Sample

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1 Problems

1. Find the number of ordered triples (a, b, c) where a, b, and c are positive integers, a is a factor of b, a is a factor of c, and $a + b + c = 100$.

2. Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following systems of equations:

$$\begin{aligned} abc &= 70, \\ cde &= 71, \\ efg &= 72. \end{aligned}$$

3. Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. The probability of obtaining a grid that does not have a 2-by-2 red square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. In triangle ABC, $AB = 125$, $AC = 117$ and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.

5. The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

6. A frog is positioned at the origin of the coordinate plane. From the point (x, y), the frog can jump to any of the points $(x + 1, y)$, $(x + 2, y)$, $(x, y + 1)$, or $(x, y + 2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0, 0)$ and ends at $(4, 4)$.

7. A group of clerks is assigned the task of sorting 1775 files. Each clerk sorts at a constant rate of 30 files per hour. At the end of the first hour, some of the clerks are reassigned to another task; at the end of the second hour, the same number of the remaining clerks are also reassigned to another task, and a similar assignment occurs at the end of the third hour. The group finishes the sorting in 3 hours and 10 minutes. Find the number of files sorted during the first one and a half hours of sorting.

8. For how many positive integers $100 < n \le 10000$ does $\lfloor \sqrt{n - 100} \rfloor$ divide n?

9. Suppose a_1, a_2, …, a_{10} are nonnegative integers such that

$$\sum_{k=1}^{10} a_k = 15 \qquad \text{and} \qquad \sum_{k=1}^{10} k a_k = 80.$$

Let M and m denote the maximum and minimum respectively of $\sum_{k=1}^{10} k^2 a_k$. Compute $M - m$.

10. For how many triples of positive integers (a, b, c) with $1 \leq a, b, c \leq 5$ is the quantity
$$(a+b)(a+c)(b+c)$$
not divisible by 4?

11. Find the least odd prime factor of $2019^8 + 1$.

12. Find the largest positive integer N satisfying the following properties: N is divisible by 7; Swapping the i^{th} and j^{th} digits of N (for any i and j with $i \neq j$) gives an integer which is *not* divisible by 7.

13. Boris plays a game in which he rolls two standard four-sided dice independently and at random, and at the end of the game receives a number of dollars equal to the product of the two rolled numbers. After the initial roll of both dice, however, he can pay two dollars to reroll one die of choice, and he is allowed to pay to reroll as many times as he wishes. If Boris plays to maximize his expected gain, how much money, in dollars, can he expect to win by playing once?

14. In $\triangle ABC$, $AB = 3$, $BC = 4$, and $CA = 5$. Circle ω intersects $\overline{AB}$ at E and B, $\overline{BC}$ at B and D, and $\overline{AC}$ at F and G. Given that $EF = DF$ and $\frac{DG}{EG} = \frac{3}{4}$, length $DE = \frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. Find $a + b + c$.

15. A regular hexagon with center at the origin in the complex plane has opposite pairs of sides one unit apart. One pair of sides is parallel to the imaginary axis. Let R be the region outside the hexagon, and let $S = \left\{\frac{1}{z} \mid z \in R\right\}$. Then the area of S has the form $a\pi + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

Time limit: 50 minutes.
Each problem is worth one point.

2 Answers

1. 200. Source: AIME 2007 II 2

   Solution: https://artofproblemsolving.com/wiki/index.php/2007_AIME_II_Problems/Problem_2

2. 96. Source: AIME 2019 II 3

   Solution: https://artofproblemsolving.com/wiki/index.php/2019_AIME_II_Problems/Problem_3

3. 929. Source: AIME 2001 II 9

   Solution: https://artofproblemsolving.com/wiki/index.php/2001_AIME_II_Problems/Problem_9

4. 56. Source: AIME 2011 I 4

   Solution: https://artofproblemsolving.com/wiki/index.php/2011_AIME_I_Problems/Problem_4

5. 46. Source: AIME 2006 I 9

   Solution: https://artofproblemsolving.com/wiki/index.php/2006_AIME_I_Problems/Problem_9

6. 556. Source: AIME 2018 II 8

   Solution: https://artofproblemsolving.com/wiki/index.php/2018_AIME_II_Problems/Problem_8

7. 945. Source: AIME 2013 II 7

   Solution: https://artofproblemsolving.com/wiki/index.php/2013_AIME_II_Problems/Problem_7

8. 205. Source: NIMO 2017 C29 4

   Solution: https://artofproblemsolving.com/community/c139h1347229p7339584

9. 286. Source: CMIMC 2017 Algebra 8

   Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_Algebra_S.pdf

10. 48. Source: CMIMC 2017 Number Theory 3

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_NumberTheory_S.pdf

11. 97. Source: AIME 2019 I 14

    Solution: https://artofproblemsolving.com/wiki/index.php/2019_AIME_I_Problems/Problem_14

12. 987546. Source: CMIMC 2017 Number Theory 6

    Solution: http://cmimc-official.herokuapp.com/docs/past-tests/2017_NumberTheory_S.pdf

13. 33/4. Source: CMIMC 2017 Combinatorics 6

    Solution: `http://cmimc-official.herokuapp.com/docs/past-tests/2017_Combinatorics_S.pdf`

14. 41. Source: AIME 2014 I 15

    Solution: `https://artofproblemsolving.com/wiki/index.php/2014_AIME_I_Problems/Problem_15`

15. 29. Source: AIME 2008 II 13

    Solution: `https://artofproblemsolving.com/wiki/index.php/2008_AIME_II_Problems/Problem_13`