Chapter 5

Asymptotic distribution

Thm 5.1.1 (Central Limit Theorem)

$X_1, \dots, X_n$: mutually indep. & identically dist.

$Var(X_1) > 0$

$E[X_1] = \mu \quad Var(X_1) = \sigma^2$

For $Z \sim N(0,1)$,

$$\lim_{n\to\infty} P\left(\frac{(X_1+\cdots+X_n)/n - \mu}{\sigma/\sqrt{n}} \le x\right) = P(Z \le x)$$

for all $x \in \mathbb{R}$

proof) $\boxed{\text{WTS}}$ $\lim_{n\to\infty} mgf_{\sqrt{n}(\bar{X}-\mu)/\sigma}(t) = mgf_Z(t) = \exp\left(\frac{t^2}{2}\right)$

$mgf_{\sqrt{n}(\bar{X}-\mu)/\sigma}(t)$

$= E\left[\exp\left(t \cdot \sqrt{n}(\bar{X}_n - \mu)/\sigma\right)\right]$

$= E\left[\exp\left(t \cdot \frac{1}{\sqrt{n}}\left(\frac{X_1-\mu}{\sigma} + \cdots + \frac{X_n - \mu}{\sigma}\right)\right)\right]$

$= E\left[\exp\left(\frac{t}{\sqrt{n}} \cdot \frac{X_1-\mu}{\sigma}\right)\right]^n$

$= mgf_{(X_1-\mu)/\sigma}(t/\sqrt{n})^n$

$\bar{X} - \mu = \frac{1}{n}\sum_{i=1} X_i - \mu$

$= \frac{1}{n}\sum_{i=1}^{n} X_i - n\mu$

$= \frac{1}{n}(X_1 - \mu_1 + \cdots + X_n - \mu_n)$

$m(s) := mgf_{(X_1-\mu)/\sigma}(s) := mgf_{(X_1-\mu)/\sigma}(t/\sqrt{n})$

$$\lim_{s\to 0} \frac{R_s}{s^2} = 0$$

$$\lim_{n\to\infty} \frac{R_{n,t}}{(t/\sqrt{n})^2} = 0$$

$$-1 < x < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

$$f(x) \in o(x)$$

$$\lim_{x\to 0} \frac{f(x)}{x} = 0$$

$$m(s) = m\left(\frac{t}{\sqrt{n}}\right) = m(0) + \frac{m'(0)}{1!}\frac{t}{\sqrt{n}} + \frac{m^{(2)}(0)}{2!}\left(\frac{t}{\sqrt{n}}\right)^2 + R_{n,t}$$

$$= 1 + E\left[\frac{X_1-\mu}{\sigma}\right]\frac{t}{\sqrt{n}} + E\left[\left(\frac{X_1-\mu}{\sigma}\right)^2\right]\frac{t^2}{2n} + R_{n,t}$$

$$= 1 + \frac{1}{\sigma}\left(E[X_1]-\mu\right)\frac{t}{\sqrt{n}} + \frac{1}{\sigma^2}E[(X_1-\mu)^2]\frac{t^2}{2n} + R_{n,t}$$

$$= 1 + \frac{t^2}{2n} + R_{n,t}$$

$$mgf_{\sqrt{n}(\bar{X}_n-\mu)/\sigma}(t) = \left(1 + \frac{t^2}{2n} + R_{n,t}\right)^n$$

$$\log mgf_{\sqrt{n}(\bar{X}_n-\mu)/\sigma}(t) = n\log\left(1 + \frac{t^2}{2n} + R_{n,t}\right)$$

$$= n\left(\frac{t^2}{2n} + r_{n,t}\right) \qquad \lim_{n\to\infty} r_{n,t} = 0$$

$$= \frac{t^2}{2} + n r_{n,t} \qquad \lim_{n\to\infty} n r_{n,t} = 0$$

$$\therefore \lim_{n\to\infty} mgf_{\sqrt{n}(\bar{X}-\mu)/\sigma}(t) = \exp(t^2/2)$$

$$\therefore \lim_{n\to\infty} P\left(\frac{(X_1+\cdots+X_n)/n-\mu}{\sigma/\sqrt{n}} \le x\right) = P(Z \le x)$$

$$Z \sim N(0,1)$$

$\square$

e.g. $X_1, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$

$E[X_1] = \lambda = \text{Var}(X_1)$ $\quad Z \sim N(0,1)$

$$\lim_{n\to\infty} P\left(\frac{(X_1+\cdots+X_n)/n - \lambda}{\sqrt{\lambda/n}} \le x \right) = P(Z \le x)$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Since, $\dfrac{(X_1+\cdots+X_n)/n - \lambda}{\sqrt{\lambda/n}} = \dfrac{Y_n/n - \lambda}{\sqrt{\lambda}/\sqrt{n}} = \sqrt{n} \cdot \dfrac{Y_n - n\lambda}{n\sqrt{\lambda}} = \dfrac{Y_n - n\lambda}{\sqrt{n\lambda}}$

and $Y_n := X_1 + \cdots + X_n \sim \text{Poisson}(n\lambda)$,

$$\lim_{n\to\infty} \sum_{k:\ \frac{k-n\lambda}{\sqrt{n\lambda}} \le x} \frac{e^{-n\lambda}(n\lambda)^k}{k!} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Since $P(a < Y_n \le b) = P\left(\dfrac{a-n\lambda}{\sqrt{n\lambda}} \le \dfrac{Y_n - n\lambda}{\sqrt{n\lambda}} \le \dfrac{b-n\lambda}{\sqrt{n\lambda}} \right)$,

$$\simeq \Phi\left(\frac{b-n\lambda}{\sqrt{n\lambda}}\right) - \Phi\left(\frac{a-n\lambda}{\sqrt{n\lambda}}\right)$$

Thm $X_1 = (X_{11}, \dots, X_{1k})', \dots, X_n = (X_{n1} \dots, X_{nk})' \overset{iid}{\sim} N(\mu, \Sigma)$

$\exists\, Var(X_1) = \Sigma \quad Z_1 = (Z_{11}, \dots, Z_{1k})' \sim N(0, \Sigma)$

$$\Rightarrow \lim_{n\to\infty} P(\sqrt{n}((X_{1j} + \dots + X_{nj})/n - \mu_j) \le x_j,\ j = 1, \dots, k)$$

$$= P(Z_{11} \le x_1, \dots, Z_{1k} \le x_k) \quad \forall x_j$$

proof is omitted.

e.g. $X_n = (X_{n1}, \dots, X_{nk})' \sim Multi(n, (p_1, \dots, p_k)')$

$X_n \overset{d}{\equiv} Z_1 + \dots + Z_n \qquad Z_i \overset{iid}{\sim} Multi(1, (p_1, \dots, p_k)')$, $Z_i = (Z_{i1}, \dots, Z_{ik})'$

$Var(Z_1) = p \qquad Var(Z_1) = diag(p_1, \dots, p_k) - pp^t$

Component wise notation →

$$\lim P(\sqrt{n}((Z_{1j} + \dots + Z_{nj})/n - p_j) \le x_j) = P(Z \le x)$$

Vector notation →

$$= \lim P(\underbrace{\sqrt{n}((Z_1 + \dots + Z_n)/n - p) \le x})$$

$$\sqrt{n}\left(\frac{X_n}{n} - p\right) \le x$$

$$= \frac{X_n - np}{\sqrt{n}}$$

where $Z \sim N(0, D(p) - pp^t)$

Rmk $D(P) - pp^t$: singular (not invertible)

proof)

$$\mathrm{diag}(P_1,\dots,P_n) - \begin{pmatrix} P_1P_1 & P_1P_2 & \cdots & P_1P_{n-1} & P_1P_n \\ P_2P_1 & P_2P_2 & \cdots & P_2P_{n-1} & P_2P_n \\ \vdots & & \ddots & \vdots & \vdots \\ P_{n-1}P_1 & P_{n-1}P_2 & \cdots & P_{n-1}P_{n-1} & P_{n-1}P_n \\ P_nP_1 & P_nP_2 & \cdots & P_nP_{n-1} & P_nP_n \end{pmatrix} = \underbrace{\begin{pmatrix} P_1(1-P_1) & -P_1P_2 & \cdots & -P_1P_{n-1} & -P_1P_n \\ -P_2P_1 & P_2(1-P_2) & \cdots & -P_2P_{n-1} & -P_2P_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -P_{n-1}P_1 & -P_{n-1}P_2 & \cdots & P_{n-1}(1-P_{n-1}) & -P_{n-1}P_n \\ -P_nP_1 & -P_nP_2 & \cdots & -P_nP_{n-1} & P_n(1-P_n) \end{pmatrix}}_{= A}$$

for each ith ($i \neq n$) row, the summation of the first $(n-1)$ entries

are $P_i(-P_1 - \cdots - P_{i-1} + 1 - P_i - P_{i+1} - \cdots - P_{n-1})$ $\because P_1 + \cdots + P_n = 1$

$= P_i(1 - P_1 - \cdots - P_{n-1}) = P_iP_n$.

$i = n$, $-P_n(P_1 + \cdots + P_{n-1}) = -P_n(1 - P_n)$

Thus $[A]^n = -\sum_{i=1}^{n-1} [A]^i$ where $[A]^i$ is the i^{th} column of A.

$\therefore \det A = \det(D(P) - pp^t) = 0$, i.e., $D(P) - pp^t$ is singular. $\square$

e.g. 5.2.1

A coin toss

head: we show a number $(0, 1+\frac{1}{n})$

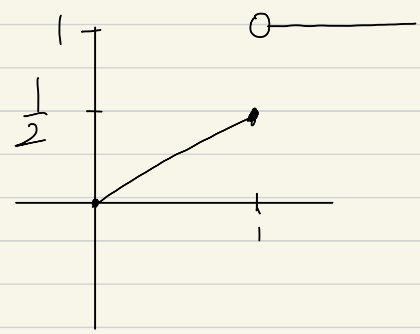
tail: $1+\frac{1}{n}$

Let X_n the number we observe after a coin toss.

$$\mathrm{cdf}_{X_n}(x) = P(X_n \le x) = \begin{cases} 0 & (x<0) \\ \frac{1}{2}\frac{x}{1+n^{-1}} & (0 \le x < 1+\frac{1}{n}) \\ 1 & x \ge 1+\frac{1}{n} \end{cases}$$

Let $G(x) := \lim_{n\to\infty} \mathrm{cdf}_{X_n}(x)$

$$= \begin{cases} 0 & (x<0) \\ \frac{1}{2}x & (0 \le x \le 1) \\ 1 & (x>1) \end{cases}$$



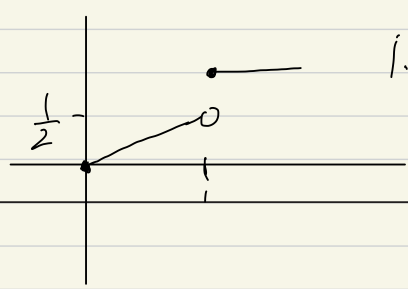
$G(x)$ is not right continuous.

