Linear Algebra Class on 26 January

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2.1 Linear Transformation

Definition. $\mathbf{V}, \mathbf{W}$ *are vector spaces over* $\mathbb{R}$*.* $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ *is* $\mathbb{R}$*-linear map if* $\mathbf{L}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{L}\mathbf{v}_1 + \mathbf{L}\mathbf{v}_2$ *and* $\mathbf{L}(c\mathbf{v}) = c\mathbf{L}\mathbf{v}$*. In other words* $\mathbf{L}$ *is called vector space homomorphism.*

Properties

1. $\mathbf{L0} = \mathbf{0}$
2. $\mathbf{L}(-\mathbf{v}) = -\mathbf{L}\mathbf{v}$
3. $\mathbf{L}(a\mathbf{v}_1 + b\mathbf{v}_2) = a\mathbf{L}\mathbf{v}_1 + b\mathbf{L}\mathbf{v}_2$

1. *Proof.* $\mathbf{L0} = \mathbf{L}(\mathbf{0} + \mathbf{0}) = \mathbf{L0} + \mathbf{L0}$ □
2. *Proof.* $\mathbf{L}(-\mathbf{v}) = \mathbf{L}(-1 \cdot \mathbf{v}) = -\mathbf{L}\mathbf{v}$ □
3. *Proof.* $\mathbf{L}(a\mathbf{v}_1 + b\mathbf{v}_2) = \mathbf{L}(a\mathbf{v}_1) + \mathbf{L}(b\mathbf{v}_2) = a\mathbf{L}\mathbf{v}_1 + b\mathbf{L}\mathbf{v}_2$ □

Remark. $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ *is a basis for* $\mathbf{V}$ *and* $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ *is linear. Then* $\mathbf{L}(\sum\limits_{i=1}^{n} a_i\mathbf{v}_i) = \sum\limits_{i=1}^{n} a_i\mathbf{L}\mathbf{v}_i$
$\therefore \mathbf{L}$ *is completely determined by* $\mathbf{L}\mathbf{v}_i$ *for* $i = 1, \dots, n$

Theorem (Linear Extension Theorem)**.** $\mathbf{V}, \mathbf{W}$ *is vector space. Let* $\mathfrak{B} \coloneqq \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ *be a basis for* $\mathbf{V}$ *and* $f : \mathfrak{B} \longrightarrow \mathbf{W}$ *and be a function. Then There is unique* $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ *linear map such that* $\mathbf{L}|_{\mathfrak{B}} = f(\iff \mathbf{L}\mathbf{v}_i = f(\mathbf{v}_i)$ *for* $i = 1, \dots, n)$

Proof. 1. existence

Define $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ by $\mathbf{L}(\sum\limits_{i=1}^{n} a_i)\mathbf{v}_i \coloneqq \sum\limits_{i=1}^{n} a_i f(\mathbf{v}_i)$. For each $i = 1, \dots, n$

$$\begin{aligned}
\mathbf{L}\mathbf{v}_i &= \mathbf{L}(0\mathbf{v}_1 + \cdots + 1\mathbf{v}_i + \cdots + 0\mathbf{v}_n) \\
&= 0f(\mathbf{v}_1) + \cdots + f(\mathbf{v}_i) + \cdots + f(\mathbf{v}_n) \\
&= f(\mathbf{v}_i) \\
&\therefore \mathbf{L}|_{\mathfrak{B}} = f
\end{aligned}$$

2. linearity of $\mathbf{L}$
   For $\sum_{i=1}^{n} a_i\mathbf{v}_i, \sum_{i=1}^{n} b_i\mathbf{v}_i \in \mathbf{V}$

$$\begin{aligned}
\mathbf{L}(\sum_{i=1}^{n} a_i\mathbf{v}_i + \sum_{i=1}^{n} b_i\mathbf{v}_i) &= \mathbf{L}(\sum_{i=1}^{n}(a_i + b_i)\mathbf{v}_i) \\
&= \sum_{i=1}^{n}(a_i + b_i)\mathbf{L}\mathbf{v}_i \\
&= \sum_{i=1}^{n} a_i\mathbf{L}\mathbf{v}_i + \sum_{i=1}^{n} b_i\mathbf{L}\mathbf{v}_i \\
&= \sum_{i=1}^{n} a_i f(\mathbf{v}_i) + \sum_{i=1}^{n} b_i f(\mathbf{v}_i) \\
&= \mathbf{L}(\sum_{i=1}^{n} a_i\mathbf{v}_i) + \mathbf{L}(\sum_{i=1}^{n} b_i\mathbf{v}_i)
\end{aligned}$$

3. uniqueness
   Let $M : \mathbf{V} \longrightarrow \mathbf{W}$ be linear map such that $\mathbf{M}|_{\mathfrak{B}} = f$. For any $\sum_{i=1}^{n} a_i\mathbf{v}_i \in \mathbf{V}$,

$$\begin{aligned}
\mathbf{L}(\sum_{i=1}^{n} a_i\mathbf{v}_i) &= \sum_{i=1}^{n} a_i f(\mathbf{v}_i) \\
&= \sum_{i=1}^{n} a_i\mathbf{M}\mathbf{v}_i \\
&= \mathbf{M}(\sum_{i=1}^{n} a_i\mathbf{v}_i) \\
&\therefore \mathbf{L} = \mathbf{M}
\end{aligned}$$

□

Definition.

