# Linear Algebra Class on 8 June

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## 6.6 Orthogonal Projection and Spectral Theorem

**Definition.** *$A \in \mathfrak{M}_{n\times n}(\mathbb{R})$, $A$ is orthogonal if $A^tA = AA^t = \mathbf{I}_n$*
*$A \in \mathfrak{M}_{n\times n}(\mathbb{C})$, $A$ is unitary if $A^*A = AA^* = \mathbf{I}_n$*

**Remark.**

$$\begin{aligned} A = \left[\mathcal{C}_1 \cdots \mathcal{C}_n\right] \textit{ is unitary} \quad \Longleftrightarrow \quad \left[A^*A\right]_{ij} &= \sum_{k=1}^{n} A^*_{ik}A_{kj} \\ &= \sum_{k=1}^{n} \overline{A}^t_{ki}A_{kj} \\ &= \mathcal{C}_i^*\mathcal{C}_j \\ &= \mathcal{C}_j \cdot \mathcal{C}_i \\ &= \delta_{ji} \end{aligned}$$

*Thus it is equivalent to $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ is an orthonormal basis for $\mathbb{C}^n(\mathbb{R}^n)$*

**Definition.**

$$\begin{aligned} A \textit{ is unitarily equivalent to } B &\Longleftrightarrow B = U^{-1}AU \textit{ for some unitary matrix } U(\because U^*U = \mathbf{I}_n \Longrightarrow U^* = U^{-1}) \\ A \textit{ is unitarily diagonalizble} &\Longleftrightarrow A \textit{ is unitarily equivalent to diagonal matrix } D, A = U^{-1}DU \end{aligned}$$

**Theorem.** *$\beta$: an orthonormal basis consisting of eigenvectors of $\mathbf{T} \Longleftrightarrow \mathbf{T}$ is unitarily diagonalizable*

*Proof.* $\Longrightarrow$ since $\mathbf{T}$ is unitarily diagonalizable, $[\mathbf{T}]_\gamma = Q^{-1}[\mathbf{T}]_\beta Q$. $Q := [\mathbf{x}_1 \cdots \mathbf{x}_n]$, $[\mathbf{T}]_\beta := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is the set of orthonormal eigenvectors.
$\therefore Q$ is unitary matrix.
$\Longleftarrow$ Since $\mathbf{T}$ is unitarily diagonalizable, $[\mathbf{T}]_\gamma = U^{-1}DU$ for some unitary matrix $U$ and some diagonal matrix $D$. Thus $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_i$ is eigenvalue and column vectors of $U$ are orthonormal basis.
$\therefore$ There exists an orthonormal basis consisting of eigenvectors of $\mathbf{T}$ □

**Definition** (Projection T)**.** *$\mathbf{W}_1, \mathbf{W}_2$ are subspaces of $\mathbf{V}$, $\mathbf{V} = \mathbf{W}_1 \bigoplus \mathbf{W}_2$, and $\mathbf{T} : \mathbf{V} \longrightarrow \mathbf{V}$ is projection on $\mathbf{W}_1$ along $\mathbf{W}_2$ if $\mathbf{Tx} = \mathbf{x}_1$ where $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \in \mathbf{W}_1 \bigoplus \mathbf{W}_2$.*

**Note.** *$im\mathbf{T} = \mathbf{W}_1 = \{\mathbf{x} \in \mathbf{V} \mid \mathbf{Tx} = \mathbf{x}\}$ and $\ker \mathbf{T} = \mathbf{W}_2$. $\therefore \mathbf{V} = im\mathbf{T} \bigoplus \ker \mathbf{T}$. Thus there is no ambiguity if we refer to $\mathbf{T}$ as a "projection on $\mathbf{W}_1$" or simply as a projection.*

**Note.** *Because* $\mathbf{V} = \mathbf{W}_1 \bigoplus \mathbf{W}_2 = \mathbf{W}_1 \bigoplus \mathbf{W}_3$ *does not imply that* $\mathbf{W}_2 = \mathbf{W}_3$, $\mathbf{W}_1$ *does not uniquely determine* $\mathbf{T}$. *However, for an orthogonal projection* $\mathbf{T}$, $\mathbf{T}$ *is uniquely determined by its range.*