So $G(x)$ is not cdf.

Suppose we perform another experiment.

Head: show the number $(0, 1)$

Tail: 1 Then $\mathrm{cdf}_X(x)$ is right continuous.



i.e., $\lim_{n\to\infty} \mathrm{cdf}_{X_n}(x) = \mathrm{cdf}_X(x)$

$\forall x \neq 1$

$\square$

Def Asymptotic Distribution

$X_n \xrightarrow[n\to\infty]{d} Z_1 \iff \lim_{n\to\infty} cdf_{X_n}(x) = cdf_{Z_1}(x)$ for every $x \in Conti(cdf_{Z_1})$

where $Conti(cdf_{Z_1})$ dentotes a set of continuos points of cdf_{Z_1}.

e.g. 5.2.2. $X_n \sim B(n, \frac{\lambda}{n})$, $X \sim Poisson(\lambda)$

$$P(X_n = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \xrightarrow[n\to\infty]{} \frac{e^{-\lambda}\lambda^k}{k!} = P(X=k)$$

$$\Rightarrow cdf_{X_n}(x) = \sum_{k:0\le k\le x} P(X_n = k) \xrightarrow[n\to\infty]{} \sum_{k:0\le k\le x} \frac{e^{-\lambda}\lambda^k}{k!} = cdf_X(x)$$ for all x.

$\because$ finite sum of $\lim_{n\to\infty} P(X_n=k) = P(X=k)$.

$\therefore B(n, \frac{\lambda}{n}) \xrightarrow[n\to\infty]{d} Poisson(\lambda)$ $\square$

$(X_n \xrightarrow[n\to\infty]{d} X)$

e.g. 5.23

$U_1, \dots, U_n \overset{iid}{\sim} U(0,1) \to U_{(1)} < \cdots < U_{(n)}$

Goal $n(1-U_{(n)}) \xrightarrow[n\to\infty]{d} \square$?

$$P(n(1-U_{(n)}) \le x) = P\left(U_{(n)} \ge 1-\frac{x}{n}\right)$$
$$= 1 - P\left(U_{(n)} \le 1-\frac{x}{n}\right)$$

$$P\left(U_{(n)} < 1-\frac{x}{n}\right) = P\left(U_1 < 1-\frac{x}{n}, \dots, U_n < 1-\frac{x}{n}\right)$$
$$= P\left(U_1 < 1-\frac{x}{n}\right)^n$$
$$= \begin{cases} 0 & (1-\frac{x}{n} < 0) \quad x > n \\ (1-\frac{x}{n})^n & (0 \le 1-\frac{x}{n} < 1) \quad 0 < x \le n \\ 1 & (1-\frac{x}{n} \ge 1) \quad x \le 0 \end{cases}$$

$$\lim_{n\to\infty} P\left(U_{(n)} < 1-\frac{x}{n}\right) = \begin{cases} e^{-x} & (x > 0) \\ 1 & (x \le 0) \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} P(n(1-U_{(n)}) \le x) = \begin{cases} 1-e^{-x} & (x > 0) \\ 0 & (x \le 0) \end{cases}$$

$$\therefore n(1-U_{(n)}) \xrightarrow[n\to\infty]{d} Z \sim Exp(1)$$

□

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$$

$$\lim_{n\to\infty} \left(1+\frac{x}{-n}\right)^{-n\cdot(-1)} = e^{-x}$$

Thm 5.2.1 $X_n \xrightarrow[n\to\infty]{d} X$, $P(X=c)=1 \iff \lim_{n\to\infty} P(|X_n-c|\geq\epsilon)=0$ for all $\epsilon>0$

proof) $|X_n-c|\geq\epsilon \iff X_n\geq c+\epsilon$ or $X_n\leq c-\epsilon$

$P(X=c)=1 \Rightarrow cdf_X(x)=\begin{cases}0 & (x<c)\\ 1 & (x\geq c)\end{cases}$

$\Rightarrow)$ $\therefore Conti(cdf_X)=\mathbb{R}\setminus\{c\}$

$\Rightarrow cdf_{X_n}(x)\xrightarrow[n\to\infty]{} cdf_X(x)$ for $x\neq c$

$\lim_{n\to\infty} P(X_n\leq c-\epsilon)=cdf_X(c-\epsilon)=0.$

$0\leq \lim_{n\to\infty} P(X_n\geq c+\epsilon)\leq \lim_{n\to\infty} P(X_n\geq c+\frac{\epsilon}{2})$

$= 1-cdf_X(c+\frac{\epsilon}{2})=0$

$\therefore \lim_{n\to\infty} P(|X_n-c|\geq\epsilon)=0,\ \forall\epsilon>0.$

$\Leftarrow)$ $\epsilon>0$,

$0< cdf_{X_n}(c-\epsilon)=P(X_n\leq c-\epsilon)\leq P(|X_n-c|\geq\epsilon)$ $\to 0$ as $n\to\infty$

$0<1-cdf_{X_n}(c+\epsilon)=P(X_n>c+\epsilon)\leq P(|X_n-c|\geq\epsilon)$

$\to 0$ as $n\to\infty$.

$\therefore cdf_{X_n}(c-\epsilon)\to 0$ as $n\to\infty$

$cdf_{X_n}(c+\epsilon)\to 1$ as $n\to\infty$

$\therefore cdf_{X_n}(x)\to cdf_X(x)$ $\forall x\neq c$ $\therefore X_n\xrightarrow[n\to\infty]{d} X$ $P(X=c)=1$

Convergence in Probability

Def $\operatorname*{plim}_{n\to\infty} X_n = c \iff \lim_{n\to\infty} P(|X_n - c| \geq \epsilon) = 0$ for all $\epsilon > 0$.

Thm (law of large numbers) $X_1, \dots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$ $E[X_1] \in \mathbb{R}$

$$\Rightarrow \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = E[X_1]$$

proof) $Var(X_1) < +\infty$.

$\overline{X_n} := (X_1 + \cdots + X_n)/n$

By Chebyshev Inequality, for all $\epsilon > 0$

$$0 \leq P(|\overline{X_n} - E[\overline{X_n}]| > \epsilon) = P(|\overline{X_n} - E[X_1]| > \epsilon)$$

$$\leq Var(\overline{X_n})/\epsilon^2$$

$$= Var(X_1)/n\epsilon^2$$

$\because E[\overline{X_n}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = E[X_1]$

$\because Var(\overline{X_n}) = Var\left(\frac{1}{n}(X_1 + \cdots + X_n)\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n} Var(X_1)$

$$\therefore \lim_{n\to\infty} P(|\overline{X_n} - E[X_1]| > \epsilon) = 0$$

$$\therefore \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = E[X_1]$$

$\square$

Rmk $X_1, \dots, X_n$

$\rightarrow$ (relative frequency of A) $= \frac{1}{n}\sum_{i=1}^{n} I_A(X_i)$ "statistical probability"

By the law of large numbers, $\operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} I_A(X_i) = E[I_A(X_1)]$

"mathematical prob." $= P(X_1 \in A)$

20.08.25

review

Convergence in probability (univariate)

⓪ $\operatorname*{plim}_{n\to\infty} X_n = C \overset{\text{def}}{\iff} \lim_{n\to\infty} P(|X_n - C| \geq \epsilon) = 0, \ \forall \epsilon > 0.$

$(X_n \xrightarrow[n\to\infty]{P} C)$

⓪ $X_n \xrightarrow[n\to\infty]{d} X \overset{\text{def}}{\iff} \lim_{n\to\infty} cdf_{X_n}(x) = cdf_X(x),$

$\forall x \in conti(cdf_X)$

⓪ $X_n \xrightarrow[n\to\infty]{d} X, \ P(X=C)=1 \iff \operatorname*{plim}_{n\to\infty} X_n = C$

Def $X_n = (X_{n1}, \dots, X_{nk})' \quad C = (C_1, \dots, C_k)'$

$\operatorname*{plim}_{n\to\infty} X_n = C \iff \lim_{n\to\infty} P(\|X_n - C\|_2 \geq \epsilon) = 0, \ \forall \epsilon > 0.$

$(X_n \xrightarrow[n\to\infty]{P} C)$

Thm 5.2.3 $\operatorname*{plim}_{n\to\infty} X_n = C$

$\overset{(a)}{\iff} \lim_{n\to\infty} P(\max_{1\leq i\leq k} |X_{ni} - C_i| \geq \epsilon) = 0 \quad \forall \epsilon > 0$

$\overset{(b)}{\iff} \operatorname*{plim}_{n\to\infty} X_{n1} = C_1, \dots, \operatorname*{plim}_{n\to\infty} X_{nk} = C_k$

proof) (a) $\max_{1\leq i\leq k} |X_{ni} - C_i| \leq \|X_n - C\|_2 \leq k \cdot \max_{1\leq i\leq k} |X_{ni} - C_i|$

$\Rightarrow$

$\epsilon > 0$,

$$\left(\max_{1\le i\le k}|X_{ni}-C_i|\ge\epsilon\right)\subseteq\left(\|X_n-C\|\ge\epsilon\right)\subseteq\left(k\cdot\max_{1\le i\le k}|X_{ni}-C_i|\ge\epsilon\right)$$

$$\Rightarrow P\left(\max_{1\le i\le k}|X_n-C_i|\ge\epsilon\right)\le P\left(\|X_n-C\|\ge\epsilon\right)\le P\left(\max_{1\le i\le k}|X_{ni}-C_i|\ge\epsilon/k\right)$$

(b) $|X_{ni}-C_i| \le \max_{1\le i\le k}|X_{ni}-C_i|$

$\epsilon > 0$, $\quad (|X_{ni}-C_i|\ge\epsilon)\subseteq\left(\max_{1\le i\le k}|X_{ni}-C_i|\ge\epsilon\right)=\bigcup_{i=1}^{k}(|X_{ni}-C_i|\ge\epsilon)$

$$P(|X_{ni}-C_i|\ge\epsilon)\le P\left(\max_{1\le i\le k}|X_{ni}-C_i|\ge\epsilon\right)=P\left(\bigcup_{i=1}^{k}|X_{ni}-C_i|\ge\epsilon\right)$$

$$\le\sum_{i=1}^{k}P(|X_{ni}-C_i|\ge\epsilon)$$

□

Thm. 5.2.4 (Law of Large Number)

$X_1=(X_{11},\dots,X_{1k})',\dots,X_n=(X_{n1},\dots,X_{nk})'$: iid.

