TITLE 20.02.01. DATE

Det.

Sample space (표본공간) : a set of all possible outcomes.

event (사건) : $\mathcal{E}$, a collection of subsets of sample space, satisfying

(1) $S \in \mathcal{E}$ where S: universal set

(2) if $A_1, A_2, \dots \in \mathcal{E}$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{E}$

(3) if $A \in \mathcal{E}$, then $A^c \in \mathcal{E}$

"probability measure"

$P : \mathcal{P}(S) \to \mathbb{R}_{\geq 0}$: set function

where $\mathcal{P}(S) := \{A \mid A \subseteq S\}$: power-set.

We call P as probability measure.

Note that probability is an example of measure.

TITLE DATE

cardinality — finite

— infinite — countable : $\exists f: A \to \mathbb{N}$ ($\mathbb{N} \to A$)

s.t f: bijective.

$\Rightarrow A = \{x_1, x_2, \ldots\}$ 와 같이 수열로 나타낼수 있는 집합.

e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$

— uncountable

e.g. $\mathbb{R}$, $\mathbb{C}$.

$\underbrace{\bigcup_{\alpha \in \Lambda} A_\alpha}_{\text{uncountable set}}$, $\underbrace{\bigcup_{i=1}^{\infty} A_i}_{\text{countable set의 union.}}$ 구분하기.

TMI : Cantor's diagonal argument를 참고하면 $[0,1]$이 uncountable 임을 알 수 있다.

Axiom of Probability. (확률의 공리)

(1) $\forall A \in \mathcal{P}(A)$, $P(A) \geq 0$.

(2) $P(S) = 1$ where S: sample space

(3) countable additivity

$$P\left(\biguplus_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

TITLE DATE

1) $P(\phi) = 0$

2) $P\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

proof) (1) Suppose that S : sample space.

$$S = \biguplus_{i=1}^{\infty} A_i \quad \text{where } A_1 := S,\ A_i := \phi \quad i \geq 2.$$

$$1 = P(S) = P\left(\biguplus_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$= P(A_1) + \sum_{i=2}^{\infty} P(A_i)$$

$$= P(S) + \sum_{i=2}^{\infty} P(A_i)$$

$$P(S) = P(S) + \sum_{i=2}^{\infty} P(A_i)$$

By cancellation law, $\sum_{i=2}^{\infty} P(A_i) = 0$

Since, $P(A) \geq 0 \quad \forall A \in \mathcal{P}(A)$,

$$P(A_i) = P(\phi) = 0 \quad \forall i \geq 2. \quad \square.$$

(2) WTS $P\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

$A_i := \phi \quad i \geq n+1$

$$P\left(\biguplus_{i=1}^{n} A_i\right) = P\left(\biguplus_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{n} P(A_i) + \sum_{j=n+1}^{\infty} P(A_j)$$

$\because P(A_j) = P(\phi) = 0.$

$$= \sum_{i=1}^{n} P(A_i)$$

$\square$.

Thm. 1.1.1

(a) $\forall A \in \mathcal{P}(A)$, $0 \leq P(A) \leq 1$ & $P(\phi) = 0$

(b) $P(A^c) = 1 - P(A)$

$\because S = A \uplus A^c$

$1 = P(S) = P(A) + P(A^c) \qquad P(A^c) = 1 - P(A)$

(c) $A \subseteq B \Rightarrow P(A) \leq P(B)$

proof) $B = A \uplus (B \setminus A)$

$$P(B) = P(A) + P(B \setminus A) \geq P(A)$$

Thm 1.1.2.

(a) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

proof)

$$A_1 = (A_1 \setminus A_2) \uplus (A_1 \cap A_2)$$

$$A_2 = (A_2 \setminus A_1) \uplus (A_1 \cap A_2)$$

$$P(A_1 \cup A_2) = P(A_1 \setminus A_2) + P(A_2 \setminus A_1) + P(A_1 \cap A_2)$$

$$\parallel$$

$$P(A_1) + P(A_2) - P(A_1 \cap A_2) = P(A_1 \setminus A_2) + P(A_2 \setminus A_1) + P(A_1 \cap A_2) \qquad \square$$

각자 해보기

(b) $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$

(c) $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - \cdots - P(A_{n-1} \cap A_n) + \cdots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$

(d) countable subadditivity

$$P(A_1 \cup A_2 \cup \cdots) \le \sum_{i=1}^{\infty} P(A_i)$$

proof)

$B_1 := A_1$

$B_2 := A_2 \setminus A_1$

$B_2 := A_3 \setminus (A_1 \cup A_2)$

$\vdots$

$B_k := A_k \setminus (A_1 \cup \cdots \cup A_{k-1})$

$$\bigcup_{i=1}^{\infty} A_i = \biguplus_{i=1}^{\infty} B_i$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\biguplus_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$$

Since $A_i = B_i \uplus (A_1 \cup \cdots \cup A_{k-1})$,

$$P(A_i) = P(B_i) + P\left(\bigcup_{i=1}^{k-1} A_i\right)$$

$$\therefore P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(B_i) \le \sum_{i=1}^{\infty} P(A_i)$$

$\square$

Thm 1.1.3. Continuity

(a) $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ ascending chain.

