Probability (Spring 2021) - Final

1. [20% = 2%*10] For each question, exactly one of the multiple-chooses is correct.
   (1) Which kind of random variables satisfy memoryless property?
   A. Exponential, B. Gaussian, C. Gamma, D. Beta, E. Cauchy.
   (2) Which function of random variables can be viewed as Laplace transform (with a reverse in the sign of the exponent) of the pdf?
   A. Probability Generating Function, B. Moment Generating Function, C. Fourier Function, D. Characteristic Function, E. Q-Function.
   (3) Which kind of random variables does not have expected (mean) value?
   A. Exponential, B. Gaussian, C. Gamma, D. Beta, E. Cauchy.
   (4) Which function of random variables can be viewed as Fourier transform (with a reverse in the sign of the exponent) of the pdf?
   A. Probability Generating Function, B. Moment Generating Function, C. Laplace Function, D. Characteristic Function, E. Q-Function.
   (5) Which inequality requires both the knowledge of mean value and variance for the random variable?
   A. Markov, B. Chebyshev, C. Chernoff, D. Hoeffding.
   (6) Which function of random variables is useful when the random variable is non-negative and integer-valued?
   A. Probability Generating Function, B. Moment Generating Function, C. Laplace Function, D. Characteristic Function.
   (7) The central limit theorem (CLT) states that, under certain conditions, the sum of a large number of random variables is approximately which distribution?
   A. Exponential, B. Gaussian, C. Gamma, D. Beta, E. Cauchy
   (8) Which function of random variables exists even if the expected value of the random variable does not exist?
   A. Probability Generating Function, B. Moment Generating Function, C. Laplace Function, D. Characteristic Function.
   (9) Q-function *Q(x)* is the probability of the "tail" of the pdf of which random variable's standard version?
   A. Exponential, B. Gaussian, C. Gamma, D. Beta, E. Cauchy.
   (10) Which inequality requires only the knowledge of mean value for the random variable?
   A. Markov, B. Chebyshev, C. Chernoff, D. Hoeffding.

(NOT the End, Turn Over to Next Page)

- **除題目有特別標注外，最後的純數學計算可以不必算出，但不能省略已知函數，如 random variables 的 pmf, pdf, CDF 等必須代入。**
- **最後一頁有合法小抄，作答時可以參考使用。**

2. [3%-3%-4%] 一家疫苗注射中心某天早上開門第一小時注射疫苗的人數為 X ，第二小時注射疫苗的人數為 Y 。這兩小時的注射人數為 a pair of random variables (X,Y) 。假設以分鐘為間距單位，在每個間距內有一人注射的機率為 $p = 0.05$ ，沒有人注射的機率為 $1-p$。已知不同間距內的注射人數相互獨立，試求：
   (1) The joint pmf of X and Y.
   (2) The marginal pmf for X and for Y.
   (3) The probability of the event $A = \{X+Y=15\}$.

3. [3%-3%-4%] A random variable X has pdf:

   $f_X(x) = cx(1-x^4)$ *for* $1 <= x <= 1$; 0 for elsewhere.

   FYI: $\boxed{\int x(1-x^4)dx = x - x^5/5}$
   (a) Find c. (**答案必須給出最後數值**)
   (b) Find the CDF of X.
   (c) Find P[|X| < 1]. (**答案必須給出最後數值**)

4. [5%-5%] 小柯與小傑為自行車運動健將。他們每週日都會進行一日雙塔，假設他們零時從富貴角燈塔出發，小柯抵達鵝鑾鼻燈塔的時間為 uniformly distributed in the interval [23:25,23:35]，而小傑抵達鵝鑾鼻的時間為 uniformly distributed in the interval [23:20, 23:40]。假設兩者抵達鵝鑾鼻的時間互相獨立，試求：
   (a) 小柯比小傑早五分鐘抵達的機率。
   (b) 小傑比小柯晚抵達的機率。

(<u>NOT the End, Turn Over to Next Page</u>)

5. [5%-5%] 比較 Markov inequality 與事件 $\{X > c\}$ 的以 c 為函數之實際機率。
   (a) 當 X 為 uniform random variable in the interval $[0,b]$。
   (b) 當 X 為 exponential random variable with parameter λ。

6. [5%-5%] 一家診所有提供疫苗接種服務，客人出現的速率是每分鐘一人，由於瓶裝疫苗開瓶後必須在短時間內用完，故診所必須等到人數湊滿五人之後才會開一瓶進行接種。假設客人的 interarrival times 為 exponential random variables，並令 X 為湊滿五人之等待時間。
   (a) 如果兩個連續發生的事件之間的時間為 exponential 分布，並且它與以前發生的事件無關，那麼該事件的總發生次數可以是什麼分布？
   (b) 試求湊滿五人的等待時間超過 15 分鐘之機率。

