Intro to Functional Equations

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Many of these problems come from Evan Chen's Functional Equation handout, found here.

1 Preview

A functional equation problem typically asks you to find all functions f satisfying a certain condition. We write $f : A \to B$ if the function f maps elements of A (the **domain**) to elements of B (the **codomain**).

Problem 1.1. Which of these are functions $f : \mathbb{Z} \to \mathbb{Z}$?

- $f(x) = x + 1$.
- $f(x) = \sqrt{x}$.
- $f(x) = x^2$.
- $f(x) = \frac{1}{x}$.

Of course, all functional equations are a two step process: name all solutions (and make sure they all work!), then show that no other solutions exist. The rules of the game:

- You must **check that all named solutions work**, or you will be docked a point. (I'm serious!)
- Substitute values (for instance, if a problem is true for all reals x, y, substitute $x = y = 0$).
- Perform analysis on the structure of f (is $f(x) \geq 2$ for all x? is f injective?).

And you are NOT allowed to:

- Substitute values for expressions, most of the time. (e.g. $y = f(x)$, then substitute $y = 1$).
- Assume that the function is well-behaved (for instance, f is not necessarily polynomial, continuous, or even "drawable").
- Make "graphical" arguments (because functions can be as weird as you want).

Here is an example functional equation.

Example 1.2 (IMO 2019/1)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(2a) + 2f(b) = f(f(a + b))$$

for all integers a and b.

(a) Determine all pairs (a, b) of integers such that $f(x) = ax + b$ is a solution.

(b) What is wrong with the following bogus proof?

Proof. Substitute $a = 0$ to get

$$f(0) + 2f(b) = f(f(b)).$$

Then substituting $x = f(b)$ yields $f(x) = 2x + f(0)$, as desired. □

2 Basic methods

Example 2.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $yf(x) = xf(y)$.

Walkthrough. This problem demonstrates a method known as separation of variables.

(a) Find all linear solutions (i.e. when is $f(x) = ax + b$ a solution?). This is a good habit to get into!

(b) Rewrite the equation to isolate x on one side, and y on the other. What does this tell us?

(c) Finish the problem. (Alternatively, why does substituting $x = 1$ immediately kill the problem?)

Example 2.2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) + 2f(\frac{1}{x}) = x$.

Walkthrough. Often, we can actually substitute *expressions* for x (that end up helping!) instead of values.

(a) Replace $x \to \frac{1}{x}$ to get a second, equivalent equation. Why can we do this?

(b) Finish the problem.

Example 2.3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) + f(n)^2 = n^2 + 3n + 3$.

Walkthrough. This problem shows the usage of induction to solve natural FEs.

(a) Determine the value of $f(1)$. (Hint: substitute $n = 1$.)

(b) Determine the value of $f(2)$, $f(3)$, and $f(4)$.

(d) Finish the problem by induction.

Example 2.4 (David Yang)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^{2018} + y) = f(x^{2019} + 2y) + f(x^{2020}).$$

Walkthrough. "DURR WE WANT STUFF TO CANCEL."

(a) Substitute a value of y that causes two terms to cancel. What does this tell you about f?

(b) Finish the problem.

Example 2.5 (Russia 2000)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$3f(x + 2y + 3z) \leq f(x + y) + f(y + z) + f(z + x).$$

Walkthrough. This is a good example of good substitutions.

(a) Show $f(x) \leq f(0)$.

(b) Show $f(x) \geq f(0)$. (Hint: Set values of x, y, z such that $x + 2y + 3z = 0$.)

(c) Conclude.

2.1 Problems for this section

You give it a try!

Problem 2.6. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(m + n) = f(m) + f(n) + mn$.

Problem 2.7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$.

3 Real methods

Unfortunately, real functional equations do not typically take this shape. Instead, we substitute values into the equation to get a handle of the "structure" of f. To clarify what this means, we present some definitions.

- A function $f : A \to B$ is **injective** or **one-to-one** if for all $s, t \in A$, $f(s) = f(t)$ if and only if $s = t$.
- A function $f : A \to B$ is **surjective** or **covering** if for all $b \in B$, there exists an $a \in A$ s.t. $f(a) = b$.
- A function $f : A \to B$ is **bijective** if it is injective and surjective.

Problem 3.1. Are these functions $f : \mathbb{R} \to \mathbb{R}$ injective? Surjective?

- $f(x) = x + 1$.
- $f(x) = \frac{1}{x}$.
- $f(x) = \lfloor x \rfloor$.
- $f(x) = x^2$.

Problem 3.2. If $f(f(x)) = x$ for all real x, show that f is bijective over the reals. (Hint: show it is both injective and surjective.)

Here's how injectivity and surjectivity can be useful in a problem. Often times, functional equation solutions look like this.

Example 3.3 (Balkan 2000/1)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(xf(x) + f(y)) = f(x)^2 + y.$$

Walkthrough. This also demonstrates the so-called **pointwise value trap**.

(a) Show that f is surjective. (Hint: vary y.)

(b) Thus, there exists c such that $f(c) = 0$. Show that $f(f(x)) = x$.

