Can You Hear Group Theory in Music?

Sean O'Reilly

June 11th, 2015

1 Introduction

1.1 Goal

The goal of this project is to use topics found in group theory, set theory, and combinatorics to systematically create maps which one can use to generate chord changes in music. An undergraduate course in algebra is all that is needed here, as the musical details will be described in the summary, and as they come. The idea is to define a set of operations that can be applied to sets of musical pitches, representing chords containing them. This set of operations can be interpreted as a group in bijection to the Dihedral Group D_n, acting on our set of pitches like a polygon[4]. Using properties of group action, we will demonstrate how to generate a network of chord changes that take a musician through an entire *family*, or orbit, of one particular chord. With this network, we build multiple maps showing how we can traverse through the chords in clever ways. This work builds on ideas first introduced by Isihara and Knapp in 1993[5], and Fiorne in 2003[4].

1.2 History

A detailed relationship between math and music dates back to 500 BC, the days of Pythagoras, who founded the first school of mathematics as a deductive science as well as one of the first schools of theoretical music[9]. In his studies, he found a way to generalize pitch intervals according to fractions, which was an incredible insight considering wave mechanics and frequency had not been discovered yet. What Pythagoras' fractions were describing, without his knowledge, was constructive and destructive interference of sound waves. An essential part of his work concerned the phenomenon of why certain tones sound the same, even if the pitches of the tones are different[3]. Ultimately, he created the first tuning system, and demonstrated how music could be studied theoretically. Over the next 2,000 years, his ideas would be developed into a systematic symbolic language, understood and studied across the world[9].

During the Renaissance, Pythagorean discoveries in music were applied to a scale choice of twelve tones. This is the scale found on most modern instruments, and demonstrated famously by the piano and its keys. Music written with the twelve-tone scale became the dominant format used by classical composers, and is consistently providing musicians more possibilities. Though music is generally interpreted as an art, and subjectively received, these

Renaissance composers used many mathematical ideas to write patterns in their music[9]. Even before the notion of a group was abstracted by mathematicians in the 19th century, Beethoven was moving through the same orbit of chords we will consider, but found through other conventions[4]. Ideas of subgroups and their cosets can be seen in centuries old compositions written for bell towers in England[6], and classical compositions called ”fugues” use similar operations to those we will define and construct.

1.3 Why do we care?

The nature of the group we build provides musicians an excellent framework for chord progression. Major and minor chords are among the most common chords played in music, with the general characteristic that major chords tend to sound happy and upbeat, and minor chords usually sound sad and mundane[8]. A musician can work with these in attempt to achieve the sound they want, but without much insight as to what pitches have changed qualitatively unless they have been studying music for years. This work will connect the major/minor chords that share pitches, and therefore qualitative similarities, in a visually pleasing way. Anytime a musician, or perhaps a musically inclined mathematician, needs a suggestion on chord choice when writing a song, they can consult the charts we build in search for new ideas.

In addition, algorithmic ways of creating music beyond the normal conventions of tonality and repetition are rarely seen today, but are becoming a rich area of study[5]. Musicians can use these methods as a guide to how one can vary their themes mathematically. Also, by using groups and modular arithmetic to interpret pitches and their movement, this project can serve as an aide to students of mathematics who are curious about music, but aren't sure how to approach it. After understanding these concepts, there will be no need to visualize an instrument as a foreign object that someone needs to train for years to understand, we will have a systematic way to produce sounds that work well together.

2 Summary

First we must establish a common ground between the two disciplines with some terminology. Using the following terms, we will be able to talk about music mathematically.

- A **pitch** is a single note, characterized by its frequency.
- Pitches with a given frequencies f and $2f$ are said to be **octaves** of one another. To our ears these sound the same due to constructive interference of the sound waves. Pitches of this type will be considered equivalent. All octaves of one frequency belong to the same **pitch class**[5].
- Stepping from octave to octave with an incremental frequency produces a sequence of pitches from different pitch classes called a **scale**. If we increment p times to reach the octave of our starting frequency, we say the scale has order p and denote it with $S = \{0, 1, 2, ..., p-1\}$. Incrementing past p lands us at the octaves of the other pitches, which means we use modular-p arithmetic when working with scales. In essence, $S \cong \mathbb{Z}_p$.

- Distances between two pitches in a scale are measured in **semitones**. It is the unit frequency of the scale. Adding n semitones to a pitch lands you n pitches down the scale. On a piano, this is the distance between any two adjacent keys, black or white.

- We will often discuss what happens to subsets of a scale. At times we will use **pitch sets** to mean subset[5]. For our purposes, pitch sets can be listed independent of ordering. This will still hold when the notion of a root is discussed in Section 4.

- These terms are derived from **atonal** music theory. As opposed to **tonal** music theory, atonal music does not feature one pitch as the center of a piece. Instead, it favors pattern and symmetry over aesthetics. Tonal music only considers a subset of pitches that resonate well with a single pitch called the *key*. Since we are describing atonal music, we don't have a key, and instead consider all pitches in a scale.[5].

Our next mission is to define the mathematics behind the movement of pitches. We will start with a set of operators T/I that can be applied to a scale of order p, and relate these to the group of rotations and reflections that can be applied to a p-sided polygon, i.e. the Dihedral Group D_p [4]. When applying the T/I group to subsets of our scale, we will use Burnside's Theorem to find the number of distinct orbits for subsets of certain sizes[5].

The idea is to use this atonal framework on tonal concepts. Subsets of a scale containing three pitches are analogous to **triads**, or chords played with three notes, which are fundamental to every tonal theorist. We will rotate and reflect the main forms of these triads to list all possibilities of movement. Then, using our basic group of operations, we will construct a group of more detailed operators called the PLR group that map triads to specific counterparts that have common pitches[4]. This group will be generated by 3 new operations that perform our usual rotations and reflections based on the input pitch set. It is bijective to both T/I and D_p, but tailored with a representation that lets us easily construct a tiling musical map. This map has the topology of a *torus*, and displays many common chord changes and compositions as lattices contained in the figure[4]. The PLR chart is derived mathematically, but can be used to construct and predict chord movement in music.

3 Technical Preparation

3.1 The T/I Group

Now, suppose a composer wishes to work with a scale S of order p mathematically. This composer chooses several pitches from S and considers the pitch set M containing these pitches. How can they operate on them? One way is to take any single pitch from S and add it to each element of the pitch set. Recall that we are working with modular arithmetic, and S equipped with semitone addition is itself a finite cyclic group, isomorphic to $\mathbb{Z}_p$. The result is a new pitch set where each element was raised by some constant pitch, preserving the semitone distances between each pitch from M, but not the pitches themselves. This type of operation is called a **transposition**[5, 6].

