}\mathbf{x}_k$. By the induction, β is an orthonormal basis consisting of eigenvectors of $\mathbf{T}$.
$\Longleftarrow$
Since there exists a basis β consisting of eigenvectors of $\mathbf{T}$,$[\mathbf{T}]_\beta$ is diagonal matrix. Since $[\mathbf{T}^*]_\beta = [\mathbf{T}]_\beta^*$, $[\mathbf{T}^*]_\beta$ is also diagonal matrix. Since matrix multiplication of diagonal matrices commute, $\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T}$.
$\therefore$ $\mathbf{T}$ is normal. □

Definition. $\mathbf{T}$ *is self-adjoint if* $\mathbf{T}^* = \mathbf{T}$.

Remark. *If* $\mathbf{T}$ *is self-adjoint operator, then* $\mathbf{T}$ *is also normal operator.*

Theorem. $\mathbf{T}$ *is self-adjoint.*

(1). Every eigenvalue of $\mathbf{T}$ *is real.*

(2). If $\mathbf{V}$ *is R-vector space, then* $\phi_{\mathbf{T}}(t)$ *splits.*

Proof.

(1) $\lambda \mathbf{x} = \mathbf{T}^* \mathbf{x} = \overline{\lambda} \mathbf{x}$. $\lambda = \overline{\lambda}$
$\therefore \lambda \in \mathbb{R}$

(2) Put $A \coloneqq [\mathbf{T}]_\beta$. Since $A = [\mathbf{T}]_\beta = [\mathbf{T}^*]_\beta = [\mathbf{T}]_\beta^* = A^*$, A is self-adjoint. Let $\mathbf{T}_A : \mathbf{x} \longmapsto A\mathbf{x}$. Note that $\mathbf{T}_A$ is self-adjoint because $[\mathbf{T}]_\gamma = A$, where γ is the standard ordered (normal) basis for $\mathbb{C}^n$. So by (1), the eigenvalues of $\mathbf{T}_A$ are real. By the fundamental theorem of algebra, the characteristic polynomial of $\phi_{\mathbf{T}_A}(t)$ splits into factors of $\lambda - t$. Since each λ is real-value, $\phi_{\mathbf{T}_A}(t)$ splits over $\mathbb{R}$. Since $\phi_{\mathbf{T}_A}(t) = \phi_A(t) = \phi_{\mathbf{T}}(t), \phi_{\mathbf{T}}(t)$ also splits over $\mathbb{R}$.

□

Theorem (Spectral Theorem of real-version)**.** $\mathbf{T}$ *is linear operator on a real inner product space with finite dimension.*
$\mathbf{T}$ *is self-adjoint* $\iff$ $\big[\beta :$ *orthonormal basis such that* $\mathbf{T}\mathbf{x}_i = \lambda_i \mathbf{x}_i \; \lambda_i \in \mathbb{R}\big]$

Proof. $\Longrightarrow$
Suppose that $\mathbf{T}$ is self-adjoint. Since $\phi_{\mathbf{T}}(t)$ splits over $\mathbb{R}$, by Schur's theorem there exists an orthonormal basis $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ such that $[\mathbf{T}]_\beta$ is upper triangular matrix. Put $A \coloneqq [\mathbf{T}]_\beta$, then $A^* = [\mathbf{T}]_\beta^* = [\mathbf{T}^*]_\beta = [\mathbf{T}]_\beta = A$. Thus A is diagonal.
$\Longleftarrow$
Let β be an orthonormal basis for $\mathbf{V}_\mathbb{R}$ consisting of eigenvectors. Put $A \coloneqq [\mathbf{T}]_\beta$, then $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i is eigenvalue. $A^* = \operatorname{diag}(\overline{\lambda}_1, \ldots \overline{\lambda}_n)$. Since $\lambda_i \in \mathbb{R}, A^* = A$. Thus, $[\mathbf{T}]_\beta = A = A^* = [\mathbf{T}^*]_\beta$.
$\therefore \mathbf{T} = \mathbf{T}^*$ □

6.6 Unitary and Orthogonal Operators

Definition. $\mathbf{T}$ *is unitary (on* $\mathbf{V}_\mathbb{C}$*) if* $\|\mathbf{T}\mathbf{x}\| = \|\mathbf{x}\| \; \forall \mathbf{x} \in \mathbf{V}$ *and* $\mathbf{T}$ *is orthogonal (on* $\mathbf{V}_\mathbb{R}$*) if* $\|\mathbf{T}\mathbf{x}\| = \|\mathbf{x}\| \; \forall \mathbf{x} \in \mathbf{V}$.

Theorem.

(1) $\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T} = \mathbf{I}$ *(*$\mathbf{T}$ *is normal and invertible)*

(2) $\langle \mathbf{T}\mathbf{x}, \, \mathbf{T}\mathbf{y} \rangle = \langle \mathbf{x}, \, \mathbf{y} \rangle$ *(preserving inner product)*

(3) β *is an orthonormal basis* $\Longrightarrow \mathbf{T}(\beta)$ *is an orthnormal basis*

(4) $\exists \beta :$ *orthonormal basis such that* $\mathbf{T}(\beta)$ *is an orthonormal basis*

(5) $\|\mathbf{T}\mathbf{x}\| = \|\mathbf{x}\| \; \forall \mathbf{x} \in V$

Proof. (1) $\Longrightarrow$ (2)

$$\begin{aligned} \langle \mathbf{T}\mathbf{x}, \, \mathbf{T}\mathbf{y} \rangle &= \langle \mathbf{x}, \, \mathbf{T}^*\mathbf{T}\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \, \mathbf{I}\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \, \mathbf{y} \rangle \end{aligned}$$