$E[|X_{1i}|]<+\infty$, $\forall i$

$$\Rightarrow \operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i=E[X_1]=(E[X_{11}],\dots,E[X_{1k}])'$$

e.g. 5.2.4

$X_1,\dots,X_n$: random sample.

$\hat{m}_r := \frac{1}{n}\sum_{i=1}^{n}X_i^r$: r^{th} sample moment.

By Lyapounov theorem $\left(E[|X|^s]<\infty,\ (E[|X|^r])^{1/r}\le(E[|X|^s])^{1/s} \text{ for } 0<r<s\right)$,

if $E[|X_1|^k]<\infty$, $E[|X_1|^r]<\infty$ for $0<r<k$.

By the law of large number, $\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(X_i,\dots,X_i^k)'=(E[X_1],\dots,E[X_1^k])'$

$$\therefore (\hat{m}_1,\dots,\hat{m}_k)'\xrightarrow[n\to\infty]{d}(E[X_1],\dots,E[X_1^k])'$$

□

Thm 5.2.5. g: continuous at C

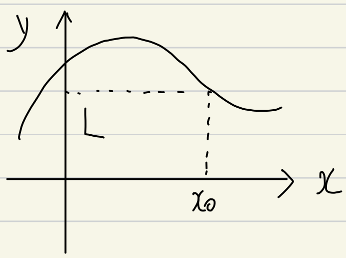
$$\underset{n\to\infty}{p\lim} X_n = C \Rightarrow \underset{n\to\infty}{p\lim}\, g(X_n) = g(C)$$

review f: continuous at x_0

x의 후보군에 x_0도 있다.

$\lim_{x\to x_0} f(x) = f(x_0)$

$$\Leftrightarrow \forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$



$$\lim_{x\to x_0} f(x) = L$$

$$\Leftrightarrow \forall \epsilon > 0,\ \exists \delta > 0 : 0 < |x - x_0| < \delta$$

$$\Rightarrow |f(x) - L| < \epsilon$$

$$\Updownarrow$$

$$L - \epsilon < f(x) < L + \epsilon$$

$$f(x) \in (L - \epsilon, L + \epsilon)$$

proof) Since g: continuous at c,

$$\forall \epsilon > 0,\ \exists \delta > 0 : \|x - c\| < \delta \Rightarrow |g(x) - g(c)| < \epsilon.$$

$$(\|X_n - C\| < \delta) \subseteq (|g(X_n) - g(C)| < \epsilon)$$

$$\Rightarrow (\|X_n - C\| \geq \delta) \supseteq (|g(X_n) - g(C)| \geq \epsilon)$$

$$P(\|X_n - C\| \geq \delta) \geq P(|g(X_n) - g(C)| \geq \epsilon)$$

Letting $n \to \infty$, since $X_n \xrightarrow[n\to\infty]{P} C$, we have

$$\lim_{n\to\infty} P(|g(X_n) - g(C)| \geq \epsilon) = 0.$$

$$\therefore \underset{n\to\infty}{p\lim}\, g(X_n) = g(C) \qquad \square$$

Thm 5.2.6. $\operatorname{plim}_{n\to\infty} X_n = a$, $\operatorname{plim}_{n\to\infty} Y_n = b$.

(a) $\operatorname{plim}_{n\to\infty}(X_n \pm Y_n) = \operatorname{plim}_{n\to\infty} X_n \pm \operatorname{plim}_{n\to\infty} Y_n = a \pm b$

(b) $\operatorname{plim}_{n\to\infty}(X_n \times Y_n) = \operatorname{plim}_{n\to\infty} X_n \cdot \operatorname{plim}_{n\to\infty} Y_n = ab$

(c) $\operatorname{plim}_{n\to\infty}(X_n \div Y_n) = \operatorname{plim}_{n\to\infty} X_n \div \operatorname{plim}_{n\to\infty} Y_n = a/b \quad (b \neq 0)$

proof) Since $\lim_{n\to\infty} P(\|X_n - C\|_2 \geq \epsilon) = 0 \Leftrightarrow \operatorname{plim}_{n\to\infty} X_{n1} = C_1, \dots, \operatorname{plim}_{n\to\infty} X_{nk} = C_k$,

$\operatorname{plim}_{n\to\infty} (X_n, Y_n)' = (a, b)'$.

Let $g_1(x,y) := x+y$, $g_2(x,y) := x-y$, $g_3(x,y) := xy$, $g_4(x,y) := \frac{x}{y}$.

Since $g_1, \dots, g_4$ are continuous function, $\operatorname{plim}_{n\to\infty} g_i(x,y) = g_i(a,b)$

$\therefore \operatorname{plim}_{n\to\infty}(X_n + Y_n) = g_1(a,b) = a+b = \operatorname{plim}_{n\to\infty} X_n + \operatorname{plim}_{n\to\infty} Y_n$

$\operatorname{plim}_{n\to\infty}(X_n - Y_n) = g_2(a,b) = a-b = \operatorname{plim}_{n\to\infty} X_n - \operatorname{plim}_{n\to\infty} Y_n$

$\operatorname{plim}_{n\to\infty}(X_n \times Y_n) = g_3(a,b) = ab = \operatorname{plim}_{n\to\infty} X_n \times \operatorname{plim}_{n\to\infty} Y_n$.

$\operatorname{plim}_{n\to\infty}(X_n \div Y_n) = g_4(a,b) = a/b = \operatorname{plim}_{n\to\infty} X_n \div \operatorname{plim}_{n\to\infty} Y_n$ $\square$

e.g. 5.2.5.

$X_1, \dots, X_n$: random sample $(n \geq 2)$ $\quad X_1, \dots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \quad \text{where } \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S_n^2 = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{2}{n}\sum_{i=1}^{n} X_i\overline{X} + \frac{1}{n}\sum_{i=1}^{n}\overline{X}^2\right)$$

$$= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\overline{X}^2 + \overline{X}^2\right)$$

$$= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2\right)$$

By the law of large number

$$\operatorname*{plim}_{n\to\infty} S_n^2 = 1 \cdot E[X_1^2] - E[X_1]^2 = \sigma^2$$

Since $x \mapsto \sqrt{x}$ is continuous, $\operatorname*{plim}_{n\to\infty} S_n = \operatorname*{plim}_{n\to\infty}\sqrt{S_n^2} = \sqrt{\sigma^2} = \sigma$

$\therefore \operatorname*{plim}_{n\to\infty} S_n = \sigma$.

(S_n : unbiased estimator).

Thm 5.2.7. $\lim_{n\to\infty} Var(X_n) = 0 \quad \lim_{n\to\infty} E[X_n] = a \Rightarrow \operatorname{plim}_{n\to\infty} X_n = a$

proof) Markov inequality,

$P(|X_n - a| \geq \epsilon) \leq E[(X_n-a)^2]/\epsilon^2$

$E[(X_n-a)^2] = E[X_n^2 - 2aX_n + a^2]$

$= E[X_n^2] - 2a E[X_n] + a^2$

$= E[X_n^2] - (E[X_n])^2 + (E[X_n])^2 - 2aE[X_n] + a^2$

$= Var(X_n) + (E[X_n] - a)^2$

$0 \leq \lim P(|X_n - a| \geq \epsilon) \leq \frac{1}{\epsilon^2} \lim_{n\to\infty} E[(X_n-a)^2] = 0$

$\therefore \operatorname{plim}_{n\to\infty} X_n = a$

e.g. 5.2.6. $U_1, \ldots, U_n \overset{iid}{\sim} U(0,1) \quad U_{(1)} < \cdots < U_{(n)}$

$\frac{n!}{(n-1)!}$ $E[U_{(n)}]$, $Var(U_{(n)})$ $\quad pdf_{U_{(n)}} = n x^{n-1} I_{(0,1)}$

$\frac{n!}{(n-1)!} x^n$ $\quad \int_0^1 n x^n dx = \frac{n}{n+1} \quad \lim E[U_{(n)}] = 1$

$Var(U_{(n)}) = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 \quad \lim_{n\to\infty} Var(U_{(n)}) = 0$ $\square$

e.g. 5.2.X

X: continuous r.v.

$F(x) = cdf_X(x)$: strictly increasing function.

$F^{-1}(\alpha)$: alpha quantile.

$X_1, \dots, X_n$: r.s. $\quad X_{(1)} < \cdots < X_{(n)}$: ordered statistic

$r_n \sim \alpha n \quad$ i.e., $\lim_{n \to \infty} \frac{r_n}{n} = \alpha$

$X_{(r_n)}$: sample quantile.

