class note 190523

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May 2019

6.2 The Gram-Schmidt Orthogonalization Process and Orthogonal Complements

Theorem (Euler's formula)**.** $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$

Theorem. *Let* $\beta := \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *orthonormal basis for* $\boldsymbol{V}$ *and (*i^{th} *Fourier Coefficient)* $:= \langle \mathbf{v}, \mathbf{v}_i \rangle$*. But more generally* $\frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{e}^{\mathrm{i}nt}dt$*.*

Proof. WTS $\beta := \{\mathrm{e}^{inx} \mid n \in \mathbb{N}_0\}$ is orthonormal subset.
$C[0, 2\pi], \langle f, g \rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx$

$$\begin{aligned} \mathrm{e}^{\overline{inx}} &= \overline{\cos nx + \mathrm{i}\sin nx} \\ &= \cos nx - \mathrm{i}\sin nx \\ &= \cos(-nx) + \mathrm{i}\sin(-nx) \\ &= \mathrm{e}^{-inx} \end{aligned}$$

$$\begin{cases} n = m & \langle f, g \rangle = 1 \\ n \neq m & \frac{1}{2\pi}\left[\frac{1}{\mathrm{i}(n-m)}\mathrm{e}^{\mathrm{i}(n-m)x}\right]_0^{2\pi} = 0 \end{cases}$$

□

Definition 6.1. $\boldsymbol{W}$ *is subspace of* $\boldsymbol{V}$, $\boldsymbol{W}^\perp := \{\mathbf{v} \in \boldsymbol{V} \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\ \mathbf{w} \in \boldsymbol{W}\}$ *is called the orthogonal complement of* $\boldsymbol{W}$

Remark. $\boldsymbol{W}^\perp$ *is subspace of* $\boldsymbol{V}$

Proof. $c \cdot \mathbf{v}_1, \mathbf{v}_2 \in \boldsymbol{W}^\perp$, $\mathbf{w} \in \boldsymbol{W}$ and $c \in \mathbb{C}$

$$\langle \mathbf{w}, c\mathbf{v}_1 + \mathbf{v}_2 \rangle = \bar{c}\langle \mathbf{w}, \mathbf{v}_1 \rangle + \langle \mathbf{w}, \mathbf{v}_2 \rangle = 0$$

□

Theorem. $\left[\mathbf{v} \in \boldsymbol{V} \implies \exists! \mathbf{w} \in \boldsymbol{W}, \mathbf{w}' \in \boldsymbol{W}^\perp \textit{ such that } \mathbf{v} = \mathbf{w} + \mathbf{w}'\right] \iff \left[\boldsymbol{V} = \boldsymbol{W} \bigoplus \boldsymbol{W}^\perp\right]$

Proof. Take an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of $\boldsymbol{W}$. Put $\mathbf{w} := \sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{v}_i \rangle \mathbf{v}_i \in \boldsymbol{W}$ and $\mathbf{w}' := \mathbf{v} - \mathbf{w}$ We want to show $\mathbf{w}' \in \boldsymbol{W}^\perp$. In other words, $\forall \mathbf{u} \in \boldsymbol{W}, \langle \mathbf{w}', \mathbf{u} \rangle = 0$. It is enough to show

$\langle \mathbf{w}', \mathbf{v}_j \rangle$ for $j = 1, \dots, k$. ($\because$ $\mathbf{u} = \sum\limits_{i=1}^{k} a_i \mathbf{v}_i$, So inner product between $\mathbf{u}$ and $\mathbf{w}'$ is $\langle \sum\limits_{i=1}^{k} a_i \mathbf{v}_i, \mathbf{w}' \rangle = \sum\limits_{i=1}^{n} a_i \langle \mathbf{v}_i, \mathbf{w}' \rangle$. If $\mathbf{v}_i$ orthgonal to $\mathbf{w}'$ for all i, then $\langle \mathbf{u}, \mathbf{w}' \rangle = 0$). So take $\mathbf{u} \coloneqq \mathbf{v}_j$.

$$\begin{aligned}
\langle \mathbf{w}', \mathbf{v}_j \rangle &= \langle \mathbf{v} - \sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{v}_i \rangle \mathbf{v}_i, \mathbf{v}_j \rangle \\
&= \langle \mathbf{v}, \mathbf{v}_j \rangle - \sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{v}_i \rangle \langle \mathbf{v}_i, \mathbf{v}_j \rangle \\
&= \langle \mathbf{v}, \mathbf{v}_j \rangle - \langle \mathbf{v}, \mathbf{v}_j \rangle \\
&= 0
\end{aligned}$$

