# Chapter 2

marginal PDF.

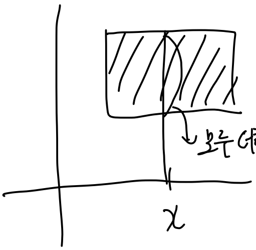
$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

**Def)** $f_1(x) := \int_{-\infty}^{+\infty} f(x,y)\,dy$

$$P(a \le X \le b) = P(a \le X \le b,\ -\infty \le Y \le +\infty,$$

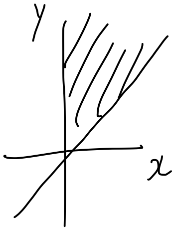
$$= \int_a^b \int_{-\infty}^{+\infty} f(x,y)\,dy\,dx$$

$$= \int_a^b f_1(x)\,dx.$$



모두 더해서 $f_1(x)$이 된다.

e.g. $f(x,y) = 2e^{-x-y} I(0 \le x \le y)$



$$f_1(x) = \int_{-\infty}^{+\infty} 2e^{-x-y} I(0 \le x \le y)\,dy$$

$$= \int_x^{+\infty} 2e^{-x-y}\,dy\, I(0 \le x)$$

$$= 2e^{-x}\left[e^{-y}\right]_x^{+\infty} I(0 \le x)$$

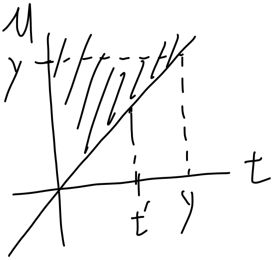
$$= 2e^{-x} e^{-x} I(0 \le x) = 2e^{-2x} I(0 \le x)$$

$$f_2(y) = \int_{-\infty}^{+\infty} 2e^{-x-y} I(0 \le x \le y)\,dx$$

$$= \int_0^y 2e^{-x-y}\,dx\, I(0 \le y)$$

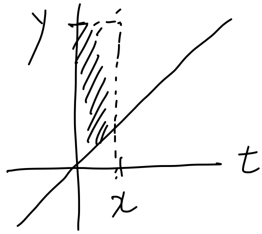
$$= 2e^{-y}\left[-e^{-x}\right]_0^y I(0 \le y)$$

$$= 2e^{-y}(1 - e^{-y})$$

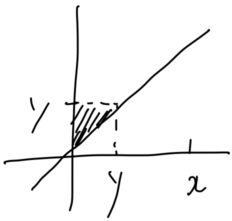



i) $x<y$

적분이 0이 아닌영역




ii) $x \geq y$




$$F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} 2e^{-t-u}\, I_{(0\leq t\leq u)}\, du\, dt.$$

$$=\int_{-\infty}^{x}\int_{t}^{y} 2e^{-t-u}\, I_{(0\leq t\leq y)}\, du\, dt$$

$$=\int_{-\infty}^{x} 2e^{-t}\left[-e^{-u}\right]_{t}^{y} I_{(0\leq t\leq y)}\, dt$$

$$=\int_{-\infty}^{x} 2e^{-t}\left(e^{-t}-e^{-y}\right) I_{(0\leq t\leq y)}\, dt$$

$$=\int_{0}^{x} 2e^{-t}\left(e^{-t}-e^{-y}\right) I_{(0\leq t\leq y)}\, dt$$

$$=\int_{0}^{\min(x,y)} \left(2e^{-2t}-2e^{-y}e^{-t}\right) I_{(\min(x,y)\geq 0)}\, dt$$

$$=\left\{\left[-e^{-2t}\right]_{0}^{\min(x,y)}-2e^{-y}\left[-e^{-t}\right]_{0}^{\min(x,y)}\right\} I_{(\min(x,y)\geq 0)}$$

$$=\left\{1-e^{-2\min(x,y)}-2e^{-y}\left(1-e^{-\min(x,y)}\right)\right\} I_{(\min(x,y)\geq 0)}$$

TITLE DATE

magInal CDF.

$F_1(x) := P(X \le x)$

$= P(X \le x, -\infty < Y < +\infty)$

$\Gamma$ $(X \le x, Y \le y) = (-\infty, x] \times (-\infty, y]$

$(X \le x, Y < +\infty) = \bigcup_{y \in \mathbb{R}} (-\infty, x] \times (-\infty, y]$

$= \lim_{y \to \infty} (-\infty, x] \times (-\infty, y]$ $\lrcorner$

$F(x,y) \cdots F_1(x)$

$f(x,y) \cdots f_1(x)$

$= P\left(\lim_{y \to \infty} (-\infty, x] \times (-\infty, y]\right)$

$= \lim_{y \to \infty} P\left((-\infty, x] \times (-\infty, y]\right)$

$= \lim_{y \to \infty} F(x,y)$.

$F_2(y) = \lim_{x \to \infty} F(x,y)$

$F(x,y) = \left(1 - e^{-2\min(x,y)} - 2e^{-y}\left(1 - e^{-\min(x,y)}\right)\right)$

$I(\min(x,y) \ge 0)$

$F_1(x) = \lim_{y \to \infty} F(x,y) = (1 - e^{-2x}) I(x \ge 0)$

$f_1(x) = 2e^{-2x} I(x \ge 0)$

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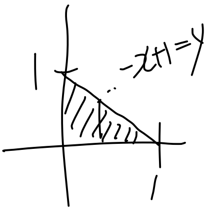
⟨Expectation⟩

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)f(x,y)\,dy\,dx$$




e.g. $f(x,y) = 120xy(1-x-y)I(x\geq 0, y\geq 0, x+y\leq 1)$

$$E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 120x^2y^2(1-x-y)I(x\geq 0, y\geq 0, x+y\leq 1)\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} 120x^2y^2(1-x-y)\,dy\,dx$$

$$= \int_0^1 120x^2\int_0^{1-x}(y^2 - xy^2 - y^3)\,dy\,dx$$

$$= \int_0^1 120x^2\left[\frac{1}{3}y^3 - \frac{x}{3}y^3 - \frac{1}{4}y^4\right]_0^{1-x} dx$$

$$= \int_0^1 120x^2\left(\frac{1}{3}(1-x)^3 - \frac{x}{3}(1-x)^3 - \frac{1}{4}(1-x)^4\right) dx$$

$$= \int_0^1 40x^2(1-x)^3 - 40x^3(1-x)^3 - 30x^2(1-x)^4\,dx$$

$$= \int_0^1 40x^2(1-x)^3(1-x) - 30x^2(1-x)^4\,dx$$

$$= \int_0^1 10x^2(1-x)^4\,dx \qquad \begin{matrix} 1-x=t \\ -dx=dt \end{matrix}$$

$$= \int_0^1 10(1-t)^2t^4\,dt$$

$$= 10\left[\frac{1}{7}t^6 - \frac{2}{6}t^6 + \frac{1}{5}t^5\right]_0^1$$

$$= 10\left(\frac{1}{7} - \frac{1}{3} + \frac{1}{5}\right)$$

$$= 2 + 10\left(\frac{-4}{21}\right) = \frac{42-40}{21} = \boxed{\frac{2}{21}}$$

TITLE DATE

$h(x,y) = g(x)$

$$E[g(X)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x) f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} g(x) \int_{-\infty}^{+\infty} f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} g(x) f_1(x)\,dx$$

TITLE DATE

2) positive-definite.

$Cov(X, X) = Var(X) \geq 0.$

$Var(X) = 0 \Longleftrightarrow X = \emptyset.$

$X := a_1 X_1 + \cdots + a_m X_n.$

By stochastically independent,

$Var(a_1 X_1 + \cdots + a_n X_n) = 0 \Rightarrow a_1 = \cdots = a_m = 0.$

Thus $X = \emptyset.$ $\square$

3) Symmetric

$$Cov(X, Y) = E[(X - \mu_1)(Y - \mu_2)]$$
$$= E[(Y - \mu_2)(X - \mu_1)]$$
$$= Cov(Y, X)$$

$\square$.

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$ ... inner produce

norm.

$$\therefore |Cov(X, Y)| \leq \sqrt{Var(X)} \cdot \sqrt{Var(Y)}$$

Cauchy-Schwarz inequality.

