# Sanitize your inputs!

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## 1 Problems

1. The sum of the areas of all triangles whose vertices are also vertices of a 1 by 1 by 1 cube is $m+\sqrt{n}+\sqrt{p}$, where $m, n$, and $p$ are integers. Find $m+n+p$.

2. Patrick tosses four four-sided dice, each numbered 1 through 4. What's the probability their product is a multiple of four?

3. Trapezoid $ABCD$ is an isosceles trapezoid with $AD = BC$. Point $P$ is the intersection of the diagonals $AC$ and $BD$. If the area of $\triangle ABP$ is 50 and the area of $\triangle CDP$ is 72, what is the area of the entire trapezoid?

4. It is known that, for all positive integers $k$, $1^2+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}$. Find the smallest positive integer $k$ such that $1^2+2^2+3^2+\ldots+k^2$ is a multiple of 200.

5. In $\triangle ABC$, $AB = AC = 10$ and $BC = 12$. Point $D$ lies strictly between $A$ and $B$ on $\overline{AB}$ and point $E$ lies strictly between $A$ and $C$ on $\overline{AC}$ so that $AD = DE = EC$. Then $AD$ can be expressed in the form $\dfrac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p+q$.

6. In rectangle $ABCD$, points $E$ and $F$ lie on sides $AB$ and $CD$ respectively such that both $AF$ and $CE$ are perpendicular to diagonal $BD$. Given that $BF$ and $DE$ separate $ABCD$ into three polygons with equal area, and that $EF = 1$, find the length of $BD$.

7. An integer is called snakelike if its decimal representation $a_1a_2a_3\cdots a_k$ satisfies $a_i < a_{i+1}$ if $i$ is odd and $a_i > a_{i+1}$ if $i$ is even. How many snakelike integers between 1000 and 9999 have four distinct digits?

8. Let $a = \pi/2008$. Find the smallest positive integer $n$ such that

$$2[\cos(a)\sin(a)+\cos(4a)\sin(2a)+\cos(9a)\sin(3a)+\cdots+\cos(n^2a)\sin(na)]$$

is an integer.

9. The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer $k$ such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find $c_k$.

10. Cyclic quadrilateral $ABCD$ satisfies $\angle ABD = 70^\circ$, $\angle ADB = 50^\circ$, and $BC = CD$. Suppose $AB$ intersects $CD$ at point $P$, while $AD$ intersects $BC$ at point $Q$. Compute $\angle APQ - \angle AQP$.

11. The decimal representation of $m/n$, where $m$ and $n$ are relatively prime positive integers and $m < n$, contains the digits $2, 5$, and 1 consecutively, and in that order. Find the smallest value of $n$ for which this is possible.

12. A frog begins at $P_0 = (0,0)$ and makes a sequence of jumps according to the following rule: from $P_n = (x_n, y_n)$, the frog jumps to $P_{n+1}$, which may be any of the points $(x_n + 7, y_n + 2)$, $(x_n + 2, y_n + 7)$, $(x_n - 5, y_n - 10)$, or $(x_n - 10, y_n - 5)$. There are $M$ points $(x, y)$ with $|x| + |y| \le 100$ that can be reached by a sequence of such jumps. Find the remainder when $M$ is divided by 1000.

13. Line segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect at $O$ such the triangles $ABO$, $CDO$, and $EFO$ are all equilateral. Let $G$, $H$, $I$, $J$, $K$, $L$, be the midpoints of segments $AB, BC, CD, DE, EF, FA$, respectively. Given $AG = EF = 4$ and $\angle GKI = 90^\circ$, compute the perimeter of $\triangle HJL$.

14. The sequence of integers $\{a_i\}_{i=0}^{\infty}$ satisfies $a_0 = 3$, $a_1 = 4$, and

$$a_{n+2} = a_{n+1}a_n + \left\lceil \sqrt{a_{n+1}^2 - 1}\sqrt{a_n^2 - 1} \right\rceil$$

for $n \ge 0$. Evaluate the sum

$$\sum_{n=0}^{\infty} \left( \frac{a_{n+3}}{a_{n+2}} - \frac{a_{n+2}}{a_n} + \frac{a_{n+1}}{a_{n+3}} - \frac{a_n}{a_{n+1}} \right).$$

15. We call a polynomial $P$ square-friendly if it is monic, has integer coefficients, and there is a polynomial $Q$ for which $P(n^2) = P(n)Q(n)$ for all integers $n$. We say $P$ is minimally square-friendly if it is square-friendly and cannot be written as the product of nonconstant, square-friendly polynomials. Determine the number of nonconstant, minimally square-friendly polynomials of degree at most 12.

*Time limit: 50 minutes.*
*Each problem is worth one point.*

## 2 Answers

1. 348. Source: AIME 2003 I 6

   Solution: https://artofproblemsolving.com/wiki/index.php/2003_AIME_I_Problems/Problem_6

2. 13/16. Source: CMIMC 2019 Combinatorics 1

   Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2019_CCS_S.pdf

3. 242. Source: NIMO 2017 C28 2

   Solution: https://artofproblemsolving.com/community/c139h1322681p7123484

4. 112. Source: AIME 2002 II 7

   Solution: https://artofproblemsolving.com/wiki/index.php/2002_AIME_II_Problems/Problem_7

5. 289. Source: AIME 2018 I 4

   Solution: https://artofproblemsolving.com/wiki/index.php/2018_AIME_I_Problems/Problem_4

6. $\sqrt{3}$. Source: HMMT 2019 Geometry 2

   Solution: https://hmmt-archive.s3.amazonaws.com/tournaments/2019/feb/geo/solutions.pdf

7. 882. Source: AIME 2004 I 6

   Solution: https://artofproblemsolving.com/wiki/index.php/2004_AIME_I_Problems/Problem_6

8. 251. Source: AIME 2008 II 8

   Solution: https://artofproblemsolving.com/wiki/index.php/2008_AIME_II_Problems/Problem_8

9. 262. Source: AIME 2016 II 9

   Solution: https://artofproblemsolving.com/wiki/index.php/2016_AIME_II_Problems/Problem_9

10. 40. Source: CMIMC 2017 Geometry 6

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_Geometry_S.pdf

11. 127. Source: AIME 2003 I 14

    Solution: https://artofproblemsolving.com/wiki/index.php/2003_AIME_I_Problems/Problem_14

12. 373. Source: AIME 2012 I 11

    Solution: https://artofproblemsolving.com/wiki/index.php/2012_AIME_I_Problems/Problem_11

13. 42. Source: FARML 2020 Individual 8

    Solution: https://drive.google.com/file/d/1QotkxlmA-pb8c8aQZFIWE7iRSeosnqTu/view

14. 14/69. Source: HMMT 2019 Algebra/NT 10

    Solution: https://hmmt-archive.s3.amazonaws.com/tournaments/2019/feb/algnt/solutions.pdf

15. 18. Source: CMIMC 2020 Algebra/NT 10

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2020_ANT_S.pdf