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1 Problems

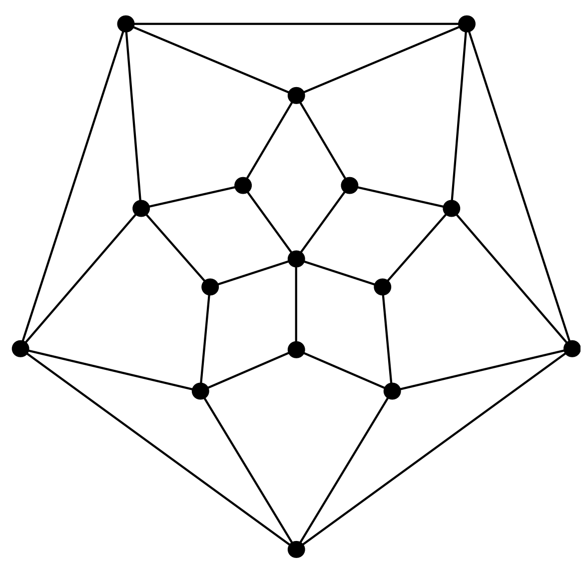
1. Find all sets of five positive integers whose mode, mean, median, and range are all equal to 5.

2. The integers 1 through 49 are to be partitioned into seven 7-element sets, A, B, C, D, E, F, and G. If a, b, c, d, e, f, and g are the medians of the seven sets, what is the largest possible median of the set $\{a, b, c, d, e, f, g\}$?

3. A sphere is inscribed in the tetrahedron whose vertices are $A = (6, 0, 0), B = (0, 4, 0), C = (0, 0, 2)$, and $D = (0, 0, 0)$. The radius of the sphere is m/n, where m and n are relatively prime positive integers. Find $m + n$.

4. A semicircle with diameter d is contained in a square whose sides have length 8. Given the maximum value of d is $m - \sqrt{n}$, find $m + n$.

5. At each of the sixteen circles in the network below stands a student. A total of 3360 coins are distributed among the sixteen students. All at once, all students give away all their coins by passing an equal number of coins to each of their neighbors in the network. After the trade, all students have the same number of coins as they started with. Find the number of coins the student standing at the center circle had originally.



6. A set $\mathcal{S}$ of distinct positive integers has the following property: for every integer x in $\mathcal{S}$, the arithmetic mean of the set of values obtained by deleting x from $\mathcal{S}$ is an integer. Given that 1 belongs to $\mathcal{S}$ and that 2002 is the largest element of $\mathcal{S}$, what is the greatest number of elements that $\mathcal{S}$ can have?

7. Triangle ABC is a right triangle with $AC = 7$, $BC = 24$, and right angle at C. Point M is the midpoint of AB, and D is on the same side of line AB as C so that $AD = BD = 15$. Given that the area of triangle CDM may be expressed as $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

8. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

9. A basketball player has a constant probability of .4 of making any given shot, independent of previous shots. Let a_n be the ratio of shots made to shots attempted after n shots. The probability that $a_{10} = .4$ and $a_n \leq .4$ for all n such that $1 \leq n \leq 9$ is given to be $p^a q^b r / (s^c)$ where p, q, r, and s are primes, and a, b, and c are positive integers. Find $(p + q + r + s)(a + b + c)$.

10. For all positive integers n, let
$$f(n) = \sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2.$$
Compute $f(2019) - f(2018)$. Here $\varphi(n)$ denotes the number of positive integers less than or equal to n which are relatively prime to n.

11. For how many triples of positive integers (a, b, c) with $1 \leq a, b, c \leq 5$ is the quantity
$$(a + b)(a + c)(b + c)$$
not divisible by 4?

12. Let ABC be a triangle with $AB = 20$ and $AC = 22$. Suppose its incircle touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively, and P is the foot of the perpendicular from D to $\overline{EF}$. If $\angle BPC = 90^\circ$, then compute BC^2.

13. Given that
$$\sum_{x=1}^{70} \sum_{y=1}^{70} \frac{x^y}{y} = \frac{m}{67!}$$
for some positive integer m, find $m \pmod{71}$.

14. Triangle AB_0C_0 has side lengths $AB_0 = 12$, $B_0C_0 = 17$, and $C_0A = 25$. For each positive integer n, points B_n and C_n are located on $\overline{AB_{n-1}}$ and $\overline{AC_{n-1}}$, respectively, creating three similar triangles $\triangle AB_nC_n \sim \triangle B_{n-1}C_nC_{n-1} \sim \triangle AB_{n-1}C_{n-1}$. The area of the union of all triangles $B_{n-1}C_nB_n$ for $n \geq 1$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find q.

15. Call a convex quadrilateral angle-Pythagorean if the degree measures of its angles are integers $w \leq x \leq y \leq z$ satisfying
$$w^2 + x^2 + y^2 = z^2.$$
Determine the maximum possible value of $x + y$ for an angle-Pythagorean quadrilateral.

Time limit: 50 minutes.
Each problem is worth one point.

2 Answers

1. $\{2,5,5,6,7\},\{3,4,5,5,8\}$. Source: CMIMC 2020 Team 2

   Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2020_Team_S.pdf`

2. 34. Source: FARML 2020 Individual 7

   Solution: `https://drive.google.com/file/d/1QotkxlmA-pb8c8aQZFIWE7iRSeosnqTu/view`

3. 5. Source: AIME 2001 I 12

   Solution: `https://artofproblemsolving.com/wiki/index.php/2001_AIME_I_Problems/Problem_12`

4. 544. Source: AIME 2005 I 11

   Solution: `https://artofproblemsolving.com/wiki/index.php/2005_AIME_I_Problems/Problem_11`

5. 280. Source: AIME 2012 I 7

   Solution: `https://artofproblemsolving.com/wiki/index.php/2012_AIME_I_Problems/Problem_7`

6. 30. Source: AIME 2002 I 14

   Solution: `https://artofproblemsolving.com/wiki/index.php/2002_AIME_I_Problems/Problem_14`

7. 578. Source: AIME 2003 II 11

   Solution: `https://artofproblemsolving.com/wiki/index.php/2003_AIME_II_Problems/Problem_11`

8. 85. Source: AIME 2020 I 14

   Solution: `https://artofproblemsolving.com/wiki/index.php/2020_AIME_I_Problems/Problem_14`

9. 660. Source: AIME 2002 II 12

   Solution: `https://artofproblemsolving.com/wiki/index.php/2002_AIME_II_Problems/Problem_12`

10. 11431. Source: CMIMC 2019 Algebra/NT 7

    Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2019_ANT_S.pdf`

11. 48. Source: CMIMC 2017 Number Theory 3

    Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2017_NumberTheory_S.pdf`

12. 84. Source: OMO 2020 Spring 15

    Solution: `https://artofproblemsolving.com/community/c1124220_2020_online_math_open_problems`

13. 12. Source: CMIMC 2016 Number Theory 8

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2016_NumberTheory_S.pdf

14. 961. Source: AIME 2013 I 13

    Solution: https://artofproblemsolving.com/wiki/index.php/2013_AIME_I_Problems/Problem_13

15. 207. Source: CMIMC 2019 Team 12

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2019_Team_S.pdf