**Theorem.** *$T$ is projection if and only if* $\mathbf{T}^2 = \mathbf{T}$

*Proof.* $\Longrightarrow$ $\mathbf{x} \coloneqq \mathbf{x}_1 + \mathbf{x}_2 \in \mathbf{W}_1 \bigoplus \mathbf{W}_2$, thus $\mathbf{Tx} = \mathbf{x}_1$. Since $\mathbf{T}^2\mathbf{x} = \mathbf{T}\mathbf{x}_1 = \mathbf{x}_1 = \mathbf{Tx}$, $\mathbf{T}^2\mathbf{x} = \mathbf{Tx}$ $\forall \mathbf{x} \in \mathbf{V}$.
$\therefore \mathbf{T}^2 = \mathbf{T}$
$\Longleftarrow$ Suppose $\mathbf{T}^2 = \mathbf{T}$. $\mathbf{x} = \mathbf{Tx} + (\mathbf{x} - \mathbf{Tx})$. Then $\mathbf{Tx} \in \mathrm{im}\mathbf{T}$. We want to show that $\mathbf{x} - \mathbf{Tx} \in \ker \mathbf{T}$.

$$\begin{aligned}
\mathbf{T}(\mathbf{x} - \mathbf{Tx}) &= \mathbf{Tx} - \mathbf{T}^2\mathbf{x} \\
&= \mathbf{Tx} - \mathbf{Tx} \\
&= \mathbf{0} \\
&\therefore \mathbf{x} - \mathbf{Tx} \in \ker \mathbf{T}
\end{aligned}$$

Now we want to show that $\mathrm{im}\mathbf{T} \cap \ker \mathbf{T} = \mathbf{O}$. Let $\mathbf{x} \in \mathrm{im}\mathbf{T} \cap \ker \mathbf{T}$ be given. Since $\mathbf{x} \in \ker T, \mathbf{Tx} = \mathbf{0}$. Since $\mathbf{x} \in \mathrm{im}\mathbf{T}, \exists \mathbf{x}_0 \in \mathbf{V}$ such that $\mathbf{x} = \mathbf{T}\mathbf{x}_0 = \mathbf{T}^2\mathbf{x}_0 = \mathbf{T}(\mathbf{T}\mathbf{x}_0) = \mathbf{Tx} = \mathbf{0}$. Thus $\mathrm{im}\mathbf{T} \cap \ker \mathbf{T} = \mathbf{0}$, i.e., $\mathbf{V} = \mathrm{im}\mathbf{T} \bigoplus \ker \mathbf{T}$. Put $\mathbf{W}_1 \coloneqq \mathrm{im}\mathbf{T}, \mathbf{W}_2 \coloneqq \ker \mathbf{T}$. $\therefore$ $\mathbf{T}$ is projection on $\mathbf{W}_1$ ☐

**Definition.** $\mathbf{T}$ *is orthogonal projection of* $\mathbf{V}$ *on* $\mathbf{W}$ *if and only if* $im\mathbf{T} = \mathbf{W}$ *and* $\ker \mathbf{T} = im\mathbf{T}^\perp, \ker \mathbf{T}^\perp = im\mathbf{T}$

**Theorem.** $\mathbf{T}$ *is orthogonal projection on* $\mathbf{W}$ $\iff$ $\exists \mathbf{T}^*$ *and* $\mathbf{T}^2 = \mathbf{T} = \mathbf{T}^*$

*Proof.* $\Longrightarrow$ $\mathbf{T}$ is projection $\iff$ $\mathbf{T}^2 = \mathbf{T}$. It is enough to show that $\exists \mathbf{T}^*$ such that $\mathbf{T}^* = \mathbf{T}$. $\mathbf{V} = \mathrm{im}\mathbf{T} \bigoplus \ker \mathbf{T}$ and $\mathrm{im}\mathbf{T}^\perp = \ker \mathbf{T}, \ker \mathbf{T}^\perp = \mathrm{im}\mathbf{T}$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{V}$ with $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ for $\mathbf{x}_1, \mathbf{y}_1 \in \mathrm{im}\mathbf{T}$ for $\mathbf{x}_2, \mathbf{y}_2 \in \ker \mathbf{T}$