Thm. ·· 기계가 하는 증명. (By definition 으로 증명)

(a) $Cov(X,Y) = Cov(Y,X)$ , $Cov(X,X) = Var(X)$

proof) $Cov(X,Y) = E[(X-\mu_1)(Y-\mu_2)]$

$= E[(Y-\mu_2)(X-\mu_1)]$

$= Cov(Y,X)$

$Cov(X,X) = E[(X-\mu_1)^2] = Var(X)$ $\square$

(b) $Cov(aX+b, cY+d)$

$= E[(aX+b-a\mu_1-b)(cY+d-c\mu_2-d)]$

$= E[a(X-\mu_1)c(Y-\mu_2)]$

$= acE[(X-\mu_1)(Y-\mu_2)]$

$= acVar(X,Y)$ $\square$

TITLE DATE

(c) $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$Cov(X,Y) = E[(X-\mu_1)(Y-\mu_2)]$$

$$= E(XY - \mu_1 Y - \mu_2 X + \mu_1\mu_2)$$

$$= E[XY] - 2\mu_1\mu_2 + \mu_1\mu_2$$

$$= E[XY] - E[X] \cdot E[Y] \quad \square.$$

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TITLE DATE

Thm. $\rho = Corr(X,Y)$

$X \sim (\mu_1, \sigma_1)$, $Y \sim (\mu_2, \sigma_2)$

(a) $Var\left(\frac{Y-\mu_2}{\sigma_2} - \rho\frac{X-\mu_1}{\sigma_1}\right) = 1-\rho^2$

(b) $-1 \le \rho \le 1$

(c) $\rho = 1 \iff P\left(\frac{Y-\mu_2}{\sigma_2} = \frac{X-\mu_1}{\sigma_1}\right) = 1$

$\rho = -1 \iff P\left(\frac{Y-\mu_2}{\sigma_2} = -\frac{X-\mu_1}{\sigma_1}\right) = 1$

proof).

$Z_1 := \frac{Y-\mu_2}{\sigma_2} - \rho\frac{X-\mu_1}{\sigma_1}$

$E[Z_1] = 0$

$$Var(Z_1) = E[Z_1^2] = E\left[\left(\frac{Y-\mu_2}{\sigma_2}\right)^2\right] - 2\rho E\left[\left(\frac{Y-\mu_2}{\sigma_2}\right)\left(\frac{X-\mu_1}{\sigma_1}\right)\right] + \rho^2 E\left[\left(\frac{X-\mu_1}{\sigma_1}\right)^2\right]$$

$$= 1 - 2\rho\frac{Cov(X,Y)}{\sigma_2\sigma_1} + \rho^2$$

$$= 1-\rho^2$$

$\square$

(b) Since $Var(Z_1) = 1-\rho^2 \ge 0$ $-1 \le \rho \le 1$

$\square$

TITLE DATE

(C) $\rho = \pm 1 \iff Var(Z) = 0$

$\iff P(Z = \mu) = 1$

where $E[Z] = \mu = 0$.

$\therefore \rho = \pm 1 \iff P\left(\frac{Y - \mu_2}{\sigma_2} - \rho \frac{X - \mu_1}{\sigma_1} = 0\right) = 1$ ☐

<Corollary.>

By the previous theorem,

$$-1 \le \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \le 1$$

$$\therefore |Cov(X, Y)| \le \sqrt{Var(X)} \cdot \sqrt{Var(Y)}$$ ☐.

TITLE 2/29. DATE

결합적률생성함수 (Joint Moment Generating Function)

$E[|X^r Y^s|] < +\infty$ ·· 절대수렴.

$$E[X^r Y^s] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^r y^s f(x,y)\,dy\,dx$$

$(r,s)$ 번째 결합적률 (joint moment)

review)

$$M(t) := E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n$$

$$M^{(k)}(0) = E[X^k]$$

$$e^{t_1 X} e^{t_2 Y} = \left(\sum_{r=0}^{\infty} \frac{(t_1 X)^r}{r!}\right)\left(\sum_{s=0}^{\infty} \frac{(t_2 Y)^s}{s!}\right)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{X^r Y^s}{r!s!} t_1^r t_2^s$$

$$M(t_1,t_2) := E[e^{t_1 X} e^{t_2 Y}] = \sum_r \sum_s \frac{E[X^r Y^s]}{r!s!} t_1^r t_2^s$$

·· 정리에 의한 결과.

$$\therefore E[X^r Y^s] = \left.\frac{\partial^{r+s} M(t_1,t_2)}{\partial t_1^r\, \partial t_2^s}\right|_{t_1=0,\ t_2=0}$$

$(\exists\ h_1, h_2 > 0)$

Thm. $E[e^{t_1 X} e^{t_2 Y}] < +\infty$, $-h_1 < t_1 < h_1$ $-h_2 < t_2 < h_2$

$\Rightarrow \forall r,s \in \mathbb{N}$, $E[X^r Y^s] < +\infty$, $E[X^r Y^s] = \left.\frac{\partial^{r+s} M(t_1,t_2)}{\partial t_1^r\, \partial t_2^s}\right|_{t_1=0,\ t_2=0}$

$$M(t_1,t_2) = \sum_r \sum_s \frac{E[X^r Y^s]}{r!s!} t_1^r t_2^s.$$

TITLE DATE

결합누율생성함수(Joint Cumulant Generating Function).

review

$$C(t) := \log E[e^{tX}] = \log\left(1+\sum_{k=1}^{\infty}\frac{m_k}{k!}t^k\right)$$

$$= \sum_{r=0}^{\infty}\frac{c_r}{r!}t^r$$

$$c_1 = m_1 = E[X]$$

$$\frac{c_2}{2!} = \frac{m_2}{2!} - \frac{m_1^2}{2!} \quad = Var(X)$$

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$$C(t_1,t_2) := \log M(t_1,t_2) = \log E[e^{t_1X}e^{t_2Y}]$$

멱급수 전개식 $$= \sum_{\substack{r=0 \\ r+s\geq 1}}^{\infty}\sum_{s=0}^{\infty}\frac{c_{r,s}}{r!s!}t_1^r t_2^s$$

$$c_{r,s} = \left.\frac{\partial^{r+s} C(t_1,t_2)}{\partial t_1^r \partial t_2^s}\right|_{\substack{t_1=0\\ t_2=0}}$$ ... $(r,s)$번째 결합누율.

$$\log\left(1+\sum_{\substack{j=0 \\ j+k\geq 1}}^{\infty}\sum_{k=0}^{\infty}\frac{m_{j,k}}{j!k!}t_1^j t_2^k\right) = \sum_{\substack{r=0 \\ r+s\geq 1}}^{\infty}\sum_{s=0}^{\infty}\frac{c_{r,s}}{r!s!}t_1^r t_2^s$$

$$A := (m_{10}t_1 + m_{01}t_2) + (m_{20}t_1^2/2 + m_{11}t_1t_2 + m_{02}t_2^2/2)$$

$$C(t_1,t_2) = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \cdots$$

$$c_{0,2} = Var(Y)$$

$$c_{1,0} = m_{10} = E[X]$$

$$c_{0,1} = m_{0,1} = E[Y]$$

$$c_{11} = m_{11} - m_{10}\cdot m_{01} = E[XY] - E[X]\cdot E[Y]$$

$$\frac{c_{2,0}}{2!} = \frac{m_{20}}{2} - \frac{(m_{10})^2}{2} = E[X^2] - (E[X])^2 = Var(X)$$

TITLE DATE

e.g. $f(x,y) = 2e^{-x-y} I(0 \le x \le y)$

$$M(t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{t_1 x} e^{t_2 y} 2e^{-x-y} I(0 \le x \le y)$$

$$= \int_0^{+\infty}\int_x^{+\infty} 2e^{(t_1-1)x} e^{(t_2-1)y} \, dy\, dx$$

$t_2 < 1$

$$= \int_0^{\infty} 2e^{(t_1-1)x} \frac{1}{t_2-1}\left[e^{(t_2-1)y}\right]_x^{+\infty} dx$$

$$= \int_0^{\infty} 2e^{(t_1-1)x} \frac{1}{1-t_2} e^{(t_2-1)x} \, dx$$

$$= \int_0^{\infty} \frac{2}{1-t_2} e^{(t_1+t_2-2)x} dx$$

$t_1+t_2 < 2$

$$= \frac{2}{1-t_2} \frac{1}{t_1+t_2-2}\left[e^{(t_1+t_2-2)x}\right]_0^{\infty}$$

$$= \frac{1}{1-t_2} \frac{2}{2-t_1-t_2}$$

$$\log M(t_1 t_2) = \log\left(\frac{1}{1-t_2}\right) + \log\left(\frac{1}{1-\left(\frac{t_1+t_2}{2}\right)}\right)$$

$A - \frac{1}{2}A^2$

$$= -\log(1-t_2) - \log\left(1 - \frac{t_1+t_2}{2}\right)$$

$$= t_2 + \frac{1}{2}t_2^2 + \frac{t_1+t_2}{2} + \frac{1}{2}\left(\frac{t_1+t_2}{2}\right)^2 + \cdots$$

$$= \frac{1}{2}t_1 + \frac{3}{2}t_2 + \frac{1}{8}t_1^2 + \frac{t_1 t_2}{4} + \frac{5}{8}t_2^2$$

$C_{1,0} = \frac{1}{2}$ $C_{1,1} = \frac{1}{4}$ $E[X] = \frac{1}{2}$ $Var(X) = \frac{1}{4}$

$C_{0,1} = \frac{3}{2}$ $C_{2,0} = \frac{1}{4}$ $E[Y] = \frac{3}{2}$ $Var(Y) = \frac{5}{4}$

$C_{0,2} = \frac{5}{4}$ $Cov(X,Y) = \frac{1}{4}$

$$M_X(s) = E[e^{sX}]$$

$$M_{X,Y}(s,t) = E[e^{sX} e^{tY}].$$

marginal moment generating function.

