Chapter 4

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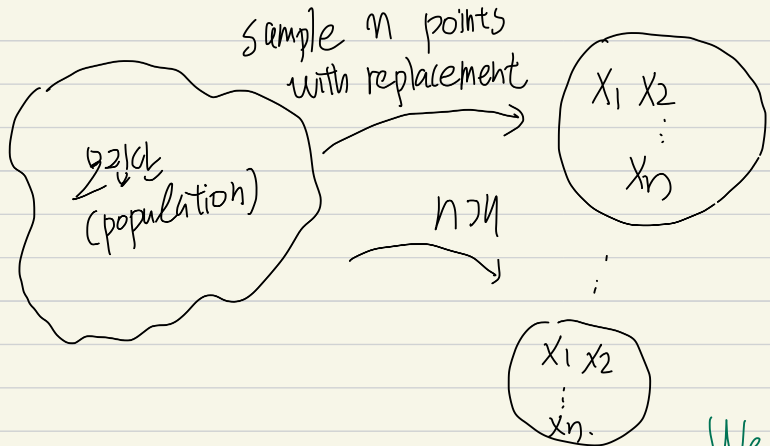


Q. random sample이 분포를 따른다는 의미는?

sampling하는 행위를 여러번 했을 때

X_i의 값은 매번 다르다.

즉 X_i는 어떠한 분포를 따르는데

그 분포는 모집단의 분포이다.

즉 $X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta)$

$\theta \in \Omega$ where θ: parameter

and Ω: parameter space

We call random sample

	X_1	X_2	X_3	$\cdots$	X_n
1	x_{11}	x_{12}	x_{13}	$\cdots$	x_{1n}
2	x_{21}	x_{22}	x_{23}	$\cdots$	x_{2n}

모두 독립이며 모집단의 분포를 따르는 random variable이다.

Def $f(x;\theta)$ $\theta \in \Omega$; probability density(mass) function of population.

$X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta)$ random sample.

⊙ Statistic(통계량) : a function w.r.t random samples.

i.e. $u(X_1, \dots, X_n)$.

Def 표본분포(sampling distribution) : 통계량의 분포.

$Y := u(X_1, \dots, X_n)$을 새로운 random sample이라 생각하면 이것이 따르는 분포를 생각하라.

e.g. $X_1, \dots, X_n \overset{iid}{\sim}$ Bernoulli(p)

(a) $\hat{p} := \frac{1}{n}(X_1 + \cdots + X_n)$ 표본비율. ... statistic

(b) $X_1, \dots, X_n$: random sample

$\bar{X} := \frac{1}{n}(X_1 + \cdots + X_n)$: 표본평균 $\longrightarrow$ 모평균추정

$S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$: 표본분산 $\longrightarrow$ 모분산추정

(c) $X_1, \dots, X_n$: r.s.

$X_{(1)} < \cdots < X_{(n)}$: 순서통계량(order stastic) 표본중앙값.

$$\underset{1 \le i \le m}{\mathrm{med}} := \begin{cases} X_{(m+1)} & n = 2m+1 \\ (X_{(m)} + X_{(m+1)})/2 & n = 2m \end{cases}$$

Example of Statistic

e.g. $X_1, \dots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$

$\Rightarrow X_1 + \cdots + X_n \sim B(n, p)$ $\qquad \hat{p} := (X_1 + \cdots + X_n)/n \swarrow$ statistic.

$$P\left(\hat{p} = \frac{k}{n}\right) = P(X_1 + \cdots + X_n = k)$$
$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Note. $\hat{p}$가 binomial을 따르는 것이 아니니 $X_1 + \cdots + X_n$이 binomial을 따른다!

$$E[\hat{p}] = \sum_{k=0}^{n} \left(\frac{k}{n}\right) \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \frac{k}{n} \frac{n!}{(n-k)!\,k!} p^{k-1} (1-p)^{n-1-(k-1)} p$$
$$= \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} (1-p)^{n-1-(k-1)} p$$
$$= (p + (1-p))^{n-1} p = p.$$

$$E[\hat{p}(\hat{p}-1)] = \sum_{k=0}^{n} \frac{k}{n}\left(\frac{k}{n} - 1\right) \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_{k=1}^{n-1} \frac{-(n-k)}{n} \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} (1-p)^{n-1-(k-1)} p$$
$$= -\frac{n-1}{n} \sum_{k=1}^{n-2} \frac{(n-2)!}{(n-k-1)!\,(k-1)!} p^{k-1} (1-p)^{n-2-(k-1)} p(1-p)$$
$$= \frac{(1-n)}{n} (p + (1-p))^{n-2} p(1-p)$$

$$\mathrm{Var}(\hat{p}) = E[\hat{p}(\hat{p}-1)] + E[\hat{p}] - E[\hat{p}]^2$$
$$= \frac{(1-n)}{n} p(1-p) + p - p^2$$
$$= p(1-p)\left(\frac{1}{n} - 1 + 1\right) = \frac{p(1-p)}{n} \qquad \square$$

Example of statistic

(b) $X_1, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$ X_i: random sample.

$X_1 + \cdots + X_n \sim \text{Poisson}(n\lambda)$

$\underbrace{T = \overline{X} := (X_1 + \cdots + X_n)/n}$

$P(\overline{X} = k/n) = P(X_1 + \cdots + X_n = k)$ $(k = 0, 1, 2, \dots)$

$= \dfrac{e^{-n\lambda}(n\lambda)^k}{k!}$

$E[\overline{X}] = \sum_{k=0}^{\infty} \dfrac{k}{n} \dfrac{e^{-n\lambda}}{k!}(n\lambda)^k = e^{-n\lambda} \sum_{k=1}^{\infty} \dfrac{(n\lambda)^{k-1}}{(k-1)!} \lambda$

$= e^{-n\lambda} e^{n\lambda} \lambda = \lambda.$

$E[\overline{X}^2] = \sum_{k=0}^{\infty} \dfrac{k^2}{n^2} \dfrac{e^{-n\lambda}}{k!}(n\lambda)^k$ $\lambda^2 + \lambda - \lambda^2$

$= \sum_{k=1}^{\infty} \dfrac{k}{n} \dfrac{e^{-n\lambda}}{(k-1)!}(n\lambda)^{k-1}\lambda$

$= \sum_{k=1}^{\infty} k \dfrac{e^{-n\lambda}}{(k-1)!}(n\lambda)^{k-2}\lambda^2$

$= \lambda^2 e^{-n\lambda} \sum_{k=1}^{\infty} k \dfrac{(n\lambda)^{k-2}}{(k-1)!}$

$\boxed{\dfrac{k}{(k-1)!} = \dfrac{1}{(k-2)!} + \dfrac{1}{(k-1)!} = \dfrac{k-1+1}{(k-1)!}}$

$= \lambda^2 e^{-n\lambda} \left(\sum_{k=2}^{\infty} \dfrac{(n\lambda)^{k-2}}{(k-2)!} + \sum_{k=1}^{\infty} \dfrac{(n\lambda)^{k-1}}{(k-1)} \dfrac{1}{n\lambda} \right)$

$= \lambda^2 e^{-n\lambda} (e^{n\lambda} + (n\lambda)^{-1} e^{n\lambda})$

$\underbrace{E[\overline{X}] = \lambda}$

unbiased estimator

$\underbrace{Var(\overline{X}) = \lambda/n}$

$\to 0$ as $n \to \infty$.

$= \lambda^2 + \dfrac{\lambda}{n}$ $\therefore Var(\overline{X}) = \lambda^2 + \dfrac{\lambda}{n} - \lambda^2$

$= \dfrac{\lambda}{n}$ $\square$

(b). $X_1, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda) \quad \lambda > 0.$

$\Rightarrow X_1 + \cdots + X_n \sim \text{Poisson}(n\lambda)$

$\Rightarrow P(\bar{X} = \frac{k}{n}) = P(X_1 + \cdots + X_k = k)$

$= \frac{e^{-n\lambda}(n\lambda)^k}{k!} \quad (k = 0, 1, \dots).$

$\underbrace{E[\bar{X}] = \lambda}_{\text{unbiased.}} \quad \underbrace{Var(\bar{X})}_{\to 0 \text{ as } n \to \infty.} = \frac{\lambda}{n}$

General Case.

Y: discrete r.V.

$Y = u(X_1, \dots, X_n)$ Y의 분포는? $X = (X_1, \dots, X_n)$ $\quad X \longmapsto u(X) = Y \sim$

$pdf_Y(y) = P(Y = y) = P(u(X) = y)$

$= \sum_{x : u(x) = y} P(X = x)$

$\therefore \underline{pdf_Y(y) = \sum_{x : u(x) = y} P(X = x)}$ $\square$

e.g. $X_1 \sim B(n_1, p) \quad X_2 \sim B(n_2, p) \quad X_1 \perp\!\!\!\perp X_2.$

(a) $Y := X_1 + X_2 \sim B(n_1 + n_2, p)$ $\quad x_2 = y - x_1$

$pdf_Y(y) = \sum_{x : x_1 + x_2 = y} P(X = x) = \sum_{x : x_1 + x_2 = y} pdf_{X_1}(x_1)\, pdf_{X_2}(x_2)$

$= \sum_{x_1 = 0}^{y} \binom{n_1}{x_1} p^{x_1} (1-p)^{n_1 - x_1} \binom{n_2}{y - x_1} p^{y - x_1} (1-p)^{n_2 + x_1 - y}$

$= \sum_{x_1 = 0}^{y} \binom{n_1}{x_1} \binom{n_2}{y - x_1} p^y (1-p)^{n_1 + n_2 - y}$

$= p^y (1-p)^{n_1 + n_2 - y} \binom{n_1 + n_2}{y}$

$\therefore Y \sim B(n_1 + n_2, p)$

$$\boxed{\begin{gathered}(1+t)^{n_1}(1+t)^{n_2} = (1+t)^{n_1+n_2} \\ \sum_{x_1=0}^{y} \binom{n_1}{x_1}\binom{n_2}{y-x_1} t^y = \binom{n_1+n_2}{y} t^y \\ t^y \text{ 계수비교!}\end{gathered}}$$

$\lambda_1, \lambda_2 > 0$.

(b). $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$ $X_1 \perp\!\!\!\perp X_2$.

$Y := X_1 + X_2$.

$$\text{pdf}_Y(y) = \sum_{x:\, x_1+x_2=y} \text{pdf}_X(x) \qquad X := (X_1, X_2)^t$$

$x_2 = y - x_1$.

$$= \sum_{x:\, x_1+x_2=y} \text{pdf}_{X_1}(x_1)\, \text{pdf}_{X_2}(x_2)$$

$$= \sum_{x:\, x_1+x_2=y} \frac{e^{-\lambda_1} \lambda_1^{x_1}}{x_1!} \frac{e^{-\lambda_2} \lambda_2^{x_2}}{x_2!}$$

$$= \sum_{x_1=0}^{y} \frac{e^{-(\lambda_1+\lambda_2)} \lambda_1^{x_1} \lambda_2^{y-x_1}}{x_1!\,(y-x_1)!}$$

$$= \sum_{x_1=0}^{y} \frac{y!}{x_1!\,(y-x_1)!} \lambda_1^{x_1} \lambda_2^{y-x_1} e^{-(\lambda_1+\lambda_2)} / y!$$

$$= \sum_{x_1=0}^{y} \binom{y}{x_1} \lambda_1^{x_1} \lambda_2^{y-x_1} e^{-(\lambda_1+\lambda_2)} / y!$$

$$= \frac{(\lambda_1+\lambda_2)^y e^{-(\lambda_1+\lambda_2)}}{y!}$$

$\therefore Y \sim \text{Poisson}(\lambda_1+\lambda_2)$

$\blacksquare$

Thm 4.1.1.

$X = (X_1, \dots, X_k)^t$ and X: continuous.

$u := (u_1, \dots, u_k)^t$: $u: \mathcal{X} \to \mathcal{Y}$

$u(x_1, \dots, x_k) = (u_1(x_1, \dots, x_k), \dots, u_k(x_1, \dots, x_k))^t$

(a) $P(X \in \mathcal{X}) = 1$

(b) u: bijection (1-1 and onto)

(c) $\mathcal{X}$: 열린집합, u: C^1 function (각 편도함수가 연속)

$$J_u(x) := \left(\det \frac{\partial u_i}{\partial x_j}(x)\right) = \det \frac{\partial(u_1, \dots, u_k)}{\partial(x_1, \dots, x_k)}(x) \neq 0 \quad \forall x \in \mathcal{X}$$

$\Rightarrow Y = u(X_1, \dots, X_k) = u(X)$의 pdf는

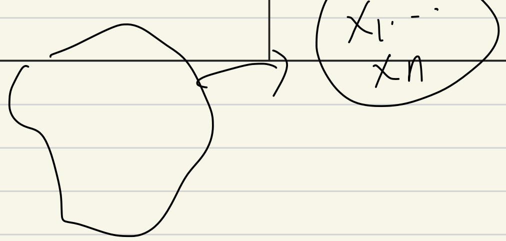
$$\Rightarrow \underline{pdf_Y(y) = pdf_X(x) \left|\det \frac{\partial y}{\partial x}\right|^{-1}} \quad y = u(x) \in \mathcal{Y}$$

proof) $P(Y \in B) = P(u(X) \in B)$

$$= \int_{x \in u^{-1}(B)} pdf_X(x)\, dx.$$

$$= \underbrace{\int_y pdf_X(u^{-1}(y)) \left|\det \frac{\partial y}{\partial x}\right|^{-1} dy} \quad y = u(x)$$

□



$Y = u(X_1, \dots, X_n) \sim pdf_Y(y)$

통계량.

$\underbrace{E[Y]} =$

$Y := \frac{X_1 + \cdots + X_n}{n}$

Z: continuous random variable.

$X = \underbrace{\sigma}_{\text{scale parameter}} Z + \underbrace{\mu}_{\text{location parameter}}$

$$\mathrm{pdf}_X(x) = f(z)\left|\det \frac{\partial x}{\partial z}\right|^{-1} \qquad x = \sigma z + \mu \qquad dx = \sigma\, dz \qquad \frac{dz}{dx} = \frac{1}{\sigma}$$

$$= \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

(a) $Z \sim N(0, 1^2)$ $\quad \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$$\mathrm{pdf}_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\therefore X \sim N(\mu, \sigma^2)$.

(e) $X \sim \mathrm{Exp}(\sigma)$

$$\mathrm{pdf}_X(x) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right) \qquad f(z) = e^{-z} I(0 \le z < +\infty)$$

$$= \frac{1}{\sigma} e^{-x/\sigma} I(0 \le x < +\infty)$$

$\therefore X \sim \mathrm{Exp}(\sigma)$

20.04.25.

Beta Distribution

e.g. 4.1.6.

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$, $X_2 \sim \text{Gamma}(\alpha_2, \beta)$ $X_1 \perp\!\!\!\perp X_2$.

$Y_1 := \dfrac{X_1}{X_1+X_2}$ $\quad Y_2 := X_1 + X_2$.

$\text{pdf}_{Y_1,Y_2}(y_1,y_2)$? $\longrightarrow$ $\text{pdf}_{Y_1}(y_1)$?

marginalization.

$X \sim \text{Gamma}(\alpha, \beta)$

$f(x) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta} I(x>0)$

$$\text{pdf}_{X_1,X_2}(x_1,x_2) = \text{pdf}_{X_1}(x_1)\,\text{pdf}_{X_2}(x_2)$$
$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} e^{-(x_1+x_2)/\beta} I(x_1>0, x_2>0)$$

Note. C^1 function

$f: U (\subset \mathbb{R}^n) \to \mathbb{R}$
open

if $\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n} : U \to \mathbb{R}$

continuous.

$F: U \to \mathbb{R}^n$

$F = (f_1, \dots, f_m)$

$F: C^1$ if all f_i are C^1.

$$\mathcal{X} = \{(x_1,x_2)^t : \text{pdf}_{X_1,X_2}(x_1,x_2) > 0\} = \{(x_1,x_2)^t : x_1>0, x_2>0\}$$

$$u : \begin{cases} y_1 = x_1/(x_1+x_2) \\ y_2 = x_1 + x_2 \end{cases} \qquad u^{-1} : \begin{cases} x_1 = y_1 y_2 > 0 \\ x_2 = (x_1+x_2) - x_1 = y_2 - y_1 y_2 = y_2(1-y_1) > 0 \end{cases}$$

i) $y_1>0, y_2>0$ $\quad 1-y_1>0$. $\quad 0<y_1<1 \quad y_2>0$

ii) $y_1<0, y_2<0$ $\quad 1-y_1<0 \not\Leftrightarrow y_1>1$ and $y_1<0$

Thm. (P.450 미적2+)

If $f: C^1 \Rightarrow f$: diff.

$$\mathcal{Y} := \{(y_1,y_2)^t : 0<y_1<1,\ y_2>0\}$$

Since $\exists u^{-1}$, u: bijection. & C^1

$\because$ $\dfrac{\partial y_1}{\partial x_1} = \dfrac{\partial}{\partial x_1}\left(1 - \dfrac{x_2}{x_1+x_2}\right) = \dfrac{-x_2}{(x_1+x_2)^2}$: continuous.

$\dfrac{\partial y_1}{\partial x_2} = \dfrac{-x_1}{(x_1+x_2)^2}$: continuous.

