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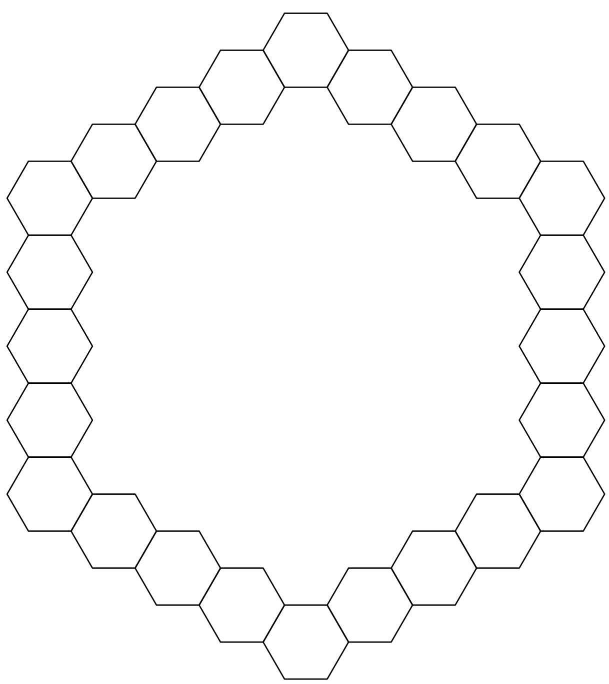
1 Problems

1. The residents of the local zoo are either rabbits or foxes. The ratio of foxes to rabbits in the zoo is $2 : 3$. After 10 of the foxes move out of town and half the rabbits move to Rabbitretreat, the ratio of foxes to rabbits is $13 : 10$. How many animals are left in the zoo?

2. Arnold is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of men. For each of the three factors, the probability that a randomly selected man in the population has only this risk factor (and none of the others) is 0.1. For any two of the three factors, the probability that a randomly selected man has exactly these two risk factors (but not the third) is 0.14. The probability that a randomly selected man has all three risk factors, given that he has A and B is $\frac{1}{3}$. The probability that a man has none of the three risk factors given that he does not have risk factor A is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

3. Patio blocks that are hexagons 1 unit on a side are used to outline a garden by placing the blocks edge to edge with n on each side. The diagram indicates the path of blocks around the garden when $n = 5$.



If $n = 202$, then the area of the garden enclosed by the path, not including the path itself, is $m\left(\sqrt{3}/2\right)$ square units, where m is a positive integer. Find the remainder when m is divided by 1000.

4. Point B is on $\overline{AC}$ with $AB = 9$ and $BC = 21$. Point D is not on $\overline{AC}$ so that $AD = CD$, and AD and BD are integers. Let s be the sum of all possible perimeters of $\triangle ACD$. Find s.

5. It is known that, for all positive integers k,
$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$. Find the smallest positive integer k such that $1^2 + 2^2 + 3^2 + \ldots + k^2$ is a multiple of 200.

6. Call a set S product-free if there do not exist $a, b, c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16, 20\}$ are product-free, whereas the sets $\{4, 16\}$ and $\{2, 8, 16\}$ are not product-free. Find the number of product-free subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

7. Triangle ABC has side lengths $AB = 12$, $BC = 25$, and $CA = 17$. Rectangle $PQRS$ has vertex P on $\overline{AB}$, vertex Q on $\overline{AC}$, and vertices R and S on $\overline{BC}$. In terms of the side length $PQ = w$, the area of $PQRS$ can be expressed as the quadratic polynomial

$$\text{Area}(PQRS) = \alpha w - \beta \cdot w^2.$$

Then the coefficient $\beta = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

8. Let $\mathcal{S}$ be the sphere with center $(0, 0, 1)$ and radius 1 in $\mathbb{R}^3$. A plane $\mathcal{P}$ is tangent to $\mathcal{S}$ at the point (x_0, y_0, z_0), where x_0, y_0, and z_0 are all positive. Suppose the intersection of plane $\mathcal{P}$ with the xy-plane is the line with equation $2x + y = 10$ in xy-space. What is z_0?

9. Suppose a, b, and c are nonzero real numbers such that

$$bc + \frac{1}{a} = ca + \frac{2}{b} = ab + \frac{7}{c} = \frac{1}{a+b+c}.$$

Find $a + b + c$.

