Exercises. Chapter 14: 1 (except the $GL_2(\mathbb{R})$ example), 2, 3;
Chapter 16: 4bd, 16 (recommended: 20, 21, 22).

Recall, if G acts on a set X and $x, y \in X$, then x is said to be G-*equivalent* to y if there exists a $g \in G$ such that $gx = y$. We write $x \sim_G y$ or $x \sim y$ if x and y are G-equivalent. In class, we proved that the G-equivalence relation is reflexive and symmetric. You should check the transitive property on your own to complete the proof that $\sim$ is an equivalence relation on X.

14.1 Each of the examples below describes an action of a group G on a set X, which will give rise to the equivalence relation defined by G-equivalence. For each example, compute the equivalence classes of the G-equivalence relation.

(a) Let $G = D_4$ be the symmetry group of a square. If $X = \{1, 2, 3, 4\}$ is the set of vertices of the square, then we can consider D_4 to consist of the following permutations:

$$\{(1), (13), (24), (1432), (1234), (12)(34), (14)(23), (13)(24)\}.$$

The elements of D_4 act on X as functions. The permutation $(13)(24)$ acts on vertex 1 by sending it to vertex 3, on vertex 2 by sending it to vertex 4, and so on.

(b) If we let $X = G$, then every group G acts on itself by the so called *left regular representation* $\lambda : G \to \mathrm{Sym}(G)$ which, for each $g \in G$, gives the function $\lambda_g : G \to G$ defined by $\lambda_g(x) = gx$. That is, G itself is a G-set under this "left multiplication" action. Alternatively, we could restrict the domain of λ to a particular subgroup, say, $H \leq G$, and then G becomes an H-set under left multiplication by elements of H.

(c) Let G be a group and suppose that $X = G$. If H is a subgroup of G, then G is an H-set under the *conjugation action*; that is, we can define an action φ of H on G with the function $\varphi : H \to (G \to G)$ where $\varphi_h(g) = hgh^{-1}$.

(d) Let H be a subgroup of G and let G/H denote the set of left cosets of H. The set G/H is a G-set under the action $\lambda : G \to (G/H \to G/H)$ given by $\lambda_g(xH) = gxH$.

14.2 Compute all X_g and all G_x for each of the following permutation groups.

(a) $X = \{1, 2, 3\}$,
$G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$

(b) $X = \{1, 2, 3, 4, 5, 6\}$,
$G = \{(1), (12), (345), (354), (12)(345), (12)(354)\}$

14.3 Compute the G-equivalence classes of X for each of the G-sets in Exercise 14.2. For each $x \in X$ verify that $|G| = |\mathcal{O}_x| \cdot |G_x|$.

16.4bd Find all of the ideals in each of the following rings. Which of these ideals are maximal and which are prime?

(b) $\mathbb{Z}_{25}$

(d) $\mathbb{M}_2(\mathbb{Z})$, the 2×2 matrices with entries in $\mathbb{Z}$.

16.16 If R is a field, show that the only two ideals of R are $\{0\}$ and R itself.

Recommended Exercises (need not be turned in)

16.20 Prove the Second Isomorphism Theorem for rings: Let I be a subring of a ring R and J an ideal in R. Then $I \cap J$ is an ideal in I and $I/(I \cap J) \cong I + J/J$.

16.21 Prove the Third Isomorphism Theorem for rings: Let R be a ring and I and J be ideals of R, where $J \subsetneq I$. Then $(R/J)/(I/J) \cong I/J$.

16.22 Prove the Correspondence Theorem: Let I be an ideal of a ring R. Then $S \to S/I$ is a one-to-one correspondence between the set of subrings S containing I and the set of subrings of R/I. Furthermore, the ideals of R correspond to ideals of R/I.