class note 181222

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1.2 Vector Space

Definition. *Cartesian product:* $A \times B := \{(a,b) | a \in A, b \in B\}$

Definition (Binary operation)**.** $\times$ *is cartesian product and* S *is a set. Then binary operation* $*$ *is defined as follows.*

$$\begin{aligned} * :\ & S \times S \longrightarrow S \\ & (a,b) \longmapsto *(a,b) =: a * b \end{aligned}$$

Definition (Scalar multiplication)**.** $\mathbf{F}$ *is field and* S *is a set. Then scalar multiplication is defined as follows.*

$$\begin{aligned} \cdot : & \mathbf{F} \times S \longrightarrow S \\ & (a,s) \longmapsto \cdot(a,s) =: a \cdot s = as \end{aligned}$$

Definition (Vector Space)**.** $\mathbf{V}$ *is non-empty set and* $\mathbf{F}$ *is a field.* $(\mathbf{V}, +, \cdot)$ *is vector space over* $\mathbf{F}$ *if the following conditions hold.*

1. *For all* $\mathbf{u}, \mathbf{v}, \mathbf{v} \in \mathbf{V}, \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *(associativity of addition).*
2. $\exists \mathbf{0} \in \mathbf{V}$ *such that* $\forall \mathbf{v} \in \mathbf{V}$, $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ *(existence of identity)*
3. $\forall \mathbf{v} \in \mathbf{V}, \mathbf{v}' \in \mathbf{V}$ *such that* $\mathbf{v} + \mathbf{v}' = \mathbf{v}' + \mathbf{v} = \mathbf{0}$ *(an inverse of* $\mathbf{v}$*)*
4. *For all* $\mathbf{v}, \mathbf{w} \in \mathbf{V}$, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ *(commutativity of addition)*
5. $(a+b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}\ \forall a, b \in \mathbf{F} \quad , \forall \mathbf{v} \in \mathbf{V}$
6. $(ab)\mathbf{v} = a(b\mathbf{v})$
7. $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$
8. $1 \cdot \mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V}$

Example 1.

1. $\mathbb{R}^n \coloneqq \{(a_1, \ldots, a_n) | \, a_i \in \mathbb{R}^n\}$

   (a) *Binary operation:* $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) \coloneqq (a_1 + b_1, \ldots, a_n + b_n)$

   (b) *Scalar multiplication:* $c \cdot (a_1, \ldots, a_n) \coloneqq (ca_1, \ldots, ca_n)$

   (c) $\mathbf{0} \coloneqq (0, \ldots, 0)$

   (d) *an inverse of* $(a_1, \ldots, a_n) = (-a_1, \ldots, -a_n) =: -(a_1, \ldots, a_n)$

2. $\mathfrak{M}_{m\times n}(\mathbb{R}) \coloneqq \{A = (a_{ij}) \mid A : m \times n \text{ matrix}, a_{ij} \in \mathbb{R}\},\ A = (a_{ij}), B = (b_{ij}) \in \mathfrak{M}_{m\times n}(\mathbb{R})$

   (a) $A + B \coloneqq (a_{ij} + b_{ij})$

   (b) $c \cdot A \coloneqq (ca_{ij})$

   (c) $\mathbf{0} \coloneqq= \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

   (d) $-A \coloneqq (-a_{ij})$

3. $\mathcal{F}(S, \mathbf{F}) \coloneqq \{\text{all functions } f : S \longmapsto \mathbf{F}\}$

   (a) $(f + g)(s) \coloneqq f(s) + g(s)\ \forall s \in S$

   (b) $(c \cdot f)(s) \coloneqq cf(s)$

   (c)

$$\begin{aligned} \mathbf{0} :\ & S \longrightarrow \mathbf{F} \\ & s \longmapsto 0 \end{aligned}$$

   (d) $(-f)(s) \coloneqq -f(s)$

4. $\mathbf{P}(\mathbb{R}) \coloneqq \{a_0 + a_1X + \cdots + a_nX^n \mid a_i \in \mathbb{R}\}$
   $f(x) \coloneqq a_0 + a_1X + \cdots + a_nX^n$, $g(x) \coloneqq b_0 + b_1X + \cdots + b_mX^m$ *and we assume* $m \geq n$ *and* $a_i = 0$ *for all* $i \geq n + 1$.

   (a) $f(x) + g(x) \coloneqq (a_0 + b_0) + (a_1 + b_1)X^1 + \cdots + (a_m + b_m)X^m$

   (b) $c \cdot f(x) = (ca_0) + (ca_1)X^1 + \cdots + (ca_n)X^n$

   (c) $\mathbf{0} \coloneqq 0 + 0 \cdot X^1 + \cdots + 0 \cdot X^n$

   (d) $-f(x) \coloneqq -a_0 - a_1X^1 - \cdots - a_nX^n$

5. *Sequence space* $V \coloneqq \{\text{all sequences } \{a_n\}_{n=1}^{\infty}\}$

$$\begin{aligned} f : \mathbb{N} & \longrightarrow \mathbb{R} \\ n & \longmapsto f(n) =: a_n \end{aligned}$$

   (a) $\{a_n\} + \{b_n\} \coloneqq \{a_n + b_n\}$

(b) $c \cdot \{a_n\} \coloneqq \{c \cdot a_n\}$

(c) $\mathbf{0} \coloneqq \{0\}$

(d) *inverse of* $\{a_n\}$*:* $\{-a_n\}$

Theorem (Cancellation law)**.** *If* $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$*, then* $\mathbf{y} = \mathbf{z}$

Proof.

$$\begin{aligned}((-\mathbf{x}) + \mathbf{x}) + \mathbf{y} &= ((-\mathbf{x}) + \mathbf{x}) + \mathbf{z} \\ \mathbf{0} + \mathbf{y} &= \mathbf{0} + \mathbf{z} \\ \mathbf{y} &= \mathbf{z}\end{aligned}$$

□

Corollary. *An identity* $\mathbf{0}$ *is unique*

Proof. Let $\mathbf{0}_1, \mathbf{0}_2$ be two different identities in $\mathbf{V}$. Then $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$ □

Theorem.

1. $0 \cdot \mathbf{x} = \mathbf{0}$

2. $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$

3. $a \cdot \mathbf{0} = \mathbf{0}$

Proof.

1.

$$\begin{aligned}\mathbf{0} + 0 \cdot \mathbf{x} &= 0 \cdot \mathbf{x} \\ &= (0 + 0)\mathbf{x} \\ &= 0 \cdot \mathbf{x} + 0 \cdot \mathbf{x} \\ \therefore 0 \cdot \mathbf{x} &= \mathbf{0}\end{aligned}$$

2.

$$\begin{aligned}a\mathbf{x} + (-a)\mathbf{x} &= (a + (-a))\mathbf{x} \\ &= 0 \cdot \mathbf{x} \\ &= \mathbf{0}\end{aligned}$$

3.

$$\begin{aligned} a\mathbf{0} &= a(\mathbf{0}+\mathbf{0}) \\ &= a\mathbf{0} + a\mathbf{0} \\ \therefore a\mathbf{0} &= \mathbf{0} \end{aligned}$$

□