Chapter 2
Signal and Linear System Analysis

Outline

- Signal Models
- Signal Classifications
- Fourier Series
- The Fourier Transform
- Power Spectral Density and Correlation
- Signals and Linear Systems
- Sampling Theory
- The Hilbert Transform

Signal Models

- Deterministic and Random Signals
- Periodic and Aperiodic Signals
- Phasor Signals and Spectra
- Singularity Functions

Deterministic and Random Signals

- In this course we are concerned with two broad classes of signals, referred to as deterministic and random.
- Deterministic signals can be modeled as completely specified functions of time.

Deterministic Signals (1/2)

- For example, the signal

$$x(t) = A\cos(\omega_0 t), -\infty < t < \infty \qquad (2.1)$$

where A and ω_0 are constants, is a familiar example of a deterministic signals.

Deterministic Signals (2/2)

- Another example of a deterministic signal is the unit rectangular pulse, denoted as $\Pi(t)$ and defined as

$$\Pi(t) = \begin{cases} 1, & |t| \leq \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Random Signals

- Random signals are signals that take on random values at any given time instant and must be modeled probabilistically.
- They will be considered in Chapters 6 and 7.
- Figure 2.1 illustrates the various types of signals just discussed.

Examples of Various Types of Signals

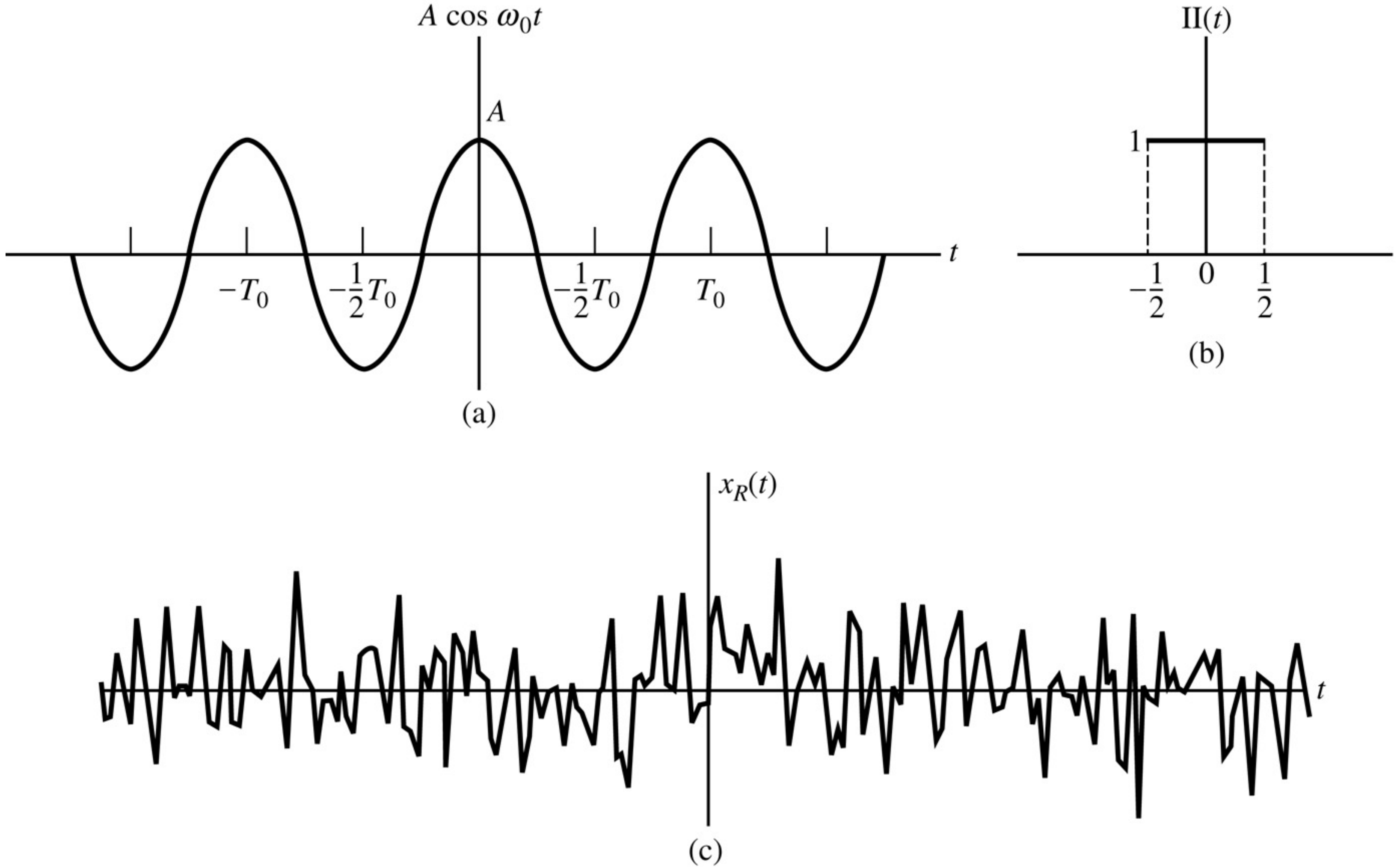


Figure 2.1 Examples of various types of signals. (a) Deterministic (sinusoidal) signal. (b) Unit rectangular pulse signal. (c) Random signal.

Periodic and Aperiodic Signals (1/2)

- The signal defined by (2.1) is an example of a periodic signal.
- A signal $x(t)$ is periodic if and only if

$$x(t + T_0) = x(t), -\infty < t < \infty \qquad (2.3)$$

where the constant T_0 is the period.

- The smallest such number satisfying (2.3) is referred to as the fundamental period (the modifier fundamental is often excluded).

Periodic and Aperiodic Signals (2/2)

- Any signal not satisfying (2.3) is called aperiodic.

Phasor Signals and Spectra (1/4)

- A useful periodic signal in system analysis is the signal

$$\tilde{x}(t) = Ae^{j(\omega_0 t+\theta)}, -\infty < t < \infty \qquad (2.4)$$

which is characterized by three parameters:

  – Amplitude A,

  – Phase θ in radians,

  – Frequency ω_0 in radians per second or $f_0 = \omega_0/2\pi$ Hz.

Phasor Signals and Spectra (2/4)

- We will refer to $\tilde{x}(t)$ as a rotating phasor to distinguish it from the phasor $Ae^{j\theta}$, for which $e^{j\omega_0 t}$ is implicit.
- Using Euler's theorem, we may readily show that $\tilde{x}(t) = \tilde{x}(t + T_0)$, where $T_0 = 2\pi/\omega_0$.
  - Recall that Euler's theorem is $e^{\pm ju} = \cos u \pm j \sin u$.
  - Also recall that $e^{j2\pi} = \cos 2\pi + j \sin 2\pi = 1 + 0 = 1$.
- Thus $\tilde{x}(t)$ is a periodic signal with period $2\pi/\omega_0$.

Phasor Signals and Spectra (3/4)

- The rotating phasor $Ae^{j(\omega_0 t+\theta)}$ can be related to a real, sinusoidal signal $A\cos(\omega_0 t+\theta)$ in two ways.
- The first is by taking its real part,

$$
\begin{aligned}
x(t) = A\cos(\omega_0 t+\theta) &= \mathrm{Re}\big(\tilde{x}(t)\big) \\
&= \mathrm{Re}\big(Ae^{j(\omega_0 t+\theta)}\big)
\end{aligned}
\qquad (2.5)
$$

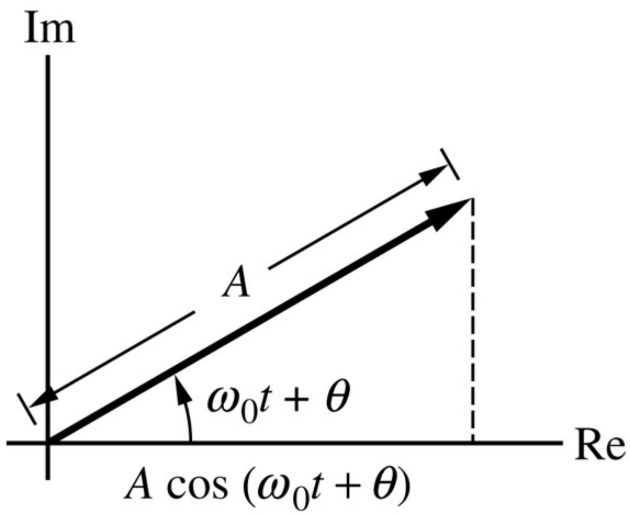
Phasor Signals and Spectra (4/4)

- The second is by taking one-half of the sum of $\tilde{x}(t)$ and its complex conjugate,

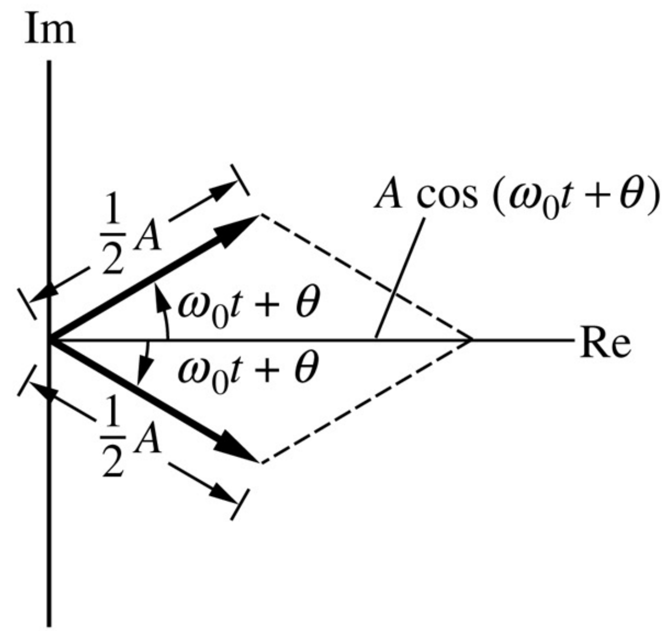
$$\begin{aligned} &A\cos(\omega_0 t + \theta) \\ &= \frac{1}{2}\tilde{x}(t) + \frac{1}{2}\tilde{x}^*(t) \\ &= \frac{1}{2}Ae^{j(\omega_0 t+\theta)} + \frac{1}{2}Ae^{-j(\omega_0 t+\theta)} \end{aligned} \qquad (2.6)$$

- Figure 2.2 illustrates these two procedures graphically.

Two Ways of Relating a Phasor Signal to a Sinusoidal Signal



(a)



(b)

Figure 2.2 Two ways of relating a phasor signal to a sinusoidal signal. (a) Projection of a rotating phasor onto the real axis. (b) Addition of complex conjugate rotating phasors.

Amplitude and Phase Spectra for the Signal $A\cos(\omega_0 t + \theta)$

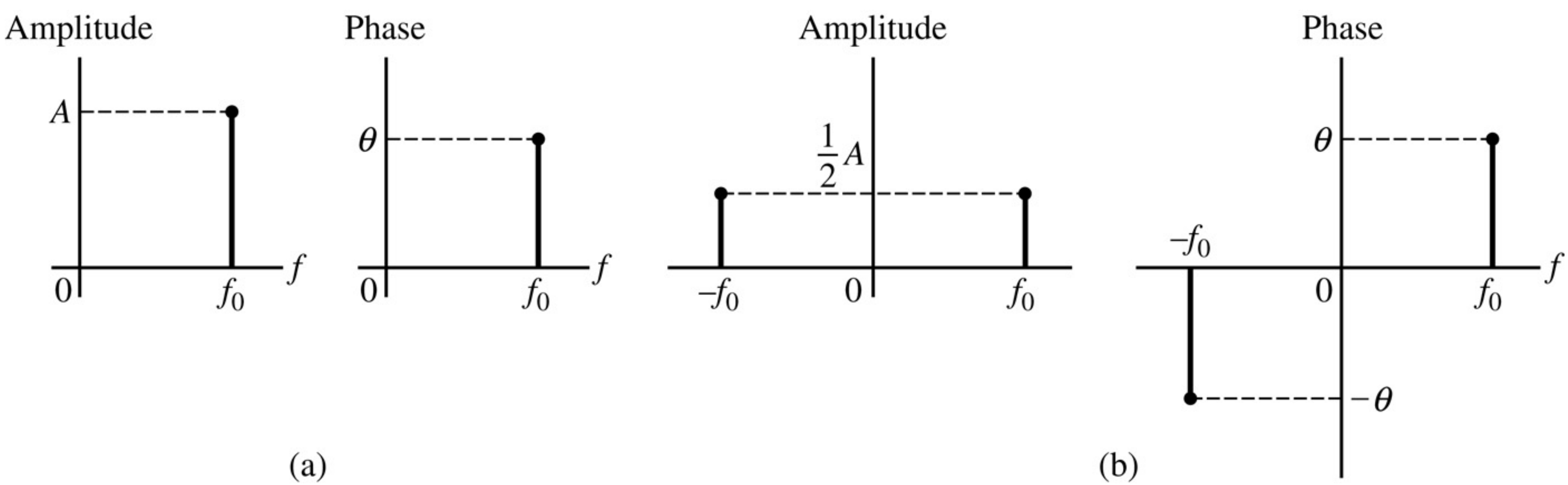


Figure 2.3 Amplitude and phase spectra for the signal $A\cos(\omega_0 t + \theta)$. (a) Single sided. (b) Double sided.

Example 2.1 (1/3)

- (a) To sketch the single-sided and double-sided spectra of

$$x(t) = 2\sin\left(10\pi t - \frac{1}{6}\pi\right) \qquad (2.7)$$

we note that $x(t)$ can be written as

$$x(t) = 2\cos\left(10\pi t - \frac{1}{6}\pi - \frac{1}{2}\pi\right)$$

$$= 2\cos\left(10\pi t - \frac{2}{3}\pi\right)$$

Example 2.1 (2/3)

$$= \mathrm{Re}\left(2e^{j(10\pi t - 2\pi/3)}\right)$$
$$= e^{j(10\pi t - 2\pi/3)} + e^{-j(10\pi t - 2\pi/3)} \qquad (2.8)$$

- Thus the single-sided and double-sided spectra are as shown in Figure 2.3, with $A = 2$, $\theta = -\frac{2}{3}\pi$ rad, and $f_0 = 5$ Hz.
- (b) If more than one sinusoidal component is present in a signal, its spectra consist of multiple lines.

Example 2.1 (3/3)

$$y(t) = 2\sin\left(10\pi t - \frac{1}{6}\pi\right) + \cos(20\pi t) \quad (2.9)$$

can be rewritten as

$$y(t) = 2\cos\left(10\pi t - \frac{2}{3}\pi\right) + \cos(20\pi t)$$

$$= \mathrm{Re}\left(2e^{j(10\pi t - 2\pi/3)} + e^{j20\pi t}\right) \quad (2.10)$$

$$= e^{j(10\pi t - 2\pi/3)} + e^{-j(10\pi t - 2\pi/3)} + (e^{j20\pi t} + e^{-j20\pi t})/2$$

Singularity Functions

- An important subclass of aperiodic signals is the singularity functions.
- In this course we will be concerned with only two:
  - The unit impulse function $\delta(t)$ (or delta function).
  - The unit step function $u(t)$.

Unit Impulse Function (1/9)

- The unit impulse function is defined in terms of the integral

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0) \qquad (2.11)$$

where $x(t)$ is any test function that is continuous at $t = 0$.

Unit Impulse Function (2/9)

- A change of variables and redefinition of $x(t)$ results in the sifting property

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0) \qquad (2.12)$$

where $x(t)$ is continuous at $t = t_0$.

- We will make considerable use of the sifting property in systems analysis.

Unit Impulse Function (3/9)

- By considering the special case $x(t) = 1$ for $t_1 \leq t \leq t_2$ and $x(t) = 0$ for $t < t_1$ or $t > t_2$, the two properties
$$\int_{t_1}^{t_2} \delta(t - t_0)dt = 1,\ t_1 \leq t_0 \leq t_2 \quad (2.13)$$
and
$$\delta(t - t_0) = 0, \qquad t \neq t_0 \qquad (2.14)$$
are obtained that provide an alternative definition of the unit impulse.

Unit Impulse Function (4/9)

- Other properties of the unit impulse function that can be proved from the definition (2.11) are the following:

  1. $\delta(at) = (\frac{1}{|a|})\delta(t)$, a is a constant.
  2. $\delta(-t) = \delta(t)$.
  3. A generalization of the sifting property,

$$\int_{t_1}^{t_2} x(t)\delta(t - t_0)dt = \begin{cases} x(t_0), & t_1 < t_0 < t_2 \\ 0, & \text{otherwise} \\ \text{undefined}, & t_0 = t_1 \text{ or } t_2 \end{cases}$$

Unit Impulse Function (5/9)

4. $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$, where $x(t)$ is continuous at $t = t_0$.

5. $\int_{t_1}^{t_2} x(t)\delta^{(n)}(t-t_0)dt = (-1)^n x^{(n)}(t_0)$, $t_1 < t_0 < t_2$. [In this equation, the superscript n denotes the nth derivative; $x(t)$ and its first n derivatives are assumed continuous at $t = t_0$.]

6. If $f(t) = g(t)$, where $f(t) = a_0\delta(t) + a_1\delta^{(1)}(t) + \cdots + a_n\delta^{(n)}(t)$ and $g(t) = b_0\delta(t) + b_1\delta^{(1)}(t) + \cdots + b_n\delta^{(n)}(t)$, this implies that $a_0 = b_0$, $a_1 = b_1$, …, $a_n = b_n$.

Unit Impulse Function (6/9)

- It is reassuring to note that (2.13) and (2.14) correspond to the intuitive notion of a unit impulse function as the limit of a suitably chosen conventional function having unity area in an infinitesimally small width.

Unit Impulse Function (7/9)

- An example is the signal

$$\delta_\epsilon(t) = \frac{1}{2\epsilon}\Pi\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & \text{otherwise} \end{cases} \qquad (2.15)$$

which is shown in Figure 2.4(a) for $\epsilon = 1/4$ and $\epsilon = 1/2$.

Unit Impulse Function (8/9)

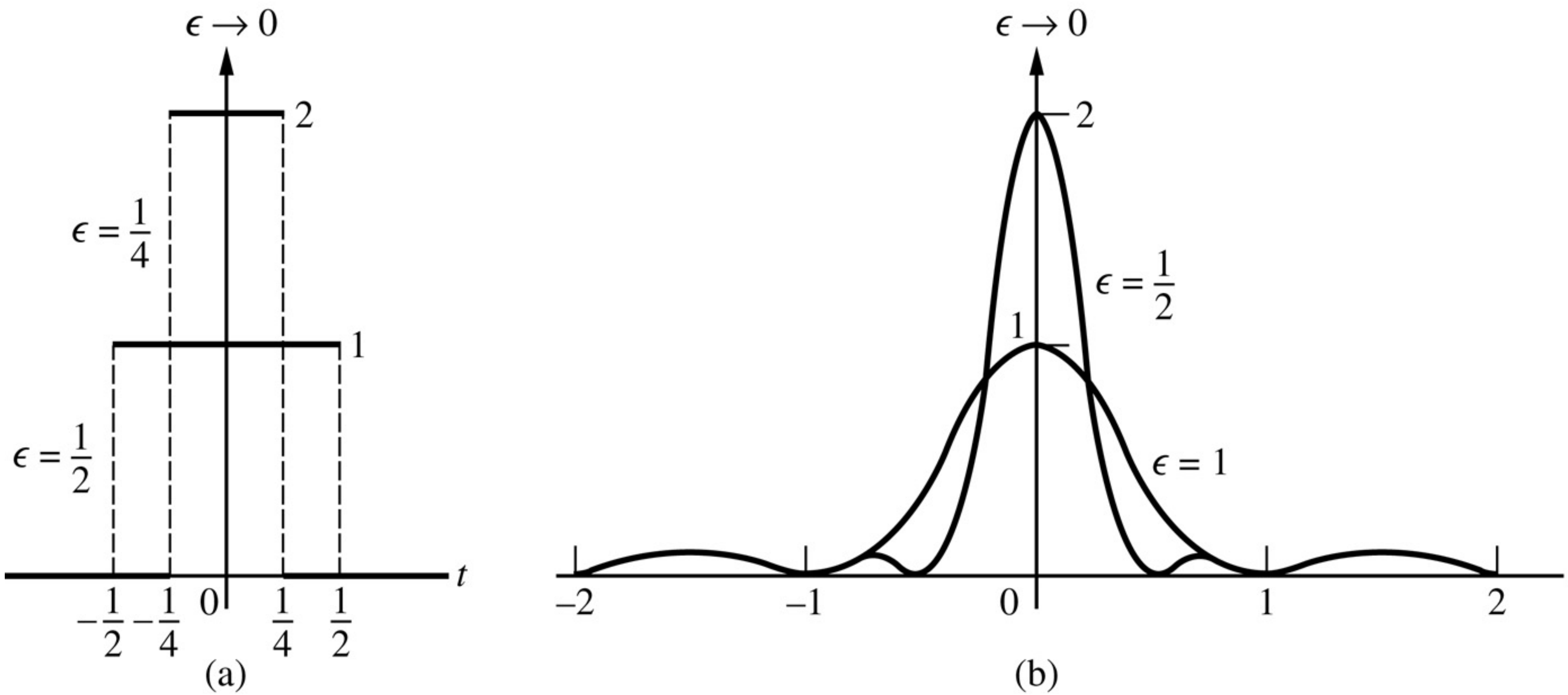


Figure 2.4 Two representations for the unit impulse function in the limit as $\epsilon \to 0$. (a) $\left(\frac{1}{2\epsilon}\right) \Pi(\frac{t}{2\epsilon})$. (b) $\epsilon \left[\left(\frac{1}{\pi t}\right) \sin\left(\frac{\pi t}{\epsilon}\right)\right]^2$.

Unit Impulse Function (9/9)

- It seems apparent that any signal having unity area and zero width in the limit as some parameter approaches zero is a suitable representation for $\delta(t)$, for example, the signal

$$\delta_{1\epsilon}(t) = \epsilon\left(\frac{1}{\pi t}\sin\frac{\pi t}{\epsilon}\right)^2 \qquad (2.16)$$

which is sketched in Figure 2.4(b).

Unit Step Function (1/3)

- Other singularity functions may be defined as integrals or derivatives of unit impulses.
- We will need only the unit step $u(t)$, defined to be the integral of the unit impulse.

Unit Step Function (2/3)

- Thus

$$u(t) \triangleq \int_{-\infty}^{t} \delta(\lambda) d\lambda = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \\ \text{undefined}, & t = 0 \end{cases} \tag{2.17}$$

or

$$\delta(t) = \frac{du(t)}{dt} \tag{2.18}$$

Unit Step Function (3/3)

- For consistency with the unit pulse function definition, we will define $u(0) = 1$.
- For example, the unit rectangular pulse function defined by (2.2) can be written in terms of unit steps as

$$\Pi(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right) \quad (2.19)$$

Signal Classifications (1/5)

- In this chapter we will be considering two signal classes, those with finite energy an those with finite power.
- As a specific example, suppose $e(t)$ is the voltage across a resistance R producing a current $i(t)$.
- The instantaneous power per ohm is $p(t) = e(t)i(t)/R = i^2(t)$.

Signal Classifications (2/5)

- Integrating over the interval $|t| \leq T$, the total energy and the average power on a per-ohm basis are obtained as the limits

$$E = \lim_{T \to \infty} \int_{-T}^{T} i^2(t)dt \qquad (2.20)$$

and

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)dt \qquad (2.21)$$

respectively.

Signal Classifications (3/5)

- For an arbitrary signal $x(t)$, which may, in general, be complex, we define total (normalized) energy as

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (2.22)$$

and (normalized) power as

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad (2.23)$$

Signal Classifications (4/5)

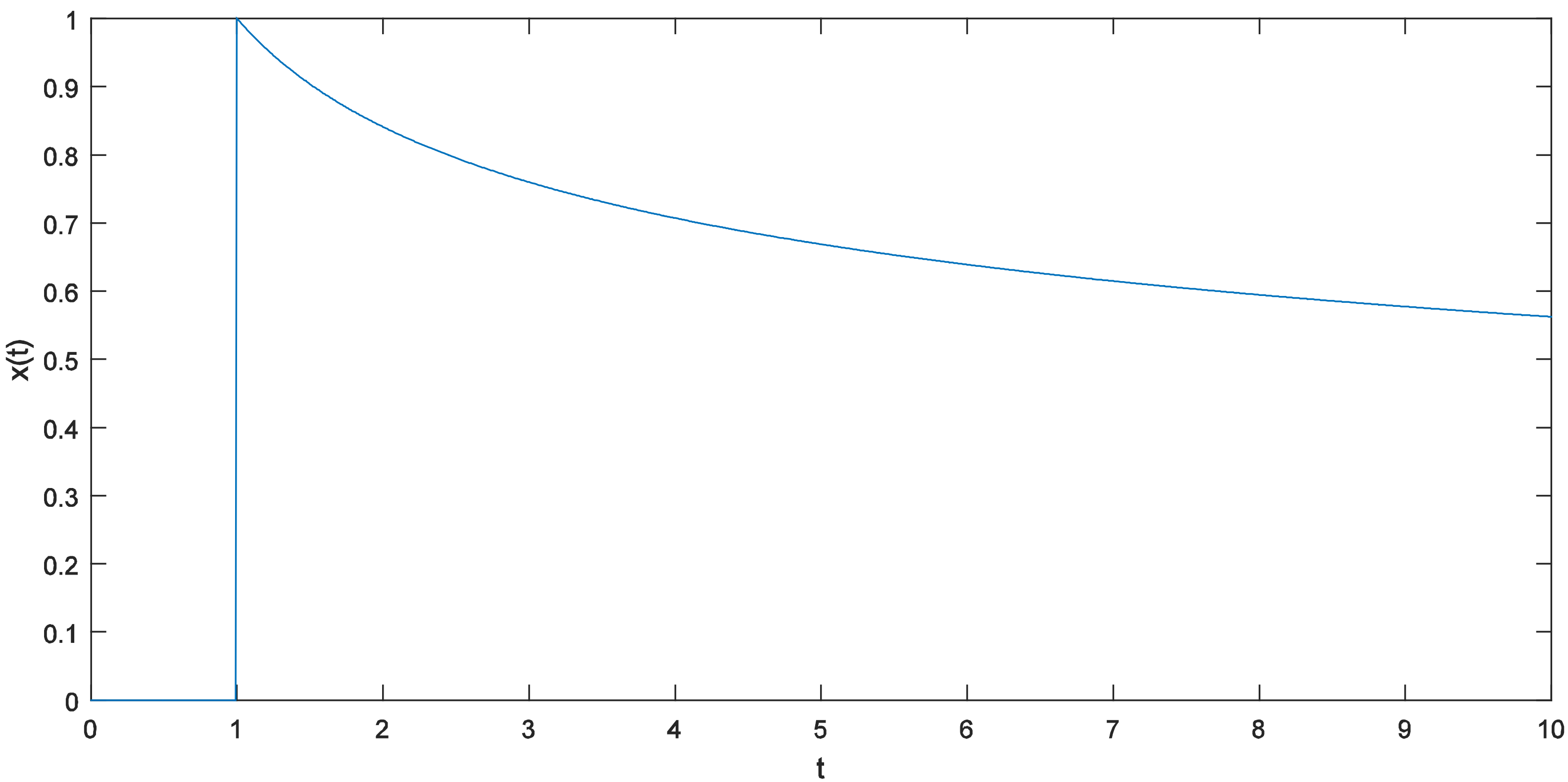
- Based on the definitions (2.22) and (2.23), we can define two distinct classes of signals:

1. We say $x(t)$ is an energy signal if and only if $0 < E < \infty$, so that $P = 0$.
2. We classify $x(t)$ as a power signal if and only if $0 < P < \infty$, so that $E = \infty$.

Signal Classifications (5/5)

- Signals that are neither energy nor power signals are easily found.
- For example, $x(t) = t^{-1/4}, t \geq t_0 > 0$, and zero otherwise.

Plot of $x(t)$ for $t_0 = 1$



Example 2.3 (1/2)

- As an example of determining the classification of a signal, consider

$$x_1(t) = Ae^{-\alpha t}u(t), \alpha > 0 \qquad (2.24)$$

  where A and α are positive constants.
- Using (2.22), we may readily verify that $x_1(t)$ is an energy signal since $E = A^2/2\alpha$ by applying (2.22).

