Roots of a Polynomial

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> When solving problems, dig at the roots instead of just hacking at the leaves.
>
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Many problems ask us to find symmetric expressions about the roots of a polynomial. For example, a common problem is to calculate the value of

$$a+b+c \quad \text{if } a, b, c \text{ are roots of } x^3-3x+1.$$

Here, we tackle the basic tools necessary to solve these problems. We will also talk about polynomial functional equations and other methods to determine the structure of a polynomial's outputs.

1 Factorization

A **polynomial** is a function of the form

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

for real numbers $a_0, a_1, \dots, a_n$. Polynomials can be expressed in terms of its roots. Note that the Fundamental Theorem of Algebra guarantees that f has n roots, say $r_1, \dots, r_n$, so we can rewrite f in the form

$$f(x)=a_n(x-r_1)(x-r_2)\dots(x-r_n).$$

Already, we can use this factorization in a multitude of ways.

Example 1.1. Let x^3-3x+1 have roots a, b, c. Compute

(a) $(a-1)(b-1)(c-1)$. (b) $(a^2-1)(b^2-1)(c^2-1)$. (c) $(a^3-1)(b^3-1)(c^3-1)$.

Note that $x^3 - 3x + 1$ can be rewritten as $(x-a)(x-b)(x-c)$, where r_i are the roots of the aforementioned polynomial. Then we can gather that, say, $-abc = (0-a)(0-b)(0-c) = (0)^3 - 3(0) + 1$ by substituting $x = 0$. This inspires the following.

Solution to Example 1.1:

(a) Note that $(a-1)(b-1)(c-1) = -(1-a)(1-b)(1-c) = -((1)^3 - 3(1) + 1) = \boxed{1}$.

(b) We factor $a^2 - 1$ as $(a-1)(a+1)$. But then

$$(a^2-1)(b^2-1)(c^2-1) = (a-1)(b-1)(c-1)(a+1)(b+1)(c+1) = (-f(1))(-f(-1))$$

which can be evaluated to be $\boxed{-3}$.

(c) We factor $a^3 - 1 = (a-1)(a-\omega)(a-\omega^2)$, then proceed as in the above solution. Alternatively, note that $a^3 - 3a + 1 = 0 \implies a^3 = 3a - 1$, so

$$(a^3-1)(b^3-1)(c^3-1) = (3a-2)(3b-2)(3c-2) = 27(a-2/3)(b-2/3)(c-2/3)$$

and proceed to get $\boxed{-\frac{19}{27}}$.

2 Vieta's Formulas

We may be familiar with the **Vieta's formulas** (or **Viète's formulas**) for quadratics: if a quadratic $q(x) = ax^2 + bx + c$ has roots r_1 and r_2, then $r_1 + r_2 = -b/a$ and $r_1 r_2 = c/a$. This can be similarly expanded to the general polynomial.

Theorem 2.1 (Vieta's Formulas)**.** Let the roots of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be $r_1, r_2, \ldots, r_n$. Then the sum of the products of the roots taken k at a time, or the ***k*-th symmetric sum**, is equal to $S_k = (-1)^{n-k} \cdot a_{n-k}/a_n$.

Proof. Simply write

$$\begin{aligned} a_n x^n + \cdots + a_0 &= a_n(x-r_1)(x-r_2)\ldots(x-r_n) \\ &= a_n(x^n - (r_1 + \cdots + r_n)x^{n-1} + (r_1 r_2 + \cdots + r_{n-1}r_n)x^{n-2} + \cdots + r_1 r_2 \ldots r_n). \end{aligned}$$

□

This can, again, be used in many ways.

Example 2.2. Let $x^3 - 3x + 1$ have roots a, b, c. Compute

(a) $a^2 + b^2 + c^2$. (b) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. (c) $a^3 + b^3 + c^3$.

Notice that all of these functions are symmetric about a, b, c, which is necessary as we do not know the exact values of a, b, c (the values are permutable). However, we know that symmetric polynomials can be written as an expression about its symmetric sums. For example, for two variables,

$$x^2 + y^2 = (x + y)^2 - 2xy = S_1^2 - 2S_2$$

Applying this philosophy to these problems yields quick solutions.

Solution to Example 2.2.

(a) Note that

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = S_1^2 - 2S_2 = 0^2 - 2(-3) = \boxed{6}.$$

(b) Combine the fractions with a common denominator, which yields

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc} = \frac{-3}{-1} = \boxed{3}.$$

(Why -1 and not 1?)

(c) We present two solutions.

