Characteristic polynomial and Diagonalizable

Seanie Lee, Jonghwan Jang

08 April 2019

Definition 1. *$\boldsymbol{x}$ is eigenvector such that $A = \lambda$ and $\neq 0$. λ is called eigenvalue*

Then how to find eigenvector and eigenvalue? get null space of $(\lambda I_n - A)$

$$
\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} \\
(\lambda I_n - A)\mathbf{x} &= 0 \\
\exists \mathbf{x} \text{ such that } \mathbf{x} \neq \mathbf{0} \quad A\mathbf{x} = \lambda\mathbf{x} &\iff \det(\lambda I_n - A) = 0
\end{aligned}
$$

Theorem 2. $A, B \in \mathfrak{M}_{n\times n}(\mathbb{R})$. $A \sim B \Longrightarrow \det A \cdot \det B$

Proof. Suppose that $B = U^{-1}AU$

$$
\begin{aligned}
\det B &= \det(U^{-1}AU) \\
&= \det U^{-1} \cdot \det A \cdot \det U \\
&= \det A
\end{aligned}
$$

$\therefore A \sim B \Longrightarrow \det A = \det B$ □

Theorem 3. *$L \in \mathfrak{L}(\boldsymbol{V}, \boldsymbol{V})$ and $\mathfrak{B}$ is a basis for $\boldsymbol{V}$. Define $\det L \coloneqq \det([L]_{\mathfrak{B}}^{\mathfrak{B}})$.*
Then $\det L$ is well-defined.

Proof. Suppose another basis $\mathfrak{C}$ for $\boldsymbol{V}$ be given. Then $[L]_{\mathfrak{B}}^{\mathfrak{B}} \sim [L]_{\mathfrak{C}}^{\mathfrak{C}}$.
By the Theorem 2. $\det([L]_{\mathfrak{B}}^{\mathfrak{B}}) = \det([L]_{\mathfrak{C}}^{\mathfrak{C}})$
$\therefore \det L$ is well-defined □

Definition 4. *$\phi_A(t) \coloneqq \det(tI_n - A) \in \boldsymbol{P}_n(\mathbb{R})$ is called characteristic polynomial of $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$.*

Rmk. Eigenvalues of A are solutions of $\phi_A(t)$

Definition 5. *$(\lambda I_n - A)\mathbf{x} = 0$ where $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$*
Null space of $(\lambda I_n - A)$ is called eigen space of A with respect to λ; $E_\lambda \coloneqq N(\lambda I_n - A)$

Observation 6. *Characteristic polynomial is invariant to similarity relation.*
Therefore $\phi_A(t)$ is well-defined.

Proof. Suppose that $B = U^{-1}AU$.

$$\begin{aligned}
\phi_B(t) &= \det(tI_n - U^{-1}AU) \\
&= \det(tU^{-1}U - U^{-1}AU) \\
&= \det(U^{-1}(tU - AU)) \\
&= \det(U^{-1}(tI_n - A)U) \\
&= \det U^{-1} \cdot \det U \cdot \det(tI_n - A) \\
&= \det(tI_n - A)
\end{aligned}$$

$\therefore \phi_A(t)$ is well defined $\square$

statement. $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. λ is an eigenvalue of A $\iff \phi_A(t) = 0$

Proof. [λ is eigenvalue of A] $\iff$ [$\exists \mathbf{x}$ such that $(\lambda I_n - A)\mathbf{x} = 0$ and $\neq 0$]
$\iff$ [$\ker(\lambda I_n - A) \neq \boldsymbol{O}$] $\iff$ [$\lambda I_n - A$ is not invertible] $\iff$ [$\det(\lambda I_n - A) = 0$] $\square$

Definition 7. *A is diagonalizable if $D \sim A$ for some diagonal matrix D*

Theorem 8. *$A \in \mathfrak{M}_{n\times n}(\mathbb{R})$ is diagonalizable if and only if [$\mathbb{R}^n$ has n-linearly independent eigen vectors. ($\iff \mathbb{R}^n$ has a basis consisting of eigen vectors of A)]*

Proof.

$$\begin{aligned}
A \text{ is diagonalizable} &\iff D = Q^{-1}AQ \text{ where } Q := [\boldsymbol{x}_1, \dots, \mathbf{x}_n] \text{ and } D := \mathrm{diag}(\lambda_1, \dots, \lambda_n) \\
&\iff QD = AQ \\
&\iff \lambda_x \mathbf{x}_j = A\mathbf{x}_j \quad \text{for } j = 1, \dots, n \quad \text{where } \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \text{ is linearly independent} \\
&(\because Q \text{ is invertible})
\end{aligned}$$

$\therefore D \sim A \iff \mathbb{R}^n$ has n-linearly independent eigenvectors. $\square$