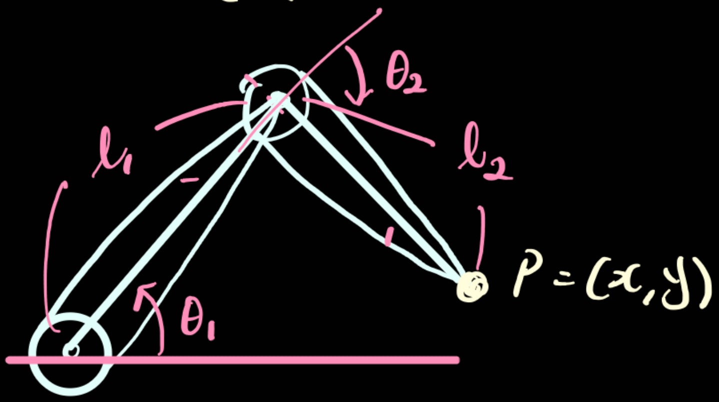
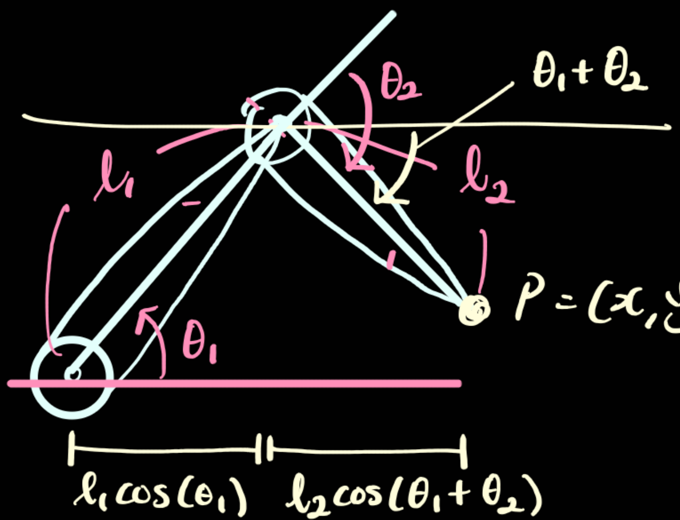
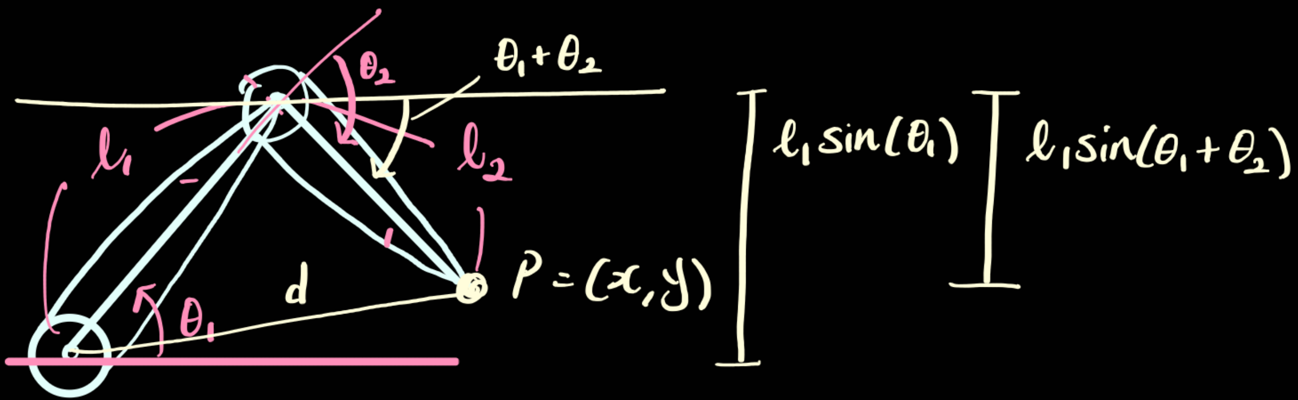
< 2-degrees of freedom >

Forward Kinematics $P = f(\theta)$, Inverse Kinematics $\theta = f^{-1}(P)$, where $\begin{cases} \text{Rotation: } \theta = (\theta_1, \theta_2) \\ \text{Position : } P = (x, y) \end{cases}$

$$P = (x,y) = \left(l_1 \cos(\theta_1) + l_2 \cos(\theta_1+\theta_2),\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1+\theta_2) \right)$$







So, Forward Kinematics $P = f(\theta) = f(\theta_1, \theta_2)$

$$P(\theta_1, \theta_2) = (x,y) = \left(l_1 \cos(\theta_1) + l_2 \cos(\theta_1+\theta_2),\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1+\theta_2) \right)$$

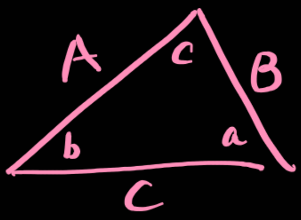
Then,

$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1+\theta_2)$$

$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1+\theta_2)$$

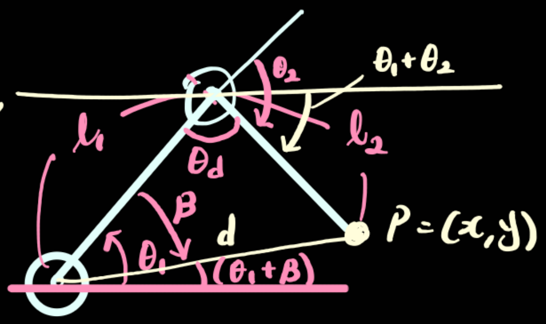
<Finding θ_2>

by Law of Cosines

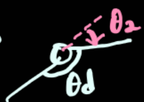


$$A^2 = B^2 + C^2 - 2BC \cdot \cos(a)$$

Then,



$$d^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cdot \cos(\theta_d)$$

<If  >

$$\theta_d = \pi + \theta_2 \equiv \theta_2 = \theta_d - \pi$$

$$d^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cdot \cos(\pi + \theta_2)$$