$\dfrac{\partial y_2}{\partial x_1} = 1 \quad \dfrac{\partial y_2}{\partial x_2} = 1 \Rightarrow$ continuous.

$$\mathrm{pdf}_{X_1,X_2}(x_1,x_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} x_1^{\alpha_1-1} x_2^{\alpha_2-1} e^{-(x_1+x_2)/\beta} I(x_1>0, x_2>0)$$

$$= \mathrm{pdf}_{X_1,X_2}(y_1y_2, y_2(1-y_1)) \left|\det \frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right|$$

$$\left(\det \frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \det\begin{pmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{pmatrix} = y_2(1-y_1)+y_1y_2 = y_2 > 0\right)$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} (y_1y_2)^{\alpha_1-1} (y_2(1-y_1))^{\alpha_2-1} e^{-y_2/\beta} I(0<y_1<1, y_2>0)\, y_2$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1} y_2^{\alpha_1+\alpha_2-1} e^{-y_2/\beta} I(0<y_1<1, y_2>0)$$

$$\mathrm{pdf}_{Y_1}(y_1) = \int_{\mathbb{R}} \mathrm{pdf}_{Y_1,Y_2}(y_1,y_2)\,dy_2.$$

$$= I(0<y_1<1)\; \frac{y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} \int_0^{+\infty} y_2^{\alpha_1+\alpha_2-1} e^{-y_2/\beta}\,dy_2$$

$$u := y_2/\beta.$$

$$\int_0^{\infty} \beta^{\alpha_1+\alpha_2-1} u^{\alpha_1+\alpha_2-1} e^{-u} \beta\,du. \qquad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx.$$

$$= \beta^{\alpha_1+\alpha_2}\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-u}\,du = \beta^{\alpha_1+\alpha_2}\,\Gamma(\alpha_1+\alpha_2)$$

$$\therefore \mathrm{pdf}_{Y_1}(y_1) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1} I(0<y_1<1)$$

$Y_1 \sim \mathrm{Beta}(\alpha_1,\alpha_2)$ $\square$

Def

$X \sim \mathrm{Beta}(\alpha_1, \alpha_2)$

$\iff X \overset{d}{=} \dfrac{Z_1}{Z_1 + Z_2}$ $\quad Z_1 \sim \mathrm{Gamma}(\alpha_1, \beta)$, $Z_2 \sim \mathrm{Gamma}(\alpha_2, \beta)$, $Z_1 \perp\!\!\!\perp Z_2$

$$\iff \mathrm{pdf}_X(x) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1} I(0<x<1)$$

e.g. 4.1.7.

Arithmetic mean of two indep. uniform dist.

$X_1, X_2 \sim I_{(0,1)}(x) \quad X_1 \perp\!\!\!\perp X_2$

$Y := \dfrac{X_1 + X_2}{2}$ $\quad \mathrm{pdf}_Y(y)$?

$Z := \dfrac{X_1 - X_2}{2}$ $\quad \mathrm{pdf}_{Y,Z}(y,z)$

marginalization

$$u = \begin{cases} y = (x_1+x_2)/2 \\ z = (x_1 - x_2)/2 \end{cases} \qquad u^{-1} = \begin{cases} x_1 = y+z \\ x_2 = y - z. \end{cases}$$

linear map.

$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \to \mathrm{rank}=2.$

$\therefore$ invertible.

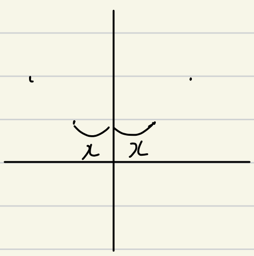
$$\det \frac{\partial(x_1, x_2)}{\partial(y, z)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0.$$

$* \ f(a-x) = f(x)$

$\Leftrightarrow x = \frac{a}{2}$ 에 대하여 대칭

$\Leftrightarrow f(\frac{a}{2}-x) = f(\frac{a}{2}+x)$

$x = \frac{a}{2}$



$$pdf_{X_1,X_2}(x_1,x_2) = pdf_{X_1}(x_1)\, pdf_{X_2}(x_1)$$

$$= pdf_{X_1}(y+z)\, pdf_{X_2}(y-z)\, |-2|$$

$$= 2\, I(0<y+z<1,\ 0<y-z<1)$$

$$\int_{-\infty}^{\infty} pdf_{Y,Z}(y,z)\, dz = \int_{-\infty}^{+\infty} 2\, I(0<y+z<1,\ 0<y-z<1)\, dz.$$

$-y < z < 1-y$

$y-1 < z < y$

$\max(-y, y-1) < z < \min(1-y, y)$

$-y = y-1$

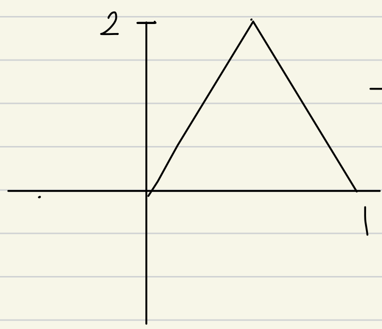
$y = \frac{1}{2}$

$\hookrightarrow 1-y = y \quad y = \frac{1}{2}$

i) $y \le \frac{1}{2}$ $\quad -y < z < y$ $\quad \int_{-y}^{y} 2\, I(0<y\le\frac{1}{2})\, dz = 4y\, I(0<y\le\frac{1}{2})$

ii) $y > \frac{1}{2}$ $\quad y-1 < z < 1-y$ $\quad \int_{y-1}^{1-y} 2\, I(\frac{1}{2}\le y<1)\, dz = 4(1-y)\, I(\frac{1}{2}<y<1)$

$$pdf_Y(y) = \left(2 - 4|y-\tfrac{1}{2}|\right) I(0<y<1) \quad \square$$



→ "삼각분포"

e.g. 4.1.8.

$X, Y \sim N(0,1) \quad X \perp\!\!\!\perp Y$

$X = R\cos\theta,\ Y = R\sin\theta \quad 0 \le R < \infty \quad 0 \le \theta < 2\pi.$

$$pdf_{R,\theta}(r,\theta) = pdf_{X,Y}(x,y) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right|$$

$$= pdf_X(r\cos\theta)\, pdf_Y(r\sin\theta)\, r\, (I(0<r<\infty, 0\le\theta<2\pi)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-r^2\cos^2\theta/2} \frac{1}{\sqrt{2\pi}} e^{-r^2\sin^2\theta/2}\, r\, I(0\le r<\infty)\, I(0\le\theta<2\pi)$$

$f(x) = xe^{-x^2/2} I(x>0)$

$$= \underbrace{re^{-r^2/2} I(r>0)}_{??} \frac{1}{2\pi} I(0\le\theta<2\pi)$$

$\boxed{\theta \sim U[0, 2\pi)}$

$$F_R(x) = \int_0^x re^{-r^2/2} dr = \int_0^{\frac{x^2}{2}} e^{-u} du$$

$u := \frac{r^2}{2}$

$$= \left[-e^{-u}\right]_0^{\frac{x^2}{2}}$$

$$= 1 - e^{-\frac{x^2}{2}}$$

$Z \sim Exp(1).$

$$F_Z(z) = \int_0^z e^{-u} du = 1 - e^{-z}$$

$\boxed{\frac{R^2}{2} \sim Exp(1)}$

Dirichelet Distribution.

$X_i \sim \text{Gamma}(\alpha_i, \beta)$. $\quad \underset{i}{\perp\!\!\!\perp} X_i \quad i=1,\dots,k+1$

$Y_1 = \dfrac{X_1}{X_1+\cdots+X_{k+1}}, \dots, Y_k := \dfrac{X_k}{X_1+\cdots+X_{k+1}}, \quad Y_{k+1} = X_1+\cdots+X_{k+1}.$

$$u: \begin{cases} x_1/(x_1+\cdots+x_{k+1}) = y_1 \\ \vdots \qquad\qquad \vdots \\ x_k/(x_1+\cdots+x_{k+1}) = y_k \\ x_1+\cdots+x_{k+1} = y_{k+1} \end{cases} \qquad u^{-1}: \begin{cases} x_1 = y_1 y_{k+1}. \\ x_2 = y_2 y_{k+1} \\ \vdots \\ x_k = y_k y_{k+1}. \\ x_{k+1} = y_{k+1}(1-y_1-\cdots-y_k) \end{cases}$$

$$\det\begin{pmatrix} y_{k+1} & 0 & \cdots & 0 & y_1 \\ 0 & y_{k+1} & \cdots & 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & y_{k+1} & y_k \\ -y_{k+1} & -y_{k+1} & & -y_{k+1} & 1-y_1-\cdots-y_k \end{pmatrix} \qquad y_1>0,\dots,y_{k+1}>0,\ y_1+\cdots+y_k<1.$$

$$= \det\begin{pmatrix} y_{k+1} & 0 & \cdots & 0 & y_1 \\ 0 & y_{k+1} & \cdots & 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & y_{k+1} & y_k \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = (y_{k+1})^k$$

$$\text{pdf}_{X_i}(x) = \frac{1}{\Gamma(\alpha_i)\beta^{\alpha_i}} x^{\alpha_i-1} e^{-x/\beta} I(x>0)$$

$$\text{pdf}_Y(y_1,\dots,y_k) = \int_{\mathbb{R}} \text{pdf}_Y(y_1,\dots,y_{k+1})\, dy_{k+1}.$$

$$\text{pdf}_Y(y_1,\dots,y_{k+1}) = \left\{\prod_{i=1}^{k+1} \frac{1}{\Gamma(\alpha_i)\beta^{\alpha_i}}\right\} \times \left\{\prod_{i=1}^{k} (y_i y_{k+1})^{\alpha_i-1}\right\} (y_{k+1}(1-y_1-\cdots-y_k))^{\alpha_{k+1}-1} e^{-y_{k+1}/\beta}\, y_{k+1}^k$$

$$= \left(\prod_{i=1}^{k} \frac{1}{\Gamma(\alpha_i)} y_i^{\alpha_i-1}\right) \times \frac{(1-y_1-\cdots-y_k)^{\alpha_{k+1}-1}}{\Gamma(\alpha_{k+1})\beta^{\alpha_1+\cdots+\alpha_{k+1}}}\, y_{k+1}^{\alpha_1+\cdots+\alpha_{k+1}-k-1} e^{-y_{k+1}/\beta} (y_{k+1})^k \; I(y_1>0,\dots,y_{k+1}>0,\ y_1+\cdots+y_k<1)$$

$$\int_{\mathbb{R}} y_{k+1}^{\alpha_1+\cdots+\alpha_{k+1}-1} e^{-y_{k+1}/\beta}\, dy_{k+1}\, I(y_{k+1}>0) \qquad u := y_{k+1}/\beta$$

$$= \int_0^\infty \beta^{\alpha_1+\cdots+\alpha_{k+1}-1} u^{\alpha_1+\cdots+\alpha_{k+1}-1} e^{-u} \beta\, du = \beta^{\alpha_1+\cdots+\alpha_{k+1}} \Gamma(\alpha_1+\cdots+\alpha_{k+1})$$

$$\therefore \text{pdf}_Y(y_1,\dots,y_k) = \frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{k+1})} y_1^{\alpha_1-1}\cdots y_k^{\alpha_k-1}(1-y_1-\cdots-y_k)^{\alpha_{k+1}-1} I(y_1>0,\dots,y_k>0,\ y_1+\cdots+y_k<1)$$

$Y \sim \text{Dirichelet}(\alpha_1,\dots,\alpha_{k+1}) \quad \alpha_i > 0.$

$$\mathrm{pdf}_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k) = \frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{k+1})} \prod_{i=1}^{k} y_i^{\alpha_i-1} (1-y_1-\cdots-y_k)^{\alpha_{k+1}-1}$$
$$I(y_1>0,\ldots,y_k>0,\ y_1+\cdots+y_k<1)$$

<u>Def</u> $Y=(Y_1,\ldots,Y_k)^t \sim \mathrm{Dirichelet}(\alpha_1,\ldots,\alpha_k,\alpha_{k+1})$

$$\Leftrightarrow Y \overset{d}{=} \left(\frac{X_1}{X_1+\cdots+X_{k+1}}, \ldots, \frac{X_k}{X_1+\cdots+X_{k+1}}\right)$$

$$X_i \sim \mathrm{Gamma}(\alpha_i, \theta) \qquad \underset{i}{\perp\!\!\!\perp} X_i$$

$$\Leftrightarrow \mathrm{pdf}_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k)$$
$$= \frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\prod_{i=1}^{k}\Gamma(\alpha_i)} y_1^{\alpha_1-1}\cdots y_k^{\alpha_k-1}(1-y_1-\cdots-y_k)^{\alpha_{k+1}-1}$$
$$I(y_1>0,\ldots,y_k>0,\ y_1+\cdots+y_k<1)$$

likelihood $\rightarrow$ $p(D|\mu) = \binom{N}{m_1 \cdots m_k} \mu_1^{m_1} \cdots \mu_k^{m_k}$.

$X \sim \mathrm{Multi}(N, (\mu_1, \dots, \mu_k)^t)$ $\quad \mu_k = (1 - \mu_1 - \cdots - \mu_{k-1})$

$\mu := (\mu_1, \dots, \mu_k) \sim \mathrm{Dirichelet}(\alpha_1, \dots, \alpha_k)$

Conjugate prior $\rightarrow$ of multinomial distribution

$$p(\mu|\alpha) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)} \prod_{i=1}^{k} \mu_i^{\alpha_i - 1} \, I(\mu_1 > 0, \dots, \mu_k > 0, \mu_1 + \cdots + \mu_k < 1)$$

posterior $\rightarrow$

$$p(\mu|D,\alpha) \propto p(D|\mu)\, p(\mu|\alpha)$$

$$\propto \prod_{i=1}^{k} \mu_i^{\alpha_i + m_i - 1}$$

$$\int_0^{\infty} \cdots \int_0^{\infty} \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)} \prod_{i=1}^{k} \mu_i^{\alpha_i - 1} = 1$$

$$\int \cdots \int \prod_{i=1}^{k} \mu_i^{\alpha_i - 1} = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)}{\Gamma(\alpha_1 + \cdots + \alpha_k)}$$

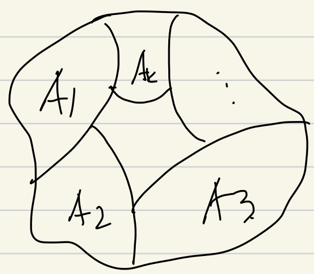
$$\therefore p(\mu|D,\alpha) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k + m_1 + \cdots + m_k)}{\Gamma(\alpha_1 + m_1) \cdots \Gamma(\alpha_k + m_k)} \prod_{i=1}^{k} \mu_i^{\alpha_i + m_i - 1} \, I(\mu_1 > 0, \dots, \mu_k > 0, \mu_1 + \cdots + \mu_k < 1).$$

Review

$\mathcal{P} = \{A_1, \dots, A_k\}$ $A_i \subset A$ partition (분할)

if $A = \biguplus \mathcal{P} = \biguplus_{i=1}^{k} A_i$

$\hookrightarrow$ disjoint union. $\left(A = \bigcup_{i=1}^{k} A_i,\ A_i \cap A_j = \emptyset\right)$
$\qquad \underbrace{i \neq j}$
mutually distinct.



Rmk. Define a relation $\sim$ on A by

$$x \sim y \Leftrightarrow x, y \in A_i \text{ for some } i = 1, \dots, k.$$

Then $\sim$ is equivalence relation on A.

Let $x \in A$ be given. Then $x \in A_i$ for some i.

$$[x] = \{y \in A : x \sim y\} = \{y \in A \mid y \in A_i\} = A_i$$

Thm 4.12

$X = (X_1, \dots, X_k)^t$: Continuous r.V.

$u : \mathcal{X} \to \mathcal{Y}$

(a) $P(X \in \mathcal{X}) = 1$

(b) $u = (u_1, \dots, u_k)^t : \mathcal{X} \to \mathcal{Y}$ $\left.\begin{matrix} m\text{-to-one} & \forall y \in \mathcal{Y} \\ \underline{\text{onto.}} & \end{matrix}\right.$

(c) $\underbrace{\mathcal{X}} : \biguplus_{i=1}^{m} \underbrace{\mathcal{X}_i}$ $\mathcal{X}_i$: open set $\forall y \in \mathcal{Y}, \ |u^{-1}(y)| = m$

$u^i := u|_{\mathcal{X}_i} : \mathcal{X}_i \to \mathcal{Y}$ bijection & $|J_{u^i}| \neq 0$. $\Rightarrow$ m-to-one.

$$\mathrm{pdf}_Y(y) = \sum_{x : u(x) = y} \mathrm{pdf}_X(x) \left| \det \frac{\partial u}{\partial x} \right|^{-1}$$

$$= \sum_{r=1}^{m} \mathrm{pdf}_X((u^r)^{-1}(y)) \left| J_{(u^r)^{-1}}(y) \right|$$

proof) $P(Y \in B) = P(u(X) \in B)$

$= P(\biguplus_{r=1}^{m} u^r(X) \in B, X \in \mathcal{X}_r)$ countable additivity

$= \sum_{r=1}^{m} P(u^r(X) \in B, X \in \mathcal{X}_r)$

$= \sum_{r=1}^{m} \int_{x \in (u^r)^{-1}(B)} \mathrm{pdf}_X(x)\, dx$ linearity of integral

$= \int_{\mathcal{Y}} \sum_{r=1}^{m} \mathrm{pdf}_X((u^r)^{-1}(y)) \left| J_{(u^r)^{-1}}(y) \right| dy$

$\therefore \ \mathrm{pdf}_Y(y) = \sum_{r=1}^{m} \mathrm{pdf}_X((u^r)^{-1}(y)) \left| J_{(u^r)^{-1}}(y) \right|$ $\square$

e.g. $X \sim U(-1,1)$ $\quad pdf_X(x) = \frac{1}{2} I(-1<x<1)$.

$Y = X^2$ ... 2-to-1 function.

sol) $x = (-1,1) = (-1,0) \uplus (0,1) \rightarrow y : (0,1)$

$X=0$인 지점은 measure-zero 이므로 무시하자.

$$pdf_Y(y) = \sum_{x:\, y=x^2} pdf_X(x) \left| \frac{1}{2x} \right| I_{(0,1)}(y)$$

$$= \sum_{x:\, y=x^2} \frac{1}{2} \left| \frac{1}{2x} \right| I_{(0,1)}(y)$$

$$= \left(\frac{1}{2} \left| \frac{1}{2\sqrt{y}} \right| + \frac{1}{2} \left| \frac{1}{-2\sqrt{y}} \right| \right) I_{(0,1)}(y)$$

$$= \frac{1}{2\sqrt{y}} I_{(0,1)}(y) \qquad \square$$

e.g. 4.1.11.