Example 2.3 (2/2)

- Letting $\alpha \to 0$, we obtain the signal $x_2(t) = Au(t)$, which has infinite energy.
- Applying (2.23), we find that $P = \frac{1}{2}A^2$ for $Au(t)$, thus verifying that $x_2(t)$ is a power signal.

Example 2.4

- Consider the rotating phasor signal given by (2.4). We may verify that $\tilde{x}(t)$ is a power signal since

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\tilde{x}(t)|^2 dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 dt = A^2 \end{aligned} \tag{2.25}$$

is finite.

For Periodic Signal (1/2)

- We note that there is no need to carry out the limiting operation to find P for a periodic signal, since an average carried out over a single period gives the same result as (2.23).

For Periodic Signal (2/2)

- That is, for a periodic signal $x_p(t)$,

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \left|x_p(t)\right|^2 dt \qquad (2.26)$$

  where T_0 is the period and t_0 is an arbitrary starting time (chosen for convenience).
- The proof of (2.26) is left to the problems.

Example 2.5 (1/2)

- The sinusoidal signal

$$x_p(t) = A\cos(\omega_0 t + \theta) \qquad (2.27)$$

has average power

$$\begin{aligned} P &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} A^2\cos^2(\omega_0 t + \theta)\,dt \\ &= \frac{\omega_0}{2\pi}\int_{t_0}^{t_0+(2\pi/\omega_0)} \frac{A^2}{2}\,dt \end{aligned} \qquad (2.28)$$

Example 2.5 (2/2)

$$+\frac{\omega_0}{2\pi}\int_{t_0}^{t_0+(2\pi/\omega_0)}\frac{A^2}{2}\cos[2(\omega_0 t+\theta)]\,dt$$

$$=\frac{A^2}{2}$$

where the identity $\cos^2 u = \frac{1}{2}+\frac{1}{2}\cos(2u)$ has been used and the second integral is zero because the integration is over two complete periods of the integrand.

Fourier Series

- Complex Exponential Fourier Series
- Symmetry Properties of the Fourier Coefficients
- Trigonometric Form of the Fourier Series
- Parseval’s Theorem
- Examples of Fourier Series
- Line Spectra

Complex Exponential Fourier Series (1/2)

- Given a signal $x(t)$ defined over the interval $(t_0, t_0 + T_0)$ with the definition

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

we define the complex exponential Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, t_0 \le t < t_0 + T_0 \quad (2.29)$$

Complex Exponential Fourier Series (2/2)

where

$$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jn\omega_0 t}dt \qquad (2.30)$$

- A useful observation about a complete orthonormal-series expansion of a signal is that the series is unique.

Example 2.6 (1/3)

- Consider the signal

$$x(t) = \cos(\omega_0 t) + \sin^2(2\omega_0 t) \quad (2.31)$$

where $\omega_0 = 2\pi/T_0$.

- Find the complex exponential Fourier series.
- Solution: We could compute the Fourier coefficients using (2.30), but by using appropriate trigonometric identities and Euler's theorem, we obtain

Example 2.6 (2/3)

$$x(t) = \cos(\omega_0 t) + \frac{1}{2} - \frac{1}{2}\cos(4\omega_0 t)$$
$$= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$
$$+ \frac{1}{2} - \frac{1}{4}e^{j4\omega_0 t} - \frac{1}{4}e^{-j4\omega_0 t} \quad (2.32)$$

- Invoking uniqueness and equating the second line term by term with $\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$, we find that

Example 2.6 (3/3)

$$X_0 = \frac{1}{2}$$
$$X_1 = \frac{1}{2} = X_{-1} \qquad (2.33)$$
$$X_4 = -\frac{1}{4} = X_{-4}$$

with all other X_n equal to zero.

- Thus considerable labor is saved by noting that the Fourier series of a signal is unique.

Symmetry Properties of the Fourier Coefficients (1/8)

- Assuming $x(t)$ is real, it follows from (2.30) that

$$X_n^* = X_{-n} \tag{2.34}$$

by taking the complex conjugate inside the integral and noting that the same result is obtained by replacing n by $-n$.

Symmetry Properties of the Fourier Coefficients (2/8)

- Writing X_n as

$$X_n = |X_n| e^{j\underline{/X_n}} \tag{2.35}$$

we obtain

$$|X_n| = |X_{-n}| \text{ and } \underline{/X_n} = -\underline{/X_{-n}} \tag{2.36}$$

- Thus, for real signals, the magnitude of the Fourier coefficients is an even function of n, and the argument is odd.

Symmetry Properties of the Fourier Coefficients (3/8)

- Several symmetry properties can be derived for the Fourier coefficients, depending on the symmetry of $x(t)$.
- For example, suppose $x(t)$ is even; that is, $x(t) = x(-t)$.
- Then, using Euler's theorem to write the expression for the Fourier coefficients as (choose $t_0 = -T_0/2$)

Symmetry Properties of the Fourier Coefficients (4/8)

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(n\omega_0 t)dt - \frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(n\omega_0 t)\,dt \quad (2.37)$$

- We see that the second term is zero, since $x(t)\sin(n\omega_0 t)$ is an odd function.

Symmetry Properties of the Fourier Coefficients (5/8)

- Thus X_n is purely real, and furthermore, X_n is an even function of n since $\cos(n\omega_0 t)$ is an even function of n.
- These consequences of $x(t)$ being even are illustrated by Example 2.6.

Symmetry Properties of the Fourier Coefficients (6/8)

- On the other hand, if $x(t) = -x(-t)$ [that is, $x(t)$ is odd], it readily follows that X_n is purely imaginary, since the first term in (2.37) is zero by virtue of $x(t)\cos(n\omega_0 t)$ being odd.
- In addition, X_n is an odd function of n, since $\sin(n\omega_0 t)$ is an odd function of n.

Symmetry Properties of the Fourier Coefficients (7/8)

- Another type of symmetry is (odd) halfwave symmetry, defined as

$$x\left(t \pm \frac{1}{2}T_0\right) = -x(t) \qquad (2.38)$$

where T_0 is the period of $x(t)$.

Symmetry Properties of the Fourier Coefficients (8/8)

- For signals with odd halfwave symmetry,

$$X_n = 0, n = 0, \pm 2, \pm 4, \ldots \qquad (2.39)$$

  which states that the Fourier series for such a signal consists only of odd-indexed terms.
- The proof of this is left to the problems.

Trigonometric Form of the Fourier Series (1/4)

- Using (2.36) and assuming $x(t)$ real, we can regroup the complex exponential Fourier series by pairs of terms of the form

$$X_n e^{jn\omega_0 t} + X_{-n} e^{-jn\omega_0 t}$$
$$= |X_n| e^{j\left(n\omega_0 t + \underline{/X_n}\right)} + |X_n| e^{-j\left(n\omega_0 t + \underline{/X_n}\right)}$$
$$= 2|X_n| \cos\left(n\omega_0 t + \underline{/X_n}\right) \qquad (2.40)$$

where the facts that $|X_n| = |X_{-n}|$ and $\underline{/X_n} = -\underline{/X_{-n}}$ have been used.

Trigonometric Form of the Fourier Series (2/4)

- Hence, (2.29) can be written in the equivalent trigonometric form:

$$x(t) = X_0 + \sum_{n=1}^{\infty} 2|X_n| \cos\left(n\omega_0 t + \underline{/X_n}\right) \quad (2.41)$$

- Expanding the cosine in (2.41), we obtain still another equivalent series of the form

$$x(t) = X_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t) \quad (2.42)$$

Trigonometric Form of the Fourier Series (3/4)

where

$$A_n = 2|X_n|\cos\underline{/X_n}$$
$$= \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(n\omega_0 t)\,dt \qquad (2.43)$$

and

$$B_n = -2|X_n|\sin\underline{/X_n}$$
$$= \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(n\omega_0 t)\,dt \qquad (2.44)$$

Trigonometric Form of the Fourier Series (4/4)

- In either the trigonometric or the exponential forms of the Fourier series, X_0 represents the average or DC component of $x(t)$.
- The term for $n = 1$ is called the fundamental, the term for $n = 2$ is called the second harmonic, and so on.

Parseval's Theorem (1/2)

- Using (2.26) for average power of a periodic signal, substituting (2.29) for $x(t)$, and interchanging the order of integration and summation, we find Parseval's theorem to be

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |X_n|^2 \quad (2.45)$$

$$= X_0^2 + \sum_{n=1}^{\infty} 2|X_n|^2 \quad (2.46)$$

Parseval's Theorem (2/2)

which is called Parseval's theorem.

Table 2.1 Fourier Series for Several Periodic Signals

Signal (one period)	Coefficients for exponential Fourier series
1. Asymmetrical pulse train; period $= T_0$: $x(t) = A\Pi\left(\frac{t-t_0}{\tau}\right), \tau < T_0$; $x(t) = x(t+T_0)$, all t	$X_n = \frac{A\tau}{T_0}\operatorname{sinc}(nf_0\tau)e^{-j2\pi nf_0t_0}$; $n = 0, \pm 1, \pm 2, \ldots$
2. Half-rectified sine wave; period $= T_0 = 2\pi/\omega_0$: $x(t) = \begin{cases} A\sin(\omega_0 t), & 0 \le t \le T_0/2 \\ 0, & -T_0/2 \le t \le 0 \end{cases}$; $x(t) = x(t+T_0)$ all t	$X_n = \begin{cases} \frac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \cdots \\ 0, & n = \pm 3, \pm 5, \cdots \\ -\frac{1}{4}jnA, & n = \pm 1 \end{cases}$
3. Full-rectified sine wave; period $= T_0 = \pi/\omega_0$: $x(t) = A\lvert\sin(\omega_0 t)\rvert$	$X_n = \frac{2A}{\pi(1-4n^2)}, \quad n = 0, \pm 1, \pm 2, \ldots$
4. Triangular wave: $x(t) = \begin{cases} -\frac{4A}{T_0}t + A, & 0 \le t \le T_0/2 \\ \frac{4A}{T_0}t + A, & -T_0/2 \le t \le 0 \end{cases}$; $x(t) = x(t+T_0)$, all t	$X_n = \begin{cases} \frac{4A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

Examples of Fourier Series (1/3)

- Table 2.1 gives Fourier series for several commonly occurring periodic waveforms.
- The left-hand column specifies the signal over one period.
- The definition of periodicity,

$$x(t) = x(t + T_0)$$

specifies it for all t.

Examples of Fourier Series (2/3)

- The derivation of the Fourier coefficients given in the right-hand column of Table 2.1 is left to the problems.
- Note that the full-rectified sine wave actually has the period $\frac{1}{2}T_0$.

Examples of Fourier Series (3/3)

- For the periodic pulse train, it is convenient to express the coefficients in terms of the sinc function, defined as

$$\operatorname{sinc} z = \frac{\sin(\pi z)}{\pi z} \tag{2.47}$$

- The sinc function is an even damped oscillatory function with zero crossings at integer values of its argument.

Example 2.7 (1/4)

- Specialize the results for the pulse train (item 1) of Table 2.1 to the complex exponential and trigonometric Fourier series of a squarewave with even symmetry and amplitudes zero and A.
- Solution: The solution proceeds by letting $t_0 = 0$ and $\tau = \frac{1}{2}T_0$ in item 1 of Table 2.1.

Example 2.7 (2/4)

- Thus,

$$X_n = \frac{1}{2} A \operatorname{sinc}\left(\frac{1}{2} n\right) \qquad (2.48)$$

- But

$$\operatorname{sinc}\left(\frac{n}{2}\right) = \frac{\sin(n\pi/2)}{n\pi/2}$$

Example 2.7 (3/4)

$$= \begin{cases} 1, & n = 0 \\ 0, & n = \text{even} \\ \left|\dfrac{2}{n\pi}\right|, & n = \pm 1, \pm 5, \pm 9, \ldots \\ -\left|\dfrac{2}{n\pi}\right|, & n = \pm 3, \pm 7, \ldots \end{cases} \qquad (2.48a)$$

Example 2.7 (4/4)

- Thus

$$x(t) = \cdots + \frac{A}{5\pi}e^{-j5\omega_0 t} - \frac{A}{3\pi}e^{-j3\omega_0 t} + \frac{A}{\pi}e^{-j\omega_0 t} + \frac{A}{2} + \frac{A}{\pi}e^{j\omega_0 t} - \frac{A}{3\pi}e^{j3\omega_0 t} + \frac{A}{5\pi}e^{j5\omega_0 t} - \cdots$$

$$= \frac{A}{2} + \frac{2A}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \cdots\right] \quad (2.49)$$

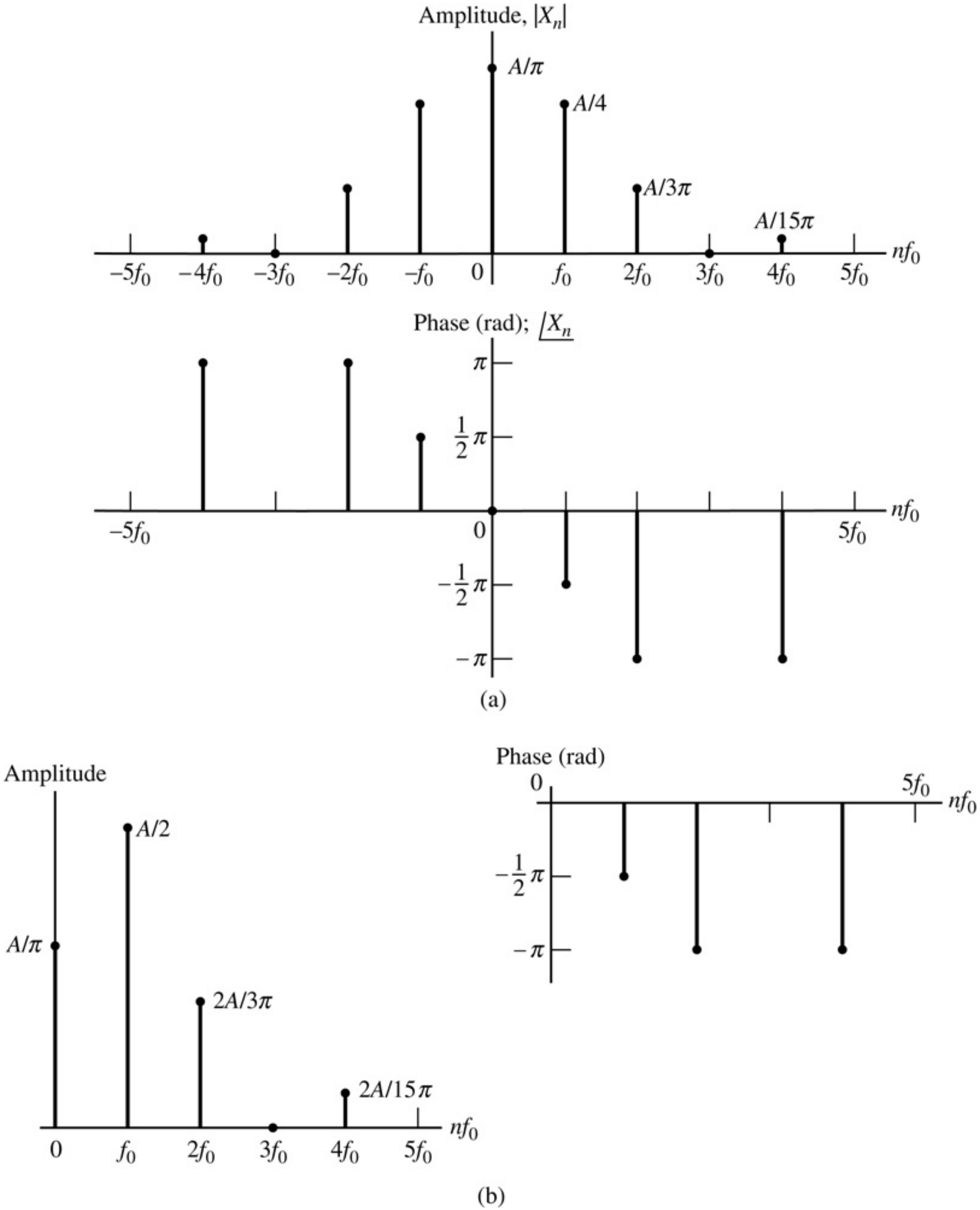


Figure 2.5 Line spectra for half-rectified sinewave.
(a) Double-sided.
(b) Single-sided.

Line Spectra (1/8)

- Figure 2.5(a) shows the double-sided spectrum for a half-rectified sine wave as plotted from the results given in Table 2.1.
- For $n = 2, 4, \ldots, X_n$ is represented as follows:

$$X_n = -\left|\frac{A}{\pi(1-n^2)}\right| = \frac{A}{\pi(n^2-1)} e^{-j\pi} \quad (2.50)$$

Line Spectra (2/8)

- For $n = -2, -4, \ldots$, it is represented as

$$X_n = -\left|\frac{A}{\pi(1-n^2)}\right| = \frac{A}{\pi(n^2-1)}e^{j\pi} \quad (2.51)$$

  to ensure that the phase is odd, as it must be (note that $e^{\pm j\pi} = -1$).
- Thus, putting this together with $X_{\pm 1} = \mp jA/4$, we get

Line Spectra (3/8)

$$|X_n| = \begin{cases} \dfrac{1}{4}A, & n = \pm 1 \\ \left|\dfrac{A}{\pi(1-n^2)}\right|, & \text{all even } n \end{cases} \tag{2.52}$$

Line Spectra (4/8)

$$\underline{/X_n} = \begin{cases} -\pi, & n = 2, 4, \ldots \\ -\frac{1}{2}\pi, & n = 1 \\ 0, & n = 0 \\ \frac{1}{2}\pi, & n = -1 \\ \pi, & n = -2, -4, \ldots \end{cases} \qquad (2.53)$$

- The single-sided spectra for the half-rectified sinewave are shown in Figure 2.5(b).

Line Spectra (5/8)

- As a second example, consider the pulse train

$$x(t) = \sum_{n=-\infty}^{\infty} A\Pi\left(\frac{t - nT_0 - \frac{1}{2}\tau}{\tau}\right) \quad (2.54)$$

- From Table 2.1 with $t_0 = \frac{1}{2}\tau$ substituted in item 1, the Fourier coefficients are

$$X_n = \frac{A\tau}{T_0}\mathrm{sinc}(nf_0\tau)e^{-j\pi nf_0\tau} \quad (2.55)$$

Line Spectra (6/8)

- The Fourier coefficients can be put in the form $|X_n| \exp(j\underline{/X_n})$, where

$$|X_n| = \frac{A\tau}{T_0}|\text{sinc}(nf_0\tau)| \qquad (2.56)$$

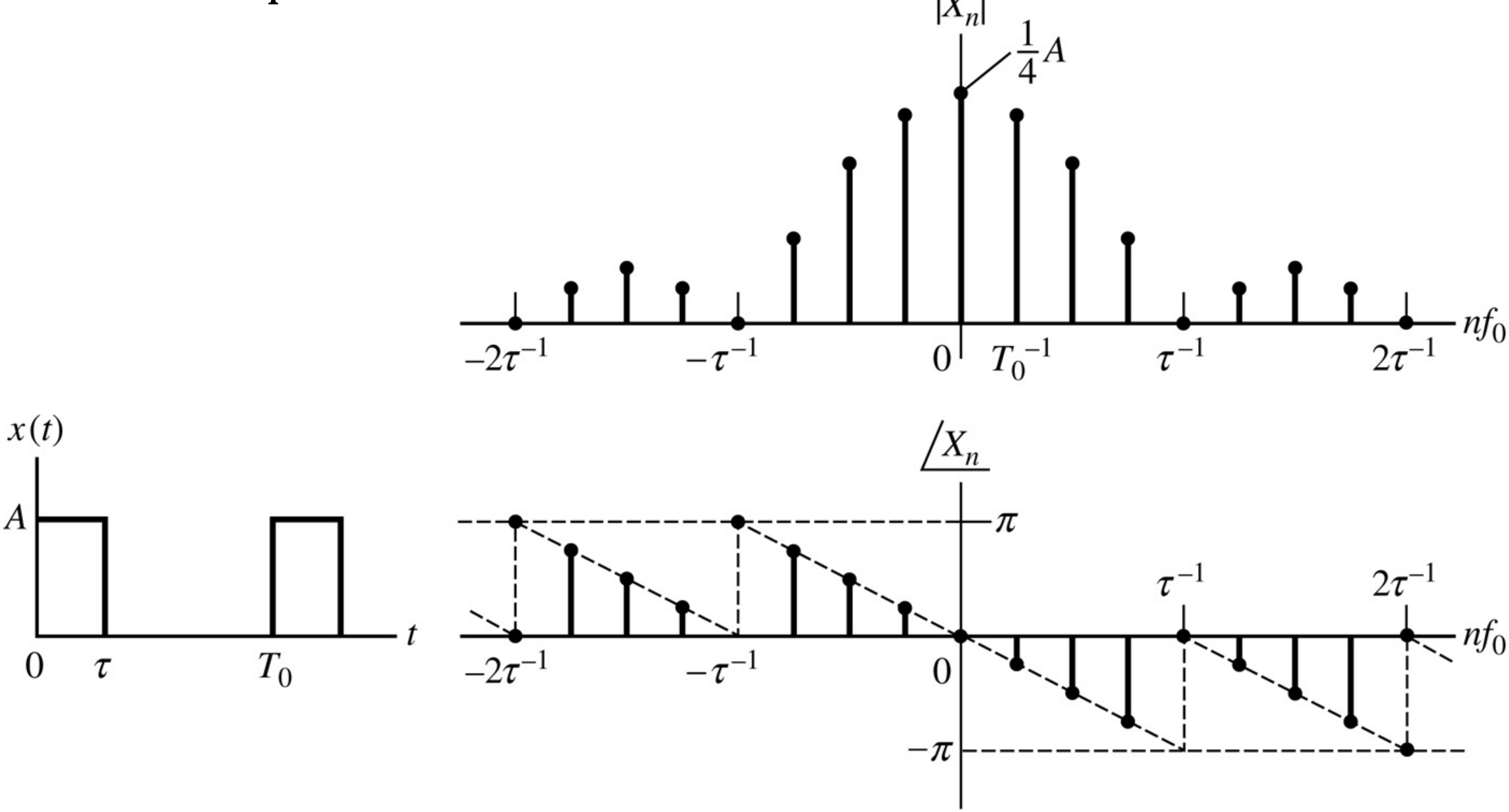
and

$$\underline{/X_n} = \begin{cases} -\pi n f_0 \tau & \text{if } \text{sinc}(nf_0\tau) > 0 \\ -\pi n f_0 \tau + \pi & \text{if } nf_0 > 0 \text{ and } \text{sinc}(nf_0\tau) < 0 \\ -\pi n f_0 \tau - \pi & \text{if } nf_0 < 0 \text{ and } \text{sinc}(nf_0\tau) < 0 \end{cases} \qquad (2.57)$$

Line Spectra (7/8)

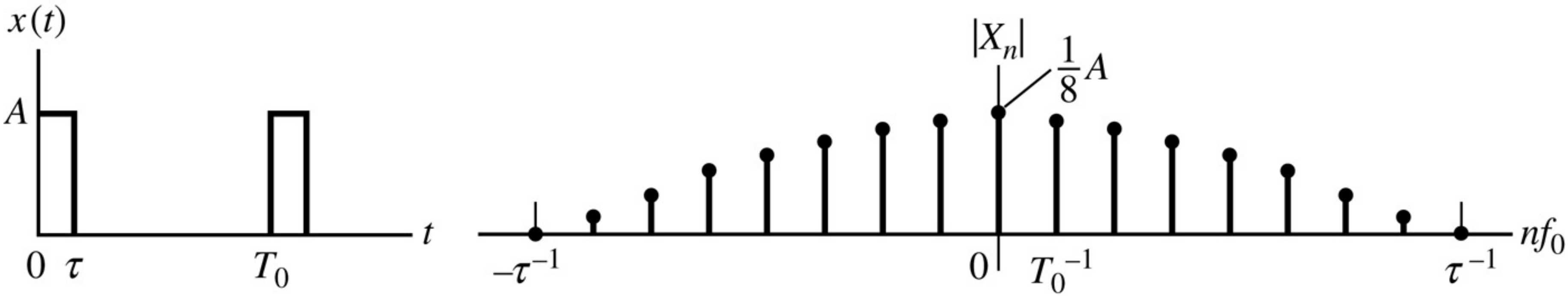
- The double-sided amplitude and phase spectra can be now be plotted.
- They are shown in Figure 2.6 for several choices of τ and T_0.

Figure 2.6 Spectra for a periodic pulse train signal.
(a) $\tau = \frac{1}{4} T_0$.

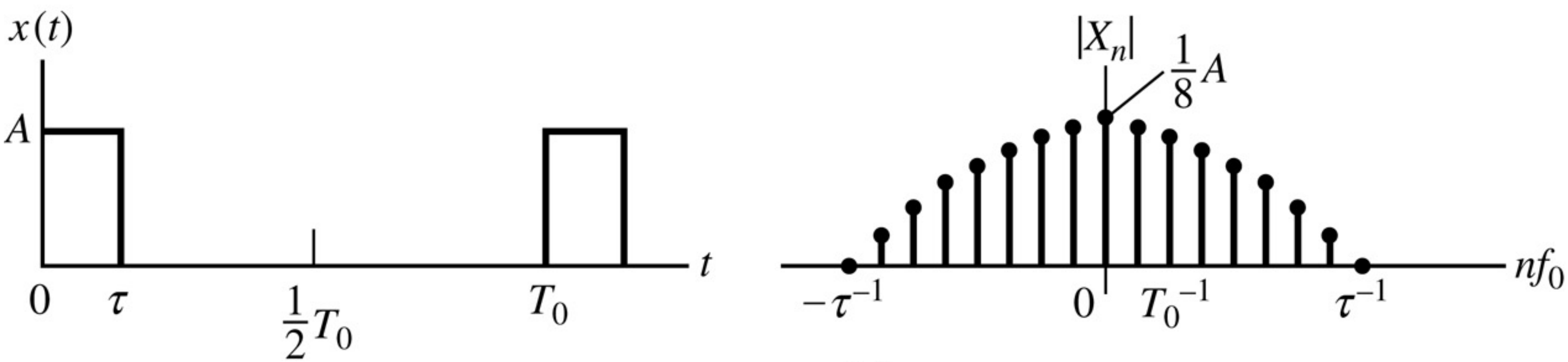


(a)

Figure 2.6 (b) $\tau = \frac{1}{8}T_0$; T_0 same as in (a). (c) $\tau = \frac{1}{8}T_0$; τ same as in (a).



(b)



(c)

Line Spectra (8/8)

- Comparing Figure 2.6(a) and (b), we note that the zeros of the envelope of the amplitude spectrum, which occur at multiples of $1/\tau$ Hz, move out along the frequency axis as the pulse width decreases.
- That is, the time duration of a signal and its spectral width are inversely proportional, a property that will be shown to be true in general later.

The Fourier Transform (1/7)

- Amplitude and Phase Spectra
- Symmetry Properties
- Energy Spectral Density
- Convolution
- Transform Theorems: Proofs and Applications
- Fourier Transforms of Periodic Signals
- Poisson Sum Formula

The Fourier Transform (2/7)

- To generalize the Fourier series representation (2.29) to a representation valid for aperiodic signals, we consider the two basic relationships (2.29) and (2.30).
- Suppose that $x(t)$ is nonperiodic but is an energy signal, so that it is integrable square in the interval $(-\infty, \infty)$.

The Fourier Transform (3/7)

- In the interval $|t| < \frac{1}{2}T_0$, we can represent $x(t)$ as the Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(\lambda) e^{-j2\pi n f_0 \lambda} d\lambda \right] e^{j2\pi n f_0 t}, \quad |t| < \frac{T_0}{2} \qquad (2.58)$$

where $f_0 = 1/T_0$.

