Linear Algebra Class on 30 March

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Definition 1. $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$

$$\det A \coloneqq \sum_{\sigma \in \mathfrak{S}} \operatorname{sgn}(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n}$$

e.g.) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\mathfrak{S} = \{\iota, (1,2)\}$

$\det A = ad - bc$

e.g.) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\mathfrak{S}_3 = \{\iota, (1,2), (2,3), (1,3), (1,2,3), (1,3,2)\}$

$Sgn = 1, -1, -1, -1, 1, 1$

$$\begin{aligned} \det A &= a_{11}a_{22}a_{33} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

<u>Our Goal</u>

1) $\det$: alternating n-linear form on $\mathbb{R}^n$

2) $\det I_n = 1$

3) (Uniqueness) μ: alternating n-linear form with $\mu(I_n) = 1 \implies \mu = \det$

Proof. 1) det : alternating n-linear form on $\mathbb{R}^n$
(1) [n-linear] For fixed $k = 1, \ldots, n$, let

$$a_{ik} = b_{ik} + l \cdot c_{ik} \quad (i = 1, \ldots, n)$$

Then, we have

$$\begin{aligned}
\det A &= \sum_{\sigma \in \mathfrak{S}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(k),k} \cdots a_{\sigma(n),n} \\
&= \sum_{\sigma \in \mathfrak{S}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots (b_{\sigma(k),k} + l \cdot c_{\sigma(k),k}) \cdots a_{\sigma(n),n} \\
&= \sum_{\sigma \in \mathfrak{S}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots b_{\sigma(k),k} \cdots a_{\sigma(n),n} + l \cdot \sum_{\sigma \in \mathfrak{S}} \operatorname{sgn}(\sigma) \cdot a_{\sigma(1),1} \cdots c_{\sigma(k),k} \cdots a_{\sigma(n),n}
\end{aligned}$$

(2) alternating

Suppose that $[A]^k = [A]^l$ (k^{th} column of $A = i^{th}$ column of A)
Let $\tau := (k, l)$ transposition
Then, $\sigma \circ \tau = \begin{pmatrix} 1 & \cdots & k & \cdots & l & \cdots & n \\ \sigma(1) & \cdots & \sigma(l) & \cdots & \sigma(k) & \cdots & \sigma(n) \end{pmatrix}$

$$\begin{aligned}
\det A &= \sum_{\sigma \in \mathfrak{S}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n} \\
&= \sum_{\sigma \in \mathfrak{A}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n} + \sum_{\sigma \in \mathfrak{B}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n} \quad (\mathfrak{S} = \mathfrak{A} \uplus \mathfrak{B}) \\
&= \sum_{\sigma \in \mathfrak{A}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n} + \sum_{\sigma \in \mathfrak{A}} Sgn(\sigma \circ \tau) \cdot a_{\sigma\circ\tau(1),1} \cdots a_{\sigma\circ\tau(n),n} \\
&= \sum_{\sigma \in \mathfrak{A}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n} + \sum_{\sigma \in \mathfrak{A}} (-1) \cdot a_{\sigma(1),1} \cdots a_{\sigma(l),k} \cdots a_{\sigma(k),l} \cdots a_{\sigma(n),n} \\
&= \sum_{\sigma \in \mathfrak{A}} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n} + \sum_{\sigma \in \mathfrak{A}} (-1) \cdot a_{\sigma(1),1} \cdots a_{\sigma(l),l} \cdots a_{\sigma(k),k} \cdots a_{\sigma(n),n} \quad (\because [A]^k = [A]^l) \\
&= 0
\end{aligned}$$

2) $\det I_n = 1$

$$\begin{aligned}
\det I_n &= \det(\mathbf{e}_1 \cdots \mathbf{e}_n) \\
&= 1
\end{aligned}$$

3) (Uniqueness) $\mu = \text{alternating n} - \text{linear form with } \mu(I_n) = 1 \implies \mu = \det$

Let $T_n \coloneqq \{\text{ all function of } \{1, \cdots, n\} \to \{1, \cdots, n\}\}$

$$\begin{aligned}
\mu(A) &= \mu(a_{11}\mathbf{e}_1 + \cdots + a_{n1}\mathbf{e}_n, \cdots, a_{n1}\mathbf{e}_1 + \cdots + a_{nn}\mathbf{e}_n) \\
&= \sum_{\sigma \in T_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \mu(\mathbf{e}_{\sigma(1)}, \cdots, \mathbf{e}_{\sigma(n)}) \\
&= \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \mu(\mathbf{e}_{\sigma(1)}, \cdots, \mathbf{e}_{\sigma(n)}) \\
&= \sum_{\sigma \in \mathfrak{S}_n} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n} \mu(\mathbf{e}_1, \cdots, \mathbf{e}_n) \\
&= \sum_{\sigma \in \mathfrak{S}_n} Sgn(\sigma) \cdot a_{\sigma(1),1} \cdots a_{\sigma(n),n} \quad (\because \mu(I_n) = \mu(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1) \\
&= \det A
\end{aligned}$$

□

Properties

- $\det A = \det A^t$
- $\det AB = \det A \cdot \det B$
- $A : invertible \Leftrightarrow \det A \neq 0$
- Gaussian elimination $3_{rd} \to$ *invariant*