$X = (X_1, X_2)^t$

$\mathrm{pdf}_{X_1,X_2}(x_1,x_2) = \frac{1}{\pi} I(0 < x_1^2 + x_2^2 < 1)$

$Y_1 := X_1^2 + X_2^2, \quad Y_2 := \frac{X_1^2}{X_1^2 + X_2^2}$

$u: \begin{cases} y_1 = x_1^2 + x_2^2 \\ y_2 = x_1^2/(x_1^2+x_2^2) \end{cases}$ $\quad u^{-1}: \begin{cases} x_1 = \pm\sqrt{y_1 y_2} \\ x_2 = \pm\sqrt{x_1^2 + x_2^2 - x_1^2} = \pm\sqrt{y_1 - y_1 y_2} \end{cases}$

$= \pm\sqrt{y_1(1-y_2)}$

$\mathcal{X} = \{(x_1,x_2)^t : 0 < x_1^2 + x_2^2 < 1, \; x_1 \neq 0, \; x_2 \neq 0\}$

$\mathcal{Y} = \{(y_1,y_2)^t : 0 < y_1 < 1, \; 0 < y_2 < 1\}$

$0 < \frac{x_1^2}{x_1^2+x_2^2} < \frac{x_1^2+x_2^2}{x_1^2+x_2^2} = 1$

$0 < y_1 y_2 + y_1(1-y_2) < 1$

$0 < y_1 < 1$

$\mathrm{pdf}_Y(y_1,y_2) = \sum_{x : u(x) = y} \frac{1}{\pi} \frac{1}{4\sqrt{y_2(1-y_2)}} I(0<y_1<1, \; 0<y_2<1)$

$$\left| \det \begin{pmatrix} 2x_1 & 2x_2 \\ \frac{2x_1 x_2^2}{(x_1^2+x_2^2)^2} & \frac{-2x_1^2 x_2}{(x_1^2+x_2^2)^2} \end{pmatrix}^{-1} \right| = \left| \frac{-4x_1^3 x_2 - 4x_1 x_2^3}{(x_1^2+x_2^2)^2} \right|$$

$$= \left| \frac{4x_1 x_2 (x_1^2+x_2^2)}{(x_1^2+x_2^2)^2} \right|$$

$$= \left| \frac{4x_1 x_2}{x_1^2+x_2^2} \right|^{-1} = \left(4 \frac{\sqrt{y_1^2 y_2(1-y_2)}}{y_1}\right)^{-1}$$

$$= \left(4\sqrt{y_2(1-y_2)}\right)^{-1}$$

$$\mathrm{pdf}_Y(y_1, y_2) = \sum_{x: u(x)=y} \frac{1}{\pi} \frac{1}{4\sqrt{y_2(1-y_2)}} I(0<y_1<1,\ 0<y_2<1)$$

$$= \sum_{x: u(x)=y} \frac{1}{\pi} \frac{1}{4\sqrt{y_2(1-y_2)}} I(0<y_1<1)\, I(0<y_2<1)$$

$$= \frac{1}{\pi} y_2^{-\frac{1}{2}} (1-y_2)^{-\frac{1}{2}} I(0<y_2<1)\, I(0<y_1<1)$$

$Y_1 \sim U(0,1)$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0<x<1)$$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx.$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}} dx \end{aligned}}$$

$$= \int_0^\infty 2 e^{-u^2} du.$$

$\because$ Gaussian integral.

$$= 2\int_0^\infty e^{-u^2} du. = \int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}.$$

$$\frac{1}{\pi} y_2^{-\frac{1}{2}} (1-y_2)^{-\frac{1}{2}} I(0<y_2<1)$$

$$= \frac{\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} y_2^{\frac{1}{2}-1}(1-y_2)^{\frac{1}{2}-1} I(0<y_2<1)$$

$Y_2 \sim \mathrm{Gamma}(\tfrac{1}{2}, \tfrac{1}{2})$. $\square$

$\because$ $I := \int_{-\infty}^{+\infty} e^{-x^2} dx$

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right)$$

$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\,dy$$

$$= \int_0^{2\pi}\int_0^\infty e^{-r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^\infty \frac{1}{2} e^{-u} du \qquad \begin{aligned} r^2 &= u \\ 2r\,dr &= du \end{aligned}$$

$$= 2\pi \cdot \left[\tfrac{1}{2} e^{-u}\right]_0^\infty = \pi$$

$$\therefore \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \qquad \square$$

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4.2. 대표적인 표본분포

- 카이제곱
- t분포
- F분포

Sol)

$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x>0)$

$= \frac{1}{\sqrt{\pi}\, 2^{\frac{1}{2}}}$

$X_1 \sim Gamma(\alpha_1, \beta)$

$X_2 \sim Gamma(\alpha_2, \beta)$

$X_1 \perp\!\!\!\perp X_2$.

$\Rightarrow X_1 + X_2 \sim Gamma(\alpha_1+\alpha_2, \beta)$

예 4.2.1 $X_1, \ldots, X_r \overset{iid}{\sim} N(0,1)$, $Y := X_1^2 + \cdots + X_r^2$

$\Rightarrow pdf_Y(y)$?

Lemma $X \sim N(0,1) \Rightarrow X^2 \sim Gamma(\frac{1}{2}, 2)$.

$Y = X^2$.

$u: y = x^2$. $\quad u^{-1}: \pm\sqrt{y}$

$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$\Rightarrow$ even function

$\mathcal{X} := (-\infty, 0) \uplus (0, \infty)$ $\quad x=0$ 인 점 제외.

$$pdf_Y(y) = \sum_{x: y=x^2} pdf_X(x = u^{-1}(y)) \left| \det \frac{\partial x}{\partial y} \right| I(y>0)$$

$$= \left(pdf_X(x=\sqrt{y}) \left| \frac{1}{2\sqrt{y}} \right| + pdf_X(x=-\sqrt{y}) \left| \frac{1}{2\sqrt{y}} \right| \right) I(y>0)$$

$\phi(z)$: even function

$$= \cancel{2} \times \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{\cancel{2}\sqrt{y}} I(y>0)$$

$$= \frac{1}{\Gamma(\frac{1}{2})2^{\frac{1}{2}}} y^{\frac{1}{2}-1} e^{-y/2} I(y>0) \quad \therefore Y \sim Gamma(\tfrac{1}{2}, 2) \quad \square$$

By the lemma, $Y = X_1^2 + \cdots + X_r^2 \sim Gamma(\frac{r}{2}, 2)$ $\square$

Def

$Y \sim \chi^2(r)$ $(r>0)$ → degree of freedom

$(r \in \mathbb{N})$ $\Leftrightarrow Y \overset{d}{=} X_1^2 + \cdots + X_r^2, \quad X_i \overset{iid}{\sim} N(0,1^2)$

$\Leftrightarrow Y \sim \text{Gamma}(\frac{r}{2}, 2) \quad (\chi^2(r) \overset{d}{=} \text{Gamma}(\frac{r}{2},2))$

$\Leftrightarrow \text{pdf}_Y(y) = \frac{1}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} y^{\frac{r}{2}-1} e^{-\frac{y}{2}} I(y>0)$

$P(Y > \chi^2_\alpha(r)) = \alpha \quad (0<\alpha<1)$

upper alpha quantile

Rmk

$\boxed{X} \sim \text{Gamma}(\nu, \beta) \Leftrightarrow \boxed{X/\beta} \sim \text{Gamma}(\nu, 1) \Leftrightarrow \boxed{2X/\beta} \sim \text{Gamm}(\nu, 2) \overset{d}{=} \chi^2(2\nu)$

$Y := X/\beta$

$\text{pdf}_Y(y) = \text{pdf}_X(x = u^{-1}(y)) \left| \det \frac{\partial x}{\partial y} \right| I(x>0)$

$= \frac{1}{\Gamma(\nu)\beta^\nu} y^{\nu-1} \beta^{\nu-1} e^{-y} \beta I(y>0)$

$= \frac{1}{\Gamma(\nu) \cdot 1} y^{\nu-1} e^{-y/1} I(y>0)$ $\quad \boxed{\therefore Y \sim \text{Gama}(\nu, 1)}$

$Z_1 := 2Y$

$\text{pdf}_{Z_1}(z) = \frac{1}{\Gamma(\nu)} \left(\frac{z}{2}\right)^{\nu-1} e^{-z/2} \frac{1}{2} I(z>0)$

$= \frac{1}{\Gamma(\nu) 2^\nu} z^{\nu-1} e^{-z/2} I(z>0)$ $\quad \boxed{\therefore Z_1 \sim \text{Gamma}(\nu, 2) \overset{d}{=} \chi^2(2\nu)}$

$P(2X/\beta > \chi^2_\alpha(2\nu)) = \alpha.$

$\Rightarrow P(X > \beta \chi^2_\alpha(2\nu)/2) = \alpha$

Thm 4.1.2

(a) $X \sim \chi^2(r) \Rightarrow E[X] = r \quad Var(X) = 2r$

(b) $Y \sim \chi^2(r) \Rightarrow mgf_Y(t) = (1-2t)^{-\frac{r}{2}} \quad \left(t < \frac{1}{2}\right)$

(c) $Y_1 \perp\!\!\!\perp Y_2$ & $Y_1 \sim \chi^2(r_1),\ Y_2 \sim \chi^2(r_2) \Rightarrow Y_1 + Y_2 \sim \chi^2(r_1+r_2)$

proof)

$\chi^2(r) \overset{d}{\equiv} Gamma\left(\frac{r}{2}, 2\right)$

(a) $E[X] = \frac{r}{2} \cdot 2 = r \quad Var(X) = \frac{r}{2} \cdot 2^2 = 2r.$

(b) $X \sim Gamma(\alpha, \beta)$

$$mgf_X(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx\, I(x>0)$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-(1/\beta - t)x}\, dx$$

$z := (1/\beta - t)x$

$dz = (1/\beta - t)dx$

$(1/\beta - t > 0)$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty (1/\beta - t)^{-\alpha+1} z^{\alpha-1} e^{-z} \cancel{(1/\beta-t)^{-1}}\, dz$$

$$= \frac{(1/\beta - t)^{-\alpha}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty z^{\alpha-1} e^{-z}\, dz$$

$$= \frac{(1/\beta - t)^{-\alpha}}{\Gamma(\alpha)\beta^\alpha}\, \Gamma(\alpha)$$

$$= \left(\beta(1/\beta - t)\right)^{-\alpha}.$$

$$= (1 - \beta t)^{-\alpha} \quad t < 1/\beta.$$

$\therefore mgf_Y(t) = (1-2t)^{-\frac{r}{2}} \quad \left(t < \frac{1}{2}\right)$

(c) $mgf_{Y_1+Y_2}(t) = mgf_{Y_1}(t) \cdot mgf_{Y_2}(t) = (1-2t)^{-\frac{r_1}{2}} \cdot (1-2t)^{-\frac{r_2}{2}}$

$= (1-2t)^{-\frac{r_1+r_2}{2}} \quad \therefore Y_1 + Y_2 \sim \chi^2(r_1+r_2)$

$\square$

e.g.

$Z \sim N(0,1) \quad V \sim \chi^2(r) \quad Z \perp\!\!\!\perp V$

$X := Z/\sqrt{V/r} \quad \mathrm{pdf}_X(x)\,?$

$Y := V$

$$u: \begin{cases} z = x\sqrt{y/r} \\ v = y \end{cases} \quad \det\frac{\partial(z,v)}{\partial(x,y)} = \det\begin{pmatrix} \sqrt{y/r} & \frac{x}{2\sqrt{y/r}\, r} \\ 0 & 1 \end{pmatrix} = \sqrt{y/r} \neq 0$$

$$\mathrm{pdf}_{X,Y}(x,y) = \mathrm{pdf}_{Z,V}(z,v)\left|\det\frac{\partial(z,v)}{\partial(x,y)}\right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} v^{\frac{r}{2}-1} e^{-v/2} \frac{\sqrt{y}}{\sqrt{r}} I(y>0)$$

$$= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})2^{\frac{r+1}{2}}\sqrt{r}} y^{\frac{r+1}{2}-1} e^{-\frac{y}{2}(1+\frac{x^2}{r})} I(y>0)$$

$$\mathrm{pdf}_X(x) = \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})2^{\frac{r+1}{2}}\sqrt{r}} \int_0^\infty y^{\frac{r+1}{2}-1} e^{-\frac{y}{2}(1+\frac{x^2}{r})}\, dy$$

$t := \frac{y}{2}(1+\frac{x^2}{r})$

$dt = \frac{1}{2}(1+\frac{x^2}{r})dy$

$$= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})\cancel{2^{\frac{r+1}{2}}}\sqrt{r}} \int_0^\infty \cancel{2^{\frac{r+1}{2}}}\cancel{{}^{-1}}\, t^{\frac{r+1}{2}-1} (1+\frac{x^2}{r})^{\cancel{1-}\frac{r+1}{2}} e^{-t}\, \cancel{2(1+\frac{x^2}{r})^{-1}}\, dt$$

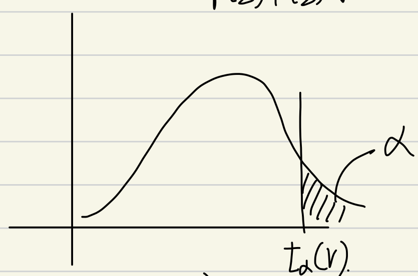
$$= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})\sqrt{r}} (1+\frac{x^2}{r})^{-\frac{r+1}{2}} \Gamma(\frac{r+1}{2})$$

$$\mathrm{pdf}_X(x) = \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})\sqrt{r}} (1+\frac{x^2}{r})^{-\frac{r+1}{2}}$$

Def $X \sim t(r) \quad (r>0)$

$\Leftrightarrow X \overset{d}{=} Z/\sqrt{V/r}$, $Z \sim N(0,1)$, $V \sim \chi^2(r)$ $\quad Z \perp\!\!\!\perp V$

$\Leftrightarrow \mathrm{pdf}_X(x) = \dfrac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\sqrt{r}} \left(1+\dfrac{x^2}{r}\right)^{-\frac{r+1}{2}}$



$P(X > \underbrace{t_\alpha(r)}) = \alpha$

$\hookrightarrow$ upper alpha quantile.

Thm 4.2.2. $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

(a) $\bar{X} := \frac{1}{n}(X_1 + \cdots + X_n) \sim N(\mu, \sigma^2/n)$

(b) $S^2 := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$, $\quad S^2 \perp\!\!\!\perp \bar{X}$

proof) $\mathrm{mgf}_{\bar{X}}(s) = E[e^{s\bar{X}}] = E[\exp(\frac{s}{n}(X_1 + \cdots + X_n))]$

$$= \mathrm{mgf}_{X_1 + \cdots + X_n}\left(\frac{s}{n}\right)$$

$$= \prod_{i=1}^{n} \mathrm{mgf}_{X_i}\left(\frac{s}{n}\right)$$

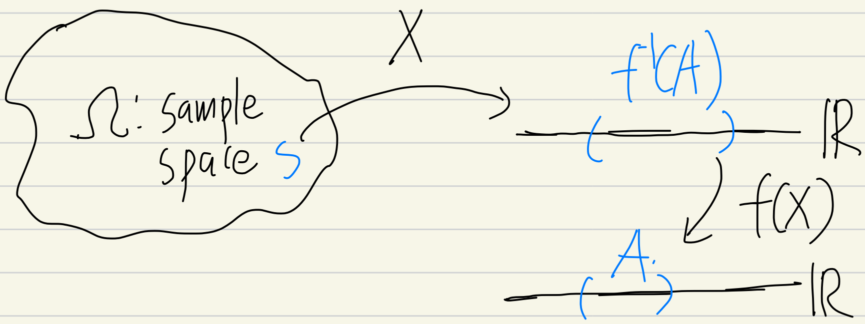
$$= \exp\left(\sum_{i=1}^{n} \mu\left(\frac{s}{n}\right) + \frac{\sigma^2}{2}\left(\frac{s^2}{n^2}\right)\right)$$

$$= \exp\left(\mu s + \frac{\sigma^2/n}{2} s^2\right)$$

$\bar{X} \sim N(\mu, \sigma^2/n)$

Lemma. $X \perp\!\!\!\perp Y \Rightarrow f(X) \perp\!\!\!\perp g(Y)$

proof) $\big(\underbrace{f(X)\in A}_{\{s\in\Omega\,:\, f(X)(s)\in A\}},\ g(Y)\in B\big) = \big(\underbrace{X\in f^{-1}(A)}_{\{s\in\Omega\,:\, X(s)\in f^{-1}(A)\}},\ Y\in g^{-1}(B)\big)$



$$P(f(X)\in A, g(Y)\in B) = P(X\in f^{-1}(A), Y\in g^{-1}(B))$$
$$= P(X\in f^{-1}(A))\,P(Y\in g^{-1}(B))$$
$$= P(f(X)\in A)\,P(g(Y)\in B)$$

$\therefore\ f(X) \perp\!\!\!\perp g(Y)$ $\square$.

proof (b) $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2 = f(X_1-\overline{X}, \dots, X_n-\overline{X})$ for some f.

$Y := (X_1-\overline{X}, \dots, X_n-\overline{X})^t$

By the previous lemma,

$\boxed{\text{ETS}}$ $\overline{X} \perp\!\!\!\perp Y$ $\left(\Rightarrow \overline{X} \perp\!\!\!\perp S^2 = f(Y)\right)$

$$mgf_{\overline{X},Y}(s,(t_1,\dots,t_n)) = E[\exp(s\overline{X} + t_1(X_1-\overline{X}) + \dots + t_n(X_n-\overline{X}))]$$

$$= E[\exp(\frac{s}{n}\sum_{i=1}^{n}X_i + \sum_{i=1}^{n}t_iX_i - \sum_{j=1}^{n}t_j\frac{(X_1+\dots+X_n)}{n})]$$

$\overline{t} := \frac{1}{n}\sum_{i=1}^{n}t_i$

$$= E[\exp(\sum_{i=1}^{n}(\frac{s}{n}+(t_i-\overline{t}))X_i)]$$

$\because X_i \overset{iid}{\sim} N(0,1)$

$$= \prod_{i=1}^{n} mgf_{X_i}(\frac{s}{n}+(t_i-\overline{t}))$$

$$= \exp(\sum_{i=1}^{n}(\mu(\frac{s}{n}+(t_i-\overline{t})) + \frac{\sigma^2}{2}(\frac{s}{n}+(t_i-\overline{t}))^2))$$

$$= \exp(\mu s + \frac{\sigma^2/n}{2}s^2)\exp(\sum_{i=1}^{n}\frac{\sigma^2}{2}(t_i-\overline{t})^2)$$

$$= mgf_{\overline{X}}(s)\, mgf_Y(t_1,\dots,t_n)$$

$\because$ $mgf_Y(t_1,\dots,t_n) = E[\exp(\sum_{i=1}^{n}(X_i-\overline{X})t_i)]$

$$= \prod_{i=1}^{n} mgf_{X_i}(t_i-\overline{t})$$

$$= \exp(\sum_{i=1}^{n}(\mu(t_i-\overline{t}) + \frac{\sigma^2}{2}(t_i-\overline{t})^2))$$

$$= \exp(\sum_{i=1}^{n}\frac{\sigma^2}{2}(t_i-\overline{t})^2)$$

$\square$

20.05.02.

