Linear Algebra Class on 9 February

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2.2 Isomorphism and Matrix

Note. *Let* $\mathbf{V}$ *be finite dimensional vector space and* $\dim \mathbf{V} = n$. *Then* $\mathbf{V} \cong \mathbb{R}^n$

Definition. *Let* $\mathfrak{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *a basis for* $\mathbf{V}$ *and* $\mathbf{v} \coloneqq a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$

1. $[\mathbf{v}]_{\mathfrak{B}} \coloneqq (a_1, \ldots, a_n) \in \mathbb{R}^n$ *is coordinate vector*
2. $[\cdot]_{\mathfrak{B}} \coloneqq \mathbf{V} \longrightarrow \mathbb{R}^n$, $\mathbf{v} \longmapsto [\mathbf{v}]_{\mathfrak{B}}$ *is coordinate map*

Theorem. $[\cdot]_{\mathfrak{B}}$ *is isomorphism*

Proof.

1. $[\cdot]_{\mathfrak{B}}$ is linear
   $\mathbf{v} \coloneqq a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, $\mathbf{w} \coloneqq b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n \quad \mathbf{v}, \mathbf{w} \in \mathbf{V}$

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= (a_1 + b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n \\
[\mathbf{v} + \mathbf{w}]_{\mathfrak{B}} &= (a_1 + b_1, \ldots, a_n + b_n) \\
&= (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \\
&= [\mathbf{v}]_{\mathfrak{B}} + [\mathbf{w}]_{\mathfrak{B}}
\end{aligned}$$

2. $[\cdot]_{\mathfrak{B}}$ is 1-1
   Let $\mathbf{v} \in \ker[\cdot]_{\mathfrak{B}}$. That is $[\mathbf{v}]_{\mathfrak{B}} = (0, \ldots, 0)$, i.e., $\mathbf{v} = \mathbf{0}$. $\therefore \ker[\cdot]_{\mathfrak{B}} = \mathbf{O}$
   $\therefore [\cdot]_{\mathfrak{B}}$ is 1-1.

3. onto
   Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$ be given. Define $\mathbf{v} \coloneqq a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. Then $(a_1, \ldots, a_n) = [\mathbf{v}]_{\mathfrak{B}} \in \operatorname{im}[\cdot]_{\mathfrak{B}}$. Thus, $\mathbb{R}^n \subset \operatorname{im}[\cdot]_{\mathfrak{B}}$. $\therefore [\cdot]_{\mathfrak{B}}$ is onto.

□

Corollary. $\dim \mathbf{V} = n \iff \mathbf{V} \cong \mathbb{R}^n$ *with* $[\cdot]_{\mathfrak{B}}$ *for some basis* $\mathfrak{B}$

Definition. *Let* $\varphi : \mathbf{V}^n \longrightarrow \mathbf{W}^m$ *linear map,* $\mathfrak{B} \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *a basis for* $\mathbf{V}$, *and* $\mathfrak{C} \coloneqq \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$. *Define* $[\varphi]^{\mathfrak{B}}_{\mathfrak{C}} \coloneqq \Big[[\varphi\mathbf{v}_1]_{\mathfrak{C}} \cdots [\varphi\mathbf{v}_n]_{\mathfrak{C}}\Big]$ *as matrix representation of* φ.

Example 1. $\mathbf{T} : P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R}),\ f(x) \longmapsto f'(x)\ \mathfrak{B} = \{1, x, x^2, x^3\}, \mathfrak{C} = \{1, x, x^2\}$.
$[\mathbf{T}]^{\mathfrak{B}}_{\mathfrak{C}} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

Theorem. *Let* $\mathbf{T} : \mathbf{V} \longrightarrow \mathbf{W}$ *linear map. Then* $[\mathbf{Tv}]_{\mathfrak{C}} = [\mathbf{T}]^{\mathfrak{B}}_{\mathfrak{C}}[\mathbf{v}]_{\mathfrak{B}}$ *for some basis* $\mathfrak{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *for* $\mathbf{V}$.

Proof. $[\mathbf{T}]^{\mathfrak{B}}_{\mathfrak{C}}[\mathbf{v}_j]_{\mathfrak{B}} = [\mathbf{T}]^{\mathfrak{B}}_{\mathfrak{C}}\mathbf{e}_j = [\mathbf{Tv}_j]_{\mathfrak{C}}$ for $j = 1, \ldots, n \therefore [\mathbf{Tv}]_{\mathfrak{C}} = [\mathbf{T}]^{\mathfrak{B}}_{\mathfrak{C}}[\mathbf{v}]_{\mathfrak{B}}$ □

Note. $A \in \mathfrak{M}_{m\times n}(\mathbb{R})$

$$\begin{aligned} \mathbf{L}_A : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \mathbf{x} &\longmapsto A\mathbf{x} \end{aligned}$$

Note. $\mathfrak{L}(\mathbf{V}, \mathbf{W}) := \{$*all linear maps* $\mathbf{V} \longrightarrow \mathbf{W}\}$, $(\mathbf{L} + \mathbf{M})\mathbf{v} := \mathbf{Lv} + \mathbf{Mv}$, *and* $(c\mathbf{L})\mathbf{v} := c\mathbf{Lv} \quad \mathbf{L}, \mathbf{M} \in \mathfrak{L}(\mathbf{V}, \mathbf{W})$. *Then* $\mathfrak{L}(\mathbf{V}, \mathbf{W})$ *is vector space over* $\mathbb{R}$

1. *identity*

$$\begin{aligned} \boldsymbol{0} : \mathbf{V} &\longrightarrow \mathbf{W} \\ \mathbf{v} &\longmapsto \boldsymbol{0} \end{aligned}$$