The Fourier Transform (4/7)

- To represent $x(t)$ for all time, we simply let $T_0 \to \infty$ such that $nf_0 = n/T_0$ becomes the continuous variable f, $1/T_0$ becomes the differential df, and the summation becomes an integral.
- Thus

$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda} d\lambda \right] e^{j2\pi f t} df \tag{2.59}$$

The Fourier Transform (5/7)

- Defining the inside integral as

$$X(f) = \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f\lambda} d\lambda \qquad (2.60)$$

we can write (2.59) as

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \qquad (2.61)$$

- The existence of these integrals is assured, since $x(t)$ is an energy signal.

The Fourier Transform (6/7)

- We note that
$$X(f) = \lim_{T_0 \to \infty} T_0 X_n \qquad (2.62)$$
which avoids the problem that $|X_n| \to 0$ as $T_0 \to \infty$.
- The frequency-domain description of $x(t)$ provided by (2.60) is referred to as the Fourier transform of $x(t)$, written symbolically as $X(f) = \mathfrak{F}[x(t)]$.

The Fourier Transform (7/7)

- Conversion back to the time domain is achieved via the inverse Fourier transform (2.61), written symbolically as $x(t) = \mathfrak{I}^{-1}[X(f)]$.
- Expressing (2.60) and (2.61) in terms of $f = \omega/2\pi$ results in easily remembered symmetrical expressions.
- Integrating (2.61) with respect to the variable ω requires a factor of $(2\pi)^{-1}$.

Amplitude and Phase Spectra (1/3)

- Writing $X(f)$ in terms of magnitude and phase as

$$X(f) = |X(f)|e^{j\theta(f)}, \theta(f) = \underline{/X(f)} \qquad (2.63)$$

we can show that for real $x(t)$,

$$|X(f)| = |X(-f)| \text{ and } \theta(f) = -\theta(-f) \qquad (2.64)$$

just as for the Fourier series.

Amplitude and Phase Spectra (2/3)

- This is done by using Euler's theorem to write

$$R = \mathrm{Re}\big(X(f)\big) = \int_{-\infty}^{\infty} x(t)\cos(2\pi f t)\,dt \quad (2.65)$$

and

$$I = \mathrm{Im}\big(X(f)\big) = -\int_{-\infty}^{\infty} x(t)\sin(2\pi f t)\,dt \quad (2.66)$$

- Thus the real part of $X(f)$ is even and the imaginary part is odd if $x(t)$ is a real signal.

Amplitude and Phase Spectra (3/3)

- Since $|X(f)|^2 = R^2 + I^2$ and $\tan\theta(f) = I/R$, the symmetry properties (2.64) follow.
- A plot of $|X(f)|$ versus f is referred to as the amplitude spectrum of $x(t)$, and a plot of $\underline{/X(f)} = \theta(f)$ versus f is known as the phase spectrum.

Symmetry Properties (1/2)

- If $x(t) = x(-t)$, that is, if $x(t)$ is even, then $x(t)\sin(2\pi f t)$ is odd in (2.66) and $\mathrm{Im}\big(X(f)\big) = 0$.
- Furthermore, $\mathrm{Re}\big(X(f)\big)$ is an even function of f because cosine is an even function.
- Thus the Fourier transform of a real, even function is real and even.

Symmetry Properties (2/2)

- On the other hand, if $x(t)$ is odd, $x(t)\cos(2\pi f t)$ is odd in (2.65) and $\mathrm{Re}\big(X(f)\big) = 0$.
- Thus the Fourier transform of a real, odd function is imaginary.
- In addition, $\mathrm{Im}\big(X(f)\big)$ is an odd function of frequency because $\sin(2\pi f t)$ is an odd function.

Example 2.8 (1/2)

- Consider the pulse

$$x(t) = A\Pi\left(\frac{t - t_0}{\tau}\right) \tag{2.67}$$

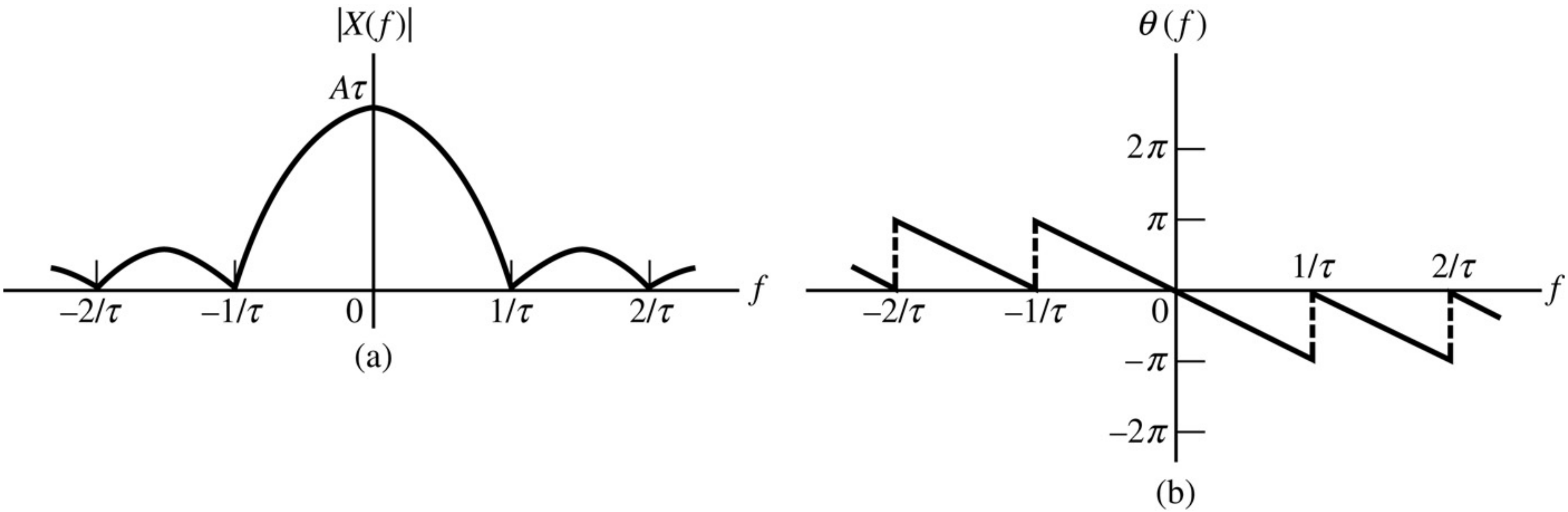
- The Fourier transform is

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} A\Pi\left(\frac{t - t_0}{\tau}\right) e^{-j2\pi f t}\, dt \\ &= A \int_{t_0-\tau/2}^{t_0+\tau/2} e^{-j2\pi f t} dt \\ &= A\tau \operatorname{sinc}(f\tau) e^{-j2\pi f t_0} \end{aligned} \tag{2.68}$$

Example 2.8 (2/2)

- The amplitude spectrum of $x(t)$ is
$$|X(f)| = A\tau|\mathrm{sinc}(f\tau)| \tag{2.69}$$
and the phase spectrum is
$$\theta(f) = \begin{cases} -2\pi t_0 f & \text{if } \mathrm{sinc}(f\tau) > 0 \\ -2\pi t_0 f \pm \pi & \text{if } \mathrm{sinc}(f\tau) < 0 \end{cases} \tag{2.70}$$
- Figure 2.7 shows the amplitude and phase spectra for the signal (2.67).
- The similarity to Figure 2.6 is to be noted, especially the inverse relationship between spectral width and pulse duration.

Figure 2.7



- Amplitude and phase spectra for a pulse signal. (a) Amplitude spectrum. (b) Phase spectrum ($t_0 = \frac{1}{2}\tau$ is assumed).

Energy Spectral Density (1/5)

- The energy of a signal, defined by (2.22), can be expressed in the frequency domain as follows:

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} x^*(t) \left[\int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \right] dt \qquad (2.71)$$

where $x(t)$ has been written in terms of its Fourier transform.

Energy Spectral Density (2/5)

- Reversing the order of integration, we obtain

$$E = \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{\infty} x^*(t) e^{j2\pi f t} dt \right] df$$

$$= \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \right]^* df$$

$$= \int_{-\infty}^{\infty} X(f) \, X^*(f) df$$

or

Energy Spectral Density (3/5)

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad (2.72)$$

- This is referred to as Rayleigh's energy theorem or Parseval's theorem for Fourier transforms.

Energy Spectral Density (4/5)

- Examining $|X(f)|^2$ and recalling the definition of $X(f)$ given by (2.60), we note that the former has the units of (volts-seconds) or, since we are considering power on a per-ohm basis, (watts-seconds)/hertz = joules/hertz.

Energy Spectral Density (5/5)

- Thus we see that $|X(f)|^2$ has the units of energy density, and we define the energy spectral density of a signal as
$$G(f) \triangleq |X(f)|^2. \quad (2.73)$$
- By integrating $G(f)$ over all frequency, we obtain the signal's total energy.

Example 2.9 (1/4)

- Rayleigh's energy theorem (Parseval's theorem for Fourier transforms) is convenient for finding the energy in a signal whose square is not easily integrated in the time domain, or vice versa.
- For example, the signal

$$x(t) = 40\text{sinc}(20t) \leftrightarrow X(f) = 2\Pi\left(\frac{f}{20}\right) \tag{2.74}$$

Example 2.9 (2/4)

has energy density

$$G(f) = |X(f)|^2 = \left[2\Pi\left(\frac{f}{20}\right)\right]^2 = 4\Pi\left(\frac{f}{20}\right) \qquad (2.75)$$

where $\Pi(\frac{f}{20})$ need not be squared because it has amplitude 1 whenever it is nonzero.

- Using Rayleigh's energy theorem, we find that the energy in $x(t)$ is

Example 2.9 (3/4)

$$E = \int_{-\infty}^{\infty} G(f)df = \int_{-10}^{10} 4\, df = 80 \text{ J} \qquad (2.76)$$

- This checks with the result that is obtained by integrating $x^2(t)$ over all t using the definite integral $\int_{-\infty}^{\infty} \text{sinc}^2 u\, du = 1$.
- The energy contained in the frequency interval $(0, W)$ can be found from the integral

Example 2.9 (4/4)

$$E_W = \int_{-W}^{W} G(f)df = 2\int_0^W \left[2\Pi\left(\frac{f}{20}\right)\right]^2 df$$

$$= \begin{cases} 8W, & W \leq 10 \\ 80, & W > 10 \end{cases} \quad (2.77)$$

which follows because $\Pi\left(\frac{f}{20}\right) = 0, |f| > 10.$

Convolution (1/3)

- We digress somewhat from our consideration of the Fourier transform to define the convolution operation and illustrate it by example.
- The convolution of two signals, $x_1(t)$ and $x_2(t)$, is a new function of time, $x(t)$, written symbolically in terms of x_1 and x_2 as

$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda \tag{2.78}$$

Convolution (2/3)

- Note that t is a parameter as for as the integration is concerned.
- The integrand is formed from x_1 and x_2 by three operations:
  1. Time reversal to obtain $x_2(-\lambda)$.
  2. Time shifting to obtain $x_2(t - \lambda)$.
  3. Multiplication of $x_1(\lambda)$ and $x_2(t - \lambda)$ to form the integrand.

Convolution (3/3)

- An example will illustrate the implementation of these operations to form $x_1 * x_2$.
- Note that the dependence on time is often suppressed.

Example 2.10 (1/3)

- Find the convolution of the two signals
$$x_1(t) = e^{-\alpha t}u(t) \text{ and } x_2(t) = e^{-\beta t}u(t),$$
$$\alpha > \beta > 0 \tag{2.79}$$
- Solution: The steps involved in the convolution are illustrated in Figure 2.8 for $\alpha = 4$ and $\beta = 2$.
- Mathematically, we can form the integrand by direct substitution:

Example 2.10 (2/3)

$$x(t) = x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} e^{-\alpha\lambda} u(\lambda) e^{-\beta(t-\lambda)} u(t-\lambda) d\lambda \qquad (2.80)$$

- But

$$u(\lambda)u(t-\lambda) = \begin{cases} 0, & \lambda < 0 \\ 1, & 0 < \lambda < t \\ 0, & \lambda > t \end{cases} \qquad (2.81)$$

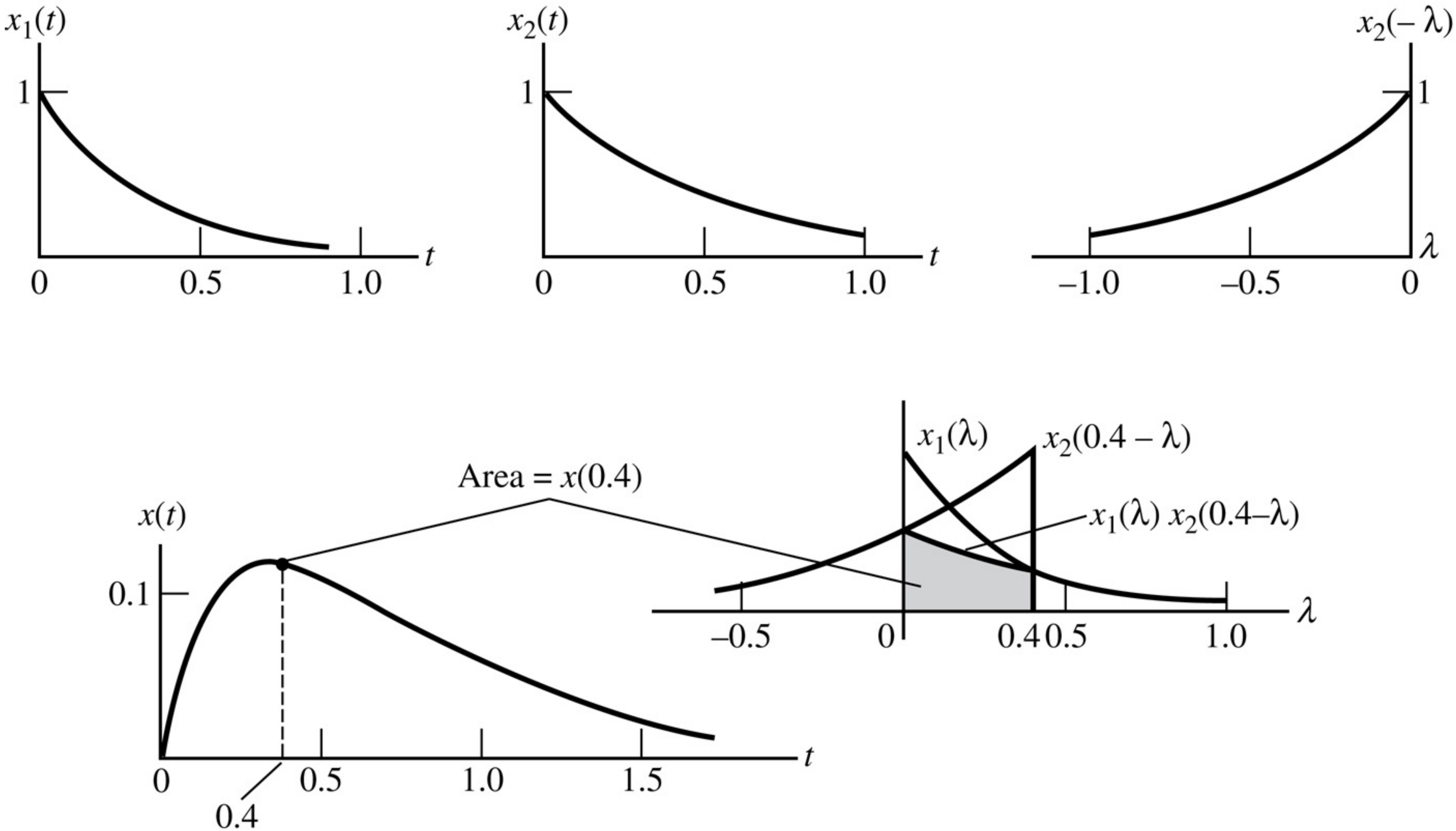
Example 2.10 (3/3)

- Thus,

$$x(t) = \begin{cases} 0, & t < 0 \\ \int_0^t e^{-\beta t} e^{-(\alpha-\beta)\lambda} d\lambda = \frac{1}{\alpha-\beta}\left(e^{-\beta t} - e^{-\alpha t}\right), & t \geq 0 \end{cases} \quad (2.82)$$

- The result for $x(t)$ is also shown in Figure 2.8.

Figure 2.8 The operations involved in the convolution of two exponentially decaying signals.



Transform Theorems: Proofs and Applications (1/2)

- Several useful theorems involving Fourier transforms can be proved.
- These are useful for deriving Fourier transform pairs as well as deducing general frequency-domain relationships.
- The notation $x(t) \leftrightarrow X(f)$ will be used to denote a Fourier transform pair.

Transform Theorems: Proofs and Applications (2/2)

- Each theorem will be stated along with a proof in most cases.
- Several examples giving applications will be given after the statements of all the theorems.
- In the statements of the theorems, $x(t)$, $x_1(t)$, and $x_2(t)$ denote signals with $X(f)$, $X_1(f)$, and $X_2(f)$ denoting their respective Fourier transforms.
- Constants are denoted by a, a_1, a_2, t_0, and f_0.

Superposition Theorem

$$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(f) + a_2 X_2(f) \quad (2.83)$$

- Proof: By the defining integral for the Fourier transform,

$$\begin{aligned}
&\Im\{a_1 x_1(t) + a_2 x_2(t)\} \\
&= \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] e^{-j2\pi f t} dt \\
&= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi f t} dt \\
&= a_1 X_1(f) + a_2 X_2(f) \qquad (2.84)
\end{aligned}$$

Time-Delay Theorem (1/2)

$$x(t - t_0) \leftrightarrow X(f)e^{-j2\pi f t_0} \qquad (2.85)$$

- Proof: Using the defining integral for the Fourier transform, we have

$$\Im\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0)e^{-j2\pi f t}dt$$

$$= \int_{-\infty}^{\infty} x(\lambda)e^{-j2\pi f(\lambda + t_0)}d\lambda$$

Time-Delay Theorem (2/2)

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda} d\lambda$$

$$= X(f) e^{-j2\pi f t_0} \qquad (2.86)$$

where the substitution $\lambda = t - t_0$ was used in the first integral.

Scale-Change Theorem (1/3)

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right) \tag{2.87}$$

- Proof: First, assume that $a > 0$. Then

$$\Im\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda / a} \frac{d\lambda}{a} = \frac{1}{a} X\left(\frac{f}{a}\right) \tag{2.88}$$

Scale-Change Theorem (2/3)

where the substitution $\lambda = at$ has been used.

- Next, considering $a < 0$, we write

$$\Im\{x(at)\} = \int_{-\infty}^{\infty} x(-|a|t)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} x(\lambda)e^{+j2\pi f\lambda/|a|}\frac{d\lambda}{|a|} \qquad (2.89)$$

$$= \frac{1}{|a|}X\left(-\frac{f}{|a|}\right) = \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

Scale-Change Theorem (3/3)

where use has been made of the relation $-|a| = a$ if $a < 0$.

Duality Theorem

$$X(t) \leftrightarrow x(-f) \qquad (2.90)$$

- The proof of this theorem follows by virtue of the fact that the only difference between the Fourier transform integral and the inverse Fourier transform integral is a minus sign in the exponent of the integrand.

Frequency Translation Theorem

$$x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0) \qquad (2.91)$$

- Proof: To prove the frequency translation theorem, note that

$$\int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi f t}dt$$
$$= \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_0)t}dt$$
$$= X(f - f_0) \qquad (2.92)$$

Modulation Theorem

$$x(t)\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0) \quad (2.93)$$

- Proof: The proof of this theorem follows by writing $\cos(2\pi f_0 t)$ in exponential form as $\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$ and applying the superposition and frequency translation theorems.

Differentiation Theorem (1/2)

$$\frac{d^n x(t)}{dt^n} \leftrightarrow (j2\pi f)^n X(f) \qquad (2.94)$$

- Proof: We prove the theorem for $n = 1$ by using integration by parts on the defining Fourier transform integral as follows:

$$\Im\left\{\frac{dx}{dt}\right\} = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j2\pi f t} dt$$

$$= x(t)e^{-j2\pi f t}\Big|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t} dt$$

Differentiation Theorem (2/2)

$$= j2\pi f X(f) \qquad (2.95)$$

where $u = e^{-j2\pi ft}$ and $dv = (dx/dt)dt$ have been used in the integration-by-parts formula, and the first term of the middle equation vanishes at each end point by virtue of $x(t)$ being an energy signal.

- The proof for values of $n > 1$ follows by induction.

Integration Theorem (1/3)

$$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow (j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f) \tag{2.96}$$

- If $X(0) = 0$ the proof of the integration theorem can be carried out by using integration by parts as in the case of the differentiation theorem.
- We obtain

Integration Theorem (2/3)

$$\Im\left\{\int_{-\infty}^{t} x(\lambda)d\lambda\right\}$$
$$=\left\{\int_{-\infty}^{t} x(\lambda)d\lambda\right\}\left(-\frac{1}{j2\pi f}e^{-j2\pi ft}\right)\Bigg|_{-\infty}^{\infty}$$
$$+\frac{1}{j2\pi f}\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \qquad (2.97)$$

Integration Theorem (3/3)

- The first term vanishes if $X(0) = \int_{-\infty}^{\infty} x(t)dt = 0$, and the second term is just $X(f)/(j2\pi f)$.
- For $X(0) \neq 0$, a limiting argument must be used to account for the Fourier transform of the nonzero average value of $x(t)$.

Convolution Theorem (1/4)

$$\int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda \triangleq \int_{-\infty}^{\infty} x_1(t-\lambda)x_2(\lambda)d\lambda$$
$$\leftrightarrow X_1(f)X_2(f) \tag{2.98}$$

- Proof: To prove the convolution theorem of Fourier transforms, we represent $x_2(t-\lambda)$ in terms of the inverse Fourier transform integral as

Convolution Theorem (2/4)

$$x_2(t-\lambda) = \int_{-\infty}^{\infty} X_2(f) e^{j2\pi f(t-\lambda)} df \quad (2.99)$$

- Denoting the convolution operation as $x_1(t) * x_2(t)$, we have

$$x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} x_1(\lambda) \left[\int_{-\infty}^{\infty} X_2(f) e^{j2\pi f(t-\lambda)} df \right] d\lambda$$

Convolution Theorem (3/4)

$$= \int_{-\infty}^{\infty} X_2(f) \left[\int_{-\infty}^{\infty} x_1(\lambda) e^{-j2\pi f\lambda} d\lambda \right] e^{j2\pi f t} df \qquad (2.100)$$

where the last step results from reversing the orders of integration.

- The bracketed term inside the integral is $X_1(f)$, the Fourier transform of $x_1(t)$.

Convolution Theorem (4/4)

- Thus

$$x_1 * x_2 = \int_{-\infty}^{\infty} X_1(f) X_2(f)\, e^{j2\pi f t} df \quad (2.101)$$

which is the inverse Fourier transform of $X_1(f)X_2(f)$.

- Taking the Fourier transform of this result yields the desired transform pair.

Multiplication Theorem

$$x_1(t)x_2(t) \leftrightarrow X_1(f) * X_2(f)$$
$$= \int_{-\infty}^{\infty} X_1(\lambda)X_2(f-\lambda)d\lambda \qquad (2.102)$$

- Proof: The proof of the multiplication theorem proceeds in a manner analogous to the proof of the convolution theorem.

Example 2.11 (1/2)

- Use the duality theorem to show that

$$2AW\text{sinc}(2Wt) \leftrightarrow A\Pi\left(\frac{f}{2W}\right) \qquad (2.103)$$

- Solution: From Example 2.8, we know that

$$x(t) = A\Pi\left(\frac{t}{\tau}\right) \leftrightarrow A\tau\text{sinc}(f\tau) = X(f) \quad (2.104)$$

- Considering $X(t)$, and using the duality theorem, we obtain

Example 2.11 (2/2)

- Considering $X(t)$, and using the duality theorem, we obtain

$$X(t) = A\tau\text{sinc}(\tau t) \leftrightarrow A\Pi\left(-\frac{f}{\tau}\right) = x(-f) \qquad (2.105)$$

  where τ is a parameter with dimension $(\mathrm{s})^{-1}$, which may be somewhat confusing at first sight!
- By letting $\tau = 2W$ and noting that $\Pi(u)$ is even, the given relationship follows.

Example 2.12 (1/3)

- Obtain the following Fourier transform pairs:

1. $A\delta(t) \leftrightarrow A$
2. $A\delta(t - t_0) \leftrightarrow Ae^{-j2\pi f t_0}$
3. $A \leftrightarrow A\delta(f)$
4. $Ae^{j2\pi f_0 t} \leftrightarrow A\delta(f - f_0)$

Example 2.12 (2/3)

- Solution:

$$\Im[A\delta(t)] = \Im\left[\lim_{\tau\to 0}\left(\frac{A}{\tau}\right)\Pi\left(\frac{t}{\tau}\right)\right]$$
$$= \lim_{\tau\to 0} A\text{sinc}(f\tau) = A \qquad (2.106)$$

or

$$\Im[A\delta(t)] = A\int_{-\infty}^{\infty}\delta(t)e^{-j2\pi f t}dt = A \qquad (2.107)$$

Example 2.12 (3/3)

- Transform pair 2 follows by application of the time-delay theorem to pair 1.
- Transform pair 3:

  $$X(t) = A \leftrightarrow A\delta(-f) = A\delta(f) = x(-f) \quad (2.108)$$

  where the evenness property of the impulse function is used.
- Transform pair 4 follows by applying the frequency-translation theorem to pair 3.

Example 2.13 (1/4)

- Use the differentiation theorem to obtain the Fourier transform of the triangular signal, defined as

$$\Lambda\left(\frac{t}{\tau}\right) \triangleq \begin{cases} 1 - \frac{|t|}{\tau}, & |t| < \tau \\ 0, & \text{otherwise} \end{cases} \tag{2.109}$$

Example 2.13 (2/4)

- Solution: Differentiating $\Lambda(t/\tau)$ twice, we obtain, as shown in Figure 2.9,

$$\frac{d^2\Lambda(t/\tau)}{dt^2} = \frac{1}{\tau}\delta(t+\tau) - \frac{2}{\tau}\delta(t) + \frac{1}{\tau}\delta(t-\tau) \tag{2.110}$$

- Using the differentiation, superposition, and time-shift theorems and the result of Example 2.12, we obtain

Example 2.13 (3/4)

$$\Im\left[\frac{d^2\Lambda(t/\tau)}{dt^2}\right] = (j2\pi f)^2 \Im\left[\Lambda\left(\frac{t}{\tau}\right)\right]$$
$$= \frac{1}{\tau}\left(e^{j2\pi f\tau} - 2 + e^{-j2\pi f\tau}\right) \qquad (2.111)$$

or, solving for $\Im[\Lambda(t/\tau)]$ and simplifying, we get

$$\Im\left[\Lambda\left(\frac{t}{\tau}\right)\right] = \frac{2\cos(2\pi f\tau) - 2}{\tau(j2\pi f)^2} = \tau\frac{\sin^2(\pi f\tau)}{(\pi f\tau)^2} \qquad (2.112)$$

Example 2.13 (4/4)

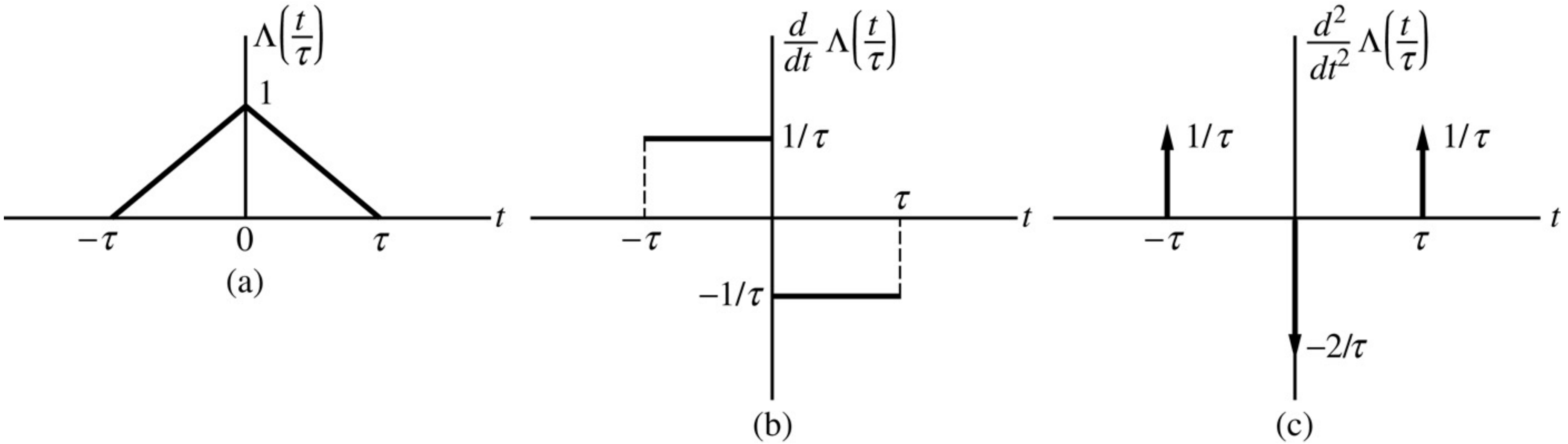
where the identity $\frac{1}{2}[1 - \cos(2\pi f t)] = \sin^2(\pi f t)$ has been used.