10. Find the least odd prime factor of $2019^8 + 1$.

11. Find the sum of all positive integers n such that, given an unlimited supply of stamps of denominations $5, n$, and $n + 1$ cents, 91 cents is the greatest postage that cannot be formed.

12. Let $\triangle PQR$ be a triangle with $\angle P = 75^\circ$ and $\angle Q = 60^\circ$. A regular hexagon $ABCDEF$ with side length 1 is drawn inside $\triangle PQR$ so that side $\overline{AB}$ lies on $\overline{PQ}$, side $\overline{CD}$ lies on $\overline{QR}$, and one of the remaining vertices lies on $\overline{RP}$. There are positive integers a, b, c, and d such that the area of $\triangle PQR$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

13. Let S be the set of all permutations of $\{1, 2, 3, 4, 5\}$. For $s = (a_1, a_2, a_3, a_4, a_5) \in S$, define $\text{nimo}(s)$ to be the sum of all indices $i \in \{1, 2, 3, 4\}$ for which $a_i > a_{i+1}$. For instance, if $s = (2, 3, 1, 5, 4)$, then $\text{nimo}(s) = 2 + 4 = 6$. Compute

$$\sum_{s \in S} 2^{\text{nimo}(s)}.$$

14. From the set of integers $\{1, 2, 3, \ldots, 2009\}$, choose k pairs $\{a_i, b_i\}$ with $a_i < b_i$ so that no two pairs have a common element. Suppose that all the sums $a_i + b_i$ are distinct and less than or equal to 2009. Find the maximum possible value of k.

15. Given that
$$\sum_{x=1}^{70} \sum_{y=1}^{70} \frac{x^y}{y} = \frac{m}{67!}$$
for some positive integer m, find $m \pmod{71}$.

Time limit: 50 minutes.
Each problem is worth one point.

2 Answers

1. 690. Source: CMIMC 2017 Algebra 1

   Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2017_Algebra_S.pdf`

2. 76. Source: AIME 2014 II 2

   Solution: `https://artofproblemsolving.com/wiki/index.php/2014_AIME_II_Problems/Problem_2`

3. 803. Source: AIME 2002 II 4

   Solution: `https://artofproblemsolving.com/wiki/index.php/2002_AIME_II_Problems/Problem_4`

4. 380. Source: AIME 2003 I 7

   Solution: `https://artofproblemsolving.com/wiki/index.php/2003_AIME_I_Problems/Problem_7`

5. 112. Source: AIME 2002 II 7

   Solution: `https://artofproblemsolving.com/wiki/index.php/2002_AIME_II_Problems/Problem_7`

6. 252. Source: AIME 2017 I 12

   Solution: `https://artofproblemsolving.com/wiki/index.php/2017_AIME_I_Problems/Problem_12`

7. 161. Source: AIME 2015 II 7

   Solution: `https://artofproblemsolving.com/wiki/index.php/2015_AIME_II_Problems/Problem_7`

8. 40/21. Source: CMIMC 2017 Geometry 4

   Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2017_Geometry_S.pdf`

9. $-\frac{\sqrt[3]{3}}{2}$. Source: CMIMC 2018 Algebra 5

   Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2018_Algebra_S.pdf`

10. 97. Source: AIME 2019 I 14

    Solution: `https://artofproblemsolving.com/wiki/index.php/2019_AIME_I_Problems/Problem_14`

11. 71. Source: AIME 2019 II 14

    Solution: `https://artofproblemsolving.com/wiki/index.php/2019_AIME_II_Problems/Problem_14`

12. 21. Source: AIME 2013 I 12

    Solution: `https://artofproblemsolving.com/wiki/index.php/2013_AIME_I_Problems/Problem_12`

13. 9765. Source: NIMO 2017 C27 8

    Solution: `https://artofproblemsolving.com/community/c139h1311411p7026706`

14. 803. Source: AIME 2009 II 12

    Solution: `https://artofproblemsolving.com/wiki/index.php/2009_AIME_II_Problems/Problem_12`

15. 12. Source: CMIMC 2016 Number Theory 8

    Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2016_NumberTheory_S.pdf`