Vandermonde's determinant and Cramer's rule

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Example 1 (Vandermonde's determinant)**.**

$$
\begin{aligned}
\det\begin{pmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^n \\
1 & x_2 & x_2^2 & \cdots & x_2^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n
\end{pmatrix}
&= \det\begin{pmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\
0 & x_2 - x_1 & x_2^2 - x_1^2 & \cdots & x_2^{n-1} - x_1^{n-1} & x_2^n - x_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & x_{n+1} - x_1 & x_{n+1}^2 - x_1^2 & \cdots & x_{n+1}^{n-1} - x_1^{n-1} & x_{n+1}^n - x_1^n
\end{pmatrix} \\
&= \det\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & x_2 - x_1 & \cdots & (x_2 - x_1)x_2^{n-2} & (x_2 - x_1)x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & x_{n+1} - x_1 & \cdots & (x_{n+1} - x_1)x_{n+1}^{n-2} & (x_{n+1} - x_1)x_{n+1}^{n-1}
\end{pmatrix} \\
&= \prod_{k=2}^{n+1}(x_k - x_1)\det\begin{pmatrix}
1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\
1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{n-1}
\end{pmatrix} \\
&= (x_2 - x_1)(x_3 - x_2)(x_3 - x_1)\cdots(x_{n+1} - x_2)(x_{n+1} - x_1) \\
&= \prod_{1 \le i < j \le n+1}(x_j - x_i)
\end{aligned}
$$

Definition 1 (adjoint matrix)**.** $adjA \coloneqq ((-1)^{i+j}\hat{A}_{ij})^t$

Theorem 2 (Cramer's rule)**.** *If the system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, then $x_i \det A = \det([A]^1, \ldots, \boldsymbol{b}, \ldots, [A]^n)$ $[A]^i$ is replaced with $\boldsymbol{b}$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$*

Proof.

$$
\begin{aligned}
\boldsymbol{b} &= x_1[A]^1 + \cdots + x_n[A]^n \\
\det([A]^1 \ldots, \boldsymbol{b}, \ldots, [A]^n) &= \det([A]^1, \ldots, x_1[A]^1 + \cdots + x_n[A]^n, \ldots, [A]^n) \\
&= \sum_{j=1}^{n} \det x_j([A]^1, \ldots, [A]^j, \ldots, [A]^n) \\
&= x_i \det([A]^1, \ldots, [A]^i, \ldots, [A]^n) \\
&= x_i \det A
\end{aligned}
$$

$\square$

Rmk. This theorem does not guarantee the existence of a solution of $A\boldsymbol{x} = \boldsymbol{b}$

Observation 3. *Suppose that $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$ is invertible. Then* $\det A \cdot I_n = A \cdot adjA$

Proof. Since A is invertible, $\exists! B$ such that $AB = BA = I_n$ where $B = (b_{ij}) \in \mathfrak{M}_{n\times n}(\mathbb{R})$. By Cramer's rule, $b_{ij} = \det \frac{([A]^1,\ldots,\boldsymbol{e}_j,\ldots,[A]^n)}{\det A} (\because A[B]^j = \boldsymbol{e}_j)$

$$\begin{aligned}\text{Since } \det A([A]^1,\ldots,\boldsymbol{e}_j,\ldots,[A]^n) &= \det\begin{pmatrix} & 0 & \\ & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \\ & 0 & \end{pmatrix} \\ &= (-1)^{i+j}\hat{A}_{ji}\end{aligned}$$

$\therefore b_{ij} = \frac{(-1)^{i+j}\hat{A}_{ji}}{\det A}$ and $A^{-1} = \frac{\text{adj}A}{\det A}$
$\therefore \det A \cdot I_n = A \cdot \text{adj}A$ □

Theorem 4. $A \cdot adjA = \det A \cdot I_n$

Proof. We want to show that $[A]_i[\text{adj}A]^j = \begin{cases} \det A & (i = j) \\ 0 & (i \neq j)\end{cases}$

$[A]_i[\text{adj}A]^j = \sum\limits_{k=1}^{n} a_{ik} \cdot (-1)^{j+k}\hat{A}_{jk}$

(1). $i = j$
$\sum\limits_{k=1}^{n} (-1)^{i+k} a_{ik}\hat{A}_{ik} = \det A \quad (\because$ cofactor expansion)

(2). $i \neq j$
We want to find B such that $a_{ik} \cdot \hat{A}_{ik} = b_{jk} \cdot \hat{B}_{jk}$

Put $B := \begin{pmatrix} \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in}\end{pmatrix}$.

Then $a_{ik} \cdot \hat{A}_{jk} = b_{jk} \cdot \hat{B}_{jk}$ and $\det B = 0$ ($\because$ det is alternating n-linear form)

□

Note. if $\det A \neq 0$, $\det(\text{adj}A) = (\det A)^{n-1}$

Proof.

$$\begin{aligned} A \cdot \text{adj}A &= \det A \cdot I_n \\ \det A \cdot \det(\text{adj}A) &= \det(\det A \cdot I_n) \\ &= (\det A)^n \cdot \det I_n \\ \det(\text{adj}A) &= (\det A)^{n-1}\end{aligned}$$

$\square$