2. *inverse*

$$\begin{aligned} -\mathbf{L} : \mathbf{V} &\longrightarrow \mathbf{W} \\ \mathbf{v} &\longmapsto -\mathbf{Lv} \end{aligned}$$

Note. $\mathbf{L} + \mathbf{M} \in \mathfrak{L}(\mathbf{V}, \mathbf{W})$

Proof.

$$\begin{aligned} (\mathbf{L} + \mathbf{M})(\mathbf{v}_1 + \mathbf{v}_2) &= \mathbf{L}(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{M}(\mathbf{v}_1 + \mathbf{v}_2) \\ &= \mathbf{Lv}_1 + \mathbf{Mv}_1 + \mathbf{Lv}_2 + \mathbf{Mv}_2 \\ &= (\mathbf{L} + \mathbf{M})\mathbf{v}_1 + (\mathbf{L} + \mathbf{M})\mathbf{v}_2 \end{aligned}$$

$$\begin{aligned} (\mathbf{L} + \mathbf{M})(c\mathbf{v}) &= \mathbf{L}c\mathbf{v} + \mathbf{M}c\mathbf{v} \\ &= c\mathbf{Lv} + c\mathbf{M}c\mathbf{v} \\ &= c(\mathbf{Lv} + \mathbf{Mv}) \\ &= c(\mathbf{L} + \mathbf{M})\mathbf{v} \end{aligned}$$

□

Theorem. $\mathbf{L} \in \mathfrak{L}(\mathbf{U}^k, \mathbf{V}^n)$, $\mathbf{M} \in \mathfrak{L}(\mathbf{V}^n, \mathbf{W}^m)$, $(\mathbf{M} \circ \mathbf{L})\mathbf{v} := \mathbf{M}(\mathbf{Lv})$. *Then* $\mathbf{M} \circ \mathbf{L} \in \mathfrak{L}(\mathbf{V}, \mathbf{W})$

Proof.

$$
\begin{aligned}
(\mathbf{M}\circ\mathbf{L})(\mathbf{v}_1+\mathbf{v}_2) &= \mathbf{M}(\mathbf{L}(\mathbf{v}_1+\mathbf{v}_2)) \\
&= \mathbf{M}(\mathbf{L}\mathbf{v}_1+\mathbf{L}\mathbf{v}_2) \\
&= \mathbf{M}(\mathbf{L}\mathbf{v}_1)+\mathbf{M}(\mathbf{L}\mathbf{v}_2) \\
&= (\mathbf{M}\circ\mathbf{L})\mathbf{v}_1+(\mathbf{M}\circ\mathbf{L})\mathbf{v}_2
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{M}\circ\mathbf{L})(c\mathbf{v}) &= \mathbf{M}(\mathbf{L}c\mathbf{v}) \\
&= \mathbf{M}(c\mathbf{L}\mathbf{v}) \\
&= c\mathbf{M}\mathbf{L}\mathbf{v}
\end{aligned}
$$

□

Note. $\mathfrak{A} \coloneqq \{\mathbf{u}_1,\ldots,\mathbf{u}_k\}, \mathfrak{B} \coloneqq \{\mathbf{v}_1,\ldots,\mathbf{v}_n\}, \mathfrak{C} \coloneqq \{\mathbf{w}_1,\ldots,\mathbf{w}_m\}$ *bases for* $\mathbf{U},\mathbf{V},\mathbf{W}$ *respectively.* $[\mathbf{M}]^{\mathfrak{B}}_{\mathfrak{C}} \coloneqq A_{m\times n}$, $[\mathbf{L}]^{\mathfrak{A}}_{\mathfrak{B}} \coloneqq B_{n\times k}$. *What is matrix representation of* $[\mathbf{M}\circ\mathbf{L}]^{\mathfrak{A}}_{\mathfrak{C}}$*?*

$$
\begin{aligned}
(\mathbf{M}\circ\mathbf{L})\mathbf{u}_j &= \mathbf{M}(\mathbf{L}\mathbf{u}_j) \\
&= \mathbf{M}(\sum_{l=1}^{n} B_{lj}\mathbf{v}_l) \\
&= \sum_{l=1}^{n} B_{lj}M\mathbf{v}_l \\
&= \sum_{l=1}^{n}\sum_{p=1}^{m} A_{pl}\mathbf{w}_p \\
&= \sum_{p=1}^{m}(\sum_{l=1}^{n} A_{pl}B_{lj})\mathbf{w}_p
\end{aligned}
$$

$$
\begin{aligned}
\big[(\mathbf{M}\circ\mathbf{L})\mathbf{u}_j\big]_{\mathfrak{C}} &= (\sum_{l=1}^{n} A_{1l}Blj, \sum_{l=1}^{n} A_{2l}B_{lj},\ldots,\sum_{l=1}^{n} A_{ml}B_{lj}) \\
\Big([\mathbf{M}\circ\mathbf{L}]^{\mathfrak{A}}_{\mathfrak{C}}\Big)_{ij} &= \sum_{l=1}^{n} A_{il}B_{lj}
\end{aligned}
$$

Definition. $A \in \mathfrak{M}_{m\times n}(\mathbb{R}), B \in \mathfrak{M}_{n\times k}(\mathbb{R})$
$[AB]_{ij} = \sum\limits_{l=1}^{n} A_{il}B_{lj}$ *for* $i = 1,\ldots,m \quad j = 1,\ldots,k$
AB*: the product of* A *and* B*,* $AB \in \mathfrak{M}_{m\times k}(\mathbb{R})$