The exam is on the next page. The time limit is 2 hours 30 minutes.

Math Practice 5.0, Quiz #1

Saturday, January 26, 2018

Version C

"According to all known laws of [mathematics], there is no way that a [student] should be able to [sweep a test]. Its [brain is] too small to get its fat little body [to solve problems]. The [student], of course, [sweeps] anyways. Because [students] don't care what [teachers] think is impossible."

V. A positive integer is *uphill* if its digits form a strictly increasing sequence. If n is uphill, find the maximum possible sum of the digits of $9 \cdot n$.

W. Let m be a positive integer. For a positive integer n, let $f(n)$ denote its units digit. A sequence $a_1, a_2, a_3, \ldots$ obeys $a_1 = m$ and

$$a_{i+1} = a_i + f(a_i)$$

for all positive integers i. Find all m such that this sequence contains infinitely many powers of 2.

X. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$.

Time limit: 2 hours 30 minutes.

Each problem is worth 7 points.

Math Practice 5.0, Quiz #1

Saturday, January 26, 2018

Version B

"According to all known laws of [mathematics], there is no way that a [student] should be able to [sweep a test]. Its [brain is] too small to get its fat little body [to solve problems]. The [student], of course, [sweeps] anyways. Because [students] don't care what [teachers] think is impossible."

W. Let m be a positive integer. A sequence $a_1, a_2, a_3, \ldots$ obeys $a_1 = m$ and

$$a_{n+1} = 2a_n - 10\left\lfloor \frac{a_n}{10} \right\rfloor$$

for all positive integers n. Find all m such that this sequence contains infinitely many powers of 2.

X. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that line AP bisects $\overline{CD}$.

Y. Let $0 < \theta < \pi/4$. Prove that

$$(\sin\theta)^{(\sin\theta)} < (\cos\theta)^{(\cos\theta)}.$$

Time limit: 2 hours 30 minutes.

Each problem is worth 7 points.

Math Practice 5.0, Quiz #1

Saturday, January 26, 2018

Version A

"According to all known laws of [mathematics], there is no way that a [student] should be able to [sweep a test]. Its [brain is] too small to get its fat little body [to solve problems]. The [student], of course, [sweeps] anyways. Because [students] don't care what [teachers] think is impossible."

X. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that line AP bisects $\overline{CD}$.

Y. Let $0 < \theta < \pi/4$. Prove that

$$(\sin\theta)^{(\sin\theta)} < (\cos\theta)^{(\cos\theta)}.$$

Z. Lux writes the number 1 a total of 2^n times on a whiteboard. Every second thereafter, Lux chooses two numbers a, b from the board, erases them, and replaces each of them with $a + b$. Prove that after $n2^{n-1}$ seconds, the sum of the numbers on the board will be at least 4^n.

Time limit: 2 hours 30 minutes.

Each problem is worth 7 points.

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1 Solutions

Problem V

A positive integer is *uphill* if its digits form a strictly increasing sequence. If n is uphill, find the maximum possible sum of the digits of $999n$.

Solution: Express $n = (a_k a_{k-1} \dots a_1 a_0)_{10}$ with $a_k < a_{k-1} < \cdots < a_1 < a_0$. Then

$$\begin{aligned} 999n &= 1000n - n \\ &= (a_k a_{k-1} \dots a_1 a_0 000)_{10} - (a_k a_{k-1} \dots a_1 a_0)_{10} \\ &= (a_k a_{k-1} a_{k-2} (a_{k-3} - a_k) \dots (a_2 - a_4)(a_1 - a_3 - 1)(9 - a_2)(9 - a_1)(10 - a_0))_{10} \end{aligned}$$

by elementary school "regrouping." Thus, $999n$ always has digit sum $-1 + 9 + 9 + 10 = 27$.

Problem W

Let m be a positive integer. For a positive integer n, let $f(n)$ denote its units digit. A sequence $a_1, a_2, a_3, \dots$ obeys $a_1 = m$ and

$$a_{i+1} = a_i + f(a_i)$$

for all positive integers i. Find all m such that this sequence contains infinitely many powers of 2.

Solution: The answer is all m not divisible by 5. Obviously, any multiples of 5 will not work (everything will be divisible by 5), so it remains to show any number that is not a multiple of 5 will work.

Casework on the last digit of a_2 shows us that $a_{k+4} = 20 + a_k$ for all $k \geq 2$. Moreover, casework on a_1 modulo 20 tells us that at least one of a_1, a_2, a_3, a_4 is divisible by 4, say a_x. Then there exist infinitely many y such that 20 divides $2^y - a_x$, and thus these powers of 2 will appear in the sequence, as desired.

Problem X

Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$.

Solution: [insert diagram]

Let AC meet BD at X and AD meet CE at Y. Moreover, let DX and CY intersect at P and AP hit CD at M.

Consider the spiral similarity about A that sends $ABCD$ to $ACDE$ (which exists, as $ABCD \sim ACDE$). In particular, this implies $\triangle AXD \sim \triangle AYE$ and $\triangle XCD \sim \triangle YDE$, both in ratio $AB : AC$. So then $AX/AY = CX/DY = AB/AC$. Then Ceva's Theorem gives us $AX/CX \cdot CM/DM \cdot DY/AY = 1 \implies CM/DM = 1$, as desired.

Problem Y

Let $0 < \theta < \pi/4$. Prove that

$$(\sin\theta)^{(\sin\theta)} < (\cos\theta)^{(\cos\theta)}.$$

Solution: Let $a = \tan\theta$. Then we rewrite our equation into

$$\left(\sin^2\theta\right)^{\tan\theta} < \cos^2\theta \implies \left(1 - \frac{1}{1+a^2}\right)^a < \frac{1}{1+a^2}.$$

But the Binomial Theorem tells us that the LHS is equal to

$$1 - \binom{a}{1}\frac{1}{1+a^2} + \binom{a}{2}\left(\frac{1}{1+a^2}\right)^2 - \binom{a}{3}\left(\frac{1}{1+a^2}\right)^3 + \dots$$

Every term after the second is < 0, so we just want to show

$$1 - \frac{a}{1+a^2} < \frac{1}{1+a^2}$$

which is obvious as $a < 1$.

Problem Z

Lux writes the number 1 a total of 2^n times on a whiteboard. Every second thereafter, Lux chooses two numbers a, b from the board, erases them, and replaces each of them with $a+b$. Prove that after $n2^{n-1}$ seconds, the sum of the numbers on the board will be at least 4^n.

Solution: Instead consider the product of the 2^n numbers on the board. When we replace a, b with $a+b, a+b$, we increase the product by a factor of

$$\frac{(a+b)^2}{ab} \geq 4.$$

So after $n2^{n-1}$ seconds the product of the numbers on the board will be at least $4^{n2^{n-1}} = 2^{n2^n}$, so by AM-GM their sum will be $\geq 2^n \sqrt[2^n]{2^{n2^n}} = 4^n$.