class note 181229

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1.3 Subspace

Theorem. *$\mathbf{V}$ is a vector space and $\mathbf{W}$ is subset of $\mathbf{V}$. $\mathbf{W}$ is subspace of $(\mathbf{V}, +, \cdot)$ if $\mathbf{W}$ is a vector space with the operations of addition and scalar multiplication defined on $\mathbf{V}$.*

Note. *Restriction of the operations of addition and scalar multiplication.*

$$\begin{aligned} + &: \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V} \\ +|_{\mathbf{W}\times\mathbf{W}} &: \mathbf{W} \times \mathbf{W} \longrightarrow \mathbf{V} \\ \cdot &: \mathbf{F} \times \mathbf{V} \longrightarrow \mathbf{V} \\ \cdot|_{\mathbf{F}\times\mathbf{W}} &: \mathbf{F} \times \mathbf{W} \longrightarrow \mathbf{V} \end{aligned}$$

So $\mathbf{W}$ becomes a subspace $\iff +|_{\mathbf{W}\times\mathbf{W}} : \mathbf{W} \longrightarrow \mathbf{W}$ and $\cdot|_{\mathbf{F}\times\mathbf{W}} : \mathbf{F} \times \mathbf{W} \longrightarrow \mathbf{W}$ and denoted by $\mathbf{W} \leq \mathbf{V}$.

Theorem (Subspace Test)**.** $\mathbf{W} \leq \mathbf{V} \iff \mathbf{x} + \mathbf{y} \in \mathbf{W}$ *and* $c\mathbf{x} \in \mathbf{W} \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{W}, \forall c \in \mathbf{F}$.

Proof. $\Longrightarrow$ It's trivial to show.
$\Longleftarrow$ $\exists \mathbf{0_v} \in \mathbf{V}$ such that $\mathbf{0_v} + \mathbf{v} = \mathbf{v} + \mathbf{0_v} = \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V}$. We want to show $\mathbf{0_v} \in \mathbf{W}$. Suppose $\mathbf{x} \in \mathbf{W}$ be given. Since $c\mathbf{x} \in \mathbf{W}, \mathbf{0_v} \in \mathbf{W}$. So define $\mathbf{0_w} := \mathbf{0}_v$.Then we want to show that $-\mathbf{x} \in \mathbf{W}$. Since $c \in \mathbf{W}$ by the assumption, take $c = -1$. Then $-1 \cdot \mathbf{x} = -\mathbf{x} \in \mathbf{W}$.Thus $\forall \mathbf{x} \in \mathbf{W}, \exists -\mathbf{x} \in \mathbf{W}$ such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0_w}$. Since the binary operation and scalar multiplication are inherited from $\mathbf{V}$, associative, commutative, and distributive property hold for $\mathbf{W}$. $\square$

Example 1.

1. $\mathbf{P}_n(\mathbb{R}) := \{f(x) \in \mathbf{P}(\mathbb{R}) \mid \deg f(x) \leq n\}$, $\mathbf{P}_n(\mathbb{R}) \leq \mathbf{P}(\mathbb{R})$
2. $C(\mathbb{R}) := \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f : \textit{continuous}\} \leq \mathcal{F}(\mathbb{R}, \mathbb{R})$
3. $\mathbf{V} := \mathfrak{M}_{m\times n}(\mathbb{R})$, *the following spaces are subspaces of* $\mathbf{V}$.
   (a) $\mathbf{U}_n(\mathbb{R}) := \{A \in \mathfrak{M}_{m\times n}(\mathbb{R}) \mid a_{ij} = 0 \, \forall i > j\}$
   (b) $\mathbf{L}_n(\mathbb{R}) := \{A \in \mathfrak{M}_{m\times n}(\mathbb{R}) \mid a_{ij} = 0 \, \forall i < j\}$
   (c) $\mathbf{D}_n(\mathbb{R}) := \{A \in \mathfrak{M}_{m\times n}(\mathbb{R}) \mid a_{ij} = 0 \, \forall i \neq j\}$

(d) $\mathbf{sl}_n(\mathbb{R}) \coloneqq \{A \in \mathfrak{M}_{m\times n}(\mathbb{R}) \mid trA \coloneqq a_{11} + \cdots + a_{nn} = 0\}$

Theorem. *$\Lambda \coloneqq$ (an index set), if $\mathbf{W}_\alpha \le \mathbf{V}, \alpha \in \Lambda$, then $\bigcap\limits_{\alpha\in\Lambda} \mathbf{W}_\alpha \le \mathbf{V}$.*

Proof. If $\mathbf{v}, \mathbf{w} \in \bigcap\limits_{\alpha\in\Lambda} \mathbf{W}_\alpha$, then $\mathbf{v}, \mathbf{w} \in \mathbf{W}_\alpha$ for all α in Λ. Since $\mathbf{W}_\alpha \le \mathbf{V}$, $\mathbf{v}+\mathbf{w} \in \mathbf{W}_\alpha$. That is $\mathbf{v}+\mathbf{w} \in \mathbf{W}_\alpha$ for all $\alpha \in \Lambda$. So $\mathbf{v}+\mathbf{w} \in \bigcap\limits_{\alpha\in\Lambda} \mathbf{W}_\alpha$. The scalar multiplication can be proved similarly. □

Note. *If $\mathbf{U}, \mathbf{W} \le \mathbf{V}$, then $\mathbf{U} \cup \mathbf{W} \le \mathbf{V}$? The answer is No. Here is the counter example. $\mathbf{V} \coloneqq \mathbb{R}^2, \mathbf{U} \coloneqq \{(x,0) \mid x \in \mathbb{R}\}, \mathbf{W} \coloneqq \{(0,y) \mid y \in \mathbb{R}\}$. $(1,0) \in \mathbf{U}, (0,1) \in \mathbf{W}$,*
but $(1,0) + (0,2) = (1,2) \notin \mathbf{U} \cup \mathbf{W}$.