Thm 4.2.2. $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

(a) $\bar{X} \sim N(\mu, \sigma^2/n)$

(b) $S^2, \bar{X}$: indep.

(c) $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$

where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

(c) $\sum_{i=1}^{n} \underline{(X_i - \bar{X})}^2/\sigma^2 = (n-1)S^2/\sigma^2$

$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(X_i - \bar{X} + \bar{X} - \mu)^2$ $\quad \because \sum_{i=1}^{n}(X_i - \bar{X}) = 0$

$= \sum_{i=1}^{n}(X_i - \bar{X})^2 + n(\bar{X}-\mu)^2$

$\sum_{i=1}^{n}(X_i-\mu)^2/\sigma^2 = \sum_{i=1}^{n}(X_i-\bar{X})^2/\sigma^2 + \frac{n}{\sigma^2}(\bar{X}-\mu)^2$

$\frac{X_i-\mu}{\sigma} \sim N(0,1) \Rightarrow \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(n)$

$\frac{n}{\sigma^2}(\bar{X}-\mu)^2 = \left(\frac{\bar{X}-\mu}{\sqrt{n}/\sigma}\right)^2 \sim \chi(1)$

$U := \sum_{i=1}^{n}(X_i-\mu)^2/\sigma^2 \sim \chi^2(n)$ $\quad V := \sum_{i=1}^{n}(X_i-\bar{X})^2/\sigma^2$ $\quad W := \left(\frac{\bar{X}-\mu}{\sqrt{n}/\sigma}\right)^2 \sim \chi(1)$

Since $V = f(S^2)$ $W = g(\bar{X})$ & $S^2 \perp\!\!\!\perp W$, $V \perp\!\!\!\perp W$.

Thus $mgf_U(t) = mgf_V(t)\, mgf_W(t)$

$(1-2t)^{-\frac{n}{2}} I(t<\tfrac{1}{2}) = mgf_V(t)\,(1-2t)^{-\frac{1}{2}} I(t<\tfrac{1}{2})$

$mgf_V(t) = (1-2t)^{-\frac{1}{2}(n-1)} I(t<\tfrac{1}{2})$

$\therefore V \sim \chi^2(n-1)$ $\quad \square$

Thm 4.2.3.

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$: random sample.

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1) \qquad V := (n-1)S^2/\sigma^2 \sim \chi^2(n-1)$$

proof)

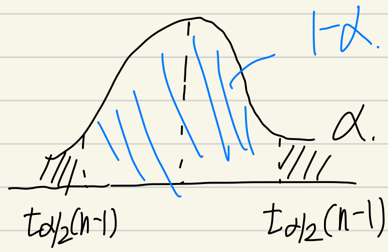
$$Z := \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \qquad \frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \times \frac{\sigma}{S} = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \frac{1}{\sqrt{V/(n-1)}}$$

$$\bar{X} \perp\!\!\!\perp S^2 \Rightarrow Z \perp\!\!\!\perp V \qquad = \frac{Z}{\sqrt{V/(n-1)}}$$

$Z = f(\bar{X})$

$V = g(S)$

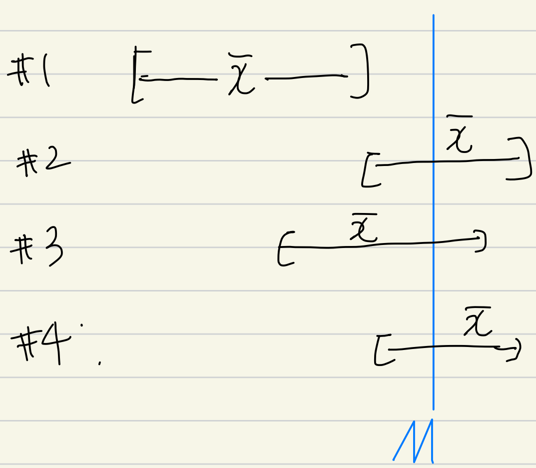
$$\therefore \frac{Z}{\sqrt{V/(n-1)}} \sim t(n-1) \ \square$$



$$P\left(\left| \frac{\bar{X}-\mu}{S/\sqrt{n}} \right| \le t_{\alpha/2}(n-1) \right) = 1-\alpha.$$

$$P\left(\bar{X} - \frac{S}{\sqrt{n}} t_{\alpha/2}(n-1) \le \mu \le \bar{X} + \frac{S}{\sqrt{n}} t_{\alpha/2}(n-1) \right) = 1-\alpha.$$

⇒ 여러번의 sampling을 통해 추정한 μ 평균의 구간에 실제 μ가 있을 확률.



↗ 모분산의 추정

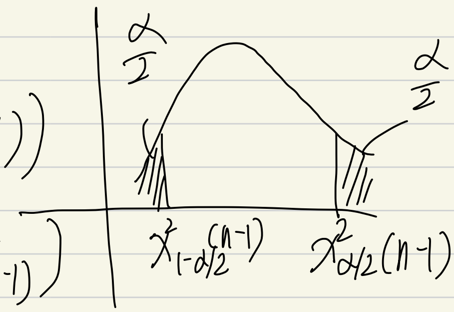
Thm 4.2.4 $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$P\left(\frac{n-1}{\chi^2_{\alpha/2}(n-1)} S^2 \le \sigma^2 \le \frac{n-1}{\chi^2_{1-\alpha/2}(n-1)} S^2\right) = 1-\alpha$$

proof) $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$

$$1-\alpha = P\left(\chi^2_{1-\alpha/2}(n-1) \le (n-1)\frac{S^2}{\sigma^2} \le \chi^2_{\alpha/2}(n-1)\right)$$

$$= P\left(\frac{n-1}{\chi^2_{\alpha/2}(n-1)} S^2 \le \sigma^2 \le \frac{n-1}{\chi^2_{1-\alpha/2}(n-1)}\right)$$



□

e.g.

$V_1 \sim \chi^2(r_1)$, $V_2 \sim \chi^2(r_2)$ $V_1 \perp\!\!\!\perp V_2$

$X := \frac{V_1/r_1}{V_2/r_2}$ $Y := V_2$ $pdf_X(x)$?

$$\left|\frac{\partial(v_1, v_2)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{r_1}{r_2}y & \frac{r_1}{r_2}x \\ 0 & 1 \end{vmatrix} = \frac{r_1}{r_2}y$$

$u: \begin{cases} v_1 = \frac{r_1}{r_2}xy \\ v_2 = y \end{cases}$

$$pdf_{X,Y}(x,y) = pdf_{V_1,V_2}(v_1,v_2)\left|\det\frac{\partial(v_1,v_2)}{\partial(x,y)}\right|$$

$$= \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{(r_1+r_2)/2}} v_1^{\frac{r_1}{2}-1} v_2^{\frac{r_2}{2}-1} e^{-(v_1+v_2)/2} \frac{r_1}{r_2}y \; I(x>0, y>0)$$

$$= \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{(r_1+r_2)/2}} \left(\frac{r_1}{r_2}xy\right)^{\frac{r_1}{2}-1} y^{\frac{r_2}{2}-1} e^{-\frac{y}{2}(\frac{r_1}{r_2}x+1)} \frac{r_1}{r_2}y$$

$$= \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{(r_1+r_2)/2}} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} x^{\frac{r_1}{2}-1} y^{\frac{r_1+r_2}{2}-1} e^{-\frac{y}{2}(\frac{r_1}{r_2}x+1)}$$

$$pdf_X(x) = \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\cancel{2^{(r_1+r_2)/2}}} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} x^{\frac{r_1}{2}-1} I(x>0) \int_0^{+\infty} y^{\frac{r_1+r_2}{2}-1} e^{-\frac{y}{2}(\frac{r_1}{r_2}x+1)} dy$$

$dt = \frac{1}{2}(\frac{r_1}{r_2}x+1)dy$

$t := \frac{1}{2}(\frac{r_1}{r_2}x+1)y$

$$\int_0^\infty t^{\frac{r_1+r_2}{2}-1} \cancel{2^{\frac{r_1+r_2}{2}-1}} \left(\frac{r_1}{r_2}x+1\right)^{-\frac{r_1+r_2}{2}+1} e^{-t}\, dt$$

$$\cancel{2\left(\frac{r_1}{r_2}x+1\right)^{-1}}$$

$$= \frac{\Gamma(\frac{r_1+r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} x^{\frac{r_1}{2}-1} \left(\frac{r_1}{r_2}x+1\right)^{-(r_1+r_2)/2} I(x>0)$$

Def	$X \sim F(r_1, r_2)$
$\Leftrightarrow$	$X \overset{d}{=} \dfrac{V_1/r_1}{V_2/r_2}$ $\quad V_1 \sim \chi^2(r_1) \quad V_2 \sim \chi^2(r_2) \quad V_1 \perp\!\!\!\perp V_2$
$\Leftrightarrow$	$pdf_X(x) = \dfrac{\Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)} \left(\dfrac{r_1}{r_2}\right)^{\frac{r_1}{2}} x^{\frac{r_1}{2}-1} \left(\dfrac{r_1}{r_2}x+1\right)^{-(r_1+r_2)/2} I(x>0)$
Thm 4.2.5	(a) $X \sim F(r_1, r_2) \Rightarrow 1/X \sim F(r_2, r_1)$
	$\&\ F_{1-\alpha}(r_1, r_2) = 1/F_\alpha(r_2, r_1)$
	(b) $X \sim t(r) \Rightarrow X^2 \sim F(1, r)$
	$\&\ t^2_{\alpha/2} = F_\alpha(1, r)$
proof) (a)	$1/X = \dfrac{V_2/r_2}{V_1/r_1} \sim F(r_2, r_1)$
	$1-\alpha = P(X \geq F_{1-\alpha}(r_1, r_2)) \Rightarrow \alpha = P(X \leq F_{1-\alpha}(r_1, r_2))$
	$\alpha = P(1/X \geq F_\alpha(r_2, r_1)) = P(X \leq 1/F_\alpha(r_2, r_1))$
	Since $P(X(\omega)) > 0 \ \forall \omega \in \Omega$, $cdf_X(x)$ is strictly increasing function
	$\Rightarrow$ cdf: one-to-one. Thus, $cdf_X(F_{1-\alpha}(r_2, r_1)) = cdf_X(1/F_\alpha(r_1, r_2))$
	$\therefore F_{1-\alpha}(r_1, r_2) = 1/F_\alpha(r_2, r_1)$

(b) $X \sim t(r)$ $\quad X = \dfrac{Z_1}{\sqrt{V/r}}$ $\quad Z_1 \sim N(0,1)$ $\quad V \sim \chi^2(r)$

$X^2 = \dfrac{Z_1^2/1}{V/r}$ $\quad Z_1^2 \sim \chi^2(1)$ $\qquad Z_1 \perp\!\!\!\perp V.$

$\therefore X^2 \sim F(1,r)$ $\quad \alpha = P(|X| \geq t_{\alpha/2}(r)) = P(X^2 \geq t^2_{\alpha/2}(r))$

$\qquad\qquad\qquad\qquad = P(X^2 \geq F_\alpha(1,r))$

두 모집단의 흩어진 정도를 비교할 수 있다.

$\therefore F_\alpha(1,r) = t^2_{\alpha/2}(r)$ $\quad \square$

Thm 4.2.6.

$$\left.\begin{array}{ll} X_{1i} \overset{iid}{\sim} N(\mu_1, \sigma_1^2) & i=1,\dots,n_i \\ X_{2j} \overset{iid}{\sim} N(\mu_2, \sigma_2^2) & j=1,\dots,n_j \end{array}\right\} \text{indep}$$

$$\Rightarrow \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(n_1-1, n_2-1)$$

proof)

$\bar{X}_i := \sum_{j=1}^{n_i} X_{ij}/n_i$ $\quad S_i^2 := \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/(n_i-1)$

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{((n_1-1)S_1^2/\sigma_1^2)/(n_1-1)}{((n_2-1)S_2^2/\sigma_2^2)/(n_2-1)} \sim F(n_1-1, n_2-1)$$

$(\because (n_1-1)S_1^2/\sigma_1^2 \sim \chi^2(n_1-1),\ (n_2-1)S_2^2/\sigma_2^2 \sim \chi^2(n_2-1)$ & indep.)

$$P\left(F_{1-\alpha/2}(n_1,n_2) \leq \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \leq F_{\alpha/2}(n_1-1, n_2-1) \right) = 1-\alpha$$

$$P\left(\frac{1}{F_{\alpha/2}(n_1-1, n_2-1)} \frac{S_1^2}{S_2^2} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{1}{F_{1-\alpha/2}(n_1-1,n_2-1)} \frac{S_1^2}{S_2^2} \right) = 1-\alpha$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad F_{\alpha/2}(n_2-1, n_1-1) \dfrac{S_1^2}{S_2^2}$

일원분류모형
(ANOVA)
분산분석

$$X_{ij} = \mu_i + e_{ij} \quad e_{ij} \overset{iid}{\sim} N(0, \sigma^2)$$

$$\bar{X}_i := \sum_{j=1}^{n_i} X_{ij} / n_i \qquad \bar{X} := \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} / n$$

$$n := \sum_{i=1}^{k} n_i \qquad \bar{\mu} := \sum_{i=1}^{k} \frac{n_i}{n} \mu_i$$

(a) $\sum_{i=1}^{k} n_i \left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2 / \sigma^2 \sim \chi^2(k-1)$

(b) $\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 / \sigma^2 \sim \chi^2(n-k)$

(c) $\sum_{i=1}^{k} n_i \left(X_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2 \perp\!\!\!\perp \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2$

$$\hat{\sigma}^2 := \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 / (n-k)$$

$$\frac{1}{k-1} \sum_{i=1}^{k} n_i \left(\bar{X}_i - \bar{X} - (\bar{\mu}_i - \bar{\mu})\right)^2 / \hat{\sigma}^2 \sim F(k-1, n-k)$$

<u>Note</u>

$X_{ij} \sim N(\mu_i, \sigma^2)$

$\because$ $mgf_{X_{ij}}(t) = E[e^{tX_{ij}}]$

$= E[\exp(t\mu_i + te_{ij})]$

$= \exp(t\mu_i) E[\exp(te_{ij})]$

$= \exp(t\mu_i)\, mgf_{e_{ij}}(t)$

$= \exp(t\mu_i) \exp(0t + \frac{\sigma^2}{2} t^2)$

$= \exp(\mu_i t + \frac{\sigma^2}{2} t^2)$

적률생성함수의 결정성에 의해

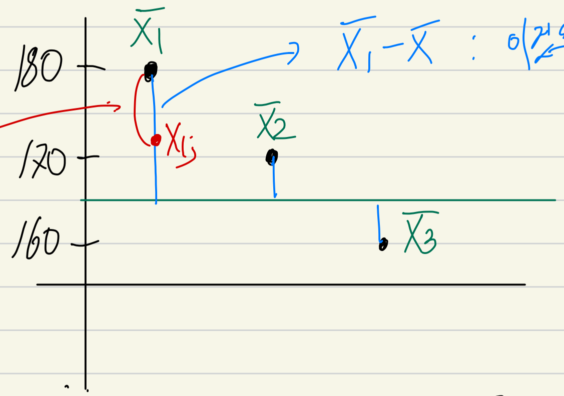
$X_{ij} \sim N(\mu_i, \sigma^2)$

□

ANOVA

분산분석.

서로 다른 정규모집단에서 모평균을 비교.



$\bar{X}_1 - \bar{X}$: 이것의 흩어진 정도 즉 $\left(\bar{X}_i - \bar{X} - E[\bar{X}_i - \bar{X}]\right)^2$ 평균이 집단간 분산이다.

$(\bar{X}_1 - X_{1j})^2$ 이것의 평균이 집단내 분산

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(X_{ij} - \bar{X} - (\mu_i - \bar{\mu})\right)^2 = \sum_{i=1}^{k} n_i(\bar{X}_i - \bar{X})^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2$$

∵ $$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(X_{ij} - \bar{X}_i + \bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2$$

$$= \underbrace{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2}_{\text{집단내 분산과 관련}} + \underbrace{\sum_{i=1}^{k} n_i\left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2}_{\text{집단간 분산과 관련.}}$$

여러 집단의 모평균이 같은지/다른지 비교하기 위해서는

$$\frac{(\text{집단간 분산})}{(\text{집단내 분산})} = \frac{\frac{1}{k-1}\sum_{i=1}^{k} n_i\left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2}{\frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2}$$ 을 보아야한다.

집단간 분산↑, 집단내 분산↓ 이면 모평균이 다르다고 할 수 있다.

즉 $$Y := \frac{\sum_{i=1}^{k} n_i\left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2 / (k-1)\sigma^2}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2 / (n-k)\sigma^2} \sim F(k-1, n-k)$$ $P(Y \geq F_\alpha(k-1, n-k))$를 보아야한다.

proof) (b) $\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij}-\overline{X}_i)^2/\sigma^2 \sim \chi^2(n-k)$

$S_i^2 := \frac{1}{n_i-1}\sum_{j=1}^{n_i}(X_{ij}-\overline{X}_i)^2$

$\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij}-\overline{X}_i)^2/\sigma^2 = \sum_{i=1}^{k}(n_i-1)S_i^2/\sigma^2.$

Since $\overline{X}_i \sim N(\mu_i, \sigma^2/n_i)$, $(n_i-1)S_i/\sigma^2 \sim \chi^2(n_i-1)$.