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

$$\lim_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} A_n$$

proof) $B_1 := A_1$

$B_2 := A_2 \setminus A_1$

$\vdots$

$B_k := A_k \setminus A_{k-1} \quad k \geq 2$

$$\bigcup_{n=1}^{\infty} A_n = \biguplus_{n=1}^{\infty} B_n \Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\biguplus_{n=1}^{\infty} B_n\right)$$

$$= \sum_{n=1}^{\infty} P(B_n)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$$

$$= \lim_{n\to\infty} \big(\cancel{P(A_1)} + \cancel{P(A_2)} - \cancel{P(A_1)} + \cancel{P(A_3)} - \cancel{P(A_2)} \cdots P(A_n) - \cancel{P(A_{n-1})}\big)$$

$$= \lim_{n\to\infty} P(A_n) \quad \square$$

(b) $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots \Rightarrow P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$

proof) $A_i := (B_i)^c$ $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq$ By (a), $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$.

Since $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n^c = \left(\bigcap_{n=1}^{\infty} B_n\right)^c$, $P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} (1 - P(B))$

$\therefore P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim P(B) \quad \square$

* f: continuous.

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

e.g. $\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \log \sqrt[n]{n}$

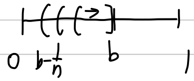
$= \log\left(\lim_{n\to\infty} \sqrt[n]{n}\right)$

$= \log 1 = 0$

e.g. $S=[0,1]$ $\quad P((a,b])=b-a$, 한점의 확률?

$$P(\{b\}) = P\left(\bigcap_{n=1}^{\infty}\left(b-\frac{1}{n}, b\right]\right) = \lim_{n\to\infty} P\left(\left(b-\frac{1}{n}, b\right]\right)$$

$$= \lim_{n\to\infty}\frac{1}{n} = 0$$



<조건부확률.>

한 사건 A가 일어났다는 전제하에 사건 B가 일어날 가능성.

(전사건은 A로 한정)

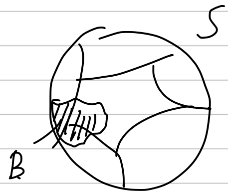
$$P(B|A) = \frac{P(A\cap B)}{P(A)} \qquad P(A)>0.$$

Thm 1. 2. 1

(a) $P(A)\,P(B) \neq 0.$

$$P(A\cap B) = P(A|B)\,P(B) = P(B|A)\,P(A)$$

(b) $S := \biguplus_{n=1}^{\infty} A_n.$



$$\Rightarrow P(B) = \sum_{n=1}^{\infty} P(B|A_n)\,P(A_n)$$

$$\begin{aligned} B &:= B\cap S \\ &= B\cap\left(\biguplus_{n=1}^{\infty} A_n\right) \\ &= \biguplus_{n=1}^{\infty} (B\cap A_n) \end{aligned}$$

$$P(B) = \sum_{n=1}^{\infty} P(B\cap A_n) = \sum_{n=1}^{\infty} P(B|A_n)\,P(A_n) \qquad \square$$

<Bayes Theorem>

$S := \biguplus_{i}^{\infty} A_i \qquad P(A_i)>0.$

$$P(B)>0 \Rightarrow P(A_j|B) \propto P(B|A_j)\,P(A_j)$$

proof) $P(A_j|B)\,P(B) = P(B\cap A_j)$

$$P(A_j|B) = \frac{P(B|A_j)\,P(A_j)}{P(B)} = \frac{P(B|A_j)\,P(A_j)}{\sum_i P(B|A_i)\,P(A_i)}$$

TITLE DATE

$$\sum_{j=1}^{\infty} P(A_j|B) = c\sum_{j=1}^{\infty} P(B|A_j)\,P(A_j)$$

$$P\Big(\biguplus_{j=1}^{\infty} A_j \,\Big|\, B\Big) = c\sum_{j=1}^{\infty} P(B|A_j)\,P(A_j)$$

$$\|$$

$$P(S|B) = 1.$$

$$\therefore\ c = 1\Big/\sum_{j=1}^{\infty} P(B|A_j)\,P(A_j) \qquad \square.$$

$P(A_i)$: prior $P(A_i|B)$: posterior.

Independence.

$$P(A\cap B) = P(A|B)\,P(B)$$

$$= P(A)\,P(B)$$

$$P(A|B) = P(A) \qquad P(B|A) = P(B)$$

Random Variable.

$X: S \to \mathbb{R}$ function.

$$P(X=1) = P(\underbrace{X^{-1}(1)}_{\text{preimage.}})$$

$f: A\to B$ $f^{-1}(b) := \{a\in A : f(a) = b\}$ $b\in B$.

MADE BY JANE

probability distribution.

- discrete ... countable.

$f(x_k) = P(X = x_k) \quad k = 1, 2, \dots$

↳ probability mass function

$$P(a \le X \le b) = \sum_{x : a \le x \le b} f(x)$$

Discrete case [edit]

Discrete probability theory needs only at most countable sample spaces Ω. Probabilities can be ascribed to points of Ω by the probability mass function $p : \Omega \to [0,1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$. All subsets of Ω can be treated as events (thus, $\mathcal{F} = 2^\Omega$ is the power set). The probability measure takes the simple form

$$(*) \qquad P(A) = \sum_{\omega \in A} p(\omega) \quad \text{for all } A \subseteq \Omega.$$

The greatest σ-algebra $\mathcal{F} = 2^\Omega$ describes the complete information. In general, a σ-algebra $\mathcal{F} \subseteq 2^\Omega$ corresponds to a finite or countable partition $\Omega = B_1 \cup B_2 \cup \dots$, the general form of an event $A \in \mathcal{F}$ being $A = B_{k_1} \cup B_{k_2} \cup \dots$. See also the examples.