- Summarizing, we have shown that

$$\Lambda\left(\frac{t}{\tau}\right) \leftrightarrow \tau \mathrm{sinc}^2(f\tau) \qquad (2.113)$$

where $[\sin(\pi f \tau)]/(\pi f \tau)$ has been replaced by $\mathrm{sinc}(f\tau)$.

Figure 2.9



- Triangular signal and its first two derivatives. (a) Triangular signal. (b) First derivative of the triangular signal. (c) Second derivative of the triangular signal.

Example 2.14 (1/8)

- As another example of obtaining Fourier transform of signals involving impulses, let us consider the signal

$$y_s(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_s) \qquad (2.114)$$

- It is a periodic waveform referred to as the ideal sampling waveform and consists of a doubly infinite sequence of impulses spaced by T_s s.

Example 2.14 (2/8)

- Solution: To obtain the Fourier transform of $y_s(t)$, we note that it is periodic and, in a formal sense, therefore can be represented by a Fourier series.
- Thus,

$$y_s(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_s) = \sum_{n=-\infty}^{\infty} Y_n e^{jn2\pi f_s t},$$

$$f_s = \frac{1}{T_s} \tag{2.115}$$

Example 2.14 (3/8)

where

$$Y_n = \frac{1}{T_s} \int_{T_s} \delta(t) e^{-jn2\pi f_s t} dt = f_s \qquad (2.116)$$

by the sifting property of the impulse function.

- Therefore,

$$y_s(t) = f_s \sum_{n=-\infty}^{\infty} e^{jn2\pi f_s t} \qquad (2.117)$$

Example 2.14 (4/8)

- Fourier transforming term by term, we obtain

$$Y_s(f) = f_s \sum_{n=-\infty}^{\infty} \Im\left[1 \cdot e^{j2\pi n f_s t}\right]$$

$$= f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s) \qquad (2.118)$$

where we have used the results of Example 2.12.

Example 2.14 (5/8)

- Summarizing, we have shown that

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_s) \leftrightarrow f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \qquad (2.119)$$

- The transform pair (2.119) is useful in spectral representations of periodic signals by the Fourier transform, which will be considered shortly.

Example 2.14 (6/8)

- A useful expression can be derived from (2.119).
- Taking the Fourier transform of the left-hand side of (2.119) yields

$$\Im\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right]$$

$$= \int_{-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] e^{-j2\pi f t} dt$$

Example 2.14 (7/8)

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t - mT_s) e^{-j2\pi f t} dt$$

$$= \sum_{m=-\infty}^{\infty} e^{-j2\pi m T_s f} \qquad (2.120)$$

where we interchanged the orders of integration and summation and used the sifting property of the impulse function to perform the integration.

Example 2.14 (8/8)

- Replacing m by $-m$ and equating the result to the right-hand side of (2.119) gives

$$\sum_{m=-\infty}^{\infty} e^{j2\pi m T_s f} = f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s) \quad (2.121)$$

- This result will be used in Chapter 7.

Example 2.15 (1/3)

- The convolution theorem can be used to obtain the Fourier transform of the triangle signal $\Lambda(t/\tau)$ defined by (2.109).
- Solution: We proceed by first showing that the convolution of two rectangular pulses is a triangle signal.
- The steps in computing

$$y(t) = \int_{-\infty}^{\infty} \Pi\left(\frac{t-\lambda}{\tau}\right) \Pi\left(\frac{\lambda}{\tau}\right) d\lambda \qquad (2.122)$$

Example 2.15 (2/3)

are carried out in Table 2.2.

- Summarizing the results, we have

$$\tau\Lambda\left(\frac{t}{\tau}\right)=\Pi\left(\frac{t}{\tau}\right)*\Pi\left(\frac{t}{\tau}\right)=\begin{cases}0, & t<-\tau\\ \tau-|t|, & |t|\le\tau\\ 0, & t>\tau\end{cases} \quad (2.123)$$

or

$$\Lambda\left(\frac{t}{\tau}\right)=\frac{1}{\tau}\Pi\left(\frac{t}{\tau}\right)*\Pi\left(\frac{t}{\tau}\right) \quad (2.124)$$

Example 2.15 (3/3)

- Using the transform pair

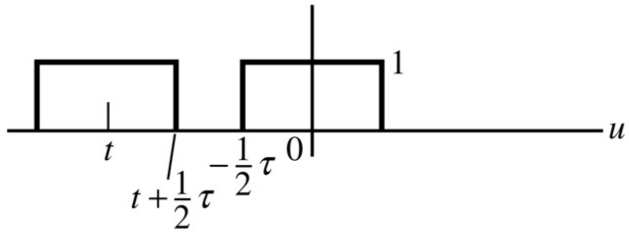
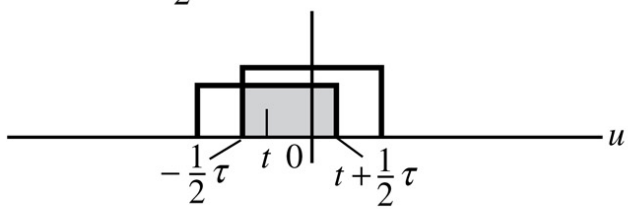
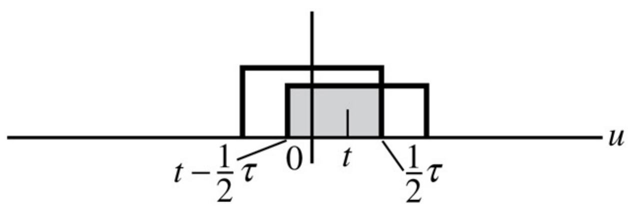
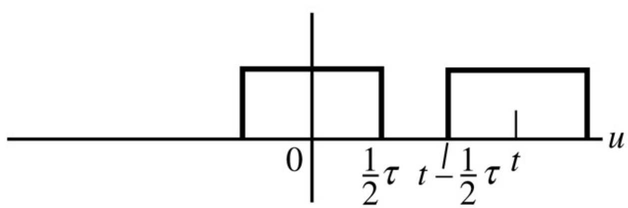
$$\Pi\left(\frac{t}{\tau}\right) \leftrightarrow \tau \operatorname{sinc}(f\tau) \tag{2.125}$$

and the convolution theorem of Fourier transforms (2.98), we obtain the transform pair

$$\Lambda\left(\frac{t}{\tau}\right) \leftrightarrow \tau \operatorname{sinc}^2(f\tau) \tag{2.126}$$

as in Example 2.13 by applying the differentiation theorem.

Table 2.2 Computation of $\Pi\left(\frac{t}{\tau}\right) * \Pi\left(\frac{t}{\tau}\right)$

Range	Integrand	Limits	Area
$-\infty < t < -\tau$			0
$-\tau < t < 0$		$-\frac{1}{2}\tau$ to $t+\frac{1}{2}\tau$	$\tau + t$
$0 < t < \tau$		$t-\frac{1}{2}\tau$ to $\frac{1}{2}\tau$	$\tau - t$
$\tau < t < \infty$			0

A Useful Result (1/2)

- A useful result is the convolution of an impulse $\delta(t - t_0)$ with a signal $x(t)$, where $x(t)$ is assumed continuous at $t = t_0$.
- Carrying out the operation, we obtain

$$\delta(t - t_0) * x(t)$$
$$= \int_{-\infty}^{\infty} \delta(\lambda - t_0) x(t - \lambda) d\lambda = x(t - t_0) \quad (2.127)$$

A Useful Result (2/2)

by the sifting property of the delta function.

- That is, convolution of $x(t)$ with an impulse occurring at time t_0 simply shifts $x(t)$ to t_0.

Example 2.16 (1/3)

- Consider the Fourier transform of the cosinusoidal pulse

$$x(t) = A\Pi\left(\frac{t}{\tau}\right)\cos(\omega_0 t)\,, \omega_0 = 2\pi f_0 \quad (2.128)$$

- Using the transform pair (see Example 2.12, item 4)

$$e^{\pm j2\pi f_0 t} \leftrightarrow \delta(f \mp f_0) \quad (2.129)$$

obtained earlier and Euler's theorem, we find that

Example 2.16 (2/3)

$$\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \quad (2.130)$$

- We have also shown that

$$A\Pi\left(\frac{t}{\tau}\right) \leftrightarrow A\tau\text{sinc}(f\tau)$$

- Therefore, using the multiplication theorem of Fourier transforms (2.102), we obtain

$$X(f) = \mathfrak{F}\left[A\Pi\left(\frac{t}{\tau}\right)\cos(\omega_0 t)\right]$$

Example 2.16 (3/3)

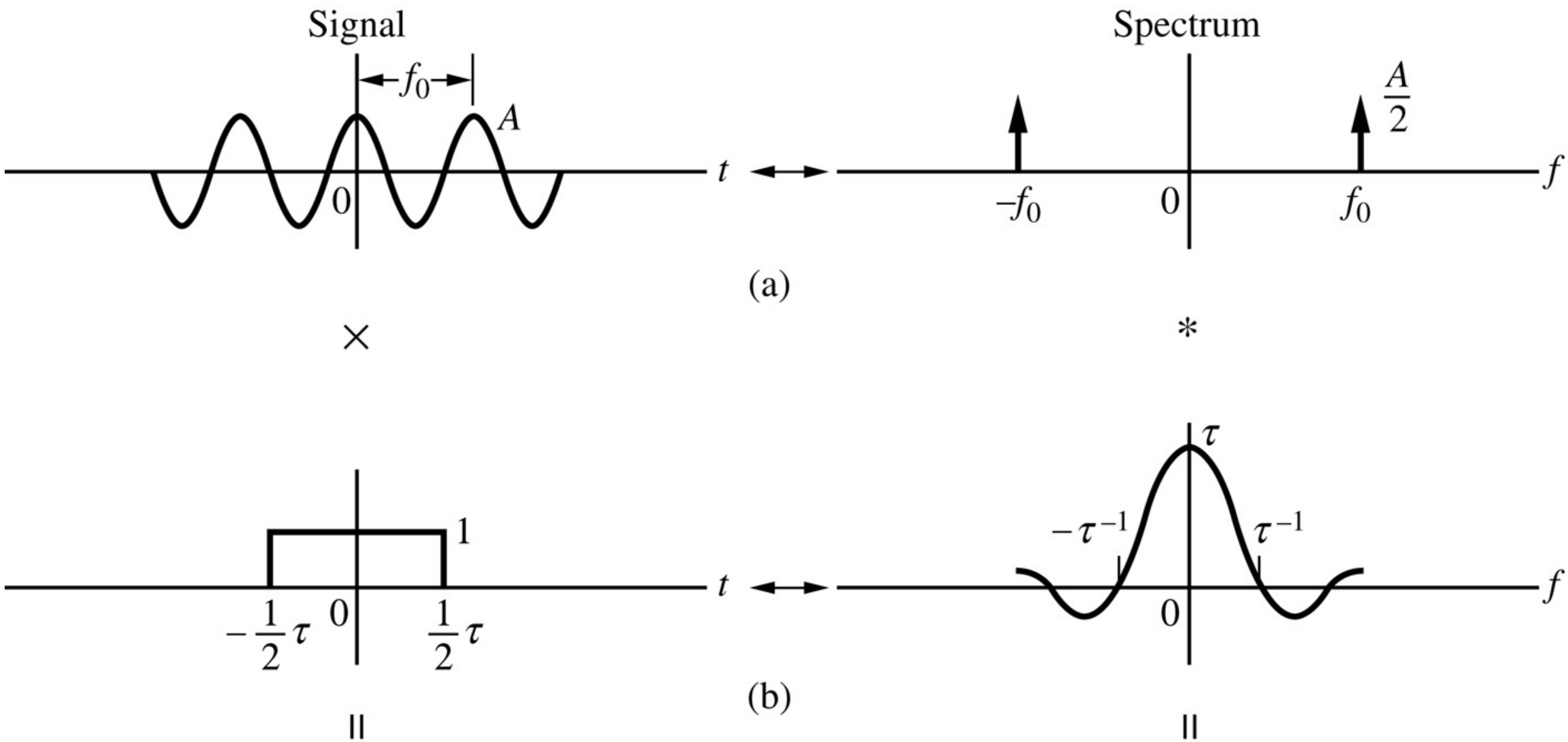
$$= [A\tau \,\mathrm{sinc}(f\tau)] * \left\{\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]\right\}$$

$$= \frac{1}{2} A\tau\{\mathrm{sinc}[(f - f_0)\tau] + \mathrm{sinc}[(f + f_0)\tau]\} \qquad (2.131)$$

where $\delta(f - f_0) * Z(f) = Z(f - f_0)$ for $Z(f)$ continuous at $f = f_0$ has been used.

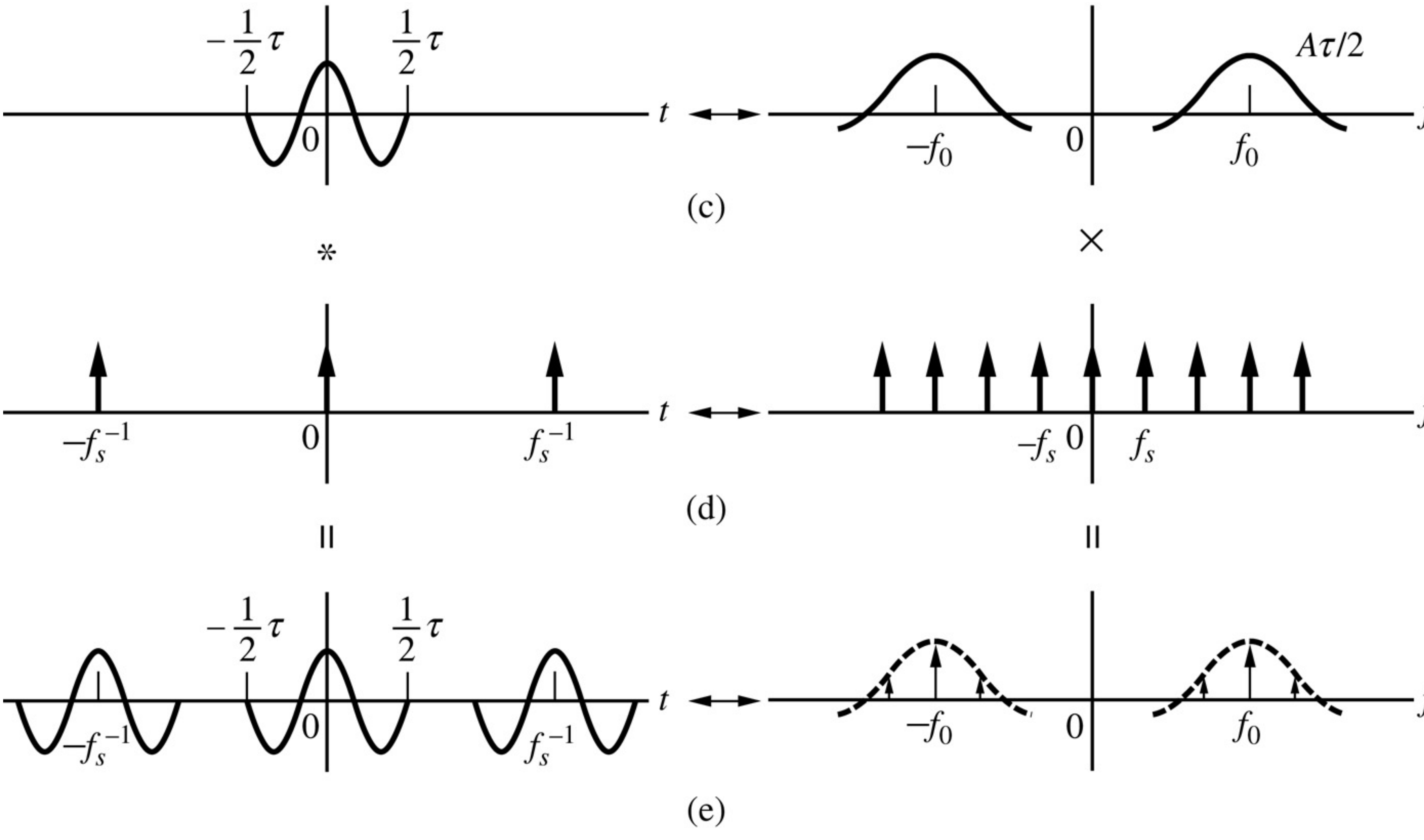
- Figure 2.10(c) shows $X(f)$.
- The same result can be obtained via the modulation theorem.

Figure 2.10 (a)(b)



(a)-(c) Application of the multiplication theorem. (c)-(e) Application of the convolution theorem.
Note: × denotes multiplication; ∗ denotes convolution, ↔ denotes transform pairs.

Figure 2.10 (c)-(e)



Fourier Transforms of Periodic Signals (1/6)

- The Fourier transform of a periodic signal, in a strict mathematical sense, does not exist, since periodic signals are not energy signals.
- However, using the transform pairs derived in Example 2.12 for a constant and a phasor signal, we could, in a formal sense, write down the Fourier transform of a periodic signal by Fourier transforming its complex Fourier series term by term.

Fourier Transforms of Periodic Signals (2/6)

- A somewhat more useful form for the Fourier transform of a periodic signal is obtained by applying the convolution theorem and the transform pair (2.119) for the ideal sampling waveform.
- To obtain it, consider the result of convolving the ideal sampling waveform with a pulse-type signal $p(t)$ to obtain a new signal $x(t)$, where $x(t)$ is a periodic power signal.

Fourier Transforms of Periodic Signals (3/6)

- This is apparent when one carries out the convolution with the aid of (2.127):

$$x(t) = \left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] * p(t)$$

$$= \sum_{m=-\infty}^{\infty} \delta(t - mT_s) * p(t)$$

$$= \sum_{m=-\infty}^{\infty} p(t - mT_s) \qquad (2.132)$$

Fourier Transforms of Periodic Signals (4/6)

- Applying the convolution theorem and the Fourier transform pair of (2.119), we find that the Fourier transform of $x(t)$ is

$$X(f) = \Im\left\{\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right\} P(f)$$

$$= \left[f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)\right] P(f)$$

Fourier Transforms of Periodic Signals (5/6)

$$= f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) P(f)$$

$$= \sum_{n=-\infty}^{\infty} f_s P(nf_s) \delta(f - nf_s) \qquad (2.133)$$

where $P(f) = \Im[p(t)]$ and the fact that $P(f)\delta(f - nf_s) = P(nf_s)\delta(f - nf_s)$ has been used.

Fourier Transforms of Periodic Signals (6/6)

- Summarizing, we have obtained the Fourier transform pair

$$\sum_{m=-\infty}^{\infty} p(t - mT_s) \leftrightarrow \sum_{n=-\infty}^{\infty} f_s P(nf_s)\delta(f - nf_s) \tag{2.134}$$

- The usefulness of (2.134) is illustrated with an example.

Example 2.17 (1/4)

- The Fourier transform of a single cosinusoidal pulse was found in Example 2.16 and is shown in Figure 2.10(c).
- The Fourier transform of a periodic cosinusoidal pulse train, which could represent the output of a radar transmitter, for example, is obtained by writing it as

Example 2.17 (2/4)

$$y(t)$$

$$= \left[\sum_{n=-\infty}^{\infty} \delta(t - mT_s) \right] * A\Pi\left(\frac{t}{\tau}\right)\cos(2\pi f_0 t),$$

$$f_0 \gg 1/\tau$$

$$= \sum_{m=-\infty}^{\infty} A\Pi\left(\frac{t - mT_s}{\tau}\right)\cos[2\pi f_0 (t - mT_s)],$$

$$f_s \leq \tau^{-1} \qquad (2.135)$$

Example 2.17 (3/4)

- This signal is illustrated in Figure 2.10(e).
- Identifying $p(t) = A\Pi\left(\frac{t}{\tau}\right)\cos(2\pi f_0 t)$, we get, by the modulation theorem, that $P(f) = \frac{A\tau}{2}[\mathrm{sinc}(f - f_0)\tau + \mathrm{sinc}(f + f_0)\tau]$.
- Applying (2.134), the Fourier transform of $y(t)$ is

Example 2.17 (4/4)

$$Y(f) = \sum_{n=-\infty}^{\infty} \frac{Af_s\tau}{2} [\mathrm{sinc}(nf_s - f_0)\tau + \mathrm{sinc}(nf_s + f_0)\tau]\delta(f - nf_s) \quad (2.136)$$

- The spectrum is illustrated on the right-hand side of Figure 2.10(e).

Poisson Sum Formula (1/3)

- We can develop the Poisson sum formula by taking the inverse Fourier transform of the right-hand side of (2.134).
- When we use the transform pair $\exp(j2\pi n f_s t) \leftrightarrow \delta(f - nf_s)$ (see Example 2.12), it follows that

$$\Im^{-1}\left\{\sum_{n=-\infty}^{\infty} f_s P(nf_s)\delta(f - nf_s)\right\}$$

$$= f_s \sum_{n=-\infty}^{\infty} P(nf_s)e^{j2\pi n f_s t} \qquad (2.137)$$

Poisson Sum Formula (2/3)

- Equating this to the left-hand side of (2.134), we obtain the Poisson sum formula:

$$\sum_{m=-\infty}^{\infty} p(t - mT_s) = f_s \sum_{n=-\infty}^{\infty} P(nf_s)e^{j2\pi n f_s t} \tag{2.138}$$

- The Poisson sum formula is useful when one goes from the Fourier transform to sampled approximations of it.

Poisson Sum Formula (3/3)

- For example, Equation (2.138) says that the sample values $P(nf_s)$ of $P(f) = \Im\{p(t)\}$ are the Fourier series coefficients of the periodic function $T_s \sum_{n=-\infty}^{\infty} p(t - mT_s)$.

Power Spectral Density and Correlation (1/3)

- Recalling the definition of energy spectral density (2.73), we see that it is of use only for energy signals for which the integral of $G(f)$ over all frequencies give total energy, a finite quantity.
- For power signals, it is meaningful to speak in terms of power spectral density.

Power Spectral Density and Correlation (2/3)

- Analogous to $G(f)$, we define the power spectral density $S(f)$ of a signal $x(t)$ as a real, even, nonnegative function of frequency, which gives total average power per ohm when integrated; that is

$$P = \int_{-\infty}^{\infty} S(f)df = \langle x^2(t) \rangle \qquad (2.139)$$

where $\langle x^2(t) \rangle = \lim_{T\to\infty} (1/2T) \int_{-T}^{T} x^2(t)dt$ denotes the time average of $x^2(t)$.

Power Spectral Density and Correlation (3/3)

- Since $S(f)$ is a function that gives the variation of density of power with frequency, we conclude that it must consist of a series of impulses for the periodic power signals that we have so far considered.
- Later, in Chapter 7, we will consider power spectra of random signals.

Example 2.18 (1/2)

- Considering the cosinusoidal signal

$$x(t) = A\cos(2\pi f_0 t + \theta) \tag{2.140}$$

  we note that its average power per ohm, $\frac{1}{2}A^2$, is concentrated at the single frequency f_0 Hz.
- However, since the power spectral density must be an even function of frequency, we split this power equally between $+f_0$ and $-f_0$ hertz.

Example 2.18 (2/2)

- Thus, the power spectral density of $x(t)$ is, from intuition, given by
$$S(f) = \frac{1}{4}A^2\delta(f - f_0) + \frac{1}{4}A^2\delta(f + f_0) \quad (2.141)$$
- Checking this by using (2.139), we see that integration over all frequencies results in the average power per ohm of $\frac{1}{2}A^2$.

The Time-Average Autocorrelation Function (1/10)

- To introduce the time-average autocorrelation function, we return to the energy spectral density of an energy signal (2.73).
- Without any apparent reason, suppose we take the inverse Fourier transform of $G(f)$, letting the independent variable be τ:

$$\begin{aligned}\phi(\tau) &\triangleq \mathfrak{I}^{-1}[G(f)] = \mathfrak{I}^{-1}[X(f)X^*(f)] \\ &= \mathfrak{I}^{-1}[X(f)] * \mathfrak{I}^{-1}[X^*(f)] \qquad (2.142)\end{aligned}$$

The Time-Average Autocorrelation Function (2/10)

- The last step follows by application of the convolution theorem.
- Applying the time-reversal theorem (item 3b in Table F.6 in Appendix F) to write $\mathfrak{F}^{-1}[X^*(f)] = x(-\tau)$ and then the convolution theorem, we obtain

$$\phi(\tau) = x(\tau) * x(-\tau) = \int_{-\infty}^{\infty} x(\lambda)x(\lambda+\tau)d\lambda$$

The Time-Average Autocorrelation Function (3/10)

$$= \lim_{T\to\infty} \int_{-T}^{T} x(\lambda)x(\lambda+\tau)d\lambda \text{ (energy signal)} \tag{2.143}$$

- Equation (2.143) will be referred to as the time-average autocorrelation function for energy signals.
- We see that it gives a measure of the similarity, or coherence, between a signal and a delayed version of the signal.

The Time-Average Autocorrelation Function (4/10)

- Note that $\phi(0) = E$, the signal energy.
- Also note the autocorrelation function and energy spectral density are Fourier transform pairs.
- We forgo further discussion of the time-average autocorrelation function for energy signals in favor of analogous results for power signals.

The Time-Average Autocorrelation Function (5/10)

- The time-average autocorrelation function $R(\tau)$ of a power signal $x(t)$ is defined as the time average

$$R(\tau) = \langle x(t)x(t+\tau)\rangle$$
$$\triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt \qquad (2.144)$$

(power signal)

The Time-Average Autocorrelation Function (6/10)

- If $x(t)$ is periodic with period T_0, the integrand of (2.144) is periodic, and the time average can be taken over a single period:

$$R(\tau) = \frac{1}{T_0}\int_{T_0} x(t)x(t+\tau)dt$$

$[x(t)$ periodic$]$

The Time-Average Autocorrelation Function (7/10)

- Just like $\phi(\tau)$, $R(\tau)$ gives a measure of the similarity between a power signal at time t and at time $t + \tau$.
- It is a function of the delay variable τ, since time t is the variable of integration.
- In addition to being a measure of the similarity between a signal and its time displacement, we note that the total power of the signal is

The Time-Average Autocorrelation Function (8/10)

$$R(0) = \langle x^2(t) \rangle \triangleq \int_{-\infty}^{\infty} S(f)df \qquad (2.145)$$

- Thus we suspect that the time-average autocorrelation function and power spectral density of a power signal are closely related, just as they are for energy signals.

The Time-Average Autocorrelation Function (9/10)

- This relationship is stated formally by the Wiener-Khinchine theorem, which says that the time-average autocorrelation function of a signal and its power spectral density are Fourier transform pairs:

$$S(f) = \Im[R(\tau)] = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi f\tau}\, d\tau \tag{2.146}$$

and

The Time-Average Autocorrelation Function (10/10)

$$R(\tau) = \mathfrak{F}^{-1}[S(f)] = \int_{-\infty}^{\infty} S(f) e^{j2\pi f\tau} df \tag{2.147}$$

- A formal proof of the Wiener-Khinchine theorem will be given in Chapter 7.
- We simply take (2.146) as the definition of power spectral density at this point.
- We note that (2.145) follows immediately from (2.147) by setting $\tau = 0$.