Since $\perp\!\!\!\perp_i \overline{X}_i$, $\perp\!\!\!\perp_i S_i^2 = f(\overline{X}_i)$

Thus, $\sum_{i=1}^{k}(n_i-1)S_i^2/\sigma^2 \sim \chi^2\left(\sum_{i=1}^{k}(n_i-1)\right) = \chi^2(n-k).$

(c) Since $\overline{X}_1, \ldots, \overline{X}_k$ and $S_1^2, \ldots, S_k^2$ independent.

$f(\overline{X}_1, \ldots, \overline{X}_k) = \sum_{i=1}^{k} n_i(\overline{X}_i - \overline{X} - (\mu_i - \overline{\mu}))^2$

$g(S_1^2, \ldots, S_k^2) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij}-\overline{X}_i)^2$

$\Rightarrow$ independent.

We only consider the special case where $n_1 = \cdots = n_k = \bar{n}$

(a) $\sum_{i=1}^{k} n_i \left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2 / \sigma^2$

proof)

$Z_i := \dfrac{\bar{X}_i - \mu_i}{\sigma/\sqrt{n_i}} \sim N(0,1)$ $\quad \bar{Z} := \dfrac{\bar{X} - \bar{\mu}}{\sigma/\sqrt{n_i}}$

$\boxed{\text{WTS}}$ $\sum_{i=1}^{k} (Z_i - \bar{Z})^2 / \sigma \sim \chi^2(k-1)$ by Thm 4.2.2

$\boxed{\text{ETS}}$ $\bar{Z} = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n_i}} = \sum_{i=1}^{k} \dfrac{n_i}{n} Z_i$ i.e., $\bar{Z}$ is a sample mean of $Z_1, \ldots, Z_k$.

$\because$ $\bar{X} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} / n$ where $n = \sum_{i=1}^{k} n_i$.

$= \sum_{i=1}^{k} \dfrac{n_i}{n} \sum_{j=1}^{n_i} X_{ij} / n_i = \sum_{i=1}^{k} \dfrac{n_i}{n} \bar{X}_i = \sum_{i=1}^{k} \dfrac{\bar{n}}{n} \bar{X}_i$ $\quad (\because n_i = \bar{n},\ i = 1, \ldots, k)$

$\bar{X} - \mu = \sum_{i=1}^{k} \dfrac{\bar{n}}{n} \bar{X}_i - \bar{\mu} = \sum_{i=1}^{k} \dfrac{\bar{n}}{n} (\bar{X}_i - \mu_i) =$

$\bar{Z}$ $= \dfrac{\bar{X} - \mu}{\sigma/\sqrt{\bar{n}}} = \dfrac{\sqrt{\bar{n}}}{\sigma} \sum_{i=1}^{k} \dfrac{\bar{n}}{n} (\bar{X}_i - \mu_i) = \sum_{i=1}^{k} \dfrac{\bar{n}}{n} \left(\dfrac{\bar{X}_i - \mu_i}{\sigma/\sqrt{\bar{n}}} \right)$

$= \sum_{i=1}^{k} \dfrac{\bar{n}}{n} Z_i$

By thm 4.2.2. ($X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ $\quad \bar{X} := (X_1 + \cdots + X_k)/k \Rightarrow \sum_{i=1}^{k} (X_i - \bar{X})^2 / \sigma \sim \chi^2(k-1)$)

$\sum_{i=1}^{k} (Z_i - \bar{Z})^2 / \sigma \sim \chi^2(k-1)$

$\square$

Finally, $\sum_{i=1}^{k} n_i \left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2 / \sigma^2 \sim \chi^2(k-1)$

$\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 / \sigma^2 \sim \chi^2(n-k)$

$$\Rightarrow \frac{\sum_{i=1}^{k} n_i \left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2 / \sigma^2 (k-1)}{\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 / \sigma^2 (n-k)} \sim F(k-1, n-k)$$

$$\hat{\sigma}^2 := \frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2$$

$$\frac{1}{k-1}\sum_{i=1}^{k} n_i\left(\bar{X}_i-\bar{X}-(\mu_i-\bar{\mu})\right)^2 \Big/ \hat{\sigma}^2$$

$$= \frac{\underbrace{\sum_{i=1}^{k} n_i\left(\bar{X}_i-\bar{X}-(\mu_i-\bar{\mu})\right)^2/\sigma^2}_{\to \chi^2(k-1)}(k-1)}{\underbrace{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2/\sigma^2}_{\to \chi^2(n-k)}(n-k)} = \frac{V_1/(k-1)}{V_2/(n-k)} \sim F(k-1, n-k)$$

$$\therefore \frac{\sum_{i=1}^{k} n_i\left(\bar{X}_i-\bar{X}-(\mu_i-\bar{\mu})\right)^2/\sigma^2(k-1)}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2/\sigma^2(n-k)} \sim F(k-1, n-k)$$

$\square$

Rmk. $\bar{X} \sim N(\bar{\mu}, \sigma^2/n)$

proof) $\bar{X} = \sum_{i=1}^{k}\frac{n_i}{n}\bar{X}_i$

$$mgf_{\bar{X}}(t) = E\left[\exp\left(\sum_{i=1}^{k}\frac{n_i}{n}t\bar{X}_i\right)\right] \quad \perp\!\!\!\perp \bar{X}_i$$

$$= \prod_{i=1}^{k} mgf_{\bar{X}_i}(t)$$

$$= \exp\left(\sum_{i=1}^{k}\mu_i\left(\frac{n_i}{n}t\right) + \sum_{i=1}^{k}\frac{\sigma^2/n_i}{2}\left(\frac{n_i}{n}t\right)^2\right)$$

$$= \exp\left(\bar{\mu}t + \sum_{i=1}^{k}\frac{\sigma^2}{2\cancel{n_i}}\frac{n_i^{\cancel{2}}}{n^2}t^2\right)$$

$$= \exp\left(\bar{\mu}t + \frac{\sigma^2/n}{2}t^2\right)$$

$\therefore \bar{X} \sim N(\bar{\mu}, \sigma^2/n)$ $\square$

Statistics 0509

§ 순서통계량의 분포 order statistic	$X_1, \dots, X_n$: random sample $X_{(1)} < \cdots < X_{(n)}$: order statistics 순서통계량. 등호가 되는 경우는 measure-zero로 생각하라.
예 4.3.1	$X_1, X_2, X_3 \overset{iid}{\sim} Exp(1)$ random sample $X_{(1)} < X_{(2)} < X_{(3)}$ $\quad Y := (X_{(1)}, X_{(2)}, X_{(3)})^t$ $pdf_Y(y)$?
sol)	$u(X_1, X_2, X_3) := (X_{(1)}, X_{(2)}, X_{(3)})^t$ $\quad$ 3! to
Symmetric group. $\mathfrak{S}_n = \{\sigma : \{1,..,n\} \to \{1,..,n\} \mid \sigma : bij\}$	$\mathcal{X} := \{(x_1, x_2, x_3)^t : x_1 \neq x_2, x_2 \neq x_3, x_3 \neq x_1\}$ $\mathcal{Y} := \{(y_1, y_2, y_3)^t : 0 < y_1 < y_2 < y_3\}$ 지수분포.
$\sigma \in \mathfrak{S}_n$, σ: permutation	$\Rightarrow u : \mathcal{X} \to \mathcal{Y}$ $\quad$ 3!-to-1 function.
$\lvert \mathfrak{S}_n \rvert = n!$	More precisely, for $\pi \in \mathfrak{S}_3$
$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix}$	$\mathcal{X}_\pi := \{(x_1, x_2, x_3)^t : 0 < x_{\pi_1} < x_{\pi_2} < x_{\pi_3}\} = \{(x_1, x_2, x_3) : x_2 < x_3 < x_1\}$ $\& \quad \mathcal{X} = \biguplus_{\pi \in \mathfrak{S}_3} \mathcal{X}_\pi$ $\qquad$ e.g. $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

u

$$\mathrm{pdf}_Y(y) = \sum_{\pi \in \mathfrak{S}_3} \mathrm{pdf}_X\left((u^{\pi})^{-1}(y)\right) |\pm 1|$$

$(u^{\pi})^{-1} = u_{\pi^{-1}}$

$u^{\pi} : \mathcal{X}_{\pi} \to \mathcal{Y}$ 1-1 function

$$J_{u^{\pi}}(x) = \det \boxed{\frac{\partial (x_{\pi_i})}{\partial (x_j)}} = \pm 1$$

↳ permutation matrix

$$\mathrm{pdf}_Y(y) = \sum_{\pi \in \mathfrak{S}_3} \mathrm{pdf}_X\left((u^{\pi})^{-1}(y)\right) |\pm 1|$$

$$= \sum_{\pi} \mathrm{pdf}_X\left(y_{\pi^{-1}(1)}, y_{\pi^{-1}(2)}, y_{\pi^{-1}(3)}\right)$$

순서를 바꿔도 더하기는 같다.

$$\mathrm{pdf}(x) = e^{-(x_1+x_2+x_3)} I(x_1>0, x_2>0, x_3>0)$$

So $\mathrm{pdf}_Y(y) = 6 e^{-(y_1+y_2+y_3)} I(0<y_1<y_2<y_3)$

In general, if we put $\mathrm{pdf}_X(x_1, \ldots, x_n) = f(x_1) \cdots f(x_n)$

then $\mathrm{pdf}_Y(y) = n!\, f(y_1) \cdots f(y_n)\, I(y_1 < \cdots < y_n)$

$pdf_{X_{(r)}}(x)$?

직관적 이해.

$X_{(r)} \in (x,\ x+|\Delta x|]$

$X_1, \dots, X_r \overset{iid}{\sim} f(x;\theta)$

$P(x \le X_{(r)} < x+|\Delta x|)$

$\doteq C_r\ P(X_1 \le x)^{r-1}\ P(x \le X_1 < x+|\Delta x|)$
$\quad P(X_1 > x+|\Delta x|)^{n-r}$

where $C_r = \dfrac{n!}{(r-1)!\,1!\,(n-r)!}$

$\Rightarrow pdf_{X_{(r)}}(x)|\Delta x| \doteq C_r\ F(x)^{r-1} f(x)\,(1-F(x))^{n-r}.$

Thm 4.3.2

(a) $X_{(r)}$

$pdf_{X_{(r)}}(x) = \dfrac{n!}{(r-1)!\,(n-r)!} F(x)^{r-1} f(x)\,(1-F(x))^{n-r}$

(b) $(X_{(r)}, X_{(s)})^t \quad r<s$

$pdf_{X_{(r)},X_{(s)}}(x,y)$

$= \dfrac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!} F(x)^{r-1} f(x)\,(F(y)-F(x))^{s-r-1}$
$\quad f(y)\,(1-F(y))^{n-s}$

$$\int_{-\infty}^{x} f(u)\,du \qquad [F(u)]_{\infty}^{x}$$

$$F(x) - F(\infty)$$

proof)

$$P(X_{(r)} \le x) = P(X_{(r)} \le x, \{X_{(r)} = X_{(1)}\} \uplus \cdots \{X_{(r)} = X_n\})$$

$$= \sum_{i=1}^{n} P(X_{(r)} \le x, \{X_{(r)} = X_i\})$$

iid.

$$= n\, P(X_{(r)} \le x, \{X_{(r)} = X_1\})$$

$$n P(X_{(r)} \le x, X_{(r)} = X_1)$$

$$= n \binom{n-1}{r-1} P(X_{(r)} \le x, X_{(r)} = X_1, \quad X_2 < X_1, \ldots, X_{r-1} < X_1$$

$$X_{r+1} > X, \ldots, X_n > X)$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \int_{-\infty}^{x} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_1} \int_{x_1}^{\infty} \cdots \int_{x_1}^{\infty} f_1(u_1) \cdots f_n(u_n)\, dV$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \int_{-\infty}^{x} \{F(x_1)\}^{r-1} \{1 - F(x_1)\}^{n-r} f_1(x_1)\, dx_1$$

$$\therefore \text{pdf}_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!} \{F(x)\}^{r-1} f_1(x) \{1 - F(x)\}^{n-r} \quad \square$$

예| 4.3.2.

$X_1, \ldots, X_n \overset{iid}{\sim} U(0,1)$

(a) $X_{(r)} \sim Beta(r, n-r+1)$

(b) $Z = (X_{(1)}, X_{(2)} - X_{(1)}, \ldots, X_{(n)} - X_{(n-1)})^t$

$\sim Dirichelet(1, \ldots, 1)$

proof) $pdf_{X_i}(x) = I_{(0,1)}(x)$

$$cdf_{X_i}(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$$pdf_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} x^{r-1} \cdot 1 \cdot (1-x)^{n-r} I_{(0,1)}(x)$$

$$= \frac{\Gamma(n-1)}{\Gamma(r)\Gamma(n-r+1)} x^{r-1} (1-x)^{n-r} I(0 < x < 1)$$

$X_{(r)} \sim Beta(r, n-r+1)$

$X_1, \ldots, X_n \sim U(0,1)$

(b) $Y := (X_{(1)}, \ldots, X_{(n)})^t \quad X = (X_1, \ldots, X_n)^t$

$Z := u(Y) = (X_{(1)},\ X_{(2)} - X_{(1)},\ \ldots,\ X_{(n)} - X_{(n-1)})^t$

$$u: \begin{cases} z_1 = y_1 \\ z_2 = y_2 - y_1 \\ \vdots \\ z_n = y_n - y_{n-1} \end{cases} \qquad u^{-1}: \begin{cases} y_1 = z_1 \\ y_2 = z_1 + z_2 \\ \vdots \\ y_n = z_1 + z_2 + \cdots + z_{n-1} + z_n \end{cases}$$

$\mathcal{Y} := \{ (y_1, \ldots, y_n)^t : 0 < y_1 < \cdots < y_n < 1 \}$

$\mathcal{Z} := \{ (z_1, \ldots, z_n)^t : z_i > 0,\ z_1 + \cdots + z_n < 1 \}$

$$\begin{aligned} pdf_Z(z) &= n!\, pdf(z_1) \cdots pdf(z_1 + \cdots + z_n) \cdot 1\, I(z_1 > 0, \ldots, z_n > 0, z_1 + \cdots + z_n < 1) \\ &= n!\, I(z_1 > 0, \ldots, z_n > 0,\ z_1 + \cdots + z_n < 1) \\ &= \frac{\Gamma(n+1)}{\Gamma(1) \cdots \Gamma(1)} z_1^{1-1} \cdots z_n^{1-1} (1 - z_1 - \cdots - z_n)^{1-1} I(z_1 > 0, \ldots, z_n > 0,\ z_1 + \cdots + z_n < 1) \end{aligned}$$

$\therefore Z \sim \text{Dirichelet}(1, \ldots, 1)$ $\square$

e.g. 4.3.3.

$X_1, \dots, X_n \overset{iid}{\sim} Exp(1)$: random sample.

$Y := (X_{(1)}, \dots, X_{(n)})^t$

$Z_1 = nX_{(1)}, \quad Z_2 = (n-1)(X_{(2)} - X_{(1)}), \dots, Z_n := X_{(n)} - X_{(n-1)}$

$\Rightarrow \quad \perp\!\!\!\perp_i Z_i \ \& \ Z_i \sim Exp(1)$

$pdf_Y(y_1, \dots, y_n) = n! \, e^{-(y_1 + \dots + y_n)} I(0 < y_1 < \dots < y_n)$

$$u: \begin{cases} z_1 = ny_1 \\ z_2 = (n-1)(y_2 - y_1) \\ \vdots \\ z_n = y_n - y_{n-1} \end{cases} \qquad u^{-1}: \begin{cases} y_1 = \frac{1}{n} z_1 \\ y_2 = \frac{1}{n} z_1 + \frac{1}{n-1} z_2 \\ y_3 = \frac{1}{n} z_1 + \frac{1}{n-1} z_2 + \frac{1}{n-2} z_3 \\ \vdots \\ y_n = \frac{1}{n} z_1 + \frac{1}{n-1} z_2 + \dots + \frac{1}{1} z_n \end{cases}$$

$\mathcal{Y} := \{(y_1, \dots, y_n)^t : 0 < y_1 < \dots < y_n\}$

$\mathcal{Z} := \{(z_1, \dots, z_n) : z_1 > 0, \dots, z_n > 0\}$

$$pdf_Z(z) = pdf_Y(y_1, \dots, y_n) \left| \det \frac{\partial(y_1, \dots, y_n)}{\partial(z_1, \dots, z_n)} \right|$$

$$= n! \, e^{-(z_1 + \dots + z_n)} \frac{1}{n!} I(z_1 > 0, \dots, z_n > 0)$$

$$= e^{-(z_1 + \dots + z_n)} I(z_1 > 0, \dots, z_n > 0)$$

$\therefore \ Z_i \sim Exp(1) \ \& \ \perp\!\!\!\perp_i Z_i$

Def) representative definition of rth order statistic of Exp(1).

$X_1, \dots, X_n \overset{iid}{\sim} Exp(1)$ random sample.

$$(X_{(r)})_{1\le r\le n} \overset{d}{\equiv} \left(\frac{1}{n}Z_1 + \cdots \frac{1}{n-r+1}Z_r\right)_{1\le r\le n}$$

$Z_r \overset{iid}{\sim} Exp(1)$ for $r=1,\dots,n$.

Thm 4.3.3 X: continuous r.v. & $x_1<x_2 \Rightarrow F(x_1)<F(x_2)$

(a) $F(X) \sim U(0,1)$

(b) $F^{-1}(U) \overset{d}{\equiv} X$

proof) Since $u_1<u_2 \Rightarrow F^{-1}(u_1) < F^{-1}(u_2)$ & F: increasing funct.

$F(x)\le u \Leftrightarrow x \le F^{-1}(u)$, $F^{-1}(u)\le x \Leftrightarrow u\le F(x)$, $F(F^{-1}(u))=u$.