Definition 1. *Let S be a scale of order p, and let $n \in \mathbb{N}$ with $0 \leq n < p$. Then define the p transposition operators that act on S*

$$\begin{aligned} T_n : S &\to S \\ x &\mapsto x + n \end{aligned}$$

The operators $\{T_0, T_1, ..., T_{p-1}\}$ can be applied to any scale of order p, or its pitch sets. Each T_n adds n semitones to an individual pitch, or to each pitch in a pitch set.

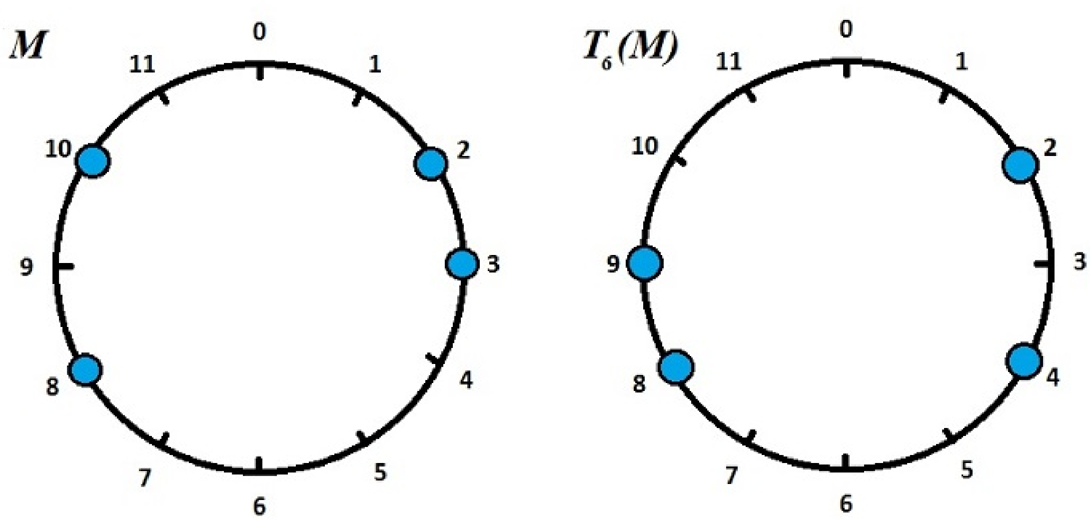
Example 1. Let $S = \{0, 1, 2, ..., 11\}$ be a scale of order 12, and let $M = \{2, 3, 8, 10\}$ be a pitch set of S. What is $T_6(M)$?

Apply the operator to each individual element of M.

$$\begin{aligned} T_6(2) &= 8 \\ T_6(3) &= 9 \\ T_6(8) &= 14 \equiv 2 \bmod 12 \\ T_6(10) &= 16 \equiv 4 \bmod 12 \end{aligned}$$

The resulting pitch set is $T_6(M) = \{2, 4, 8, 9\}$.

Due to the cyclic nature of pitches, adding 6 semitones to each element of M produced a pitch set that was very close to M. Two pitches remained constant while the other two pitches differed from M by only one semitone each. When operating on a pitch set rather than an individual pitch, *rotational* symmetries are possible. This is easily made visible by use of circles. With $p = 12$, such as in the example and most music, imagine a clock face with 0,...,11 listed around the perimeter[5].



This visualization is a key idea of our task. Taking a pitch set and transposing it by some number of semitones is really a rotation through a scale. Any simple melody or chord can be rotated alike, since the individual pitches are found with in some pitch set of S.

With attention towards what happened to M in the example, we must discuss the idea of a non-identity transposition fixing a certain pitch set of a scale. Since any scale of order p combined with semitone addition is in bijection with $\mathbb{Z}_p$, we can use our knowledge of groups to make some conclusions.

First of all, since S is cyclic, all subgroups of S are cyclic with generator d. By Cauchy's Theorem, $d \mid p$ [5]. What this means in the context of semitone addition is that a generator d places pitches evenly around the circle. So any transposition $T_d, ..., T_{kd}$ fixes a subgroup generated by d, *and* its cosets[5].

Secondly, the reverse logic implies no T_n can fix a pitch set if n is relatively prime to p, unless it is the identity T_0. These facts give us the following *fixtures* for transpositions $T_0, ..., T_{11}$ in a scale of order 12 [5].

Fixtures of T.

We are using $p = 12$ which means nontrivial subgroups of S are generated by either 2, 3, 4, or 6.

$$\begin{aligned}
T_2, T_4, T_6, T_8, T_{10} &: \{0,2,4,6,8,10\}, \{1,3,5,7,9,11\} \\
T_3, T_6, T_9 &: \{0,3,6,9\}, \{1,4,7,10\}, \{2,5,8,11\} \\
T_4, T_8 &: \{0,4,8\}, \{1,5,9\}, \{2,6,10\}, \{3,7,11\} \\
T_6 &: \{0,6\}, \{1,7\}, \{2,8\}, \{3,9\}, \{4,10\}, \{5,11\}
\end{aligned}$$

Each transposition operator T_n fixes the listed pitch sets of S, as well as the pitch sets fixed by operators T_k where $k \mid n$.

In group theory, the notion of a rotation is often coupled with the notion of a reflection. Another operation we will define is called an **inversion**. With this operation, we invert the pitches in a pitch set by subtracting their semitone value from p, and then transposing them with a common pitch[5, 6]. This is equivalent to reflecting a pitch set across the vertical line connecting 0 to 6 on our musical clock, and then performing a transposition of n semitones[5].

