Differential Equations **Name:** ______________________

Exam III, Fall 2019 **ID:** ______________________

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1. (10%) **Laplace Transform and Convolution**
   Use the *Laplace transform* to solve the integral equation

$$f(t) = \cos t + \int_0^t e^{-\tau} f(t-\tau)d\tau$$

Solution: $f(t) = \cos t + \sin t$

2. (10%) **Dirac Delta Function**
   Use the Laplace transform to solve the initial-value problem

$$y'' + 2y' + y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0$$

Solution: $y = (t-1)e^{-(t-1)}\mathscr{U}(t-1)$

3. (10%) **Repeated Eigenvalues**
   Solve the initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\mathbf{X}, \quad \mathbf{X}(1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Solution: $\mathbf{X}(t) = \begin{pmatrix} -4te^{3(t-1)} + 5e^{3(t-1)} \\ 4te^{3(t-1)} - e^{3(t-1)} \end{pmatrix}$

4. (10%) **Variation of Parameters**
   Solve the initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\mathbf{X} - \begin{pmatrix} 15 \\ 4 \end{pmatrix}te^{-2t}, \quad \mathbf{X}(0) = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

using *variation of parameters*

Solution: $\mathbf{X}(t) = \frac{1}{14}\begin{pmatrix} (6+28t-7t^2)e^{-2t}+92e^{5t} \\ (-4+14t+21t^2)e^{-2t}+46e^{5t} \end{pmatrix}$

5. (10%) **Matrix Exponential**
   Use the Laplace transform to compute $e^{\mathtt{A}t}$ for

$$\mathtt{A} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$$

Solution: $\frac{1}{7}\begin{pmatrix} e^{-2t}+6e^{5t} & -2e^{-2t}+2e^{5t} \\ -3e^{-2t}+3e^{5t} & 6e^{-2t}+e^{5t} \end{pmatrix}$

6. (10%) **Improved Euler's Method**
   Use the improved Euler's method to obtain a two-decimal approximation of the indicated value with the interval $h = 0.1$.

$$y' = 4x - 2y, \quad y(0) = 2; \quad y(0.2)$$

Solution: $y(0.2) = 1.41$ (or 1.42)

7. (10%) **Orthogonal Functions**
   Show the given functions are orthogonal

$$f_1(x) = e^x, f_2(x) = xe^{-x} - e^{-x}$$

   on the interval $[0, 2]$.

Solution: The students should verify $\int_0^2 e^x(xe^{-x} - e^{-x})dx = 0$

8. (10%) **Fourier Series Expansion**
   Expand the function

$$f(x) = \begin{cases} 1, & -1 \le x < 0 \\ x, & 0 \le x < 1 \end{cases}$$

   in a Fourier series.

Solution: $\frac{3}{4}+\sum_{n=1}^{\infty}\frac{-1+(-1)^n}{n^2\pi^2}\cos n\pi x-\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin n\pi x$

9. (10%) **Sturm-Liouville Problem**
   Find the eigenvalues and and associated eigenfunctions of the Sturm-Liouville problem

$$\begin{array}{ll} y''+\lambda y=0, & (0<x<L) \\ y'(0)=0, & y(L)=0 \end{array}$$

Solution: Eigenvalues: $\lambda_n=\frac{(2n-1)^2\pi^2}{4L^2}$, eigenfunctions: $y_n(x)=\cos\frac{(2n-1)\pi x}{2L}$ for $n=1,2,3,\cdots$

10. (10%) **Partial Differential Equation**
    Use separation of variables to solve the partial differential equation

$$y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=0$$

Solution: $u(x,y)=ce^{\lambda(x^2-y^2)/2}$

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