7. [4%-3%-3%] 一台數據機輸入+2 電位之訊號到頻道中。此頻道有雜訊干擾；雜訊從集合 {0,-1,-2,-3} （對應機率為 {5/10,2/10,1/10,2/10}）中隨機產生。
   (a) 試求頻道最後輸出訊號的 PMF。
   (b) 頻道最後輸出訊號等於輸入訊號的機率? **(答案必須給出最後數值)**
   (c) 頻道最後輸出訊號為正電位的機率? **(答案必須給出最後數值)**

8. [10%] 已知 X 為 Poisson random variable with parameter $\alpha=1$，比較 Chernoff bound 與 $P[X >= 10]$ 的實際數值。

9. [10%] 一家醫院的冰箱內裝有 AZ 疫苗 40 瓶與 Moderna 疫苗 60 瓶。今天我們從冰箱中隨機拿 10 瓶出來使用。令 X,Y 分別為挑選出 AZ 疫苗與 Moderna 疫苗的瓶數。試求 X,Y 的 joint PMF。

(<u>End of Question Sheet, Next Page is Cheat Sheet</u>)

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For Your Information

Bernoulli Random Variable

$S_X = \{0, 1\}$

$p_0 = q = 1 - p \qquad p_1 = p \qquad 0 \le p \le 1$

$E[X] = p \quad \text{VAR}[X] = p(1-p) \qquad G_X(z) = (q + pz)$

Remarks: The Bernoulli random variable is the value of the indicator function I_A for some event A; $X = 1$ if A occurs and 0 otherwise.

Binomial Random Variable

$S_X = \{0, 1, \ldots, n\}$

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n$$

$E[X] = np \quad \text{VAR}[X] = np(1-p) \qquad G_X(z) = (q + pz)^n$

Remarks: X is the number of successes in n Bernoulli trials and hence the sum of n independent, identically distributed Bernoulli random variables.

Geometric Random Variable

First Version: $S_X = \{0, 1, 2, \ldots\}$

$p_k = p(1-p)^k \qquad k = 0, 1, \ldots$

$$E[X] = \frac{1-p}{p} \qquad \text{VAR}[X] = \frac{1-p}{p^2} \qquad G_X(z) = \frac{p}{1-qz}$$

Remarks: X is the number of failures before the first success in a sequence of independent Bernoulli trials. The geometric random variable is the only discrete random variable with the memoryless property.

Second Version: $S_{X'} = \{1, 2, \ldots\}$

$p_k = p(1-p)^{k-1} \qquad k = 1, 2, \ldots$

$$E[X'] = \frac{1}{p} \qquad \text{VAR}[X'] = \frac{1-p}{p^2} \qquad G_{X'}(z) = \frac{pz}{1-qz}$$

Remarks: $X' = X + 1$ is the number of trials until the first success in a sequence of independent Bernoulli trials.

Poisson Random Variable

$S_X = \{0, 1, 2, \ldots\}$

$$p_k = \frac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, \ldots \quad \text{and } \alpha > 0$$

$E[X] = \alpha \qquad \text{VAR}[X] = \alpha \qquad G_X(z) = e^{\alpha(z-1)}$

Remarks: X is the number of events that occur in one time unit when the time between events is exponentially distributed with mean $1/\alpha$.

Uniform Random Variable

$S_X = \{1, 2, \ldots, L\}$

$$p_k = \frac{1}{L} \qquad k = 1, 2, \ldots, L$$

$$E[X] = \frac{L+1}{2} \quad \text{VAR}[X] = \frac{L^2 - 1}{12} \qquad G_X(z) = \frac{z}{L}\frac{1 - z^L}{1 - z}$$

Remarks: The uniform random variable occurs whenever outcomes are equally likely. It plays a key role in the generation of random numbers.

The exponential random variable X with parameter λ has pdf

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \ge 0 \end{cases}$$

and CDF

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

The Markov inequality:

– If X is non-negative, then $\boxed{P[X \ge a] \le \frac{\mathrm{E}\{X\}}{a}}$

Chernoff bound:

$$\boxed{P[X \ge a] \le \min_{s>0} e^{-as}\mathrm{E}\{e^{sX}\}}$$

The Chebyshev inequality:

$$\boxed{P\left[|X - m| \ge a\right] \le \frac{\sigma^2}{a^2}}$$