Properties of $R(\tau)$ (1/3)

- The time-average autocorrelation function has several useful properties, which are listed below:

1. $R(0) = \langle x^2(t) \rangle \geq |R(\tau)|$, for all τ; that is, a relative maximum of $R(\tau)$ exists at $\tau = 0$.
2. $R(-\tau) = \langle x(t)x(t-\tau) \rangle = R(\tau)$; that is, $R(\tau)$ is even.

Properties of $R(\tau)$ (2/3)

3. $\lim_{|\tau|\to\infty} R(\tau) = \langle x(t)\rangle^2$ if $x(t)$ does not contain periodic components.
4. If $x(t)$ is periodic in t with period T_0, then $R(\tau)$ is periodic in τ with period T_0.
5. The time-average autocorrelation function of any power signal has a Fourier transform that is nonnegative.

Properties of $R(\tau)$ (3/3)

- Property 5 results by virtue of the fact that normalized power is a nonnegative quantity.
- These properties will be proved in Chapter 7.
- The autocorrelation function and power spectral density are important tools for systems analysis involving random signals.

Example 2.19

- We skip this example because it is too long.

Example 2.20

- We skip this example because it is too long.

Signals and Linear Systems (1/6)

- Definition of a Linear Time-Invariant System
- Impulse Response and the Superposition Integral
- Stability
- Transfer (Frequency-Response) Function
- Causality
- Symmetry Properties of $H(f)$

Signals and Linear Systems (2/6)

- Input-Output Relationships for Spectral Densities
- Response to Periodic Inputs
- Distortionless Transmission
- Group and Phase Delay
- Nonlinear Distortion
- Ideal Filters

Signals and Linear Systems (3/6)

- Approximation of Ideal Lowpass Filters by Realizable Filters
- Relationship of Pulse Resolution and Risetime to Bandwidth

Signals and Linear Systems (4/6)

- In this section we are concerned with the characterization of systems and their effects on signals.
- In system modeling, the actual elements, such as resistors, capacitors, inductors, springs, and masses, that compose a particular system are usually not concern.

Signals and Linear Systems (5/6)

- Rather, we view a system in terms of the operation it performs on an input to produce an output.
- Symbolically, for a single-input, single-output system, this is accomplished by writing

$$y(t) = \mathcal{H}[x(t)] \tag{2.150}$$

where $\mathcal{H}[\cdot]$ is the operator that produces the output $y(t)$ from the input $x(t)$, as illustrated in Figure 2.12.

Signals and Linear Systems (6/6)

- We now consider certain classes of systems, the first of which is linear time-invariant systems.

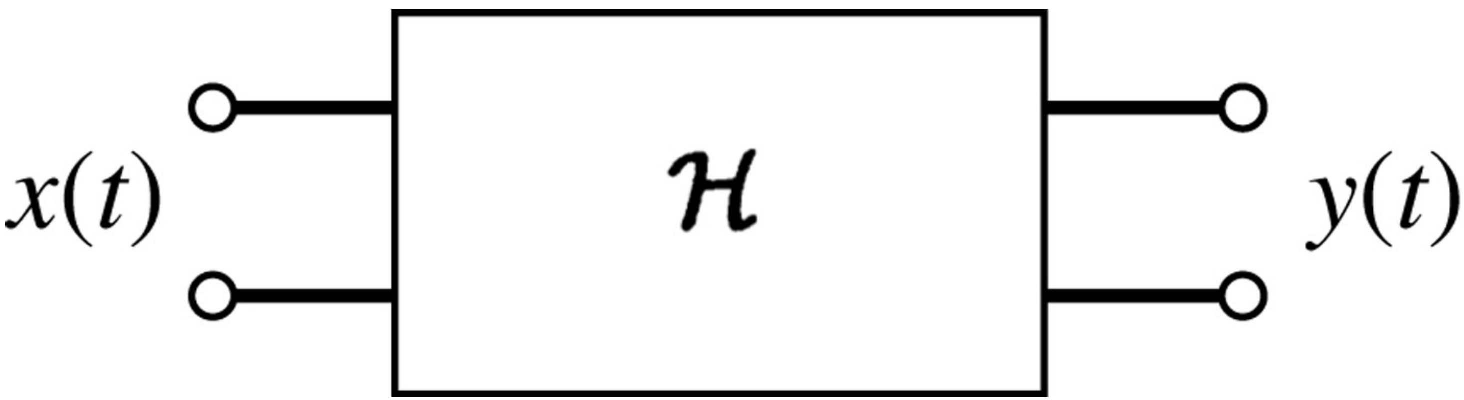


Figure 2.12 Operator representation of a linear system.

Definition of a Linear Time-Invariant System (1/2)

- If a system is linear, superposition holds.
- That is, if $x_1(t)$ results in the output $y_1(t)$ and $x_2(t)$ results in the output $y_2(t)$, then the output due to $\alpha_1 x_1(t) + \alpha_2 x_2(t)$, where α_1 and α_2 are constants, is given by

$$\begin{aligned} y(t) &= \mathcal{H}[\alpha_1 x_1(t) + \alpha_2 x_2(t)] \\ &= \alpha_1 \mathcal{H}[x_1(t)] + \alpha_2 \mathcal{H}[x_2(t)] \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned} \qquad (2.151)$$

Definition of a Linear Time-Invariant System (2/2)

- If the system is time invariant, or fixed, the delayed input $x(t - t_0)$ gives the delayed output $y(t - t_0)$; that is

$$y(t - t_0) = \mathcal{H}[x(t - t_0)] \qquad (2.152)$$

- With these properties explicitly stated, we are now ready to obtain more concrete descriptions of linear time-invariant (LTI) systems.

Impulse Response and the Superposition Integral (1/10)

- The impulse response $h(t)$ of an LTI system is defined to be the response of the system to an impulse applied at $t = 0$, that is

$$h(t) \triangleq \mathcal{H}[\delta(t)] \tag{2.153}$$

Impulse Response and the Superposition Integral (2/10)

- By the time-invariant property of the system, the response to an impulse applied at any time t_0 is $h(t - t_0)$, and the response to the linear combination of impulses $\alpha_1 \delta(t - t_1) + \alpha_2 \delta(t - t_2)$ is $\alpha_1 h(t - t_1) + \alpha_2 h(t - t_2)$ by the superposition property and time invariance.

Impulse Response and the Superposition Integral (3/10)

- Through induction, we may therefore show that the response to the input

$$x(t) = \sum_{n=1}^{N} \alpha_n \delta(t - t_n) \tag{2.154}$$

is

$$y(t) = \sum_{n=1}^{N} \alpha_n h(t - t_n) \tag{2.155}$$

Impulse Response and the Superposition Integral (4/10)

- We will use (2.155) to obtain the superposition integral, which expresses the response of an LTI system to an arbitrary input (with suitable restrictions) in terms of the impulse response of the system.
- Considering the arbitrary input signal $x(t)$ of Figure 2.13(a), we can represent it as

$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda \qquad (2.156)$$

Impulse Response and the Superposition Integral (5/10)

by the sifting property of the unit impulse.

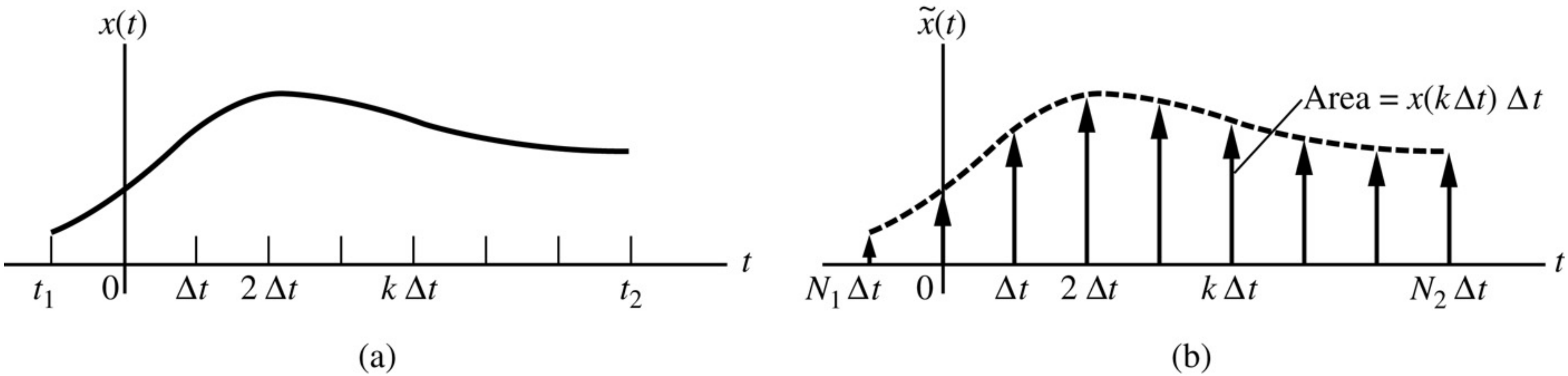


Figure 2.13 A signal and an approximate representation. (a) Signal. (b) Approximation with a sequence of impulses.

Impulse Response and the Superposition Integral (6/10)

- Approximating the integral of (2.156) as a sum, we obtain

$$x(t) \cong \sum_{n=N_1}^{N_2} x(n\,\Delta t)\delta(t - n\Delta t)\Delta t\,, \Delta t \ll 1 \quad (2.157)$$

where $t_1 = N_1\Delta t$ is the starting time of the signal and $t_2 = N_2\Delta t$ is the ending time.

Impulse Response and the Superposition Integral (7/10)

- The output, using (2.155) with $\alpha_n = x(n\Delta t)\Delta t$ and $t_n = n\Delta t$, is

$$\tilde{y}(t) = \sum_{n=N_1}^{N_2} x(n\,\Delta t)h(t - n\,\Delta t)\,\Delta t \qquad (2.158)$$

where the tilde denotes the output resulting from the approximation to the input given by (2.157).

Impulse Response and the Superposition Integral (8/10)

- In the limit as Δt approaches $d\lambda$ and $n\,\Delta t$ approaches the continuous variable λ, the sum becomes an integral, and we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \qquad (2.159)$$

where the limits have been changed to $\pm\infty$ to allow arbitrary starting and ending times for $x(t)$.

Impulse Response and the Superposition Integral (9/10)

- Making the substitution $\sigma = t - \lambda$, we obtain the equivalent result

$$y(t) = \int_{-\infty}^{\infty} x(t-\sigma)h(\sigma)d\sigma \qquad (2.160)$$

- Because these equations were obtained by superposition of a number of elementary responses due to each individual impulse, they are referred to as superposition integrals.

Impulse Response and the Superposition Integral (10/10)

- A simplification results if the system under consideration is causal, that is, is a system that does not respond before an input is applied.
- For a causal system, $h(t-\lambda)=0$ for $t<\lambda$, and the upper limit (2.159) can be set equal to t.
- Furthermore, if $x(t)=0$ for $t<0$, the lower limit becomes zero.

Stability

- A fixed, linear system is bounded-input, bounded-output (BIBO) stable if every bounded input results in a bounded output.
- It can be shown that a system is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty \qquad (2.161)$$

Transfer (Frequency-Response) Function (1/3)

- Applying the convolution theorem of Fourier transforms (item 8 of Table F.6 in Appendix F) to either (2.159) or (2.160), we obtain

$$Y(f) = H(f)X(f) \tag{2.162}$$

where $X(f) = \mathfrak{I}\{x(t)\}$, $Y(f) = \mathfrak{I}\{y(t)\}$, and

$$H(f) = \mathfrak{I}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \tag{2.163}$$

or

Transfer (Frequency-Response) Function (2/3)

$$h(t) = \mathfrak{I}^{-1}\{H(f)\} = \int_{-\infty}^{\infty} H(f) e^{j2\pi f t} df \quad (2.164)$$

- $H(f)$ is referred to as the transfer (frequency-response) function of the system.
- We see that either $h(t)$ or $H(f)$ is an equally good characterization of the system.
- By an inverse Fourier transform on (2.162), the output becomes

Transfer (Frequency-Response) Function (3/3)

$$y(t) = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df \qquad (2.165)$$

Causality (1/6)

- A system is causal if it does not anticipate the input.
- In terms of the impulse response, it follows that for a time-invariant causal system,

$$h(t) = 0, t < 0. \tag{2.166}$$

- When causality is viewed from the standpoint of the frequency-response function of the system, a celebrated theorem by Wiener and Paley states that if

Causality (2/6)

$$\int_{-\infty}^{\infty} |h(t)|^2 \, dt = \int_{-\infty}^{\infty} |H(f)|^2 df < \infty \quad (2.167)$$

with $h(t) \equiv 0$ for $t < 0$, it is then necessary that

$$\int_{-\infty}^{\infty} \frac{|\ln|H(f)||}{1 + f^2} df < \infty. \quad (2.168)$$

Causality (3/6)

- Conversely, if $|H(f)|$ is square integrable and if the integral in (2.168) is unbounded, then we cannot make $h(t) \equiv 0, t < 0$ no matter what we choose for $\underline{/H(f)}$.
- Consequences of (2.168) are that no filter can have $|H(f)| \equiv 0$ over a finite band of frequencies (i.e., a filter cannot perfectly reject any band of frequencies).

Causality (4/6)

- In fact, the Paley-Wiener criterion restricts the rate at which $|H(f)|$ for a linear causal time-invariant system can vanish.
- For example,

$$|H_1(f)| = e^{-k_1|f|} \Rightarrow |\ln|H_1(f)|| = k_1|f| \quad (2.169)$$

and

$$|H_2(f)| = e^{-k_2 f^2} \Rightarrow |\ln|H_2(f)|| = k_2 f^2 \quad (2.170)$$

Causality (5/6)

where k_1 and k_2 are positive constants, are not allowable amplitude responses for causal filters because (2.168) does not give a finite result in either case.

- The sufficiency statement of the Paley-Wiener criterion is stated as follows:

Causality (6/6)

- Given any square-integrable function $|H(f)|$ for which (2.168) is satisfied, there exists an $\underline{/H(f)}$ such that $H(f) = |H(f)| \exp\left[j\underline{/H(f)}\right]$ is the Fourier transform of $h(t)$ for a causal filter.

Symmetry Properties of $H(f)$ (1/3)

- The frequency response function of an LTI system $H(f)$ is, in general, a complex quantity.
- We therefore write it in terms of magnitude and argument as

$$H(f) = |H(f)| \exp\left[j \underline{/H(f)}\right] \tag{2.171}$$

where $|H(f)|$ is called the amplitude- (magnitude-) response function and $\underline{/H(f)}$ is called the phase-response function of the LTI system.

Symmetry Properties of $H(f)$ (2/3)

- Also, $H(f)$ is the Fourier transform of a real-time function $h(t)$.
- Therefore, it follows that

$$|H(f)| = |H(-f)| \tag{2.172}$$

and

$$\underline{/H(f)} = -\underline{/H(-f)} \tag{2.173}$$

Symmetry Properties of $H(f)$ (3/3)

- That is, the amplitude response of a system with real-valued impulse response is an even function of frequency and its phase response is an odd function of frequency.

Example 2.22

- We skip this example because it is too long.

Input-Output Relationships for Spectral Densities (1/2)

- Consider a fixed linear two-port system with frequency-response function $H(f)$, input $x(t)$, and output $y(t)$.
- If $x(t)$ and $y(t)$ are energy signals, their energy spectral densities are $G_x(f) = |X(f)|^2$ and $G_y(f) = |Y(f)|^2$, respectively.
- Since $Y(f) = H(f)X(f)$, it follows that

$$G_y(f) = |H(f)|^2 G_x(f) \tag{2.183}$$

Input-Output Relationships for Spectral Densities (2/2)

- A similar relationship holds for power signals and spectra:

$$S_y(f) = |H(f)|^2 S_x(f) \qquad (2.184)$$

- This will be proved in Chapter 7.

Response to Periodic Inputs (1/5)

- Consider the steady-state response of a fixed linear system to the complex exponential input signal $Ae^{j2\pi f_0 t}$.
- Using the superposition integral, we obtain

$$y_{ss}(t) = \int_{-\infty}^{\infty} h(\lambda) A e^{j2\pi f_0 (t-\lambda)} d\lambda$$

$$= Ae^{j2\pi f_0 t} \int_{-\infty}^{\infty} h(\lambda) e^{-j2\pi f_0 \lambda} d\lambda$$

$$= H(f_0) A e^{j2\pi f_0 t} \tag{2.185}$$

Response to Periodic Inputs (2/5)

- That is, the output is a complex exponential signal of the same frequency but with amplitude scaled by $|H(f_0)|$ and phase shifted by $\underline{/H(f_0)}$ relative to the amplitude and phase of the input.

Response to Periodic Inputs (3/5)

- Using superposition, we conclude that the steady-state output due to an arbitrary periodic input is represented by the complex exponential Fourier series

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(nf_0) e^{jn2\pi f_0 t} \quad (2.186)$$

Response to Periodic Inputs (4/5)

or

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n||H(nf_0)| \exp\left\{j[2\pi n f_0 t + /\underline{X_n} + /\underline{H(nf_0)}]\right\}$$

$$= X_0 H(0) + 2\sum_{n=1}^{\infty} |X_n||H(nf_0)| \cos\left[2\pi n f_0 t + /\underline{X_n} + /\underline{H(nf_0)}\right] \quad (2.188)$$

(2.187)

where (2.172) and (2.173) have been used to get the second equation.

Response to Periodic Inputs (5/5)

- Thus, for a periodic input, the magnitude of each spectral component of the input is attenuated (or amplified) by the amplitude-response function at the frequency of the particular spectral component, and the phase of each spectral component is shifted by the value of the phase-shift function of the system at the frequency of the particular spectral component.

Example 2.23 (1/3)

- Consider the response of a filter having the frequency-response function

$$H(f) = 2\Pi\left(\frac{f}{42}\right)e^{-j\pi f/10} \qquad (2.189)$$

to a unit-amplitude triangular signal with period 0.1s.

- From Table 2.1 and (2.29), the exponential Fourier series of the input signal is

$$x(t) = \cdots \frac{4}{25\pi^2}e^{-j100\pi t} + \frac{4}{9\pi^2}e^{-j60\pi t} + \frac{4}{\pi^2}e^{-j20\pi t} + \frac{4}{\pi^2}e^{j20\pi t} + \frac{4}{9\pi^2}e^{j60\pi t} + \frac{4}{25\pi^2}e^{j100\pi t} + \cdots$$

$$= \frac{8}{\pi^2}\left[\cos(20\pi t) + \frac{1}{9}\cos(60\pi t) + \frac{1}{25}\cos(100\pi t) + \cdots\right] \qquad (2.190)$$

Example 2.23 (2/3)

- The filter eliminates all harmonics above 21 Hz and passes all those below 21 Hz, imposing an amplitude scale factor of 2 and a phase shift of $-\pi f/10$ rad.
- The only harmonic of the triangular wave to be passed by the filter is the fundamental, which has a frequency of 10 Hz, giving a phase shift of $-\pi(10)/10 = -\pi$ rad.

Example 2.23 (3/3)

- The output is therefore

$$y(t) = \frac{16}{\pi^2}\cos\left[20\pi\left(t - \frac{1}{20}\right)\right] \quad (2.191)$$

where the phase shift is seen to be equivalent to a delay of $\frac{1}{20}$ s.

Distortionless Transmission (1/5)

- Equation (2.188) shows that both the amplitudes and phases of the spectral components of a periodic input signal will, in general, be altered as the signal is sent through a two-port LTI system.
- This modification may be desirable in signal processing applications, but it amounts to distortion in signal transmission applications.

Distortionless Transmission (2/5)

- While it may appear at first that ideal signal transmission results only if there is no attenuation and phase shift of the spectral components of the input, this requirement is too stringent.
- A system will be classified as distortionless if it introduces the same attenuation and time delay to all spectral components of the input, for then the output looks like the input .

Distortionless Transmission (3/5)

- In particular, if the output of a system is given in terms of the input as
$$y(t) = H_0 x(t - t_0) \qquad (2.192)$$
where H_0 and t_0 are constants, the output is a scaled, delayed replica of the input ($t_0 > 0$ for causality).

Distortionless Transmission (4/5)

- Employing the time-delay theorem to Fourier transform (2.85) and using the definition $H(f) = Y(f)/X(f)$, we obtain

$$H(f) = H_0 e^{-j2\pi f t_0} \qquad (2.193)$$

as the frequency-response function of a distortionless system; that is, the amplitude response of a distortionless system is constant, and the phase shift is linear with frequency.

Distortionless Transmission (5/5)

- Of course, these restrictions are necessary only within the frequency ranges where the input has significant spectral content.
- Figure 2.17 and Example 2.24, considered shortly, will illustrate these comments.

Distortion (1/3)

- In general, we can isolate three types of distortion.
- First, if the system is linear but the amplitude response is not constant with frequency, the system is said to introduce amplitude distortion.
- Second, if the system is linear but the phase shift is not a linear function of frequency, the system introduces phase, or delay distortion.
- Third, if the system is not linear, we have nonlinear distortion.

Distortion (2/3)

- Of course, these three types of distortion may occur in combination with one another.

Distortion (3/3)

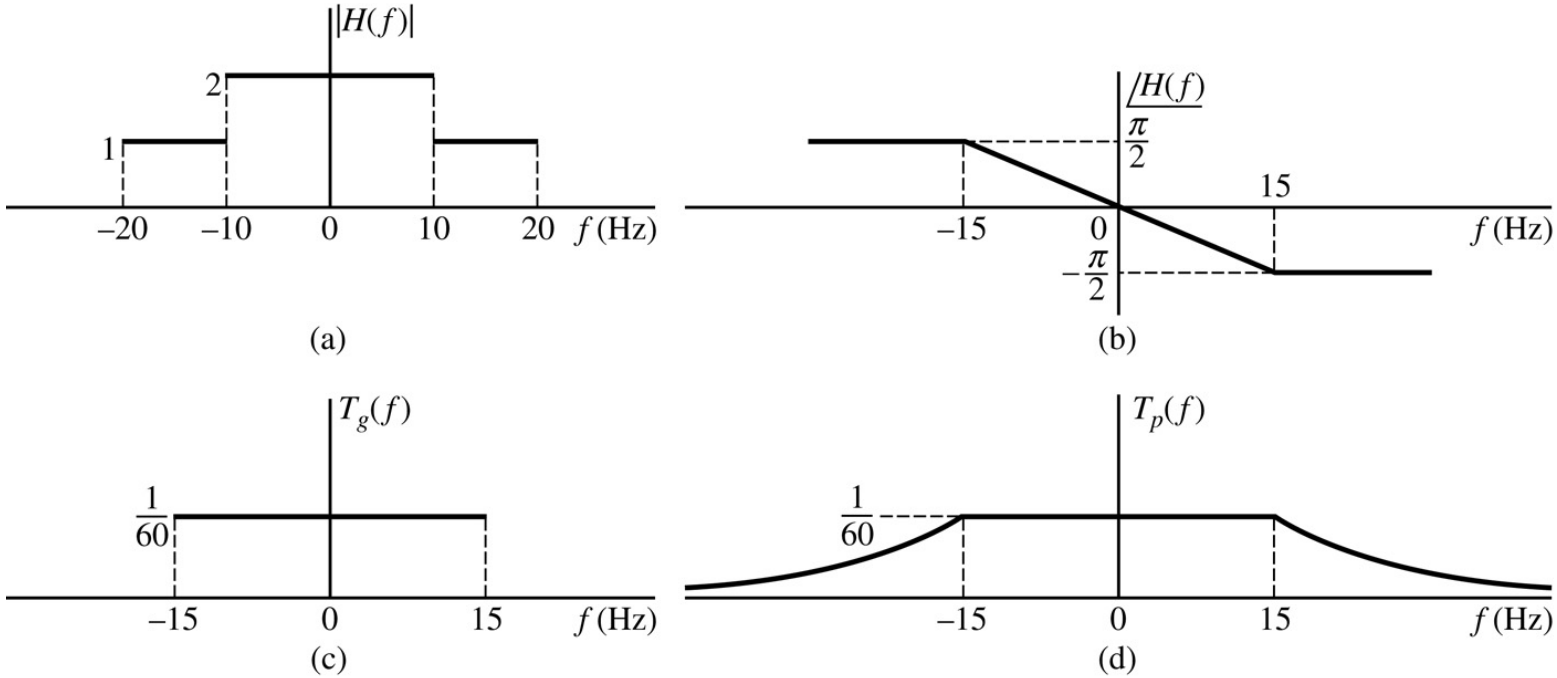


Figure 2.17
Amplitude and phase response and group and phase delays of the filter for Example 2.24. (a) Amplitude response. (b) Phase response. (c) Group delay. (d) Phase delay.

Group and Phase Delay (1/7)

- One can often identify phase distortion in a linear system by considering the derivative of phase with respect to frequency.
- A distortionless system exhibits a phase response in which phase is directly proportional to frequency.
- Thus the derivative of phase-response function with respect to frequency of a distortionless system is a constant.

Group and Phase Delay (2/7)

- The negative of this constant is called the group delay of the LTI system.
- In other words, the group delay is defined by the equation

$$T_g(f) = -\frac{1}{2\pi}\frac{d\theta(f)}{df} \qquad (2.194)$$

in which $\theta(f)$ is the phase response of the system.

Group and Phase Delay (3/7)

- For a distortionless system, the phase-response function is given by (2.193) as

$$\theta(f) = -2\pi f t_0. \qquad (2.195)$$

- This yields a group delay of

$$T_g(f) = -\frac{1}{2\pi}\frac{d}{df}(-2\pi f t_0)$$

or

$$T_g(f) = t_0. \qquad (2.196)$$

Group and Phase Delay (4/7)

- This confirms the preceding observation that the group delay of a distortionless LTI system is a constant.
- Group delay is the delay that a group of two or more frequency component undergo in passing through a linear system.
- If a linear system has a single-frequency component as the input, the system is always distortionless, since the output can be written as an amplitude-scaled and phase-shifted (time-delay) version of the input.

Group and Phase Delay (5/7)

- As an example, assume that the input to a linear system is given by
$$x(t) = A\cos(2\pi f_1 t). \qquad (2.197)$$
- It follows from (2.188) that the output can be written as
$$y(t) = A|H(f_1)|\cos[2\pi f_1 t + \theta(f_1)] \qquad (2.198)$$
where $\theta(f_1)$ is the phase response of the system evaluated at $f = f_1$.

Group and Phase Delay (6/7)

- Equation (2.198) can be written as

$$y(t) = A|H(f_1)|\cos\left\{2\pi f_1\left[t + \frac{\theta(f_1)}{2\pi f_1}\right]\right\}. \quad (2.199)$$

- The delay of the single component is defined as the phase delay:

$$T_P(f) = -\frac{\theta(f)}{2\pi f}. \quad (2.200)$$

- Thus (2.199) can be written as

$$y(t) = A|H(f_1)|\cos\{2\pi f_1[t - T_P(f_1)]\}. \quad (2.201)$$

Group and Phase Delay (7/7)

- Use of (2.195) shows that for a distortionless system, the phase delay is given by

$$T_P(f) = -\frac{1}{2\pi f}(-2\pi f t_0) = t_0 \qquad (2.202)$$

- Thus, we see that distortionless systems have equal group and phase delays.
- The following example should clarify the preceding definitions.