$\boxed{\text{WTS}}$ $\quad \operatorname*{plim}_{n \to \infty} X_{(r_n)} = F^{-1}(\alpha)$

$$X_{(r_n)} \overset{d}{=} h\Big(\underbrace{\frac{1}{n} Z_1 + \cdots + \frac{1}{n-r_n+1} Z_{r_n}}_{= Y_n}\Big), \quad Z_i \overset{iid}{\sim} Exp(1)$$

$$h(y) := F^{-1}(1 - e^{-y})$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} f\Big(a + \frac{b-a}{n} k\Big) \frac{b-a}{n} = \int_a^b f(x)\, dx$$

Since $X \sim Exp(1) \Rightarrow E[X] = 1 = Var(X)$

$$E[Y_n] = \frac{1}{n} + \cdots + \frac{1}{n-r_n+1} = \frac{1}{n} \sum_{k=0}^{r_n-1} \frac{1}{1-k/n} \approx \int_0^\alpha \frac{1}{1-x} dx$$

$$Var(Y_n) = \frac{1}{n^2} + \cdots + \frac{1}{(n-r_n+1)^2} = \frac{1}{n^2} \sum_{k=0}^{r_n-1} \frac{1}{(1-k/n)^2} \approx \frac{1}{n} \int_0^\alpha \frac{dx}{(1-x)^2}$$

$$\because \quad \frac{1}{n} \sum_{k=0}^{r_n-1} \frac{1}{1-k/n} = \frac{r_n}{n} \sum_{k=0}^{r_n-1} \frac{1}{1 - \frac{k \cdot r_n}{r_n \cdot n}} \frac{1}{r_n}$$

$$\approx \sum_{k=0}^{r_n-1} \frac{1}{1 - \frac{\alpha k}{r_n}} \frac{\alpha}{r_n} \approx \int_0^\alpha \frac{dx}{1-x}.$$

$$X_n \xrightarrow[n\to\infty]{d} X \ \& \ P(X=c)=1 \iff \operatorname*{plim}_{n\to\infty} X_n = c$$

$$\lim_{n\to\infty} E[Y_n] = -\log(1-\alpha)$$

$$\frac{1}{n}\int_0^{\alpha} \frac{1}{(1-x)^2}dx = \frac{1}{n}\int_1^{1-\alpha} -\frac{1}{t^2}dt$$

$1-x=t$

$-dx=dt$

$$= -\frac{1}{n}\left[-t^{-1}\right]_1^{1-\alpha}$$

$$= -\frac{1}{n}\left(1-\frac{1}{1-\alpha}\right) = -\frac{1}{n}\times\frac{-\alpha}{1-\alpha} = \frac{1}{n}\times\frac{\alpha}{1-\alpha}$$

$$\therefore \lim_{n\to\infty} Var(Y_n) = 0.$$

$$\therefore \operatorname{plim} Y_n = -\log(1-\alpha).$$

$$\Rightarrow \operatorname*{plim}_{n\to\infty} h(Y_n) = F^{-1}\left(1-e^{+\log(1-\alpha)}\right) = F^{-1}(\alpha)$$

($\because$ We assume h: continuous)

Q. $X_{(rn)} \xrightarrow[n\to\infty]{d} F^{-1}(\alpha)$?

$$\Rightarrow X_{(rn)} \overset{d}{=} h(Y_n), \quad X_{(rn)} \xrightarrow[n\to\infty]{d} X \ \& \ P(X=F^{-1}(\alpha))=1$$

$$\therefore \operatorname*{plim}_{n\to\infty} X_{(rn)} = F^{-1}(\alpha).$$

$$\Rightarrow h(Y_n) \xrightarrow[n\to\infty]{d} F^{-1}(\alpha)$$

$$\begin{array}{c} \parallel\!\!| \\ X_{(r_n)} \end{array}$$

$$\Rightarrow X_{(r_n)} \xrightarrow[n\to\infty]{d} F^{-1}(\alpha)$$

$$\therefore \underset{n\to\infty}{\text{plim}}\, X_{(r_n)} = F^{-1}(\alpha) \quad \square$$

$\boxed{\text{limsup, liminf}}$

20.08.01. Assume cdf_Z is continuous

Slutsky Theorem.

$X_n \xrightarrow[n\to\infty]{d} Z,\quad \underset{n\to\infty}{\mathrm{plim}}\, Y_n = c$

(a) $X_n + Y_n \xrightarrow[n\to\infty]{d} Z + c$

(b) $X_n - Y_n \xrightarrow[n\to\infty]{d} Z - c$

(c) $Y_n X_n \xrightarrow[n\to\infty]{d} cZ$

(d) $X_n / Y_n \xrightarrow[n\to\infty]{d} Z/c \quad (c \neq 0)$

$\alpha < \beta + \epsilon \ \ \forall \epsilon > 0$
$\Rightarrow \alpha \le \beta$

proof) $\forall \epsilon > 0$

(a) $P(X_n + Y_n \le z) = P(X_n + Y_n \le z,\ |Y_n - c| \ge \epsilon)$
$\qquad + P(X_n + Y_n \le z,\ |Y_n - c| < \epsilon)$

n에 대한 sequence로 생각.
이 수열의 limit가 존재하는가?

$X_n \le z - Y_n$
$c - \epsilon < Y_n < c + \epsilon$
$\Rightarrow X_n \le z - Y_n < z - c + \epsilon$

$\le P(|Y_n - c| \ge \epsilon) + P(X_n < z - c + \epsilon)$

$\Rightarrow \limsup_{n\to\infty} P(X_n + Y_n \le z) \le P(Z < z - c + \epsilon) \quad \forall \epsilon > 0$

Since cdf is right continuous, $\downarrow \epsilon \to 0^+$

$\limsup_{n\to\infty} P(X_n + Y_n \le z) \le P(Z \le z - c)$
$\qquad \le P(Z + c \le z)$

$$P(X_n+Y_n>z) = P(X_n+Y_n>z,\ |Y_n-c|<\epsilon)$$
$$+P(X_n+Y_n>z,\ |Y_n-c|\geq\epsilon)$$
$$\leq P(|Y_n-c|\geq\epsilon)+P(X_n>z-c-\epsilon)$$
$$\forall\epsilon>0.$$

$X_n>z-Y_n>z-c-\epsilon$

$c-\epsilon<Y_n<c+\epsilon$

$\Rightarrow \limsup_{n\to\infty} P(X_n+Y_n>z)\leq P(Z>z-c-\epsilon)\ \forall\epsilon>0$

$\Rightarrow \limsup_{n\to\infty}(1-P(X_n+Y_n\leq z))\leq 1-P(Z\leq z-c-\epsilon)$

$\Rightarrow \liminf_{n\to\infty} P(X_n+Y_n\leq z)\geq P(Z\leq z-c-\epsilon)$

Since we assume cdf_Z: continuous

$$\liminf_{n\to\infty} P(X_n+Y_n\leq z)\geq P(Z\leq z-c)$$

Since $\liminf_{n\to\infty} P(X_n+Y_n\leq z)\leq \limsup_{n\to\infty}(X_n+Y_n\leq z)$

$\leq P(Z+c\leq z)\leq \liminf(X_n+Y_n\leq z)$,

$\lim_{n\to\infty} P(X_n+Y_n\leq z) = P(Z+c\leq z)$

$\therefore X_n+Y_n \xrightarrow[n\to\infty]{d} Z+c$

(b) $\operatorname*{plim}_{n\to\infty} Y_n = C \iff \lim_{n\to\infty} P(|Y_n - C| \geq \epsilon) = 0 \quad \forall \epsilon > 0$

$\iff \lim_{n\to\infty} P(|-Y_n - (-C)| \geq \epsilon) = 0 \quad \forall \epsilon > 0$

$\iff \operatorname*{plim}_{n\to\infty} (-Y_n) = -C$

$\therefore X_n - Y_n \xrightarrow[n\to\infty]{d} Z - C$

(c) Suppose $C = 0$.

$\forall \epsilon, k > 0$

$k|Y_n| \geq |Y_n X_n| \geq \epsilon$

$$P(|Y_n X_n| \geq \epsilon) = P(|Y_n X_n| \geq \epsilon,\ |X_n| > k) + P(|Y_n X_n| \geq \epsilon,\ |X_n| \leq k)$$

$$\leq P(|Y_n X_n| \geq \epsilon,\ |X_n| > k) + P(|Y_n| \geq \epsilon/k)$$

$\because P(A \cap B) \leq P(A) + P(B)$

$$\leq P(|Y_n X_n| \geq \epsilon) + P(X_n > k) + P(X_n < -k) + P(|Y_n| \geq \epsilon/k)$$

$\because cdf_Z$: cont.

$$0 \leq \limsup_{n\to\infty} P(|Y_n X_n| \geq \epsilon) \leq \underbrace{P(Z > k) + P(Z < -k)}_{\xrightarrow[k\to\infty]{} 0} \quad \forall k > 0$$

$\therefore \lim_{n\to\infty} P(|Y_n X_n| \geq \epsilon) = 0$, i.e., $\operatorname*{plim}_{n\to\infty} |Y_n X_n| = 0$.

For a general c, $\operatorname*{plim}_{n\to\infty}(Y_n - c) = 0$, $X_n \xrightarrow[n\to\infty]{d} Z_1$

$\left(\because \lim_{n\to\infty} P(|Y_n - c - 0| \geq \epsilon) = 0\right)$

By the above proof, $\operatorname*{plim}_{n\to\infty}(Y_n - c)X_n = 0$

$\therefore Y_nX_n = (Y_n - c)X_n + cX_n \xrightarrow[n\to\infty]{d} cZ_1$ by (a)

$\because X_n \xrightarrow[n\to\infty]{d} Z_1$

$\Leftrightarrow \lim_{n\to\infty} P(X_n \leq z) = P(Z_1 \leq z)$

$\Leftrightarrow \lim_{n\to\infty} P(X_n \leq z/c) = P(Z_1 \leq z/c)$ $(c \neq 0)$

$\Leftrightarrow \lim_{n\to\infty} P(cX_n \leq z) = P(cZ_1 \leq z)$

(d) $\operatorname*{plim}_{n\to\infty} Y_n = c$ $(c \neq 0)$

Since $\operatorname*{plim}_{n\to\infty} 1 = 1$, $\operatorname*{plim}_{n\to\infty} 1/Y_n = 1/c$ by thm 5.2.6. (d)

$\therefore X_n/Y_n \xrightarrow[n\to\infty]{d} Z_1/c$ $(c \neq 0)$ $\square$

e.g. 5.3.1

$X_1, \dots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$ $\quad \overline{X}_n := \sum_{i=1}^{n} X_i / n$, $S_n := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$

$\frac{\overline{X} - \mu}{S_n/\sqrt{n}}$: studentized sample mean.

By central limit theorem, $\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow[n\to\infty]{d} Z$, $Z \sim N(0,1)$

By law of large number $\operatorname{plim}_{n\to\infty} S_n = \sigma$.