(t=0 으로 생각)

t=0 이라 가정

$$M_{X,Y}(s,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{sx} e^{ty} f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} e^{sx} \int_{-\infty}^{+\infty} e^{ty} f(x,y)\,dy\,dx.$$

$$= \int_{-\infty}^{+\infty} e^{sx} \int_{-\infty}^{+\infty} f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} e^{sx} f_1(x)\,dx$$

$$=: M_X(s)$$

TITLE DATE

$$M_X(s) = E[e^{sX}]$$

$$M_{XY}(s,t) = E[e^{sX} e^{tY}]$$

$$M_X(s) = E[e^{sX}] = \int_{-\infty}^{+\infty} e^{sx} f_1(x)\,dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{sx} f(x,y)\,dy\,dx$$

$$M_X(s) = M_{X,Y}(s,0) \qquad e^{ty} \quad t=0$$

$$M_{X,Y}(s,0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{sx} e^{0y} f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} e^{sx} \int_{-\infty}^{+\infty} e^{0y} f(x,y)\,dy\,dx.$$

$$= \int_{-\infty}^{+\infty} e^{sx} f_1(x)\,dx$$

$$=: M_X(s).$$

TITLE Conditional joint distribution. DATE

$X, Y$: discrete random variable.

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

$$f_{2|1}(y|x) := \frac{f(x,y)}{f_1(x)} \quad (f_1(x) > 0)$$

$X, Y$: Continuous

$P(c \le Y \le d \mid X=x) = 0$ $P(X=x)=0$ 이므로

앞의 정의를 사용할 수 없다. 대신 확률밀도함수의 연속성을 활용.

$h > 0$.

$$P(c \le Y \le d \mid x \le X \le x+h) = \frac{P(c \le Y \le d, x \le X \le x+h)/h}{P(x \le X \le x+h)/h}$$

$$P(c \le Y \le d, x \le X \le x+h)/h$$

$$= \frac{1}{h}\int_x^{x+h}\int_c^d f(x,y)\,dy\,dx \quad g(x) := \int_c^d f(x,y)\,dy$$

$f(x,y)$가 연속이므로, $g(x)$도 연속. 따라서 $g(x)$의 anti-derivative $G(x)$가 존재. $G'(x) = g(x)$

$$= \frac{G(x+h) - G(x)}{h} \quad \text{(미적분학 기본정리)}$$

마찬가지로, $P(x \le X \le x+h)/h = \frac{1}{h}\int_x^{x+h} f_1(x)\,dx = \frac{F(x+h) - F(x)}{h}$

$$\lim_{h \to 0^+} P(c \le Y \le d \mid x \le X \le x+h) = \frac{\int_c^d f(x,y)\,dy}{f_1(x)} =: P(c \le Y \le d \mid X=x)$$

$$\therefore f_{2|1}(y|x) = \frac{f(x,y)}{f_1(x)}$$

TITLE DATE

Thm 2.3.1

$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b P(c \le Y \le d \mid X = x) f_1(x)\,dx$$

proof)

$$\int_a^b P(c \le Y \le d \mid X = x) f_1(x)\,dx$$

$$= \int_a^b \left( \int_c^d f_{2|1}(y|x)\,dy \right) f_1(x)\,dx$$

$$= \int_a^b \int_c^d f_{2|1}(y|x) f_1(x)\,dy\,dx$$

$$= P(a \le X \le b,\ c \le Y \le d) \qquad \square$$

e.g.

$$f(x,y) = 2e^{-x-y} I_{(0 \le x \le y)}$$

$$f_1(x) = \int_{-\infty}^{+\infty} 2e^{-x-y} I_{(0 \le x \le y)} = \int_x^{+\infty} 2e^{-x} e^{-y} I_{(x \ge 0)}\,dx$$

$$= 2e^{-x}\left[-e^{-y}\right]_x^{+\infty} I_{(x \ge 0)} = 2e^{-2x} I_{(x \ge 0)}$$

$x \ge 0$,

$$f_{2|1}(y|x) = \frac{2e^{-x-y} I_{(0 \le x \le y)}}{2e^{-2x} I_{(x \ge 0)}} = e^{x-y} I_{(y \ge x)}$$

$$P(Y \ge c \mid X = x) = \int_c^{\infty} e^{x-y} I_{(y \ge x)}\,dy$$

$$= \int_{\max(c,x)}^{\infty} e^{x-y}\,dy$$

$$= e^{x - \max(c,x)}$$

$$P(X\geq 2, Y\geq 3) = \int_2^{+\infty} P(Y\geq 3 \mid X=x) f_1(x)\,dx$$

$$= \int_2^{+\infty} e^{x-\max(3,x)}\, 2e^{-2x} I(x\geq 0)\,dx$$

$$= \int_2^{+\infty} e^{x-\max(3,x)}\, 2e^{-2x}\,dx.$$

$$= \int_2^{3} 2e^{-3-x}\,dx + \int_3^{+\infty} 2e^{-2x}\,dx$$

$$= \left[-2e^{-3-x}\right]_2^3 + \left[e^{-2x}\right]_3^{+\infty}$$

$$= -2e^{-6} + 2e^{-5} + e^{-6}$$

$$= 2e^{-5} + e^{-6} \qquad \square$$

TITLE Conditional Expectation, Variance. DATE

$$\mu_{2|1}(x) := E[Y \mid X=x]$$

$$= \int_{-\infty}^{+\infty} f_{2|1}(y|x)\, dy$$

y에 대한 적분이므로 x만 남아 x에 대한 함수이다.

$$E[g(X,Y) \mid X=x] = \int_{-\infty}^{+\infty} g(x,y) f_{2|1}(y|x)\, dy$$

$$\mathrm{Var}(Y \mid X=x) = E[(Y-\mu_{2|1}(x))^2 \mid X=x]$$

$$= E[Y^2 \mid X=x] - 2\mu_{2|1}(x) E[Y \mid X=x] + (\mu_{2|1}(x))^2$$

$$= E[Y^2 \mid X=x] - (\mu_{2|1}(x))^2$$

$$= E[Y^2 \mid X=x] - (E[Y \mid X=x])^2 \quad \square$$

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Thm 2.3.4

$x \mapsto \mu_{2|1}(x)$, 즉 $\mu_{2|1}(X)$를 새로운 random variable
$g(X) = E[Y|X]$라 생각하자.

(a) $E[E[Y|X]] = E[Y]$

(b) $Cov(Y - E[Y|X], v(X)) = 0 \quad \forall v(X)$.

covariance를 free vector space에서 inner product라 생각하면 X와 수직인 성분으로 Y를 보내는 것.

proof) $E[E[Y|X]] = \int_{-\infty}^{+\infty} E[Y|X=x] f_1(x)\,dx$

$$= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} y\, f_{2|1}(y|x)\,dy \right) f_1(x)\,dx$$

$$= \int_{-\infty}^{+\infty} y \int_{-\infty}^{+\infty} f_{2|1}(y|x) f_1(x)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} y \int_{-\infty}^{+\infty} f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} y\, f_2(y)\,dy = E[Y].$$

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(b) $Z_1 := Y - E[Y|X]$

$E[Z_1|X] = E[Y - E[Y|X]|X]$

$= E[Y|X] - E[E[Y|X]|X]$ $\because$ (a)

$= E[Y|X] - E[Y|X] = 0$

$E[Z_1] = E[Y - E[Y|X]] = E[Y] - E[E[Y|X]]$

$= E[Y] - E[Y] = 0$.

$Cov(Z_1, v(X)) = E[Z_1 v(X)] - E[Z_1]E[v(X)]$

$= E[Z_1 v(X)]$

$= E[E[Z_1 v(X)|X]]$

$= E[v(X) E[Z_1|X]] = 0$

$\because$ $E[g(X=x)|X=x] = \int_{-\infty}^{+\infty} g(x) f_{2|1}(y|x)\,dy$

$= g(x) \int_{-\infty}^{+\infty} f_{2|1}(y|x)\,dy$

$= g(x)$

$E[g(X)|X] = g(X)$

$\square$.