(a) $P(F(X)\le u) = P(X\le F^{-1}(u)) = F(F^{-1}(u)) = u \quad \forall u : 0\le u\le 1$

$\therefore F(X)\sim U(0,1)$

(b) $P(F^{-1}(U)\le x) = P(U\le F(x)) = F(x) \quad \forall -\infty<x<+\infty$

$\therefore F^{-1}(U) \overset{d}{\equiv} X$

$\square$

e.g. 4.3.4

$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$ $\quad U_{(1)} < \cdots < U_{(r)}$

$Y_1, \dots, Y_n \overset{iid}{\sim} Exp(1)$ $\quad Y_{(1)} < \cdots < Y_{(r)}$

$\Rightarrow U_{(r)} \overset{d}{\equiv} (1 - e^{-Y_{(r)}})$ for $1 \le r \le n$.

sol)

$$G(y) = \begin{cases} 1 - e^{-y}, & y \ge 0 \\ 0 & y < 0 \end{cases}$$

Since $y_1 > y_2 \Rightarrow G(y_1) > G(y_2)$ & Y: continuous,

$G(Y_i) \overset{iid}{\sim} U(0,1)$.

$$pdf_{U_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} f(x) (1 - F(x))^{n-r} I(0<x<1)$$

$$= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} x^{r-1} (1-x)^{n-r} I(0<x<1).$$

$U_{(r)} \sim Beta(r, n-r+1)$

Note that G: increasing function. i.e.,

$Y_1 < \cdots < Y_{(n)} \Rightarrow G(Y_{(1)}) < \cdots < G(Y_{(n)})$

Thus, $G(Y)_{(r)} = G(Y_{(r)})$.

Since $G(Y_i) \overset{iid}{\sim} U(0,1)$,

$$G(Y_{(r)}) \sim Beta(r, n-r+1).$$

$$\therefore \underset{\substack{\| \\ 1 - e^{-Y_{(r)}}}}{G(Y_{(r)})} \overset{d}{\equiv} U_{(r)} \quad \text{for } r = 1, \dots, n. \qquad \square$$

Thm 4.3.4.

X: continuous r.v. $X_{(1)} < \cdots < X_{(n)}$

$h(y) := F^{-1}(1-e^{-y}) I(y>0)$

$\Rightarrow X_{(r)} \overset{d}{=} h\left(\frac{1}{n}Z_1 + \cdots + \frac{1}{n-r+1}Z_r\right)$ $\quad Z_i \overset{iid}{\sim} Exp(1)$ for $i=1,\dots,n$.

proof) $U_i \overset{iid}{\sim} U(0,1)$ for $i=1,\dots,n$.

$X_{(r)} \overset{d}{=} F^{-1}(U_{(r)})$

$\because P(F^{-1}(U_{(r)}) \le x) = P(U_{(r)} \le F(x))$

By the previous example, $U_{(r)} \sim Beta(r, n-r+1)$

$$P(U_{(r)} \le F(x)) = \int_0^{F(x)} \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} t^{r-1}(1-t)^{n-r}\,dt$$

$$pdf_{U_{(r)}}(x) = \frac{d}{dx}\int_0^{F(x)} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r}\,dt$$

$\because$ Fundamental Theorem of calculus

$$= \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1}(1-F(x))^{n-r}\frac{d}{dx}F(x)$$

$$= \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} f(x)(1-F(x))^{n-r}$$

$$= pdf_{X_{(r)}}(x)$$

$\therefore X_{(r)} \overset{d}{=} F^{-1}(U_{(r)})$

By the previous thm., $Y_1,\dots,Y_n \sim Exp(1)$ $\quad U_{(r)} \overset{d}{=} (1-e^{-Y_{(r)}})$

$$\therefore X_{(r)} \overset{d}{=} F^{-1}(1-e^{-Y_{(r)}}) \equiv h(Y_{(r)})$$

Since $Y_{(r)} \overset{d}{=} \left(\frac{1}{n}Z_1 + \cdots + \frac{1}{n-r+1}Z_r\right)$ $\quad Z_i \overset{iid}{\sim} Exp(1)$,

$\therefore X_{(r)} \overset{d}{=} h\left(\frac{1}{n}Z_1 + \cdots + \frac{1}{n-r+1}Z_r\right)$ $\quad \square$

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review

$X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta)$

$X_{(1)} < \cdots < X_{(n)}$, $\quad Y := (X_{(1)}, \dots, X_{(n)})^t$

$\text{pdf}_Y(y_1, \dots, y_n) = n!\, f(y_1) \cdots f(y_n)\, I(y_1 < \cdots < y_n)$

$f_r(x) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{\infty} f_{1,\dots,n}(y_1, \dots, y_{r-1}, x, y_{r+1}, \dots, y_n)\, dy_1 \cdots dy_{r-1}\, dy_{r+1} \cdots dy_n$

$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(y_1) \cdots f(y_{r-1})\, I(y_1 < \cdots < y_{r-1})\, dy_1 \cdots dy_{r-1}$

$\iint f(x,y)\, dx\, dy$

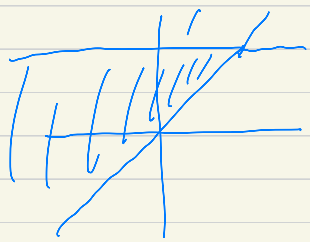
$f(x,y) = g_1(x) g_2(y)$

$\iint g_1(x) g_2(y)\, dx\, dy$

$= \int g_2(y) \left(\int g_1(x)\, dx \right) dy$

$= \left(\int g_1(x)\, dx \right) \left(\int g_2(y)\, dy \right)$

$f(x) f(y) I(x < y < 1)$



$f(x) f(y)$

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$U_1, \ldots, U_n \overset{iid}{\sim} U(0,1)$

$F^{-1}(U_r) \overset{d}{=} X_r$.

$\therefore X_{(r)} \overset{d}{=} F^{-1}(U_{(r)})$

4.4 다변량 정규분포 (Multivariate Normal Distribution)

Thm 4.4.1 $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$ $A = (a_{ij}) \in M_n(\mathbb{R})$

$\mu := (\mu_1, \dots, \mu_n)' \in \mathbb{R}^n$

$$X_1 = a_{11}Z_1 + \cdots + a_{1n}Z_n + \mu_1$$
$$\vdots$$
$$X_n = a_{n1}Z_1 + \cdots + a_{nn}Z_n + \mu_n$$

$X = AZ + \mu$ $X = (X_1, \dots, X_n)'$ $Z := (Z_1, \dots, Z_n)'$

$\Rightarrow$ (a) A: invertible (non-singular)

$\Rightarrow \mathrm{pdf}_X(x) = \left(\det(2\pi\Sigma)\right)^{-1/2} \exp\left(-(x-\mu)'\Sigma^{-1}(x-\mu)/2\right\}$

where $x \in \mathbb{R}^n$ $\Sigma = AA'$

(b) $\mathrm{mgf}_X(t) = \exp(\mu' t + \frac{1}{2} t'\Sigma t)$ $t \in \mathbb{R}^n$

proof)

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^{-1})^t = (A^t)^{-1}$

$\det(cA) = c^n \det A$

$\det A' = \det A$

$\det(2\pi\Sigma) = (2\pi)^n \det(AA') = (2\pi)^n \det A^2$

$\mathrm{pdf}_{Z}(z) = (2\pi)^{-\frac{n}{2}} \exp(-\frac{1}{2}(z_1^2 + \cdots + z_n^2))$

(a) $= (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}z'z}$

$\mathrm{mgf}_{Z}(s) = \exp(\frac{1}{2}(s_1^2 + \cdots + s_n^2)) = e^{\frac{1}{2}s's}$

$X = AZ + \mu = u(Z)$ $u^{-1}(x) = A^{-1}(x-\mu)$ $\Sigma^{-1} = (AA')^{-1} = (A^{-1})'A$

$\mathrm{pdf}_X(x) = \mathrm{pdf}_{Z}(z = u^{-1}(x)) |\det A|^{-1}$

$= (2\pi)^{-\frac{n}{2}} \exp(-\frac{1}{2}(A^{-1}(x-\mu))' A^{-1}(x-\mu)) \cdot |\det A|^{-1}$

$\Rightarrow \det(2\pi\Sigma)^{-\frac{1}{2}} = (2\pi)^{-\frac{n}{2}} |\det A|^{-1}$

$= (2\pi)^{-n/2} \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)) |\det A|^{-1}$

$= \det(2\pi\Sigma)^{-1/2} \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu))$

$X = AZ + \mu$

(b) $mgf_X(t) = E[\exp(t'(AZ+\mu))]$ $\quad t \in \mathbb{R}^n, \mu \in \mathbb{R}^n$

$= E[\exp((A't)'Z)] \cdot \exp(t'\mu)$

$= mgf_Z(A't) \cdot \exp(t'\mu)$

$= \exp(\frac{1}{2} t'AA't) \cdot \exp(t'\mu)$

$= \exp(\mu't + \frac{1}{2} t'\Sigma t)$ $\square$

Thm 4.4.2

(a) $X \sim N(\mu, \Sigma) \Rightarrow E[X] = \mu$, $Var(X) = \Sigma$

(b) $X \sim N(\mu, \Sigma)$, $\Sigma = AA'$, $A = A'$ $(A := \Sigma^{1/2})$

$\Leftrightarrow X \overset{d}{=} \Sigma^{1/2} Z + \mu$ $\quad Z \sim N(0, I_n)$

proof)

(a) $cgf_X(t) = \log(\exp(\mu't + \frac{1}{2} t'\Sigma t))$

$= \mu't + \frac{1}{2} t'\Sigma t$

$\nabla_t \, cgf_X(t) = \mu + \Sigma t$

$\nabla_t^2 \, cgf_X(t) = \Sigma.$

$\therefore E[X] = \nabla_t \, cgf_X(t) \big|_{t=0} = \mu$

$Var(X) = \nabla_t^2 \, cgf_X(t) \big|_{t=0} = \Sigma.$

Note. $A = A^t \in M_n(\mathbb{R})$

$$\mathbf{x}^t A \mathbf{x} = (x_1, \dots, x_n) \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{pmatrix}$$

$$= x_1 \sum_{j=1}^{n} a_{1j} x_j + \cdots + x_n \sum_{j=1}^{n} a_{nj} x_j$$

$$\frac{\partial}{\partial x_i} \mathbf{x}^t A \mathbf{x} = \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij} x_j + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ji} x_i + 2 a_{ii} x_i$$

$\because A^t = A$

$$= 2 \sum_{j=1}^{n} a_{ij} x_j$$

$$\therefore \nabla_{\mathbf{x}} (\mathbf{x}^t A \mathbf{x}) = 2 A \mathbf{x}$$

(b) Since Σ : symmetric, $\Sigma = P \operatorname{diag}(\lambda_1, \dots, \lambda_n) P^t$

where $\lambda_1, \dots, \lambda_n$ are eigen values of Σ and

P : orthogonal matrix. $(P'P = PP' = I)$

Since Σ : semi-positive definite, $\lambda_i \geq 0$.

Thus, $\Sigma = P \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) P^t$

$$= \underbrace{P \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) P^t}_{A = \Sigma^{1/2}} P \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) P^t$$

$$mgf_{\Sigma^{1/2}Z_1+\mu}(t) = E[\exp(t'\Sigma^{1/2}Z_1 + t'\mu)] \qquad t\in\mathbb{R}^n$$

$$= E[\exp((\Sigma^{1/2}t)'Z_1)]\exp(t'\mu)$$

$$= mgf_{Z_1}(\Sigma^{1/2}t)\exp(t'\mu)$$

$$= \exp(\tfrac{1}{2}t'\Sigma^{1/2}\Sigma^{1/2}t)\exp(t'\mu)$$

$$= \exp(\mu't + \tfrac{1}{2}t'\Sigma t)$$

$$\therefore \Sigma^{1/2}Z_1+\mu \sim N(\mu,\Sigma)$$

□

e.g. 4.4.1

$$N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right)$$

$$X=(X_1,X_2)' \sim N(\mu_1,\mu_2;\sigma_1^2,\sigma_2^2,\rho)$$

$$-1 \le \rho = \frac{Cov(X_1,X_2)}{\sqrt{Var(X_1)Var(X_2)}} \le 1 \qquad Cov(X_1,X_2)=\rho\sigma_1\sigma_2$$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \times \begin{pmatrix}\sigma_2^2 & -\rho\sigma_1\sigma_2\\ -\rho\sigma_1\sigma_2 & \sigma_1^2\end{pmatrix}$$

$$\det\Sigma = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$$

if $\sigma_1>0$, $\sigma_2>0$ and $-1<\rho<1$,

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\Big(-\frac{1}{2}\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \times$$

$$\sigma_2^2(x_1-\mu_1)^2 - 2\rho\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2)$$

$$2 \times \sigma_1^2(x_2-\mu_2)^2$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\Big(-\frac{1}{2}\Big(\frac{x_1-\mu_1}{\sigma_1}\Big)^2$$

$$+\rho\Big(\frac{x_1-\mu_1}{\sigma_1}\Big)\Big(\frac{x_2-\mu_2}{\sigma_2}\Big) - \frac{1}{2}\Big(\frac{x_2-\mu_2}{\sigma_2}\Big)$$

Thm 4.4.3

(a) $X \sim N(\mu, \Sigma)$

$\Rightarrow AX+b \sim N(A\mu+b,\ A\Sigma A^t)$

(b) $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$

$Cov(X_1, X_2) = 0 \Rightarrow X_1 \perp\!\!\!\perp X_2.$

(c) $X \sim N(\mu, \Sigma)$

$Cov(AX, BY) = A\Sigma B^t = 0 \Rightarrow AX \perp\!\!\!\perp BX$

proof)

(a) $X \overset{d}{\equiv} \Sigma^{1/2} Z_1 + \mu,\quad Z_1 \sim N(0, I),\quad \Sigma^{1/2}\Sigma^{1/2} = \Sigma,$

$\Sigma^{1/2} = (\Sigma^{1/2})^t$

$\therefore AX+b \overset{d}{\equiv} (A\Sigma^{1/2})Z_1 + (A\mu+b)$

$AX+b \sim N(A\mu+b,\ A\Sigma^{1/2}\Sigma^{1/2}A^t) \overset{d}{\equiv} N(A\mu+b,\ A\Sigma A^t)$

$\mu_1, t_1 \in \mathbb{R}^n$
$\mu_2, t_2 \in \mathbb{R}^m$

(b) $mgf_{X_1,X_2}(t) = \exp\left\{ (\mu_1, \mu_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \frac{1}{2}(t_1\ t_2)\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right\}$

Since $Cov(X_1, X_2) = \Sigma_{12} = 0,\ \Sigma_{21} = (\Sigma_{12})^t = 0$

$\therefore mgf_{X_1,X_2}(t) = \exp(\mu_1' t_1 + \mu_2' t_2 + \frac{1}{2} t_1' \Sigma_{11} t_1 + \frac{1}{2} t_2' \Sigma_{22} t_2)$

$= mgf_{X_1}(t_1) \times mgf_{X_2}(t_2)$

$\therefore X_1 \perp\!\!\!\perp X_2$

(c) $\begin{pmatrix} AX \\ BX \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} X \sim N\left(\begin{pmatrix} A\mu \\ B\mu \end{pmatrix}, \begin{pmatrix} A\Sigma A^t & A\Sigma B^t \\ B\Sigma A^t & B\Sigma B^t \end{pmatrix}\right)$

Since $Cov(AX, BX) = A\Sigma B^t = 0,$

$AX \perp\!\!\!\perp BX.$ $\square$

Thm 4.4.4

(a) $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$

$\Rightarrow X_1 \sim N(\mu_1, \Sigma_{11})$

(b) $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$

$\Rightarrow X_2 | X_2 = x_1 \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$

proof)

(a) $X_1 = (I, 0)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$X_1 \sim N\left((I, 0)\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, (I, 0)\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix}\right)$

$\therefore X_1 \sim N(\mu_1, \Sigma_{11})$

(b) $\mathrm{pdf}_{X_1}(x_1) = \det(2\pi\Sigma_{11})^{-1/2}\exp\left(-\frac{1}{2}(x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1)\right)$

$\mathrm{pdf}_X(x_1,x_2) = \det(2\pi\Sigma)^{-1/2}\exp\left(-\frac{1}{2}(x_1-\mu_1, x_2-\mu_2)'\Sigma^{-1}(x_1-\mu_1, x_2-\mu_2)\right)$

Since $\det\Sigma = \det\Sigma_{11}\cdot\det(\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$,

$\det(2\pi\Sigma)^{-1/2}/\det(2\pi\Sigma_{11})^{-1/2}$

$= \det(\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$

$\Sigma_{22\cdot 1} := \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

$$\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}^{-1} = \begin{pmatrix}\Sigma_{11}^{-1}+\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & -\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot 1}^{-1}\\ -\Sigma_{22\cdot 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22\cdot 1}^{-1}\end{pmatrix}$$

$$(x_1-\mu_1,\ x_2-\mu_2)'\begin{pmatrix}\Sigma_{11}^{-1}+\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & -\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot 1}^{-1}\\ -\Sigma_{22\cdot 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22\cdot 1}^{-1}\end{pmatrix}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix},$$

$$= (x_1-\mu_1,\ x_2-\mu_2)'\begin{pmatrix}\Sigma_{11}^{-1}(x_1-\mu_1)+\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)-\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot 1}^{-1}(x_2-\mu_2)\\ -\Sigma_{22\cdot 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)+\Sigma_{22\cdot 1}^{-1}(x_2-\mu_2)\end{pmatrix}$$

$$= (x_1-\mu_1,\ x_2-\mu_2)' \begin{pmatrix} \Sigma_{11}^{-1}(x_1-\mu_1) + \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1) - \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot1}^{-1}(x_2-\mu_2) \\ -\Sigma_{22\cdot1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1) + \Sigma_{22\cdot1}^{-1}(x_2-\mu_2) \end{pmatrix}$$

$$-(x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1)$$

$$= \cancel{(x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1)} + (x_1-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)$$

$$- (x_1-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot1}^{-1}(x_2-\mu_2) - (x_2-\mu_2)'\Sigma_{22\cdot1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)$$

$$+ (x_2-\mu_2)'\Sigma_{22\cdot1}^{-1}(x_2-\mu_2)$$

$$\Rightarrow \left((x_1-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12} - (x_2-\mu_2)'\right)\Sigma_{22\cdot1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)$$

$$- \left((x_1-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12} - (x_2-\mu_2)'\right)\Sigma_{22\cdot1}^{-1}(x_2-\mu_2)$$

$$= \left((x_1-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12} - (x_2-\mu_2)'\right)\Sigma_{22\cdot1}^{-1}\left(\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1) - (x_2-\mu_2)\right)$$

$$= \left((x_2-\mu_2)' - (x_1-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12}\right)\Sigma_{22\cdot1}^{-1}\left(x_2-\mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)\right)$$

$$= \left((x_2-\mu_2)' - (x_1-\mu_1)'(\Sigma_{21}\Sigma_{11}^{-1})^t\right)\Sigma_{22\cdot1}^{-1}\left(x_2-\mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)\right)$$

$$\therefore X_2|X_1=x_1 \sim N\left(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1),\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)$$

Thm 4.4.5. (a) $X \sim N_k(\mu, \Sigma)$ Σ: non-singular.