Definition 2. *Let S be a scale of order p, and let $n \in \mathbb{N}$ with $0 \leq n < p$. Then define the p inversion operators on S*

$$\begin{aligned}
I_n : S &\to S \\
x &\mapsto p - x + n
\end{aligned}$$

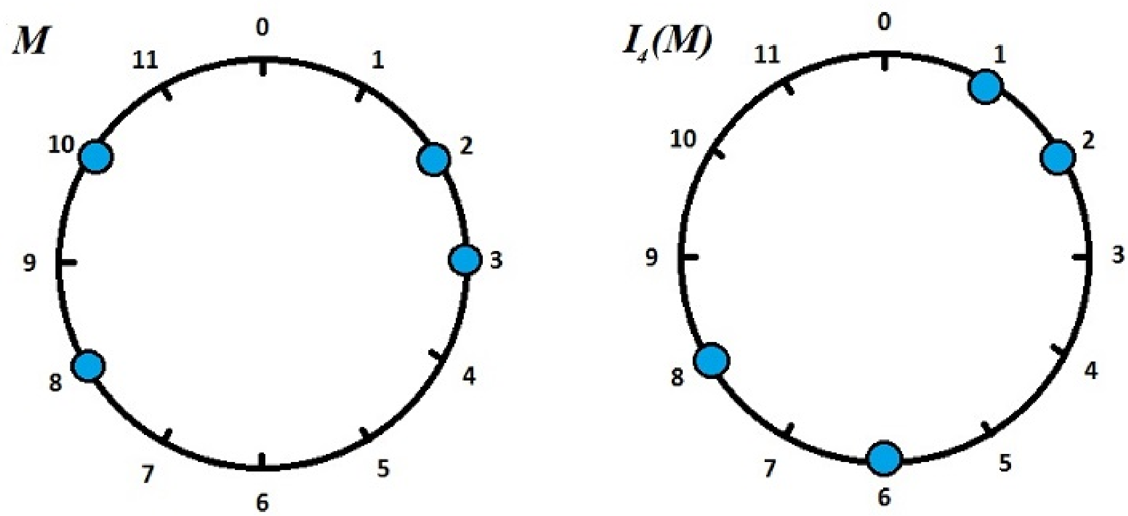
$\{I_0, I_1, ..., I_{p-1}\}$ can be applied to any scale of order p, or its pitch sets. The operator I_n subtracts each pitch it is applied to from p, and adds n semitones to the result.

Example: Let $S = \{0, 1, 2, ..., 11\}$ be a scale of order 12, and let $M = \{2, 3, 8, 10\}$ be the same pitch set as in Example 1. What is $I_4(M)$?

Apply the operator to each individual element of M.

$$\begin{aligned} I_4(2) &= 12 - 2 + 4 = 14 \equiv 2 \bmod 12 \\ I_4(3) &= 12 - 3 + 4 = 13 \equiv 1 \bmod 12 \\ I_4(8) &= 12 - 8 + 4 = 8 \\ I_4(10) &= 12 - 10 + 4 = 6 \end{aligned}$$

The resulting pitch set is $I_4(M) = \{1, 2, 6, 8\}$.



Fixtures of I.

Determining the fixture of an inversion operator is a little trickier, but determining how many elements are in the fixture is relatively simple. First we notice that $I_n(I_n(x)) = x$ for any inversion[5]. This tells us that each I_n fixes a pair of pitches $\{x, I_n(x)\}$ for any $x \in S$, or a single pitch in the case $x = I_n(x)$. We must find out how many single pitches and how many pairs of pitches are invariant with a given I_n. Beginning with single pitches, we check the case $x = I_n(x)$. This is only true when

$$x = (p - x + n) \bmod p$$

which is the same as saying

$$n + kp = 2x.$$

Assuming $x < p$, we are only concerned with $k = 0$ and $k = 1$. Since the right hand side is even, the left hand side must also be even. With $p = 12$, we can conclude n is even, and this gives us two solutions for x, one for each value of k. (If n is odd, our equation tells us I_n fixes no single pitch, just pairs[5].)

Example: Which single pitches in a scale S of order 12 does the operator I_8 fix?

With $n = 8$ and $p = 12$, by the equation above we have that any pitch fixed by I_8 satisfies

$$8 + 12k = 2x$$

for $k = 0$ and $k = 1$.

$$2x = 8 + 12(0) = 8$$
$$2x = 8 + 12(1) = 20$$

$$\Longrightarrow x = 4, x = 10$$

Thus two single pitches, $x = 4$ and $x = 10$ are invariant under operation by I_8. The remaining five pairs of pitches in S of the form $\{x, I_8(x)\}$ are also invariant. We can use this information to count how many pitch sets of a given size are fixed by inversion operators.

Now we turn our attention to the combined set of operators $T_n \cup I_n$ that can be applied to a scale of size p. If we compose two or more of our operations, what happens? We should expect that any combination of transpositions and inversions can be represented as a single transposition or inversion, since these are all possible rotations and reflections of a finite set of pitches arranged around a circle. As it turns out, this set which we will call T/I forms a group under functional composition[4].

Proposition 1. *The set T/I forms a group.*

Proof. First we check that T/I is closed under functional composition. Let M be a pitch set.

Case 1 Two Transpositions of n and m semitones.

$$T_n(T_m(M)) = T_{n+m}(M)$$

This is true since rotations can be taken in any order to achieve the same state, and the result will be a rotation by the sum of n and m. If $n + m = p$, then since $p \equiv 0 \bmod p$, $T_{n+m} = T_0$ and $T_n = (T_m)^{-1}$.

Case 2 Two Inversions of n and m semitones.

$$I_n(I_m(M)) = T_{n-m}(M)$$

In the case of inversions, two performed in a row undo the reflection, and we are left with a transposition by the difference of n and m. Note that if $n = m$, the result is the identity, T_0. Each of these operators I_n is its own inverse $(I_n)^{-1}$.

Case 3 An Inversion and a Transposition of n and m semitones, in each direction.

$$I_n(T_m(M)) = I_{n-m}(M)$$
$$T_n(I_m(M)) = I_{n+m}(M)$$

In each subcase, the reflection is preserved but the inversion is different based on the order of operation.

These three cases together show that all combinations of transpositions and inversions can be represented by a single transposition or inversion, so T/I is closed under the binary operation. In the process, we have also shown that each operation has an inverse in T/I such that the two performed together are equivalent to the identity transposition T_0. Lastly, to qualify T/I as a group we need to show that it is associative under the binary operation. We can do this for transpositions and inversions separately.

Transpositions:

$$T_k(T_n(T_m(M))) = T_k(T_{n+m}(M)) = T_{k+n}(T_m(M)) = T_{k+n+m}(M)$$

Inversions:

$$I_k(I_n(I_m(M))) = I_k(T_{n-m}(M)) = I_{k-n+m}(M) = T_{k-n}(I_m(M)) = I_k(I_n(I_m(M)))$$

Then, by Case 2, $T_k = I_a(I_b)$ for some $a, b < p$. So we can take any combination of T's and I's and represent them entirely with I's. This forces the associativity to carry over to any general case, completing the proof.

