Linear Algebra Class on 12 January

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1.5 Basis Extension Theorem

Lemma 1.1. *$\mathfrak{B}$ is a basis for $\mathbf{W} \leq \mathbf{V}$. $\mathbf{v} \in \mathbf{V}, \mathbf{v} \neq \mathbf{0}$. $\mathfrak{B} \cup \{\mathbf{v}\}$ is linearly independent $\iff$ $\mathbf{v} \notin span\mathfrak{B}$*

Proof. We want to prove contrapositive of the statement. That is $\mathbf{v} \in \text{span}\mathfrak{B} \iff \mathfrak{B} \cup \{\mathbf{v}\}$ is linearly dependent set.
$\Longrightarrow$ Suppose $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for $\mathbf{W}$ and $\mathbf{v} \in \text{span}\mathfrak{B}$ be given. Since $\mathbf{v} \in \text{span}\mathfrak{B}$, $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$. i.e. $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k - \mathbf{v} = \mathbf{0}$. There is a non-trivial representation of $\mathbf{0}$. Therefore $\mathfrak{B} \cup \{\mathbf{v}\}$ is linearly dependent.
$\Longleftarrow$ Let's assume $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + c\mathbf{v} = 0$. Since $\mathfrak{B} \cup \{\mathbf{v}\}$ is linearly dependent, there are some non-zero coefficient. Assume $c = 0$. Then $\forall a_i = 0$ because $\mathfrak{B}$ is linearly independent, which implies $\mathfrak{B} \cup \{\mathbf{v}\}$ is linearly independent. But it contradicts to the assumption. So c is non-zero. Then $v = c^{-1}(-a_1\mathbf{v}_1 - \cdots - a_l\mathbf{v}_k)$. $\therefore \mathbf{v} \in \text{span}\mathfrak{B}$ □

Theorem (Basis Extension Theorem)**.** *$\mathbf{W}$ is a subspace of $\mathbf{V}$ and $\mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for $\mathbf{W}$. Then there exists $\mathfrak{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\} \supset \mathfrak{B}$ and $\mathfrak{C}$ is a basis for $\mathbf{V}$.*

Proof. Choose $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \notin \text{span}\mathfrak{B}$. By the lemma 1.1, $\mathfrak{B}' := \mathfrak{B} \cup \{\mathbf{v}\}$ is linearly independent. Since $\mathbf{V}$ is finite vector space, this process terminates in finite step. Thus repeat this process until the cardinality of $\mathfrak{B}' = \dim \mathbf{V}$ □

Theorem. *$\mathbf{W}_1, \mathbf{W}_2$ are subspace of $\mathbf{V}$. The followings are equivalent*

(1) $\mathbf{V} = \mathbf{W}_1 \bigoplus \mathbf{W}_2$

(2) $\mathfrak{B}, \mathfrak{B}_1$, *and* $\mathfrak{B}_2$ *are bases for* $\mathbf{V}, \mathbf{W_1}, \mathbf{W_2}$ *respectively.* $\mathfrak{B} = \mathfrak{B}_1 \biguplus \mathfrak{B}_2$