$\Rightarrow (X-\mu)^t \Sigma^{-1} (X-\mu) \sim \chi^2(k)$

(b) $Z_1 \sim N(0, I)$, $A^2 = A$

$\Rightarrow Z_1^t A Z_1 \sim \chi^2(r)$ $r = trA$.

proof) (a) Since Σ: symmetric matrix (L_Σ: self-adjoint) & semi-positive definite, by Spectral Theorem,

$\Sigma = P \, diag(\lambda_1, \dots, \lambda_n) P^t$ where P: orthogonal matrix ($PP^t = P^tP = I$) and $\lambda_1, \dots, \lambda_n$ are eigen-values of Σ. ($\lambda_i \geq 0$)

$\Sigma = P \, diag(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) P^t P \, diag(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) P^t$

$\Sigma^{1/2} := P \, diag(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) P^t$. $(\Sigma^{1/2})^t = \Sigma^{1/2}$

$(\Sigma^{1/2})^{-1} \Sigma (\Sigma^{1/2})^{-1} = P \, diag(\frac{1}{\sqrt{\lambda_1}}, \dots, \frac{1}{\sqrt{\lambda_n}}) P^t P \, diag(\lambda_1, \dots, \lambda_n) P^t$

$\cdot P \, diag(1/\sqrt{\lambda_1}, \dots, 1/\sqrt{\lambda_n}) P^t = I$.

$\therefore (\Sigma^{1/2})^{-1}(X-\mu) \sim N(0, I_n)$

$Z_1 = (Z_{11}, \dots, Z_{1k})' = \Sigma^{-1/2}(X-\mu)$

$(X-\mu)' \Sigma^{-1} (X-\mu) = Z_1' Z_1 = Z_{11}^2 + \dots + Z_{1k}^2 \sim \chi^2(k)$ $Z_{1i} \sim N(0,1)$

(b) Since $A^t = A \in M_n(\mathbb{R})$,

$A = P^t \operatorname{diag}(\lambda_1, \dots, \lambda_n) P$ $\quad$ P: orthogonal matrix.

Since $A^2 = A$, $\lambda_i^2 x_i = \lambda x_i$ $\quad$ $\lambda_i = 1$ or 0.

W.L.O.G, $\lambda_1 = \lambda_2 = \dots = \lambda_r = 1$, $\lambda_{r+1} = \dots = \lambda_n = 0$.

$$Z^t A Z = Z^t P^t \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P Z$$
$$= (PZ)^t \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P Z$$

$X = (X_1, \dots, X_r)^t := PZ$

$Z^t A Z = X^t \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} X = X_1^2 + \dots + X_r^2$, $X \sim N(0, PP^t) = N(0, I)$

$\therefore Z^t A Z \sim \chi^2(r)$

Since $A \sim \operatorname{diag}(\lambda_1, \dots, \lambda_n)$,

$\operatorname{tr} A = \operatorname{tr}(\operatorname{diag}(\lambda_1, \dots, \lambda_n)) = r$

$\square$

e.g. 4.4.4

$X_{ij} \overset{iid}{\sim} N(\mu_i, \sigma^2)$ $\quad \bar{X}_i := \sum_{j=1}^{n_i} X_{ij}/n_i$

$\bar{X} := \sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}/n$ $\quad n := \sum_{i=1}^{k} n_i$

$\bar{\mu} := \sum_{i=1}^{k} \frac{n_i}{n}\mu_i$ $\quad E[Y] = (E[Y_1], \dots, E[Y_r])$

$Y_i := \bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})$ $\quad Y := (Y_1, \dots, Y_r)' \sim\ ?$ $\quad r = k-1$

(1) $E[Y_i] = E[\bar{X}_i] - E[\bar{X}] - \mu_i + \bar{\mu}$

$\bar{X} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}/n = \sum_{i=1}^{k} \frac{n_i}{n}\bar{X}_i$

$= \mu_i - \sum_{i=1}^{k} \frac{n_i}{n} E[\bar{X}_i] - \mu_i + \bar{\mu}$

$= \bar{\mu} - \sum_{i=1}^{k} \frac{n_i}{n}\mu_i = 0.$

(2) $Var(Y_i) = \underbrace{E[Y_i^2] - E[Y_i]^2}_{} = E[Y_i^2]$

$= E[(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu}))^2]$

$= E[(\bar{X}_i - \bar{X})^2] + E[(\mu_i - \bar{\mu})^2]$

$-2E[(\bar{X}_i - \bar{X})\underline{(\mu_i - \bar{\mu})}]$ $\quad E[(\bar{X}_i - \bar{X})] = \underline{\mu_i - \bar{\mu}}$

$= E[\bar{X}_i^2] + E[\bar{X}^2] - 2E[\bar{X}_i\bar{X}]$

$+ (\mu_i - \bar{\mu})^2 - 2(\mu_i - \bar{\mu})^2$

$= E[\bar{X}_i^2] + E[\bar{X}^2] - 2E[\bar{X}_i\bar{X}] - (\mu_i - \bar{\mu})^2$

$\bar{X} = \sum_{i=1}^{k} \frac{n_i}{n} \bar{X}_i \qquad \perp\!\!\!\perp_i \bar{X}_i$

$\bar{X} \sim N(\bar{\mu}, \sigma^2/n)$

$\because mgf_{\bar{X}}(t) = E[\exp(\sum_{i=1}^{k} \frac{n_i}{n} \bar{X}_i t)] \qquad \bar{X}_i \sim N(\mu_i, \frac{\sigma^2}{n_i})$

$= \prod_{i=1}^{k} mgf_{\bar{X}_i}(\frac{n_i}{n} t)$

$= \exp(\sum_{i=1}^{k} \frac{n_i}{n} t\mu_i + \sum_{i=1}^{k} \frac{1}{2} (\frac{n_i}{n} t)^2 \frac{\sigma^2}{n_i})$

$\sum_{i=1}^{k} \frac{n_i}{n} \mu_i = \bar{\mu}$

$= \exp(\bar{\mu} t + \frac{1}{2} \sum_{i=1}^{k} \frac{n_i}{n^2} \sigma^2 t^2)$

$= \exp(\bar{\mu} t + \frac{1}{2} \frac{\sigma^2}{n} t^2)$

$\therefore \bar{X} \sim N(\bar{\mu}, \sigma^2/n)$

$Cov(\bar{X}_i, \bar{X}) = Cov(\bar{X}_i, \sum_{j=1}^{k} \frac{n_j}{n} \bar{X}_j) \quad \because \bar{X}_i \perp\!\!\!\perp \bar{X}_j \Rightarrow Cov(\bar{X}_i, \bar{X}_j) = 0$

$= Cov(\bar{X}_i, \frac{n_i}{n} \bar{X}_i) = \frac{n_i}{n} Var(\bar{X}_i)$

$= E[\bar{X}_i \bar{X}] - E[\bar{X}_i] E[\bar{X}]$

$E[\bar{X}_i^2] = Var(\bar{X}_i) + E[\bar{X}_i]^2$

$E[\bar{X}_i \bar{X}] = \frac{\sigma^2}{n} + \mu_i \bar{\mu}$

$E[\bar{X}^2] = Var(\bar{X}) + E[\bar{X}]^2$

$\therefore E[\bar{X}_i^2] + E[\bar{X}^2] - 2E[\bar{X}_i \bar{X}] - (\mu_i - \bar{\mu})^2$

$= \underline{\mu_i^2 + \frac{\sigma^2}{n_i}} + \underline{\bar{\mu}^2 + \frac{\sigma^2}{n}} - 2\frac{\sigma^2}{n} - 2\mu_i \bar{\mu} - \mu_i^2 - \bar{\mu}^2 + 2\mu_i \bar{\mu}$

$= (n_i^{-1} - n^{-1}) \sigma^2$

$Var(\bar{X}_i) = (n_i^{-1} - n^{-1}) \sigma^2$

$\because\ Cov(X+b,\ Y+c) = Cov(X,Y)$

$(i \neq j)$

(3) $Cov(Y_i, Y_j) = Cov(\bar{X}_i - \bar{X} - \mu_i + \bar{\mu},\ \bar{X}_j - \bar{X} - \mu_j + \bar{\mu})$

$= Cov(\bar{X}_i - \bar{X},\ \bar{X}_j - \bar{X})$

$\because\ Cov(X, Y+Z) = Cov(X,Y) + Cov(X,Z)$

$= Cov(\bar{X}_i, \bar{X}_j) - Cov(\bar{X}_i, \bar{X}) + Cov(\bar{X}, \bar{X}) - Cov(\bar{X}_j, \bar{X})$

$= Var(\bar{X}) - Cov(\bar{X}_i, \bar{X}) - Cov(\bar{X}_j, \bar{X})$

$= \frac{\sigma^2}{n} - \frac{n_i}{n} \times \frac{\sigma^2}{n_i} - \frac{n_j}{n} \times \frac{\sigma^2}{n_j}$

$= -\frac{\sigma^2}{n}$

$\therefore\ Var(Y) = \underline{\left[diag(n_1^{-1}, \dots, n_r^{-1}) - n^{-1} \mathbb{1}\mathbb{1}^t\right]\sigma^2}$

$\mathbb{1}^t := (1, \dots, 1)^t$

$\therefore\ Y \sim N(0, \Sigma),\ \Sigma := Var(Y)$

$c = -1/(1+b^t a)$

$Y := (Y_1, \ldots, Y_r)$

$(I + ab^t)^{-1} = I + cab^t$

$(I + ab^t)^{-1} = I + cab^t$

$c = -1/(1+b^t a)$

$a := -\frac{1}{n}(n_1, \ldots, n_r)$

$b := (1, \ldots, 1)$

$1 + b^t a = 1 - \frac{n_1 + \cdots + n_r}{n}$

$= 1 - \frac{n - n_k}{n}$

$= + \frac{n_k}{n}$

$r = k-1$

$n_1 + \cdots + n_k = n$

$n_1 + \cdots + n_r = n - n_k$

$$\left(D(n_i^{-1}) - n^{-1} 11'\right)^{-1} = \left[I_n - n^{-1} D(n_i) 11' \right]^{-1} D(n_i^{-1})^{-1}$$

$$= D(n_i) - \frac{n}{n_k} \times \left(-\frac{1}{n}\right) \begin{pmatrix} n_1 \\ \vdots \\ n_r \end{pmatrix} (1, \ldots, 1) \begin{pmatrix} n_1 & & 0 \\ & \ddots & \\ 0 & & n_r \end{pmatrix}$$

$$= D(n_i) + \frac{1}{n_k} \times \begin{pmatrix} n_1 \\ \vdots \\ n_r \end{pmatrix} (n_1 \cdots n_r)$$

$$= D(n_i) + n_k^{-1} \times (n_1, \ldots, n_r)^t (n_1, \ldots, n_r)$$

$E(Y_i) = 0$

$$Y \sim N\left(0, \Sigma = \left[D(n_i^{-1}) - n^{-1} 11'\right] \sigma^2\right)$$

$$Y' \Sigma Y = \sum_{i=1}^{r} n_i Y_i^2 / \sigma^2 + \frac{1}{n_k} \left(\sum_{i=1}^{r} n_i Y_i\right)^2 / \sigma^2$$

$$\sum_{i=1}^{k} n_i Y_i = \sum_{i=1}^{k} n_i (\bar{X}_i - \bar{X} - \mu_i + \bar{\mu}) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} - n\bar{X} - \sum_{i=1}^{k} n_i \mu_i + n\bar{\mu}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} - \sum_{i=1}^{k} n_i \bar{X}_i - \sum_{i=1}^{k} n_i \mu_i + \sum_{i=1}^{k} n_i \mu_i$$

$$= 0$$

$$\boxed{n_k Y_k = -\sum_{i=1}^{r} n_i Y_i}$$

$$\therefore \sum_{i=1}^{r} n_i Y_i^2 / \sigma^2 + \frac{1}{n_k} \left(\sum_{i=1}^{r} n_i Y_i\right)^2 / \sigma^2 = \sum_{i=1}^{k} n_i Y_i^2 / \sigma^2$$

By the thm 4.4.5-(a)

$$\therefore \sum_{i=1}^{k} n_i Y_i^2 / \sigma^2 \sim \chi^2(r) \overset{d}{=} \chi^2(k-1)$$

$$\therefore \sum_{i=1}^{k} n_i \left(\bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})\right)^2 \sim \chi^2(k-1)$$

$\square$

6.20.

Lemma.

$X_i \sim N(\mu_i, \sigma^2/n_i)$ & $\underset{i}{\perp\!\!\!\perp} X_i$ for $i=1,\dots,k$

$\Rightarrow X := (X_1,\dots,X_k)' \sim N((\mu_1,\dots,\mu_k)', \sigma^2 \operatorname{diag}(n_1^{-1},\dots,n_k^{-1}))$

$Z_i := \dfrac{X_i - \mu_i}{\sigma/\sqrt{n_i}} \sim N(0,1)$

$$X = \left(\frac{\sigma}{\sqrt{n_1}} Z_1 + \mu_1, \dots, \frac{\sigma}{\sqrt{n_k}} Z_k + \mu_k\right)'$$

$$= \sigma \operatorname{diag}(n_1^{-\frac{1}{2}},\dots,n_k^{-\frac{1}{2}})(Z_1,\dots,Z_k) + (\mu_1,\dots,\mu_k)'$$

By the definition of multivariate normal distribution,

$$X \sim N\left((\mu_1,\dots,\mu_k)', \left(\sigma \operatorname{diag}(n_1^{-\frac{1}{2}},\dots,n_k^{-\frac{1}{2}})\right) \cdot \left(\sigma \operatorname{diag}(n_1^{-\frac{1}{2}},\dots,n_k^{-\frac{1}{2}})\right)'\right)$$

$$\therefore X \sim N((\mu_1,\dots,\mu_k)', \sigma^2 \operatorname{diag}(n_1^{-1},\dots,n_k^{-1}))$$

$\square$

$\bar{X}_i \sim N(\mu_i, \sigma^2/n_i)$ & $\underset{i}{\perp\!\!\!\perp} \bar{X}_i$. By the previous lemma,

$D^{-1}(n_i^{-1}) := \begin{pmatrix} n_1^{-1} & & 0 \\ & \ddots & \\ 0 & & n_k^{-1} \end{pmatrix}$

$\Sigma_X := \sigma^2 D(n_i^{-1})$

$\mu_X := (\mu_1, \dots, \mu_k)'$

$$X := (\bar{X}_1, \dots, \bar{X}_k)' \sim N((\mu_1, \dots, \mu_k)', \sigma^2 D(n_i^{-1}))$$

$\Leftrightarrow X \overset{d}{=} AZ + \mu_X$ where $AA' = \Sigma_X$, $Z \sim N(0, I)$

$Y_i = \bar{X}_i - \bar{X} - (\mu_i - \bar{\mu})$ $(i = 1, \dots, r,\ r = k-1)$

$Y = (Y_1, \dots, Y_r)'$

WTS $Y \sim N(0, \Sigma)$ where $\Sigma = [D(n_i^{-1}) - n^{-1}11']\sigma^2$

$$\begin{aligned} Y_i &= \bar{X}_i - \bar{X} - (\mu_i - \bar{\mu}) \\ &= \bar{X}_i - \sum_{j=1}^{k} \frac{n_j}{n} \bar{X}_j - (\mu_i - \bar{\mu}) \\ &= \left(-\frac{n_1}{n}\bar{X}_1, \dots, \left(1 - \frac{n_i}{n}\right)\bar{X}_i, \dots, -\frac{n_k}{n}\bar{X}_k\right) - (\mu_i - \bar{\mu}) \\ &= \left(-\frac{n_1}{n}, \dots, \left(1 - \frac{n_i}{n}\right), \dots, -\frac{n_k}{n}\right)' X - (\mu_i - \bar{\mu}) \end{aligned}$$

$$Y = \underbrace{\begin{pmatrix} 1 - \frac{n_1}{n} & -\frac{n_2}{n} & \cdots & -\frac{n_r}{n} & -\frac{n_k}{n} \\ -\frac{n_1}{n} & 1 - \frac{n_2}{n} & \cdots & -\frac{n_r}{n} & -\frac{n_k}{n} \\ \vdots & \vdots & & & \\ -\frac{n_1}{n} & -\frac{n_2}{n} & \cdots & 1 - \frac{n_r}{n} & -\frac{n_k}{n} \end{pmatrix}}_{=: B \in M_{r \times k}(\mathbb{R})} X - \begin{pmatrix} \mu_1 - \bar{\mu} \\ \mu_2 - \bar{\mu} \\ \vdots \\ \mu_k - \bar{\mu} \end{pmatrix}$$