Example 2.24 (1/5)

- Consider a system with amplitude response and phase shift as shown in Figure 2.17 and the following four inputs:

  1. $x_1(t) = \cos(10\pi t) + \cos(12\pi t)$

  2. $x_2(t) = \cos(10\pi t) + \cos(26\pi t)$

  3. $x_3(t) = \cos(26\pi t) + \cos(34\pi t)$

  4. $x_4(t) = \cos(32\pi t) + \cos(34\pi t)$

Example 2.24 (2/5)

- Although this system is somewhat unrealistic from a practical standpoint, we can use it to illustrate various combinations of amplitude and phase distortion.
- Using (2.188) and superposition, we obtain the following corresponding outputs:

1. $y_1(t) = 2\cos\left(10\pi t - \frac{1}{6}\pi\right) + 2\cos(12\pi t - \frac{1}{5}\pi)$

$$= 2\cos\left[10\pi\left(t - \frac{1}{60}\right)\right] + 2\cos\left[12\pi\left(t - \frac{1}{60}\right)\right]$$

Example 2.24 (3/5)

2. $y_2(t) = 2\cos\left(10\pi t - \frac{1}{6}\pi\right) + \cos(26\pi t - \frac{13}{30}\pi)$

$$= 2\cos\left[10\pi\left(t - \frac{1}{60}\right)\right] + \cos\left[26\pi\left(t - \frac{1}{60}\right)\right]$$

3. $y_3(t) = \cos\left(26\pi t - \frac{13}{30}\pi\right) + \cos\left(34\pi t - \frac{1}{2}\pi\right)$

$$= \cos\left[26\pi\left(t - \frac{1}{60}\right)\right] + \cos\left[34\pi\left(t - \frac{1}{68}\right)\right]$$

4. $y_4(t) = \cos\left(32\pi t - \frac{1}{2}\pi\right) + \cos\left(34\pi t - \frac{1}{2}\pi\right)$

$$= \cos\left[32\pi\left(t - \frac{1}{64}\right)\right] + \cos\left[34\pi\left(t - \frac{1}{68}\right)\right]$$

Example 2.24 (4/5)

- Checking these results with (2.192), we see that only the input $x_1(t)$ is passed without distortion by the system.
- For $x_2(t)$, amplitude distortion results, and for $x_3(t)$ and $x_4(t)$, phase (delay) distortion is introduced.
- The group delay and phase delay are also illustrate in Figure 2.17.

Example 2.24 (5/5)

- It can be seen that for $|f| \leq 15\text{Hz}$, the group and phase delays are both equal to $\frac{1}{60}$s.
- For $|f| > 15\text{Hz}$, the group delay is zero, and the phase delay is

$$T_p(f) = \frac{1}{4|f|}, |f| > 15\text{Hz} \qquad (2.203)$$

Nonlinear distortion (1/9)

- To illustrate the idea of nonlinear distortion, let us consider a nonlinear system with the input-output characteristic
$$y(t) = a_1 x(t) + a_2 x^2(t) \qquad (2.204)$$
where a_1 and a_2 are constants, and with the input
$$x(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t) \qquad (2.205)$$

Nonlinear distortion (2/9)

- The output is therefore

$$y(t) = a_1[A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t)] + a_2[A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t)]^2 \quad (2.206)$$

- Using trigonometric identities, we can write the output as $y(t) = a_1[A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t)] + \frac{1}{2}a_2\left(A_1^2 + A_2^2\right) + \frac{1}{2}a_2\left[A_1^2\cos(2\omega_1 t) + A_2^2\cos(2\omega_2 t)\right] + a_2A_1A_2\{\cos[(\omega_1+\omega_2)\,t] + \cos[(\omega_1-\omega_2)t]\}$ (2.207)

Nonlinear distortion (3/9)

- As can be from (2.207) and as shown in Figure 2.18, the system has produced frequencies in the output other than the frequencies of the input.

Nonlinear distortion (4/9)

- In addition to the first term in (2.207), which may be considered the desired output, there are distortion terms at harmonics of the input frequencies (in this case, second) as well as distortion terms involving sums and differences of the harmonics (in this case, third) of the input frequencies.

Nonlinear distortion (5/9)

- The former are referred to as harmonic distortion terms, and the latter are referred to as intermodulation distortion terms.
- Note that a second-order nonlinearity could be used as a device to double the frequency of an input sinusoid.
- Third-order nonlinearities can be used as triplers, and so forth.

Nonlinear distortion (6/9)

- A general input signal can be handled by applying the, multiplication theorem given in Table F.6 in Appendix F.
- Thus, for the nonlinear system with the transfer characteristic given by (2.204), the output spectrum is

$$Y(f) = a_1 X(f) + a_2 X(f) * X(f) \quad (2.208)$$

Nonlinear distortion (7/9)

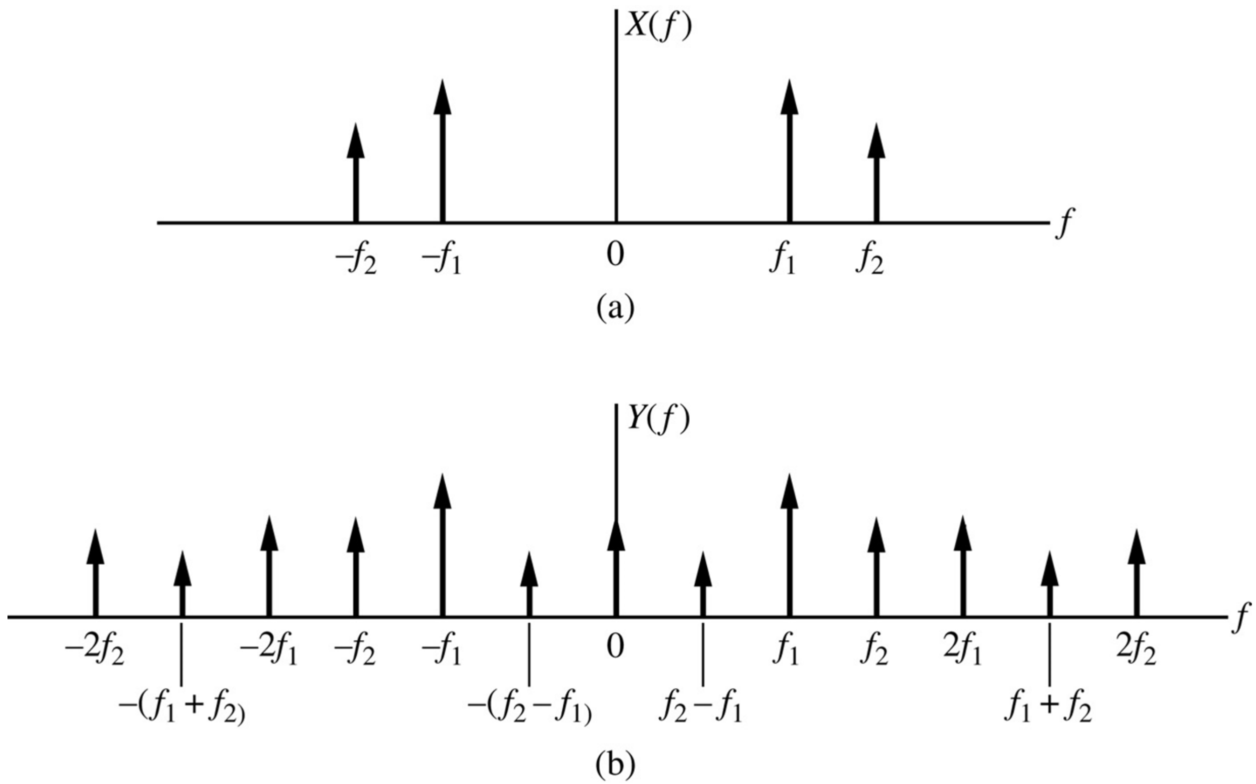


Figure 2.18
Input and output spectra for a nonlinear system with discrete frequency input.
(a) Input spectrum. (b) Output spectrum.

Nonlinear distortion (8/9)

- The second term is considered distortion and is seen to give interference at all frequencies occupied by the desired output (the first term).
- It is impossible to isolate harmonic and intermodulation distortion components as before.
- For example, if

$$X(f) = A\Pi\left(\frac{f}{2W}\right) \tag{2.209}$$

Nonlinear distortion (9/9)

- Then the distortion terms is

$$a_2 X(f) * X(f) = 2a_2 W A^2 \Lambda\left(\frac{f}{2W}\right) \quad (2.210)$$

- The input and output spectra are shown in Figure 2.19.
- Note that the spectral width of the distortion term is double that of the input.

Fig. 2.19

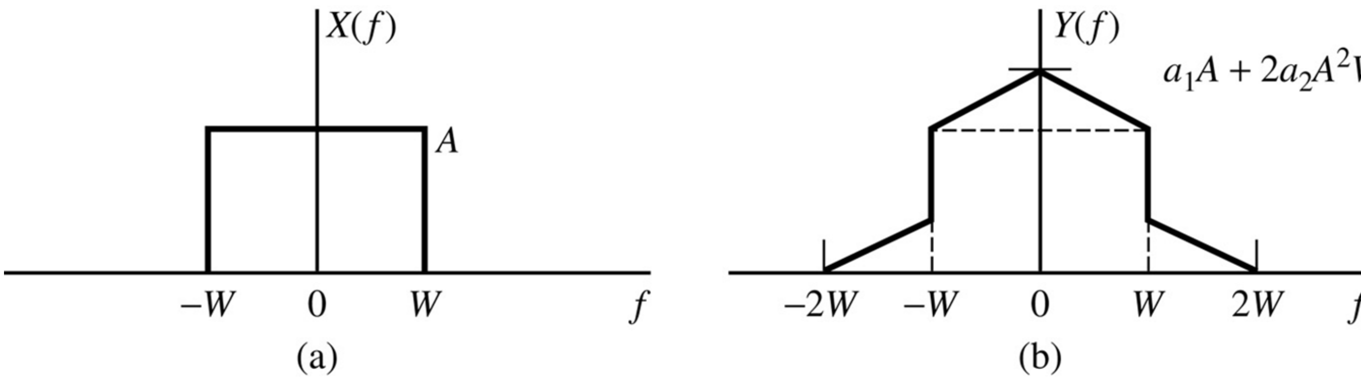


Figure 2.19
Input and output spectra for a nonlinear system with continuous frequency input.
(a) Input spectrum. (b) Output spectrum.

Ideal filter (1/9)

- It is often convenient to work with filters having idealized transfer functions with rectangular amplitude-response functions that are constant within the passband and zero elsewhere.

Ideal filter (2/9)

- We will consider three general types of ideal filter: lowpass, highpass, and bandpass.
- Within the passband, a linear phase-response characteristic is assumed.
- Thus, if B is the single-sided bandwidth (width of the stopband for the highpass filter) of the filter in question, the transfer functions of ideal lowpass, highpass and bandpass filters are easily written.
- -The stopband of a filter will be defined here as the frequency range(s) for which $|H(f)|$ is below 3dB of its maximum value.

Ideal filter (3/9)

- 1. For the ideal lowpass filter

$$H_{LP}(f) = H_0 \prod({}^{f}/_{2B})e^{-j2\pi f t_0} \quad (2.211)$$

- 2. For the ideal highpass filter

$$H_{HP}(f) = H_0[1 - \prod({}^{f}/_{2B})]e^{-j2\pi f t_0} \quad (2.212)$$

- 3. For the ideal bandpass filter

$$H_{BP}(f) = [H_1(f - f_0) + H_1(f + f_0)]e^{-j2\pi f t_0} \quad (2.213)$$

Where $H_1(f) = H_0 \prod(f/B)$

Ideal filter (4/9)

- The amplitude-response and phase-response functions for these filters are shown in Figure 2.20.
- The corresponding impulse responses are obtained by inverse Fourier transformation of the respective frequency-response function.
- For example, the impulse response of an ideal lowpass filter is, from Example 2.11 and the time-delay theorem, given by

$$h_{LP}(t) = 2BH_0\text{sinc}[2B(t - t_0)] \quad (2.214)$$

Ideal filter (5/9)

- Since $h_{LP}(t)$ is not zero for t<0, we see that an ideal lowpass filter is noncausal.
- Nevertheless, ideal filters are useful concepts because they simplify calculations and can give satisfactory results for spectral considerations.
- Turning to the ideal bandpass filter, we may use the modulation theorem to write its impulse as

$$h_{BP}(t) = 2h_1(t - t_0)\cos[2\pi f_0(t - t_0)] \quad (2.215)$$

Ideal filter (6/9)

- Where

$$h_1(t) = \mathfrak{I}^{-1}[H_1(f)] = H_0 Bsinc(Bt) \quad (2.216)$$

- Thus the impulse response of an ideal bandpass filter is the oscillatory signal

$$h_{BP}(t) = 2H_0 Bsinc[B(t - t_0)]\cos[2\pi f_0(t - t_0)] \quad (2.217)$$

Ideal filter (7/9)

- Figure 2.21 illustrates $h_{LP}(t)$ and $h_{BP}(t)$.
- If $f_0 \gg B$, it is convenient to view $h_{BP}(t)$ as the slowly varying envelope $2H_0 sinc(Bt)$ modulating the high-frequency oscillatory signal $\cos(2\pi f_0 t)$ and shifted to the right by t_0 s.
- Derivation of the impulse response of an ideal high pass filter is left to the problems (Problem 2.63).

Ideal filter (8/9)

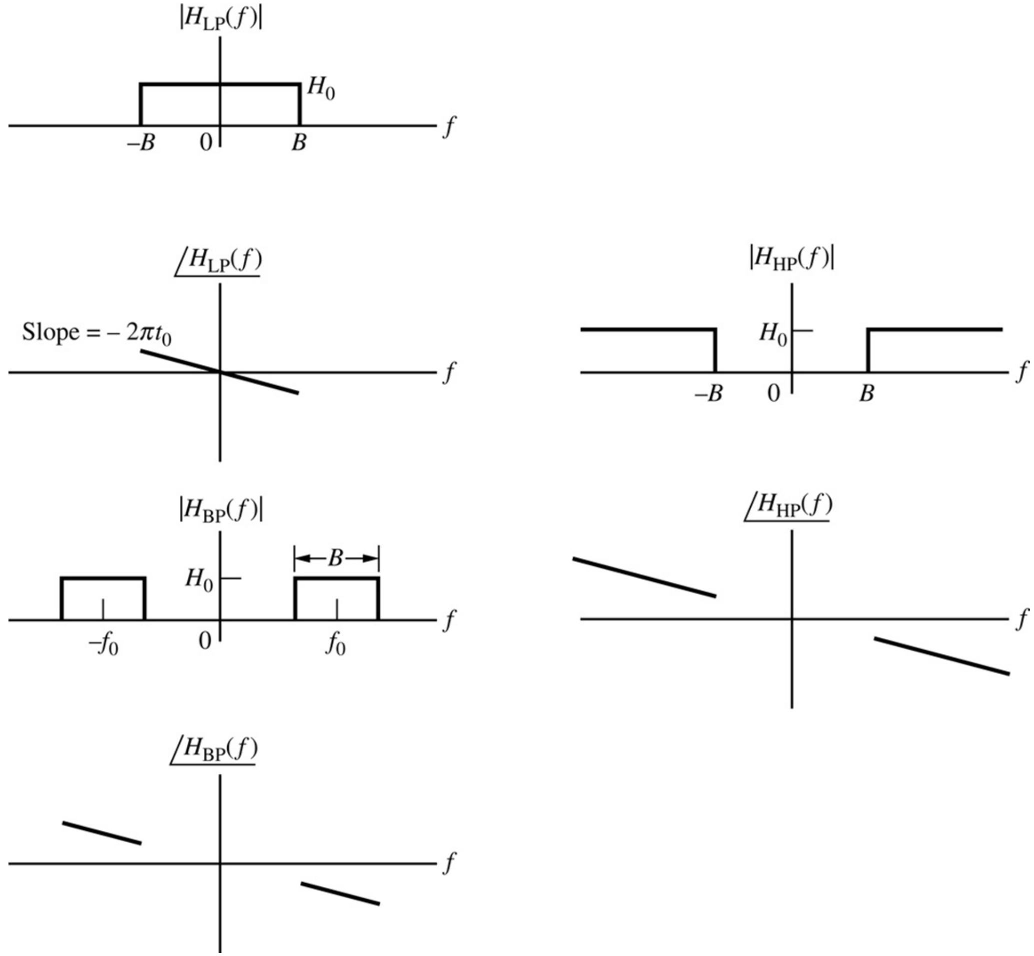


Figure 2.20
Amplitude-response and phase-response functions for ideal filters.

Ideal filter (9/9)

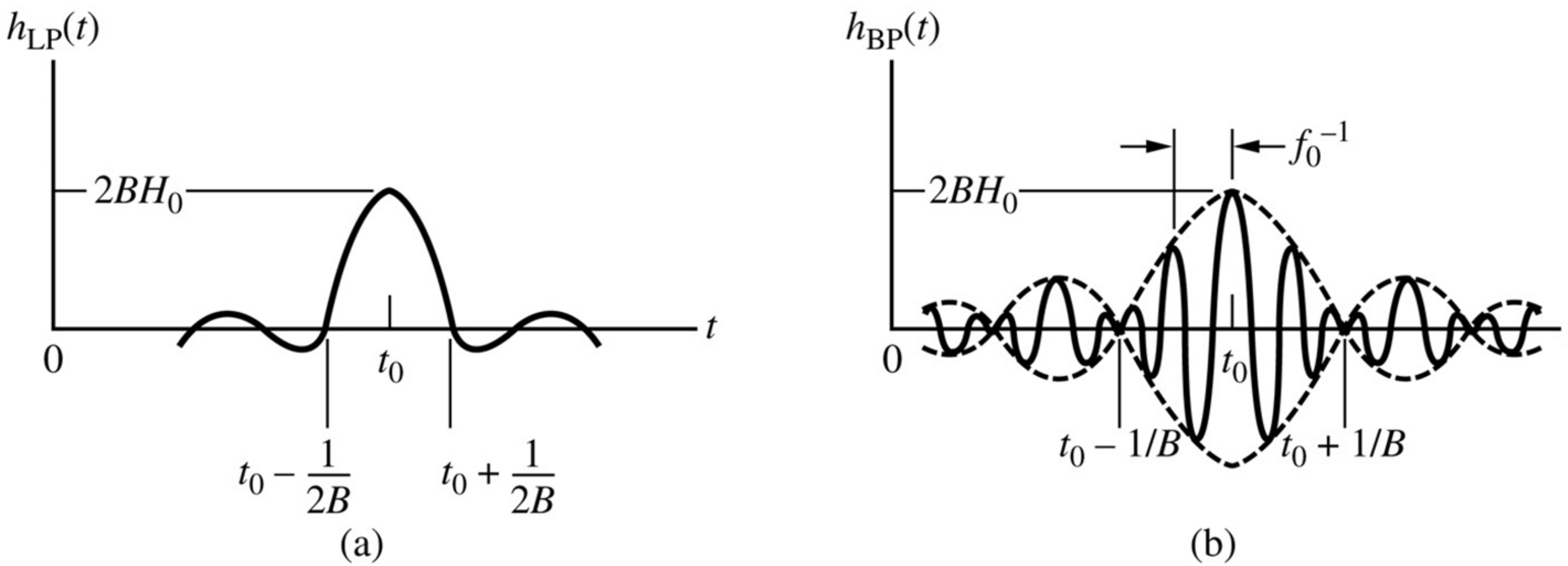


Figure 2.21
Impulse responses for ideal lowpass and bandpass filters.
(a) $h_{LP}(t)$. (b) $h_{BP}(t)$.

Approximation of Ideal Lowpass Filters by Realizable Filters (1/16)

- Although ideal filters are noncausal and therefore unrealizable devices, there are several practical filter types that may be designed to approximate ideal filter characteristic as closely as desired.
- In this section we consider three such approximations for the lowpass case.

Approximation of Ideal Lowpass Filters by Realizable Filters (2/16)

- Bandpass and highpass approximations may be obtained through suitable frequency transformation.
- The three filter types to be considered are

(1) Butterworth.

(2) Chebyshev.

(3) Bessel.

Approximation of Ideal Lowpass Filters by Realizable Filters (3/16)

- The Butterworth filter is a filter design chosen to maintain a constant amplitude response in the passband at the cost of less stopband attenuation.
- An nth-order Butterworth filter is characterized by a transfer function, in terms of the complex frequency s, of the form

$$H_{BW}(s) = \frac{\omega_3^n}{(s - s_1)(s - s_2)\dots(s - s_n)} \quad (2.218)$$

Approximation of Ideal Lowpass Filters by Realizable Filters (4/16)

- Where the poles s_1, s_2, …, s_n are symmetrical with respect to the real axis and equally spaced about a semicircle of radius ω_3 in the left half *s* plane and $f_3 = {}^{\omega_3}/_{2\pi}$ is the 3-dB cutoff frequency.

-From the basic circuit theory courses you will recall that the poles and zeros of a rational function of s, $H(s) = N(s)/D(S)$, are those values of complex frequency $s \triangleq \sigma + j\omega$ for which $D(s) = 0$ and $\mathrm{N}(s) = 0$,respectively.

Approximation of Ideal Lowpass Filters by Realizable Filters (5/16)

- Typical pole locations are shown in Figure 2.22(a).

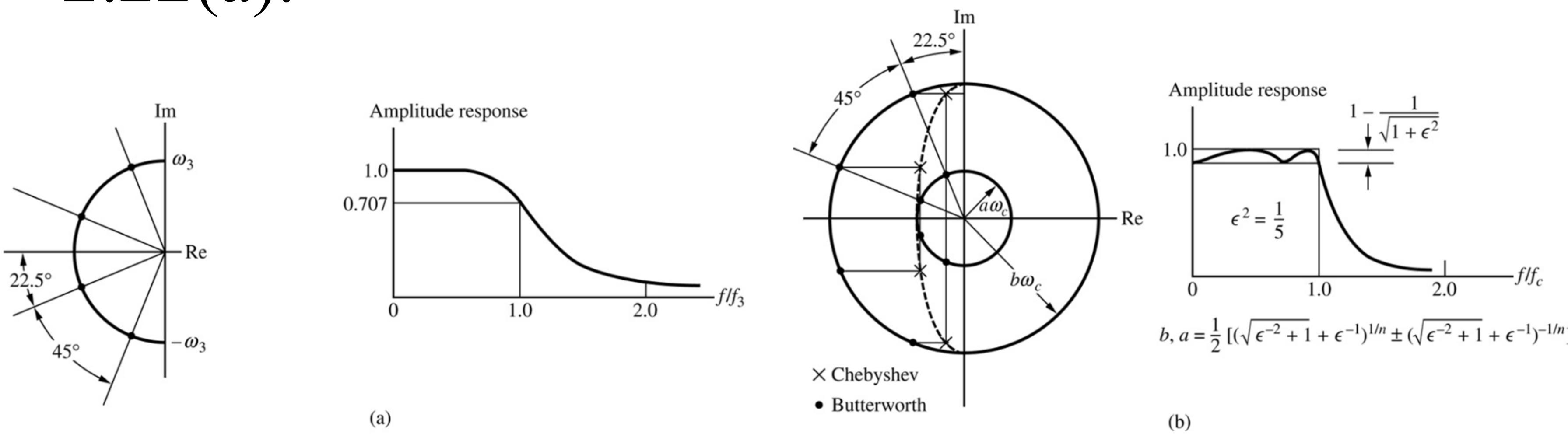


Figure 2.22
Pole locations and amplitude response for fourth-order Butterworth and Chebyshev filters.
(a) Butterworth filter. (b) Chebyshev filter.

Approximation of Ideal Lowpass Filters by Realizable Filters (6/16)

- For example, the system function of a second-order Butterworth filter is

$$H_{2nd-order\ BW}(S)$$
$$= \frac{\omega_3^2}{(s + [(1+j)/\sqrt{2}]\omega_3)(s + [(1-j)/\sqrt{2}]\omega_3)}$$
$$= \frac{\omega_3^2}{s^2 + \sqrt{2}\omega_3 s + \omega_3^2} \qquad (2.219)$$

Where $f_3 = \omega_3/2\pi$ is the 3-dB cutoff frequency in hertz.

Approximation of Ideal Lowpass Filters by Realizable Filters (7/16)

- The amplitude response for an nth-order Butterworth filter is of the form

$$|H_{BU}(f)| = \frac{1}{\sqrt{1 + (f/f_3\ )^{2n}}} \qquad (2.220)$$

Note that as n approaches infinity, $|H_{BU}(f)|$ approaches an ideal lowpass filter characteristic.

- However, the filter delay also approaches infinity.

Approximation of Ideal Lowpass Filters by Realizable Filters (8/16)

- The Chebyshev lowpass filter has am amplitude response chosen to maintain a minimum allowable attenuation in the passband while maximizing the attenuation in the stopband.
- A typical pole-zero diagram is shown in Figure 2.22(b).

Approximation of Ideal Lowpass Filters by Realizable Filters (9/16)

- The amplitude response of a Chebyshev filter is of the form

$$|H_C(f)| = \frac{1}{\sqrt{1+\epsilon^2 C_n^2(f)}} \qquad (2.221)$$

The parameter ϵ is specified by the minimum allowable attenuation in the passband, and $C_n(f)$,

known as a Chebyshev polynomial, is given by the recursion relation

$$C_n(f) = 2\left(\frac{f}{f_c}\right)C_{n-1}(f) - C_{n-2}(f), \qquad \mathrm{n} = 2,3,\ldots \qquad (2.222)$$

Approximation of Ideal Lowpass Filters by Realizable Filters (10/16)

- Where

$$C_1(f) = \frac{f}{f_c} \text{ and } C_0(f) = 1 \qquad (2.223)$$

Regardless of the value of n, it turns out that $C_n(f_c) = 1$, so that $H_C(f_c) = (1+\epsilon^2)^{-1/2}$. (Note that f_c is not necessarily the 3-dB frequency here.)

Approximation of Ideal Lowpass Filters by Realizable Filters (11/16)

- The Bessel lowpass filter is a design that attempts to maintain a linear phase response in the passband at the expense of the amplitude response.
- The cutoff frequency of a Bessel filter is define by

$$f_c = (2\pi t_0\ )^{-1} = \frac{\omega_c}{2\pi} \qquad (2.224)$$

Where t_0 is the nominal delay of the filter.

Approximation of Ideal Lowpass Filters by Realizable Filters (12/16)

- The frequency response function of an nth-order Bessel filter is given by

$$H_{BE}(f) = \frac{K_n}{B_n(f)} \qquad (2.225)$$

Where K_n is a constant chosen to yield $H(0) = 1$, and $B_n(f)$ is a Bessel polynomial of order *n* defined by

$$B_n(f) = (2n-1)B_{n-1}(f) - \left(\frac{f}{f_c}\right)^2 B_{n-2}(f) \qquad (2.226)$$

Approximation of Ideal Lowpass Filters by Realizable Filters (13/16)

- Where

$$B_0(f) = 1 \; and \; B_1(f) = 1 + j(\frac{f}{f_c}) \quad (2.227)$$

- Figure 2.23 illustrates the amplitude-response and group-delay characteristics of third-order Butterworth, Bessel, and Chebyshev filter.
- All four filters are normalized to have 3-dB amplitude attenuation at frequency of $f_c = 1$ Hz.

Approximation of Ideal Lowpass Filters by Realizable Filters (14/16)

- The amplitude responses show that the Chebyshev filters have more attenuation than the Butterworth and Bessel filters do for frequencies exceeding the 3-dB frequency.
- Increasing the passband ($f < f_c$) ripple of a Chebyshev filter increases the stopband ($f > f_c$) attenuation.

Approximation of Ideal Lowpass Filters by Realizable Filters (15/16)

- The group-delay characteristics shown in Figure 2.23(b) illustrate, as expected, that the Bessel filter has the most constant group delay.
- Comparison of the Butterworth and the 0.1-dB ripple Chebyshev group delays shows that although the group delay of the Chebyshev filter has a higher peak, it has a more constant group delay for frequencies less than about $0.4f_c$.