Definition. *$\mathbf{W}_1, \mathbf{W}_2 \le \mathbf{V}$, $\mathbf{W}_1 + \mathbf{W}_2 \coloneqq \{\mathbf{w}_1 + \mathbf{w}_2 \mid \mathbf{w}_1 \in \mathbf{W}_1, \mathbf{w}_2 \in \mathbf{W}_2\}$ is the sum of $\mathbf{W}_1$ and $\mathbf{W}_2$.*

Definition. *$\mathbf{W}_1 \bigoplus \mathbf{W}_2 \coloneqq$ the direct sum if it is the sum of $\mathbf{W}_1$ and $\mathbf{W}_2 = \mathbf{V}$ with $\mathbf{W}_1 \cap \mathbf{W}_2 = \mathbf{O}$*

Theorem. $\mathbf{W}_1 + \mathbf{W}_2 \le \mathbf{V}$

Proof. Since $\mathbf{w}_1 \in \mathbf{W}_1, \mathbf{w}_2 \in \mathbf{W}_2$, $\mathbf{W}_1 + \mathbf{W}_2 \subset \mathbf{V}$. Let $\mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}'_1 + \mathbf{w}'_2 \in \mathbf{W}_1 + \mathbf{W}_2$ and $c \in \mathbf{F}$ be given. Then we want to show $\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}'_1+\mathbf{w}'_2 \in \mathbf{W}_1+\mathbf{W}_2$. Since $\mathbf{w}_1+\mathbf{w}'_1 \in \mathbf{W}_1$ and $\mathbf{w}_2+\mathbf{w}'_2 \in \mathbf{W}_2$, $\mathbf{w}_1 + \mathbf{w}'_1 + \mathbf{w}_2 + \mathbf{w}'_2 \in \mathbf{W}_1 + \mathbf{W}_2$. Proof for scalar multiplication is similar. □

Theorem. $\mathbf{V} = \mathbf{W}_1 \bigoplus \mathbf{W}_2 \iff \big[$*Every $\mathbf{v} \in \mathbf{V}$ can be written as $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ for a unique* $\mathbf{w}_1 \in \mathbf{W}_1, \mathbf{w}_2 \in \mathbf{W}_2\big]$

Proof. $\implies$ For any $\mathbf{v} \in \mathbf{V}$, $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ for some $\mathbf{w}_1 \in \mathbf{W}_1, \mathbf{w}_2 \in \mathbf{W}_2$ because $\mathbf{V} = \mathbf{W}_1 + \mathbf{W}_2$. We want to show the uniqueness. Suppose $\mathbf{w}_1+\mathbf{w}_2 = \mathbf{w}'_1+\mathbf{w}'_2$. Then $\mathbf{w}_1-\mathbf{w}'_1 = \mathbf{w}'_2 = \mathbf{w}_2 \in \mathbf{W}_1\cap\mathbf{W}_2 = \mathbf{O}$. That is $\mathbf{w}_1 - \mathbf{w}'_1 = \mathbf{w}'_2 - \mathbf{w}_2 = \mathbf{0}$.
$\therefore \mathbf{w}_1 = \mathbf{w}'_1$ and $\mathbf{w}_2 = \mathbf{w}'_2$
$\impliedby \mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 \quad \mathbf{w}_1 \in \mathbf{W}_1, \mathbf{w}_2 \in \mathbf{W}_2$ and $\mathbf{w}_1, \mathbf{w}_2$ are unique. We want to show that $\mathbf{W}_1 \cap \mathbf{W}_2 = \mathbf{O}$. Let's assume that $\mathbf{W}_1 \cap \mathbf{W}_2 \ne \mathbf{O}$. That is $\exists \mathbf{w} \in \mathbf{W}_1 \cap \mathbf{W}_2$ such that $\mathbf{w} \ne \mathbf{0}$. Since $\mathbf{w} \in \mathbf{W}_1 \cap \mathbf{W}_2, \mathbf{w}_1 + \mathbf{w} \in \mathbf{W}_1$ and $\mathbf{w}_2 - \mathbf{w} \in \mathbf{W}_2$. Thus $\mathbf{v} = (\mathbf{w}_1 + \mathbf{w}) + (\mathbf{w}_2 - \mathbf{w}) = \mathbf{w}_1 + \mathbf{w}_2$. It contradicts to the assumption that $\mathbf{v}$ can be uniquely written as sum of two vectors from $\mathbf{W}_1, \mathbf{W}_2$.
$\therefore \mathbf{W}_1 \cap \mathbf{W}_2 = \mathbf{O}$ □