$\mu := -(\mu_1 - \bar{\mu}, \dots, \mu_k - \bar{\mu})'$

$Y \overset{d}{=} BX + \mu$

$Y \overset{d}{=} B(AZ + \mu_X) + \mu \overset{d}{=} (BA)Z + (B\mu_X + \mu)$

$BAA'B' = B\Sigma_X B'$ $\quad B(\mu_1, \dots, \mu_k)' + \mu$

$$\underbrace{B(\mu_1,\dots,\mu_k)'}_{} - (\mu_1-\bar\mu,\dots,\mu_k-\bar\mu)'$$

$$\left(\mu_1-\sum_{i=1}^{k}\frac{n_i}{n}\mu_i,\ \dots,\ \mu_k-\sum_{i=1}^{k}\frac{n_i}{n}\mu_i\right)$$

$$-(\mu_1-\bar\mu,\dots,\mu_k-\bar\mu) = 0$$

$$BAA'B' = B\Sigma_X B' = B\sigma^2 D(n_i^{-1})B'$$

$$\sigma^2 \underbrace{\begin{pmatrix} 1-\frac{n_1}{n} & -\frac{n_2}{n} & \cdots & -\frac{n_r}{n} & -\frac{n_k}{n} \\ -\frac{n_1}{n} & 1-\frac{n_2}{n} & \cdots & -\frac{n_r}{n} & -\frac{n_k}{n} \\ \vdots & \vdots & & & \\ -\frac{n_1}{n} & -\frac{n_2}{n} & \cdots & 1-\frac{n_r}{n} & -\frac{n_k}{n} \end{pmatrix}}_{r\times k} \begin{pmatrix} n_1^{-1} & & & 0 \\ & n_2^{-1} & & \\ & & \ddots & \\ 0 & & & n_k^{-1} \end{pmatrix} B'$$

$$= \sigma^2\left[\begin{pmatrix} n_1^{-1} & & & 0 \\ & n_2^{-1} & & \\ & & \ddots & \\ 0 & & n_r^{-1} & 0 \end{pmatrix} - n^{-1}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\right] B'$$

$$= \sigma^2 \begin{pmatrix} n_1^{-1} & 0 & \cdots & 0 & 0 \\ 0 & n_2^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & n_r^{-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} 1-\frac{n_1}{n} & -\frac{n_1}{n} & \cdots & -\frac{n_1}{n} \\ -\frac{n_2}{n} & 1-\frac{n_2}{n} & \cdots & -\frac{n_2}{n} \\ \vdots & \vdots & & \\ -\frac{n_r}{n} & -\frac{n_r}{n} & & 1-\frac{n_r}{n} \\ -\frac{n_k}{n} & -\frac{n_k}{n} & & -\frac{n_k}{n} \end{pmatrix}$$

$$= \sigma^2\left(D(n_i^{-1}) - n^{-1}11'\right) \quad \therefore\ Y \sim N\left(0, [D(n_i^{-1}) - n^{-1}11']\sigma^2\right) \qquad \square$$

e.g. 4.4.5

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\Rightarrow \bar{X},\ S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2, \quad \bar{X} \perp\!\!\!\perp S$

$\&\ (n-1)S^2/\sigma^2 \sim \chi^2(n-1)$

proof)

$X := (X_1, \dots, X_n)' \sim N(\mu 1, \sigma^2 I)$

$\bar{X} = n^{-1} 1' X, \quad (n-1)S^2 = \sum_{i=1}^n (X_i - \bar{X})^2 = X'(I - n^{-1}11')X$

$\because \sum_{i=1}^n (X_i - \bar{X})^2 = (X - \bar{X}1)'(X - \bar{X}1)$

$$= X'X - X'\bar{X}1 - 1'\bar{X}X + \bar{X}^2 1'1$$

$$= X'X - n^{-1}X'\underbrace{1'X}_{\text{scalar}}1 - n^{-1}1'\underbrace{1'X}_{\text{transpose}}X + n^{-2}1'X1'X\underbrace{1'1}_{n}$$

$$= X'X - n^{-1}X'11'X - n^{-1}1'\underbrace{X'1}_{\text{scalar}}X + n^{-1}X'11'X$$

$$= X'X - n^{-1}X'11'X - n^{-1}X'11'X + n^{-1}X'11'X$$

$$= X'X - n^{-1}X'11'X = X'(I - n^{-1}11')X$$

$I - n^{-1}11'$: symmetric $\&$ $1'(I - n^{-1}11') = 1' - 1' = 0$

$\Rightarrow Cov(1'X, (I - n^{-1}11')X) = 1' Cov(X, X)(I - n^{-1}11')'$

$$= 1'(\sigma^2 I)(I - n^{-1}11')$$

$$= \sigma^2 1'(I - n^{-1}11') = 0$$

$\therefore \bar{X} (= n^{-1}1'X) \perp\!\!\!\perp (I - n^{-1}11')X$

$(I - n^{-1}11')^2 = (I - n^{-1}11')(I - n^{-1}11') = I - 2n^{-1}11' + n^{-2}11'11'$

$$= I - 2n^{-1}11' + n^{-1}11'$$

$$= I - n^{-1}11'$$

$\Rightarrow X'(I - n^{-1}11')X = X'(I - n^{-1}11')'(I - n^{-1}11')X = ((I - n^{-1}11')X)'((I - n^{-1}11')X)$

$\therefore \bar{X} \perp\!\!\!\perp (n-1)S^2$

$(n-1)S^2/\sigma^2 = X'(I - n^{-1}11')X/\sigma^2 = (X-\mu 1)'(I-n^{-1}11')(X-\mu 1)/\sigma^2$

$\because$ $(I - n^{-1}11')X = X - n^{-1}11'X$

$$= X - \frac{1}{n}\begin{pmatrix}1&\cdots&1\\ \vdots&\ddots&\vdots\\ 1&\cdots&1\end{pmatrix}\begin{pmatrix}X_1\\ \vdots\\ X_n\end{pmatrix} = X - \frac{1}{n}(X_1+\cdots+X_n, \ldots, X_1+\cdots+X_n)$$

$$= X - \bar{X}(1,\ldots,1)$$

$(I-n^{-1}11')\mu 1 = \mu 1 - \mu 1 = 0$

Since $(X-\mu 1)/\sigma^2 \sim N(0,I)$ & $tr(I-n^{-1}11') = n - \frac{1}{n}\times n = n-1$,

$(n-1)S^2/\sigma^2 = (X-\mu 1)'(I-n^{-1}11')(X-\mu 1)/\sigma^2 \sim \chi^2(n-1)$

Rmk 1. $\boxed{T^2 = T \iff T\text{: projection. where } T: V\to V \quad V = \mathrm{im}T \oplus \ker T}$

proof) $\Rightarrow$ Suppose $T^2=T$.

$W_1 := \{w - Tw : w\in V\}$

Since $T^2=T$, $T(w-Tw)=0$. i.e., $W_1 \subset \ker T$.

Conversely, $\forall w\in\ker T$, $w = w - Tw$

i.e., $w\in W_1$. Thus $W_1 = \ker T$.

$\forall w\in V$, $w = (w-Tw)+Tw \in \ker T + \mathrm{im} T$.

Suppose $w \in \ker T\cap \mathrm{im}T$ be given. $w = T\omega$ for some $\omega \in V$

$0 = Tw = T^2\omega = T\omega = w$ $\therefore$ $V = \ker T \oplus \mathrm{im}T$.

$\Leftarrow$ Suppose T: projection on W_1 along W_2.

$w = w_1 + w_2 \quad w_1 \in W_1, \ w_2 \in W_2$.

$T^2 v = T w_1 = w_1 = T v$

$\therefore T^2 = T$. $\square$

Rmk

$n^{-1} 11'$: projection

$\because (n^{-1} 11')(n^{-1} 11') = n^{-2} \underbrace{1'1}_{n} 1' = n^{-1} 11'$

$\mathrm{im}(n^{-1} 11') = \mathrm{span}\{(1, \dots, 1)\}$: one dimensional subspace of $\mathbb{R}^n$.

$$n^{-1} 11' (x_1, \dots, x_n)' = \frac{x_1 + \cdots + x_n}{n} (1, \dots, 1)'$$

Linear regression models.

rank $X = p+1$
X has column full rank.

$x_0, \dots, x_p$: explanatory variable
y : response variable.

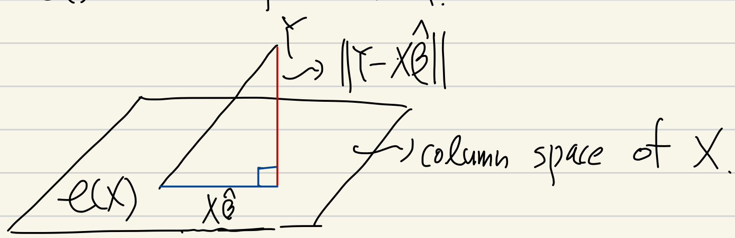
$Y_i := x_{i0}\beta_0 + \cdots + x_{ip}\beta_p + e_i$ (observation noise.) $\quad e_i \overset{iid}{\sim} N(0, \sigma^2)$

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \quad X = \begin{pmatrix} x_{10} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{n0} & \cdots & x_{np} \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_p \end{pmatrix} \quad e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

$\Rightarrow Y = X\beta + e, \quad e \sim N(0, \sigma^2 I_n)$

We want to estimate $\hat{\beta}$ s.t. $\min_{\beta} \|Y - X\beta\|$.

$X\beta \in \mathcal{C}(X)$: column space of X.



Note that
$V = \mathbb{R}^n$ &
$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$
$x, y \in \mathbb{R}^n$

Let $X\hat{\beta}$ be an orthogonal projection of Y onto $\mathcal{C}(X)$.
Then $\|Y - X\hat{\beta}\| \le \|Y - X\beta\|$ for any β.
Since $X\beta \perp Y - X\hat{\beta}$,

$$\langle X\beta, Y - X\hat{\beta} \rangle = \langle \beta, X'(Y - X\hat{\beta}) \rangle = 0 \quad \text{for any } \beta.$$

Thus, $X'(Y - X\hat{\beta}) = 0$. $\quad X'X\hat{\beta} = X'Y$.
Since, $\text{rank}(X'X) = \text{rank}\, X = p+1$, $X'X$ is invertible.
(by the Lemma 2.)
$\therefore \hat{\beta} = (X'X)^{-1}X'Y$.

$$\hat{\sigma}^2 := (Y - X\hat{\beta})'(Y - X\hat{\beta}) / (n-p-1)$$

평균오차 제곱합.

Lemma 2 $\operatorname{rank} A = r \Rightarrow \operatorname{rank} A^*A = r$. where $A^* := \bar{A}^t$

proof) By the dimension theorem, it is enough to show that

$\ker L_A = \ker L_{A^*A}$.

$\Rightarrow)$ suppose $x \in \ker L_A$ be given. Then $L_{A^*A} x = A^*Ax = 0$.

i.e., $x \in \ker L_{A^*A}$.

$\Leftarrow)$ suppose $x \in \ker L_{A^*A}$ be given. $0 = \langle A^*Ax, x \rangle = x^* A^* Ax = \langle Ax, Ax \rangle$
$= \|Ax\|^2$

Thus $Ax = 0$. i.e., $x \in \ker L_A$.

$\therefore \operatorname{rank} A = \operatorname{rank} A^*A$.

Thm 4.4.6.

$Y = X\beta + e$, rank $X = p+1$

$e \sim N(0, \sigma^2 I_n)$ $\quad \hat{\sigma}^2 = (Y - X\hat{\beta})'(Y - X\hat{\beta})/(n-p-1)$

$\Rightarrow$ (a) $\hat{\beta} \sim N_{p+1}(\beta, \sigma^2(X'X)^{-1})$

(b) $\hat{\beta} \perp\!\!\!\perp \hat{\sigma}^2$

(c) $(n-p-1)\hat{\sigma}^2/\sigma^2 \sim \chi^2(n-p-1)$

proof (a) $\hat{\beta} = (X'X)^{-1}X'Y \sim N_{p+1}\left((X'X)^{-1}X'X\beta, \underbrace{((X'X)^{-1}X')\sigma^2 I_n ((X'X)^{-1}X')'}\right)$

$Y \sim N(X\beta, \sigma^2 I_n)$

$$= \sigma^2 (X'X)^{-1}X'X(X'X)^{-1}$$
$$= \sigma^2 (X'X)^{-1}$$

$\therefore \hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1})$

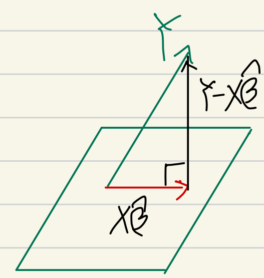
(b) $\Pi := X(X'X)^{-1}X'$

1) $\Pi' = (X(X'X)^{-1}X')' = X(X'X)^{-1}X' = \Pi$

2) $\Pi^2 = (X(X'X)^{-1}X')(X(X'X)^{-1}X') = X(X'X)^{-1}X' = \Pi$

3) $\Pi X = X(X'X)^{-1}X'X = X$

$\hat{\beta} = (X'X)^{-1}X'Y$, $\hat{\sigma}^2 = (Y - X\hat{\beta})'(Y - X\hat{\beta})/(n-p-1)$



$Y - X\hat{\beta} = Y - \Pi Y = (I - \Pi)Y$

$$\mathrm{Cov}((X'X)^{-1}X'Y, (I-\Pi)Y) = (X'X)^{-1}X' \underset{\sigma^2 I}{\underset{\shortparallel}{\mathrm{Cov}(Y)}} (I-\Pi)'$$
$$= (X'X)^{-1}X'(\sigma^2 I)(I - \Pi)$$

$\therefore \hat{\beta} \perp\!\!\!\perp (I-\Pi)Y$ $\quad = \sigma^2 \cdot (X'X)^{-1}X' - (X'X)^{-1}X'X(X'X)^{-1}X'$

$\therefore \hat{\beta} \perp\!\!\!\perp \hat{\sigma}^2$ $\quad = 0$

$\hat{\beta} = (X'X)^{-1}X'Y$

$X\hat{\beta} = X(X'X)^{-1}X'Y$

$X = (c_0 \cdots c_p)$

$\hat{\beta} = (\beta_0, \ldots, \beta_p)'$

$\Rightarrow X\hat{\beta} = \beta_0 c_0 + \cdots + \beta_p c_p$

$= \mathcal{C}(X)$

$X(X'X)^{-1}X'Y$: orthogonal projection onto $\mathcal{C}(X)$

$\Pi = X(X'X)^{-1}X'$

(I-Π): orthogonal projection.
and $X\beta \perp (I-\Pi)Y$.
Thus $(I-\Pi)X\beta = 0$ for any $\beta \in \mathbb{R}^{p+1}$

(c) $(I-\Pi)^2 = I-\Pi$, $\quad (I-\Pi)X = \underbrace{X - X(X'X)^{-1}X'X}_{=0}$

$$(Y-X\hat\beta)'(Y-X\hat\beta) = ((I-\Pi)Y)'((I-\Pi)Y)$$

$\because (I-\Pi)(X\beta) = 0$

$$= Y'(I-\Pi)'(I-\Pi)Y$$
$$= (Y-X\beta)'(I-\Pi)'(I-\Pi)(Y-X\beta)$$
$$= (Y-X\beta)'(I-\Pi)(Y-X\beta)$$

Since $\frac{1}{\sigma}(Y-X\beta) \sim N(0, I_n)$ & $(I-\Pi)^2 = (I-\Pi)$,

$$\frac{(n-p-1)\hat\sigma^2}{\sigma} = (Y-X\hat\beta)'(Y-X\hat\beta)/\sigma^2$$
$$= (Y-X\beta)'(I-\Pi)(Y-X\beta)/\sigma^2$$
$$\sim \chi^2(tr(I-\Pi))$$

$$tr(I_n-\Pi) = n - tr(X(X'X)^{-1}X')$$
$$= n - tr((X'X)^{-1}X'X)$$
$$= n - p - 1$$

$\therefore (n-p-1)\hat\sigma^2/\sigma \sim \chi^2(n-p-1)$ $\quad\square$

Simple linear regression

$$Y_i = \beta_0 + \beta_1 x_{i1} + e_i \quad e_i \overset{iid}{\sim} N(0, \sigma^2)$$

$$X = \begin{pmatrix} 1 & x_{11} \\ \vdots & \vdots \\ 1 & x_{n1} \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \Rightarrow Y = X\beta + e \quad e \sim N(0, I_n)$$

$$\hat{\beta} = (X'X)^{-1}X'Y \qquad X'Y = \begin{pmatrix} \sum_{i=1}^n Y_i \\ \sum_{i=1}^n x_{i1}Y_i \end{pmatrix}$$

$$X'X = \begin{pmatrix} 1 & \cdots & 1 \\ x_{11} & \cdots & x_{n1} \end{pmatrix} \begin{pmatrix} 1 & x_{11} \\ \vdots & \vdots \\ 1 & x_{n1} \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n x_{i1} \\ \sum_{i=1}^n x_{i1} & \sum_{i=1}^n (x_{i1})^2 \end{pmatrix}$$

$$(X'X)^{-1} = \frac{1}{n\sum_{i=1}^n (x_{i1})^2 - \left(\sum_{i=1}^n x_{i1}\right)^2} \begin{pmatrix} \sum_{i=1}^n x_{i1}^2 & -\sum_{i=1}^n x_{i1} \\ -\sum_{i=1}^n x_{i1} & n \end{pmatrix}$$

$$\bar{x}_1 = \frac{1}{n}\sum_{i=1}^n x_{i1} \quad \bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$$

$$\hat{\beta}_1 = \frac{1}{n\sum_{i=1}^n (x_{i1})^2 - n^2\bar{x}_1^2} \times \left(-\left(\sum_{i=1}^n Y_i\right) \times \left(\sum_{i=1}^n x_{i1}\right) + n\sum_{i=1}^n x_{i1}Y_i\right)$$

$$= \frac{1}{\cancel{n}\left(\sum_{i=1}^n x_{i1}^2 - n\bar{x}_1^2\right)} \times \left(\cancel{n}\sum_{i=1}^n x_{i1}Y_i - \cancel{n}\bar{x}_1\sum_{i=1}^n Y_i\right)$$

$$= \sum_{i=1}^n (x_{i1} - \bar{x}_1)Y_i \Big/ \left(\sum_{i=1}^n (x_{i1}^2 - \bar{x}_1^2)\right)$$

$$\hat{Y} = X\hat{\beta} = \begin{pmatrix} 1 & x_{11} \\ \vdots & \vdots \\ 1 & x_{n1} \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (\hat{\beta}_0 + \hat{\beta}_1 x_{11}, \ldots, \hat{\beta}_0 + \hat{\beta}_1 x_{n1})$$

$$\bar{Y} = \frac{1}{n}(\hat{Y}_1 + \cdots + \hat{Y}_n) = \frac{1}{n}(n\hat{\beta}_0 + \hat{\beta}_1 \times n\overline{x_1})$$

$$\bar{Y} = \hat{\beta}_0 + \hat{\beta}_1 \overline{x_1}$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \overline{x_1}$$

$$\hat{\beta}_0 + \hat{\beta}_1 x_{i1} = \bar{Y} + (x_{i1} - \overline{x_1})\hat{\beta}_1$$

$\square$