Sample

Made with TiKT

Foo Bar

Generated June 11, 2020

1 Problems

1. Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

2. Let $ABCD$ be an isosceles trapezoid with $AD = BC = 15$ such that the distance between its bases AB and CD is 7. Suppose further that the circles with diameters $\overline{AD}$ and $\overline{BC}$ are tangent to each other. What is the area of the trapezoid?

3. Ten identical crates each of dimensions 3 ft $\times$ 4 ft $\times$ 6 ft. The first crate is placed flat on the floor. Each of the remaining nine crates is placed, in turn, flat on top of the previous crate, and the orientation of each crate is chosen at random. Let $\frac{m}{n}$ be the probability that the stack of crates is exactly 41 ft tall, where m and n are relatively prime positive integers. Find m.

4. A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

   - Any cube may be the bottom cube in the tower.
   - The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

   Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

5. There are $2n$ complex numbers that satisfy both $z^{28} - z^8 - 1 = 0$ and $|z| = 1$. These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \leq \theta_1 < \theta_2 < \ldots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \ldots + \theta_{2n}$.

6. The sequences of positive integers $1, a_2, a_3, ...$ and $1, b_2, b_3, ...$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k.

7. For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

8. The Annual Interplanetary Mathematics Examination (AIME) is written by a committee of five Martians, five Venusians, and five Earthlings. At meetings, committee members sit at

a round table with chairs numbered from 1 to 15 in clockwise order. Committee rules state that a Martian must occupy chair 1 and an Earthling must occupy chair 15, Furthermore, no Earthling can sit immediately to the left of a Martian, no Martian can sit immediately to the left of a Venusian, and no Venusian can sit immediately to the left of an Earthling. The number of possible seating arrangements for the committee is $N(5!)^3$. Find N.

9. For integers a, b, c and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

10. Cyclic quadrilateral $ABCD$ satisfies $\angle ABD = 70^\circ$, $\angle ADB = 50^\circ$, and $BC = CD$. Suppose AB intersects CD at point P, while AD intersects BC at point Q. Compute $\angle APQ - \angle AQP$.

11. There are 15 cities, and there is a train line between each pair operated by either the Carnegie Rail Corporation or the Mellon Transportation Company. A tourist wants to visit exactly three cities by travelling in a loop, all by travelling on one line. What is the minimum number of such 3-city loops?

12. For $\pi \le \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \cdots$$
$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \cdots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Then $\sin\theta = -\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

13. Let ABC be a triangle with $BC = 49$ and circumradius 25. Suppose that the circle centered on BC that is tangent to AB and AC is also tangent to the circumcircle of ABC. Then

$$\frac{AB \cdot AC}{-BC + AB + AC} = \frac{m}{n}$$

where m and n are relatively prime positive integers. Compute $100m + n$.

14. Call a polynomial P prime-covering if for every prime p, there exists an integer n for which p divides $P(n)$. Determine the number of ordered triples of integers (a, b, c), with $1 \le a < b < c \le 25$, for which $P(x) = (x^2 - a)(x^2 - b)(x^2 - c)$ is prime-covering.

15. Let $\phi(n)$ denote the number of positive integers less than or equal to n that are coprime to n. Compute

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{5^n + 1}.$$

Time limit: 50 minutes.
Each problem is worth one point.

2 Answers

1. 72. Source: AIME 2009 I 5

   Solution: https://artofproblemsolving.com/wiki/index.php/2009_AIME_I_Problems/Problem_5

2. 105. Source: CMIMC 2016 Geometry 2

   Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2016_Geometry_S.pdf

3. 190. Source: AIME 2008 I 9

   Solution: https://artofproblemsolving.com/wiki/index.php/2008_AIME_I_Problems/Problem_9

4. 458. Source: AIME 2006 I 11

   Solution: https://artofproblemsolving.com/wiki/index.php/2006_AIME_I_Problems/Problem_11

5. 840. Source: AIME 2001 II 14

   Solution: https://artofproblemsolving.com/wiki/index.php/2001_AIME_II_Problems/Problem_14

6. 262. Source: AIME 2016 II 9

   Solution: https://artofproblemsolving.com/wiki/index.php/2016_AIME_II_Problems/Problem_9

7. 25. Source: AIME 2005 I 12

   Solution: https://artofproblemsolving.com/wiki/index.php/2005_AIME_I_Problems/Problem_12

8. 346. Source: AIME 2009 I 10

   Solution: https://artofproblemsolving.com/wiki/index.php/2009_AIME_I_Problems/Problem_10

9. 510. Source: AIME 2020 I 11

   Solution: https://artofproblemsolving.com/wiki/index.php/2020_AIME_I_Problems/Problem_11

10. 40. Source: CMIMC 2017 Geometry 6

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_Geometry_S.pdf

11. 88. Source: CMIMC 2019 Combinatorics 9

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2019_CCS_S.pdf

12. 36. Source: AIME 2013 I 14

    Solution: https://artofproblemsolving.com/wiki/index.php/2013_AIME_I_Problems/Problem_14

13. 250049. Source: NIMO 2017 C28 8

    Solution: `https://artofproblemsolving.com/community/c139h1322689p7123504`

14. 1194. Source: CMIMC 2019 Team 15

    Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2019_Team_S.pdf`

15. 65/288. Source: CMIMC 2018 Number Theory 9

    Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2018_NumberTheory_S.pdf`