The case $p(\omega) = 0$ is permitted by the definition, but rarely used, since such ω can safely be excluded from the sample space.

p. 29. 1.4.3.

$$I_A(x) := \begin{cases} 1 & (x \in A) \\ 0 & (x \notin A) \end{cases}$$

- continuous ... uncountable

$$P(a \le X \le b) = \int_a^b f(x)\,dx.$$

Mean (평균)

$$\mu := \int_{-\infty}^{\infty} x f(x)\,dx$$

Expectation

$$E[X] := \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E[g(X)] := \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

TITLE DATE

Variance.

$$Var(X) = E[(X-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

Thm 1.4.1.

(a) $E[aX+b] = aE[X]+b.$

($\because \int, \Sigma$: linear)

(b) $E[c_1 g_1(X) + c_2 g_2(X)]$

$= c_1 E[g_1(X)] + c_2 E[g(X)]$

(c) $g_1(X) \le g_2(X) \Rightarrow E[g_1(X)] \le E[g_2(X)]$

Thm 1.4.2.

(a) $Var(aX+b) = a^2 Var(X)$

$Y := aX+b$ $\quad E[Y] = a\mu + b$ $\quad \mu = E[X]$

$$Var(Y) = E[(aX+b-(a\mu+b))^2]$$

$$= E[(aX-a\mu)^2]$$

$$= E[a^2(X-\mu)^2]$$

$$= a^2 E[(X-\mu)^2]$$

$$= a^2 Var(X) \quad \square$$

TITLE DATE

(b) $Var(X) = E(X^2) - \{E(X)\}^2$

$$
\begin{aligned}
Var(X) &= E[(X-\mu)^2] \\
&= E[X^2 - 2\mu X + \mu^2] \\
&= E[X^2] - 2\mu E[X] + E[\mu^2] \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2 \\
&= E[X^2] - \{E[X]\}^2 \qquad \square
\end{aligned}
$$

TITLE 2/8. DATE

P : probability $\quad S$: sample space.

axiom
$$\begin{cases} (1)\ P(A) \geq 0 \quad \forall A \subseteq \mathcal{P}(S) = 2^S \\ (2)\ P(S) = 1 \\ (3)\ P\left(\biguplus_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \end{cases}$$

$\{A_i\}_{i=1}^{\infty} \subset \mathcal{P}(S)$

① $A \subseteq B \Rightarrow P(A) \leq P(B)$

$A \qquad P(B) = P(A) + P(B \setminus A)$

$B = A \uplus (B \setminus A)$

$P(B) = P(A \uplus (B \setminus A)) = P(A) + P(B \setminus A) \geq P(A)$

Continuity

$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$.

$P(\lim_{n\to\infty} A_n)$

$= P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$

$= \lim_{n\to\infty} P(A_n)$

$\bigcup_{n=1}^{\infty} A_n =: \lim_{n\to\infty} A_n$

$\mathrm{pdf}_X(x) = P(X = x)$

$\mathrm{cdf}_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

확률생성함수.

power-series 복습. vs generating function.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

$$g(x) = \sum_{n=0}^{\infty} f(n) x^n$$

$$= f(0)x^0 + f(1)x^1 + f(2)x^2 + \cdots$$

$f(0) = a_0.$

$f'(0) = a_1$

$f''(0) = 2! a_2$

$\vdots$

$f^{(n)}(0) = n! a_n.$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$g(0) = f(0)$

$g'(0) = f(1)$

$g''(0) = 2! f(2)$

$g^{(n)}(0) = n! f(n)$

$$f(n) = \frac{g^{(n)}(0)}{n!}$$

확률생성함수

$X \subseteq \mathbb{N} \cup \{0\}.$ $\quad f(x) = P(X = x)$

$$G(s) := \sum_{x=0}^{\infty} f(x) s^x = E[s^X]$$

$$\frac{d^k}{ds^k} \left\{ \sum_{x=0}^{\infty} f(x) s^x \right\} \Bigg|_{s=0} / k! = f(k)$$

$G(s)$로부터 $f(k)$를 알아낼수 있어 확률생성함수.

$\exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

적률생성함수.

$$G(s) = E[s^X] = \sum_{x=0}^{\infty} s^x f(x)$$

$s>0$ 이라 하자. $t=\log s.$

$$G(s) = E[s^X] = E[e^{X\log s}] = E[e^{tX}]$$

Def) $M(t) := E[e^{tX}]$: Moment Generating Function

where $E[e^{tX}]<+\infty$ $\forall t: -h<t<h$ $(\exists h>0)$

$$E[e^{tX}] = E\left[\sum_{n=0}^{\infty}\frac{(tX)^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{E[X^n]}{n!}t^n$$

과연 성립하는가?

즉 적률생성함수로부터 적률을 구할수 있는가? Yes.

다음의 두

Def) $E[|X|^k]<+\infty$

$$m_k = E[X^k] = \begin{cases} \sum_x x^k f(x) \\ \int_{-\infty}^{+\infty} x^k f(x)\,dx. \end{cases}$$

m_k는 X의 k차 적률.

