Linear Algebra Class on 18 May

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6.1 INNER PRODUCTS AND NORMS

Definition. *Let* V *be a vector space over* F*, where* F *is either* $\mathbb{R}$ *or* $\mathbb{C}$*. Regardless of whether* V *is or is not an inner product space, we may still define a "norm"* $\|\cdot\| : \mathsf{V} \to R_{\geq 0}$ *as a real-valued function on* V *satisfying the following three conditions for all* $\mathbf{x}, \mathbf{y} \in \mathsf{V}$ *and* $a \in F$
(1) *: (positive-definite)* $\|\mathbf{x}\| \geq 0$*, and* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$
(2) *:* $\|a\mathbf{x}\| = |a| \cdot \|\mathbf{x}\|$
(3) *: (Triangle Inequality)* $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

Definition. *A function* $d(\cdot, \cdot) : \mathsf{V} \times \mathsf{V} \to R_{\geq 0}$ *is a "metric" if satisfying the following three conditions for all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathsf{V}$
(1) *: (positive-definite)* $d(\mathbf{x}, \mathbf{x}) \geq 0$*, and* $d(\mathbf{x}, \mathbf{x}) = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$
(2) *: (symmetric)* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$
(3) *: (Triangle Inequality)* $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$

Remark. *Is this statement equivalent?*

$$(positive-definite)\big(\,\|\mathbf{x}\| \geq 0, \forall \mathbf{x} \in \mathsf{V}, \|\mathbf{x}\| = \mathbf{0}\big) \text{ if and only if } \big(\mathbf{x} = \mathbf{0} \Leftrightarrow \|\mathbf{x}\| > 0 \text{ if } \mathbf{x} \neq \mathbf{0}\big)$$

Definition. *Let* V *be an inner product space. For* $\mathbf{x} \in \mathsf{V}$*, we define the* ***norm*** *of* ***length*** *of* $\mathbf{x}$ *by* $\|x\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Theorem. *Let* V *be an inner product space over* F*. Then for all* $\mathbf{x}, \mathbf{y} \in \mathsf{V}$ *and* $c \in F$*, the following statements are true.*

(1) *(Cauchy - Schwarz Inequality)* $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$

(2) *(Triangle Inequality)* $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

Note. *Let* $z := a + bi$*. Then* $\mathrm{Re}z := a$, $\mathrm{Im}z := b$

Proof. (1) Cauchy-Schwarz Inequality
(case 1) $\mathbf{y} = \mathbf{0}$: done

(case 2) $\mathbf{y} \neq \mathbf{0}$

$$\begin{aligned}
&0 \le \|\mathbf{x} - c\mathbf{y}\| \\
\Rightarrow\ &0 \le \|\mathbf{x} - c\mathbf{y}\|^2 = \langle \mathbf{x} - c\mathbf{y},\, \mathbf{x} - c\mathbf{y} \rangle \\
&\qquad\qquad\quad = \langle \mathbf{x},\, \mathbf{x} \rangle - \bar{c}\langle \mathbf{x},\, \mathbf{y} \rangle - c\langle \mathbf{y},\, \mathbf{x} \rangle + c\bar{c}\langle \mathbf{y},\, \mathbf{y} \rangle
\end{aligned}$$

Taking $c \coloneqq \dfrac{\langle \mathbf{x},\, \mathbf{y} \rangle}{\langle \mathbf{x},\, \mathbf{y} \rangle}$, we have

$$\begin{aligned}
\Rightarrow\ &0 \le \|\mathbf{x}\|^2 - \frac{\overline{\langle \mathbf{x},\, \mathbf{y} \rangle}}{\langle \mathbf{y},\, \mathbf{y} \rangle} \cdot \langle \mathbf{x},\, \mathbf{y} \rangle - \frac{\langle \mathbf{x},\, \mathbf{y} \rangle}{\langle \mathbf{y},\, \mathbf{y} \rangle} \cdot \langle \mathbf{y},\, \mathbf{x} \rangle + \frac{\langle \mathbf{x},\, \mathbf{y} \rangle \cdot \overline{\langle \mathbf{x},\, \mathbf{y} \rangle}}{\langle \mathbf{y},\, \mathbf{y} \rangle^2} \cdot \langle \mathbf{y},\, \mathbf{y} \rangle \\
\Rightarrow\ &0 \le \|\mathbf{x}\|^2 - \frac{|\langle \mathbf{x},\, \mathbf{y} \rangle|^2}{\|\mathbf{x}\|^2} \\
\Rightarrow\ &|\langle \mathbf{x},\, \mathbf{y} \rangle| \le \|\mathbf{x}\| \cdot \|\mathbf{y}\|
\end{aligned}$$