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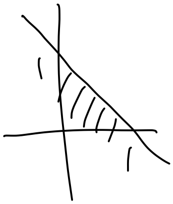



e.g. $f(x,y) = 120xy(1-x-y)\, I(x\ge 0, y\ge 0, x+y\le 1)$

$$f_1(x) = \int_{-\infty}^{+\infty} 120xy(1-x-y)\, I(x\ge 0, y\ge 0, x+y\le 1)\, dy$$

$$= \int_0^{1-x} 120xy(1-x-y)\, I(0\le x\le 1)\, dy$$

$$= 120x \int_0^{1-x} ((1-x)y - y^2)\, dy$$

$$= 120x \left[ \frac{1-x}{2} y^2 - \frac{1}{3} y^3 \right]_0^{1-x} I(0\le x\le 1)$$

$$60x(1-x)^3 - 40x(1-x)^3 = 20x(1-x)^3 I(0\le x\le 1)$$

$$f_{2|1}(y|x) = \frac{120xy(1-x-y)\ \overbrace{}^{I(x\ge 0, y\ge 0, x+y\le 1)}}{20x(1-x)^3 I(0\le x\le 1)}$$

$$= \frac{6y(1-x-y)}{(1-x)^3} I(0\le y\le 1-x)$$

$$E[Y|X=x] = \int_{-\infty}^{+\infty} y \cdot f_{2|1}(y|x)\, dy \quad I(0\le x\le 1)$$

$$= \int_0^{1-x} \frac{6y^2(1-x-y)}{(1-x)^3}\, dy \quad I(0\le x\le 1)$$

$$= \frac{6}{(1-x)^3} \int_0^{1-x} ((1-x)y^2 - y^3)\, dy \quad I(0\le x\le 1)$$

$$= \frac{6}{(1-x)^3} \left[ \frac{1-x}{3} y^3 - \frac{1}{4} y^4 \right]_0^{1-x} \quad I(0\le x\le 1)$$

$$= \frac{1}{(1-x)^3} \left( 2(1-x)^4 - \frac{3}{2}(1-x)^4 \right)$$

$$2(1-x) - \frac{3}{2}(1-x) = \frac{1}{2}(1-x)$$

$$E[Y|X] = \frac{1}{2}(1-X)$$

$$E[E[Y|X]] = \int_{-\infty}^{+\infty} \frac{1}{2}(1-x)\, 20x(1-x)^3\, I(0\le x\le 1)\, dx$$

$$= \int_0^1 10x(1-x)^4\, dx$$

$$= \int_0^1 10(1-t)t^4\, dt$$

$$= 10\left[ \frac{1}{5}t^5 - \frac{1}{6}t^6 \right]_0^1$$

$$= \frac{1}{3} = E[Y].$$

$$E[Y] = \int_{-\infty}^{+\infty} y f_2(y)\, dy = \int_{-\infty}^{+\infty} y\, 20y(1-y)^3 I(0\le y\le 1)\, dy$$

$$= \int_0^1 20y^2(1-y)^3\, dy$$

$$= \int_0^1 20(1-t)^2 t^3\, dt$$

$$= 20\int_0^1 (t^5 - 2t^4 + t^3)\, dt$$

$$= 20\left[ \frac{1}{6}t^6 - \frac{2}{5}t^5 + \frac{1}{4}t^4 \right]_0^1 = 20\left( \frac{1}{6} - \frac{2}{5} + \frac{1}{4} \right) = \frac{1}{3}$$

$E[X] = \frac{1}{3}$

$Cov(Y - E[Y|X], X) = 0.$

$E[(Y - (1-X)/2)X] - E[Y - E[Y|X]] \cdot E[X]$

$= E[XY] - E[X]/2 + E[X^2]/2$

$E[X^2] = \int_{-\infty}^{+\infty} 20x^3(1-x)^3 I(0 \le x \le 1)\,dx$

$= \int_0^1 20x^3(1-x)^3\,dx = \int_0^1 20x^3(1-3x+3x^2-x^3)\,dx$

$= 20\int_0^1 (x^3 - 3x^4 + 3x^5 - x^6)\,dx$

$= 20\left[\frac{1}{4}x^4 - \frac{3}{5}x^5 + \frac{1}{2}x^6 - \frac{1}{7}x^7\right]_0^1$

$= 5 - 12 + 10 - \frac{20}{7}$

$= 3 - \frac{20}{7} = \frac{1}{7}$

$E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 120x^2y^2(1-x-y) I(0 \le x, 0 \le y, x+y \le 1)\,dy\,dx$

$= \int_0^1 \int_0^{1-x} 120x^2\{(1-x)y^2 - y^3\}\,dy\,dx$

$= \int_0^1 120x^2\left[\frac{1-x}{3}y^3 - \frac{1}{4}y^4\right]_0^{1-x} dx$

$= \int_0^1 40x^2(1-x)^4 - 30x^2(1-x)^4\,dx$

$= \int_0^1 10x^2(1-x)^4\,dx = \int_0^1 10(1-t)^2t^4\,dt$

$= 10\int_0^1 (t^6 - 2t^5 + t^4)\,dt$

$= 10\left[\frac{1}{7}t^7 - \frac{2}{6}t^6 + \frac{1}{5}t^5\right]_0^1$

$\frac{10}{7} - \frac{10}{3} + 2 = \frac{10}{7} - \frac{4}{3}$

$= \frac{2}{21}$

$\therefore Cov(Y - E[Y|X], X) = \frac{2}{21} - \frac{1}{3} \times \frac{1}{2} + \frac{1}{7} \times \frac{1}{2}$

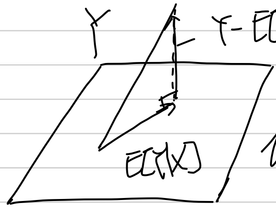
$= \frac{4 - 7 + 3}{42} = 0.$

free vector space에서 내적을 $Cov(\cdot,\cdot)$으로 정의한다.

$Cov(Y-E[Y|X], v(X))=0$ 이라는 것은 $Y-E[Y|X]$와 $v(X)$가 orthogonal 함을 의미한다.

Review $E[E[Y|X]] = E[Y]$

$$Cov(Y-E[Y|X], v(X)) = 0, \quad \forall v(X)$$




$Y-E[Y|X] \in$ orthogonal complement of $v(X)$ $\{v(x) \mid v: \text{function}\}$

Thm. 2.3.5.

"$E[(Y-E[Y|X])^2] \le E[(Y-v(X))^2], \ \forall v(X)$

& $E[(Y-E[Y|X])^2] = E[Var(Y|X)]$"

proof). $Y-v(X) = (Y-E[Y|X]) \oplus (E[Y|X]-v(X))$

$$(Y-v(X))^2 = (Y-E[Y|X])^2 + (E[Y|X]-v(X))^2 + 2(Y-E[Y|X])(E[Y|X]-v(X))$$

expectation.

$$E[(Y-E[Y|X])(E[Y|X]-v(X))] = Cov(Y-E[Y|X], E[Y|X]-v(X))$$

$\because E[Y-E[Y|X]] = E[Y]-E[E[Y|X]] = E[Y]-E[Y]=0.$

$$E[(Y-v(X))^2] = E[(Y-E[Y|X])^2 + (E[Y|X]-v(X))^2] \ge E[(Y-E[Y|X])^2]$$

$$E[(Y-E[Y|X])^2|X] = Var(Y|X)$$

$$\therefore\ E[(Y-E[Y|X])^2] = E[E[(Y-E[Y|X])^2|X]]$$
$$= E[Var(Y|X)] \qquad \square$$

Thm 2.3.6. (분산분해)

$$Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$$

$\mu := E[Y]$

$$Y-\mu = \left(\boxed{Y-E[Y|X]}\right) + \left(\boxed{E[Y|X]-\mu}\right) \cdots (*)$$

$$Var(Y) = E[(Y-\mu)^2] \quad \text{(by definition)}$$
$$= E[(Y-E[Y|X])^2] + E[(E[Y|X]-\mu)^2]$$

$$\left( E[(Y-E[Y|X])(E[Y|X]-\mu)] = Cov(Y-E[Y|X],\ E[Y|X]-\mu) = 0 \right.$$

Let $E[Y|X] := v(X)$

Since $\mu = E[Y] = E[E[Y|X]]$,

$$E[(v(X)-E[v(X)])^2] = Var(v(X))$$

$$= E[\underbrace{Var(Y|X)}]+Var(E[Y|X])$$

$$\because\ \underset{\substack{\parallel \\ E[Var(Y|X)]}}{E[E[(Y-E[Y|X])^2|X]]} = E[(Y-E[Y|X])^2]$$

$$\therefore\ Var(Y) = E[Var(Y|X)] + Var(E[Y|X]) \qquad \square$$

예제 23.8. $f(x,y) = 120xy(1-x-y) I(x\ge0, y\ge0, x+y\le1)$

① $E[Y|X]$, $Var(Y|X)$

② $Var(E[Y|X]) + E[Var(Y|X)]$

$$E[Y|X=x] = \int_{-\infty}^{+\infty} y f_{2|1}(y|x)\, dy$$

$$f_1(x) = \int_{-\infty}^{+\infty} 120xy(1-x-y) I(x\ge0, y\ge0, x+y\le1)$$

$$= 120x \int_0^{1-x} y(1-x-y)\, dy\, I(0\le x\le1) = 120x\left[\frac{(1-x)}{2}y^2 - \frac{1}{3}y^3\right]_0^{1-x} I(0\le x\le1)$$