<u>Thm.</u>

(a) $M(t) = E[e^{tX}] < +\infty$

$\Rightarrow \forall n \in \mathbb{N}$, $\exists E[X^n]$ s.t

$$M^{(n)}(0) = E[X^n] \;\&\; M(t) = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n.$$

(b). $M_X(t)$, $M_Y(t)$ exists and $M_X(t) = M_Y(t)$ for some interval I containing zero.

$\Rightarrow$ $pdf_X(x) = pdf_Y(x)$

$(cdf_X(x) = cdf_Y(x))$

proof of (b) is omitted. $\square$

TITLE DATE

(a) proof) $\frac{|tx|^k}{k!} \le e^{|tx|} \le e^{tx} + e^{-tx}$

existence.

$\left(\because e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{, i.e. } e^{|tx|} = \sum_{n=0}^{\infty} \frac{|tx|^n}{n!} \right)$

$E\left[\frac{|tX|^k}{k!}\right] \le E\left[e^{tX} + e^{-tX}\right]$

$\Rightarrow \frac{E[|X|^k]\, t^k}{k!} \le M(t) + M(-t) < +\infty$

$\therefore \forall k \in \mathbb{N}, \exists E[X^k] < +\infty$.

$e^{tx} - \sum_{k=0}^{n-1} \frac{(tx)^k}{k!} = \frac{1}{(n-1)!} \int_0^1 (1-u)^{n-1} e^{utx} du\, (tx)^n$

(테일러 정리에 의해)

$\left| \int f(x)dx \right| \le \int |f(x)| dx$

$\left| e^{tx} - \sum_{k=0}^{n-1} \frac{(tx)^k}{k!} \right| = \left| \frac{1}{(n-1)!} \int_0^1 (1-u)^{n-1} e^{utx} du\, (tx)^n \right|$

$\le \frac{1}{(n-1)!} \int_0^1 \left| (1-u)^{n-1} e^{utx} du\, (tx)^n \right|$

$\because$ $0 < u < 1 \Rightarrow (1-u) > 0$.

$u|tx| \ge utx$ & e^x: increasing fnt.

$e^{u|tx|} \ge e^{utx}$.

$\le \frac{1}{(n-1)!} \int_0^1 (1-u)^{n-1} e^{u|tx|} du\, (|tx|)^n$

$= \int_0^1 (1-u)^{n-1} e^{u|tx|} du \frac{|tx|^{n-1}}{(n-1)!} \frac{|tx|}{1!}$

$\le \int_0^1 (1-u)^{n-1} du\, e^{|tx|} \cdot e^{|tx|} e^{|tx|}$

$= \frac{1}{n} e^{3|tx|} \le \frac{1}{n} \left(e^{3tx} + e^{-3tx} \right)$

TITLE DATE

$$\left| e^{tx} - \sum_{k=0}^{n-1} \frac{(tx)^k}{k!} \right| \leq \frac{1}{n}\left(e^{3tx} + e^{-3tx}\right)$$

$$\left| E[e^{tX}] - \sum_{k=0}^{n-1} \frac{E[X^k]}{k!} t^k \right| \leq \frac{1}{n}\left(M(3t) + M(-3t)\right)$$

Since $M(3t) + M(-3t) < +\infty$,

$n \to \infty$ $$\left| E[e^{tX}] - \sum_{k=0}^{\infty} \frac{E[X^k]}{k!} t^k \right| = 0$$

$$\therefore M(t) = E[e^{tX}] = \sum_{k=0}^{\infty} \frac{E[X^k]}{k!} t^k$$

$M(0) = E[1] = 1.$

$M'(0) = \frac{E[X]}{1!} 1!$

$M^{(2)}(0) = \frac{E[X^2]}{2!} 2!$

$\vdots$

$M^{(n)}(0) = \frac{E[X^n]}{n!} n!$

$\therefore M^{(n)}(0) = E[X^n]$ $\square$.

e.g. $f(x) = e^{-x} I_{(x>0)}$

$$M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} e^{-x} I_{(x>0)}\, dx$$

$$= \int_{0}^{\infty} e^{(t-1)x} dx$$

$$= \frac{1}{(t-1)} e^{(t-1)x} \Big|_0^{\infty}$$

$$= \frac{1}{1-t} \qquad t<1$$

$-1 < t < 1$ 이어야하지 않나? $\rightarrow$ $M(t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} (t)^n = \sum_{n=0}^{\infty} \frac{n!}{n!} t^n$

$$E[X^n] = n!$$

$$M'(0) = \frac{1}{(1-t)^2}\Big|_{t=0} = 1$$

$$M^{(2)}(0) = \frac{\cancel{+}2\cancel{(1-t)}}{(1-t)^{\cancel{4}3}}\Big|_{t=0} = \frac{2}{(1-t)^3}\Big|_{t=0} = 2.$$

$$M^{(3)}(0) = \frac{3!}{(1-t)^5}\Big|_{t=0} = 3!$$

$\vdots$

$$M^{(n)}(0) = n!$$

Def) $C(t) = \log M(t) = \log E[e^{tX}]$ … Cumulant Generating Function (누율생성함수)