(2) Triangle Inequality

$$\begin{aligned}
&(\|\mathbf{x}\| + \|\mathbf{y}\|)^2 - \|\mathbf{x} + \mathbf{y}\|^2 \\
&= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\,\|\mathbf{x}\| \cdot \|\mathbf{y}\| - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 - \langle \mathbf{x},\, \mathbf{y} \rangle - \langle \mathbf{y},\, \mathbf{x} \rangle \\
&= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\,\|\mathbf{x}\| \cdot \|\mathbf{y}\| - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 - 2\mathrm{Re}\langle \mathbf{x},\, \mathbf{y} \rangle \quad (\because \langle \mathbf{x},\, \mathbf{y} \rangle + \overline{\langle \mathbf{x},\, \mathbf{y} \rangle} = 2\mathrm{Re}\langle \mathbf{x},\, \mathbf{y} \rangle) \\
&\ge 2(\|\mathbf{x}\| \cdot \|\mathbf{y}\| - |\langle \mathbf{x},\, \mathbf{y} \rangle|) \\
&\ge 0 \quad (\because Cauchy - Shwarz \ \ Inequality)
\end{aligned}$$

$\square$

6.2 GRAM-SCHMIDT ORTHOGONALIZATION PROCESS

Definition. *Lev* V *be an inner product space. Vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* V *are* ***orthogonal (perpendicular)*** *if* $\langle \mathbf{x},\, \mathbf{y} \rangle = 0$. *A subset* S *of* V *is* ***orthogonal*** *if any two distinct vectors in* S *are orthogonal. A vector* $\mathbf{x}$ *in* V *is a* ***unit vector*** *if* $\|\mathbf{x}\| = 1$. *Finally, a subset* S *of* V *if* S *is orthogonal and consists entirely of unit vectors.*
$\mathbf{x}, \mathbf{y}$ *: "orthogonal" (perpendicular) if* $\langle \mathbf{x},\, \mathbf{y} \rangle = 0$
$\mathbf{x}, \mathbf{y}$ *: "orthonomal" if* $\mathbf{x}, \mathbf{y}$ *: orthogonal &* $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$

Theorem. *Let* V *be an inner product space, and let* S *be an orthogonal subset of* V *consisting of nonzero vectors. Then* S *is linearly independent.(Corollary 2 in the book)*
$S : \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \Rightarrow S$ *is linearly independent.*

Proof. Let $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = 0$.

$$\begin{aligned}\forall i, 0 &= \langle \mathbf{0},\, \mathbf{v}_i \rangle \\ &= \langle a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k,\, \mathbf{v}_i \rangle \\ &= \sum_{j=1}^{n} a_j \langle \mathbf{v}_j,\, \mathbf{v}_i \rangle \\ &= a_i \left\|\mathbf{v}_i\right\|^2 (\because S : \text{orthogonal subset})\end{aligned}$$

Since $\left\|\mathbf{v}_i\right\|^2 \geq 0$, $a_i = 0$ for all $i = 1, 2, \ldots, k$. So S is linearly independent. $\square$

