Hello World

Made with TIKT

Sean Li

Generated June 9, 2020

1 Problems

1. In order to complete a large job, 1000 workers were hired, just enough to complete the job on schedule. All the workers stayed on the job while the first quarter of the work was done, so the first quarter of the work was completed on schedule. Then 100 workers were laid off, so the second quarter of the work was completed behind schedule. Then an additional 100 workers were laid off, so the third quarter of the work was completed still further behind schedule. Given that all workers work at the same rate, what is the minimum number of additional workers, beyond the 800 workers still on the job at the end of the third quarter, that must be hired after three-quarters of the work has been completed so that the entire project can be completed on schedule or before?

2. An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

3. In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

4. Find the least positive integer k for which the equation $\left\lfloor \frac{2002}{n} \right\rfloor = k$ has no integer solutions for n. (The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x.)

5. It is known that, for all positive integers k,
   $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$. Find the smallest positive integer k such that $1^2 + 2^2 + 3^2 + \ldots + k^2$ is a multiple of 200.

6. Let $a > 1$ be a positive integer. The sequence of natural numbers $\{a_n\}_{n \geq 1}$ is defined such that $a_1 = a$ and for all $n \geq 1$, a_{n+1} is the largest prime factor of $a_n^2 - 1$. Determine the smallest possible value of a such that the numbers a_1, a_2,..., a_7 are all distinct.

7. Charles has two six-sided die. One of the die is fair, and the other die is biased so that it comes up six with probability $\frac{2}{3}$ and each of the other five sides has probability $\frac{1}{15}$. Charles chooses one of the two dice at random and rolls it three times. Given that the first two rolls are both sixes, the probability that the third roll will also be a six is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

8. The polynomial $P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$ has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

9. Call a positive integer n k-pretty if n has exactly k positive divisors and n is divisible by k. For example, 18 is 6-pretty. Let S be the sum of positive integers less than 2019 that are 20-pretty. Find $\frac{S}{20}$.

10. A 5×5 grid of squares is filled with integers. Call a rectangle corner-odd if its sides are grid lines and the sum of the integers in its four corners is an odd number. What is the maximum possible number of corner-odd rectangles within the grid?

    Note: A rectangle must have four distinct corners to be considered corner-odd; i.e. no $1 \times k$ rectangle can be corner-odd for any positive integer k.

11. Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

    $$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

    Find $|f(0)|$.

12. Let ABC be an equilateral triangle, and let D and F be points on sides BC and AB, respectively, with $FA = 5$ and $CD = 2$. Point E lies on side CA such that angle $DEF = 60^\circ$. The area of triangle DEF is $14\sqrt{3}$. The two possible values of the length of side AB are $p \pm q\sqrt{r}$, where p and q are rational, and r is an integer not divisible by the square of a prime. Find r.

13. Beatrix is going to place six rooks on a 6×6 chessboard where both the rows and columns are labeled 1 to 6; the rooks are placed so that no two rooks are in the same row or the same column. The *value* of a square is the sum of its row number and column number. The *score* of an arrangement of rooks is the least value of any occupied square.The average score over all valid configurations is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

14. Points A, B, and C lie in the plane such that $AB = 13$, $BC = 14$, and $CA = 15$. A peculiar laser is fired from A perpendicular to $\overline{BC}$. After bouncing off BC, it travels in a direction perpendicular to CA. When it hits CA, it travels in a direction perpendicular to AB, and after hitting AB its new direction is perpendicular to BC again. If this process is continued indefinitely, the laser path will eventually approach some finite polygonal shape T_∞. What is the ratio of the perimeter of T_∞ to the perimeter of $\triangle ABC$?

15. In an election for the Peer Pressure High School student council president, there are 2019 voters and two candidates Alice and Celia (who are voters themselves). At the beginning, Alice and Celia both vote for themselves, and Alice's boyfriend Bob votes for Alice as well. Then one by one, each of the remaining 2016 voters votes for a candidate randomly, with probabilities proportional to the current number of the respective candidate's votes. For example, the first undecided voter David has a $\frac{2}{3}$ probability of voting for Alice and a $\frac{1}{3}$ probability of voting for Celia.

    What is the probability that Alice wins the election (by having more votes than Celia)?

Time limit: 50 minutes.
Each problem is worth one point.

2 Answers

1. 766. Source: AIME 2004 II 5

   Solution: `https://artofproblemsolving.com/wiki/index.php/2004_AIME_II_Problems/Problem_5`

2. 144. Source: AIME 2014 I 2

   Solution: `https://artofproblemsolving.com/wiki/index.php/2014_AIME_I_Problems/Problem_2`

3. 120. Source: AIME 2019 I 3

   Solution: `https://artofproblemsolving.com/wiki/index.php/2019_AIME_I_Problems/Problem_3`

4. 49. Source: AIME 2002 II 8

   Solution: `https://artofproblemsolving.com/wiki/index.php/2002_AIME_II_Problems/Problem_8`

5. 112. Source: AIME 2002 II 7

   Solution: `https://artofproblemsolving.com/wiki/index.php/2002_AIME_II_Problems/Problem_7`

6. 46. Source: CMIMC 2018 Number Theory 4

   Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2018_NumberTheory_S.pdf`

7. 167. Source: AIME 2014 II 6

   Solution: `https://artofproblemsolving.com/wiki/index.php/2014_AIME_II_Problems/Problem_6`

8. 482. Source: AIME 2004 I 13

   Solution: `https://artofproblemsolving.com/wiki/index.php/2004_AIME_I_Problems/Problem_13`

9. 472. Source: AIME 2019 II 9

   Solution: `https://artofproblemsolving.com/wiki/index.php/2019_AIME_II_Problems/Problem_9`

10. 60. Source: HMNT 2018 Team 7

    Solution: `https://hmmt-archive.s3.amazonaws.com/tournaments/2018/nov/team/solutions.pdf`

11. 72. Source: AIME 2015 I 10

    Solution: `https://artofproblemsolving.com/wiki/index.php/2015_AIME_I_Problems/Problem_10`

12. 989. Source: AIME 2007 I 15

    Solution: `https://artofproblemsolving.com/wiki/index.php/2007_AIME_I_Problems/Problem_15`

13. 371. Source: AIME 2016 II 13

    Solution: `https://artofproblemsolving.com/wiki/index.php/2016_AIME_II_Problems/Problem_13`

14. 168/295. Source: CMIMC 2019 Team 13

    Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2019_Team_S.pdf`

15. 1513/2017. Source: HMMT 2019 Combo 7

    Solution: `https://hmmt-archive.s3.amazonaws.com/tournaments/2019/feb/comb/solutions.pdf`