$\Big(\because$ $S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$

$= \frac{n}{n-1} \left(\frac{1}{n} \sum_{i=1}^{n} X_i^2 - \frac{1}{n} 2 \sum_{i=1}^{n} X_i \overline{X} + \frac{1}{n} \sum_{i=1}^{n} \overline{X}_n^2 \right)$

$= \frac{n}{n-1} \left(\frac{1}{n} \sum_{i=1}^{n} X_i^2 - (\overline{X}_n)^2 \right)$

$\operatorname{plim}_{n\to\infty} S_n^2 = E[X_1^2] - E[X_1]^2 = \sigma^2$

Since $x \mapsto \sqrt{x}$: continuous, $\operatorname{plim}_{n\to\infty} S_n = \sigma$ $\Big)$

$\therefore$ $\frac{\overline{X}_n - \mu}{S_n/\sqrt{n}} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \Big/ \frac{S_n}{\sigma} \xrightarrow[n\to\infty]{d} Z/1 = Z$

by Slutsky theorem.

In particular, $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}} \sim t(n-1)$

$\therefore$ $t(n-1) \underset{n\to\infty}{\approx} N(0,1)$ $\square$

$X_1,\dots,X_n \overset{iid}{\sim} (\mu,\sigma^2)$ $\quad\bar{X}_n := \sum_{i=1}^{n} X_i/n$

e.g. 5.3.2 | $S_n^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2$

Asymptotic distribution of S_n^2 ?

$\sqrt{n}(S_n^2-\sigma^2)$

$=\sqrt{n}(n\times\frac{1}{n}S_n^2-\sigma^2)$

$=\sqrt{n}\left(\frac{n-1}{n}S_n^2-\sigma^2\right)+\frac{1}{\sqrt{n}}S_n^2$

$=\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2-\sigma^2\right)+\frac{S_n^2}{\sqrt{n}}$

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu-(\bar{X}_n-\mu))^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-2\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)(\bar{X}_n-\mu)+\frac{1}{n}\sum_{i=1}^{n}(\bar{X}_n-\mu)^2$$

$$=\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-2(\bar{X}_n-\mu)^2+(\bar{X}_n-\mu)^2$$

$$=\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-(\bar{X}_n-\mu)^2$$

$\Rightarrow \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)-\sigma^2\right)+\frac{S_n^2}{\sqrt{n}}$

$=\sqrt{n}\left(\boxed{\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\sigma^2}-\boxed{(\bar{X}_n-\mu)^2}\right)+\frac{S_n^2}{\sqrt{n}}$

The first term | Let $Y_i := (X_i-\mu)^2$. Then $Y_1,\dots,Y_n$: iid.

By CLT, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}Y_i-\underbrace{E[Y_1]}_{=\sigma^2}\right)\xrightarrow[n\to\infty]{d} W\sim N(0, Var(Y_1))$

$E[Y_1]=E[(X_1-\mu)^2]=\sigma^2$

$Var(Y_1)=E[Y_1^2]-E[Y_1]^2=E[(X_1-\mu)^4]-\sigma^4$

$\Leftrightarrow \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\sigma^2\right)\xrightarrow[n\to\infty]{d} W\sim N(0, E[(X_1-\mu)^4]-\sigma^4)$

The second term. | $\sqrt{n}(\bar{X}_n-\mu)\xrightarrow[n\to\infty]{d} Z\sim N(0,\sigma^2)$ by CLT

$\left(\because\ E[\bar{X}_n]=\frac{1}{n}\sum_{i=1}^{n}E[X_i]=\frac{1}{n}n\mu=\mu\right)$

& $\operatorname{plim}_{n\to\infty}(\bar{X}_n-\mu)=0$ by LLN.

By slutsky thm, $\sqrt{n}(\bar{X}_n-\mu)^2\xrightarrow[n\to\infty]{P} 0$.

Combining these two, $\sqrt{n}(S_n^2-\sigma^2)=\underbrace{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\sigma^2\right)}_{\text{분포수렴}}-\underbrace{\sqrt{n}(\bar{X}_n-\sigma^2)}_{\text{확률수렴}}+S_n^2/\sqrt{n}$

$\xrightarrow[n\to\infty]{d} N(0, E[(X_1-\mu)^4]-\sigma^4)$ □

by slutsky theorem

Multivariate.

$X_n = (X_{n1}, \dots, X_{nk})' \quad n = 1, 2, \dots$

$Z_1 = (Z_{11}, \dots, Z_{1k})'$

$$X_n \xrightarrow[n\to\infty]{d} Z_1 \iff \lim_{n\to\infty} P(X_{n1} \le x_1, \dots, X_{nk} \le x_k) = P(Z_{11} \le x_1, \dots, Z_{1k} \le x_k)$$

for all $(x_1, \dots, x_k)' \in \text{Conti}(\text{cdf}_{Z_1})$

Thm $X_n \xrightarrow[n\to\infty]{d} Z_1$, g: conti. $\Rightarrow g(X_n) \xrightarrow[n\to\infty]{d} g(Z_1)$

e.g. 5.3.3. $X_n \sim B(n, p) \quad 0 < p < 1$

$$\Rightarrow \frac{X_n - np}{\sqrt{np(1-p)}} \xrightarrow[n\to\infty]{d} Z_1 \sim N(0,1) \quad \text{by CLT.}$$

$$\Rightarrow \left(\frac{X_n - np}{\sqrt{np(1-p)}}\right)^2 \xrightarrow[n\to\infty]{d} Z_1^2 \overset{d}{\equiv} W \sim \chi^2(1)$$

$X_{n1} := X_n, \quad X_{n2} := n - X_n \quad p_1 := p \quad p_2 := 1 - p.$

$\Rightarrow (X_{n1}, X_{n2}) \sim \text{Multi}(n, (p_1, p_2)')$

$$\frac{(X_{n1} - np_1)^2}{np_1} + \frac{(X_{n2} - np_2)^2}{np_2} = \frac{(X_n - np)^2}{np} + \frac{(\overbrace{n - X_n - n(1-p)}^{(X_n - np)^2})^2}{n(1-p)}$$

$$\therefore \sum_{i=1}^{2} \frac{(X_{ni} - np_i)^2}{np_i} \xrightarrow[n\to\infty]{d} W \sim \chi^2(1)$$

$$= \frac{(X_n - np)^2}{np(1-p)}$$

$\square$

Thm 5.3.3.

$$\sqrt{n}(X_n - \theta) \xrightarrow[n\to\infty]{d} Z, \quad g \in C^1$$

$$\Rightarrow \sqrt{n}(g(X_n) - g(\theta)) \xrightarrow[n\to\infty]{d} \dot{g}(\theta)^t Z \qquad \dot{g}(\theta) := \nabla g(\theta)$$

proof)

$$g(x) = g(\theta) + (\dot{g}(\theta) + r(x))(x-\theta) \qquad r(x) \xrightarrow{x\to\theta} 0$$

$\lim_{x\to a} f(x) = L$

$\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0$

$: 0 < |x-a| < \delta$

$\Rightarrow |f(x) - L| < \epsilon$

For any $\epsilon > 0$, there is a $\delta > 0$ s.t

$$0 < |x-\theta| < \delta \Rightarrow |r(x)| < \epsilon \qquad (\because g \in C^1)$$

$$\Rightarrow (|X_n - \theta| < \delta) \subseteq (|r(X_n)| < \epsilon)$$

$$\Rightarrow (|X_n - \theta| \geq \delta) \supseteq (|r(X_n)| \geq \epsilon)$$

$$\therefore P(|X_n - \theta| \geq \delta) \geq P(|r(X_n)| \geq \epsilon)$$

Since $\sqrt{n}(X_n - \theta) \xrightarrow[n\to\infty]{d} Z$,

$$X_n - \theta = \frac{1}{\sqrt{n}} \times \sqrt{n}(X_n - \theta) \xrightarrow[n\to\infty]{P} 0$$ by Slutsky's theorem.

$$\therefore \lim_{n\to\infty} P(|r(X_n)| \geq \epsilon) = 0$$

$$\therefore \sqrt{n}(g(X_n) - \dot{g}(\theta))$$

$$= (\dot{g}(\theta) + r(X_n)) \cdot \sqrt{n}(X_n - \theta) \xrightarrow[n\to\infty]{d} \dot{g}(\theta) Z$$

by Slutsky's theorem $\square$

e.g. 5.3.5.

$\sqrt{n}(S_n^2 - \sigma^2) \xrightarrow[n\to\infty]{d} W \sim N(0, (\ell_4+2)\sigma^4)$

$g(x) := \sqrt{x} \in C^1 \Rightarrow g'(x) = \frac{1}{2\sqrt{x}}$

So, $\sqrt{n}(S_n - \sigma) \xrightarrow[n\to\infty]{d} \frac{1}{2\sqrt{\sigma^2}} W \stackrel{d}{\equiv} N(0, (\ell_4+2)\sigma^2/4)$

$$P(Z \le z-c) \le P(Z < z-c+\epsilon)$$

---

cdf : 오른쪽 연속성.

$$\Rightarrow \limsup_{n\to\infty} P(X_n+Y_n > z) \le P(Z > z-c-\epsilon)$$

$$\shortparallel$$

$$\limsup_{n\to\infty} (1-P(X_n+Y_n \le z)) \le 1-P(Z \le z-c-\epsilon)$$

$$\Rightarrow \liminf_{n\to\infty} P(X_n+Y_n \le z) \ge P(Z \le z-c-\epsilon) \quad \forall \epsilon>0.$$

$cdf_Z(\alpha-\epsilon)$

$cdf_Z(\alpha)$

$\downarrow$ cdf_Z : 연속.

$$\Rightarrow \liminf_{n\to\infty} P(X_n+Y_n \le z) \ge P(Z \le z-c)$$

$$\shortparallel$$

$$P(Z+c \le z)$$

$$\therefore \lim_{n\to\infty} P(X_n+Y_n \le z) = P(Z+c \le z)$$

(limsup과 liminf이 같으므로 $\lim P(X_n+Y_n \le z)$ 존재)

$$\therefore X_n+Y_n \xrightarrow[n\to\infty]{d} Z+c.$$

(b) $\operatorname{plim}_{n\to\infty} Y_n = c \Rightarrow \operatorname{plim}(-Y_n) = -c$

$$\Longleftrightarrow P(|Y_n-c| \ge \epsilon) = 0 \quad \forall \epsilon>0$$

$$\Longleftrightarrow P(|-Y_n-(-c)| \ge \epsilon) = 0. \quad \forall \epsilon>0.$$

$$\Longleftrightarrow \operatorname{plim}(-Y_n) = -c$$

(c) Assume $c=0$.

