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1 Problems

1. Let a_1, a_2, …, a_n be a geometric progression with $a_1 = \sqrt{2}$ and $a_2 = \sqrt[3]{3}$. What is
$$\frac{a_1 + a_{2013}}{a_7 + a_{2019}}?$$

2. Let B be the set of all binary integers that can be written using exactly 5 zeros and 8 ones where leading zeros are allowed. If all possible subtractions are performed in which one element of B is subtracted from another, find the number of times the answer 1 is obtained.

3. Let S be the set of points whose coordinates x, y, and z are integers that satisfy $0 \le x \le 2$, $0 \le y \le 3$, and $0 \le z \le 4$. Two distinct points are randomly chosen from S. The probability that the midpoint of the segment they determine also belongs to S is m/n, where m and n are relatively prime positive integers. Find $m + n$.

4. Given that $\log_{10} \sin x + \log_{10} \cos x = -1$ and that $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, find n.

5. In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. Then $K = \dfrac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

6. It is known that, for all positive integers k, $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$. Find the smallest positive integer k such that $1^2 + 2^2 + 3^2 + \ldots + k^2$ is a multiple of 200.

7. A pyramid has a triangular base with side lengths 20, 20, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. The volume of the pyramid is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

8. For positive integer n, let $s(n)$ denote the sum of the digits of n. Find the smallest positive integer satisfying $s(n) = s(n + 864) = 20$.

9. For nonnegative integers a and b with $a + b \le 6$, let $T(a, b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a, b)$, where a and b are nonnegative integers with $a + b \le 6$. Find the remainder when S is divided by 1000.

10. A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to the point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. The coordinates of the vertices of isosceles trapezoid $ABCD$ are all integers, with $A = (20, 100)$ and $D = (21, 107)$. The trapezoid has no horizontal or vertical sides, and $\overline{AB}$ and $\overline{CD}$ are the only parallel sides. The sum of the absolute values of all possible slopes for $\overline{AB}$ is m/n, where m and n are relatively prime positive integers. Find $m + n$.

12. Tessa has a unit cube, on which each vertex is labeled by a distinct integer between 1 and 8 inclusive. She also has a deck of 8 cards, 4 of which are black and 4 of which are white. At each step she draws a card from the deck, and

    - if the card is black, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance 1 away from this vertex;
    - if the card is white, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance $\sqrt{2}$ away from this vertex.

    When Tessa finishes drawing all cards of the deck, what is the maximum possible value of a number that is on the cube?

13. Let A, B, C, and D be points in the plane with $AB = AC = BC = BD = CD = 36$ and such that $A \neq D$. Point K lies on segment AC such that $AK = 2KC$. Point M lies on segment AB, and point N lies on line AC, such that D, M, and N are collinear. Let lines CM and BN intersect at P. Then the maximum possible length of segment KP can be expressed in the form $m + \sqrt{n}$ for positive integers m and n. Compute $100m + n$.

14. Let N be the number of ordered triples $(a, b, c) \in \{1, \ldots, 2016\}^3$ such that $a^2 + b^2 + c^2 \equiv 0 \pmod{2017}$. What are the last three digits of N?

15. In triangle ABC with $AB = 23$, $AC = 27$, and $BC = 20$, let D be the foot of the A altitude. Let $\mathcal{P}$ be the parabola with focus A passing through B and C, and denote by T the intersection point of AD with the directrix of $\mathcal{P}$. Determine the value of $DT^2 - DA^2$. (Recall that a parabola $\mathcal{P}$ is the set of points which are equidistant from a point, called the *focus* of $\mathcal{P}$, and a line, called the *directrix* of $\mathcal{P}$.)

Time limit: 50 minutes.
Each problem is worth one point.

2 Answers

1. 8/9. Source: CMIMC 2019 Algebra/NT 1

   Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2019_ANT_S.pdf

2. 330. Source: AIME 2012 I 5

   Solution: https://artofproblemsolving.com/wiki/index.php/2012_AIME_I_Problems/Problem_5

3. 200. Source: AIME 2001 I 10

   Solution: https://artofproblemsolving.com/wiki/index.php/2001_AIME_I_Problems/Problem_10

4. 12. Source: AIME 2003 I 4

   Solution: https://artofproblemsolving.com/wiki/index.php/2003_AIME_I_Problems/Problem_4

5. 23. Source: AIME 2018 II 4

   Solution: https://artofproblemsolving.com/wiki/index.php/2018_AIME_II_Problems/Problem_4

6. 112. Source: AIME 2002 II 7

   Solution: https://artofproblemsolving.com/wiki/index.php/2002_AIME_II_Problems/Problem_7

7. 803. Source: AIME 2017 I 4

   Solution: https://artofproblemsolving.com/wiki/index.php/2017_AIME_I_Problems/Problem_4

8. 695. Source: AIME 2015 I 8

   Solution: https://artofproblemsolving.com/wiki/index.php/2015_AIME_I_Problems/Problem_8

9. 564. Source: AIME 2017 I 7

   Solution: https://artofproblemsolving.com/wiki/index.php/2017_AIME_I_Problems/Problem_7

10. 103. Source: AIME 2020 I 8

    Solution: https://artofproblemsolving.com/wiki/index.php/2020_AIME_I_Problems/Problem_8

11. 131. Source: AIME 2000 II 11

    Solution: https://artofproblemsolving.com/wiki/index.php/2000_AIME_II_Problems/Problem_11

12. 42648. Source: HMNT 2018 Team 8

    Solution: https://hmmt-archive.s3.amazonaws.com/tournaments/2018/nov/team/solutions.pdf

13. 1632. Source: OMO 2019 Fall 15

    Solution: https://artofproblemsolving.com/community/c951538_2019_online_math_open_problems

14. 192. Source: CMIMC 2017 Number Theory 8

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_NumberTheory_S.pdf

15. 96. Source: CMIMC 2017 Geometry 8

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_Geometry_S.pdf