Approximation of Ideal Lowpass Filters by Realizable Filters (16/16)

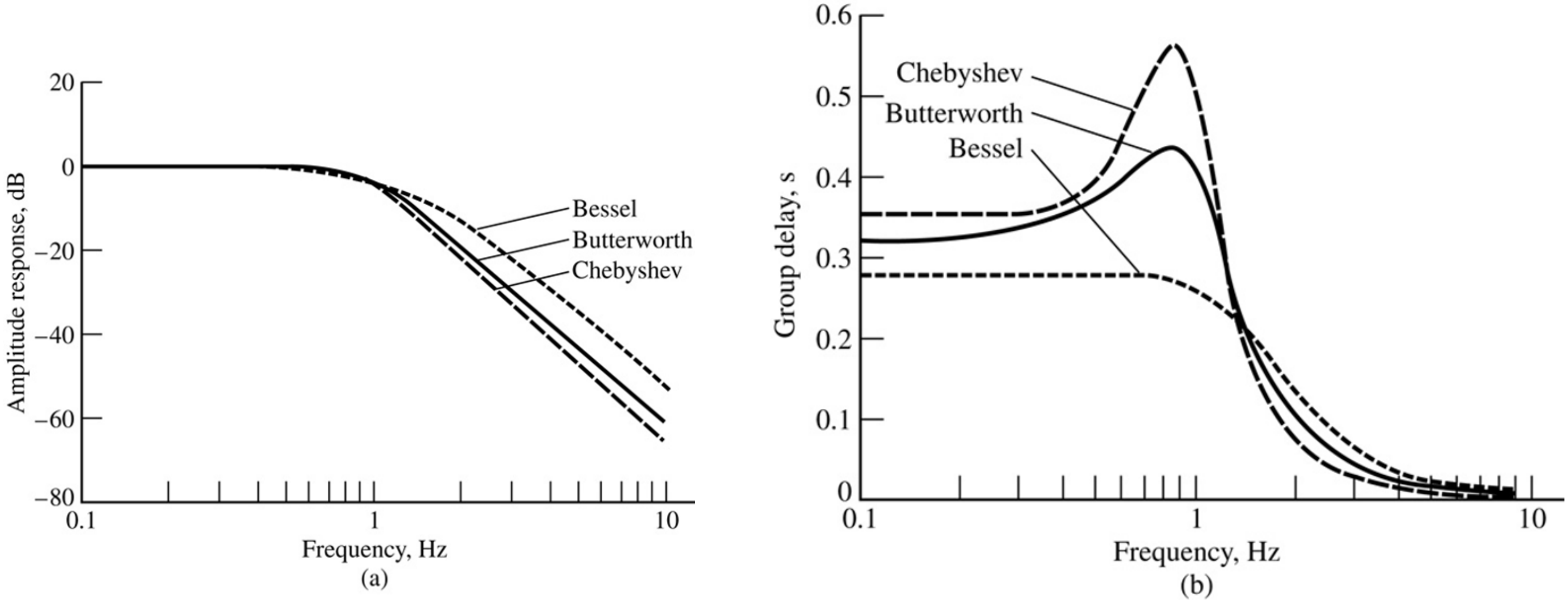


Figure 2.23
Comparison of third-order Butterworth, Chebyshev (0.1-dB ripple), and Bessel filters.
(a) Amplitude response. (b) Group delay.
All filters are designed to have a 1-Hz, 3-dB bandwidth.

Computer example 2.2

Relationship of Pulse Resolution and Risetime to Bandwidth (1/12)

- In our consideration of signal distortion, we assumed bandlimited signal spectra.
- We found that the input signal to a filter is merely delayed and attenuated if the filter has constant amplitude response and linear phase response throughout the passband of the signal.
- But suppose the input signal is not bandlimited.
- What rule of thumb can we use to estimate the required bandwidth?

Relationship of Pulse Resolution and Risetime to Bandwidth (2/12)

- This is a particularly important problem in pulse transmission, where the detection and resolution of pulses at a filter output are of interest.
- A satisfactory definition for pulse duration and bandwidth, and the relationship between them, is obtained by consulting Figure 2.24.
- In Figure 2.24(a), a pulse with a single maximum, taken at $t = 0$ for convenience, is shown with a rectangular approximation of height $x(0)$ and duration T.

Relationship of Pulse Resolution and Risetime to Bandwidth (3/12)

- It is required that the approximating pulse and $|x(t)|$ have equal areas. Thus

$$Tx(0) = \int_{-\infty}^{\infty} |x(t)| dt \geq \int_{-\infty}^{\infty} x(t) dt = X(0) \tag{2.228}$$

where we have used the relationship

$$X(0) = \Im[x(t)]|_{f=0} = \int_{-\infty}^{\infty} x(t) e^{-j2\pi t \cdot 0} dt \tag{2.229}$$

Relationship of Pulse Resolution and Risetime to Bandwidth (4/12)

- Turning to Figure 2.24(b), we obtain a similar inequality for the rectangular approximation to the pulse spectrum.
- Specifically, we may write

$$2WX(0) = \int_{-\infty}^{\infty} |X(f)| df \geq \int_{-\infty}^{\infty} X(f) df$$

$$= x(0) \qquad (2.230)$$

Relationship of Pulse Resolution and Risetime to Bandwidth (5/12)

where we have used the relationship

$$x(0) = \Im^{-1}[X(f)]|_{t=0} = \int_{-\infty}^{\infty} X(f)e^{j2\pi f\cdot 0}df \qquad (2.231)$$

- Thus we have the pair of inequalities

$$\frac{x(0)}{X(0)} \geq \frac{1}{T} \text{ and } 2W \geq \frac{x(0)}{X(0)} \qquad (2.232)$$

Relationship of Pulse Resolution and Risetime to Bandwidth (6/12)

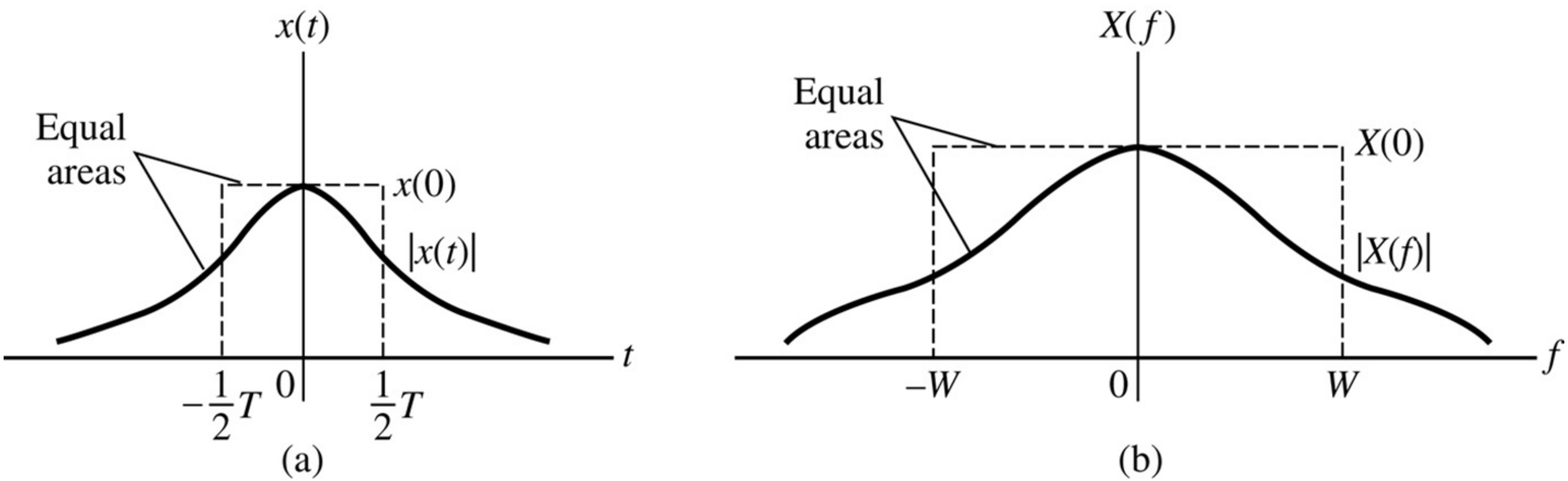


Figure 2.24
Arbitrary pulse signal and spectrum.
(a) Pulse and rectangular approximation.
(b) Amplitude spectrum and rectangular approximation.

Relationship of Pulse Resolution and Risetime to Bandwidth (7/12)

which, when combined, result in the relationship of pulse of duration and bandwidth

$$2W \geq \frac{1}{T} \qquad (2.233)$$

or

$$W \geq \frac{1}{2T}\,\mathrm{Hz} \qquad (2.234)$$

Relationship of Pulse Resolution and Risetime to Bandwidth (8/12)

- Other definitions of pulse duration and bandwidth could have been used, but a relationship similar to (2.233) and (2.234) would have resulted.
- This inverse relationship between pulse duration and bandwidth has been illustrated by all the example involving pulse spectra that we have considered so far (such as Example 2.8, 2.11, and 2.13).

Relationship of Pulse Resolution and Risetime to Bandwidth (9/12)

- If pulses with bandpass spectra are considered, the relationship is

$$W \geq \frac{1}{T}\,\mathrm{Hz} \qquad (2.235)$$

- This is illustrated by Example 2.16.
- A result similar to (2.233) and (2.234) also holds between the risetime T_R and bandwidth of a pulse.

Relationship of Pulse Resolution and Risetime to Bandwidth (10/12)

- A suitable definition of risetime is the time required for a pulse's leading edge to go from 10% to 90% of its final value.
- For the bandpass case, (2.235) holds with T replaced by T_R, where T_R is the risetime of the envelope of the pulse.
- Risetime can be used as a measure of a system's distortion.

Relationship of Pulse Resolution and Risetime to Bandwidth (11/12)

- To see how this is accomplished, we will express the step response of a filter in terms of its impulse response.
- From the superposition integral of (2.160), with $x(t-\sigma)=u(t-\sigma)$, the step response of a filter with impulse response $h(t)$ is

$$y_s(t)=\int_{-\infty}^{\infty} h(\sigma)u(t-\sigma)d\sigma=\int_{-\infty}^{t} h(\sigma)d\sigma \tag{2.236}$$

Relationship of Pulse Resolution and Risetime to Bandwidth (12/12)

- This follows because $u(t - \sigma) = 0$ for $\sigma > t$.
- Therefore, the step response of a linear system is the integral of its impulse response.
- This is not too surprising, since the unit step function is the integral of a unit impulse function.
- This result is a special case of a more general result for an LTI system: If the response of a system to a given input is known and that input is modified through a linear operation, such as integration, then the output to the modified input is obtained by performing the same linear operation on the output due to the original input.

Example 2.25 (1/3)

- The impulse response of a lowpass RC filter is given by

$$h(t) = \frac{1}{RC} e^{-t/_{RC}} u(t) \qquad (2.237)$$

For which the step response is found to be

$$y_s(t) = \left(1 - e^{-2\pi f_3 t}\right) u(t) \qquad (2.238)$$

where the 3-dB bandwidth of the filter, defined following (2.175), has been used.

Example 2.25 (2/3)

- The step response is plotted in Figure 2.25(a), where it is seen that the 10% to 90% risetime is approximately

$$T_R = \frac{0.35}{f_3} = 2.2RC \tag{2.239}$$

Which demonstrates the inverse relationship between bandwidth and risetime.

Example 2.25 (3/3)

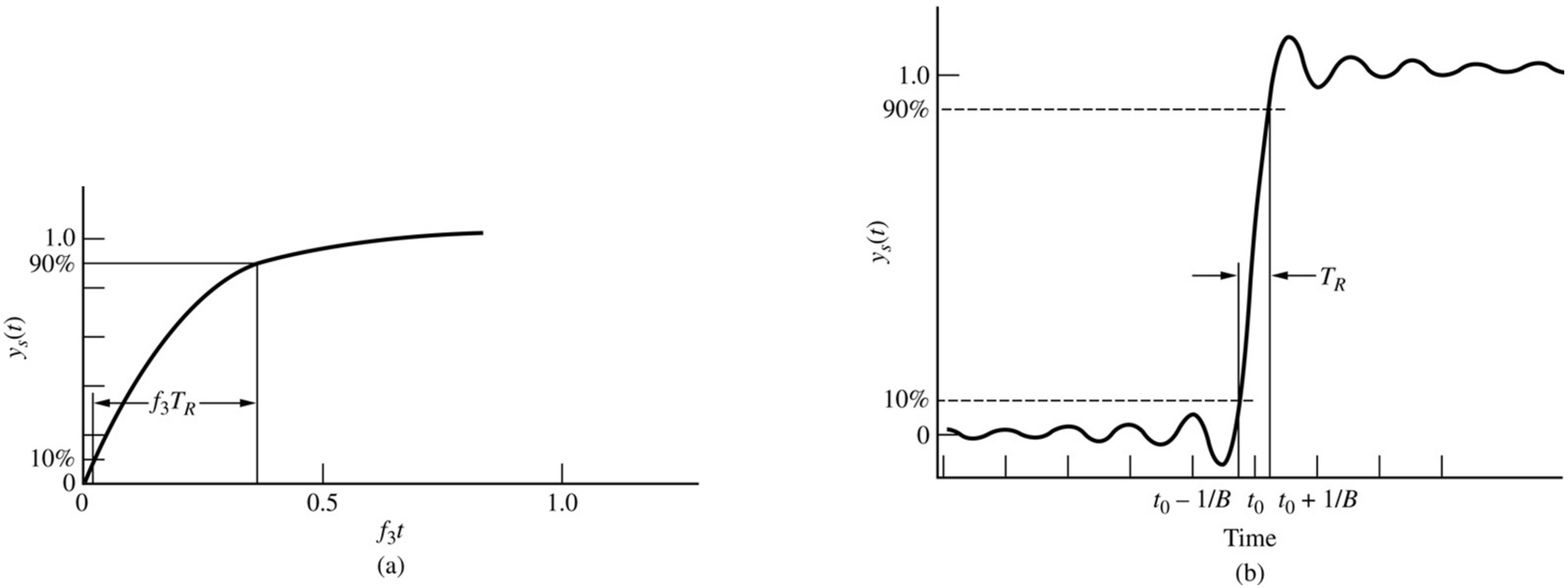


Figure 2.25
Step response of (a) a lowpass RC filter and (b) an ideal lowpass filter, illustrating 10% to 90% risetime of each.

Example 2.26 (1/3)

- Using (2.214) with $H_0 = 1$, the step response of an ideal lowpass, filter is

$$y_s(t) = \int_{-\infty}^{t} 2B\text{sinc}[2B(\sigma - t_0)]d\sigma$$

$$= \int_{-\infty}^{t} 2B \frac{\sin[2\pi B(\sigma - t_0)]}{2\pi B(\sigma - t_0)} d\sigma \qquad (2.240)$$

- By changing variables in the integrand to $u = 2\pi B(\sigma - t_0)$, the step response becomes

Example 2.26 (2/3)

$$y_s(t) = \frac{1}{2\pi}\int_{-\infty}^{2\pi B(t-t_0)} \frac{\sin u}{u}\,du$$

$$= \frac{1}{2} + \frac{1}{\pi} Si[2\pi B(t - t_0)] \qquad (2.241)$$

where $Si(x) = \int_0^x \left(\frac{\sin u}{u}\right) du = -Si(x)$ is the sine-integral function.

Example 2.26 (3/3)

- A plot of $y_s(t)$ for an ideal lowpass filter, such as is shown in Figure 2.25(b), reveals that the 10% to 90% risetime is approximately

$$T_R \cong \frac{0.44}{B} \tag{2.242}$$

- Again, the inverse relationship between bandwidth and risetime is demonstrated.

Sampling theory (1/17)

- In many applications it is useful to represent a signal in terms of sample values taken at appropriately spaced intervals.
- Such sample-data systems find application in feedback control, digital computer simulation, and pulse-modulation communication systems.
- In this section we consider the representation of a signal $x(t)$ by a so-called ideal instantaneous sample waveform of the form

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \qquad (2.243)$$

where T_s is the sampling interval.

Sampling theory (2/17)

- Two questions to be answered in connection with such sampling are
  - What are the restrictions on $x(t)$ and T_s to allow perfect recovery of $x(t)$ from $x_\delta(t)$?
  - How is $x(t)$ recovery from $x_\delta(t)$?
- Both questions are answered by the uniform sampling theorem for lowpass signals, which may be state as follows:

Sampling theory (3/17)

- **Theorem**
- If a signal $x(t)$ contains no frequency components for frequencies above $f = W$ Hz, then it is completely described by instantaneous sample values uniformly spaced in time with period $T_s < \frac{1}{2W}$.
- The signal can be exactly reconstructed from the sampled waveform given by (2.243) by passing it through an ideal lowpass filter with bandwidth B, where $W < B < f_s - W$ with $f_s = T_s^{-1}$.

Sampling theory (4/17)

- The frequency $2W$ is referred to as the Nyquist frequency.
- To prove the sampling theorem, we find the spectrum of (2.243).
- Since $\delta(t - nT_s)$ is zero everywhere except at $t = nT_s$, (2.243) can be written as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{2.244}$$

Sampling theory (5/17)

- Applying the multiplication theorem of Fourier transforms (2.102), the Fourier transform of (2.244) is

$$X_\delta(f) = X(f) * \left[f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \qquad (2.245)$$

where the transform pair (2.119) has been used.

- Interchanging the order of summation and convolution and noting that

$$X(f) * \delta(f - nf_s) = \int_{-\infty}^{\infty} X(u)\delta(f - u - nf_s)du$$
$$= X(f - nf_s) \qquad (2.246)$$

Sampling theory (6/17)

- By the sifting property of the delta function, we obtain

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad (2.247)$$

- Thus, assuming that the spectrum of $x(t)$ is bandlimited to W Hz and that $f_s > 2W$ as stated in the sampling theorem, we may readily sketch $X_\delta(f)$.

Sampling theory (7/17)

- Figure 2.26 shows a typical choice for $X(f)$ and the corresponding $X_\delta(f)$.
- We note that sampling simply results in a periodic repetition of $X(f)$ in the frequency domain with a spacing f_s.
- If $f_s < 2W$, the separate terms in (2.247) overlap, and there is no apparent way to recover $x(t)$ from $x_\delta(t)$ without distortion.

Sampling theory (8/17)

- On the other hand, if $f_s > 2W$, the term in (2.247) for $n = 0$ is easily separated from the rest by ideal lowpass filtering.
- Assuming an ideal lowpass filter with the frequency-response function

$$H(f) = H_0 \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}, W \leq B \leq f_s - W \tag{2.248}$$

- The output spectrum, with $x_\delta(t)$ at the input, is

$$Y(f) = f_s H_0 X(f) e^{-j2\pi f t_0} \tag{2.249}$$

Sampling theory (9/17)

- And by the time-delay theorem, the output waveform is

$$y(t) = f_s H_0 x(t - t_0) \qquad (2.250)$$

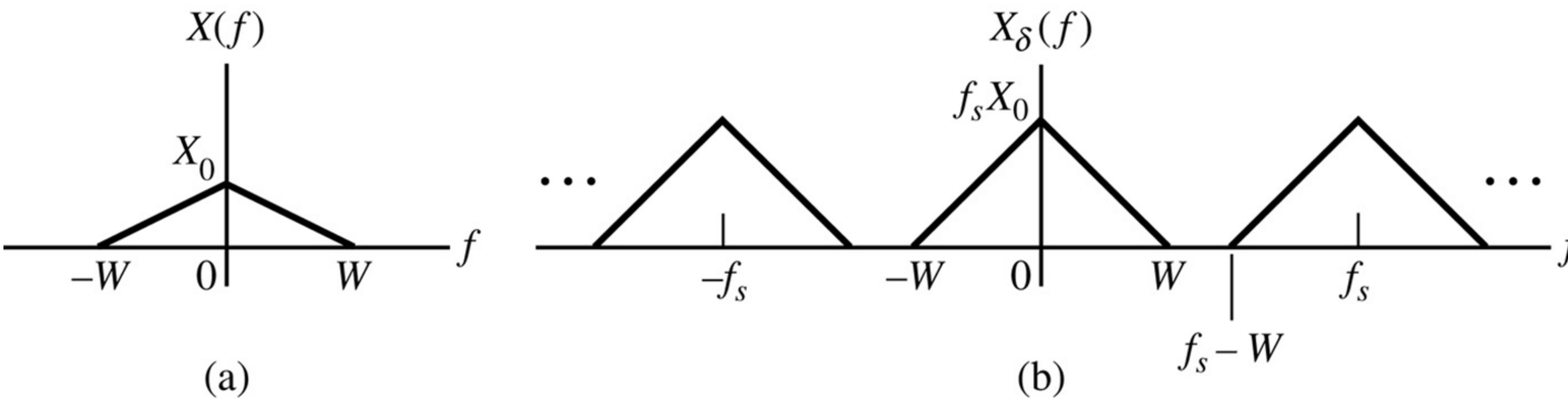


Figure 2.26
Signal spectra for lowpass sampling.
(a) Assumed spectrum for $x(t)$. (b) Spectrum of the sampled signal.

Sampling theory (10/17)

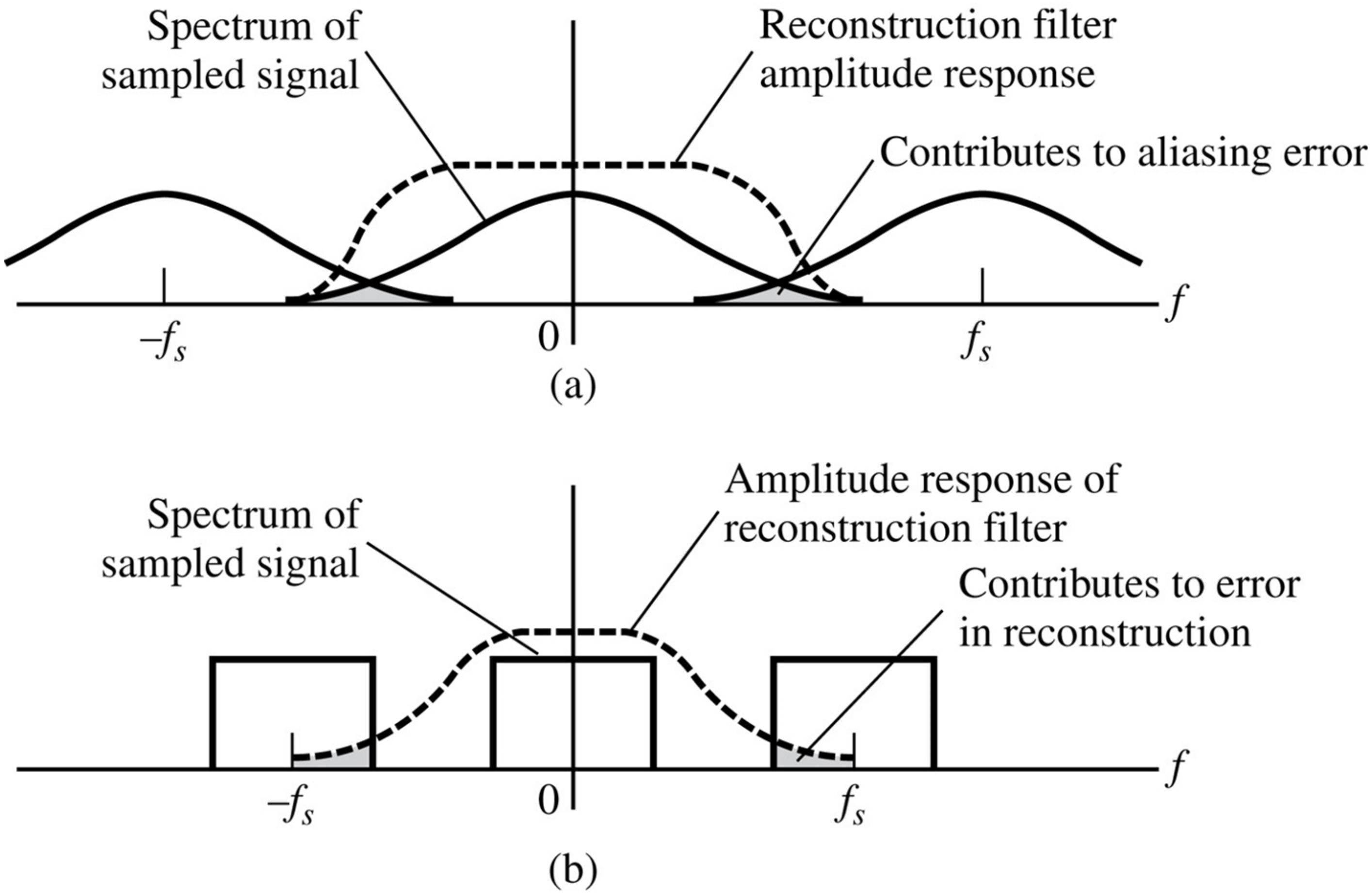


Figure 2.27
Spectra illustrating two types of errors encountered in reconstruction of sampled signals.
(a) Illustration of aliasing error in the reconstruction of sampled signals.
(b) Illustration of error due to nonideal reconstruction filter.

Sampling theory (11/17)

- Thus, if the conditions of the sampling theorem are satisfied, we see that distortionless recovery of $x(t)$ from $x_\delta(t)$ is possible.
- Conversely, if the conditions of the sampling theorem are not satisfied, either because $x(t)$ is not bandlimited or because $f_s < 2W$, we see that distortion at the output of the reconstruction filter is inevitable.
- Such distortion, referred to as aliasing, is illustrated in Figure 2.27(a).

Sampling theory (12/17)

- It can be combated by filtering the signal before sampling or by increasing the sampling rate.
- A second type of error, illustrated in Figure 2.27(b), occurs in the reconstruction process and is due to the nonideal frequency response characteristics of practical filters.
- This type of error can be minimized by choosing reconstruction filters with sharper roll-off characteristics or by increasing the sampling rate.

Sampling theory (13/17)

- Note that the error due to aliasing and the error due to imperfect reconstruction filters are both proportional to signal level.
- An alternative expression for the reconstructed output from the ideal lowpass filter can be obtained by noting that when (2.243) is passed through a filter with impulse response $h(t)$, the output is

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h(t - nT_s) \qquad (2.251)$$

Sampling theory (14/17)

- But $h(t)$ corresponding to (2.248) is given by (2.214). Thus

$$y(t) = 2BH_0 \sum_{n=-\infty}^{\infty} x(nT_s)\operatorname{sinc}[2B(t - t_0 - nT_s)] \tag{2.252}$$

- And we see that just as a periodic signal can be completely represented by its Fourier coefficients, a bandlimited signal can be completely represented by its sample values.

Sampling theory (15/17)

- By setting $B = \frac{1}{2} f_s$, $H_0 = T_s$, and $t_0 = 0$ for simplicity, (2.252) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\mathrm{sinc}(f_s t - n) \qquad (2.253)$$

- This expansion is equivalent to a generalized Fourier series, for we may show that

$$\int_{-\infty}^{\infty} \mathrm{sinc}(f_s t - n)\,\mathrm{sinc}(f_s t - m)dt = T_s \delta_{nm} \qquad (2.254)$$

where $\delta_{nm} = 1$, $n = m$, and is 0 otherwise.

Sampling theory (16/17)

- Turning next to bandpass spectra, for which the upper limit on frequency f_u is much larger than the single-sided bandwidth W, one may naturally inquire as to the feasibility of sampling at rates less than $f_s > 2f_u$.
- The uniform sampling theorem for bandpass signals gives the conditions for which this is possible.

Sampling theory (17/17)

- **Theorem**
- If a signal has a spectrum of bandwidth W Hz and upper frequency limit f_u, then a rate f_s at which the signal can be sampled is $2f_u/m$, where m is the largest integer not exceeding f_u/W.
- All higher sampling rates are not necessarily usable unless they exceed $2f_u$.

Example 2.27 (1/5)

- Consider the bandpass signal $x(t)$ with the spectrum shown in Figure 2.28.
- According to the bandpass sampling theorem, it is possible to reconstruct $x(t)$ from sample values taken at a rate of

$$f_s = \frac{2f_u}{m} = \frac{2(3)}{2} = 3 \text{ samples per second} \tag{2.255}$$

whereas the lowpass sampling theorem requires 6 samples per second.