$\epsilon>0$ $\qquad \geq |Y_nX_n| \geq \epsilon$

$$P(|Y_nX_n|\geq\epsilon) = P(|Y_nX_n|\geq\epsilon,\ |X_n|>k) \qquad k>0.$$
$$+P(|Y_nX_n|\geq\epsilon,\ |X_n|\leq k)$$

$$\leq P(|Y_n|\geq \epsilon/k) + P(X_n>k) + P(X_n<-k)$$

$$\Rightarrow 0\leq \limsup_{n\to\infty} P(|Y_nX_n|\geq\epsilon) \leq P(Z_1>k) + P(Z_1<-k) \xrightarrow[k\to\infty]{} 0$$ cdf Z_1: cont.

$\therefore \lim_{n\to\infty} P(|Y_nX_n|\geq\epsilon)=0$

$\therefore \operatorname{plim}_{n\to\infty} Y_nX_n=0$.

For a general c, $\operatorname{plim}_{n\to\infty}(Y_n-c)=0$ $\quad X_n \xrightarrow[n\to\infty]{d} Z_1$.

By the above proof, $\operatorname{plim}_{n\to\infty}(Y_n-c)X_n=0$.

$\therefore Y_nX_n=(Y_n-c)X_n+cX_n \xrightarrow[n\to\infty]{d} cZ_1$. by (a).

(d) $\operatorname{plim}_{n\to\infty} Y_n=c \Rightarrow \operatorname{plim}_{n\to\infty}(1/Y_n)=1/c$

$\therefore X_n/Y_n \xrightarrow[n\to\infty]{d} Z_1/c \quad (c\neq 0)$ $\qquad \square$.

Thm 5.3.3.

$\sqrt{n}(X_n-\theta) \xrightarrow[n\to\infty]{d} Z$ $\quad g \in C^1$ (편도함수가 연속)

$\Rightarrow \sqrt{n}(g(X_n)-g(\theta)) \xrightarrow[n\to\infty]{d} \dot{g}(\theta)^t Z$

proof) $g(x) = g(\theta) + (\dot{g}(\theta) + r(x))(x-\theta)$ $\quad r(x) \xrightarrow{x\to\theta} 0$

$\lim_{x\to a} f(x) = L$

$\Leftrightarrow \forall \epsilon>0, \exists \delta>0$

$: 0<|x-a|<\delta$

$\Rightarrow |f(x)-L|<\epsilon.$

For any $\epsilon>0$, there is a $\delta>0$ s.t

$0<|x-\theta|<\delta \Rightarrow |r(x)|<\epsilon.$??

$\Rightarrow (|X_n-\theta|<\delta) \subseteq (|r(X_n)|<\epsilon)$

$\Rightarrow |X_n-\theta| \geq \delta \supseteq (|r(X_n)| \geq \epsilon)$

$\therefore P(|r(X_n)| \geq \epsilon) \leq P(|X_n-\theta| \geq \delta)$

Since $\sqrt{n}(X_n-\theta) \xrightarrow[n\to\infty]{d} Z$,

$X_n-\theta = \frac{1}{\sqrt{n}} \times \sqrt{n}(X_n-\theta) \xrightarrow[n\to\infty]{d} 0\cdot Z \xrightarrow[n\to\infty]{P} 0.$

by Slutsky thm.

$\therefore \operatorname{plim} r(X_n) = 0.$

$\therefore \sqrt{n}(g(X_n)-g(\theta))$

$= (\dot{g}(\theta) + r(X_n))(X_n-\theta)\sqrt{n}$

$\dot{g}(\theta)+0$

$\xrightarrow{d} g$

e.g. 5.3.1 $X_1, \dots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$

$\dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}}$: stundentized sample mean.

$\dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow[n\to\infty]{d} Z$, $Z \sim N(0,1)$, $\operatorname{plim}_{n\to\infty} S_n = \sigma$

$\Rightarrow \dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \Big/ \dfrac{S_n}{\sigma} \xrightarrow[n\to\infty]{d} Z/1 = Z$

(By Slutsky thm.)

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\Rightarrow \dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t(n-1)$ (p.167)

$\therefore t(n-1) \overset{n\to\infty}{\approx} N(0,1)$

예 5.3.2 $S_n^2 = \dfrac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

$= \dfrac{1}{n-1} \sum_{i=1}^{n} \left((X_i - \mu) - (\bar{X} - \mu) \right)^2$

$= \dfrac{n}{n-1} \left\{ \dfrac{1}{n} \sum_{i=1}^{n} \underbrace{(X_i - \mu)^2}_{\overset{!!}{Y_n}} - (\bar{X} - \mu)^2 \right\}$

$\Rightarrow Y_1, \dots, Y_n$: iid

By CLT $\sqrt{n}\left(\dfrac{1}{n} \sum_{i=1}^{n} Y_i - E[Y_1]\right) \xrightarrow[n\to\infty]{d} W \sim N(0, Var(Y_1))$

$E[Y_1] = E[(X_1 - \mu)^2] = Var(X_1) = \sigma^2$

$Var(Y_1) = E[Y_1^2] - E[Y_1]^2 = E[(X_1 - \mu)^4] - \sigma^4$

$\rho_4 := E\left[\left(\frac{X_1-\mu}{\sigma}\right)^4\right]-3$

$\Rightarrow E[(X_1-\mu)^4]-\sigma^4$

$= \sigma^4\left(E\left[\left(\frac{X_1-\mu}{\sigma}\right)^4\right]-1\right)$

$= \sigma^4(\rho_4+2)$

$\sqrt{n}(\bar{X}_n-\mu) \xrightarrow[n\to\infty]{d} Z, \quad Z \sim N(0,\sigma^2)$ by CLT.

and $\operatorname*{plim}_{n\to\infty}(\bar{X}_n-\mu)=0$ by LLN.

$\Rightarrow$ By slutsky thm. $\sqrt{n}(\bar{X}_n-\mu)^2 \xrightarrow[n\to\infty]{p} 0$

$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2-\sigma^2\right)$

$= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-(\bar{X}_n-\mu)^2-\sigma^2\right)$

$= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\sigma^2\right)-\sqrt{n}(\bar{X}_n-\mu)^2$

$\xrightarrow[n\to\infty]{d} W \sim N(0, \underline{E[(X_1-\mu)^4]-\sigma^4})$

by slutsky's thm. $\quad = (\rho_4+2)\sigma^4$

$\therefore \sqrt{n}(S_n^2-\sigma^2)$

$= \sqrt{n}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2-\sigma^2\right)$

$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2-\sigma^2\right)+\frac{1}{\sqrt{n}}S_n^2$

$)S^2$

$= (n-1 \quad n$

$= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2-\sigma^2+\frac{1}{n}S_n^2\right)$

$= \sqrt{n}\left(\frac{n-1}{n}S_n^2-\sigma^2+\frac{1}{n}S_n^2\right)$

$= \sqrt{n}(S_n^2-\sigma^2)$

다차원 경우의 극한분포.

$X_n = (X_{n1}, \dots, X_{nk})' \quad n = 1, 2, \dots\dots$

$Z = (Z_1, \dots, Z_k)'$

$$X_n \xrightarrow[n\to\infty]{d} Z \iff \lim_{n\to\infty} P(X_{n1} \le x_1, \dots, X_{nk} \le x_k) = P(Z_1 \le x_1, \dots, Z_k \le x_k)$$

for all $(x_1, \dots, x_k)' \in \text{Conti}(\text{cdf}_{Z})$

<u>Thm.</u>

$$X_n \xrightarrow[n\to\infty]{d} Z, \quad g : \text{conti} \Rightarrow g(X_n) \xrightarrow[n\to\infty]{d} g(Z)$$

e.g. 5.3.3.

$X_n \sim Bin(n, p)$. $0 < p < 1$.

$$\Rightarrow \frac{X_n - np}{\sqrt{np(1-p)}} \xrightarrow[n\to\infty]{d} Z \sim N(0,1) \quad \text{by CLT.}$$

$$\Rightarrow \frac{X_n - np}{\sqrt{np(1-p)}} \xrightarrow[n\to\infty]{d} Z^2 \overset{d}{=} W \sim \chi^2(1)$$

$X_{n1} := X_n, \quad X_{n2} = n - X_n \qquad p_1 := p \quad p_2 := 1-p.$

$\Rightarrow (X_{n1}, X_{n2})' \sim Multi(n, (p_1, p_2)')$

$$\frac{(X_{n1} - np_1)^2}{np_1} + \frac{(X_{n2} - np_2)^2}{np_2} = \frac{(X_n - np)^2}{np} + \frac{(\overbrace{n - X_n - n(1-p)}^{X_n - np})^2}{n(1-p)}$$

$$= \frac{(X_n - np)^2}{np(1-p)}$$

$$\therefore \sum_{i=1}^{2} \frac{(X_{ni} - np_i)^2}{np_i} \xrightarrow[n\to\infty]{d} W \sim \chi^2(1)$$

e.g. 5.3.4

$(X_{n1}, \dots, X_{nk})' \sim Multi(n, (p_1, \dots, p_k)')$ $\quad \sum_{i=1}^{k} p_i = 1 \quad p_i > 0$.

$\boxed{\text{WTS}}$ $\sum_{j=1}^{k} \frac{(X_{nj} - np_j)^2}{np_j} \xrightarrow[n\to\infty]{d} W, \quad W \sim \chi^2(k-1)$

proof)

Let $X_n := (X_{n1}, \dots, X_{nk})'$, $\quad p := (p_1, \dots, p_k)'$ $\qquad E[X_n] = np$

$Var(X_n) = D(p_j) - pp^t$

$X_n \overset{d}{=} Z_1 + \cdots + Z_n \qquad Z_i \overset{iid}{\sim} Multi(1, (p_1, \dots, p_k)') \qquad D(p_j) := diag(p_1, \dots, p_k)$

By CLT, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i - p\right) \xrightarrow[n\to\infty]{d} X, \quad X \sim N_k(0, D(p_j) - pp^t)$

$= \frac{1}{\sqrt{n}} X_n - \sqrt{n}p$

$= (X_n - np)/\sqrt{n} \xrightarrow[n\to\infty]{d} X$

Let $g(X) := \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix} X$ where $X \in \mathbb{R}^k$.

