Sample

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1 Problems

1. Find the number of five-digit positive integers, n, that satisfy the following conditions:
   (a) the number n is divisible by 5,
   (b) the first and last digits of n are equal, and
   (c) the sum of the digits of n is divisible by 5.

2. A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in AIME and the four digits in 2007. No character may appear in a sequence more times than it appears among the four letters in AIME or the four digits in 2007. A set of plates in which each possible sequence appears exactly once contains N license plates. Find N/10.

3. For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots .$$

   Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

4. Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the least positive integer n such that the probability that the two selected unit squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

5. Let ABC be a triangle with sides 3, 4, and 5, and $DEFG$ be a 6-by-7 rectangle. A segment is drawn to divide triangle ABC into a triangle U_1 and a trapezoid V_1 and another segment is drawn to divide rectangle $DEFG$ into a triangle U_2 and a trapezoid V_2 such that U_1 is similar to U_2 and V_1 is similar to V_2. The minimum value of the area of U_1 can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

6. Circle C with radius 2 has diameter $\overline{AB}$. Circle D is internally tangent to circle C at A. Circle E is internally tangent to circle C, externally tangent to circle D, and tangent to $\overline{AB}$. The radius of circle D is three times the radius of circle E, and can be written in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$.

7. Let k be a positive integer. Marco and Vera play a game on an infinite grid of square cells. At the beginning, only one cell is black and the rest are white. A turn in this game consists of the following. Marco moves first, and for every move he must choose a cell which is black and which has more than two white neighbors. (Two cells are neighbors if they share an edge, so every cell has exactly four neighbors.) His move consists of making the chosen black cell white and turning all of its neighbors black if they are not already. Vera then performs

the following action exactly k times: she chooses two cells that are neighbors to each other and swaps their colors (she is allowed to swap the colors of two white or of two black cells, though doing so has no effect). This, in totality, is a single turn. If Vera leaves the board so that Marco cannot choose a cell that is black and has more than two white neighbors, then Vera wins; otherwise, another turn occurs. Let m be the minimal k value such that Vera can guarantee that she wins no matter what Marco does. For $k = m$, let t be the smallest positive integer such that Vera can guarantee, no matter what Marco does, that she wins after at most t turns. Compute $100m + t$.

8. For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

9. Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2a)\sin(na)]$$

is an integer.

10. Over all natural numbers n with 16 (not necessarily distinct) prime divisors, one of them maximizes the value of $s(n)/n$, where $s(n)$ denotes the sum of the divisors of n. What is the value of $d(d(n))$, where $d(n)$ is the the number of divisors of n?

Time limit: 50 minutes.
Each problem is worth one point.

2 Answers

1. 200. Source: AIME 2013 I 2

   Solution: https://artofproblemsolving.com/wiki/index.php/2013_AIME_I_Problems/Problem_2

2. 372. Source: AIME 2007 II 1

   Solution: https://artofproblemsolving.com/wiki/index.php/2007_AIME_II_Problems/Problem_1

3. 336. Source: AIME 2016 I 1

   Solution: https://artofproblemsolving.com/wiki/index.php/2016_AIME_I_Problems/Problem_1

4. 90. Source: AIME 2015 II 5

   Solution: https://artofproblemsolving.com/wiki/index.php/2015_AIME_II_Problems/Problem_5

5. 35. Source: AIME 2004 I 9

   Solution: https://artofproblemsolving.com/wiki/index.php/2004_AIME_I_Problems/Problem_9

6. 254. Source: AIME 2014 II 8

   Solution: https://artofproblemsolving.com/wiki/index.php/2014_AIME_II_Problems/Problem_8

7. 103. Source: OMO 2019 Fall 10

   Solution: https://artofproblemsolving.com/community/c951538_2019_online_math_open_problems

8. 25. Source: AIME 2005 I 12

   Solution: https://artofproblemsolving.com/wiki/index.php/2005_AIME_I_Problems/Problem_12

9. 251. Source: AIME 2008 II 8

   Solution: https://artofproblemsolving.com/wiki/index.php/2008_AIME_II_Problems/Problem_8

10. 54. Source: CMIMC 2020 Team 9

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2020_Team_S.pdf