Proof. $\Longrightarrow$ Suppose that $\mathfrak{B}_1 := \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and let $\mathfrak{B}_2 := \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be given and let them be bases for $\mathbf{W}_1, \mathbf{W}_2$ respectively. We want to show $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$. Assume $\mathfrak{B}_1 \cap \mathfrak{B}_2 \neq \emptyset$ and $\mathbf{w} \in \mathfrak{B}_1 \cap \mathfrak{B}_2$ such that $\mathbf{w} \neq \mathbf{0}$ be given. Since $\mathfrak{B}_1 \subset \mathbf{W}_1$ and $\mathfrak{B}_2 \subset \mathbf{W}_2$, $\mathbf{w} \in \mathbf{W}_1, \mathbf{W}_2$. So $\mathbf{w} \in \mathbf{W}_1 \cap \mathbf{W}_2$, i.e. $\mathbf{w} = \mathbf{0}$. But it contradicts to the assumption. Therefore $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$. Since $\mathfrak{B}_1, \mathfrak{B}_2$ are bases for $\mathbf{W}_1, \mathbf{W}_2$ and $\mathbf{W}_1, \mathbf{W}_2$ are direct sum of $\mathbf{V}$, $\mathfrak{B}_1 \cup \mathfrak{B}_2$ span $\mathbf{V}$. Thus it is enough to show that $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is linearly independent subset of $\mathbf{V}$. Suppose that $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + b_1\mathbf{u}_1 + \cdots + b_m\mathbf{u}_m = \mathbf{0}$ $\mathbf{v} := a_1\mathbf{v}_1 + \cdots a_k\mathbf{v}_k = -(b_1\mathbf{u}_1 + \cdots b_m\mathbf{u}_m)$ Since $\mathbf{v} \in \text{span}\mathfrak{B}_1 \cap \mathfrak{B}_2, \mathbf{v} \in \mathbf{W}_1 \cap \mathbf{W}_2$. Thus $\mathbf{v} = \mathbf{0}$, which implies that $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = b_1\mathbf{u}_1 + b_m\mathbf{u}_m = \mathbf{0}$. Since $\mathfrak{B}_1, \mathfrak{B}_2$ are bases for $\mathbf{W}_1, \mathbf{W}_2$, $a_i = 0$ for $i = 1, \dots, k$ and $b_j = 0$ for $j = 1, \dots, m$. Therefore $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is linearly independent.
$\Longleftarrow$ Suppose that $\mathfrak{B}_1 := \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $\mathfrak{B}_2 := \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be given and let them be bases for

$\mathbf{W}_1, \mathbf{W}_2$ respectively. Since it's trivial to show that $\mathbf{W}_1 + \mathbf{W}_2 \subset \mathbf{V}$, it is enough to show that $\mathbf{V} \subset \mathbf{W}_1 + \mathbf{W}_2$. Suppose $\mathbf{v} \in \mathbf{V}$ be given. Since $\mathbf{v} \in \mathbf{W}_1 + \mathbf{W}_2$, $\mathbf{v} = \sum_{i=1}^{k} a_i \mathbf{v}_i + \sum_{j=1}^{l} b_j \mathbf{u}_j \in \mathbf{W}_1 + \mathbf{W}_2$. Therefore $\mathbf{V} \subset \mathbf{W}_1 + \mathbf{W}_2$ Finally we want to show that $\mathbf{W}_1 \cap \mathbf{W}_2 = \mathbf{O}$. Let's assume that $\exists \mathbf{w}$ such that $\mathbf{w} \neq \mathbf{0}$, $\mathbf{w} \in \mathbf{W}_1 \cap \mathbf{W}_2$. Since $\mathbf{w} \in \mathbf{W}_1 \cap \mathbf{W}_2$, $\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_n = b_1\mathbf{u}_1 + \cdots + b_m\mathbf{u}_m$. i.e. $\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_n - b_1\mathbf{u}_1 + \cdots + b_m\mathbf{u}_m = 0$ Since $\mathfrak{B}$ is a basis, $\forall a_i, b_j = 0$. Therefore $\mathbf{v}$ is always zero vector. $\square$

1.6 Quotient Space

Definition (Coset)**.** $\mathbf{W}$ *is a subspace of* $\mathbf{V}$
$\mathbf{v} \in \mathbf{V}, \mathbf{v} + \mathbf{W} := \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in \mathbf{W}\}$

Definition (Quotient Space)**.** $\mathbf{V}/\mathbf{W} := \{\mathbf{v} + \mathbf{W} \mid \mathbf{v} \in \mathbf{V}\}$

Theorem. $\mathbf{v} + \mathbf{W} \leq \mathbf{V} \iff \mathbf{v} \in \mathbf{W}$