Let $\mathbf{L} : \mathbf{V} \longrightarrow \mathbf{W}$ be linear map. $\ker\mathbf{L} := \{\mathbf{v} \in \mathbf{V} \mid \mathbf{L}\mathbf{v} = \mathbf{0}\}$ $\mathrm{im}\mathbf{L} := \{\mathbf{L}\mathbf{v} \in \mathbf{W} \mid \mathbf{v} \in \mathbf{V}\}$

Theorem. *Let $\varphi : \mathbf{V} \longrightarrow \mathbf{W}$ be linear map. Then $\ker\varphi = \mathbf{O} \iff \varphi$ is one to one.*

Proof. $\Longrightarrow$ Suppose that $\varphi\mathbf{v}_1 = \varphi\mathbf{v}_2$ $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$. By linearity $\varphi(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$, which implies that $\mathbf{v}_1 - \mathbf{v}_2 \in \ker\varphi$. Since $\ker\varphi = \mathbf{O}$, $\mathbf{v}_1 = \mathbf{v}_2$
$\Longleftarrow$ Let $\mathbf{v} \in \ker\varphi$ be given. Since φ is linear map, $\varphi\mathbf{0} = \mathbf{0}$. Also $\varphi\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \in \ker\varphi$. Since φ is one to one, $\varphi\mathbf{v} = \varphi\mathbf{0} \Longrightarrow \mathbf{v} = \mathbf{0}$ $\therefore \varphi$ is one to one. □

Theorem. *Let $\varphi : \mathbf{V} \longrightarrow \mathbf{W}$ be linear map. φ is one-to-one $\iff$ $[\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent subse $\{\varphi\mathbf{v}_1, \ldots, \varphi\mathbf{v}_k\}$ is linearly independent subset of $\mathbf{W}$*

Proof. $\Longrightarrow$ Suppose that $\sum_{i=1}^{k} a_i\varphi(\mathbf{v}_i) = \mathbf{0}$. Then left-hand-side is $\varphi(\sum_{i=1}^{k} a_i\mathbf{v}_i) = \mathbf{0}$. Since φ is one-to-one $\sum_{i=1}^{k} a_i\mathbf{v}_i = \mathbf{0}$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, all a_i is zero. $\therefore$ $\{\varphi\mathbf{v}_1, \ldots, \varphi\mathbf{v}_k\}$ is linearly independent.

$\Longleftarrow$ Suppose $\varphi\mathbf{v} = \varphi\mathbf{v}'$

$$\begin{aligned} \varphi\mathbf{v} - \varphi\mathbf{v}' &= \mathbf{0} \\ \varphi(\mathbf{v} - \mathbf{v}') &= \mathbf{0} \\ \mathbf{v} - \mathbf{v} &= \mathbf{0} \end{aligned}$$

Suppose $\mathbf{v} - \mathbf{v}' \neq \mathbf{0}$. Then $\{\mathbf{v} - \mathbf{v}'\}$ is linearly independent subset of $\mathbf{V}$, but $\{\varphi(\mathbf{v} - \mathbf{v}')\} = \{\mathbf{0}\}$ is linearly dependent subset of $\mathbf{W}$. But it contradicts to the assumption. $\therefore \mathbf{v} = \mathbf{v}'$

φ is one-to-one. □

Theorem. *Let* $\varphi : \mathbf{V} \longrightarrow \mathbf{W}$ *be linear map.*

φ *is onto* $\iff \big[span S = \mathbf{V} \varphi span = \mathbf{W} S\big]$

Proof. $\Longrightarrow$ Put $S \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ then $\varphi S \coloneqq \{\varphi\mathbf{v}_i, \ldots, \varphi\mathbf{v}_k\}$

Suppose $\mathbf{w} \in \mathbf{W}$ be given. Since φ is onto, there are some $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{w} = \varphi\mathbf{v}$. Since S is spanning set, so $\mathbf{v} = \sum\limits_{i=1}^{k} a_i\mathbf{v}_i$ and $\mathbf{w} = \varphi(\sum\limits_{i=1}^{k} a_i\mathbf{v}_i)$. Since φ is linear map, $\mathbf{w} = \sum\limits_{i=1}^{k} a_i\varphi\mathbf{v}_i$

$\therefore \mathbf{w} \in$ spanφS

$\Longleftarrow$ Suppose $S \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and span$S = \mathbf{V}$. Then span$\varphi S = \mathbf{W}$ by the assumption. Since φS is spanning set of $\mathbf{W}$, $\mathbf{w} = \sum\limits_{i=1}^{k} a_\varphi\mathbf{v}_i = \varphi(\sum\limits_{i=1}^{k} a_i\mathbf{v}_i)$