Q. 어떻게 급수형태로 나올수 있을까?

$$C(t) = \sum_{r=0}^{\infty} \frac{C^{(r)}(0)}{r!} t^r = \sum_{r=0}^{\infty} \frac{c_r(X)}{r!} t^r$$

$C^{(r)}(0)$: X의 r차 누율.

$$\log E[e^{tX}] = \log\left(\sum_{k=0}^{\infty} \frac{m_k}{k!} t^k\right) = \log\left(1 + \sum_{k=1}^{\infty} \frac{m_k}{k!} t^k\right)$$

$$A := \sum_{k=1}^{\infty} \frac{m_k}{k!} t^k.$$

$$\frac{1}{1+A} = 1 - A + A^2 - A^3 + \cdots \quad -1 < A < 1$$

$$\log(1+A) = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \cdots$$

$C(0) = 0$

1차항의 계수이므로 A의 1차항 계수를 보면된다.

$C^{(1)}(0) = m_k.$

$C^{(2)}(0) = (A$의 1차항과 A^2의 2차항 계수 $\times (-\frac{1}{2}))$

$$\frac{C^{(2)}(0)}{2!} = \frac{m_2}{2!} - \frac{1}{2} m_1^2$$

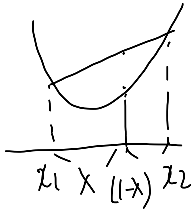
$$C^{(2)}(0) = m_2 - m_1^2.$$

$$= E[X^2] - (E[X])^2$$

$$= Var(X)$$

Jensen's Inequality

Convex function. ϕ



$\phi((1-\lambda)x_1 + \lambda x_2)$

$\le (1-\lambda)\phi(x_1) + \lambda\phi(x_2)$

ϕ: convex function

$\Rightarrow E[\phi(X)] \ge \phi(E[X])$

proof) Suppose ϕ : differentiable at μ.

$$\phi(x) = \phi(\mu) + \phi'(\mu)(x-\mu) + \phi''(\xi)(x-\mu)^2/2!$$

for some $\xi \in (x, \mu)$.

Since ϕ: convex function, $\phi'' > 0$.

Thus, $\phi(x) \ge \phi(\mu) + \phi'(\mu)(x-\mu)$

$$E[\phi(X)] \ge E[\phi(\mu)] + E[\phi'(\mu)(X-\mu)]$$

$$= \phi(\mu) + \phi'(\mu)(E[X-\mu])$$

$$= \phi(\mu)$$

$\therefore E[\phi(X)] \ge \phi(E[X])$

□

e.g. $\phi(x) = x^2$: convex.

$E[\phi(X)] \ge \phi(E[X])$

$E[X^2] \ge (E[X])^2$.

$E[X^2] - (E[X])^2 \ge 0$

$\therefore Var(X) \ge 0$ Var : well-defined.

TITLE Lyapounov's inequality.

DATE

Hölder's inequality로 증명

$E[|X|^s] < +\infty \qquad 0 < r < s$

$\Rightarrow \quad E[|X|^r]^{1/r} \le E[|X|^s]^{1/s}$

proof) For $p > 1 \quad \phi(y) := y^p I_{(y \ge 0)}$

$\phi''(y) = p(p-1)y^{p-2} I_{(y \ge 0)} > 0$ for $p > 1$.

$\Rightarrow \phi$: convex function.

By Jensen's inequality, $(E[Y])^p \le E[Y^p]$

$p := \frac{s}{r} > 1 \quad Y := |X|^r \ge 0$

$$E[|X|^r]^{\frac{s}{r}} = (E[Y])^p \le E[Y^p] = E[|X|^s]$$

$$\therefore E[|X|^r]^{\frac{1}{r}} \le E[|X|^s]^{\frac{1}{s}}$$

$\square$

Corollary

(a) Markov's inequality.

$$E[|Z|^r] < +\infty \quad (r>0)$$

$$\Rightarrow \forall k>0 \quad P(|Z| \geq k) \leq E[|Z|^r]/k^r$$

(b) Chebyshev's inequality

$$\Rightarrow Var(X) < +\infty$$

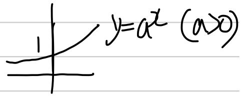
$$P(|X - E[X]| \geq k) \leq Var(X)/k^2$$

proof) (a)

$$P(|Z| \geq k) = E[I_{(|Z| \geq k)}]$$

$$I_{(|Z| \geq k)} = 1 \cdot I_{(|Z|/k \geq 1)} \leq \left(\frac{|Z|}{k}\right)^r I_{(|Z|/k \geq 1)}$$

$\because \left(\frac{|Z|}{k}\right)^r \geq 1$ for $\frac{|Z|}{k} \geq 1$, $r>0$



$$P(|Z| \geq k) = E[I_{(|Z| \geq k)}] \leq E\left[\frac{|Z|^r}{k^r}\right] = E[|Z|^r]/k^r$$

$$\therefore P(|Z| \geq k) \leq E[|Z|^r]/k^r$$

$\square$

TITLE DATE

(b) $Z := X - E[X]$ $r=2$.

$$P(|X-E[X]| \geq k) \leq E[|X-E[X]|^2]/k$$
$$= E[(X-\mu)^2]/k$$
$$= Var(X)/k. \quad \square$$