Theorem. *Let* V *be an inner product space and* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *be an orthogonal subset of* V *consisting of nonzero vectors. If* $\mathbf{v} \in span(S)$, *then*

$$\begin{aligned}\mathbf{v} &= \sum_{i=1}^{k} \frac{\langle \mathbf{v},\, \mathbf{v}_i \rangle}{\left\|\mathbf{v}_i\right\|^2} v_i \\ &= \langle \mathbf{v},\, \mathbf{v}_1 \rangle \frac{\mathbf{v}_1}{\left\|\mathbf{v}_1\right\|^2} + \langle \mathbf{v},\, \mathbf{v}_2 \rangle \frac{\mathbf{v}_2}{\left\|\mathbf{v}_2\right\|^2} + \cdots + \langle \mathbf{v},\, \mathbf{v}_k \rangle \frac{\mathbf{v}_k}{\left\|\mathbf{v}_k\right\|^2}\end{aligned}$$

We can compute a_i *for all* $i = 1, \ldots, k$

Proof. Let $\mathbf{v} = a_i\mathbf{v}_i + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$. Then

$$\begin{aligned}\langle \mathbf{v}, \mathbf{v}_i, \rangle &= \sum_{j=1}^{k} a_j \langle \mathbf{v}_j,\, \mathbf{v}_i \rangle \\ &= a_i \left\|\mathbf{v}_i\right\|^2 (\because S : \mathit{orthogonal})\end{aligned}$$

So

$$a_i = \frac{\langle \mathbf{v},\, \mathbf{v}_i \rangle}{\left\|\mathbf{v}_i\right\|^2}$$

$\square$

Theorem (Gram-Schmidt Process)**.** *Let* V *be an inner product space and* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *be a linearly independent subset of* V. *Define* $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, *where* $\mathbf{w}_1 = \mathbf{v}_1$ *and*

$$\mathbf{w}_k = \mathbf{v}_k - \sum_{j=1}^{k-1} \frac{\langle \mathbf{v}_k,\, \mathbf{w}_j \rangle}{\left\|\mathbf{w}_j\right\|^2} \mathbf{w}_k \quad \mathit{for} \quad 2 \leq k \leq n$$

Then S' *is an orthogonal set of nonzero vectors such that* $\operatorname{span}(S') = \operatorname{span}(S)$

Proof. (Idea) Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be linearly independent subset. We want to find an orthogonal subset $\{\mathbf{w}_1, \mathbf{w}_2\}$ such that $\operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \operatorname{span}(\{\mathbf{w}_1, \mathbf{w}_2\})$ where $\mathbf{v}_1 = \mathbf{w}_1$.

Since $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, we can construct $\mathbf{w}_2$ such that $\mathbf{w}_2 = a_1\mathbf{v}_1 + \mathbf{v}_2$. To find a_1,

$$\begin{aligned} \mathbf{w}_2 &\coloneqq a_1\mathbf{v}_1 + \mathbf{v}_2 \\ \implies 0 = \langle \mathbf{w}_2, \mathbf{w}_1 \rangle &= \langle a_1\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 \rangle \\ &= a_1\langle \mathbf{v}_1, \mathbf{w}_1 \rangle + \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \\ &= a_1\langle \mathbf{w}_1, \mathbf{w}_1 \rangle + \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \end{aligned}$$

$$\therefore a_1 = -\frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} = -\frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}$$

So, we can construct $\mathbf{w}_2$ such that $\mathbf{w}_2 \coloneqq \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$.
Repeat this process on $\mathbf{w}_3, \mathbf{w}_4, \ldots, \mathbf{w}_k$ for $2 \le k \le n$, then we have

$$\begin{aligned} \mathbf{w}_3 &\coloneqq \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 \\ \mathbf{w}_4 &\coloneqq \mathbf{v}_4 - \frac{\langle \mathbf{v}_4, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_4, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \frac{\langle \mathbf{v}_4, \mathbf{w}_3 \rangle}{\langle \mathbf{w}_3, \mathbf{w}_3 \rangle}\mathbf{w}_3 \\ &\vdots \\ \mathbf{w}_k &\coloneqq \mathbf{v}_k - \frac{\langle \mathbf{v}_k, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_k, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \cdots - -\frac{\langle \mathbf{v}_k, \mathbf{w}_{k-1} \rangle}{\langle \mathbf{w}_{k-1}, \mathbf{w}_{k-1} \rangle}\mathbf{w}_{k-1} \\ &\coloneqq \mathbf{v}_k - \sum_{i=1}^{k-1} \frac{\langle \mathbf{v}_k, \mathbf{w}_i \rangle}{\langle \mathbf{w}_i, \mathbf{w}_i \rangle}\mathbf{w}_i \end{aligned}$$