$\square$

Not only can we now call T/I a group, but it is actually an important group in disguise. Mathematicians refer to the group of all possible rotations and reflections of a p sided polygon as the Dihedral Group D_p, and it is abstractly represented here as the group of transpositions and inversions that can be applied to a set of pitches in a scale of size p. In our case, the unit rotation is T_1. Our unit inversion is I_0, which reflects a pitch or a pitch set across the vertical line connecting 0 to 6 in our examples. Together, these generate the group, which has $2p$ elements[4].

$$D_p = \{1, x, x^2, ..., x^{p-1}, y, xy, ..., x^{p-1}y\}$$
$$x^p = 1 \quad , \quad y^2 = 1 \quad , \quad yx = x^{p-1}y$$

$$\begin{aligned} T/I &= \{T_0, T_1, {T_1}^2, ..., {T_1}^{p-1}, I_0, T_1 I_0, {T_1}^2 I_0, ..., {T_1}^{p-1} I_0\} \\ &= \{T_0, T_1, ..., T_{p-1}, I_0, I_1, ..., I_{p-1}\} \end{aligned}$$
$${T_1}^p = T_0 \quad , \quad I_0^2 = T_0 \quad , \quad I_0 T_1 = {T_1}^{p-1} I_0$$

3.2 Group Action and Set Classes

Using T/I to operate on a finite set of pitches in a scale defines a **group action** of T/I on the set of subsets of the scale S [5].

Definition 3 (Group Action)**.** *Let G be a group and let X be a non-empty set. An action of G on X is a function*

$$\begin{aligned} * : G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

that preserves the following two properties:

1) *$e * x = x$ for all $x \in X$ and identity e of G.*

2) *$(gh) * x = g * (h * x)$ for all $x \in X$ and for all $g, h \in G$.*

If such a function exists for some set X, we call X a G-set. With a group action, any $g \in G$ will take something in X and map it to something else in X. Depending on the choices of G and X, many different phenomena can arise here. For instance, taking all elements of G and acting on a single element $x \in X$ could give us all other elements in X, or maybe just some or them, or perhaps x itself. In order to more effectively describe this sort of behavior, we define the concept of an **orbit**[1].

Definition 4. *Let G be a group that acts on G-set S, and let $s \in S$. We define the orbit of s under G as the set*

$$\mathcal{O}_s = \{s' \in S \mid g \cdot s' = s \text{ for some } g \in G\}$$

Note: s is always in its own orbit since G has an identity element.

Additionally, we can now define the concept of a fixture, which was already discussed for transpositions and inversions separately.

Definition 5. *Let G be a group that acts on G-set S, and let $s \in S$. We define the fixture of element $g \in G$ as the set*

$$F(g) = \{s \in S \mid g \cdot s = s\}$$

Applying these concepts to singleton pitches will be overkill. After a few moments we can convince ourselves that the orbit of any singleton pitch under D_p must be every other pitch in the scale. It is easy enough to work like this without orbits, or even group theory. So rather than consider the group action of D_p on a scale S, we will instead look at the group action of D_p on the set of *all subsets* of S, or all pitch sets. When considering pitch sets of size $k < p$, we find that their orbits are the set of their symmetries around a circle. We will refer to the orbit of a pitch set as its **set class**[5], and are interested in counting how many set classes there are for pitch sets of given order.

3.3 Burnside's Theorem

The following theorem was proved by William Burnside in 1904 [2].

Theorem 1 (Burnside's Theorem)**.** *Let finite group G act on finite set S. The number of orbits n that partition S is the average number of elements $s \in S$ fixed by an element $g \in G$.*

$$n = \frac{1}{|G|} \sum_{g \in G} |F(g)|$$

Tonal music theorists have been interested in the different chords that can be played with three notes for centuries, and the families of chords that share the same qualities. The three note chords, called *triads*, are analogous to pitch sets of order 3 in a scale of order 12. By applying Burnside's theorem to these, we can count the different set classes that music theorists have famously sorted by more tedious methods[3].

Example: Counting set classes of triads

First we need to know the order of the fixtures of the elements in T/I. The set we are acting on is pitch sets of order 3. From Section 3.1, the only transposition operators beside T_0 that fix pitch sets of order 3 are

$$T_4, T_8 : \{0, 4, 8\}, \{1, 5, 9\}, \{2, 6, 10\}, \{3, 7, 11\}.$$

T_4 and T_8 each fix 4 triads for a total of 8.

The identity T_0 fixes all triads. With $p = 12$ in our scale, the number of possible triads is

$$\binom{12}{3} = \frac{12!}{3! \cdot 9!} = 220.$$

Therefore transpositions fix a total of $220 + 8 = 228$ triads.

For inversions, recall that I_n fixes 5 pairs of pitches and two singleton pitches if n is even. When n is odd, I_n fixes 6 pairs of pitches. We cannot build pitch sets of order 3 with pairs alone. Rather, we need one pair and one singleton pitch. Because of this, no odd inversion can fix pitch sets of order 3 [5].

For each even inversion we select one pair from the five that it fixes, and place it with one of the two singleton pitches it fixes, giving us a total of

$$\binom{2}{1} \cdot \binom{5}{1} = 10$$

triads fixed by each even inversion operator.

Therefore inversions fix a total of $6 \cdot 10 = 60$ triads.

Plugging these results into Burnside's Theorem gives us

$$n = \frac{1}{24} \sum_{g \in T/I} \{228 + 60\} = \frac{288}{24} = 12$$

So Burnside's Theorem tells us that there are 12 distinct set classes that partition the 220 pitch sets of order 3, each relating to a family of triads in music theory[5]. The operations defined in the next section are applied to only one of these set classes, but any of the other 11 set classes can be treated similarly, and fashioned into a map of triad changes.

4 Building PLR

We now take this framework a step further and build a new group out of the T/I operations. This group will be called PLR, and its elements will perform specific inversions on triads, based on the pitches within their pitch set. These operations take our triads to others that share common pitches, and represent fundamental changes in music theory. The pattern in which they traverse the available triads will help us create a torus of triad changes that can be used to predict or create music[4].

4.1 Major and Minor Triads

For the remainder of the section we will focus on the pitch set $M = \{0, 4, 7\}$. The reason for restricting our attention here will be more obvious to musicians than mathematicians. This pitch set represents an example of a **major** triad. A major triad is characterized by the intervals between pitches, and generally brings an upbeat or happy sound to music[4].

