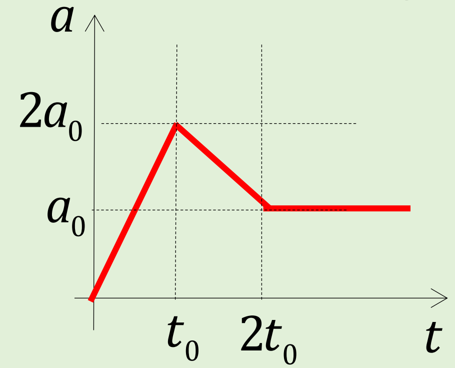
1. Consider a particle moving with the acceleration vs. time graph as in Fig. 1. $\boldsymbol{a_0}$ and $\boldsymbol{t_0}$ are constants with unit in $\boldsymbol{m/s^2}$ and $\boldsymbol{s}$. Assume the particle is at rest and at origin (原點) at $\boldsymbol{t = 0}$.

(a) Find the velocity $\boldsymbol{v(t)}$ for each interval (i.e. $\boldsymbol{0 < t < t_0, t_0 < t < 2t_0, 2t_0 < t}$

(b) Find the displacement $\boldsymbol{x(t)}$ of the particle for each interval.

(c) What are the velocity and displacement of the particle at $\boldsymbol{t = 3t_0}$?

Fig. 1



Sol:

For $0 < t < t_0 \,; \quad a = \dfrac{2a_0}{t_0} t \,; \quad v_0 = 0$

$$\int_{v_0}^{v} dv = v(t) - v_0 = \int_0^t a dt$$

$$v(t) - v_0 = \int_0^t \frac{2a_0}{t_0} t dt = \frac{a_0}{t_0} t^2 \to v(t) = \frac{a_0}{t_0} t^2$$

$$v(t) = \frac{a_0}{t_0} t^2 \,; \quad x(0) = 0$$

$$\int_{x_0}^{x} dx = \int_0^t v dt$$

$$x(t) - x_0 = \int_0^t \left(\frac{a_0}{t_0} t^2 \right) dt \to x(t) = \frac{a_0}{3t_0} t^3$$

For $t_0 < t < 2t_0 \,; \quad a = -\dfrac{a_0}{t_0} t + 3a_0 \,; \quad v_{t_0} = a_0 t_0$

$$\int_{v_0}^{v} dv = v(t) - v_{t_0} = \int_{t_0}^{t} a dt = \int_{t_0}^{t} (-\frac{a_0}{t_0} t + 3a_0) dt$$

$$v(t) - v_{t_0} = (-\frac{a_0}{2t_0})(t^2 - t_0{}^2) + 3a_0 (t - t_0)$$

$$\to v(t) = -\frac{a_0}{2t_0} t^2 + 3a_0 t - \frac{3}{2} a_0 t_0$$

$$v(t) = -\frac{a_0}{2t_0}t^2 + 3a_0 t - \frac{3}{2}a_0 t_0\,; \quad x_{t_0} = \frac{a_0 {t_0}^2}{3}$$

$$\int_{x_{t_0}}^{x} dx = x(t) - x_{t_0} = \int_{t_0}^{t} v dt = \int_{t_0}^{t}\left(-\frac{3}{2}a_0 t_0 + 3a_0 t - \frac{a_0}{2t_0}t^2\right)dt$$

$$x(t) - x_{t_0} = -\frac{3}{2}a_0 t_0 (t - t_0) + \frac{3}{2}a_0 (t^2 - {t_0}^2) - \frac{a_0}{6t_0}(t^3 - {t_0}^3)$$

$$\rightarrow x(t) = -\frac{3}{2}a_0 t_0 (t - t_0) + \frac{3}{2}a_0 (t^2 - {t_0}^2) - \frac{a_0}{6t_0}(t^3 - {t_0}^3) + \frac{a_0 {t_0}^2}{3}$$