$$= 60x(1-x)^3 - 40x(1-x)^3$$

$$= 20x(1-x)^3 I(0\le x\le1)$$

$$f_{2|1}(y|x) = \frac{120xy(1-x-y) I(x\ge0, y\ge0, x+y\le1)}{20x(1-x)^3 I(0\le x\le1)}$$

$$= \frac{6y(1-x-y)}{(1-x)^3} I(0\le y\le1-x)$$

$$E[Y|X=x] = \int_0^{1-x} \frac{6y^2(1-x-y)}{(1-x)^3}\, dy\ I(0\le x\le1) = \frac{6}{(1-x)^3}\left[\frac{(1-x)}{3}y^3 - \frac{1}{4}y^4\right]_0^{1-x}$$

$$= 2(1-x) - \frac{3}{2}(1-x) = \frac{1}{2}(1-x)$$

$$E[Y|X] = \frac{1-X}{2}$$

$$Var(Y|X) = E[(Y-E[Y|X])^2|X] = E[Y^2|X] + E\left[\left(\frac{1-X}{2}\right)^2\Big|X\right]$$

$$-2E[Y E[Y|X]|X]$$

$$\frac{6}{(1-x)^3}\left[\frac{1-x}{4}y^4 - \frac{1}{5}y^5\right]_0^{1-x}$$

$$= E[Y^2|X] + \left(\frac{1-X}{2}\right)^2 - 2(E[Y|X])^2$$

$$\frac{3}{2}(1-x)^2 - \frac{6}{5}(1-x)^2$$

$$\frac{15-12}{10} \quad \frac{3}{10}(1-x)^2$$

$$= \frac{3}{10}(1-X)^2 + \frac{(1-X)^2}{4} - 2\cdot\frac{(1-X)^2}{4}$$

$$= \frac{3}{10}(1-X)^2 - \frac{(1-X)^2}{4}$$

$$= \frac{(1-X)^2}{20}$$

$$\therefore Var(Y|X) = \frac{(1-X)^2}{20}$$

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$$Var\left(E[Y|X]\right) + E[Var(Y|X)]$$

$Var(Y|X) = \frac{(1-X)^2}{20}$ $E[Y|X] = \frac{1-X}{2}$

$E[X] = \frac{1}{3}$ $E[X^2] = \frac{1}{7}$ $Var(X) = \frac{2}{63}$

$$Var\left(\frac{1-X}{2}\right) + E\left[\frac{(1-X)^2}{20}\right]$$

$$= \frac{1}{4}Var(X) + \frac{1}{20}\left(E[X^2] - 2E[X] + 1\right)$$

$$= \frac{1}{4} \times \frac{2}{63} + \frac{1}{20}\left(\frac{1}{7} - \frac{2}{3} + 1\right)$$

$$= \frac{1}{2\times 63} + \frac{1}{20}\left(\frac{1}{7} + \frac{1}{3}\right) = \frac{1}{2\times 63} + \frac{1}{\cancel{20}_{2}} \times \frac{\cancel{10}}{21}$$

$$= \frac{1+3}{2\times 63} = \boxed{\frac{2}{63}}$$

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24 Def. $P(a \le X \le b,\ c \le Y \le d) = P(a \le X \le b)\,P(c \le Y \le d)$

$\Rightarrow X, Y$: mutually independent.

Thm. TFAE.

(a) $\text{cdf}(x,y) = \text{cdf}_1(x)\,\text{cdf}_2(y)$

(b) $\text{pdf}(x,y) = \text{pdf}_1(x)\,\text{pdf}_2(y)$

분포결정성.

(c) $\text{mgf}_{1,2}(t_1,t_2) = \text{mgf}_1(t_1)\,\text{mgf}_2(t_2)$

(d) $P(X\in A,\ Y\in B) = P(X\in A)\,P(X\in B)\ \ ^\forall A, B.$

(e) $X, Y$: mutually independent.

Rmk. $\text{pdf}_{1,2}(x,y) = g_1(x)\,g_2(y) \Rightarrow$ (b)

$\because \int_{-\infty}^{+\infty} g_1(x)dx \int_{-\infty}^{+\infty} g_2(y)dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \text{pdf}_{1,2}(x,y)\,dy\,dx = 1$

$$g_1(x)g_2(y) = \frac{g_1(x)}{\int_{-\infty}^{+\infty} g_1(x)dx}\,\frac{g_2(y)}{\int_{-\infty}^{+\infty} g_2(y)dy} = \text{pdf}_1(x)\,\text{pdf}_2(y).$$

$$\text{pdf}_1(x) = \int_{-\infty}^{+\infty} \text{pdf}_{1,2}(x,y)\,dy$$
$$= \int_{-\infty}^{+\infty} g_1(x)g_2(y)\,dy = g_1(x)\int_{-\infty}^{+\infty} g_2(y)\,dy.$$

$$\text{pdf}_2(y) = g_2(y)\int_{-\infty}^{+\infty} g_1(x)\,dx$$

$$g_1(x)g_2(y) = \frac{\text{pdf}_1(x)\,\text{pdf}_2(y)}{\int_{-\infty}^{+\infty} g_1(x)dx\int_{-\infty}^{+\infty} g_2(y)dy} = \text{pdf}_1(x)\,\text{pdf}_2(y)$$

$\left(\because \int_{-\infty}^{+\infty} g_1(x)dx \int_{-\infty}^{+\infty} g_2(y)dy = 1\right)$ $\square$

$f$: function $X \to Y$.

$f^{-1}(A) := \{x \in X \mid f(x) \in A\}$

e.g. $A = [1,3)$

$X \in A \Leftrightarrow X \in [1,3)$

$\Leftrightarrow 1 \le X \le 3$

Thm 2.4.2. $X, Y$: indep. $\Rightarrow g_1(X), g_2(Y)$: indep.

proof) $(g_1(X) \in A,\ g_2(Y) \in B) = (X \in g_1^{-1}(A),\ Y \in g_2^{-1}(B))$

$\{s \in S : g_1(X)(s) \in A\}$

$g_1^{-1}, g_2^{-1}$: preimage.

$$P(\underbrace{g_1(X) \in A}_{\{s\in S:\ g_1(X)(s)\in A\}},\ g_2(Y) \in B)$$

$$= P(\underbrace{X \in g_1^{-1}(A),\ Y \in g_2^{-1}(B)}_{\{s \in S:\ X(s) \in g^{-1}(A)\}})$$

$$= P(X \in g_1^{-1}(A))\, P(Y \in g_2^{-1}(B))$$

$$= P(g_1(X) \in A)\, P(g_2(Y) \in B) \quad \forall A, B \qquad \square.$$

Thm 2.4.3. $X, Y$: indep.

$$E[g_1(X) g_2(Y)] = E[g_1(X)]\, E[g_2(Y)]$$

proof) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g_1(x) g_2(y)\, pdf_{1,2}(x,y)\, dy\, dx$

$$= \int_{-\infty}^{+\infty} g_1(x)\, pdf_1(x)\, dx \int_{-\infty}^{+\infty} g_2(y)\, pdf_2(y)\, dy$$

$$= E[g_1(X)]\, E[g_2(Y)] \qquad \square.$$

Thm 2.4.4. $X, Y$: indep. $\Rightarrow Cov(X, Y) = 0$

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

$$= E[X]E[Y] - E[X]E[Y]$$

$$= 0. \qquad \square.$$

Thm 2.4.5.

(a) $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

(b) $X, Y$: indep. $\Rightarrow Var(X+Y) = Var(X) + Var(Y)$

proof) $Var(X+Y) = E[(X+Y-E[X+Y])^2]$

$$E[X+Y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x+y) f_{1,2}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f_{1,2}(x,y)\,dy\,dx + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y f_{1,2}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} x \int_{-\infty}^{+\infty} f_{1,2}(x,y)\,dy\,dx + \int_{-\infty}^{+\infty} y \int_{-\infty}^{+\infty} f_{1,2}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} x f_1(x)\,dx + \int_{-\infty}^{+\infty} y f_2(y)\,dy$$

$= E[X] + E[Y]$. 이므로

$$E[(X+Y-E[X+Y])^2] = E[(X-E[X]+Y-E[Y])^2]$$

$$= E[(X-E[X])^2] + E[(Y-E[Y])^2] + 2E[(X-E[X])(Y-E[Y])]$$

$$= Var(X) + Var(Y) + 2Cov(X,Y)$$

If $X, Y$: mutually independent $\Rightarrow Cov(X,Y) = 0$

$\therefore Var(X+Y) = Var(X) + Var(Y)$

$\square$.