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{Ty} \rangle &= \langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{T}(\mathbf{y}_1 + \mathbf{y}_2 \rangle \\
&= \langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{T}\mathbf{y}_1 \rangle \\
&(\because \mathbf{y}_2 \in \ker \mathbf{T}) \\
&= \langle \mathbf{x}_1, \mathbf{T}\mathbf{y}_1 \rangle + \langle \mathbf{x}_2, \mathbf{T}\mathbf{y}_1 \rangle \\
&= \langle \mathbf{x}_1, \mathbf{y}_1 \rangle + \langle \mathbf{x}_2, \mathbf{y}_1 \rangle \\
&(\because \mathbf{y}_1 \in \mathrm{im}\mathbf{T}, \mathbf{T}^2 = \mathbf{T}, \mathbf{y}_1 = \mathbf{T}\mathbf{y}' = \mathbf{T}^2\mathbf{y}' = \mathbf{T}\mathbf{y}_1)
\end{aligned}$$

Since $\ker \mathbf{T} = \mathrm{im}\mathbf{T}^\perp, \langle \mathbf{x}_2, \mathbf{y}_1 \rangle = 0$. i.e., $\langle \mathbf{x}_1, \mathbf{Ty} \rangle = \langle \mathbf{x}_1, \mathbf{y}_1 \rangle$

$$\begin{aligned}
\langle \mathbf{Tx}, \mathbf{y} \rangle &= \langle \mathbf{T}\mathbf{x}_1 + \mathbf{T}\mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2 \rangle \\
&= \langle \mathbf{T}\mathbf{x}_1, \mathbf{y}_1 + \mathbf{y}_2 \rangle \\
&= \langle \mathbf{x}_1, \mathbf{y}_1 \rangle + \langle \mathbf{x}_1, \mathbf{y}_2 \rangle \\
&= \langle \mathbf{x}_1, \mathbf{y}_1 \rangle \\
\langle \mathbf{x}, \mathbf{Ty} \rangle &= \langle \mathbf{Tx}, \mathbf{y} \rangle
\end{aligned}$$

Put $\mathbf{T}^* \coloneqq \mathbf{T}$. $\therefore \exists \mathbf{T}^*$ such that $\mathbf{T}^* = \mathbf{T}$
$\Longleftarrow$ since $\mathbf{T}^2 = \mathbf{T}$, $\mathbf{T}$ is projection. We want to show that $\mathrm{im}\mathbf{T} = \ker \mathbf{T}^\perp$ and $\ker \mathbf{T} = \mathrm{im}\mathbf{T}^\perp$. Let

$\mathbf{x} \in \mathrm{im}\mathbf{T}, \mathbf{y} \in \ker \mathbf{T}$. Since $\mathbf{T}^2 = \mathbf{T}$ and $\mathbf{x} \in \mathrm{im}\mathbf{T}$, $\mathbf{Tx} = \mathbf{T}^2\mathbf{x}_0 = \mathbf{Tx}_0 = \mathbf{x}$. Then $\mathbf{x} = \mathbf{Tx} = \mathbf{T}^*\mathbf{x}$

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{Tx}, \mathbf{y} \rangle \\ &= \langle \mathbf{T}^*\mathbf{x}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{Ty} \rangle \\ &= \langle \mathbf{x}, \mathbf{0} \rangle \\ &= 0 \\ \therefore \mathbf{x} &\in \ker \mathbf{T}^\perp \end{aligned}$$

Let $\mathbf{y} \in \ker \mathbf{T}^\perp$. We want to show that $\mathbf{Ty} = \mathbf{y}$

$$\begin{aligned} \|\mathbf{y} - \mathbf{Ty}\|^2 &= \langle \mathbf{y} - \mathbf{Ty}, \mathbf{y} - \mathbf{Ty} \rangle \\ &= \langle \mathbf{y}, \mathbf{y} - \mathbf{Ty} \rangle - \langle \mathbf{Ty}, \mathbf{y} - \mathbf{Ty} \rangle \end{aligned}$$

