Hello World

Made with TIKT

Sean Li

Generated June 10, 2020

1 Problems

1. For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series
$$12 + 12r + 12r^2 + 12r^3 + \cdots .$$
Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

2. What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?

3. For how many ordered pairs (x, y) of integers is it true that $0 < x < y < 10^6$ and that the arithmetic mean of x and y is exactly 2 more than the geometric mean of x and y?

4. Consider the set of points that are inside or within one unit of a rectangular parallelepiped (box) that measures 3 by 4 by 5 units. Given that the volume of this set is $\frac{m+n\pi}{p}$, where m, n, and p are positive integers, and n and p are relatively prime, find $m + n + p$.

5. In rectangle $ABCD$, points E and F lie on sides AB and CD respectively such that both AF and CE are perpendicular to diagonal BD. Given that BF and DE separate $ABCD$ into three polygons with equal area, and that $EF = 1$, find the length of BD.

6. Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. A rectangular piece of paper measures 4 units by 5 units. Several lines are drawn parallel to the edges of the paper. A rectangle determined by the intersections of some of these lines is called basic if

   (i) all four sides of the rectangle are segments of drawn line segments, and

   (ii) no segments of drawn lines lie inside the rectangle.

   Given that the total length of all lines drawn is exactly 2007 units, let N be the maximum possible number of basic rectangles determined. Find the remainder when N is divided by 1000.

8. Anders is solving a math problem, and he encounters the expression $\sqrt{15!}$. He attempts to simplify this radical as $a\sqrt{b}$ where a and b are positive integers. The sum of all possible values of ab can be expressed in the form $q \cdot 15!$ for some rational number q. Find q.

9. Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2 a)\sin(na)]$$

is an integer.

10. A triangular array of numbers has a first row consisting of the odd integers $1, 3, 5, \ldots, 99$ in increasing order. Each row below the first has one fewer entry than the row above it, and the bottom row has a single entry. Each entry in any row after the top row equals the sum of the two entries diagonally above it in the row immediately above it. How many entries in the array are multiples of 67?

11. Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

13. A regular hexagon with center at the origin in the complex plane has opposite pairs of sides one unit apart. One pair of sides is parallel to the imaginary axis. Let R be the region outside the hexagon, and let $S = \left\{\frac{1}{z} \mid z \in R\right\}$. Then the area of S has the form $a\pi + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

14. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\dfrac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

15. Let c denote the largest possible real number such that there exists a nonconstant polynomial P with

$$P(z^2) = P(z - c)P(z + c)$$

for all z. Compute the sum of all values of $P(\frac{1}{3})$ over all nonconstant polynomials P satisfying the above constraint for this c.

Time limit: 50 minutes.
Each problem is worth one point.

2 Answers

1. 336. Source: AIME 2016 I 1

   Solution: https://artofproblemsolving.com/wiki/index.php/2016_AIME_I_Problems/Problem_1

2. 180. Source: AIME 2000 II 4

   Solution: https://artofproblemsolving.com/wiki/index.php/2000_AIME_II_Problems/Problem_4

3. 997. Source: AIME 2000 I 6

   Solution: https://artofproblemsolving.com/wiki/index.php/2000_AIME_I_Problems/Problem_6

4. 505. Source: AIME 2003 I 5

   Solution: https://artofproblemsolving.com/wiki/index.php/2003_AIME_I_Problems/Problem_5

5. $\sqrt{3}$. Source: HMMT 2019 Geometry 2

   Solution: https://hmmt-archive.s3.amazonaws.com/tournaments/2019/feb/geo/solutions.pdf

6. 52. Source: AIME 2010 II 4

   Solution: https://artofproblemsolving.com/wiki/index.php/2010_AIME_II_Problems/Problem_4

7. 896. Source: AIME 2007 II 8

   Solution: https://artofproblemsolving.com/wiki/index.php/2007_AIME_II_Problems/Problem_8

8. 4. Source: HMNT 2018 General 7

   Solution: https://hmmt-archive.s3.amazonaws.com/tournaments/2018/nov/gen/solutions.pdf

9. 251. Source: AIME 2008 II 8

   Solution: https://artofproblemsolving.com/wiki/index.php/2008_AIME_II_Problems/Problem_8

10. 17. Source: AIME 2008 I 6

    Solution: https://artofproblemsolving.com/wiki/index.php/2008_AIME_I_Problems/Problem_6

11. 97. Source: AIME 2011 II 12

    Solution: https://artofproblemsolving.com/wiki/index.php/2011_AIME_II_Problems/Problem_12

12. 32. Source: AIME 2019 I 13

    Solution: https://artofproblemsolving.com/wiki/index.php/2019_AIME_I_Problems/Problem_13

13. 29. Source: AIME 2008 II 13

    Solution: https://artofproblemsolving.com/wiki/index.php/2008_AIME_II_Problems/Problem_13

14. 647. Source: AIME 2018 II 13

    Solution: https://artofproblemsolving.com/wiki/index.php/2018_AIME_II_Problems/Problem_13

15. 13/23. Source: CMIMC 2017 Algebra 10

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_Algebra_S.pdf