⓪ $\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(x_1,\dots,x_k)\,dx_k\cdots dx_1 = 1$

⓪ $\int_{a_1}^{b_1}\cdots\int_{a_k}^{b_k} f(x_1,\dots,x_k)\,dx_k\cdots dx_1$

$= P(a_1 \le X_1 \le b_1, \dots, a_k \le X_k \le b_k)$

marginal prob: ⓪ $f_{1,2}(x,y) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(x,y,x_3,\dots,x_k)\,dx_k\cdots dx_3$

⓪ $E[g(X_1,\dots,X_k)]$ $g: \mathbb{R}^k \to \mathbb{R}$

$:= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} g(x_1,\dots,x_k) f(x_1,\dots,x_k)\,dx_k\cdots dx_1$

**Def** $X = (X_1,\dots,X_k)^t \in \mathbb{R}^k$

평균벡터 .... $E[X] := (E[X_1],\dots,E[X_k])^t \in \mathbb{R}^k$

분산행렬 ... $Var(X) := \left(Cov(X_i,X_j)\right)_{\substack{1\le i\le k\\1\le j\le k}} \in M_k(\mathbb{R})$

(symmetric)

행렬은 선형사상이므로 이 선형사상은 self-adjoint.

(self-adjoint $\Leftrightarrow T^* = T$)

$(W_{ij}) \in M_k(\mathbb{R})$ .. 행렬의 expectation은 entry 마다 expectation으로 정의.

$E[(W_{ij})] = (E[W_{ij}]) \in M_k(\mathbb{R})$

Thm 25.2 $V=(V_{ij})$, $W=(W_{ij})$ : $k \times k$ matrices for r.V

(a) $E[CWD] = CE[W]D$ $C, D$: constant matrices.

(b) $E[V+W] = E[V] + E[W]$

proof)

(a) $\big[E[CWD]\big]_{ij} = E\big[[CWD]_{ij}\big]$

$= E\Big[\sum_{l=1}^{k} [CW]_{il} [D]_{lj}\Big]$

$= E\Big[\sum_{l=1}^{k} \Big(\sum_{m=1}^{k} [C]_{im} [W]_{ml}\Big)[D]_{lj}\Big]$

constant matrices

$= \sum_{l=1}^{k} \Big(\sum_{m=1}^{k} [C]_{im} E\big[[W]_{ml}\big]\Big)[D]_{lj}$

$= \sum_{l=1}^{k} [CE[W]]_{il} [D]_{lj} = [CE[W]D]_{ij}$

(b) $E[V+W] = E\big[[V+W]_{ij}\big]$

$= E\big[[V]_{ij}\big] + E\big[[W]_{ij}\big]$

$= E[V] + E[W]$ $\square$

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$X - E[X]$

$= (X_1 - E[X_1], \ldots, X_k - E[X_k])^t$

$\mathbb{V} = (v_1, \ldots, v_k)$

$\mathbb{V} \cdot \mathbb{V}' = \begin{pmatrix} v_1 v_1' & \cdots & v_1 v_k' \\ \vdots & \ddots & \vdots \\ v_k v_1' & \cdots & v_k v_k' \end{pmatrix}$

$= (v_1 \mathbb{V}' \cdots v_k \mathbb{V}')$

$\mathrm{rank}(\mathbb{V} \cdot \mathbb{V}') = 1$

<u>Note</u> $\mathrm{Var}(X) = (\mathrm{Cov}(X_i, X_j))$

$= (E[(X_i - E[X_i])(X_j - E[X_j])])$

$= E[(X - E[X])(X - E[X])^t] \in M_k(\mathbb{R})$

$X = (X_1, \ldots, X_k)^t \quad Y = (Y_1, \ldots, Y_l)^t$

$\mathrm{Cov}(X, Y) = (\mathrm{Cov}(X_i, Y_j)) \in M_{k \times l}(\mathbb{R})$

공분산 행렬.

<u>Rmk.</u> ① $Y = X \Rightarrow \mathrm{Cov}(X, X) = \mathrm{Var}(X)$

② $\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])^t]$

$A \in M_{n\times k}(\mathbb{R})$, $b \in \mathbb{R}^n$

Thm 2.5.3. $X = (X_1, \dots, X_k) \in \mathbb{R}^k$

(a) $E[AX+b] = A\,E[X] + b$

proof) $E[AX+b] = \left(E\left[\sum_{j=1}^{k} A_{1j}X_j\right], \dots, E\left[\sum_{j=1}^{k} A_{nj}X_j\right]\right)^t + b$

$$= \left(\sum_{j=1}^{k} A_{1j}E[X_j], \dots, \sum_{j=1}^{k} A_{nj}E[X_j]\right)^t + b$$

$$= A\,E[X] + b \quad \square$$

quadratic form ···· (b) $Var(AX+b) = A\,Var(X)A^t$ $\quad E[AX+b] = AE[X]+b$

proof) $Var(AX+b) = E[(AX+b-AE[X]-b)(AX+b-AE[X]-b)^t]$

$$= E[A(X-E[X])(X-E[X])^tA^t]$$

$$= A\,E[(X-E[X])(X-E[X])^t]A^t$$

$$= A\,Var(X)A^t \quad \square$$

(c) $Cov(AX+b, CY+d) = A\,Cov(X,Y)C^t$

proof) $Cov(AX+b, CY+d) = E[(AX+b-AE[X]-b)(CY+d-CE[Y]-d)^t]$

$$= E[A(X-E[X])(Y-E[Y])C^t]$$

$$= A\,E[(X-E[X])(Y-E[Y])]C^t$$

$$= A\,Cov(X,Y)C^t \quad \square$$

Covariance는 free vector space의 내적으로 이해한다면 bilinear가 되는 것은 "자연스럽다". $\rightarrow$

(d) $\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$

$\mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$

proof) $E[X+Y] = (E[X_1+Y_1], \dots, E[X_k+Y_k])$

$= (E[X_1], \dots, E[X_k]) + (E[Y_1], \dots, E[Y_k])$

$= E[X] + E[Y]$

$$\begin{aligned}
\mathrm{Cov}(X+Y, Z) &= E[(X+Y-E[X+Y])(Z-E[Z])^t] \\
&= E[(X-E[X]+Y-E[Y])(Z-E[Z])^t] \\
&= E[(X-E[X])(Z-E[Z])^t] \\
&\quad + E[(Y-E[Y])(Z-E[Z])^t] \\
&= \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z) \qquad \square
\end{aligned}$$

(e) $\mathrm{Cov}(Y, X) = \mathrm{Cov}(X, Y)^t$

proof)

$$\begin{aligned}
\mathrm{Cov}(X, Y)^t &= E[((X-E[X])(Y-E[Y])^t)^t] \\
&= E[(Y-E[Y])(X-E[X])^t] \\
&= \mathrm{Cov}(Y, X) \qquad \square
\end{aligned}$$

(f) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X)$

proof)

$$\begin{aligned}
\mathrm{Var}(X+Y) &= E[(X-E[X]+Y-E[Y])(X-E[X]+Y-E[Y])^t] \\
&= E[(X-E[X])(X-E[X])^t] + E[(Y-E[Y])(Y-E[Y])^t] \\
&\quad + E[(X-E[X])(Y-E[Y])^t] \\
&\quad + E[(Y-E[Y])(X-E[X])^t] \\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X) \qquad \square
\end{aligned}$$

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Thm 2.5.4. $X=(X_1,\dots,X_k)^t$

$\Rightarrow Var(X)$: positive semi definite

i.e $\begin{cases} Var(X) = Var(X)^t \\ a^t Var(X) a \geq 0 \quad \forall a \in \mathbb{R}^k. \end{cases}$

proof) $a^t Var(X) a = Var(a^t X) \geq 0.$

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Def) $E[X_1^{r_1}\cdots X_k^{r_k}] := \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} x_1^{r_1}\cdots x_k^{r_k} f(x_1,\dots,x_k)\,dx_k\cdots dx_1.$

"joint moment"

$M(t_1,\dots,t_k) := E[e^{t_1X_1+\cdots+t_kX_k}]$

Thm 25.5. $E[e^{t_1X+\cdots+t_kX_k}] < +\infty$, $-h_i < t_i < h_i \ (\exists h_i > 0)$

$\Rightarrow \forall r_1,\dots,r_k \in \mathbb{N}_0$, $\exists E[X_1^{r_1}\cdots X_k^{r_k}]$

$$M(t_1,\dots,t_k) = \sum_{r_1=0}^{\infty}\cdots\sum_{r_k=0}^{\infty} \frac{E[X_1^{r_1}\cdots X_k^{r_k}]}{r_1!\cdots r_k!} t_1^{r_1}\cdots t_k^{r_k}$$

$$\therefore E[X_1^{r_1}\cdots X_k^{r_k}] = \left[\frac{\partial^{r_1+\cdots+r_k}}{\partial t_1^{r_1}\cdots\partial t_k^{r_k}} M(t_1,\dots,t_k)\right]_{t_1=\cdots=t_k=0}$$

Def) $C(t_1,\dots,t_k) := \log E[e^{t_1X_1+\cdots+t_kX_k}]$; "cgf"

$$E[X_i] = \left.\frac{\partial C(t_1,\dots,t_k)}{\partial t_i}\right|_{t_i=0}.$$

$$Cov(X_i,X_j) = \left.\frac{\partial^2 C(t_1,\dots,t_k)}{\partial t_i \partial t_j}\right|_{t_i=t_j=0}.$$

예2.5.9.

$$f(x_1, x_2, x_3) = 6e^{-x_1-x_2-x_3} I(0 \le x_1 \le x_2 \le x_3)$$

$$X = (X_1, X_2, X_3)^t$$

$M_X(t)$, $C_X(t)$, $E[X]$?