If $\mathbf{w} \in \boldsymbol{W} \cap \boldsymbol{W}^{\perp}$, then $\langle \mathbf{w}, \mathbf{w} \rangle = 0$, which implies that $\mathbf{w} = \mathbf{0}$. Thus $\boldsymbol{W} \cap \boldsymbol{W}^{\perp} = \mathbf{O}$.
Let $\mathbf{w}_1 + \mathbf{w}_1' = \mathbf{w}_2 + \mathbf{w}_2'$ for some $\mathbf{w}_1, \mathbf{w}_2 \in \boldsymbol{W}$ and $\mathbf{w}_1', \mathbf{w}_2' \in \boldsymbol{W}^{\perp}$.
Then $\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{w}_2' - \mathbf{w}_1' = \mathbf{0} \in \boldsymbol{W} \cap \boldsymbol{W}^{\perp}$. Thus $\mathbf{w}_1 = \mathbf{w}_2$, $\mathbf{w}_1' = \mathbf{w}_2'$.
$\therefore$ $\mathbf{v}$ is uniquely written in sum of $\mathbf{w}, \mathbf{w}'$ $\square$

6.3 The Adjoint of a Linear Operator

Note. *With inner product, we can define natural isomorphism(not dependent to any basis)*

Theorem.

(1) $g_{\mathbf{y}}$ is a linear functional ($g_{\mathbf{y}} \in \boldsymbol{V}^*$)

$$\begin{aligned}
g_{\mathbf{y}} : \boldsymbol{V} &\to \mathbb{R} \quad \mathbf{y} \in \boldsymbol{V} \\
\mathbf{x} &\mapsto \langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}$$

(2) The following map is natural isomorphism

$$\begin{aligned}
\boldsymbol{V} &\to \boldsymbol{V}^* \\
\mathbf{y} &\mapsto g_{\mathbf{y}}
\end{aligned}$$

Proof.

(1) We want to show $\langle c\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y} \rangle = c\langle \mathbf{x}_1, \mathbf{y} \rangle + \langle \mathbf{x}_2, \mathbf{y} \rangle$ $\quad c \in \mathbb{R}$ and $c \cdot \mathbf{x}_1, \mathbf{x}_2 \in \boldsymbol{V}$. It is trivial to show because inner product has linearity on the first component.

(2) We want to show that the map is linear and bijective. First for linearity.

$$\begin{aligned} g_{\mathbf{y}_1+\mathbf{y}_2}(\mathbf{x}) &= \langle \mathbf{x},\, \mathbf{y}_1 + c\mathbf{y}_2 \rangle \\ &= \langle \mathbf{x},\, \mathbf{y}_1 \rangle + \bar{c}\langle \mathbf{x},\, \mathbf{y}_2 \rangle \\ &= g_{\mathbf{y}_1}(\mathbf{x}) + \bar{c} g_{\mathbf{y}_2}(\mathbf{x}) \\ &= g_{\mathbf{y}_1}(\mathbf{x}) + c g_{\mathbf{y}_2}(\mathbf{x}) \end{aligned}$$

Then we show the map is one-to-one. Suppose that $g_{\mathbf{y}_1} = g_{\mathbf{y}_2}$, i.e., $g_{\mathbf{y}_1}(\mathbf{x}) = g_{\mathbf{y}_2}(\mathbf{x})$ for all $\mathbf{x} \in \boldsymbol{V}$.

$$\begin{aligned} \langle \mathbf{x},\, \mathbf{y}_1 \rangle &= \langle \mathbf{x},\, \mathbf{y}_2 \rangle \text{ for all } \mathbf{x} \in \boldsymbol{V} \\ \langle \mathbf{x},\, \mathbf{y}_1 \rangle - \langle \mathbf{x},\, \mathbf{y}_2 \rangle &= 0 \\ \langle \mathbf{x},\, \mathbf{y}_1 - \mathbf{y}_2 \rangle &= 0 \\ \text{Take } \mathbf{x} &= \mathbf{y}_1 - \mathbf{y}_2 \\ \langle \mathbf{y}_1 - \mathbf{y}_2,\, \mathbf{y}_1 - \mathbf{y}_2 \rangle &= 0 \\ \mathbf{y}_1 - \mathbf{y}_2 &= \mathbf{0} \\ \mathbf{y}_1 &= \mathbf{y}_2 \\ \therefore \mathbf{y} &\mapsto g_{\mathbf{y}} \text{ is 1-1} \end{aligned}$$

Finally we show the map is onto. Suppose that $f \in \boldsymbol{V}^*$ be given. Take orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\boldsymbol{V}$. Let $\mathbf{y} := \sum_{i=1}^{n} f(\mathbf{v}_i)\mathbf{v}_i$ and $g_{\mathbf{y}} := \langle \mathbf{x},\, \mathbf{y} \rangle \; \forall \mathbf{x} \in \boldsymbol{V}$. We want to show that $f = g_{\mathbf{y}}$. Since f and $g_{\mathbf{y}}$ is linear map, it suffices to show that $f(\mathbf{v}_i) = g_{\mathbf{y}}(\mathbf{v}_j)$ for $j = 1, \ldots, n$.