(c) Show $f(0) = 0$.

(d) Show $f(x)^2 = x^2$. (Hint: substitute $x \to f(x)$. This is a common trick.) Does this imply $f(x) = x$ or $f(x) = -x$?

(e) Show that there cannot exist a, b such that $f(a) = a$ and $f(b) = -b$.

(f) Conclude.

4 Problem break

Here are some problems for you to try.

Problem 4.1. Find all functions $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$ such that

$$f(x) + 2f\left(\frac{1}{1-x}\right) = x.$$

Problem 4.2 (Iran TST 1996)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y.$$

Problem 4.3 (IMO 2010/1)**.** Find all function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor .$$

Problem 4.4 (IMO 2008/4)**.** Find all functions $f : (0, \infty) \mapsto (0, \infty)$ such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 4.5 (Pan-African 2018/1)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + y)^2 = f(x^2) + f(y^2)$.

5 Cauchy and Jensen

We begin with a classic problem.

Example 5.1 (Cauchy)**.** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + y) = f(x) + f(y)$.

Walkthrough.

(a) Optionally, note that, if f is a solution, then so is $c \cdot f$ for any real c (i.e. the equaiton is "homogenous" in f). Thus, we can technically assume without loss of generality that $f(1) = 1$.

(b) Show $f(0) = 0$.

(c) Compute $f(n)$ in terms of $f(1)$ for natural numbers n.

(d) Compute $f(n)$ in terms of $f(1)$ for integers n.

(e) Compute $f(\frac{1}{2})$ and $f(\frac{3}{5})$ in terms of $f(1)$.

(f) Complete the proof.

Example 5.2 (Jensen)**.** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$.

Walkthrough.

(a) What is wrong with the following bogus proof?

Proof. Imagine the graph of f. The condition is equivalent to if two points are on the graph, then its midpoint must be on the graph. Thus, f must be linear. □

(b) Optionally, note that, if f is a solution, then $f + c$ is also a solution for any real number c. Thus, we can "without loss of generality" assume $f(0) = 0$.

(c) Substitute $g(x) = f(x) - f(0)$. Thus, we gain the info that $g(0) = 0$.

(d) Show $g(x) = 2g(x/2)$.

(e) Reduce the problem to Cauchy.

You can try for a bit!

Problem 5.3 (JMO 2015/4)**.** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression.

Example 5.4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$.

Walkthrough. Huh.

(a) Try to find $f(\sqrt{2})$ in terms of $f(1)$.

(b) Try harder.

(c) Cry.

Theorem 5.5. A nonlinear solution to the Cauchy equation from $\mathbb{R}$ to $\mathbb{R}$ is dense. In particular, a solution is linear if

- it is increasing,
- it is bounded on some nontrivial interval (e.g. positive over positives), or
- it is continuous or differentiable on any interval.

Example 5.6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) \quad \text{and} \quad f(xy) = f(x)f(y).$$

Walkthrough.

(a) Show that $x > 0$ implies $f(x) > 0$.

(b) Conclude.

5.1 Problem for this section

Problem 5.7 (USAMO 2002/1)**.** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

6 Advanced techniques

6.1 Structural analysis

Example 6.1 (Shortlist 2005 A2)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)f(y) = 2f(x + yf(x)).$$

Walkthrough. This is a prime example of looking at the structure of f.

(a) Find all linear solutions.

(b) Plug in something that will cause terms to cancel. When does this fail?

(c) Show that $f(x) \geq 1$ for all x.

(d) Show that $f(x) \geq 2$ for all x. (Hint: show that if c in the range of f, so is $c^2/2$. How can we use this?)

(e) Show that f is nondecreasing.

(f) Show that if $f(c) = 2$ for some c, then $f(x) \equiv 2$.

(g) Show that $f(x) = 2$ or f is injective.

(h) Conclude. (Hint: swap x and y.)

6.2 Monsters

Example 6.2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(x)) = x + 2$.

(a) Show that $f(x + 2) = f(x) + 2$.

(b) Express the function f in terms of $f(0)$ and $f(1)$.

(c) Find all pairs $(f(0), f(1))$ that make a valid solution and conclude. Is the answer surprising?

7 More problems

Here are some harder problems for you to try.

Problem 7.1 (IMO 1983/1)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy

- $f(xf(y)) = yf(x)$ for all x, y, and
- $f(x) \to 0$ as $x \to \infty$.

Problem 7.2 (IMO 2019/1)**.** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a + b)).$$

Problem 7.3 (APMO 2019/1)**.** Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a, b.

Problem 7.4 (Shortlist 1980)**.** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and

$$f(xy) = f(x)f(y) - f(x + y) + 1.$$

Problem 7.5 (Shortlist 2018 A1)**.** Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

Problem 7.6 (IMO 2012/4)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

Problem 7.7 (Shortlist 2016 A4)**.** Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

Problem 7.8 (Shortlist 2013 N6)**.** Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying

$$f\left(\frac{f(x) + a}{b}\right) = f\left(\frac{x + a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$.