MATH 3140 Homework 10

Exercises. 1 below plus Judson Ch 10. 1abe, 5, 10, 11, 13acd, 14.
Due date: Friday, 11/09

1. Let $\mathbf{G} = \langle G, \cdot, ^{-1}, e\rangle$ be a finite group of order n. Take the set G (the elements of $\mathbf{G}$) and consider the group of all permutations of these elements. This group is sometimes denoted by $\mathrm{Sym}(G)$; note that it is isomorphic to the symmetric group S_n of permutations of an n-element set. Now fix an element $a \in G$ and recall that the function $\lambda_a : G \to G$, defined by $\lambda_a(g) = a \cdot g$, is a permutation of the set G. That is, λ_a belongs to the permutation group $\mathrm{Sym}(G)$.

(a) Prove that the function $\lambda : G \to \mathrm{Sym}(G)$ is a group homomorphism.

(b) What is the kernel of λ?[1]

(c) Let N denote the equivalence class of $\ker\lambda$ that contains the identity element e of G. Prove that N is a normal subgroup of G.

10.1 For each of the following groups G, determine whether H is a normal subgroup of G. If H is a normal subgroup, write out a Cayley table for the factor group G/H.

(a) $G = S_4$ and $H = A_4$
(b) $G = A_5$ and $H = \{(1), (123), (132)\}$
(e) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

10.5. Show that the intersection of two normal subgroups is a normal subgroup.

10.10. Let H be a subgroup of index 2 of a group G. Prove that H must be a normal subgroup of G. Conclude that S_n is not simple for $n \geq 3$.

10.11. If a group G has exactly one subgroup H of order k, prove that H is normal in G.

10.13. Recall that the **center** of a group G is the set

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\ \}.$$

(a) Calculate the center of S_3.
(c) Show that the center of any group G is a normal subgroup of G.
(d) If $G/Z(G)$ is cyclic, show that G is abelian.[2]

[1]Recall that the kernel of a function $f : X \to Y$ is the subset of $X \times X$ defined by

$$\ker f = \{(x_1, x_2) : f(x_1) = f(x_2)\}.$$

As you have already proved, the kernel is an equivalence relation on X.

[2]Hint: Let $Z := Z(G)$. If G/Z is cyclic then there exists $x \in G$ such that for each $a \in G$ there exists $m \in \mathbb{N}$ such that $aZ = x^m Z$. Fix $a, b \in G$ and show $ab = ba$ using the fact that $aZ = x^m Z$ and $bZ = x^n Z$ for some m and n.

10.14. Let G be a group. The *commutator subgroup* of G, denoted by G', consists of all finite products of elements in G of the form $aba^{-1}b^{-1}$, as a and b range over elements of G. That is, $G' = \langle \{aba^{-1}b^{-1} \mid a, b \in G\} \rangle$.

(a) Show that G' is a normal subgroup of G.

(b) Let N be a normal subgroup of G. Prove that G/N is abelian if and only if N contains the commutator subgroup of G.