Since $\mathbf{y} - \mathbf{Ty} \in \ker \mathbf{T}$, $\langle \mathbf{y}, \mathbf{y} - \mathbf{Ty} \rangle = 0$.

$$\begin{aligned} \langle \mathbf{Ty}, \mathbf{y} - \mathbf{Ty} \rangle &= \langle \mathbf{y}, \mathbf{T}^*(\mathbf{y} - \mathbf{Ty}) \rangle \\ &= \langle \mathbf{y}, \mathbf{T}(\mathbf{y} - \mathbf{Ty}) \rangle \\ &= \langle \mathbf{y}, \mathbf{0} \rangle \\ &= 0 \\ \therefore \mathbf{y} &= \mathbf{Ty} \end{aligned}$$

Then we want to show that $\ker \mathbf{T} = \mathrm{im}\mathbf{T}^\perp$. $\ker \mathbf{T} \subset (\ker \mathbf{T}^\perp)^\perp = \mathrm{im}\mathbf{T}^\perp$.
Let $\mathbf{x} \in \mathrm{im}\mathbf{T}^\perp$. $\forall \mathbf{y} \in \mathbf{V}$,

$$\begin{aligned} \langle \mathbf{T}, \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{T}^*\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{Ty} \rangle \\ &= 0 \\ \therefore \mathbf{Tx} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{x} \in \ker \mathbf{T}, \therefore \ker \mathbf{T} = \mathrm{im}\mathbf{T}^\perp$ $\square$

**Note.** $\mathbf{T}$ *is orthogonal projection on* $\mathbf{W}$. $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is an orthonormal basis of* $\mathbf{V}$ *such that* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is an orthonormal basis for* $\mathbf{W} \leq \mathbf{V}$. *Then* $[\mathbf{T}]_\beta = diag(1, \ldots, 1, 0, \ldots, 0)$.

$$\mathbf{Tv}_i = \begin{cases} \mathbf{v}_i & (i \leq k) \\ \mathbf{0} & (i > k) \end{cases}$$

**Theorem** (Spectral Theorem). $\mathbf{V}$ *is finite-dimensional vector space over* $\mathbf{F}$ *and* $\mathbf{T}$ *is linear operator on* $\mathbf{V}$ *with distinct eigenvalue* $\lambda_1, \ldots, \lambda_k$. $\mathbf{T}$ *is normal if* $\mathbf{F} = \mathbb{C}$ *or* $\mathbf{T}$ *is self-adjoint if* $\mathbf{F} = \mathbb{R}$. $\mathbf{W}_i \coloneqq \mathbf{E}_{\lambda_i}$ *for* $\mathbf{T}_i$ *is orthogonal projection on* $\mathbf{W}_i$.

*(a).* $\mathbf{V} = \mathbf{W}_1 \bigoplus \cdots \bigoplus \mathbf{W}_k$ *(eigenspace decomposition)*

*(b).* $\mathbf{W}_i' \coloneqq \bigoplus_{j \neq i} \mathbf{W}_j \implies \mathbf{W}_i' = \mathbf{W}_i^\perp$

*(c).* $\mathbf{T}_i \circ \mathbf{T}_j = \delta_{ij}\mathbf{T}_i$

*(d).* $\mathbf{I} = \mathbf{T}_1 \cdots + \mathbf{T}_k$

*(e).* $\mathbf{T} = \lambda_1 \mathbf{T}_1 + \cdots + \lambda_k \mathbf{T}_k$ *(Spectral decomposition)*

(b). *Proof.* For $\mathbf{x} \in \mathbf{W}_i, \mathbf{y} \in \mathbf{W}_j$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Since $\mathbf{T}$ is normal or self-adjoint, two eigenvectors corresponding each to two different eigenvalues are orthogonal. $\mathbf{W}_i' \subset \mathbf{W}_i^{\perp}$

$$\begin{aligned}
\dim \mathbf{W}_i' &= \sum_{i \neq j} \dim \mathbf{W}_j \\
&= \dim \mathbf{V} - \dim \mathbf{W}_i \\
&= \dim \mathbf{W}_i^{\perp} \\
\mathbf{W}_i' &= \mathbf{W}_i^{\perp}
\end{aligned}$$