Proof. $\Longrightarrow$ There exists $\mathbf{w} \in \mathbf{W}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$ ($\because \mathbf{v} + \mathbf{W} \leq \mathbf{V}$).
$\therefore \mathbf{v} = -\mathbf{w} \in \mathbf{W}$
$\Longleftarrow$ Suppose $\mathbf{v} + \mathbf{w}_1, \mathbf{v} + \mathbf{w}_2 \in \mathbf{v} + \mathbf{W}$ be given. We want to show that $\mathbf{v} + \mathbf{w}_1 + \mathbf{v} + \mathbf{w}_2 \in \mathbf{v} + \mathbf{W}$. $\mathbf{v}+\mathbf{w}_1+\mathbf{v}+\mathbf{w}_2 = \mathbf{v}+(\mathbf{v}+\mathbf{w}_1+\mathbf{w}_2) \in \mathbf{v}+W$. For scalar multiplication, $c(\mathbf{v}+\mathbf{w}) = \mathbf{v}+(1-c)\mathbf{v}+\mathbf{w} \in \mathbf{W}$.
$\therefore$ By the subspace test, $\mathbf{v} + \mathbf{W}$ is a subspace of $\mathbf{V}$. $\square$

Theorem. $\mathbf{v}_1 + \mathbf{W} = \mathbf{v}_2 + \mathbf{W} \iff \mathbf{v}_1 - \mathbf{v}_2 \in \mathbf{W}$

Proof. $\Longrightarrow$ $\mathbf{v} \in \mathbf{v}_1 + \mathbf{W}, \mathbf{v}_2 + \mathbf{W}$. Then $\mathbf{v} = \mathbf{v}_1 + \mathbf{w}_1 = \mathbf{v}_2 + \mathbf{w}_2$ for some $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$.
$\therefore \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{w}_2 - \mathbf{w}_1 \in \mathbf{W}$
$\Longleftarrow$ Let $\mathbf{v}_1 + \mathbf{w}_1 \in \mathbf{v}_1 + \mathbf{W}$. Since $\mathbf{v}_1 - \mathbf{v}_2 \in \mathbf{W}$, $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{w}$ for some $\mathbf{w} \in \mathbf{W}$.
$\mathbf{v}_1 + \mathbf{w}_1 = (\mathbf{v}_2 + \mathbf{w}) + \mathbf{w}_1 = \mathbf{v}_2 + (\mathbf{w} + \mathbf{w}_1) \in \mathbf{v}_2 + \mathbf{W}$ Conversely, let $\mathbf{v}_2 + \mathbf{w}_2 \in \mathbf{v}_w + \mathbf{W}$ be given. Then $\mathbf{v}_2 + \mathbf{w}_2 = (\mathbf{v}_1 - \mathbf{w}) + \mathbf{w}_2 = \mathbf{v}_1 + (-\mathbf{w} + \mathbf{w}_2) \in \mathbf{v}_1 + \mathbf{W}$
$\therefore \mathbf{v}_1 + \mathbf{W} = \mathbf{v}_2 + \mathbf{W}$ $\square$

Theorem. *Define a binary operation and a scalar multiplication as follows*

$$(\mathbf{v}_1 + \mathbf{W}_1) + (\mathbf{v}_2 + \mathbf{W}_2) := (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{W}$$
$$c(\mathbf{v} + \mathbf{W}) := c\mathbf{v} + \mathbf{W}$$

Then these two binary operations are well-defined.

Proof. Let $\mathbf{v}_1 + \mathbf{W} = \mathbf{v}_1' + \mathbf{W}$ and $\mathbf{v}_2 + \mathbf{W} = \mathbf{v}_2' + \mathbf{W}$ be given. By the previous theorem, $\mathbf{v}_1 - \mathbf{v}_1' \in \mathbf{W}$ and $\mathbf{v}_2 - \mathbf{v}_2' \in \mathbf{W}$. $(\mathbf{v}_1+\mathbf{v}_2)-(\mathbf{v}_1'+\mathbf{v}_2') = (\mathbf{v}_1-\mathbf{v}_1')+(\mathbf{v}_2-\mathbf{v}_2') \in \mathbf{W}$ Thus, $(\mathbf{v}_1+\mathbf{v}_2)+\mathbf{W} = (\mathbf{v}_1'+\mathbf{v}_2')+\mathbf{W}$ For scalar multiplication, $c\mathbf{v}_1 - c\mathbf{v}_1' \in \mathbf{W}$. Thus,$c\mathbf{v}_1 + \mathbf{W} = c\mathbf{v}_1' + \mathbf{W}$.
$\therefore$ The two binary operations are well-defined. $\square$

Theorem.