(i) We use the classic factorization

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

which yields

$$a^3 + b^3 + c^3 = S_1(S_1^2 - 3S_2) + 3S_3 = 0(0^2 - 3(-3)) + 3(-1) = \boxed{-3}.$$

(ii) Note that, because a is a root, we have $a^3 - 3a + 1 = 0$ or $a^3 = 3a - 1$. So we get

$$a^3 + b^3 + c^3 = (3a - 1) + (3b - 1) + (3c - 1) = 3(a + b + c) - 3 = \boxed{-3}.$$

With these basic tools, we can still solve an enormous number of problems.

2.1 Problems for this section

Problem 2.3 (AIME 2008)**.** Let r, s, and t be the three roots of the equation $8x^3 + 1001x + 2008 = 0$. Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

Problem 2.4 (USAMO 2014)**.** Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

Problem 2.5 (CHMMC 2017)**.** The equation

$$(x - \sqrt[3]{13})(x - \sqrt[3]{53})(x - \sqrt[3]{103}) = \frac{1}{3}$$

has three distinct real solutions r, s, t for x. Calculate the value of $r^3 + s^3 + t^3$.

3 Transformation

Solving for roots of a polynomial are not typically difficult when those roots are rational (i.e. use Rational Root Theorem). However, when we don't know the exact values of these roots, we can still *theoretically* solve for symmetric expressions about these roots. For example,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc}.$$

However, when expressions become more complicated, these expressions become unwieldy. For example,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{(ab+bc+ca)^2 - 2abc(a+b+c)}{(abc)^2}.$$

This difficulty only grows as we encounter larger numbers and powers.

However, the concept of the **transformation of roots** makes this idea much easier. Say we wanted to calculate the value of

$$\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} \quad \text{where } a, b, c \text{ are roots of } x^3 - 3x + 1.$$

Then a typical way to solve this problem would be to create a polynomial with roots $a-1, b-1, c-1$, then use Vieta's formulas. In other words, notice that $x^3 + 3x^2 - 1 = (x+1)^3 - 3(x+1) + 1$ has roots $a-1, b-1, c-1$ (why?). Thus,

$$\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} = \frac{(a-1)(b-1) + (b-1)(c-1) + (c-1)(a-1)}{(a-1)(b-1)(c-1)} = \frac{0}{1} = 0.$$

In a similar fashion, we get the following tool.

Theorem 3.1 (Most common transformations)**.** Given a polynomial $p(x)$ with roots $r_1, r_2, \ldots, r_n$,

- the polynomial $q(x) = p(x - k)$ has roots $r_1 + k, r_2 + k, \ldots, r_n + k$,
- the polynomial $q(x) = p(x/m)$ has roots $mr_1, mr_2, \ldots, mr_n$, and
- the polynomial $q(x) = p(\sqrt{x})p(-\sqrt{x})$ has roots $r_1^2, r_2^2, \ldots, r_n^2$.

There are, of course, a few more transformations (cubing roots?), however, these are the tools that you'll need the most.

Example 3.2. Let $x^3 - 3x + 1$ have roots a, b, c. Compute

(a) $1/(a+1) + 1/(b+1) + 1/(c+1)$.

(b) $a^2b^2 + b^2c^2 + c^2a^2$.

(c) $(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$.

Solution to Example 3.2.

(a) The polynomial with roots $a+1, b+1, c+1$ is $(x-1)^3 - 3(x-1) + 1 = x^3 - 3x^2 - 3$, so

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = \frac{(a+1)(b+1) + (b+1)(c+1) + (c+1)(a+1)}{(a+1)(b+1)(c+1)} = \boxed{0}.$$

(b) The polynomial with roots a^2, b^2, c^2 is

$$-(x^{3/2} - 3x^{1/2} + 1)(-x^{3/2} + 3x^{1/2} + 1) = x^3 - 6x^2 + 9x - 1,$$

so by Vieta's formulas our answer is $\boxed{9}$.

(c) Note that $a^2 + b^2 = 6 - c^2$ and similar, so

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) = (6 - a^2)(6 - b^2)(6 - c^2)$$

which corresponds to the *monic* polynomial with roots a^2, b^2, c^2 (Note: people forget the "monic" part ALL of the time. Don't be fooled!). So our answer is

$$(6)^3 - 6(6)^2 + 9(6) - 1 = \boxed{53}.$$

3.1 Problems for this section

Problem 3.3. Prove all of the statements in Theorem 2.1. (Hint for the last one: factor $q(x)$ in terms of its roots.)

Problem 3.4. Given a polynomial $p(x)$ with roots $r_1, \ldots, r_n$, find a polynomial with roots $r_1^3, \ldots, r_n^3$.

Problem 3.5 (CMIMC 2016)**.** Let r_1, r_2, …, r_{20} be the roots of the polynomial $x^{20} - 7x^3 + 1$. If

$$\frac{1}{r_1^2+1} + \frac{1}{r_2^2+1} + \cdots + \frac{1}{r_{20}^2+1}$$

can be written in the form $\frac{m}{n}$ where m and n are positive coprime integers, find $m+n$.