□

(c). *Proof.* $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ where $\mathbf{x}_i \in \mathbf{W}_i$. $\mathbf{T}_i\mathbf{T}_j(\mathbf{x}) = \mathbf{T}_i\mathbf{x}_j$

$$\begin{cases} i = j & \mathbf{T}_i\mathbf{x}_j = \mathbf{x}_i = \mathbf{T}_i\mathbf{x} \\ i \neq j & \mathbf{T}_i\mathbf{x}_j = \mathbf{0} \end{cases}$$

$\therefore \mathbf{T}_i\mathbf{T}_j = \delta_{ij}\mathbf{T}_i$ □

(d). *Proof.* $\ker \mathbf{T}_i = \mathrm{im}\mathbf{T}_i^{\perp} = \mathbf{W}_i^{\perp} = \mathbf{W}_i'$. $\mathbf{V} = \mathrm{im}\mathbf{T}_i + \ker \mathbf{T}_i (= \mathbf{W}_i')$

For all $\mathbf{x} \in \mathbf{V}$,

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}_i + (\mathbf{x}_1 + \cdots + \mathbf{x}_{i-1} + \mathbf{x}_{i+1} + \cdots + \mathbf{x}_k) \\
&= \mathbf{T}_i\mathbf{x} + (\mathbf{T}_1\mathbf{x} + \cdots + \mathbf{T}_{i-1}\mathbf{x} + \mathbf{T}_{i+1}\mathbf{x} + \cdots + \mathbf{T}_k\mathbf{x}) \\
&= (\mathbf{T}_1 + \cdots + \mathbf{T}_k)\mathbf{x} \\
\therefore \mathbf{I} &= \mathbf{T}_1 + \cdots + \mathbf{T}_k
\end{aligned}$$

□

(e). *Proof.* $\forall \mathbf{x} \in \mathbf{V}, \mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ for $\mathbf{x}_i \in \mathbf{W}_i$

$$\begin{aligned}
\mathbf{T}\mathbf{x} &= \mathbf{T}(\mathbf{x}_1 + \cdots + \mathbf{x}_k) \\
&= \mathbf{T}\mathbf{x}_1 + \cdots + \mathbf{T}\mathbf{x}_k \\
&= \lambda_1\mathbf{x}_1 + \cdots + \lambda_k\mathbf{x}_k \\
&= \lambda_1\mathbf{T}_1\mathbf{x} + \cdots + \lambda_k\mathbf{T}_k\mathbf{x} \\
&= (\lambda_1\mathbf{T}_1 + \cdots + \lambda_k\mathbf{T}_k)\mathbf{x} \\
\therefore &= \lambda_1\mathbf{T}_1 + \cdots + \lambda_k\mathbf{T}_k
\end{aligned}$$

□

**Note.** *If $A$ is unitarily diagonalizable, $A = Q^{-1}AQ = Q^*AQ$. Thus getting inverse of $Q$ is just $\overline{Q}^t$. It's much computationally cheaper than usual case.*

**Note.** *If* $\mathbf{T}$ *is normal(self-adjoint) and* $\beta$ *is the union of orthonormal bases of the* $\mathbf{W}_i$ *and let* $m_i = \dim \mathbf{W}_i$*, then* $[\mathbf{T}]_\beta = \begin{pmatrix} \lambda_1 \mathbf{I}_{m_1} & & & 0 \\ & \lambda_2 \mathbf{I}_{m_2} & & \\ & & \ddots & \\ 0 & & & \lambda_k \mathbf{I}_{m_k} \end{pmatrix}$

**Corollary.** $\mathbf{T}$ *is unitary* $\iff$ $\mathbf{T}$ *is normal and* $|\lambda_i| = 1$ *for* $i = 1, \ldots, k$

*Proof.* $\implies$ Suppose that $\mathbf{T}$ is unitary. Since $\mathbf{T}^*\mathbf{T} = \mathbf{T}\mathbf{T}^*$, $\mathbf{T}$ is normal.

