

Sample Exam

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1 Problems

1. The residents of the local zoo are either rabbits or foxes. The ratio of foxes to rabbits in the zoo is $2 : 3$. After 10 of the foxes move out of town and half the rabbits move to Rabbitretreat, the ratio of foxes to rabbits is $13 : 10$. How many animals are left in the zoo?
2. Six congruent circles form a ring with each circle externally tangent to two circles adjacent to it. All circles are internally tangent to a circle C with radius 30. Let K be the area of the region inside circle C and outside of the six circles in the ring. Find $\lfloor K \rfloor$.
3. When each of 702, 787, and 855 is divided by the positive integer m , the remainder is always the positive integer r . When each of 412, 722, and 815 is divided by the positive integer n , the remainder is always the positive integer $s \neq r$. Find $m + n + r + s$.
4. A frog is positioned at the origin of the coordinate plane. From the point (x, y) , the frog can jump to any of the points $(x + 1, y)$, $(x + 2, y)$, $(x, y + 1)$, or $(x, y + 2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0, 0)$ and ends at $(4, 4)$.
5. Reimu invented a new number base system that uses exactly five digits. The number 0 in the decimal system is represented as 00000, and whenever a number is incremented, Reimu finds the leftmost digit (of the five digits) that is equal to the “units” (rightmost) digit, increments this digit, and sets all the digits to its right to 0. (For example, an analogous system that uses three digits would begin with 000, 100, 110, 111, 200, 210, 211, 220, 221, 222, 300, ...) Compute the decimal representation of the number that Reimu would write as 98765.
6. In triangle ABC the medians \overline{AD} and \overline{CE} have lengths 18 and 27, respectively, and $AB = 24$. Extend \overline{CE} to intersect the circumcircle of ABC at F . The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.
7. In the middle of a vast prairie, a firetruck is stationed at the intersection of two perpendicular straight highways. The truck travels at 50 miles per hour along the highways and at 14 miles per hour across the prairie. Consider the set of points that can be reached by the firetruck within six minutes. The area of this region is m/n square miles, where m and n are relatively prime positive integers. Find $m + n$.
8. Let x_n be the smallest positive integer such that 7^n divides $x_n^2 - 2$. Find $x_1 + x_2 + x_3$.
9. Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

10. Let c denote the largest possible real number such that there exists a nonconstant polynomial P with

$$P(z^2) = P(z - c)P(z + c)$$

for all z . Compute the sum of all values of $P(\frac{1}{3})$ over all nonconstant polynomials P satisfying the above constraint for this c .

*Time limit: 50 minutes.
Each problem is worth one point.*

2 Answers

1. 690. Source: CMIMC 2017 Algebra 1
Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_Algebra_S.pdf
2. 942. Source: AIME 2005 I 1
Solution: https://artofproblemsolving.com/wiki/index.php/2005_AIME_I_Problems/Problem_1
3. 62. Source: AIME 2017 I 2
Solution: https://artofproblemsolving.com/wiki/index.php/2017_AIME_I_Problems/Problem_2
4. 556. Source: AIME 2018 II 8
Solution: https://artofproblemsolving.com/wiki/index.php/2018_AIME_II_Problems/Problem_8
5. 1727. Source: OMO 2020 Spring 20
Solution: https://artofproblemsolving.com/community/c1124220_2020_online_math_open_problems
6. 63. Source: AIME 2002 I 13
Solution: https://artofproblemsolving.com/wiki/index.php/2002_AIME_I_Problems/Problem_13
7. 731. Source: AIME 2000 I 13
Solution: https://artofproblemsolving.com/wiki/index.php/2000_AIME_I_Problems/Problem_13
8. 121. Source: CMIMC 2019 Algebra/NT 5
Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2019_ANT_S.pdf
9. 158. Source: AIME 2010 I 9
Solution: https://artofproblemsolving.com/wiki/index.php/2010_AIME_I_Problems/Problem_9
10. 13/23. Source: CMIMC 2017 Algebra 10
Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_Algebra_S.pdf