Signals and Systems – Quiz 3 05/11/2020

Name: Student ID:

1. (30%) Consider a causal discrete-time LTI system whose input $x[n]$ and output $y[n]$ are related by the following difference equation:

$$y[n]-\frac{1}{3}y[n-1]=x[n]$$

(a) Determine the frequency response of this system.

(b) Find the Fourier series representation of the output $y[n]$ when the input

$$x[n]=sin\left(\frac{\pi}{4}n\right)+2\cos(\frac{\pi}{2}n)$$

Solution:

(a) Consider an input $x[n]$ of the form $e^{j\omega n}$. $y[n]=H\left(e^{j\omega}\right)e^{j\omega n}$

$$H\left(e^{j\omega}\right)e^{j\omega n}-\tfrac{1}{3}e^{-j\omega}e^{j\omega n}H\left(e^{j\omega}\right)=e^{j\omega n}.$$

Therefore,

$$H\left(e^{j\omega}\right)=\frac{1}{1-\frac{1}{3}e^{-j\omega}}.$$

From eq.(3.131), we know that

$$y[n]=\sum_{k=<N>}a_kH(e^{j2\pi k/N})\,e^{jk\left(\frac{2\pi}{N}\right)n}$$

When the input is $x[n]$ and $x[n]$ has the Fourier series coefficients a_k and fundamental frequency $2\pi/N$. The Fourier series coefficients of $y[n]$ are $a_kH(e^{\frac{j2\pi k}{N}})$.

Here, N = 8 and the nonzero FS coefficients of $x[n]$ are $a_1=a_{-1}^*=1/2j, a_2=a_{-2}=1$.

Therefore, the nonzero FS coefficients of $y[n]$ are

$$b_1=a_1H\left(e^{\frac{j\pi}{4}}\right)=\frac{1}{2j\left(1-\left(\frac{1}{3}\right)e^{\frac{-j\pi}{4}}\right)},\qquad b_{-1}=a_{-1}H\left(e^{-\frac{j\pi}{4}}\right)=\frac{-1}{2j\left(1-\left(\frac{1}{3}\right)e^{\frac{j\pi}{4}}\right)}$$

$$b_2=a_2H\left(e^{\frac{j\pi}{2}}\right)=\frac{1}{\left(1-\left(\frac{1}{3}\right)e^{\frac{-j\pi}{2}}\right)},\qquad b_{-2}=a_{-2}H\left(e^{-\frac{j\pi}{2}}\right)=\frac{1}{\left(1-\left(\frac{1}{3}\right)e^{\frac{j\pi}{2}}\right)},$$

2. (30%) Suppose that we are given the following information about a signal $x(t)$:

$a. x(t)$ is real and odd.

$b. x(t)$ is periodic with period T=2 and has Fourier coefficients a_k

c. $a_k = 0$ for $|k| > 1$

$d. \frac{1}{2}\int_0^2 |x(t)|^2 dt = 1$

Specify two different signals that satisfy these conditions.

Solution:

From clue a, we know $x(t)$ is real and odd.

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}}$$

$x(t)$ is real => $x(t) = x^*(t)$, $a_k = a_{-k}^*$

$x(t)$ is odd => $x(t) = -x(-t)$, $a_k = -a_{-k}$

It indicates that its Fourier series coefficients a_k are purely imaginary and odd. Therefore

$a_k = -a_{-k}$ and $a_0 = 0$.

Then from clue b, we know the only unknown Fourier series coefficients are a_1 and a_{-1}

Using Parseval's relation,

$$\frac{1}{2}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$=> |a_1|^2 + |a_{-1}|^2 = 1, => 2|a_1|^2 = 1$$

Therefore

$$a_1 = -a_{-1} = \frac{1}{\sqrt{2}} j, \text{or } a_1 = -a_{-1} = \frac{-1}{\sqrt{2}} j$$

The two possible signals that satisfy the given information are

$$x_1(t) = \frac{1}{\sqrt{2}} j e^{j\left(\frac{2\pi}{2}\right)t} - \frac{1}{\sqrt{2}} j e^{-j\left(\frac{2\pi}{2}\right)t} = -\sqrt{2}\sin(\pi t)$$

$$and\ x_2(t) = \frac{-1}{\sqrt{2}} j e^{j\left(\frac{2\pi}{2}\right)t} + \frac{1}{\sqrt{2}} j e^{-j\left(\frac{2\pi}{2}\right)t} = \sqrt{2}\sin(\pi t)$$

3. (40%) Figure 1 depicts a first-order RC circuit. If we take the voltage across the capacitor $v_c(t)$ as the output.
   (a) Determine the linear constant-coefficients differential equation of this system.
   (b) Determine the frequency response of this system
   (c) Determine and plot the step response of this system.

   (hint: $H(j\omega) = \frac{1}{1+aj\omega}$ => $h(t) = \frac{1}{a}e^{-\frac{t}{a}}u(t)$)

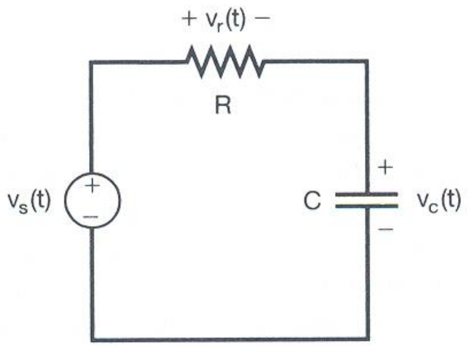


Figure 1.

Solution:

(a) Constant-coefficients differential equation

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

Let input voltage $v_s(t) = e^{j\omega t}$ output voltage $v_c(t) = H(j\omega)e^{j\omega t}$

$$\Rightarrow \quad RC\frac{d}{dt}\left[H(j\omega)e^{j\omega t}\right] + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

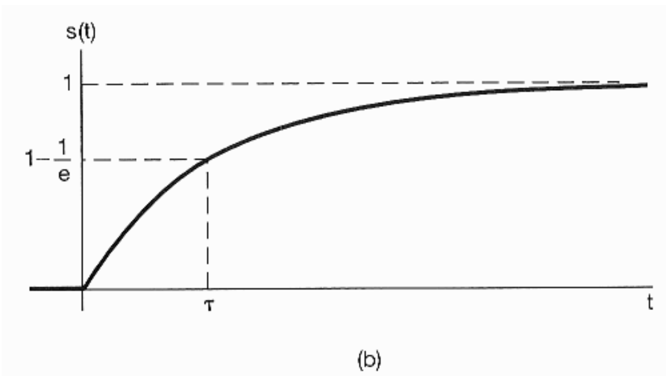
$$RCj\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

(b) Frequency response $H(j\omega)$

$$H(j\omega) = \frac{1}{1 + RCj\omega}$$

(c) $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$,

$$=> s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = \frac{1}{RC}(-RC)e^{-\frac{\tau}{RC}}\Big|_0^t = [1 - e^{-\frac{t}{RC}}]u(t)$$



$\tau = RC$