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} (t)^n \qquad -1<t<1$$

$1+t+t^2+$

$$-\log(1-t) = t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots$$

TITLE 2/22 DATE

Thm. $\mu = E[X]$

$Var(X) = 0 \iff P(X=\mu) = 1$

Note. $X^{-1}(\mu) \neq S$ 을 기억하라.

measure zero-set이 있을 수 있다.

e.g (유리수는 $\mathbb{R}$에 대해서 measure-zero set)

$P(\mathbb{Q}) = 0$ but $\mathbb{Q} \neq \phi$.

$$(|X-\mu| > 0) = \bigcup_{n=1}^{\infty} \left(|X-\mu| \geq \frac{1}{n}\right)$$

$$P((|X-\mu| > 0)) = P\left(\bigcup_{n=1}^{\infty} |X-\mu| \geq \frac{1}{n}\right)$$

$$\leq \sum_{n=1}^{\infty} P\left(\left(|X-\mu| \geq \frac{1}{n}\right)\right)$$

$\because$ Chebysev's inequality

$$0 \leq P\left(|X-\mu| \geq \frac{1}{n}\right) \leq Var(X) / \left(\frac{1}{n}\right)^2 = 0$$

$$P\left(|X-\mu| \geq \frac{1}{n}\right) = 0 \quad n = 1, 2, \ldots$$

$$0 \leq P(|X-\mu| > 0) \leq \sum_{n=1}^{\infty} P\left(|X-\mu| \geq \frac{1}{n}\right) = 0$$

$\therefore P(|X-\mu| > 0) = 0$

$P(X=\mu) = 1 - P(|X-\mu| > 0) = 1$

TITLE DATE

$\Leftarrow P(X=\mu)=1$

discrete R.V.

i) $E[X] = \sum x f(x) I(x=\mu)$

$= \mu$

$E[X^2] = \sum x^2 f(x) I(x=\mu)$

$= \mu^2$

$Var(X) = 0.$

continuous R.V.

ii) $E[X] = \int_{-\infty}^{\infty} x f(x) I(x=\mu)\, dx$

$= \int_{\mu}^{\mu} \mu \cdot 1\, dx = 0$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) I(x=\mu)\, dx$

$= \int_{\mu}^{\mu} \mu^2 \cdot 1 \cdot dx = 0$

$\therefore Var(X) = 0.$ $\square$

TITLE DATE

$f(x,y) \geq 0$.

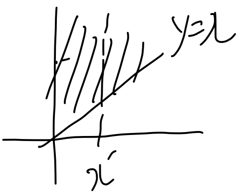
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

x,y에 의존하는 indicator func.

e.g. $f(x,y) = ce^{-x-y} I(0 \leq x \leq y)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ce^{-x-y} I(0 \leq x \leq y)\,dy\,dx$$

x를 하나 고정하고 y에 대한 적분



$$= \int_{-\infty}^{\infty}\int_{x}^{\infty} ce^{-x-y}\,dy\, I(0 \leq x)\,dx$$

$$= \int_{-\infty}^{\infty} ce^{-x}\left[-e^{-y}\right]_{x}^{\infty} I(0 \leq x)\,dx$$

$$= \int_{-\infty}^{\infty} ce^{-2x} I(0 \leq x)\,dx$$

$$= \int_{0}^{\infty} ce^{-2x}\,dx$$

$$= c\left[-\frac{1}{2}e^{-2x}\right]_{0}^{\infty}$$

$$= c/2 = 1$$

$$\boxed{c = 2}$$

TITLE DATE

$f(x,y) = 2e^{-x-y} I(0 \le x \le y)$

$P(X \ge 2, Y \ge 3)$

$\int_2^\infty \int_3^\infty 2e^{-x-y} I(0 \le x \le y)\, dy\, dx$

$= \int_2^\infty \int_{\max(x,3)}^\infty 2e^{-x-y}\, dy\, I(0 \le x)\, dx$

$= \int_2^\infty 2e^{-x} [-e^{-y}]_{\max(x,3)}^\infty I(0 \le x)\, dx$

$= \int_2^\infty 2e^{-x} e^{-\max(x,3)} I(0 \le x)\, dx$

$= \int_2^3 2e^{-x-3}\, dx + \int_3^\infty 2e^{-2x}\, dx$

$= 2e^{-3}[-e^{-x}]_2^3 + 2[-\frac{1}{2}e^{-2x}]_3^\infty$

$2e^{-3}(-e^{-3} + e^{-2}) + e^{-6}$

$\boxed{2e^{-5} - e^{-6}}$

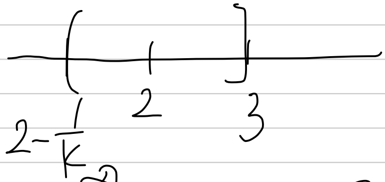
1.5_ 양의 실수 전체의 부분집합 A에 대하여

$$\mathrm{P}(A) = \int_A e^{-x}dx$$

로 확률이 정의되었을 때, $A_k = \left\{x : 2 - \frac{1}{k} < x \le 3\right\}$ $(k = 1, 2, \cdots)$에 대하여 다음 사건과 확률을 구하여라.