□

Proof. (2) $\implies$ (3)
Let $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an orthonormal basis. $\langle \mathbf{T}\mathbf{x}_i,\, \mathbf{T}\mathbf{x}_j \rangle = \langle \mathbf{x}_i,\, \mathbf{x}_j \rangle = \delta_{ij}$
$\therefore$ $\mathbf{T}(\beta)$ is an orthonormal basis. ☐

Proof. (3) $\implies$ (4)
Let β be an orthnormal basis. Then by (3), $\mathbf{T}(\beta)$ is an orthonormal basis.
$\therefore \exists \beta$ such that $\mathbf{T}(\beta)$ is an orthonormal basis. ☐

Proof. (4) $\implies$ (5)
Let β be an orthonormal basis such that $\mathbf{T}(\beta)$ is an orthonormal basis.
Put $\mathbf{x} \coloneqq \sum_{i=1}^{n} a_i \mathbf{x}_i$ where $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$

$$\begin{aligned}
\|\mathbf{x}\|^2 &= \langle \mathbf{x},\, \mathbf{x} \rangle \\
&= \langle \sum_{i=1}^{n} a_i \mathbf{x}_i,\, \sum_{j=1}^{n} a_j \mathbf{x}_j \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a}_j \langle \mathbf{x}_i,\, \mathbf{x}_j \rangle \\
&= \sum_{i=1}^{n} |a_i|^2 \\
\langle \mathbf{T}\mathbf{x},\, \mathbf{T}\mathbf{x} \rangle &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a}_j \langle \mathbf{T}\mathbf{x}_i,\, \mathbf{T}\mathbf{x}_j \rangle \\
&= \sum_{i=1}^{n} |a_i|^2 \\
\therefore \|\mathbf{T}\mathbf{x}\| &= \|\mathbf{x}\|
\end{aligned}$$

☐

Lemma. $\mathbf{U}$ *is self-adjoint. If* $\langle \mathbf{x},\, \mathbf{U}\mathbf{x} \rangle = 0\, \forall \mathbf{x} \in \mathbf{V}$*, then* $\mathbf{U} = \boldsymbol{0}$*.*

Proof. By the spectral theorem, there is an orthonormal basis $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ such that $\mathbf{U}\mathbf{x}_i = \lambda_i \mathbf{x}_i$.

$$\begin{aligned}
\forall i\, \langle \mathbf{x}_i,\, \mathbf{U}\mathbf{x}_i \rangle &= \langle \mathbf{x}_i,\, \lambda_i \mathbf{x}_i \rangle \\
&= \overline{\lambda}_i \langle \mathbf{x}_i,\, \mathbf{x}_i \rangle \\
&= \overline{\lambda}_i \|\mathbf{x}_i\|^2 \\
&= 0 \\
\overline{\lambda}_i &= 0\, \forall i
\end{aligned}$$

$$
\begin{aligned}
\forall \mathbf{x} &= \sum_{i=1}^{n} a_i \mathbf{x}_i \\
\mathbf{U}\mathbf{x} &= \sum_{i=1}^{n} a_i \mathbf{U}\mathbf{x}_i \\
&= \sum_{i=1}^{n} a_i \lambda_i \mathbf{x}_i \\
&= \mathbf{0}
\end{aligned}
$$

$\square$

Proof. (5) $\Longrightarrow$ (1)
Suppose $\forall \mathbf{x}\ \|\mathbf{T}\mathbf{x}\| = \|\mathbf{x}\|$.

$$
\begin{aligned}
\|\mathbf{x}\|^2 &= \langle \mathbf{x},\, \mathbf{x} \rangle \\
&= \langle \mathbf{T}\mathbf{x},\, \mathbf{T}\mathbf{x} \rangle \\
&= \langle \mathbf{x},\, \mathbf{T}^*\mathbf{T}\mathbf{x} \rangle \\
\langle \mathbf{x},\, \mathbf{x} \rangle - \langle \mathbf{x},\, \mathbf{T}^*\mathbf{T}\mathbf{x} \rangle &= 0 \\
\langle \mathbf{x},\, (\mathbf{I} - \mathbf{T}^*\mathbf{T})\mathbf{x} \rangle &= 0
\end{aligned}
$$

Since $(\mathbf{I} - \mathbf{T}^*\mathbf{T})^* = \mathbf{I} - \mathbf{T}^*\mathbf{T}$, $\mathbf{I} - \mathbf{T}^*\mathbf{T}$ is self-adjoint operator. By the previous lemma, $\mathbf{I} - \mathbf{T}^*\mathbf{T} = \mathbf{0}$. Thus $\mathbf{T}^*\mathbf{T} = \mathbf{I}, \mathbf{T}^*$ is onto and $\mathbf{T}$ is one-to-one. Since $\dim \mathbf{V} < +\infty$, by the pigeonhole's principle $\mathbf{T}^*$ is also one-to-one and $\mathbf{T}$ is also onto. Thus $\mathbf{T}, \mathbf{T}^*$ are invertible and $\mathbf{T}^{-1} = \mathbf{T}^*$. $\therefore \mathbf{T}\mathbf{T}^* = \mathbf{I}$ $\square$

Definition (Inner Product Isomorphism)**.** $\mathbf{V}$ *is vector space with inner product* $\langle \cdot,\, \cdot \rangle_1$ *and* $\mathbf{W}$ *is vector space with another inner product* $\langle \cdot,\, \cdot \rangle_2$. $\mathbf{T} : \mathbf{V} \longrightarrow \mathbf{W}$ *is inner product isomorphism* $\Longleftrightarrow$ $\mathbf{T}$ *is vector space isomorphism and* $\langle \mathbf{T}\mathbf{x},\, \mathbf{T}\mathbf{y} \rangle_2 = \langle \mathbf{x},\, \mathbf{y} \rangle_1$