We say any major triad is of the form

$$\{x, x+4, x+7\} = \{x, x-8, x-5\}$$

with root x.

In addition, another important triad is called a **minor** triad. This is very related to the major triad, but brings a much sadder and less resolved sound to music. Its pitch set is of the similar form

$$\{x, x+8, x+5\} = \{x, x-4, x-7\}$$

with root $x+5$.

Note: The root will be used for naming purposes, and does not imply an ordering on pitch sets.

The set class $\mathcal{O}_M$ produced by applying all elements of the T/I group to the major triad $M = \{0, 4, 7\}$ is shown below.

Transpositions:

$$\{0,4,7\}, \{1,5,8\}, \{2,6,9\}, \{3,7,10\}, \{4,8,11\}, \{5,9,0\}$$

$$\{6,10,1\}, \{7,11,2\}, \{8,0,3\}, \{9,1,4\}, \{10,2,5\}, \{11,3,6\}$$

Inversions:

$$\{0,8,5\}, \{1,9,6\}, \{2,10,7\}, \{3,11,8\}, \{4,0,9\}, \{5,1,10\}$$

$$\{6,2,11\}, \{7,3,0\}, \{8,4,1\}, \{9,5,2\}, \{10,6,3\}, \{11,7,4\}$$

Notice that the transpositions give us all possible major triads, and the inversions give us all possible *minor* triads. Even though we can explain this coincidence rather easily, this interpretation of the relationship between major and minor is cleaner and easier than memorizing pitch interval charts. Now, we can make a clean transition to musical notation with these definitions. To name the set of major/minor triads with standard conventions of music means renaming the elements of our scale S [4].

Let

$$S = \{0,1,2,3,4,5,6,7,8,9,10,11\} = \{C, C^{\#}, D, D^{\#}, E, F, F^{\#}, G, G^{\#}, A, A^{\#}, B\}$$

Using this new system, we update our chart for the set class of M above.

Transpositions(major):

C	C$^\#$	D	D$^\#$	E	F
{0,4,7}	{1,5,8}	{2,6,9}	{3,7,10}	{4,8,11}	{5,9,0}

F$^\#$	G	G$^\#$	A	A$^\#$	B
{6,10,1}	{7,11,2}	{8,0,3}	{9,1,4}	{10,2,5}	{11,3,6}

Inversions(minor):

Fm	F$^\# m$	Gm	G$^\# m$	Am	A$^\# m$
{0,8,5}	{1,9,6}	{2,10,7}	{3,11,8}	{4,0,9}	{5,1,10}

Bm	Cm	C$^\# m$	Dm	D$^\# m$	Em
{6,2,11}	{7,3,0}	{8,4,1}	{9,5,2}	{10,6,3}	{11,7,4}

The triads produced by action with the T/I group above are given their name based on the corresponding major/minor root pitch, with inclusion of a lower-case "m" for minor.

i.e. The symbol C represents the major triad rooted at 0, which is the pitch set $\{0, 0 + 4, 0 + 7\} = \{0, 4, 7\}$. The symbol Dm represents the minor triad rooted at 2, the pitch set $\{9, 9 + 8, 9 + 5\} = \{9, 5, 2\}$.

4.2 The PLR Operators

We have already shown that the 24 transposition and inversion operators build a group that is in bijection with D_{12}. However, our notation imposes a structure on T_n and I_n that is based solely on how many semitones are transposed or inverted on a pitch set. There is no qualitative interpretation here, so we want to redefine our group elements to perform T/I operations that represent common triad changes.

Definition 6 (PLR Operations)**.** *Let $\mathcal{O}_M$ be the set class of $M = \{0, 4, 7\}$, which is the set of major and minor triads. Let the functions P, L, R represent the following inversions.*

(1) *Define P*

$$\begin{aligned} P: \quad & \mathcal{O}_M \to \mathcal{O}_M \\ & \{x, y, z\} \mapsto I_{x+z}\{x, y, z\} \end{aligned}$$

This inversion takes any triad to its **parallel** major/minor triad. This is equivalent to taking a major triad to a minor triad rooted at the same pitch, and vice versa[4].

i.e. The triad C is the pitch set {0,4,7}.

$$P(\{0,4,7\}) = I_7\{0,4,7\} = \{7,3,0\}$$

and $\{7,3,0\}$ is precisely the pitch set for Cm.

(2) *Define L*

$$\begin{aligned} L: \quad & \mathcal{O}_M \to \mathcal{O}_M \\ & \{x,y,z\} \mapsto I_{y+z}\{x,y,z\} \end{aligned}$$

This inversion performs a **leading** tone change on our original triad. Given a major triad $\{x,y,z\}$, we reduce our root x by one semitone, and are left with a minor triad rooted at y [7].

i.e.

$$L(\{0,4,7\}) = I_{11}\{0,4,7\} = \{11,7,4\}$$

So in the case of C, the leading tone change sends C to Em.

(3) *Define R*

$$\begin{aligned} R: \quad & \mathcal{O}_M \to \mathcal{O}_M \\ & \{x,y,z\} \mapsto I_{x+y}\{x,y,z\} \end{aligned}$$

This inversion takes any triad to its **relative** major/minor triad. For a major triad $\{x,y,z\}$, this is equivalent to adding two semitones to z, which gives us a minor triad rooted at $z+2$. This chord change is important to tonal theorists and the studying of *keys* [4].

i.e.

$$R(\{0,4,7\}) = I_4\{0,4,7\} = \{4,0,9\}$$

The relative minor of C is Am.

Notice that all the operations bring us to a triad that shares two of three pitches with our original triad. These are fundamental chord changes since these types of triads resonate well together. They produce friendlier transitions than triads with three completely different pitches.

Proposition 2. *The set of all combinations of P,L, and R that can be performed on major/minor triads forms a group isomorphic to T/I.*

Proof. We will show PLR is a group by showing any combination of the basis operators can be reduced to simple transpositions and inversions from T/I on the pitch set $M = \{x, y, z\} = \{x, x+4, x+7\}$.