$Y_n := (Y_{n1}, \dots, Y_{nr}) := g(X_n)$

$= (X_{n1}, \dots, X_{nk-1})$

$\mu := (\mu_1, \dots, \mu_r) := (p_1, \dots, p_r)$ $\qquad$ where $r = k-1$.

Since $[\ (X_{n1}, \dots, X_{nk})' \xrightarrow[n\to\infty]{d} (Z_{1}, \dots, Z_{k})$ & g: conti.

$\Rightarrow g(X_{n1}, \dots, X_{nk}) \xrightarrow[n\to\infty]{d} g(Z_{1}, \dots, Z_{k})\ ]$

$g\left(\frac{X_n - np}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}}\left(g(X_n) - g(np)\right)$ $\quad (\because g$: linear$)$

$= \frac{1}{\sqrt{n}}(Y_n - n\mu) \xrightarrow[n\to\infty]{d} Z, \quad Z \sim N_r(0, \Sigma)$

$\Sigma = D(\mu_j) - \mu\mu^t$

Since $z \mapsto z^t \Sigma^{-1} z$: continuous,

$$\left((Y_n - n\mu)/\sqrt{n}\right)^t \Sigma^{-1} \left((Y_n - n\mu)/\sqrt{n}\right) \xrightarrow[n\to\infty]{d} Z^t \Sigma^{-1} Z \quad Z \sim N_r(0, \Sigma)$$

By thm 4.4.5 $\left[X \sim N_k(\mu, \Sigma) \ \& \ \Sigma : \text{non-singular} \Rightarrow (X-\mu)^t \Sigma^{-1} (X-\mu) \sim \chi^2(k) \right]$,

$Z^t \Sigma^{-1} Z \sim \chi^2(r)$ where $r = k-1$.

i.e., $\left((Y_n - n\mu)/\sqrt{n}\right)^t \Sigma^{-1} \left((Y_n - n\mu)/\sqrt{n}\right) \xrightarrow[n\to\infty]{d} W$, $W \sim \chi^2(k-1)$

$$\Sigma^{-1} = \left(D(\mu_j) - \mu\mu^t\right)^{-1} = \left\{ D(\mu_j) \left(I_r - D^{-1}(\mu_j)\mu\mu^t\right)\right\}^{-1}$$
$$= \left(I_r - D^{-1}(\mu_j)\mu\mu^t\right)^{-1} D^{-1}(\mu_j)$$

Since $1 + b^t a \neq 0 \Rightarrow (I + ab^t)^{-1} = I + c\, ab^t$, $c = -1/(1 + b^t a)$,

$a := -D^{-1}(\mu_j)\mu$, $b := \mu$ $\quad c = -1/\left(1 - \mu^t D^{-1}(\mu_j)\mu\right)$

$$ab^t = -D^{-1}(\mu_j)\mu\mu^t = -\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (\mu_1 \cdots \mu_r)$$

$$c = -1/\left(1 - \mu^t (1, \dots, 1)^t\right) = -1/\left(1 - (\mu_1 + \cdots + \mu_r)\right) = -1/p_k \quad (\because \mu_1 + \cdots + \mu_r + p_k = p_1 + \cdots + p_k = 1)$$

$$\Rightarrow \left(I_r - D^{-1}(\mu_j)\mu\mu^t\right)^{-1} D^{-1}(\mu_j) = D^{-1}(\mu_j) + \frac{1}{p_k}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \mu^t D^{-1}(\mu_j)$$
$$= D^{-1}(\mu_j) + \frac{1}{p_k}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (1 \cdots 1)$$
$$= D^{-1}(\mu_j) + \frac{1}{p_k} \mathbb{1}\mathbb{1}^t$$

$$\left((Y_n - n\mu)/\sqrt{n}\right)^t \left(D^{-1}(\mu_j) + \frac{1}{p_k} 11^t\right)\left((Y_n - n\mu)/\sqrt{n}\right)$$

$$= \left((Y_n - n\mu)/\sqrt{n}\right)^t D^{-1}(\mu_j)\left((Y_n - n\mu)/\sqrt{n}\right)$$

$$+ \frac{1}{p_k}\left((Y_n - n\mu)/\sqrt{n}\right)^t 11^t \left((Y_n - n\mu)/\sqrt{n}\right)$$

$$= \sum_{j=1}^{k-1} \frac{(Y_{nj} - n\mu_j)^2}{n\mu_j} + \frac{1}{np_k} \underbrace{(Y_n - n\mu)^t 1}_{\parallel} 1^t (Y_n - n\mu)$$

$$\sum_{i=1}^{k-1} (Y_{ni} - n\mu_i) = n - X_{nk} - n(1 - p_k) = np_k - X_{nk}.$$

$$Y_{n1} + \cdots + Y_{nk-1} + X_{nk} = n$$

$$\mu_1 + \cdots + \mu_k + p_k = 1$$

$$= \sum_{j=1}^{k-1} \frac{(Y_{nj} - n\mu_j)^2}{n\mu_j} + \frac{(X_{nk} - np_k)^2}{np_k}$$

$$= \sum_{j=1}^{k} \frac{(X_{nj} - np_j)^2}{np_j} \xrightarrow[n\to\infty]{d} W \sim \chi^2(k-1) \qquad \square$$

e.g. 5.3.5.

$\sqrt{n}(S_n^2-\sigma^2) \xrightarrow{d} W \quad W \sim N(0, (\beta_4+2)\sigma^4)$

$g(x) := \sqrt{x} \in C^1 \Rightarrow \dot{g}(x) = \frac{1}{2\sqrt{x}}$

$\therefore \sqrt{n}(S_n - \sigma) \xrightarrow[n\to\infty]{d} \frac{1}{2\sigma} W$

$\therefore \sqrt{n}(S_n - \sigma) \xrightarrow[n\to\infty]{d} Z, \quad Z \sim N(0, (\beta_4+2)\sigma^2/4)$

e.g 5.3.6

$X_i \overset{iid}{\sim} (\mu_1, \sigma_1^2)$ $\quad Y_i \overset{iid}{\sim} (\mu_2, \sigma_2^2)$

$$\rho_n := \frac{\sum_{i=1}^{n} (X_i - \overline{X_n})(Y_i - \overline{Y_n})}{\sqrt{\sum_{i=1}^{n} (X_i - \overline{X_n})^2} \sqrt{\sum_{i=1}^{n} (Y_i - \overline{Y_n})^2}}$$

Assume $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$

$E[X_1 Y_1] = \rho$

$\overline{X_n} := \sum_{i=1}^{n} X_i / n$, $\overline{Y_n} := \sum_{i=1}^{n} Y_i / n$, $\overline{X_n Y_n} := \sum_{i=1}^{n} X_i Y_i / n$

$\overline{X_n^2} := \sum_{i=1}^{n} X_i^2 / n$, $\overline{Y_n^2} := \sum_{i=1}^{n} Y_i^2 / n$.

$$\hat{\rho}_n = \frac{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X_n})(Y_i - \overline{Y_n})}{\sqrt{\sum_{i=1}^{n} (X_i - \overline{X_n})^2 / n} \sqrt{\sum_{i=1}^{n} (Y_i - \overline{Y_n})^2 / n}}$$

$$= \frac{\frac{1}{n} \sum_{i=1}^{n} X_i Y_i - \frac{1}{n} \overline{X_n} \sum_{i=1}^{n} Y_i - \frac{1}{n} \overline{Y_n} \sum_{i=1}^{n} X_i + \frac{1}{n} \sum_{i=1}^{n} \overline{X_n}\,\overline{Y_n}}{\left(\sum_{i=1}^{n} X_i^2 / n - 2 \overline{X_n} \sum_{i=1}^{n} X_i / n + \sum_{i=1}^{n} (\overline{X_n})^2 / n \right)^{1/2} \cdot \left(\sum_{i=1}^{n} Y_i^2 / n - 2 \overline{Y_n} \sum_{i=1}^{n} Y_i / n + \sum_{i=1}^{n} (\overline{Y_n})^2 / n \right)^{1/2}}$$

$$= \frac{\overline{X_n Y_n} - \overline{X_n}\,\overline{Y_n}}{\left(\overline{X_n^2} - (\overline{X_n})^2 \right)^{1/2} \cdot \left(\overline{Y_n^2} - (\overline{Y_n})^2 \right)^{1/2}}$$

$g(t_1, t_2, t_3, t_4, t_5) := (t_3 - t_1 t_2)(t_4 - t_1^2)^{-1/2} (t_5 - t_2^2)^{-1/2}$

$g(\overline{X_n}, \overline{Y_n}, \overline{X_n Y_n}, \overline{X_n^2}, \overline{Y_n^2}) = \hat{\rho}_n$

$Z_i := (X_i, Y_i, X_iY_i, X_i^2, Y_i^2)'$: iid

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i - E[Z_1]\right) \xrightarrow[n\to\infty]{d} V, \quad V \sim N_5(0, Var(Z_1))$$

By CLT.

Assume $Var(Z_1) < +\infty$.

