T-invariant and T-cyclic subspace

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Theorem 1. *Let* $\mathbf{T}$ *be a linear operator on a finite-dimensional vector space* $\mathbf{V}$ *such that the characteristic polynomial of* $\mathbf{T}$ *splits. Let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the distinct eigenvalues of* $\mathbf{T}$*. Then,*

(1). $\mathbf{T}$ *is diagonalizable if and only if the multiplicity of* λ_i *is equal to dim*$(\mathbf{E}_{\lambda_i})$ *for all* $i = 1, \ldots, k$

(2). If $\mathbf{T}$ *is diagonalizable and* β_i *is an ordered basis for* $\mathbf{E}_{\lambda_i}$*, for each* i*, then* $\beta = \beta_1 \cup \cdots \cup \beta_k$ *is an ordered basis for* $\mathbf{V}$ *consisting of eignevectors of* $\mathbf{T}$

Note. $[\mathbf{T}]_\beta = \begin{pmatrix} \lambda_1 I_{n_1} & & & 0 \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ 0 & & & \lambda_k I_{n_k} \end{pmatrix}$: block diagonal matrix

Definition 2. $\mathbf{T} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ *is linear operator and* $\boldsymbol{W} \leq \boldsymbol{V}$*. If* $\boldsymbol{TW} \subseteq W$*, then* $\boldsymbol{W}$ *is called T-invariant subspace.* $\mathbf{0}, \boldsymbol{V}$*, and* $\boldsymbol{E}_\lambda$ *are examples of it.*

Lemma 3. $M \coloneqq \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, $\det M = \det A \det C$

Proof. 1. if $\det A = 0$

Then there are some linearly dependent column vectors in A, so $\det M = \det A \det C = 0$

2. if $\det A \neq 0$

$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix} \det \begin{pmatrix} I_n & A^{-1}B \\ 0 & C \end{pmatrix} = \det A \det C$

$\square$

Note. $\boldsymbol{V} = \boldsymbol{W}_1 \bigoplus \cdots \bigoplus \boldsymbol{W}_k$, $\boldsymbol{T} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$, and β_i is a basis for $\boldsymbol{W}_i$. Then $\beta = \beta_1 \biguplus \cdots \biguplus \beta_k$ is a basis for $\boldsymbol{V}$. But $[\boldsymbol{T}]_\beta$ is not a block matrix because $\boldsymbol{T}(\beta_i) \not\subset \boldsymbol{W}_i$. But if $\boldsymbol{W}_1$ is T-invariant, then $[\boldsymbol{T}]_\beta = \begin{pmatrix} [\boldsymbol{T}|_{\boldsymbol{W}_1}]_{\beta_1} & B \\ 0 & C \end{pmatrix}$

Moreover if $\boldsymbol{W}_i$ is T-invariant for all i the, $[T]_\beta = \begin{pmatrix} [\boldsymbol{T}|_{\boldsymbol{W}_1}]_{\beta_1} & & & 0 \\ & [\boldsymbol{T}|_{\boldsymbol{W}_2}]_{\beta_2} & & \\ & & \ddots & \\ 0 & & & [\boldsymbol{T}|_{\boldsymbol{W}_k}]_{\beta_k} \end{pmatrix}$

and by Lemma 3. $\phi_{\boldsymbol{T}}(t) = \phi_{\boldsymbol{T}_1}(t) \times \cdots \times \phi_{\boldsymbol{T}_k}(t)$; where $\boldsymbol{T}_i \coloneqq \boldsymbol{T}|_{\boldsymbol{W}_1}$

Definition 4. *$\boldsymbol{T}:\boldsymbol{V}\longrightarrow\boldsymbol{V}$ linear. span$\{\boldsymbol{v},\boldsymbol{T}\boldsymbol{v},\boldsymbol{T}^2\boldsymbol{v},\dots,\}$ is $\boldsymbol{T}$-cyclic subspace of $\boldsymbol{v}\in\boldsymbol{V}$*

Note. $\boldsymbol{T}$-cyclic subspace of $\boldsymbol{v}\in\boldsymbol{V}$ is the smallest $\boldsymbol{T}$-invariant of $\boldsymbol{V}$ containing $\boldsymbol{v}$.

Proof. 1. $\boldsymbol{T}$-invariant.

$\boldsymbol{W}:=$ span$\{\boldsymbol{v},\boldsymbol{T}\boldsymbol{v},\boldsymbol{T}^2\boldsymbol{v},\dots,\}$ and $\boldsymbol{w}=\sum_{i=0}^{\infty}a_i\boldsymbol{T}^i\boldsymbol{v}\in\boldsymbol{W}$

$\boldsymbol{T}\boldsymbol{w}=\sum_{i=0}^{\infty}a_i\boldsymbol{T}^{i+1}\boldsymbol{v}\in\boldsymbol{W}$

$\therefore\boldsymbol{W}$ is $\boldsymbol{T}$-invariant.

