# Properties of determinant and cofactor expansion

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**Properties of determinant**

1. $\det A = \det A^t$

   *Proof.*

$$\begin{aligned}\det A^t &= \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)}\cdots a_{n,\sigma(n)}\\ &= \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) a_{1,\sigma^{-1}(1)}\cdots a_{n,\sigma^{-1}(n)}\\ \text{Since } \sigma \text{ is bijectvie },&\exists! i \text{ such that } \sigma i = j \text{ for } j = 1,\dots,n\\ &= \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1}\cdots a_{\sigma(n),n}\\ &= \det A\end{aligned}$$

□

2. $\det AB = \det A \cdot \det B \quad A, B \in \mathfrak{M}_{n\times n}(\mathbb{R})$

   *Proof.* Put $C \coloneqq AB$. Since $A\boldsymbol{b}_1 = \boldsymbol{c}_1, \dots, A\boldsymbol{b}_n = \boldsymbol{c}_n$, $[C]^k = b_{1k}[A]^1 + \cdots + b_{nk}[A]^n$

$$\begin{aligned}\det C &= \det([C]^1,\dots,[C]^n)\\ &= \det(b_{11}[A]^1 + \cdots + b_{n1}[A]^n, \dots, b_{1n}[A]^1 + \cdots + b_{nn}[A]^n)\\ &= \sum_{\sigma\in\mathfrak{S}_n} b_{\sigma(1),1}\cdots b_{\sigma(n),1} \det([A]^{\sigma(1)},\dots,[A]^{\sigma(n)})\\ &= \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) b_{\sigma(1),1}\cdots b_{\sigma(n),n} \det([A]^1,\dots,[A]^n)\\ &= \det A \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) b_{\sigma(1),1}\cdots b_{\sigma(n),n}\\ &= \det A \cdot \det B\end{aligned}$$

□

3. $A$ is invertible $\iff \det A \neq 0$

   **Lemma 1.** $A \in \mathfrak{M}_{n\times n}(\mathbb{R}).\{[A]^1,\dots,[A]^n\}$ *is linearly dependent* $\Longrightarrow \det A = 0$

*Proof.* Since $\{[A]^1, \dots, [A]^n\}$ is linearly dependent, there is at least one non-zero coefficient in $a_1, \dots, a_n \in \mathbb{R}$ such that $a_1[A]^1 + \cdots + a_n[A]^n = 0$. Without the loss of generality, suppose $a_1 = 0$. Then $[A]^1 = b_2[A]^2 + \cdots + b_n[A]^n$

$$\begin{aligned}
\det A &= \det([A]^1, \dots, [A]^n) \\
&= \det(b_2[A]^2 + \cdots + b_n[A]^n, [A]^2, \dots, [A]^n) \\
&= \sum_{j=2}^{n} b_j \det([A]^j, \dots, [A]^j, \dots, [A]^n) \\
&= 0
\end{aligned}$$

$\therefore \det A = 0$ $\square$

*Proof.* $\Longrightarrow$ Since $A$ is invertible $\exists! B$ such that $AB = BA = I_n$
Since $\det AB = \det A \cdot \det B$, $\det A \cdot \det B = \det AB = \det I_n = 1$
$\therefore \det A \neq 0$
$\Longleftarrow$ Suppose that $\det A \neq 0$. By the Lemma 1, $\{[A]^1, \dots, [A]^n\}$ is linearly independent, which implies that $A$ is full rank. Since $A$ is full rank $\iff$ $A$ is invertible, $A$ is invertible.
$\therefore \det A \neq 0 \Longrightarrow A$ is invertible. $\square$

4. $\det(\dots, [A]^i + c[A]^j, \dots, [A]^j, \dots) = \det A$

   *Proof.*

$$\begin{aligned}
\det(\dots, [A]^i + c[A]^j, \dots, [A]^j, \dots) &= \det(\dots, [A]^i, \dots, [A]^j, \dots) + c\det(\dots, [A]^j, \dots, [A]^j, \dots) \\
&= \det(\dots, [A]^i, \dots, [A]^j, \dots) \\
&= \det A
\end{aligned}$$

