Linear Algebra Exam

April 20, 2019

1. (By Seanie Lee) Let V, W be finite dimensional $\mathbb{R}$-vector spaces and let $L : V \to W$ be a linear map. Show that L is one-to-one if and only if L carries linearly independent subsets of V to linearly independent subsets of W, i.e. $\left[\{\mathbf{v}_1, \dots, \mathbf{v}_k\}\text{: lin. indep. in } V \implies \{L\mathbf{v}_1, \dots, L\mathbf{v}_k\}\text{: lin. indep. in } W\right]$.

2. (By Dongsu Kang) State and prove the dimension theorem.

3. (By Jonghwan Jang) Give at least 4 properties of determinant and prove two of them.

4. (By Youngjun Kwon) Let V, W be finite dimensional $\mathbb{R}$-vector spaces. Prove that V is isomorphic to W if and only if $\dim V = \dim W$.

5. Find all real numbers k such that the following matrix is invertible:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$$

6. (a) Find a 3×3 nonsingular real matrix A satisfying $3A = A^2 + AB$, where

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

   (b) Find the inverse of A.

7. Let G be a group and let $H \leq G$ be given.

   (a) For fixed $g \in G$, show that $g^{-1}Hg := \{g^{-1}hg \in G \,:\, h \in H\}$ is a subgroup of G,

   (b) We say that H is normal in G if $g^{-1}Hg \subset H$ for all $g \in G$. Prove that if G is abelian, then H is always normal in G.

8. Consider the general linear group $\mathrm{GL}_n(\mathbb{R})$ which consists of all $n \times n$ invertible matrices over $\mathbb{R}$. It is a group under matrix multiplication.

   (a) Prove or disprove that $\{A \in \mathrm{GL}_n(\mathbb{R}) \,:\, \det A = 2\}$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$,

(b) Show that the special linear group $\mathrm{SL}_n(\mathbb{R})$ is normal in $\mathrm{GL}_n(\mathbb{R})$.

9. Let A, $B \in M_{n\times n}(\mathbb{R})$ be given. Write the definition of the followings, respectively:

   (a) A is similar to B,

   (b) A is diagonalizable,

   (c) the characteristic polynomial $\phi_A(t)$ of A.

Answer

1. (By Seanie Lee) Let V, W be finite dimensional $\mathbb{R}$-vector spaces and let $L : V \to W$ be a linear map. Show that L is one-to-one if and only if L carries linearly independent subsets of V to linearly independent subsets of W

   proof) $\to$). Suppose that $\sum_{i=1}^{k} a_i L(\mathbf{v_i}) = 0$
   $then, (LHS) = L(\sum_{i=1}^{k} a_i \mathbf{v_i} = 0$
   since $L : 1-1,\ \sum_{i=1}^{k} a_i \mathbf{v_i} = 0$
   since $\mathbf{v_1}, \dots, \mathbf{v_k}$lin. indep, all a_i are 0
   $\therefore \mathbf{L}(\mathbf{v_i}), \dots, \mathbf{L}(\mathbf{v_k})$: lin.indep

2. (By Dongsu Kang)Let V, W are vector spaces and $T : V \to W$, T is linear. Then,

$$dim(V) = dim(kerT) + dim(imT)$$

   proof) Let $\dim(V) = n$, $\dim(\ker T) = k$, $\{\mathbf{v_1}, \dots, \mathbf{v_k}\}$ is basis for $kerT$, and it could be extended to $\{\mathbf{v_1}, \dots, \mathbf{v_n}\}$ by basis extended theorem and it is basis for V. then, we want to show that $S := \{T\mathbf{v_{k+1}}, \dots, T\mathbf{v_n}\}$ is basis for im$T$

   1) linear indepence

   Suppose that $\sum_{i=k+1}^{n} a_i \mathbf{v_1} = 0, \forall a_i = 0, i = k+1, \dots, n.$ by linearity, $T(\sum_{i=k+1}^{n} a_i \mathbf{v_1}) \in kerT.$ and we can write $\sum_{i=1}^{k} b_i v_i = \mathbf{v}, \mathbf{v} \in kerT.$ since $b_1\mathbf{v_1} + \ldots + b_k\mathbf{v_k} + \ldots + a_{k+1}\mathbf{v_{k+1}} + \ldots + a_n\mathbf{v_n} = 0, \forall a_i = 0, \forall b_i = 0.$ therefore, all coefficients are zero scalar and it is linear independence.

   2) span(S) = imT

   $\mathbf{v} \in V$, $T(\mathbf{v}) \in imT,$

$$\begin{aligned}
\mathbf{v} &= a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n} \\
T(\mathbf{v}) &= T(a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n}) \\
&= a_1T(\mathbf{v_1}) + \cdots + a_{k+1}T(\mathbf{v_{k+1}}) + \cdots + a_nT(\mathbf{v_n}) (\because \ by\ linearity)
\end{aligned}$$

   $a_i\mathbf{v_i} \in kerT$ $, \forall a_i = 0, i = 1, \dots, k$

   $\therefore span(S) = imT$

   $\therefore set\ S\ is\ basis\ for\ imT$ ■

3. (By Jonghwan Jang)

4. (By Youngjun Kwon) Let V, W be finite dimensional $\mathbb{R}$-vector spaces. Prove that V is isomorphic to W if and only if $\dim V = \dim W$.

   proof)

   $\rightarrow$) Suppose that V is isomorphic to W such that $T : V \rightarrow W$ is an isomorphism from V to W. Isomorphism represent linear and bijection which leads to that both vector spaces have same dimension. so $dim(V) = dim(W)$

   $\leftarrow$) Suppose that $dim(V) = dim(W)$ and let $= \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ and $\gamma = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}\}$ be bases for V and W. respectively. There exists $T : V \rightarrow W$ such that T is linear and $T(v_i) = w_i,\ for\ i = 1, 2, \ldots, n.$

$$dim(imT) = span(T(\mathcal{B})) = span(\gamma) = W$$

   So T is onto. Since $R(T)$ is subspace of W, we have that T is also one-to-one.

   $\therefore T$ is an isomorphism ■

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7. (a) $h_1,\ h_2 \in H, g^{-1}h_1g \cdot (g^{-1}h_2g)^{-1} \ = \ g^{-1}(h_1h_2^{-1})g \in g^{-1}Hg$

   (b) $\forall g \in G,\ g^{-1}hg = g^{-1}gh = h \in H.$

   $\therefore abelian$ ■

8. (a) It is easy to solve it

   (b)$A \in GL_n(\mathbb{R}),\ B \in SL_n(\mathbb{R})\ det(A^{-1}BA) = det(A^{-1})det(B)det(A) = 1$

   $\therefore\ A^{-1}SL_n(\mathbb{R})A \subset SL_n(\mathbb{R})$

9. (a) $A, B \in M_{n\times n}(\mathbb{R})$ "A is similar to B" if $\exists Q \in M_{n\times n}(\mathbb{R}).\ inv.\ such\ that\ B = Q^{-1}AQ$

   (b) A linear operator T on a finite-dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is called **diagonalizable** if L_A is diagonalizable.

   (c) Let $A \in M_{n\times n}(F)$. The polynomial $f(t) = det(A - tI_n)$ is called the **characteristic polynomial** of A