(Find two ways to do this one!)

Problem 3.6 (NIMO 15)**.** Let r, s, t be the roots of the polynomial $x^3 + 2x^2 + x - 7$. Then

$$\left(1 + \frac{1}{(r+2)^2}\right)\left(1 + \frac{1}{(s+2)^2}\right)\left(1 + \frac{1}{(t+2)^2}\right) = \frac{m}{n}$$

for relatively prime positive integers m and n. Compute $100m + n$.

Problem 3.7 (CHMMC 2018)**.** Suppose r, s, t are the roots of the polynomial $x^3 - 2x + 3$. Find

$$\frac{1}{r^3-2} + \frac{1}{s^3-2} + \frac{1}{t^3-2}.$$

4 Polynomial Functional Equations

Rather than focusing on arbitrary expression in terms of the roots of a polynomial, this section focuses on the *structure* of a polynomial in its relation to its roots and outputs.

A **polynomial functional equation** is a problem where we try to find all polynomials that satisfy a given property. For example, finding all polynomials $p(x)$ such that

$$p(x)p(y) = p(xy) \quad \text{for all reals } x, y$$

is a polynomial functional equation problem.

We have a couple of methods to show that two polynomials are equal or not. For example, we right off the bat there is no nonconstant polynomial p such that

$$p(x) = p(x^2 + x)$$

as the RHS has twice the degree of the LHS. Similarly, we know that there are no nonconstant solutions to the equation

$$p(x) = p(2x + 1)$$

as their leading coefficients are wildly different. These are both equivalent to comparing the coefficients of a polynomial. Plugging in $p(x) = a_n x^n + \cdots + a_0$ into the equation and solving by equating coefficients is a process known as **undetermined coefficients**.

However, often times this (again) is extremely impractical. A key idea in problems is that when two polynomials are equal, **their roots must also be equal**.

This is best illustrated with an example.

Example 4.1 (AIME 2016)**.** Let $P(x)$ be a nonzero polynomial such that $(x-1)P(x+1) = (x+2)P(x)$ for every real x, and $(P(2))^2 = P(3)$. Compute $P\left(\frac{7}{2}\right)$.

Immediately, by plugging in $x = 2$ we get

$$P(3) = 4P(2) \implies (P(2), P(3)) = (4, 16).$$

However, solely plugging in values of x and using recursion only gets us P over the integers. To find $P\left(\frac{7}{2}\right)$, we have to find P explicitly (this is common in polynomial functional equations).

Solution to Example 2.1.

We analyze the roots of P. Notice that $x = -2, 1$ imply that $P(1) = P(-1) = 0$, respectively. Thus, we can factor

$$P(x) = (x-1)(x+1)Q(x)$$

for a new polynomial Q. Substituting back into the equation, we receive

$$(x-1)(x)(x+2)Q(x+1) = (x+2)(x-1)(x+1)Q(x) \implies xQ(x+1) = (x+1)Q(x)$$

for $x \neq 1, -1, 2$. Plugging in $x = 0$ gives $Q(0) = 0$, so

$$Q(x) = xR(x)$$

for a polynomial R. Plugging this in again gives us

$$x(x+1)R(x+1) = (x+1)(x)R(x) \implies R(x) = R(x+1).$$

We now prove that R must be a constant c. Assume not. Then R must have a least root r. But then $x = r - 1$ gives that r is a root, a contradiction. So

$$R(x) = c \implies Q(x) = cx \implies P(x) = cx(x-1)(x+1).$$

Plugging in $x = 2, 3$ gives $(6c)^2 = 24c \implies c = \frac{2}{3}$, and

$$P\left(\frac{7}{2}\right) = \frac{2}{3} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{9}{2} = \boxed{\frac{105}{4}}.$$

Occasionally, we will have to do more in-depth analysis of the roots of a polynomial, especially when looking at complex roots. See Problems 4.3 and 4.4.

4.1 Problems for this section

Problem 4.2 (HMMT 2003)**.** Suppose $P(x)$ is a polynomial such that $P(1) = 1$ and

$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$

for all real x for which both sides are defined. Find $P(-1)$.

Problem 4.3 (PUMaC 2017)**.** Let S_P be the set of all polynomials P with complex coefficients, such that $P(x^2) = P(x)P(x-1)$ for all complex numbers x. Suppose P_0 is the polynomial in S_P of maximal degree such that $P_0(1) \mid 2016$. Find $P_0(10)$.

Problem 4.4 (AIME 2007)**.** Let $f(x)$ be a polynomial with real coefficients such that $f(0) = 1$, $f(2) + f(3) = 125$, and for all x, $f(x)f(2x^2) = f(2x^3 + x)$. Find $f(5)$.