First of all, we recognize the fact that the identity operation T_0 exists in the available combinations of PLR because any inversion operation is its own inverse.

$$P \circ P(M) = L \circ L(M) = R \circ R(M) = M$$

Now, using P,L and R, we find combinations that are equivalent to the unit transposition T_1 and the unit inversion I_0, since these generate the group T/I.

a) Performing the combination $LPRP(M)$ gives us our unit transposition.

$$\begin{aligned} L \circ P \circ R \circ P(M) &= I_{y+z}(I_{x+z}(I_{x+y}(I_{x+z}(\{x, y, z\})))) \\ &= T_{y-x}(T_{y-z}(\{x, y, z\}) \\ &= T_{2y-x-z}(\{x, y, z\}). \end{aligned}$$

Using the fact $\{x, y, z\} = \{x, x+4, x+7\}$, we have

$$\begin{aligned} T_{2y-x-z}(\{x, y, z\}) &= T_{2(x+4)-x-(x+7)}(\{x, x+4, x+7\}) \\ &= T_{2x+8-2x-7}(\{x, x+4, x+7\}) \\ \\ &= T_1(M). \end{aligned}$$

b) Performing the combination $RLP(M)$ gives us our unit inversion.

$$\begin{aligned} R \circ L \circ P(M) &= I_{x+y}(I_{y+z}(I_{x+z}(\{x, y, z\}))) \\ &= I_{x+y}(T_{y-x}(\{x, y, z\}) \\ &= I_{2x}(\{x, y, z\}) \\ &= I_{2x}(\{x, x+4, x+7\}) \\ &= \{12 + x, 12 + x - 4, 12 + x - 7\} \\ &= \{x, x+8, x+5\} \\ \\ &= I_0(M). \end{aligned}$$

This means PLR operations can generate the 24 transpositions and inversions in the T/I group with these versions of T_1 and I_0. Considering the set class $\mathcal{O}_M$ only contains 24 triads under action with T/I, and that PLR is defined with inversions from T/I, PLR cannot possibly have any more than the 24 operations that give us all of the triads in $\mathcal{O}_M$. This is a bijection, and we can conclude $PLR \cong T/I$.

□

The fact that $PLR \cong D_{12}$ follows directly from the proof [4]. However, the unit rotation and reflection

$$(LPRP)^{12}(M) = T_0 \quad , \quad (RLP)^2(M) = T_0$$

with consistency condition

$$(RLP)(LPRP)(M) = (LPRP)^{11}(RLP)(M)$$

provides an awfully complicated description of how these operations traverse the 24 triads in the set class of M. We would like to generate D_{12} in a simpler way with PLR. As it turns out, we will only need L and R, and can cite a common example for how this works.

Example: Beethoven's Ninth Symphony

In bars 143-176 of Beethoven's Ninth Symphony, the bass line contains the following triads in order[4].

$$C, Am, F, Dm, A^{\#}, Gm, D^{\#}, Cm, G^{\#}, Fm, C^{\#}, A^{\#}m, F\#, D^{\#}m, B, G^{\#}m, E, C^{\#}m, A$$

This is the sequence obtained from alternatively applying the operators R and L to the starting major triad $C = \{0, 4, 7\}$. Notice Beethoven's example only contains 19 triads, and we want to generate the group of 24. Triad A must have been a nice place to stop for his composition, but we can continue where he left off (with R), and see that the remaining five are found.

$$A \xrightarrow{R} F^{\#}m, D, Bm, G, Em, C$$

Therefore, L and R give us all 24 triads, and generate our PLR group alone. In particular, the combined operation (LR) acts like our rotation, whereas L acts like our reflection [4].

$$(LR)^{12} = L^2 = T_0 \quad , \quad L(LR) = (LR)^{11}L$$

This implies that the operation P can be defined completely in terms of R and L.

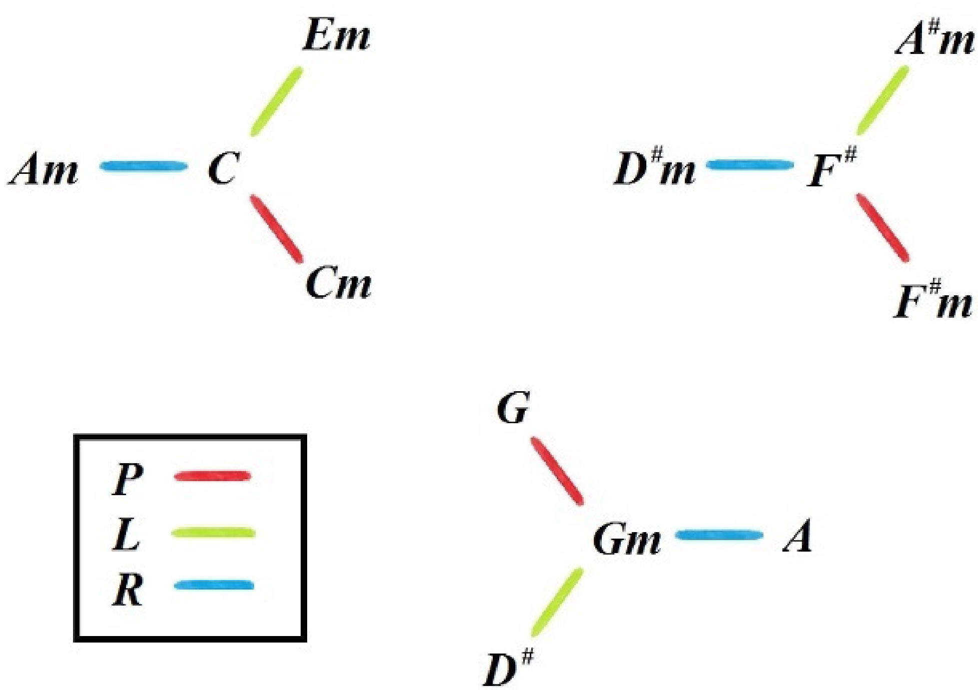
$$C \xrightarrow{R} Am \xrightarrow{L} F \xrightarrow{R} Dm \xrightarrow{L} A^{\#} \xrightarrow{R} Gm \xrightarrow{L} D^{\#} \xrightarrow{R} Cm$$

$$P = R(LR)^3 \in PLR$$

4.3 Application to Musical Thought

By associating major/minor triads in $\mathcal{O}_M$ with the others that share two of three pitches, we can create a network of smooth triad changes that reach every triad in the set class [4].

In order to do this, we need to think of the 24 different triads in $\mathcal{O}_M$ graphically. Let the triads be nodes, and the PLR operations be the edges that connect them. Assign each PLR operation a specific direction, and map each triad to its associated triads. For instance,C, $F^{\#}$, and Gm have the following connections.