$\therefore \varphi$ is onto. □

Corollary. *Let* $\varphi : \mathbf{V} \longrightarrow \mathbf{W}$ *be linear map.*

φ *is bijective* $\iff \big[\mathfrak{B}$ *is a basis for* $\mathbf{V} \Longrightarrow \varphi\mathfrak{B}$ *is a basis for* $\mathbf{W}\big]$

Proof. By previous two theorems, it's trivial to show that φ is bijective $\iff \big[\mathfrak{B}$ is a basis for $\mathbf{V} \Longrightarrow \varphi\mathfrak{B}$ is a basis for $\mathbf{W}\big]$ □

Theorem (Dimension Theorem)**.** *Let* $\varphi : \mathbf{V} \longrightarrow \mathbf{W}$ *be linear map. Then* $\dim \mathbf{V} = \dim(\ker\varphi) = \dim(im\varphi)$

Proof. Take a basis $\mathfrak{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $\ker\varphi$. Extend $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots \mathbf{v}_n\}$ for $\mathbf{V}$ by the basis extension theorem. We want to show $\mathfrak{C} = \{\varphi\mathbf{v}_{k+1}, \ldots, \varphi\mathbf{v}_n\}$ is a basis for imφ. First we show that $\mathfrak{C}$ is linearly independent subset of $\mathbf{W}$. Suppose that $\sum\limits_{i=k+1}^{n} a_i\varphi\mathbf{v}_i = \mathbf{0}$. Since φ is linear map, $\sum\limits_{i=k+1}^{n} a_i\varphi\mathbf{v}_i = \varphi(\sum\limits_{i=k}^{n} a_i\mathbf{v}_i) = \mathbf{0}$. $\sum\limits_{i=k+1}^{n} a_i\mathbf{v}_i \ker\varphi$,

$$\begin{aligned} a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n &= b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k \\ b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k - a_{k+1}\mathbf{v}_{k+1} - \cdots - a_n\mathbf{v}_n &= \mathbf{0} \end{aligned}$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbf{V}$, all $a_i, b_j = 0$ for $i = k+1, \ldots, n$ and for $j = 1, \ldots, n$.

$\therefore \{\varphi\mathbf{v}_{k+1}, \ldots, \varphi\mathbf{v}_n\}$ is linearly independent. Next, we want to show that im$\varphi =$ span$\{bv_{k+1}, \mathbf{v}_n\}$. Let

$\varphi\mathbf{v} \in \text{ im}\varphi$ be given. Put $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$

$$
\begin{aligned}
\varphi\mathbf{v} &= \varphi(\sum_{i=1}^{n} a_i\mathbf{v}_i) \\
&= a_1\varphi\mathbf{v}_1 + \cdots + a_k\varphi\mathbf{v}_k + a_{k+1}\varphi\mathbf{v}_{k+1} + \cdots + a_n\varphi\mathbf{v}_n \\
&(\because a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \in \ker\varphi) \\
&= a_{k+1}\varphi\mathbf{v}_{k+1} + \cdots + a_n\varphi\mathbf{v}_n \in \text{ span}\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}
\end{aligned}
$$

$\therefore \dim(\ker\varphi) + \dim(\text{im}\varphi) = k + (n-k) = n = \dim\mathbf{V}$ $\square$

Theorem (Pigeon hole principle for finite dimensional vector space)**.** *Let* $\varphi : \mathbf{V} \longrightarrow \mathbf{W}$ *be linear map with* $\dim\mathbf{V} = \dim\mathbf{W}$*. Then the followings are equivalent.*

(1). φ *is one-to-one(* $\iff \dim\mathbf{V} = \dim(im\varphi)$)

(2). φ *is onto(* $\iff$ $im\varphi = \mathbf{W}$)

(3). φ *is bijection*

Proof. (1) $\Longrightarrow$ (2) Since φ is linear map, if S is spanning set for $\mathbf{V}$, φS is spanning set for $\text{im}\varphi$($\because S \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, $\forall\mathbf{v} \in \mathbf{V}, \mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$, $\varphi\mathbf{v} = a_1\varphi\mathbf{v}_1 + \cdots a_k\varphi\mathbf{v}_k$). Let $\mathfrak{B}$ be a basis for $\mathbf{V}$. Since φ is one-to-one, $\varphi\mathfrak{B}$ is spanning set and linearly independent subset for $\text{im}\varphi$. i.e.,
$\therefore \dim\mathbf{W} = \dim\mathbf{V} = \dim\text{im}\varphi$
$\therefore \varphi$ is onto.
(2) $\Longrightarrow$ (2) $\dim\mathbf{W} = \dim\mathbf{V} = \dim(\ker\varphi) + \dim(\text{im}\varphi) = \dim\ker\varphi + \dim\mathbf{W}$. Since φ is onto, $\dim\text{im}\varphi = \dim\mathbf{W}, \dim(\ker\varphi) = 0$. i.e., $\ker\varphi = \mathbf{O}$
$\therefore \varphi$ is one-to-one. $\square$