1. $(\mathbf{v}_1 + \mathbf{W}) \cap (\mathbf{v}_2 + \mathbf{W}) \neq \emptyset \Longrightarrow \mathbf{v}_1 + \mathbf{W} = \mathbf{v}_2 + W$

2. $\mathbf{V}/\mathbf{W} = \biguplus_{\mathbf{v} \in \mathbb{R}} \mathbf{v} + \mathbf{W}$ *where* $\mathbb{R} :=$ *the set of representatives.*

3. $\forall \mathbf{v} \in \mathbf{V}$, *the following map is bijection. Thus,* $|\mathbf{W}| = |\mathbf{v} + \mathbf{W}|$

$$\begin{aligned} \mathbf{W} &\longrightarrow \mathbf{v} + \mathbf{W} \\ \mathbf{w} &\longmapsto \mathbf{v} + \mathbf{w} \end{aligned}$$

Theorem. *$\mathbf{W}$ is a subspace of $\mathbf{V}$. $\mathfrak{B} := \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for $\mathbf{W}$ and $\mathfrak{C} := \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ is a basis for $\mathbf{V}$ by extending $\mathfrak{B}$. Then $\{\overline{\mathbf{v}}_{k+1}, \dots, \overline{\mathbf{v}}_n\}$ is a basis for $\mathbf{V}/\mathbf{W}$ where $\overline{\mathbf{v}}_j := \mathbf{v}_j + \mathbf{W}$.*

Proof. $\overline{\mathfrak{D}} := \{\overline{\mathbf{v}}_{k+1}, \dots, \overline{\mathbf{v}}_n\}$ where $\mathfrak{B} \uplus \mathfrak{D} = \mathfrak{C}$ Let $\overline{\mathbf{v}} \in \mathbf{V}/\mathbf{W}$ be given. Since $\mathfrak{C}$ is a basis for $\mathbf{V}$, $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n$ for some $a_i \in \mathbb{R}$. Then

$$\begin{aligned} \overline{\mathbf{v}} &= \overline{a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n} \\ &= \overline{a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k} + \overline{a_{k+1}\mathbf{v}_{k+1} \cdots + a_n\mathbf{v}_n} \\ &= \overline{a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n} \quad (\because \mathfrak{B} \text{ is a basis for } \mathbf{W}) \\ &= a_{k+1}\overline{\mathbf{v}}_{k+1} + \cdots + a_n\overline{\mathbf{v}}_n \in \operatorname{span}\overline{\mathfrak{D}} \end{aligned}$$

We want to show that $\overline{\mathfrak{D}}$ is linearly independent subset. Suppose $a_{k+1}\overline{\mathbf{v}}_{k+1} + \cdots + a_n\overline{\mathbf{v}}_n = \overline{\mathbf{0}}$ Then $(a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n - \mathbf{0}) \in \mathbf{W}$. Put $\mathbf{v} := a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n$, then $\mathbf{v} \in \operatorname{span}\mathfrak{B}$. So, $a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k$. $a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n - b_1\mathbf{v}_1 - \cdots - b_k\mathbf{v}_k = \mathbf{0}$. Since $\mathfrak{C}$ is a basis for $\mathbf{V}$, so $a_i, b_j = 0$ for $i = k+1, \dots, n$ and for $j = 1, \dots, k$. Thus $\mathfrak{D}$ is linearly independent.
$\overline{\mathfrak{D}}$ is a basis for $\{\overline{\mathbf{v}}_{k+1}, \dots, \overline{\mathbf{v}}_n\}$ $\square$

Corollary. $\dim \mathbf{V}/\mathbf{W} = \dim \mathbf{V} - \dim \mathbf{W}$