$$\begin{aligned} \mathbf{T}\mathbf{T}^*\mathbf{v}_i &= \mathbf{T}(\overline{\lambda}_i \mathbf{v}_i) \\ &= |\lambda_i|^2 \mathbf{v}_i \\ &= \mathbf{v}_i \ (\because \mathbf{T}\mathbf{T}^* = \mathbf{I}) \\ \therefore |\lambda_i|^2 &= 1 \end{aligned}$$

$\impliedby$ Suppose that $\mathbf{T}$ is normal and $|\lambda_i| = 1$. Since $\mathbf{T}_i$ is orthogonal projection, $\mathbf{T}_i^2 = \mathbf{T}_i = \mathbf{T}_i^*$.

$$\begin{aligned} \mathbf{T}^*\mathbf{T} &= (\overline{\lambda}_1 \mathbf{T}_1 + \cdots + \overline{\lambda}_k \mathbf{T}_k)(\lambda_1 \mathbf{T}_1 + \cdots + \lambda_k \mathbf{T}_k) \\ &= |\lambda_1|^2 \mathbf{T}_1^2 + \cdots + |\lambda_k|^2 \mathbf{T}_k^2 \\ &= \mathbf{T}_1 + \cdots + \mathbf{T}_k \\ &= \mathbf{I} \end{aligned}$$

$\therefore$ $\mathbf{T}$ is unitary □

**Corollary.** $\mathbf{T}$ *is normal and* $\mathbf{F} = \mathbb{C}$*. Then* $\mathbf{T}$ *is self-adjoint if and only if* $\forall \lambda \in \mathbb{R}$

*Proof.* $\implies$ Suppose that $\mathbf{T}$ is self-adjoint. For all $i$

$$\begin{aligned} \lambda_i \mathbf{x}_i &= \mathbf{T}\mathbf{x}_i \\ &= \mathbf{T}^*\mathbf{x}_i \\ &= \overline{\lambda}_i \mathbf{x}_i \\ \therefore \overline{\lambda}_i &= \lambda_i \\ \therefore \lambda_i &\in \mathbb{R} \end{aligned}$$

$\impliedby$

$$\begin{aligned} \mathbf{T} &= \lambda_1 \mathbf{T}_1 + \lambda_2 \mathbf{T}_2 + \cdots + \lambda_k \mathbf{T}_k \\ \mathbf{T}^* &= \overline{\lambda}_1 \mathbf{T}_1 + \cdots + \overline{\lambda}_k \mathbf{T}_k \\ &= \lambda_1 \mathbf{T}_1 + \cdots + \lambda_k \mathbf{T}_k \\ &= \mathbf{T} \end{aligned}$$

$\therefore$ $\mathbf{T}$ is self-adjoint □

**Definition** (Singular Value Decomposition)**.** $\mathbf{V}, \mathbf{W}$ *are finite-dimensional inner product space.* $\mathbf{T} : \mathbf{V} \longrightarrow \mathbf{W}$*: linear transformation of rank* $r$*. Then* $\exists \beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$*: an orthonormal basis for* $\mathbf{V}$*,* $\gamma = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ *: an orthonormal basis of* $\mathbf{W}$*, and* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ *such that*

$$\mathbf{T}\mathbf{v}_i = \begin{cases} \sigma_i \mathbf{u}_i & (i \leq r) \\ \mathbf{0} & (i > r) \end{cases}$$

*Conversely if the preceding conditions are satisfied, then*

$$\begin{cases} \mathbf{T}^*\mathbf{T}\mathbf{v}_i = \sigma_i^2 \mathbf{u}_i & (i \leq r) \\ \mathbf{T}^*\mathbf{T}\mathbf{v}_i = \mathbf{0} & (i > r) \end{cases}$$

$\therefore$ *the scalars* $\sigma_1, \ldots, \sigma_r$ *are uniquely determined by* $\mathbf{T}$