□

Theorem. *Let $\mathfrak{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis and $A = [\mathsf{T}]_{\mathfrak{B}}$. Then $A_{ij} = \langle \mathsf{T}\mathbf{v}_j, \mathbf{v}_i \rangle$*

Proof.

$$\mathsf{T}\mathbf{v}_j = \langle \mathsf{T}\mathbf{v}_j, \mathbf{v}_1 \rangle\mathbf{v}_1 + \langle \mathsf{T}\mathbf{v}_j, \mathbf{v}_2 \rangle\mathbf{v}_2 + \cdots + \langle \mathsf{T}\mathbf{v}_j, \mathbf{v}_n \rangle\mathbf{v}_n$$

$$\therefore A_{ij} = \langle \mathsf{T}\mathbf{v}_j, \mathbf{v}_i \rangle$$

□

Example 1. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset \mathbb{R}^4$
$\mathbf{v}_1 = (1, 0, 1, 0)$, $\mathbf{v}_2 = (1, 1, 1, 1)$, $\mathbf{v}_3 = (0, 1, 2, 1)$.
Then $\mathbf{w}_1 = \mathbf{v}_1 = (1, 0, 1, 0)$,

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{w}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 \\ &= (1, 1, 1, 1) - \frac{2}{2}(1, 0, 1, 0) \\ &= (0, 1, 0, 1) \end{aligned}$$

$$
\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{w}_1,\, \mathbf{v}_3\rangle}{\langle \mathbf{w}_1,\, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2,\, \mathbf{v}_3\rangle}{\langle \mathbf{w}_2,\, \mathbf{w}_2\rangle}\mathbf{w}_2 \\
&= (0,1,2,1) - \frac{2}{2}(1,0,1,0) - \frac{2}{2}(0,1,0,1) \\
&= (-1,0,1,0)
\end{aligned}
$$

Example 2. $\mathsf{V} = \mathbf{P}_2(\mathbb{R})$,

$$
\langle f(x),\, g(x)\rangle \coloneqq \int_{-1}^{1} f(x)g(x)dx
$$

Let $\mathfrak{B} = \{1, x, x^2\}$ *be standard basis for* V. *Find orthonomal basis.*
$\mathbf{w}_1 = \mathbf{v}_1 = 1$

$$
\begin{aligned}
\mathbf{w}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{w}_1,\, \mathbf{v}_2\rangle}{\langle \mathbf{w}_1,\, \mathbf{w}_1\rangle}\mathbf{w}_1 \\
&= x - \frac{\int_{-1}^{1} 1\cdot x dx}{\int_{-1}^{1} 1\cdot 1 dx}\cdot 1 \\
&= x - \frac{\left[\frac{1}{2}x^2\right]_{-1}^{1}}{\left[x\right]_{-1}^{1}}\cdot 1 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{w}_1,\, \mathbf{v}_3\rangle}{\langle \mathbf{w}_1,\, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2,\, \mathbf{v}_3\rangle}{\langle \mathbf{w}_2,\, \mathbf{w}_2\rangle}\mathbf{w}_2 \\
&= x^2 - \frac{\int_{-1}^{1} 1\cdot x^2 dx}{\int_{-1}^{1} 1\cdot 1 dx}\cdot 1 - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx}\cdot x \\
&= x^2 - \frac{\left[\frac{1}{3}x^3\right]_{-1}^{1}}{\left[x\right]_{-1}^{1}} - \frac{\left[\frac{1}{4}x^4\right]_{-1}^{1}}{\left[\frac{1}{3}x^3\right]_{-1}^{1}}\cdot x \\
&= x^2 - \frac{1}{3}
\end{aligned}
$$