(a) $\lim\limits_{k\to\infty} A_k$와 $\mathrm{P}(\lim\limits_{k\to\infty} A_k)$ (b) $\mathrm{P}(A_k)$와 $\lim\limits_{k\to\infty}\mathrm{P}(A_k)$

(a)



By definition, $\lim\limits_{k\to\infty} A_k = \bigcap\limits_{k=1}^{\infty} A_k = [2,3]$.

$\left(\because \ \forall k,\ 2-\frac{1}{k} \le 2\right.$ 이므로 $\left. 2 \in A_k.\right)$

$P(X = x \in [2,3])$.

$= \int_2^3 e^{-x}dx = \left[-e^{-x}\right]_2^3$

$[a]_2^3$

$= e^{-2} - e^{-3}$ □.

(b) $P(A_k) = \int_{2-\frac{1}{k}}^{3} e^{-x}dx$

$= e^{-2+\frac{1}{k}} - e^{-3}$.

By continuity $\lim\limits_{k\to\infty} P(A_k) = P(\lim\limits_{k\to\infty} A_k) = e^{-2} - e^{-3}$.

1.6_ 확률변수 X의 누적분포함수가

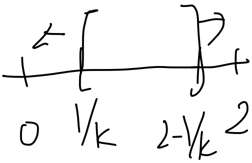
$$F(x)=\begin{cases}0 & x\le 0\\ x^2/8 & 0\le x<2\\ 1 & x\ge 2\end{cases} \text{ 이고}$$

$$C_k=\{x:1/k\le x\le 2-1/k\},$$

$$D_k=\{x:2-1/k<x<2+1/k\}\ (k=1,2,\cdots)$$

일 때 다음 사건과 확률을 구하여라.

(a) $\lim_{k\to\infty} C_k$와 $P(X\in\lim_{k\to\infty} C_k)$ (b) $\lim_{k\to\infty} D_k$와 $P(X\in\lim_{k\to\infty} D_k)$



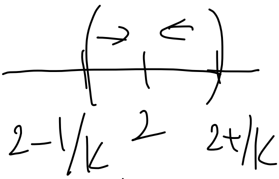
ascending chain

(a) $\lim_{k\to\infty} C_k = \bigcup_{k=1}^{\infty} C_k = (0,2)$

$P(X=x\in(0,2))$

$= F(2)-F(0)$

$= F(2) = \frac{1}{2}$



descending chain.

(b). $\lim_{k\to\infty} D_k = \bigcap_{k=1}^{\infty} D_k = \{2\}$.

$P(X=2\}$

$F(2+)-F(2-)$

$= 1-\frac{1}{2}$

$= \frac{1}{2}$

(c) $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^2} & x \geq 0 \end{cases}$

좌미분계수:
$$\lim_{x \to 0+} \frac{F(x) - F(0)}{x} = \lim_{x \to 0+} \frac{1 - e^{-x^2}}{x}$$

l'Hôpital thm.

$$= \lim_{x \to 0+} \frac{2x e^{-x^2}}{1}$$

$$= 0$$

우미분계수: 0.

$\therefore$ $x=0$에서 $F(x)$가 미분가능.

$$f(x) = \frac{dF(x)}{dx} = 2x e^{-x^2}$$

$$f(x) = \begin{cases} 0 & (x < 0) \\ 2x e^{-x^2} & (x \geq 0) \end{cases}$$

TITLE DATE

cdf?

(c) $f(x)=ce^{-2|x|},\ -\infty<x<+\infty$

even function.

$$c\int_{-\infty}^{\infty} e^{-2|x|}dx=1.$$

$$2c\int_{0}^{\infty} e^{-2x}dx=1$$

$$2c\left[-\frac{1}{2}e^{-2x}\right]_0^{\infty}=1$$

$$c=1.$$

i) $x<0$

$$F(x)=\int_{-\infty}^{x} e^{2t}dt=\left[\frac{1}{2}e^{2t}\right]_{-\infty}^{x}$$

$$=\frac{1}{2}e^{2x}$$

ii) $x\geq 0$

$$F(x)=\int_{-\infty}^{0} e^{2t}dt+\int_{0}^{x} e^{-2t}dt$$

$$=\frac{1}{2}+\left[-\frac{1}{2}e^{-2t}\right]_0^{x}$$

$$=\frac{1}{2}+\left(-\frac{1}{2}e^{-2x}+\frac{1}{2}\right)$$

$$=1-\frac{1}{2}e^{-2x}.$$

TITLE DATE

1.11_ (a) 확률변수 X와 임의의 양수 k에 대하여 다음 부등식이 성립하는 것을 밝혀라.

$$kP(X > k) \le E(X\,I(X > k))$$

(b) 0 또는 양수의 값을 갖는 연속형 확률변수 X의 기댓값이 존재할 때, 즉 $E(X) < +\infty$일 때 다음이 성립하는 것을 밝혀라.

$$\lim_{x \to +\infty} xP(X > x) = 0$$

(c) 0 또는 양수의 값을 갖는 연속형 확률변수 X의 기댓값이 존재하고 그 누적분포함수 F가 모든 점에서 미분가능할 때 다음이 성립하는 것을 밝혀라.

$$EX = \int_0^{+\infty} (1 - F(x))dx$$

$\forall k > 0$

(a) $kP(X>k) = k\int_k^{+\infty} f(x)\,dx.$

$$E[X\,I(X>k)] = \int_{-\infty}^{\infty} x I_{(x>k)} f(x)\,dx$$

$$= \int_k^{\infty} x f(x)\,dx.$$

$$\ge \int_k^{\infty} k f(x)\,dx.$$

$$= kP(X>k)$$

$\therefore kP(X>k) \le E[X\,I(X>k)]$. 요

TITLE DATE

(b) $X \geq 0$, $E[X] < +\infty$.