$$\begin{aligned} g_{\mathbf{y}}(\mathbf{v}_j) &= \langle \mathbf{v}_j,\, \mathbf{y} \rangle \\ &= \langle \mathbf{v}_j,\, \sum_{i=1}^{n} f(\mathbf{v}_i)\mathbf{v}_i \rangle \\ &= \sum_{i=1}^{n} f(\mathbf{v}_i) \langle \mathbf{v}_j,\, \mathbf{v}_i \rangle \\ (\because f(\mathbf{v}_i) &\in \mathbb{R} \text{ for } i = 1, \ldots, n) \\ &= \sum_{i=1}^{n} f(\mathbf{v}_i)\delta_{ji} \\ &= f(\mathbf{v}_j) \text{ for } j = 1, \ldots, n \\ \therefore g_{\mathbf{y}} &= f \end{aligned}$$

For every $f \in \boldsymbol{V}^*, \exists \mathbf{y}$ such that $f = \langle \mathbf{x},\, \mathbf{y} \rangle \; \forall \mathbf{x} \in \boldsymbol{V}$. $\therefore g \mapsto g_{\mathbf{y}}$ is onto.

$\square$

Theorem. $\boldsymbol{T} : \boldsymbol{T} \to \boldsymbol{V}$*: linear operator. Then* $\exists! \boldsymbol{T}^* : \boldsymbol{V} \to \boldsymbol{V}$ *such that* $\langle \boldsymbol{T}\mathbf{x},\, \mathbf{y} \rangle = \langle \mathbf{x},\, \boldsymbol{T}^*\mathbf{y} \rangle$ *and* $\boldsymbol{T}^*$ *is linear.*

Proof. By the previous theorem, there is a unique $\mathbf{y}' \in \boldsymbol{V}$ such that $g(\mathbf{x}) = \langle \mathbf{x},\, \mathbf{y}' \rangle$. I.e., $\langle \boldsymbol{T}\mathbf{x},\, \mathbf{y} \rangle = \langle \mathbf{x},\, \mathbf{y}' \rangle$. Define $\boldsymbol{T}^* : \boldsymbol{V} \to \boldsymbol{V}$ by $\boldsymbol{T}^*\mathbf{y} \coloneqq \mathbf{y}'$. Then $\langle \boldsymbol{T}\mathbf{x},\, \mathbf{y} \rangle = \langle \mathbf{x},\, \boldsymbol{T}^*\mathbf{y} \rangle$. Then we show $\boldsymbol{T}^*$ is linear.

$$\begin{aligned}
&\forall \mathbf{x} \in \boldsymbol{V} \\
\langle \mathbf{x},\, \boldsymbol{T}^*(\mathbf{y}_1 + \mathbf{y}_2) \rangle &= \langle \boldsymbol{T}\mathbf{x},\, \mathbf{y}_1 + \mathbf{y}_2 \rangle \\
&= \langle \boldsymbol{T}\mathbf{x},\, \mathbf{y}_1 \rangle + \langle \boldsymbol{T}\mathbf{x},\, \mathbf{y}_2 \rangle \\
&= \langle \mathbf{x},\, \boldsymbol{T}^*\mathbf{y}_1 \rangle + \langle \mathbf{x},\, \boldsymbol{T}^*\mathbf{y}_2 \rangle \\
&= \langle \mathbf{x},\, \boldsymbol{T}^*\mathbf{y}_1 + \boldsymbol{T}^*\mathbf{y}_2 \rangle \\
\langle \mathbf{x},\, \boldsymbol{T}^*(\mathbf{y}_1 + \mathbf{y}_2) \rangle - \langle \mathbf{x},\, \boldsymbol{T}^*\mathbf{y}_1 + \boldsymbol{T}^*\mathbf{y}_2 \rangle &= 0 \\
\boldsymbol{T}^*(\mathbf{y}_1 + \mathbf{y}_2) - \boldsymbol{T}^*\mathbf{y}_1 - \boldsymbol{T}^*\mathbf{y}_2 &= \mathbf{0} \\
\therefore \boldsymbol{T}^*(\mathbf{y}_1 + \mathbf{y}_2) &= \boldsymbol{T}^*\mathbf{y}_1 + \boldsymbol{T}^*\mathbf{y}_2
\end{aligned}$$

Finally, we need to show that $\boldsymbol{T}^*$ is unique. Suppose that $\boldsymbol{U} : \boldsymbol{V} \to \boldsymbol{V}$ is linear and that it satisfies $\langle \boldsymbol{T}\mathbf{x},\, \mathbf{y} \rangle = \langle \mathbf{x},\, \boldsymbol{U}\mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \boldsymbol{V}$. Then $\langle \mathbf{x},\, \boldsymbol{T}^*\mathbf{y} \rangle = \langle \mathbf{x},\, \boldsymbol{U}\mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \boldsymbol{V}$, so $\boldsymbol{T^*} = \boldsymbol{U}$. $\square$