We want to find an orthonomal basis, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are orthogonal vectors &* $\|\mathbf{u}_1\|^2 = \|\mathbf{u}_2\|^2 = \|\mathbf{u}_3\|^2 = 1$

$$\begin{aligned}
\|\mathbf{w}_1\|^2 &= \langle \mathbf{w}_1,\, \mathbf{w}_1\rangle \\
&= \int_{-1}^{1} 1\cdot 1 dx \\
&= \Big[x\Big]_{-1}^{1} = \frac{1}{2} \\
\|\mathbf{w}_2\|^2 &= \langle \mathbf{w}_2,\, \mathbf{w}_2\rangle \\
&= \int_{-1}^{1} x\cdot x dx \\
&= \Big[\frac{1}{3}x^3\Big]_{-1}^{1} \\
&= \frac{2}{3} \\
\|\mathbf{w}_3\|^2 &= \langle \mathbf{w}_3,\, \mathbf{w}_3\rangle \\
&= \int_{-1}^{1} (x^2-\frac{1}{3})(x^2-\frac{1}{3}) dx \\
&= \int_{-1}^{1} x^4-\frac{2}{3}x^2+\frac{1}{9} dx \\
&= \Big[\frac{1}{5}x^5-\frac{2}{9}x^3+\frac{1}{9}x\Big]_{-1}^{1} \\
&= \frac{8}{45}
\end{aligned}$$

$\therefore$ *orthonormal basis* $= \{\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2-\frac{1}{3})\}$

Example 3. $f(x) = 1 + 2x + 3x^2 \in \mathrm{P}_2(\mathbb{R})$. *Express a linear combination of orthonomal basis* $\{\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2-\frac{1}{3})\}$.
Let $\mathbf{u}_1 = \sqrt{\frac{1}{2}}, \mathbf{u}_2 = \sqrt{\frac{3}{2}}x, \mathbf{u}_3 = \sqrt{\frac{45}{8}}(x^2-\frac{1}{3})$.
Then by Theorem,

$$a_i = \frac{\langle f(x),\, \mathbf{v}_i\rangle}{\|\mathbf{v}_i\|^2} = \langle f(x),\, \mathbf{v}_i\rangle \quad (\because \mathbf{v}_i \textit{ is orthonormal vector, } \|\mathbf{v}_i\|^2 = 1)$$

$$
\begin{aligned}
a_1 &= \langle f(x),\, \mathbf{v}_1 \rangle \\
&= \int_{-1}^{1} \sqrt(\frac{1}{2}) + \sqrt{2}x + \sqrt{\frac{9}{2}}x^2 dx \\
&= \Big[\sqrt{\frac{1}{2}}x + \frac{2}{\sqrt{2}}x^2 + \sqrt{\frac{1}{2}}x^3 \Big]_{-1}^{1} \\
&= 2\sqrt{2} \\
a_2 &= \langle f(x),\, \mathbf{v}_2 \rangle \\
&= \int_{-1}^{1} \frac{\sqrt{6}}{2}x + \sqrt{6}x^2 + \frac{3\sqrt{3}}{2}x^3 dx \\
&= \Big[\frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{6}}{3}x^3 + \frac{3\sqrt{3}}{8}x^4 \Big]_{-1}^{1} \\
&= \frac{2\sqrt{6}}{3} \\
a_3 &= \langle f(x),\, \mathbf{v}_3 \rangle \\
&= \int_{-1}^{1} \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})(1 + 2x^2 + 3x^3)dx \\
&= \int_{-1}^{1} \sqrt{\frac{5}{8}}(3x^2 - 1)(1 + 2x^2 + 3x^3)dx \\
&= \sqrt{\frac{5}{8}} \int_{-1}^{1} -1 - 2x + 6x^3 + 9x^5 dx \\
&= \sqrt{\frac{5}{8}} \Big[-x + x^2 + \frac{3}{2}x^4 + \frac{9}{5}x^5 \Big] \\
&= \sqrt{\frac{5}{8}} \times \frac{8}{5} \\
&= \frac{2\sqrt{10}}{5}
\end{aligned}
$$

$$
\therefore f(x) = 2\sqrt{2}\mathbf{u}_1 + \frac{2\sqrt{6}}{3}\mathbf{u}_2 + \frac{2\sqrt{10}}{5}\mathbf{u}_3
$$