   $\therefore \det A$ is invariant to $3^{rd}$ elementary row operation $\square$

**Theorem 2** (cofactor expansion)**.** $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \hat{A}_{ij}$ *for* $i = 1, \dots, n$
*where* $\hat{A_{ij}} := \det M_{ij}$ *and* $M_{ij}$ *is a matrix that is obtained from* $A$ *by deleting i-th row and j-th column of* $A$. $\hat{A}_{ij}$ *is called (i,j) minor of* $A$

**Definition 3.**

$$\begin{aligned}
D_i(A) &:= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \hat{A}_{ij} \quad \textit{for } i = 1, \dots, n \\
D^j(A) &:= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \hat{A}_{ij} \quad \textit{for } j = 1, \dots, n
\end{aligned}$$

**Theorem 4.** $D_i(A), D^j(A) : \mathfrak{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ *are all alternating n-linear form on* $\mathbb{R}^n$ *with value 1 at* $\boldsymbol{I}_n$*. i.e.* $D_i(A) = D^j(A) = \det A$

*Proof.* (1). n-linear form on $\mathbb{R}^n$

Suppose that $a_{ik} = b_{ik} + lc_{ik}$ for $i = 1, \dots, n$ and $\mathrm{D}_i(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \hat{A}_{ij}$

$$\begin{cases} \text{if } j = k, a_{ij} \text{ is linear but } \hat{A}_{ij} \text{ is invariant to } [A]^k \text{ because } [A]^k \text{is excluded from } M_{ij} \\ \text{else if } j \neq k, a_{ij} \text{ is constant and } \hat{A}_{ij} \text{ is linear with respect to } [A]^k (\because [A]^k \text{ is included in } M_{ij}) \end{cases}$$

$\therefore$ all summand of $\mathrm{D}_i(A)$ is linear with respect to $[A]^k$

$\therefore$ $\mathrm{D}_i(A)$ is n-linear form on $\mathbb{R}^n$

(2). $D_i(A)$ is alternating

Suppose that $[A]^k = [A]^{k+1}$ We want to show that $\mathrm{D}_i(A) = 0$

if $j \neq k$ and $j \neq k+1$, the two same columns are included in $M_{ij}$. i.e., $\hat{A}_{ij} = 0$

$\therefore \mathrm{D}_i(A) = (-1)^{i+k} a_{ik} \hat{A}_{ik} + (-1)^{i+k+1} a_{ik+1} \hat{A}_{ik+1}$

Since $[A]^k = [A]^{k+1}, a_{ik} = a_{ik+1}$ and $\hat{A}_{ik} = \hat{A}_{ik+1}$

$\therefore \mathrm{D}_i(A) = 0$

(3). $\mathrm{D}_i(\boldsymbol{I}_n) = 1$

$\because \mathrm{D}_i(\boldsymbol{I}_n) = (-1)^{i+i} \det(\boldsymbol{I}_{n-1}) = 1$

By (1), (2), and (3) $\mathrm{D}_i(A) = \det A$. Since $\det A = \det A^t$, $\mathrm{D}_i(A) = \mathrm{D}_i(A^t) = \mathrm{D}^i(A)$

$\therefore \mathrm{D}_i(A) = \mathrm{D}^j(A)$ are all alternating n-linear form on $\mathbb{R}^n$ with value 1 at $\boldsymbol{I}_n$ □

**Example 1** (Cartan Matrix of type A)**.**

$$A_n = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & otherwise \end{cases}$$

$$= \begin{pmatrix} 2 & -1 & & & & \mathit{0} \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ \mathit{0} & & & & -1 & 2 \end{pmatrix}$$

*Put* $a_n \coloneqq \det A_n$.

$$a_n = 2a_{n-1} - (-1)\dot{(} - 1)a_{n-2}$$

$$a_n - a_{n-1} = a_{n-1} - a_{n-2}$$

*Since* $a_1 = 2, a_2 = 3, a_n = n + 1$