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{t_1x_1+t_2x_2+t_3x_3}\, 6e^{-x_1-x_2-x_3}\, I \, dx_3 dx_2 dx_1$$

$$= 6\int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty e^{(t_1-1)x_1} e^{(t_2-1)x_2} e^{(t_3-1)x_3} \, dx_3 dx_2 dx_1$$

$t_3 < 1$ $\qquad \left[\frac{1}{t_3-1} e^{(t_3-1)x_3}\right]_{x_2}^{\infty} \qquad t_3 - 1 < 0$

$t_2 + t_3 < 2$

$$= \frac{6}{1-t_3} \int_0^\infty \int_{x_1}^\infty e^{(t_1-1)x_1} e^{(t_2+t_3-2)x_2} \, dx_2 dx_1$$

$$= \frac{6}{1-t_3} \int_0^\infty e^{(t_1-1)x_1} \frac{1}{2-t_2-t_3} e^{(t_2+t_3-2)x_1} \, dx_1$$

$$= \frac{6}{(1-t_3)(2-t_2-t_3)} \int_0^\infty e^{(t_1+t_2+t_3-3)x_1} \, dx_1$$

$$M(t_1, t_2, t_3) = \frac{6}{(1-t_3)(2-t_2-t_3)(3-t_1-t_2-t_3)}$$

$$= \frac{6}{1-t_3} \times \frac{1}{1-\frac{1}{2}(t_2+t_3)} \times \frac{1}{1-\frac{1}{3}(t_1+t_2+t_3)}$$

$\log(1+A) = A - \frac{1}{2}A^2$

$$C(t_1,t_2,t_3) = \log 6 - \log(1-t_3) - \log\left(1-\frac{1}{2}(t_2+t_3)\right) - \log\left(1-\frac{1}{3}(t_1+t_2+t_3)\right)$$

$$+\left(+t_3+\frac{1}{2}t_3^2\right) + \left(+\frac{t_2+t_3}{2} + \frac{1}{2}\times\frac{(t_2+t_3)^2}{4}\right)$$

$$+\left(\frac{t_1+t_2+t_3}{3} + \frac{1}{2}\times\frac{(t_1+t_2+t_3)^2}{9}\right)$$

$$E[X] = \left(\frac{1}{3}, \frac{5}{6}, \frac{11}{6}\right)$$

$$Var(X) = \left(Cov(X_i, X_j)\right)$$

$$\frac{1}{2}\left(\frac{1}{9}t_1^2 + \frac{13}{36}t_2^2 + \frac{49}{36}t_3^2 + 2\left(\frac{t_1t_2}{9} + \frac{13}{36}t_2t_3 + \frac{t_1t_3}{9}\right)\right)$$

$$Var(X) = \frac{1}{36}\begin{pmatrix} 4 & 4 & 4 \\ 4 & 13 & 13 \\ 4 & 13 & 49 \end{pmatrix}$$ ... symmetric.

**Def** $X, Y$: multivariate random variables.

$$f_{Y|X}(y|x) := \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$g: \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$

$X = (X_1, \dots, X_k)$

$Y = (Y_1, \dots, Y_l)$

$E[g(X,Y) \mid X = x]$ $\qquad x \in \mathbb{R}^k$

다차원적분. $$:= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g(x, y_1, \dots, y_l) f_{Y|X}(y_1, \dots, y_l \mid x)\, dy_l \cdots dy_1$$

$$\int_{\mathbb{R}^l} g(x,y) f_{Y|X}(y|x)\, dy$$

Fubini's theorem

**Def** $X = (X_1, \dots, X_k)$, $Y = (Y_1, \dots, Y_l)$

$$E[Y|X] := \left(E[Y_1|X], \dots, E[Y_l|X]\right)^t = \left(g_1(X), \dots, g_l(X)\right)^t$$

$$Var(Y|X) := \left(Cov(Y_i, Y_j \mid X)\right)$$

where $Cov(Y_i, Y_j | X) := E[Y_i Y_j | X] - E[Y_i|X]E[Y_j|X]$

$$\parallel$$

$$E[(Y_i - E[Y_i|X])(Y_j - E[Y_j|X])]$$

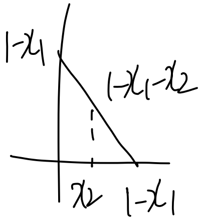
$\because$ $E[Y_iY_j|X] + E[E[Y_i|X]E[Y_j|X]] - E[Y_jE[Y_i|X]|X] - E[Y_iE[Y_j|X]|X]$

$= E[Y_iY_j|X] + E[Y_i|X]E[Y_j|X] - E[Y_i|X]E[Y_j|X] - E[Y_j|X]E[Y_i|X]$

$= E[Y_iY_j|X] - E[Y_i|X]E[Y_j|X]$ $\quad \square$.

e.g. $f_{2,3|1}(x_2,x_3|x_1) = \frac{6(1-x_1-x_2-x_3)}{(1-x_1)^3} I_{x_2\ge 0,\, x_3\ge 0,\, x_2+x_3\le 1-x_1}$

$E[(X_2,X_3)|X_1]$




$$E[X_2|X_1=x_1] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_2 \frac{6(1-x_1-x_2-x_3)}{(1-x_1)^3} I \, dx_3 dx_2$$

$$= \int_0^{1-x_1}\int_0^{1-x_1-x_2} \frac{6x_1}{(1-x_1)^3} (1-x_1-x_2)-x_3 \, dx_3 dx_2$$

$$= \int_0^{1-x_1} \frac{6x_1}{(1-x_1)^3} \left[(1-x_1-x_2)x_3 - \frac{1}{2}x_3^2\right]_0^{1-x_1-x_2} dx_2$$

$$= \int_0^{1-x_1} \frac{6x_2}{(1-x_1)^3} \frac{1}{2}(1-x_1-x_2)^2 \, dx_2$$

$$= \frac{3}{(1-x_1)^3}\int_0^{1-x_1} x_2(1-x_1-x_2)^2 \, dx_2$$

$1-x_1-x_2=t$

$-dx_2=dt$

$$= \frac{3}{(1-x_1)^3}\int_0^{1-x_1} (1-x_1-t)t^2 \, dt$$

$$\left[\frac{(1-x_1)}{3}t^3 - \frac{1}{4}t^4\right]_0^{1-x_1}$$

$$\frac{\cancel{3}}{(1-x_1)^3} \frac{(1-x_1)^4}{\cancel{12}\,4} = \frac{1-x_1}{4}$$

$\therefore E[X_2|X_1] = \frac{1-X_1}{4}$, $E[X_3|X_1] = \frac{1-X_1}{4}$

$\therefore E[(X_2,X_3)^t|X_1] = \frac{1-X_1}{4}(1,1)$. □

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$$Var((X_2,X_3)^t|X_1) = \left(Cov(X_i,X_j|X_1)\right)_{i,j=2,3}.$$

$$E[X_2^2|X_1],\ E[X_3^2|X_1] \quad E[X_2X_3|X_1].$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_2^2 \frac{6(1-x_1-x_2-x_3)}{(1-x_1)^3} I(x_2\geq 0, x_3\geq 0, x_2+x_3 \leq 1-x_1)$$

$$= \int_0^{1-x_1}\int_0^{1-x_1-x_2} \frac{6x_2^2}{(1-x_1)^3}(1-x_1-x_2-x_3)\,dx_3\,dx_2$$

$$= \int_0^{1-x_1} \frac{6x_2^2}{(1-x_1)^3}\left[(1-x_1-x_2)x_3-\frac{1}{2}x_3^2\right]_0^{1-x_1-x_2} dx_2$$

$$\frac{1}{2}(1-x_1-x_2)^2$$

$$= \int_0^{1-x_1} \frac{3x_2^2}{(1-x_1)^3}(1-x_1-x_2)^2\,dx_2$$

$$= \frac{3}{(1-x_1)^3}\int_0^{1-x_1} x_2^2\left\{(1-x_1)^2-2(1-x_1)x_2+x_2^2\right\}dx_2$$