2. Let $\boldsymbol{U}$ be any $\boldsymbol{T}$-invariant subspace of $\boldsymbol{V}$ containing $\boldsymbol{v}$. Since $\boldsymbol{U}$ is $\boldsymbol{T}$-invariant and $\boldsymbol{v}\in\boldsymbol{V}$, $\boldsymbol{T}\boldsymbol{v}\in\boldsymbol{V}$.Repeatedly $\boldsymbol{T}^k\boldsymbol{v}\in\boldsymbol{V}$ for all k. Thus $\{\boldsymbol{v},\boldsymbol{T}\boldsymbol{v},\boldsymbol{T}^2\boldsymbol{v},\dots\}\subset\boldsymbol{U}$ $\therefore$ span$\{\boldsymbol{v},\boldsymbol{T}\boldsymbol{v},\boldsymbol{T}^2\boldsymbol{v},\dots\}\leq\boldsymbol{U}$

□

Theorem 5. *$\boldsymbol{W}$ is $\boldsymbol{T}$-cyclic subspace of $\boldsymbol{V}$ generated by a nonzero vector $\boldsymbol{v}\in\boldsymbol{V}$ and dim$\boldsymbol{W}=k$*

1. *$\{\boldsymbol{v},\boldsymbol{T}\boldsymbol{v},\dots,\boldsymbol{T}^{k-1}\boldsymbol{v}\}$ is a basis for $\boldsymbol{W}$*
2. *$a_0\boldsymbol{v}+a_1\boldsymbol{T}\boldsymbol{v}+\cdots+a_{k-1}\boldsymbol{T}^{k-1}\boldsymbol{v}+\boldsymbol{T}^k\boldsymbol{v}=0\Longrightarrow\phi_{\boldsymbol{T}|\boldsymbol{W}}(t)=t^k+a_{k-1}t^{k-1}+\cdots+a_1t+a_0$*

Proof. 1. Let l be the largest integer such that $\beta:=\{\boldsymbol{v},\boldsymbol{T}\boldsymbol{v},\dots,\boldsymbol{T}^{l-1}\boldsymbol{v}\}$ is linearly independent. Let $\boldsymbol{Z}:=$ spanβ. Then β is a basis for $\boldsymbol{Z}$. Since β is linearly independent set and $\beta\cup\{\boldsymbol{T}^i\boldsymbol{v}\}$ is linearly dependent, $\boldsymbol{T}^i\boldsymbol{v}\in$ spanβ for $i=l,l+1,\dots$. So, $\boldsymbol{T}^l\boldsymbol{v}\in\boldsymbol{Z}$. Note that $\boldsymbol{Z}$ is $\boldsymbol{T}$-invariant because of the following reason.

$$\boldsymbol{w}=b_0\boldsymbol{v}+b_1\boldsymbol{T}\boldsymbol{v}+\cdots+b_{l-1}\boldsymbol{T}^{l-1}\in\boldsymbol{Z}$$
$$\boldsymbol{T}\boldsymbol{w}=b_0\boldsymbol{T}\boldsymbol{v}+b_1\boldsymbol{T}^2\boldsymbol{v}+\cdots+b_{l-2}\boldsymbol{T}^{l-1}\boldsymbol{v}+b_{l-1}\boldsymbol{T}^l\boldsymbol{v}\in\boldsymbol{Z}$$

Moreover $\boldsymbol{Z}$ is $\boldsymbol{T}$-invariant subspace containing $\boldsymbol{v}$ and $\boldsymbol{W}$ is the smallest $\boldsymbol{T}$-invariant subspace containing $\boldsymbol{v}$, $\boldsymbol{W}\leq\boldsymbol{Z}$ which implies that $k\leq l$. But dim$\boldsymbol{W}=k$ and $l\leq k$, thus $k=l$
$\therefore\beta$ is linearly independent subset of $\boldsymbol{W}$.
Since β is linearly independent and $\beta\cup\{\boldsymbol{T}^i\boldsymbol{v}\}$ is linearly dependent $\Longrightarrow\boldsymbol{T}^i\boldsymbol{v}\in$ spanβ for $i\geq k$.
$\therefore\beta$ is a basis for $\boldsymbol{W}$