WTS. $\Rightarrow \lim_{x \to +\infty} x P(X > x) = 0$.

$$0 \leq k P(X > k) \leq E[X I(X > k)].$$

ETS. $\lim_{k \to +\infty} E[X I(X > k)] = 0$.

Rmk. 엄밀한 증명은 (불편하더라도) 해석학을 배우고 하도록하라.... (1번 뒤에...)

$$E[X] = \int_0^\infty x f(x) dx$$

$$\forall k > 0 \quad = \int_0^k x f(x) dx + \int_k^{+\infty} x f(x) dx$$

$E[X] < +\infty$ 이므로 $\forall k > 0$ 에 대하여

(RHS)의 각 항이 수렴하는 것으로 알려져 있다.

$\lim_{k \to +\infty} \int_0^k x f(x) dx = E[X]$ 가 되므로

$$\lim_{k \to +\infty} E[X I(X > k)] = \lim_{k \to +\infty} \int_k^\infty x f(x) dx = 0$$

$$\therefore \lim_{k \to +\infty} k P(X > k) = 0 \qquad \square$$

1.16_ X의 확률밀도함수가 다음과 같을 때 X의 적률생성함수와 평균, 분산을 구하여라.

$$f(x) = \frac{1}{2}e^{-|x|},\ -\infty < x < +\infty$$

$$M(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2}\left(\int_{0}^{\infty} e^{(t-1)x} dx + \int_{-\infty}^{0} e^{(t+1)x} dx \right)$$

$$= \frac{1}{2}\left(\left[\frac{1}{t-1} e^{(t-1)x} \right]_{0}^{\infty} + \left[\frac{1}{t+1} e^{(t+1)x} \right]_{-\infty}^{0} \right)$$

$-1 < t < 1$

$$= \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t} \right)$$

$$= \frac{1}{1-t^2}$$

$$E[X] = M'(0) = \frac{d}{dt}\left(\frac{1}{1-t^2} \right)\Big|_{t=0}.$$

$$E[X] = M^{(2)}(0)$$

1.17_ 확률변수 X의 적률생성함수가 존재하는 것이 알려져 있고

$$E(X^{2r}) = (2r)!/2^r r!, \ r = 1, 2, 3, \cdots$$

$$E(X^{2r-1}) = 0, \ r = 1, 2, 3, \cdots$$

일 때 X의 적률생성함수를 구하여라.

$$M(t) = \sum_{k=0}^{\infty} \frac{E[X^k]}{k!} t^k \text{ 이므로}$$

$$= \sum_{r=0}^{\infty} \frac{(2r)!}{(2r)!\, 2^r r!} t^{2r} + \sum_{r=1}^{\infty} \frac{0}{(2r-1)!} t^{(2r-1)}$$

$$= \sum_{r=0}^{\infty} \frac{1}{2^r r!} t^{2r}$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{t^2}{2}\right)^r = \exp\left(\frac{t^2}{2}\right)$$

TITLE DATE

1.19_ 표준화된 확률변수 $Z = (X - \mu)/\sigma$의 적률 $m_r(Z)$와 누율 $c_r(Z)$의 관계로 부터

$$c_3(Z) = m_3(Z),\ c_4(Z) = m_4(Z) - 3$$

임을 밝혀라. Z의 3차 누율 $c_3(Z) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$와 4차 누율 $c_4(Z) = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] - 3$을 각각 X의 왜도(歪度, skewness)와 첨예도(尖銳度, kurtosis)라고 한다.

$$E\left[\left(\frac{X-\mu}{\sigma}\right)\right] = \frac{1}{\sigma}\left(E[X] - \mu\right) = 0$$

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}\left(E[X^2] - 2\mu E[X] + \mu^2\right)$$

$$= \frac{1}{\sigma^2}\left(E[X^2] - \mu^2\right)$$

$$= \frac{1}{\sigma^2}\cdot\sigma^2 = 1$$

$$C(t) = \log\left(\sum_{k=0}^{\infty}\frac{m_k}{k!}t^k\right) = \log\left(1 + \sum_{k=1}^{\infty}\frac{m_k}{k!}t^k\right)$$

$$A := \sum_{k=1}^{\infty}\frac{m_k}{k!}t^k = \frac{1}{2}t^2 + \frac{m_3}{3!}t^3 + \frac{m_4}{4!}t^4 + \cdots$$

$$= \log(1+A) = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4.$$

$$C(t) = \sum_{r=0}^{\infty}\frac{c_r(Z)}{r!}t^r$$

$$\frac{c_3(Z)}{3!} = \frac{m_3}{3!} \qquad \frac{c_4(Z)}{4!} = \frac{m_4}{4!} - \frac{1}{2}\times\left(\frac{1}{2}\times\frac{1}{2}\right)$$

$$c_4(Z) = m_4 - 3$$

$$\therefore\ c_3(Z) = m_3 \qquad c_4(Z) = m_4 - 3 \qquad \text{Q.}$$