Since $g \in C^1$, $\sqrt{n}\left(g\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) - g(E[Z_1])\right) \xrightarrow[n\to\infty]{d} \dot{g}(E[Z_1])^t V$

$g(E[Z_1]) = g(0,0,\rho,1,1) =: g(\theta)$

$\Rightarrow \sqrt{n}(\hat{\rho}_n - \rho) = \sqrt{n}\left(g\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) - g(E[Z_1])\right)$

$$\partial t_1 = -t_2(t_4 - t_1^2)^{-1/2}(t_5 - t_2^2)^{-1/2} + (t_3 - t_1t_2)(t_4 - t_1^2)^{-3/2} \cdot t_1 (t_5 - t_2^2)^{-1/2}$$

$$\partial t_2 = -t_1(t_4 - t_1^2)^{-1/2}(t_5 - t_2^2)^{-1/2} + (t_3 - t_1t_2)(t_4 - t_1^2)^{-1/2} t_2 (t_5 - t_2^2)^{-3/2}$$

$$\partial t_3 = (t_4 - t_1^2)^{-1/2}(t_5 - t_2^2)^{-1/2}$$

$$\partial t_4 = (t_3 - t_1t_2)(t_4 - t_1^2)^{-3/2} \cdot \left(-\frac{1}{2}\right)(t_5 - t_2^2)^{-1/2}$$

$$\partial t_5 = (t_3 - t_1t_2)(t_4 - t_1^2)^{-1/2}(t_5 - t_2^2)^{-3/2} \cdot \left(-\frac{1}{2}\right)$$

$$\dot{g}(0,0,\rho,1,1) = (0, 0, 1, -\rho/2, -\rho/2)'$$

$$\dot{g}(\theta)^t V \sim N(0, \underbrace{\dot{g}(\theta)^t Var(Z_1) \dot{g}(\theta)}_{= Var(\dot{g}(\theta)^t Z_1)})$$

$$\dot{g}(\theta)^t Z_1 = X_1Y_1 - \frac{\rho}{2}(X_1^2 + Y_1^2)$$

$\therefore \sqrt{n}(\hat{\rho}_n - \rho) \xrightarrow[n\to\infty]{d} W \quad W \sim N\left(0, Var\left(X_1Y_1 - \frac{\rho}{2}(X_1^2 + Y_1^2)\right)\right)$

(Assume $E[X_1^4] < +\infty$, $E[Y_1^4] < +\infty$)

e.g. 5.3.7.	$(X_i, Y_i) \overset{iid}{\sim} N(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$
	By e.g. 4.4.3,
	$Y_1 - \rho X_1 \mid X_1 = x_1 \sim N(0, 1-\rho^2)$.
	$\Rightarrow$ Given $X_1 = x_1$, $Y_1 - \rho X_1 \mid X_1 = x_1 \perp\!\!\!\perp X_1$
	Assume $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$.
	$T := \dfrac{Y_1 - \rho X_1}{\sqrt{1-\rho^2}}$
	Since $Y_1 - \rho X_1 \perp\!\!\!\perp X_1$, $T \perp\!\!\!\perp X_1$
	and $T \sim N(0, 1^2)$, $X_1 \sim N(0, 1^2)$
	$Y_1 = \rho X_1 + \sqrt{1-\rho^2}\, T$
From e.g. 5.3.6	$X_1 Y_1 - \frac{\rho}{2}(X_1^2 + Y_1^2) = \frac{\rho}{2}(1-\rho^2) X_1^2 + (1-\rho^2)^{3/2} X_1 T - \frac{\rho}{2}(1-\rho^2) T^2$

$$Var\left(X_1Y_1 - \frac{\rho}{2}(X_1^2+Y_1^2)\right) = Var\left(\frac{\rho}{2}(1-\rho^2)X_1^2 + (1-\rho^2)^{3/2}X_1T - \frac{\rho}{2}(1-\rho^2)Y_1^2\right)$$

$X_1 \perp\!\!\!\perp T \Rightarrow X_1^3 \perp\!\!\!\perp T$

$\because E[X_1^3T] = E[X_1^3]E[T]$

$= E[T_1^3]E[X_1]$

$= E[X_1T_1^3]$

$E[X_1^4] = E[T^4]$

$E[X_1^2T^2] = E[X_1^2]E[T^2]$

$= 1$

$$mgf_{X_1}(t) = mgf_T(t) = \exp\left(\frac{1}{2}t^2\right)$$

$$\partial t = t\exp\left(\frac{1}{2}t^2\right)$$

$$\partial^2 t = \exp\left(\frac{1}{2}t^2\right) + t^2\exp\left(\frac{1}{2}t^2\right)$$

$$\partial^3 t = t\exp\left(\frac{1}{2}t^2\right) + 2t\exp\left(\frac{1}{2}t^2\right) + t^3\exp\left(\frac{1}{2}t^2\right)$$

$$\partial^4 t = \exp\left(\frac{1}{2}t^2\right) + t^2\exp\left(\frac{1}{2}t^2\right) + 2\exp\left(\frac{1}{2}t^2\right) + 2t^2\exp\left(\frac{1}{2}t^2\right) + 3t^2\exp\left(\frac{1}{2}t^2\right) + t^4\exp\left(\frac{1}{2}t^2\right)$$

$E[X_1] = 0$, $E[X_1^2] = 1$ $E[X_1^3] = 0$ $E[X_1^4] = 3$

$$E\left[\frac{\rho}{2}(1-\rho^2)X_1^2 + (1-\rho^2)^{3/2}X_1T - \frac{\rho}{2}(1-\rho^2)T\right]$$

$$= \frac{\rho}{2}(1-\rho^2)E[X_1^2] - \frac{\rho}{2}(1-\rho^2)E[T] + (1-\rho^2)^{3/2}E[X_1T]$$

$$= (1-\rho)^{3/2}E[X_1T] = (1-\rho^2)^{3/2}\left(E[X_1]\cdot E[T]\right) = 0$$

$$\therefore Var\left(\frac{\rho}{2}(1-\rho^2)X_1^2 + (1-\rho^2)^{3/2}X_1T - \frac{\rho}{2}(1-\rho^2)Y_1^2\right) = E\left[\left(\frac{\rho}{2}(1-\rho^2)X_1^2 + (1-\rho^2)^{3/2}X_1T - \frac{\rho}{2}(1-\rho^2)Y_1^2\right)^2\right]$$

$$= \underline{\frac{\rho^2}{4}(1-\rho^2)^2E[X_1^4]} + \underline{\frac{\rho^2}{4}(1-\rho^2)^2E[T^4]} + (1-\rho^2)^3E[X_1^2T^2] + \rho(1-\rho^2)\cancel{(1-\rho^2)^{3/2}E[X_1^3T]}$$

$$- \rho(1-\rho^2)\cancel{(1-\rho^2)^{3/2}}E[X_1T^3] - \frac{\rho^2}{2}(1-\rho^2)^2E[X_1^2T^2]$$

$$= \frac{\rho^2}{2}(1-\rho^2)^2 \times 3 + (1-\rho^2)^3 - \frac{\rho^2}{2}(1-\rho^2)^2$$

$$= (1-\rho^2)^2\left(\frac{3}{2}\rho^2 + 1 - \rho^2 - \frac{\rho^2}{2}\right) = (1-\rho^2)^2 \therefore Var\left(X_1Y_1 - \frac{\rho}{2}(X_1^2+Y_1^2)\right) = (1-\rho^2)^2$$

$$\therefore \sqrt{n}(\hat{\rho}_n - \rho) \xrightarrow[n\to\infty]{d} W, \quad W \sim N(0, (1-\rho^2)^2)$$

By the previous theorem, $g \in C^1 \Rightarrow \sqrt{n}(g(\hat{\rho}_n) - g(\rho)) \xrightarrow[n\to\infty]{d} \dot{g}(\rho)W$

$\dot{g}(\rho)W \sim N(0, (\dot{g}(\rho)^t)^2 (1-\rho^2)^2)$

Fisher transformation $\Rightarrow$ $g(\hat{\rho}_n) := \frac{1}{2} \log \frac{1+\hat{\rho}_n}{1-\hat{\rho}_n}$

$\frac{d}{dx} g(x) = \frac{1}{2} \frac{1-x}{1+x} \times \frac{(1-x) - (1+x)\cdot(-1)}{(1-x)^2} = \frac{1}{2} \frac{1}{1+x} \times \frac{1-x+1+x}{1-x} = \frac{1}{1-x^2}$

$\therefore \dot{g}(\rho) = \frac{1}{1-\rho^2}$

$\therefore \dot{g}(\rho) W \sim N(0, 1)$ $\square$

e.g.5.3.8

sample quantile $X_{(r_n)}$ ($r_n \sim \alpha n$, $0<\alpha<1$)

F^{-1}: inverse of CDF

$h(y) := F^{-1}(1-e^{-y})$ $(y>0)$

$X_{(r_n)} \overset{d}{=} h\left(\frac{1}{n}Z_1 + \cdots + \frac{1}{n-r_n+1}Z_{r_n}\right)$, $Z_i \overset{iid}{\sim} Exp(1)$

$Y_n := \frac{1}{n}Z_1 + \cdots + \frac{1}{n-r_n+1}Z_{r_n}$

$E[Y_n] \simeq -\log(1-\alpha)$ (from e.g.5.2.7)

$Var(Y_n) \simeq \frac{1}{n}\frac{\alpha}{1-\alpha}$

$(n\to\infty)$

$W_n := \sqrt{n}\,\dfrac{Y_n - (-\log(1-\alpha))}{\sqrt{\alpha/(1-\alpha)}}$

By the exercise 5.15, $\lim_{n\to\infty} mgf_{W_n}(t) = \exp(\frac{1}{2}t^2)$

$= mgf_W(t)$ $W \sim N(0,1)$

$\therefore W_n \xrightarrow[n\to\infty]{d} W$ $W \sim N(0,1)$

$\therefore \sqrt{n}(Y_n - (-\log(1-\alpha))) \xrightarrow[n\to\infty]{d} \sqrt{\alpha/(1-\alpha)}\,W$ by CLT.

Assume h: differentiable.

$\sqrt{n}(h(Y_n) - h(-\log(1-\alpha))) \xrightarrow[n\to\infty]{d} \dot{h}(-\log(1-\alpha))\sqrt{\alpha/(1-\alpha)}\,W$

$(f\circ g)(x) = x$

$f'(g(x))\cdot g'(x) = 1$

$g'(x) = \dfrac{1}{f'(g(x))}$

$(f \circ g)(x) = x$

$f'(g(x)) \cdot g'(x) = 1$

$g'(x) = 1 / f'(g(x))$

$$\dot{h}(y) = \frac{1}{f(F^{-1}(1-e^{-y}))} \times e^{-y}$$

$$\dot{h}(-\log(1-\alpha)) = \frac{1-\alpha}{f(F^{-1}(\alpha))}$$

Since $X_{(rn)} \overset{d}{\equiv} h(Y_{rn})$

$$\sqrt{n}(X_{(rn)} - F^{-1}(\alpha)) \overset{d}{\equiv} \sqrt{n}(h(Y_{rn}) - h(-\log(1-\alpha))) \xrightarrow[n\to\infty]{d} Z_1$$

$$Z_1 = \dot{h}(-\log(1-\alpha))^t \sqrt{\alpha/(1-\alpha)}\, W$$

$$\therefore Z_1 \sim N\left(0, \frac{(1-\alpha)^2}{\{f(F^{-1}(\alpha))\}^2} \times \frac{\alpha}{1-\alpha}\right)$$

$$\therefore Z_1 \sim N(0, \alpha(1-\alpha)/\{f(F^{-1}(\alpha))\}^2)$$

$\square$