Probability
Final Exam, June 20, 2017

1. (30%) Let X and Y be two random variables that are jointly Gaussian with the following statistics:

$$E\{X\} = 1, \ E\{Y\} = 2, \ E\{X^2\} = 5, \ E\{Y^2\} = 13, \ E\{XY\} = 2.$$

   (a) (5%) Find the correlation coefficient of X and Y.

   (b) (5%) Are X and Y independent? Please explain your answer (0 point if the explanation is wrong).

   (c) (5%) Are X and Y i.i.d. (independent and identically distributed)? Please explain your answer (0 point if the explanation is wrong).

   (d) (5%) Find the joint pdf $f_{X,Y}(x,y)$.

   (e) (5%) Let $Z = X + 2Y$. Find the probability density function of Z.

   (f) (5%) Let $Z = X + 2Y$. Find the conditional pdf $f_{Z|X}(z|1)$.

2. (20%) Let X and Y be two random variables with the following joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} x + y & , \ 0 \le x \le 1, \ 0 \le y \le 1. \\ 0 & , \ \text{otherwise}. \end{cases}$$

   (a) (10%) Find the probability $P(Y > X^3)$.

   (b) (10%) Find the marginal pdf $f_X(x)$.

3. (30%) Let X and Y be two random variables with the following joint cumulative distribution function (CDF):

$$F_{X,Y}(x,y) = \frac{1}{1 + e^{-2x} + e^{-y}}, \quad -\infty < x < \infty, \ -\infty < y < \infty.$$

   (a) (10%) Is X and Y dependent or independent? Please show and explain your answer. (0 point if the explanation is wrong.)

   (b) (10%) Find the join pdf $f_{X,Y}(x,y)$.

   (c) (10%) Let $W = X + Y$ and $V = X^2 - Y^2$. Find the joint pdf $f_{W,V}(w,v)$.

4. (10%) In many engineering problems, the thermal noise is often modelled as a Gaussian random variable. Please explain why.

5. (10%) Let X and Y be two random variables. The probability mass function of the random variable X is given by

$$p_X(k) = \begin{cases} 1/2 & , \ k = 1 \\ 1/3 & , \ k = 2 \\ 1/6 & , \ k = 3 \\ 0 & , \ \text{otherwise}. \end{cases}$$

Suppose $E\{Y|X = 1\} = 1$, $E\{Y|X = 2\} = 0$, and $E\{Y|X = 3\} = -1$. Please find $E\{Y\}$.