For $2t_0 < t; \quad a = a_0\,; \quad v_{2t_0} = \frac{5}{2}a_0 t_0$

$$\int_{v_{2t_0}}^{v} dv = v(t) - v_{2t_0} = \int_{2t_0}^{t} a dt = a_0 (t - 2t_0)$$

$$\rightarrow v(t) = \frac{5}{2}a_0 t_0 + a_0 (t - 2t_0)$$

$$v(t) = \frac{5}{2}a_0 t_0 + a_0 (t - 2t_0); \quad x_{2t_0} = \frac{13}{6}a_0 {t_0}^2$$

$$\int_{x_{2t_0}}^{x} dx = x(t) - x_{2t_0} = \int_{2t_0}^{t} v dt = \int_{2t_0}^{t}\left(\frac{5}{2}a_0 t_0 + a_0 (t - 2t_0)\right)dt$$

$$x(t) - x_{2t_0} = \frac{1}{2}a_0 t_0 (t - 2t_0) + \frac{a_0}{2}(t^2 - 2{t_0}^2) \rightarrow x(t) = \frac{1}{2}a_0 t_0 (t - 2t_0) + \frac{a_0}{2}(t^2 - 2{t_0}^2) + \frac{13}{6}a_0 {t_0}^2$$

$$\rightarrow \quad v(3t_0) = \frac{7}{2}a_0 t_0\,; \quad x(3t_0) = 4a_0 {t_0}^2 \longrightarrow 4a_0{t_0}^2 + \frac{13}{6}a_0{t_0}^2 = \frac{37}{6}a_0{t_0}^2$$

2. Find the answer for the following.

$$(a) \quad \int_0^{\pi} \sin(3x - \frac{\pi}{2}) dx$$

$$u = 3x - \frac{\pi}{2}; \quad dx = \frac{du}{3}$$

$$\int_0^{\pi} \sin(3x - \frac{\pi}{2}) dx = \frac{1}{3} \int_{-\frac{1}{2}\pi}^{\frac{5}{2}\pi} \sin u du$$

$$= -\frac{1}{3}\left(\cos\frac{5}{2}\pi - \cos(-\frac{1}{2}\pi) \right) = 0$$

$$(b) \quad \int_{t_1}^{t_2} \frac{1}{2t-5} dt$$

$$= \frac{1}{2} \int_{t_1}^{t_2} \frac{1}{2t-5} d(2t) = \frac{1}{2} \ln|2t-5| \Big|_{t_1}^{t_2}$$

$$= \frac{1}{2} \ln\left| \frac{2t_2 - 5}{2t_1 - 5} \right|$$

$$(c) \quad \int_0^3 e^{-5x+2} dx$$

$$u = -5x + 2$$

$$dx = \frac{1}{-5} du$$

$$\frac{1}{-5} \int_2^{-13} e^u du$$

$$= \frac{1}{5}(e^2 - e^{-13})$$

$$(d) \quad \int_5^6 \frac{3}{\sqrt{2x-9}} dx$$

$$= 3\int_5^6 (2x-9)^{-\frac{1}{2}} dx$$

$$= \frac{3}{2} \frac{(2x-9)^{\frac{1}{2}}}{1/2} \Big|_{x=5}^{x=6}$$

$$= 3\sqrt{3} - 3$$

3. The acceleration of an object is given by ***a = 4 -2t² (m/s²)*** . The position of the object is ***2m*** at ***t = 1 sec*** , and the velocity of the object is ***0 m/s*** at ***t = 1 sec*** .

(a) Write expressions for the position and velocity of the object as functions of time.

(b) Find the position and velocity of the object at t = 4 sec.

Sol: (a) $a = 4 - 2t^2; \quad v_1 = 0; \quad x_1 = 2$

$$\int_{v_1}^{v} dv = v(t) - v_1 = \int_1^t \left(4 - 2t^2\right) dt = -\frac{2}{3}t^3 + 4t - \frac{10}{3}$$

$$\rightarrow v(t) = -\frac{2}{3}t^3 + 4t - \frac{10}{3}$$

$$\int_{x_1}^{x} dx = x(t) - x_1 = \int_1^t v dt = \int_1^t \left(-\frac{2}{3}t^3 + 4t - \frac{10}{3}\right) dt$$

$$x(t) - x_1 = -\frac{t^4}{6} + 2t^2 - \frac{10}{3}t + \frac{3}{2} \rightarrow x(t) = -\frac{t^4}{6} + 2t^2 - \frac{10}{3}t + \frac{7}{2}$$

(b) $$\begin{cases} v(t) = -\frac{2}{3}t^3 + 4t - \frac{10}{3} \\ x(t) = -\frac{t^4}{6} + 2t^2 - \frac{10}{3}t + \frac{7}{2} \end{cases}$$

$$t_2 = 4\sec \rightarrow \begin{cases} v(t_2) = -30\,(m/s) \\ x(t_2) = -\frac{41}{2}\,(m) \end{cases}$$