2. $[\boldsymbol{T}|_{\boldsymbol{W}}]_\beta=\begin{pmatrix}0&0&\cdots&0&-a_0\\1&0&\cdots&0&-a_1\\0&1&\cdots&0&-a_2\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&-a_{k-1}\end{pmatrix}$ and $\phi_{[\boldsymbol{T}|_{\boldsymbol{W}}]_\beta}(t)=\det\begin{pmatrix}t&0&\cdots&0&a_0\\-1&t&\cdots&0&a_1\\0&-1&\cdots&0&a_2\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&-1&t+a_{k-1}\end{pmatrix}$

$$\text{Put } b_i \coloneqq \det \begin{pmatrix} t & 0 & \cdots & 0 & a_i \\ -1 & t & \cdots & 0 & a_{i+1} \\ 0 & -1 & \cdots & 0 & a_{i+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t+a_{k-1} \end{pmatrix}_{(k-i)\times(k-i)}$$

$$b_0 = \begin{pmatrix} t & 0 & \cdots & 0 & a_0 \\ -1 & t & \cdots & 0 & a_1 \\ 0 & -1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t+a_{k-1} \end{pmatrix}$$

$$b_0 = t\det \begin{pmatrix} t & 0 & \cdots & 0 & a_1 \\ -1 & t & \cdots & 0 & a_2 \\ 0 & -1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t+a_{k-1} \end{pmatrix} + (-1)^{k+1}a_0 \det \begin{pmatrix} -1 & t & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

$$\begin{aligned} b_0 &= tb_1 + (-1)^{k+1+k-1}a_0 \\ b_0 &= tb_1 + a_0 \\ b_1 &= tb_2 + a_1 \\ &\vdots \\ b_{k-2} &= tb_{k-1} + a_{k-2} \\ b_{k-1} &= t + a_{k-1} \\ b_0 &= t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0 \end{aligned}$$

$$\therefore \phi_{[\boldsymbol{T}|\boldsymbol{W}}(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0$$

□

Theorem 6. *If $\boldsymbol{F} = \mathbb{C}$ and $\boldsymbol{T} \in \mathfrak{L}(\boldsymbol{V}, \boldsymbol{V})$, then $\exists \mathfrak{B}$ such that $\mathfrak{B}$ is a basis for $\boldsymbol{V}$ and $[\boldsymbol{T}]_{\mathfrak{B}}$ is upper-triangular matrix.*

Proof. Induction on matrix size n. Since (1x1) matrix is upper-triangular matrix, let's assume $n \geq 2$. Since $\boldsymbol{F} = \mathbb{C}$, $\exists \boldsymbol{v}_1$ such that $\boldsymbol{T}\boldsymbol{v}_1 = \lambda \boldsymbol{v}_1$. Construct a basis $\mathfrak{C} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ by Basis Extension Theorem. Then $[\boldsymbol{T}]_{\mathfrak{C}} = \left(\begin{array}{c|c} \lambda & * \\ \hline 0 & B \end{array}\right)$, where $B \in \mathfrak{M}_{n-1,n-1}(\mathbb{C})$. By Induction hypothesis, $\exists P \in \mathfrak{M}_{n-1,n-1}(\mathbb{C})$ such that $p^{-1}BP$ is upper-triangular matrix. Put $U \coloneqq \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right)$.

Since, $U^{-1} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^{-1} \end{array}\right)$, $[\boldsymbol{T}]_{\mathfrak{C}} \sim U^{-1}\left(\begin{array}{c|c} \lambda & * \\ \hline 0 & B \end{array}\right)U = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^{-1} \end{array}\right)\left(\begin{array}{c|c} \lambda & * \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right) = \left(\begin{array}{c|c} \lambda & * \\ \hline 0 & P^{-1}BP \end{array}\right)$

Since $P^{-1}BP$ is upper-triangular matrix, $[\boldsymbol{T}]_{\mathfrak{C}}$ is similar to upper-triangular matrix. □

Example 1. *Let* $A \in \mathbf{M}_{n\times n}(\mathbb{R})$ *be the matrix defined by* $A_{ij} = 1$ *for all* i *and* j. *Find the characteristic polynomial of* A. $\phi_A(t) = t^{n-1}(t-n)$

Proof. Since, rank$A = 1$, dim$(kerA)$ = n-1, which implies that $\exists \boldsymbol{x}$ such that $A\boldsymbol{x} = 0\boldsymbol{x}$ and $\boldsymbol{x} \neq 0$.Thus dimE_0 = n-1. Since multiplicty of eignevalue 0 is greater than or equal to n-1, $\phi_A(t) = t^{n-1}(t-\lambda)$. Since there exists at least eigenvalue, which is 0, of A, A is similar to upper triangular matrix U by Theorem 6. Since U is upper-triangular matrix and trace A = trace $U = n, -(\lambda + 0) = -n$.

$$U = \begin{pmatrix} 0 & & & & \\ & 0 & & * & \\ & & \ddots & & \\ & \mathbf{0} & & \ddots & \\ & & & & \lambda \end{pmatrix}$$

$\therefore \phi_A(t) = \phi_U(t) = t^{n-1}(t-n)$ □