$$\left[\frac{(1-x_1)^2}{3}x_2^3-\frac{1}{2}(1-x_1)x_2^4+\frac{1}{5}x_2^5\right]_0^{1-x_1}$$

$$(1-x_1)^5\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)$$

$$\frac{(1-x_1)^2}{10} \quad E[X_2^2|X_1]=\frac{(1-X_1)^2}{10},\ E[X_3^2|X_1]=\frac{(1-X_1)^2}{10}$$

$$E[X_2X_3|X_1] = \int_0^{1-x_1}\int_0^{1-x_1-x_2} \frac{6x_2x_3}{(1-x_1)^3}(1-x_1-x_2-x_3)\,dx_3\,dx_2$$

$$= \int_0^{1-x_1} \frac{6x_2}{(1-x_1)^3}\int_0^{1-x_1-x_2} x_3(1-x_1-x_2-x_3)\,dx_3\,dx_2$$

$$\left[\frac{1}{2}(1-x_1-x_2)x_3^2 - \frac{1}{3}x_3^3\right]_0^{1-x_1-x_2}$$

$$\frac{\cancel{6}x_2}{(1-x_1)^3} \times \frac{1}{\cancel{6}}(1-x_1-x_2)^3$$

$$= \frac{1}{(1-x_1)^3}\int_0^{1-x_1} x_2(1-x_1-x_2)^3\,dx_2 \qquad 1-x_1-x_2=t$$

$$= \frac{1}{(1-x_1)^3}\int_0^{1-x_1}(1-x_1-t)t^3\,dt$$

$$\left[\frac{(1-x_1)t^4}{4} - \frac{1}{5}t^5\right]_0^{1-x_1}$$

$$\frac{(1-x_1)^{\cancel{5}\,2}}{20} \times \frac{1}{\cancel{(1-x_1)^3}}$$

$$\frac{(1-x_1)^2}{20}$$

$$E[X_2X_3|X_1] = \frac{(1-X_1)^2}{20} \qquad E[X_2^2|X_1] = \frac{(1-X_1)^2}{10}$$

$$E[X_2|X_1] = \frac{1-X_1}{4}$$

$$Var(X_3|X_1) = Var(X_2|X_1) = \frac{(1-X_1)^2}{10} - \frac{(1-X_1)^2}{16}$$

$$= \frac{3}{80}(1-X_1)^2$$

$$Cov(X_2,X_3|X_1) = \frac{(1-X_1)^2}{20} - \frac{(1-X_1)^2}{16} = \frac{-(1-X_1)^2}{80}.$$

$$\therefore Var((X_2,X_3)^t|X_1) = \frac{(1-X_1)^2}{80}\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix} \quad \square$$

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**Thm.** $X=(X_1,\dots,X_k)$ $Y=(Y_1,\dots,Y_l)$

(a) $E[E[Y|X]]=E[Y]$.

(b) $Cov(Y-E[Y|X], v(X))=0 \quad \forall v(X)$

(a) proof) $E[E[Y|X]]$

$$=\left(E[E[Y_1|X]],\dots,E[E[Y_l|X]]\right)^t$$

for $i=1,\dots,l$

$$E[E[Y_i|X]]=\int_{\mathbb{R}^k} E[Y_i|X=x]f_X(x)\,dx$$

$$=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} E[Y_i|X=x]f(x_1,\dots,x_k)\,dx_k\cdots dx_1$$

$$=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} y_i f_{Y_i|X}(y_i|x_1,\dots,x_k)f(x_1,\dots,x_k)\,dx_k\cdots dx_1\,dy_i$$

$$=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} y_i f_{X,Y_i}(x,y_i)\,dx_k\cdots dx_1\,dy_i$$

$$=\int_{-\infty}^{+\infty} y_i f_{Y_i}(y_i)\,dy_i=E[Y_i].$$

$\therefore E[E[Y|X]]=E[Y]$.

(b) $\mathrm{Cov}(Y - E[Y|X],\ v(X)) = 0 \in M_{l\times k}(\mathbb{R})$

$$E[Y|X] = (E[Y_1|X], \dots, E[Y_l|X])^t = (g_1(X), \dots, g_l(X))$$

$$E[Y_1|X=x] = \int_{-\infty}^{+\infty} y_1 f_{Y_1|X}(y_1|x)\,dy_1$$

$$\left(\mathrm{Cov}(Y - E[Y|X], v(X))\right)_{ij}$$

$$= \mathrm{Cov}(Y_i - E[Y_i|X],\ v(X_j)).$$

$$Z_i := Y_i - E[Y_i|X].$$

$$E[Z_i] = E[Y_i] - E[E[Y_i|X]].$$

$$= E[Y_i] - E[Y_i] = 0.$$

$$E[Z_i|X] = E[Y_i|X] - E[E[Y_i|X]\,|\,X]]$$

$$= E[Y_i|X] - E[Y_i|X] = 0.$$

$\because$ $E[E[Y_i|X]|X_j]$. $g_i(X) := E[Y_i|X]$

$$= E[g_i(X)\,|\,X_j = x] \quad x \in \mathbb{R}.$$

$$= \int_{-\infty}^{+\infty} g_i(x) f_{Y_i|X}(y_i|x)\,dy_i$$

$$= g_i(x) \int_{-\infty}^{+\infty} f_{Y_i|X}(y_i|x)\,dy_i = g_i(x)$$

$$E[g_i(X)\,|\,X] = g_i(X)$$

i.e. $E[E[Y_i|X]] = E[Y_i|X]$. $\lrcorner$

$$\mathrm{Cov}(Y_i - E[Y_i|X],\ v(X_j)) = E[Z_i v(X_j)] - E[Z_i]E[v(X_j)]$$

$$= E[E[Z_i v(X_j)\,|\,X]]$$

$$= E[v(X_j) E[Z_i|X]] = 0$$

Q

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**Thm.** $E[\|Y - r(X)\|^2] \geq E[\|Y - E[Y|X]\|^2]$

proof).

M A D E B Y J A N E

**Def** $X_1, \dots, X_k$: "mutually independent"

$\iff P(X_1 \in A_1, \dots, X_k \in A_k) = \prod_{i=1}^{k} P(X_i \in A_i)$

**Thm** $pdf(x_1, \dots, x_n) = pdf_1(x_1) \cdots pdf_n(x_n)$

$mgf(t_1, \dots, t_n) = mgf_1(t_1) \cdots mgf_n(t_n)$

**Rmk** $X_1, \dots, X_n$: Mutually independat.

(a)' $\iff pdf(x_1, \dots, x_n) = g_1(x_1) \cdots g_n(x_n)$

(b)' $mgf(t_1, \dots, t_n) = m_1(t_1) \cdots m_n(t_n)$

**Thm.** (a) $X_1, \dots, X_n$: mutually independent

$\Rightarrow E[g_1(X_1) \cdots g(X_n)] = E[g_1(X_1)] \cdots E[g_n(X_n)]$

(b) // $\Rightarrow Cov(X_i, X_j) = 0 \quad (i \neq j)$

(c) // $\Rightarrow g(X_1), \dots, g(X_n)$: mutually independent.

proof) $E[g_1(X_1) \cdots g_n(X_n)] = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g_1(x_1) \cdots g_n(x_n)$
$f(x_1, \dots, x_n)\, dx_n \cdots dx_1$

$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g_1(x_1) \cdots g_n(x_n) f_1(x_1) \cdots f_n(x_n)\, dx_n \cdots dx_1$

$= \left(\int_{-\infty}^{+\infty} g_1(x_1) f_1(x_1)\, dx_1\right) \cdots \left(\int_{-\infty}^{+\infty} g_n(x_n) f_n(x_n)\, dx_n\right)$

$= E[g_1(X_1)] \cdots E[g_n(X_n)]$

$Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j]$

$= E[X_i]E[X_j] - E[X_i]E[X_j] = 0$ ∎

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Thm 2.5.11. $X_1, \dots, X_n$: mutually independent. $X_i \in \mathbb{R}^k$

(a) $Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n)$

(b) $mgf_{X_1 + \cdots + X_n}(t) = mgf_{X_1}(t) + \cdots + mgf_{X_n}(t)$

proof) (a) $Var(X_1 + \cdots + X_n) = Var(\sum_{i=1}^{n} X_i)$

$$= Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, X_j)$$

$Cov(X_i, X_j) = 0$ $i \neq j$.

$$= \sum_{i=1}^{n} Cov(X_i, X_i)$$

$$= \sum_{i=1}^{n} Var(X_i)$$

(b) $mgf_{X_1 + \cdots + X_n}(t) = E[e^{t \cdot (X_1 + \cdots + X_n)}]$

$t := (t_1, \dots, t_k)$ $\quad = E[e^{t \cdot X_1}] + \cdots + E[e^{t \cdot X_n}]$

$$= mgf_{X_1}(t) + \cdots + mgf_{X_n}(t) \quad \square$$