There is only one solution it seems

<observe ..>

$$\cos(\pi+\theta_2) = \frac{d^2 - l_1^2 - l_2^2}{-2 l_1 l_2}$$

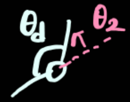
$\cos(\theta_1 + \theta_2) =$

$\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

$$\underbrace{\cos(\pi)}_{-1}\cos(\theta_2) - \underbrace{\sin(\pi)}_{0}\sin(\theta_2) = \frac{d^2 - l_1^2 - l_2^2}{-2 l_1 l_2}$$

$$-\cos(\theta_2) = \frac{d^2 - l_1^2 - l_2^2}{-2 l_1 l_2}$$

Same thing as

<If  >

$$\theta_d = \pi - \theta_2 \equiv \theta_2 = \pi - \theta_d$$

$$d^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cdot \cos(\pi - \theta_2)$$

There are two solutions it seems.

<1st>

$$\cos(\pi-\theta_2) = \frac{d^2 - l_1^2 - l_2^2}{-2 l_1 l_2}$$

$\cos(\theta_1 - \theta_2) =$

$\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$

$$\underbrace{\cos(\pi)}_{-1}\cos(\theta_2) + \underbrace{\sin(\pi)}_{0}\sin(\theta_2) = \frac{d^2 - l_1^2 - l_2^2}{-2 l_1 l_2}$$

$$-\cos(\theta_2) = \frac{d^2 - l_1^2 - l_2^2}{-2 l_1 l_2}$$

Since $d^2 = x^2 + y^2$

$$-\cos(\theta_2) = \frac{x^2 + y^2 - l_1^2 - l_2^2}{-2 l_1 l_2}$$

$$\cos(\theta_2) = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\boxed{\theta_2 = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}\right)}$$

⟨1st⟩

$$d^2 = l_1^2 + l_2^2 - 2l_1l_2\cdot\cos(\theta_d)$$

Since, $\cos(\theta_d) = \dfrac{d^2-l_1^2-l_2^2}{-2l_1l_2}$

$$\theta_d = \cos^{-1}\left(\frac{d^2-l_1^2-l_2^2}{-2l_1l_2}\right)$$

since $d^2 = x^2+y^2$

$$\theta_d = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{-2l_1l_2}\right)$$

since $\theta_2 = \theta_d - \pi$ or

$$\boxed{\theta_2 = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{-2l_1l_2}\right)-\pi}$$

⟨2nd⟩

$$d^2 = l_1^2 + l_2^2 - 2l_1l_2\cdot\cos(\theta_d)$$

Since, $\cos(\theta_d) = \dfrac{d^2-l_1^2-l_2^2}{-2l_1l_2}$

$$\theta_d = \cos^{-1}\left(\frac{d^2-l_1^2-l_2^2}{-2l_1l_2}\right)$$

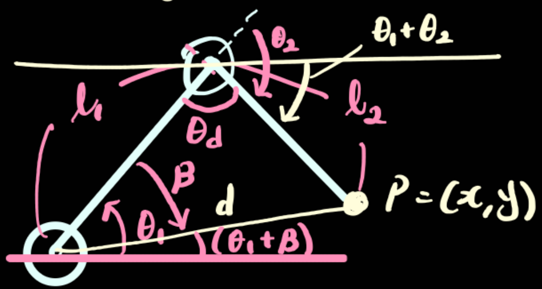
since $d^2 = x^2+y^2$

$$\theta_d = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{-2l_1l_2}\right)$$

since $\theta_2 = \pi - \theta_d$ or

$$\boxed{\theta_2 = \pi - \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{-2l_1l_2}\right)}$$

< Finding θ_1 >

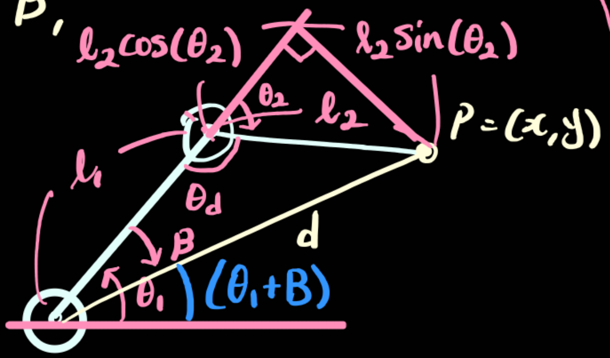


So, $\tan(\theta_1+\beta) = \frac{y}{x}$

$$\theta_1+\beta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \beta$$

To find β,



$$\tan(B) = \frac{l_2\sin(\theta_2)}{l_1\cos(\theta_2)}$$

$$\beta = \tan^{-1}\left(\frac{l_2\sin(\theta_2)}{l_1+l_1\cos(\theta_2)}\right)$$

★

$$\boxed{\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2\sin(\theta_2)}{l_1+l_1\cos(\theta_2)}\right)}$$