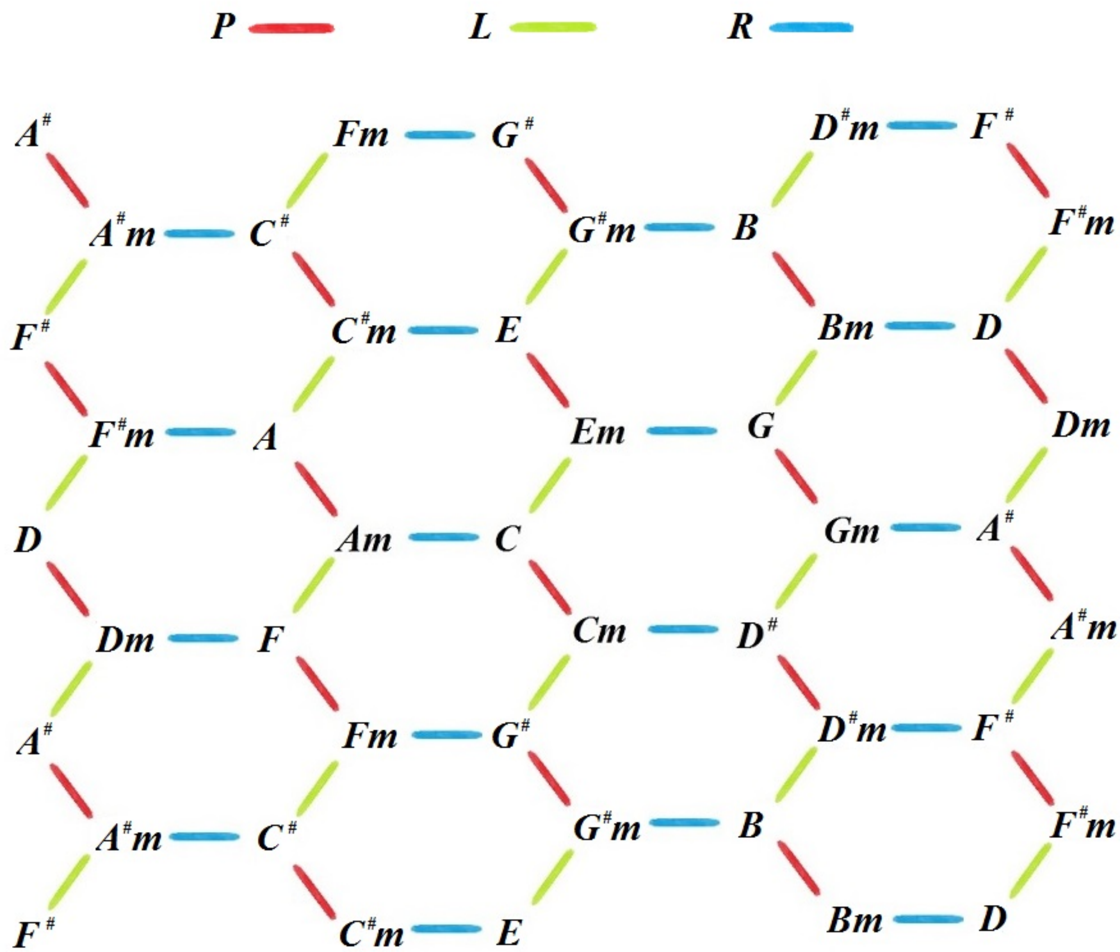


Now, look at just one of these pieces. The three triads associated with C each share two different pitches with C. However, these associated triads, Em, Am, and Cm, do not fall within each others associated triads.

Cm	**Am**	**Em**
P(Cm) = C	P(Am) = A	P(Em) = E
L(Cm) = G#	L(Am) = F	L(Em) = C
R(Cm) = D#	R(Am) = C	R(Em) = G

The same is true for $F^{\#}$ and Gm. We cannot connect these and create small networks, so we are left to expand outward. With the same pattern as above, we connect each node to its own PLR triads, and continue like this in all directions[4]. As it turns out, this larger network builds a hexagonal tiling of our triads, called a tessellation, that resembles a honeycomb patterning.

