Characteristic polynomial and Diagonalizable

Seanie Lee, Jonghwan Jang

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Definition 1. *$\boldsymbol{x}$ is eigen-vector such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $\boldsymbol{x} \neq 0$. λ is called eigen-value*

Then how to find eigen vector and eigen value? get null space of $(\lambda I_n - A)$

$$\begin{aligned} A\boldsymbol{x} &= \lambda\boldsymbol{x} \\ (\lambda I_n - A)\boldsymbol{x} &= 0 \\ \exists\boldsymbol{x} \text{ such that } \boldsymbol{x} \neq 0 &\iff \det(\lambda I_n - A) = 0 \end{aligned}$$

Theorem 2. $A, B \in M_{n\times n}.A \sim B \Longrightarrow \det A \cdot \det B$

Proof. Suppose that $B = U^{-1}AU$

$$\begin{aligned} \det B &= \det(U^{-1}AU) \\ &= \det U^{-1} \cdot \det A \cdot \det U \\ &= \det A \end{aligned}$$

$\therefore A \sim B \Longrightarrow \det A = \det B$ □

Theorem 3. *$L \in \mathfrak{L}(\boldsymbol{V}, \boldsymbol{V})$ and $\mathfrak{B}$ is a basis for $\boldsymbol{V}$. Define $\det L := \det([L]_{\mathfrak{B}}^{\mathfrak{B}})$.*
Then $\det L$ is well-defined.

Proof. Suppose another basis $\mathfrak{C}$ for $\boldsymbol{V}$ be given. Then $[L]_{\mathfrak{B}}^{\mathfrak{B}} \sim [L]_{\mathfrak{C}}^{\mathfrak{C}}$.
By the Theorem 2. $\det([L]_{\mathfrak{B}}^{\mathfrak{B}}) = \det([L]_{\mathfrak{C}}^{\mathfrak{C}})$
$\therefore \det L$ is well-defined □

Definition 4. *$\phi_A(t) := \det(tI_n - A) \in \boldsymbol{P}_n(\mathbb{R})$ is called characteristic polynomial of A.*

Rmk. Eigen values of A are solutions of $\phi_A(t)$

Definition 5. $(\lambda I_n - A)\mathbf{x} = 0$ *where $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$*
Null space of $(\lambda I_n - A)$ is called eigen space of A with respect to λ; $E_\lambda := N(\lambda I_n - A)$

Observation 6. *Characteristic polynomial is invariant to similarity relation.*
Therefore $\phi_A(t)$ is well-defined.

Proof. Suppose that $B = U^{-1}AU$.

$$\begin{aligned}\phi_B(t) &= \det(tI_n - U^{-1}AU)\\ &= \det(tU^{-1}U - U^{-1}AU)\\ &= \det(U^{-1}(tU - AU))\\ &= \det(U^{-1}(tI_n - A)U)\\ &= \det U^{-1} \cdot \det U \cdot \det(tI_n - A)\\ &= \det(tI_n - A)\end{aligned}$$

$\therefore \phi_A(t)$ is well defined □

statement. $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. λ is egien value of A $\iff$ $\phi_A(t) = 0$

Proof. [λ is eigen value of A] $\iff$ [$\exists \boldsymbol{x}$ such that $(\lambda I_n - A)\boldsymbol{x} = 0$ and $\boldsymbol{x} \neq 0$]
$\iff$ [$ker(\lambda I_n - A) \neq \boldsymbol{O}$] $\iff$ [$\lambda I_n - A$ is not invertible] $\iff$ [$\det(\lambda I_n - A) = 0$] □

Definition 7. *A is diagonalizable if $D \sim A$ for some diagonal matrix D*

Theorem 8. *$A \in \mathfrak{M}_{n\times n}(\mathbb{R})$ is diagonalizable if and only if [$\mathbb{R}^n$ has n-linearly independent eigen vectors. ($\iff$ $\mathbb{R}^n$ has a basis consisting of eigen vectors of A)]*

Proof.

$$\begin{aligned}A \text{ is diagonalizable} &\iff D = Q^{-1}AQ \text{ where } Q := [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n] \text{ and } D := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)\\ &\iff QD = AQ\\ &\iff \lambda_x \boldsymbol{x}_j = A\boldsymbol{x}_j \quad \text{for } j = 1, \ldots, n \quad \text{where } \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \text{ is linearly independent}\\ &(\because Q \text{ is invertible})\end{aligned}$$

$\therefore D \sim A \iff \mathbb{R}^n$ has n-linearly independent eigen vectors. □