Figure 1. PLR Network

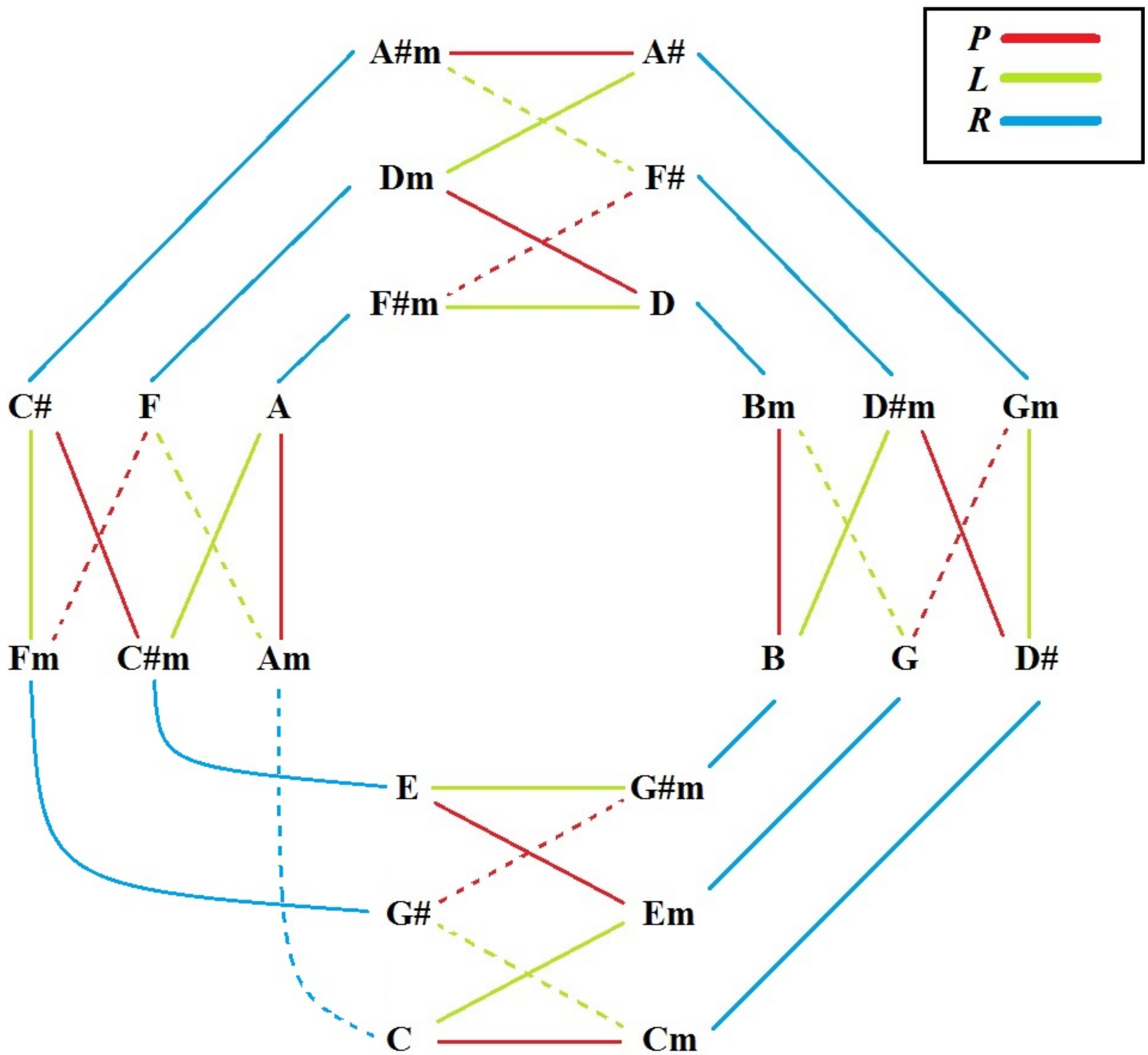


As a tiling implies, this pattern repeats in both directions. Horizontally, the pattern is offset, or twisted. Applying (LR), the unit rotation, twelve times to a starting triad gives us the same triad again, traversing diagonally through the "blue-green" lattice. Vertically, we traverse the "green-red" lattice, and notice that the same triad appears after applying LP three times. We can see that this is consistent with $P = R(LR)^3$ since

$$(LP)^3 = (L(R(LR)^3))^3 = ((LR)^4)^3 = (LR)^{12} = T_0.$$

This sort of tiling represents a common topology that mathematicians refer to as $\mathbb{S}_1 \times \mathbb{S}_1$, or a *torus*[4]. The shape looks like a donut in three dimensions, and we can visualize its construction by molding the network. Imagine connecting the top edge of the diagram to the bottom edge where triads align. This builds a cylinder with open ends on the left and right. Then, we can bend and connect these open ends where triads align again, being careful for the twist, and making a rigid musical donut. This torus will only have each of the 24 triads in our set class represented once. Our vertical and diagonal lattices will be become cycles that take us around the torus in different directions, and the next figure shows what this might look like in two dimensions, with dotted lines for clarity.

Figure 2. PLR Torus



Both the PLR Network and PLR Torus show us visually how the orbit generated by a major/minor triad is traversed by PLR operations. Each of the 24 triads is only connected to its parallel major/minor, relative major/minor, and leading tone change.

As a quick exercise, recall the 19 triad sequence that is part of the bass line in Beethoven's Ninth Symphony

$$C, Am, F, Dm, A^{\#}, Gm, D^{\#}, Cm, G^{\#}, Fm, C^{\#}, A^{\#}m, F\#, D^{\#}m, B, G^{\#}m, E, C^{\#}m, A.$$

Starting from C, find your way through Beethoven's masterpiece. Do you see any patterns? Could other musicians easily employ similar strategies?

5 Remarks

The PLR Network and PLR torus do more for the eye than provide insight on the connection of major and minor triads. There is a musical cloak over the real phenomenon, which lies in the inner-mechanisms of group action. Both groups that we produced, T/I and PLR, are isomorphic to each other[4]. One would think a visual demonstration of their action on sets would look similar, if not the same. However, due to large differences in how we chose to do our "bookkeeping", the representations of identical elements between the two groups appear to obey different structures. Indeed, a torus pattern does exist in some interpretation of the same action with T/I. However, without an embedded concept that makes it easier to grasp, such as music, the interpretation is very non-obvious.

The process described in this project provides plenty of opportunities for future projects or explorations. Certainly, the same concepts could be applied to any family of chords. A good candidate is the family of *seventh* chords, another very common type of chord that uses four pitches. Using the $p = 12$ scale, these chords have the following forms

with root x:

Major:

$$\{x, x+4, x+7, x+11\} = \{x, x-8, x-5, x-1\}$$

Minor:

$$\{x, x+3, x+7, x+10\} = \{x, x-9, x-5, x-2\}$$

Dominant:

$$\{x, x+4, x+7, x+10\} = \{x, x-8, x-5, x-2\}$$

Diminished:

$$\{x, x+3, x+6, x+9\} = \{x, x-9, x-6, x-3\}$$

Notice that there are more than just the major/minor chords in this family, which are also a product of tonal music theory. Furthering the understanding we have of the relationships between these is of great interest to musicians, as these four chords have more depth and diversity than the major/minor triads alone. A student could try to define qualitative operations out of the dihedral group (or maybe the symmetric group) that place these four chords in a set class together[5].

In conclusion, the PLR diagrams provide an extremely useful tool that reveal hidden relationships in the music we hear everywhere. By exposing the cycles and loops that exist in the diagrams, we find that the most common chord progressions and riffs often contain pitches that are adjacent or symmetrically placed around the PLR torus[4]. This gives us a strategy that can be used to predict or create our very own music! After plotting and observing this for ourselves, coming up with new ideas for songs is as simple as connecting the dots.

References

[1] Michael Artin, *Algebra*, Pearson Prentice Hall, 2011

[2] Burnside, William, *Theory of Groups of Finite Order*, University, 1911

[3] Fauvel, John, Raymond Flood, and Robin J. Wilson., *Music and Mathematics: from Pythagoras to Fractals*, Oxford University Press, 2006

[4] Fiore, Thomas M., *Music and Mathematics*, University of Michigan, Michigan, 2003

[5] Graham, Jeff, *An Application of Burnside's Theorem to Music Theory*, UMAP, vol. 29, p45, 2008

[6] Harkleroad, Leon, *The Math Behind the Music*, AMC, vol. 10, p12, 2006

[7] Kopp, David, *Chromatic Transformations in Nineteenth-Century Music*, Cambridge University Press, vol. 17, 2006

[8] Mazzola, Guerino and others, *The Topos of Music*, 2002, Birkhäuser, Basel

[9] Papadopoulos, Athanase, *Mathematics and Group Theory in Music*, 2014 journal=arXiv preprint arXiv:1407.5757