Example 2.27 (2/5)

- To show that this is possible, we sketch the spectrum of the sampled signal.
- According to (2.247), which holds in general,

$$X_\delta(f) = 3 \sum_{n=-\infty}^{\infty} X(f - 3n) \qquad (2.256)$$

Example 2.27 (3/5)

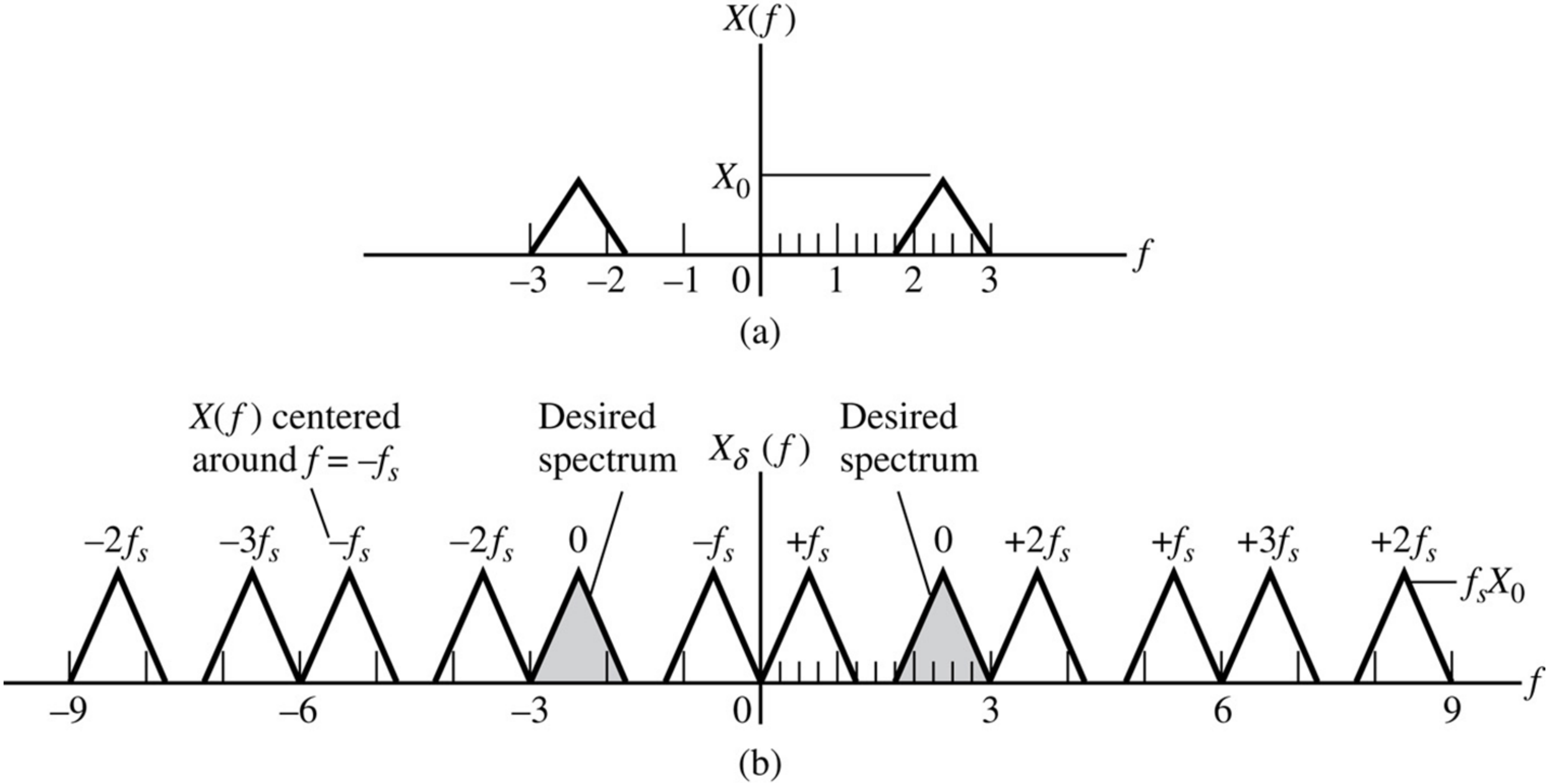


Figure 2.28
Signal spectra for bandpass sampling.
(a) Assumed bandpass signal spectrum. (b) Spectrum of the sampled signal.

Example 2.27 (4/5)

- The resulting spectrum is shown in Figure 2.28(b), and we see that it is theoretically possible to recover $x(t)$ from $x_\delta(t)$ by bandpass filtering.
- Another way of sampling a bandpass signal of bandwidth W is to resolve it into two lowpass quadrature signals of bandwidth $\frac{1}{2}W$.

Example 2.27 (5/5)

- Both of these may then be sampled at a minimum rate of $2\left(\frac{1}{2}W\right) = W$ samples per second, thus resulting in an overall minimum sampling rate of $2W$ samples per second.

The Hilbert Transform

- It may be advantageous to postpone this section until consideration of single-sideband systems in Chapter 3.

Definition (1/6)

- Consider a filter that simply phase shifts all positive frequency components of its input by $-\frac{1}{2}\pi$ rad and all negative frequency components of its input by $\frac{1}{2}\pi$ rad; that is, its frequency response function is

$$H(f) = -j\mathrm{sgn}f \tag{2.257}$$

where the sgn function (read "signum f") is defined as

$$\mathrm{sgn}f = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases} \tag{2.258}$$

Definition (2/6)

- We note that $|H(f)| = 1$ and $\underline{/H(f)}$ is odd, as it must be.
- If $X(f)$ is the input spectrum to the filter, the output spectrum is $-j(\mathrm{sgn} f)X(f)$, and the corresponding time function is (2.259)
$$\hat{x}(t) = \Im^{-1}[-j(\mathrm{sgn} f)X(f)] = h(t) * x(t)$$
where $-j\Im^{-1}[\mathrm{sgn} f]$ is the impulse response of the filter.

Definition (3/6)

- To obtain $\mathfrak{I}^{-1}[\mathrm{sgn} f]$ without resorting to contour integration, we consider the inverse transform of the function

$$G(f;\alpha) = \begin{cases} e^{-\alpha f}, f > 0 \\ -e^{\alpha f}, f < 0 \end{cases} \qquad (2.260)$$

- We note that $\lim_{\alpha \to 0} G(f;\alpha) = \mathrm{sgn} f$.
- Thus our procedure will be to inverse Fourier transform $G(f;\alpha)$ and take the limit of the result as α approaches zero.

Definition (4/6)

- Performing the inverse transformation, we obtain

$$
\begin{aligned}
g(t;\alpha) &= \mathfrak{I}^{-1}[G(f;\alpha)] \\
&= \int_0^{\infty} e^{-\alpha f} e^{j2\pi f t} df - \int_{-\infty}^{0} e^{\alpha f} e^{j2\pi f t} df \\
&= \frac{j4\pi t}{\alpha^2 + (2\pi t)^2} \qquad (2.261)
\end{aligned}
$$

Definition (5/6)

- Taking the limit as α approaches zero, we get the transform pair

$$\frac{j}{\pi t} \leftrightarrow \operatorname{sgn} f \qquad (2.262)$$

- Using the result in (2.259), we obtain the output of the filter:

$$\hat{x}(t) = \int_{-\infty}^{\infty} \frac{x(\lambda)}{\pi(t-\lambda)} d\lambda = \int_{-\infty}^{\infty} \frac{x(t-\eta)}{\pi\eta} d\eta \qquad (2.263)$$

Definition (6/6)

- The signal $\hat{x}(t)$ is defined as the Hilbert transform of $x(t)$.
- Since the Hilbert transform corresponds to a phase shift of $\pm\frac{1}{2}\pi$, we note that the Hilbert transform of $\hat{x}(t)$ corresponds to the frequency-response function $(-j \operatorname{sgn} f)^2 = -1$, or a phase shift of π radians.
- Thus

$$\hat{\hat{x}}(t) = -x(t) \tag{2.264}$$

Example 2.28 (1/4)

- For an input to a Hilbert transform filter of
$$x(t) = \cos(2\pi f_0 t) \tag{2.265}$$
which has a spectrum given by
$$X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \tag{2.266}$$
- We obtain an output spectrum from the Hilbert transform of
$$\hat{X}(f) = \frac{1}{2}\delta(f - f_0)e^{-j\pi/2} + \frac{1}{2}\delta(f + f_0)e^{j\pi/2} \tag{2.267}$$

Example 2.28 (2/4)

- Taking the inverse Fourier transform of (2.267), we find the output signal to be

$$\hat{x}(t) = \frac{1}{2}e^{j2\pi f_0 t}e^{-j\pi/2} + \frac{1}{2}e^{-j2\pi f_0 t}e^{j\pi/2}$$
$$= \cos(2\pi f_0 t - \pi/2) \qquad (2.268)$$

or

$$\cos\widehat{(2\pi f_0 t)} = \sin(2\pi f_0 t)$$

Example 2.28 (3/4)

- Of course, the Hilbert transform could have been found by inspection in this case by adding $-\frac{1}{2}\pi$ to the argument of the cosine.
- Doing this for the signal $\sin\omega_0 t$, we find that

$$\sin\widehat{(2\pi f_0 t)} = \sin\left(2\pi f_0 t - \frac{1}{2}\pi\right)$$
$$= -\cos(2\pi f_0 t) \tag{2.269}$$

Example 2.28 (4/4)

- We may use the two results obtain to show that
$$\widehat{e^{j2\pi f_0 t}} = -j\mathrm{sgn}(2\pi f_0)e^{j2\pi f_0 t} \qquad (2.270)$$
- This is done by considering the two cases $f_0 > 0$ and $f_0 < 0$ and using Euler's theorem in conjunction with the results of (2.268) and (2.269).
- The result (2.270) also follows directly by considering the response of a Hilbert transform filter with frequency response $H_{\mathrm{HT}}(f) = -j\mathrm{sgn}(2\pi f)$ to the input $x(t) = e^{j2\pi f_0 t}$.

Properties (1/9)

- The Hilbert transform has several useful properties that will be illustrated later.
- Three of these properties will be proved here:

1. The energy (or power) in a signal $x(t)$ and its Hilbert transform $\hat{x}(t)$ are equal. To show this, we consider the energy spectral densities at the input and output of a Hilbert transform filter.

Properties (2/9)

- Since $H(f) = -j\text{sgn}f$, these densities are related by

$$\left|\hat{X}(f)\right|^2 \triangleq |\Im[\hat{x}(t)]|^2 = |-j\text{sgn}(f)|^2|X(f)|^2$$
$$= |X(f)|^2 \qquad (2.271)$$

  where $\hat{X}(f) = \Im[\hat{x}(t)] = -j\text{sgn}(f)X(f)$.

- Thus, since the energy spectral densities at input and output are equal, so are the total energies.
- A similar proof holds for power signals.

Properties (3/9)

2. A signal and its Hilbert transform are orthogonal; that is

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0 \text{ (energy signals)} \quad (2.272)$$

or

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\hat{x}(t)dt = 0 \text{ (power signals)} \quad (2.273)$$

Properties (4/9)

- Considering (2.272), we note that the left-hand side can be written as

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = \int_{-\infty}^{\infty} X(f)\hat{X}^*(f)df \qquad (2.274)$$

- By Parseval's theorem generalized, where $\hat{X}(f) = \Im[\hat{x}(t)] = -j\text{sgn}(f)X(f)$.
- It therefore follows that

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = \int_{-\infty}^{\infty} (+j\text{sgn}f)|X(f)|^2 df \; (2.275)$$

Properties (5/9)

- However, the integrand of the right-hand side of (2.275) is odd, being the product of the even function $|X(f)|^2$ and the odd function $j\,\mathrm{sgn}\,f$.
- Therefore, the integral is zero, and (2.272) is proved.
- A similar proof holds for (2.273).

Properties (6/9)

3. If $c(t)$ and $m(t)$ are signals with nonoverlapping spectra, where $m(t)$ is lowpass and $c(t)$ is highpass, then

$$\widehat{m(t)c}(t) = m(t)\hat{c}\,(t) \tag{2.276}$$

To prove this relationship, we use the Fourier integral to represent $m(t)$ and $c(t)$ in terms of their spectra $M(f)$ and $C(f)$, respectively. Thus

$$m(t)c(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(f)C(f')\exp[j2\pi(f+f')t]\,df\,df' \tag{2.277}$$

Where we assume $M(f) = 0$ for $|f| > W$ and $C(f') = 0$ for $|f'| < W$.

Properties (7/9)

- The Hilbert transform of (2.277) is

$$\widehat{m(t)c(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(f)C(f') \exp[j2\pi\widehat{(f+f')t}]\, df\, df' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(f)C(f')[-j\mathrm{sgn}(f+f')] \exp[j2\pi(f+f')t]\, df\, df' \quad (2.278)$$

where (2.270) has been used.

Properties (8/9)

- However, the product $M(f)C(f')$ is nonvanishing only for $|f| < W$ and $|f'| > W$, and we may replace $\text{sgn}(f + f')$ by $\text{sgn}f'$ in this case. Thus

$$\widehat{m(t)c}(t) = \int_{-\infty}^{\infty} M(f)\exp(j2\pi ft)\,df \int_{-\infty}^{\infty} C(f')\,[-j(\text{sgn}f')]\exp(j2\pi f't)\,df' \tag{2.279}$$

- However, the first integral on the right-hand side is just $m(t)$, and the second integral is $\hat{c}(t)$, since

$$c(t) = \int_{-\infty}^{\infty} C(f')\,\exp(j2\pi f't)\,df'$$

Properties (9/9)

And

$$\hat{c}(t) = \int_{-\infty}^{\infty} C(f') \exp(\widehat{j2\pi f' t})\, df' \qquad (2.280)$$

$$= \int_{-\infty}^{\infty} C(f') \left[-j(\mathrm{sgn} f')\right] \exp(j2\pi f' t)\, df'$$

Hence (2.279) is equivalent to (2.276), which was the relationship to be proved.

Example 2.29

- Given that $m(t)$ is a lowpass signal with $M(f) = 0$ for $|f| > W$, we may directly apply (2.276) in conjunction with (2.275) and (2.269) to show that

$$m(t)\widehat{\cos(}\omega_0 t) = m(t)\sin(\omega_0 t) \quad (2.281)$$

And

$$m(t)\widehat{\sin(}\omega_0 t) = -m(t)\cos(\omega_0 t) \quad (2.282)$$

If $f_0 = {}^{\omega_0}/_{2\pi} > W$.

Analytic Signals (1/8)

- An analytic signal $x_p(t)$, corresponding to the real signal $x(t)$, is defined as

$$x_p(t) = x(t) + j\hat{x}(t) \qquad (2.283)$$

  where $\hat{x}(t)$ is the Hilbert transform of $x(t)$.
- We now consider several properties of an analytic signal.

Analytic Signals (2/8)

- We used the term envelope in connection with the ideal bandpass filter.
- The envelope of a signal is defined mathematically as the magnitude of the analytic signal $x_p(t)$.
- The concept of an envelope will acquire more importance when we discuss modulation in Chapter 3.

Example 2.30 (1/3)

- In section 2.6.12, (2.217), we showed that the impulse response of an ideal bandpass filter with bandwidth B, delay t_0, and center frequency f_0 is given by (2.284)

$$h_{\mathrm{BP}}(t) = 2H_0 B \operatorname{sinc}[B(t - t_0)]\cos[\omega_0(t - t_0)]$$

- Assuming that $B < f_0$, we can use the result of Example 2.29 to determine the Hilbert transform of $h_{\mathrm{BP}}(t)$.

Example 2.30 (2/3)

- The result is
$$\hat{h}_{\mathrm{BP}}(t) = 2H_0B\mathrm{sinc}[B(t-t_0)]\sin[\omega_0(t-t_0)] \tag{2.285}$$
- The envelope is
$$\left|h_{\mathrm{BP}}(t) + j\hat{h}_{\mathrm{BP}}(t)\right| \tag{2.286}$$
$$= \sqrt{[h_{\mathrm{BP}}(t)]^2 + \left[\hat{h}_{\mathrm{BP}}(t)\right]^2}$$
$$= \sqrt{\{2H_0B\mathrm{sinc}[B(t-t_0)]\}^2\{\cos^2[\omega_0(t-t_0)] + \sin^2[\omega_0(t-t_0)]\}}$$
or
$$2H_0B|\mathrm{sinc}[B(t-t_0)]| \tag{2.287}$$

Example 2.30 (3/3)

- The envelope is obviously easy to identify if the signal is composed of a lowpass signal multiplied by a high-frequency sinusoid.
- Note, however, that the envelope is mathematically defined for any signal.

Analytic Signals (3/8)

- The spectrum of the analytic signal is also of interest.
- We will use it to advantage in Chapter 3 when we investigate single-sideband modulation.
- Since the analytic signal, from (2.283), is defined as

$$x_p(t) = x(t) + j\hat{x}(t)$$

Analytic Signals (4/8)

it follows that the Fourier transform of $x_p(t)$ is

$$X_p(f) = X(f) + j[-j\text{sgn}(f)X(f)] \quad (2.288)$$

where the term in brackets is the Fourier transform of $\hat{x}(t)$. Thus

$$X_p(f) = X(f)[1 + \text{sgn}f] \quad (2.289)$$

or

$$X_p(f) = \begin{cases} 2X(f), f > 0 \\ 0, \quad\quad f < 0 \end{cases} \quad (2.290)$$

Analytic Signals (5/8)

- The subscript p is used to denote that the spectrum is nonzero only for positive frequencies.
- Similarly, we can show that the signal

$$x_n(t) = x(t) - j\hat{x}(t) \qquad (2.291)$$

is nonzero only for negative frequencies.

- Replacing $\hat{x}(t)$ by $-\hat{x}(t)$ in the preceding discussion result in

Analytic Signals (6/8)

$$X_n(f) = X(f)[1 - \operatorname{sgn} f] \qquad (2.292)$$

or

$$X_n(f) = \begin{cases} 0, & f > 0 \\ 2X(f), & f < 0 \end{cases} \qquad (2.293)$$

- These spectra are illustrated in Figure 2.29.

Analytic Signals (7/8)

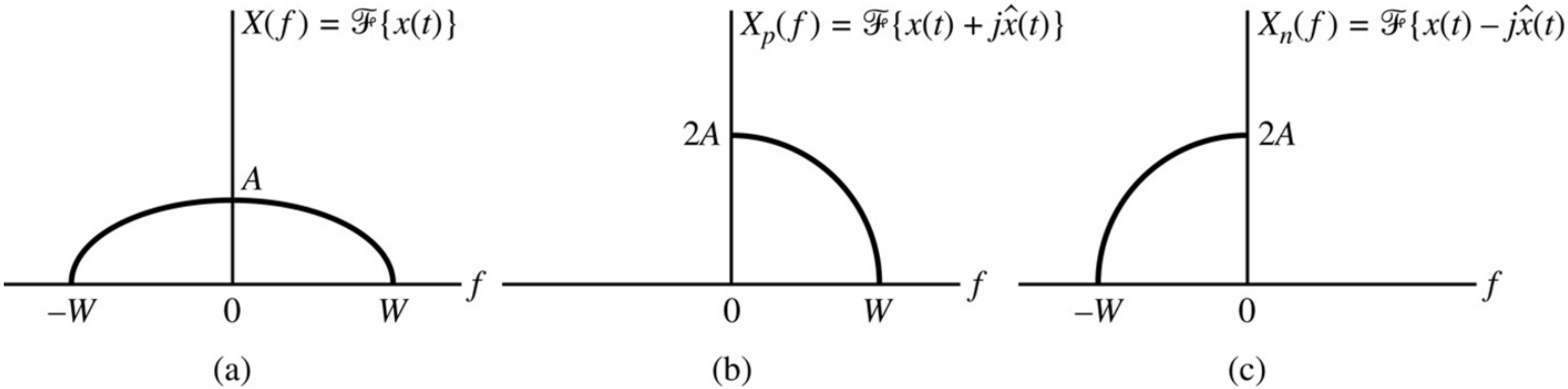


Figure 2.29
Spectra of analytic signal.
(a) Spectrum of $x(t)$. (b) Spectrum of $x(t) + j\hat{x}(t)$. (c) Spectrum of $x(t) - j\hat{x}(t)$

Analytic Signals (8/8)

- The observations may be made at this point.
- First, if $X(f)$ is nonzero at $f = 0$, then $X_p(f)$ and $X_n(f)$ will be discontinuous at $f = 0$.
- Also, we should not be confused that $|X_n(f)|$ and $\left|X_p(f)\right|$ are not even, since the corresponding time-domain signals are not real.

Complex Envelope Representation of Bandpass Signals (1/8)

- If $X(f)$ in (2.288) corresponds to a signal with a bandpass spectrum, as shown in Fig. 2.30(a), it then follows by (2.290) that $X_p(f)$ is just twice the positive frequency portion of $X(f) = \mathfrak{F}\{x(t)\}$, as shown in Fig. 2.30(b).
- By the frequency-translation theorem, it follows that $x_p(t)$ can be written as

$$x_p(t) = \tilde{x}(t)e^{j2\pi f_0 t} \tag{2.294}$$

Complex Envelope Representation of Bandpass Signals (2/8)

- Where $\tilde{x}(t)$ is a complex-values lowpass signal (hereafter referred to as the complex envelope) and f_0 is a reference frequency chosen for convenience.
- If the spectrum of $x_p(t)$ has a center of symmetry, a natural choice for f_0 would be this point of symmetry, but it need not be.
- The spectrum (assumed to be real for ease of plotting) of $\tilde{x}(t)$ is shown in Figure 2.30(c).

Complex Envelope Representation of Bandpass Signals (3/8)

- To find $\tilde{x}(t)$, we may proceed along one of two paths [note that simply taking the magnitude of (2.294) gives only $|\tilde{x}(t)|$ but not its argument].

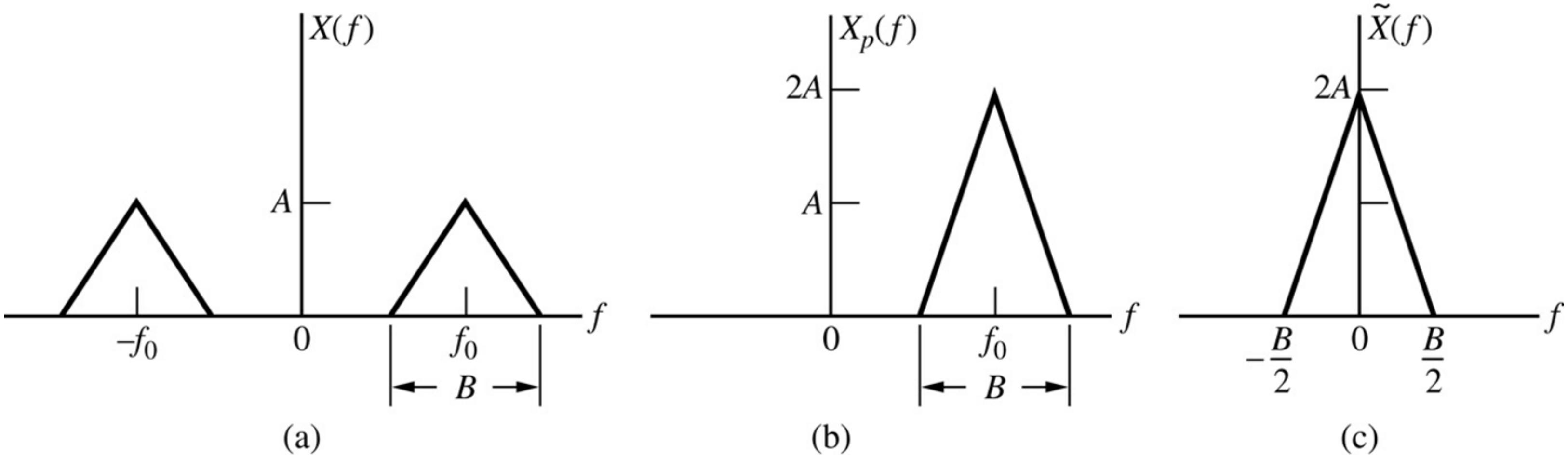


Figure 2.30
Spectra pertaining to the formation of a complex envelope of a signal $x(t)$.
(a) A bandpass signal spectrum. (b) Twice the positive-frequency portion of $X(f)$ corresponding to $\Im\{x(t) + j\hat{x}(t)\}$. (c) Spectrum of $\tilde{x}(t)$.

Complex Envelope Representation of Bandpass Signals (4/8)

- First, using (2.283), we can find the analytic signal $x_p(t)$ and then solve (2.294) for $\tilde{x}(t)$. That is,

$$\tilde{x}(t) = x_p(t)e^{-j2\pi f_0 t} \tag{2.295}$$

- Second, we can find $\tilde{x}(t)$ by using a frequency-domain approach to obtain $X(f)$, then scale its positive frequency components by a factor of 2 to give $X_p(f)$, and translate the resultant spectrum by f_0 Hz to the left.

Complex Envelope Representation of Bandpass Signals (5/8)

- The inverse Fourier transform of this translated spectrum is then $\tilde{x}(t)$.
- For example, for the spectra shown in Figure 2.30, the complex envelope, using Figure 2.30(c), is

$$\tilde{x}(t) = \mathfrak{I}^{-1}\left[2A\Lambda\left(\frac{2f}{B}\right)\right] = AB\mathrm{sinc}^2(Bt/2) \tag{2.296}$$

Complex Envelope Representation of Bandpass Signals (6/8)

- The complex envelope is real in this case because the spectrum $X(f)$ is symmetrical around $f = f_0$.
- Since $x_p(t) = x(t) + j\hat{x}(t)$,where $x(t)$ and $\hat{x}(t)$ are the real and imaginary parts, respectively, of $x_p(t)$, it follows from (2.294) that

$$x_p(t) = \tilde{x}(t)e^{j2\pi f_0 t} \triangleq x(t) + j\hat{x}(t) \quad (2.297)$$

Complex Envelope Representation of Bandpass Signals (7/8)

or

$$x(t) = \mathrm{Re}(\tilde{x}(t)e^{j2\pi f_0 t}) \qquad (2.298)$$

and

$$\hat{x}(t) = \mathrm{Im}(\tilde{x}(t)e^{j2\pi f_0 t}) \qquad (2.299)$$

- Thus, from (2.298), the real signal $x(t)$ can be expressed in terms of its complex envelope as

$$x(t) = \mathrm{Re}\left(\tilde{x}(t)e^{j2\pi f_0 t}\right)$$
$$= \mathrm{Re}\left(\tilde{x}(t)\right)\cos(2\pi f_0 t) - \mathrm{Im}\left(\tilde{x}(t)\right)\sin(2\pi f_0 t)$$
$$= x_R(t)\cos(2\pi f_0 t) - x_I(t)\sin(2\pi f_0 t) \qquad (2.300)$$

Complex Envelope Representation of Bandpass Signals (8/8)

where

$$\tilde{x}(t) \triangleq x_R(t) + jx_I(t) \tag{2.301}$$

- The signal $x_R(t)$ and $x_I(t)$ are known as the inphase and quadrature components of $x(t)$.

Example 2.31 (1/3)

- Consider the real bandpass signal

$$x(t) = \cos(22\pi t) \tag{2.302}$$

- It Hilbert transform is

$$\hat{x}(t) = \sin(22\pi t) \tag{2.303}$$

- So the corresponding analytic signal is

$$\begin{aligned} x_p(t) &= x(t) + j\hat{x}(t) \\ &= \cos(22\pi t) + j\sin(22\pi t) \\ &= e^{j22\pi t} \end{aligned} \tag{2.304}$$

Example 2.31 (2/3)

- In order to find the corresponding complex envelope, we need to specify f_0, which for the purposes of this example, we take as $f_0 = 10$ Hz.
- Thus, from (2.295), we have

$$\tilde{x}(t) = x_p(t)e^{-j2\pi f_0 t} = e^{j22\pi t}e^{-j20\pi t} = e^{j2\pi t}$$
$$= \cos(2\pi t) + j\sin(2\pi t) \qquad (2.305)$$

Example 2.31 (3/3)

- So that, from (2.301), we obtain

$$x_R(t) = \cos(2\pi t) \text{ and } x_I(t) = \sin(2\pi t) \quad (2.306)$$

- Putting these into (2.300), we get

$$x(t) = x_R(t)\cos(2\pi f_0 t) - x_I(t)\sin(2\pi f_0 t)$$
$$= \cos(2\pi t)\cos(20\pi t) - \sin(2\pi t)\sin(20\pi t)$$
$$= \cos(22\pi t) \quad (2.307)$$

which is, not surprisingly, what we began with in (2.302).

Complex Envelope Representation of Bandpass Systems (1/5)

- Consider a bandpass system with impulse response $h(t)$ that is represented in terms of a complex envelope $\tilde{h}(t)$ as
$$h(t) = \mathrm{Re}(\tilde{h}(t)e^{j2\pi f_0 t}) \qquad (2.308)$$
where $\tilde{h}(t) = h_R(t) + jh_I(t)$.
- Assume that the input is also bandpass with representation (2.298).
- The output, by the superposition integral, is
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda \qquad (2.309)$$

Complex Envelope Representation of Bandpass Systems (2/5)

- By Euler's theorem, we can represent $h(t)$ and $x(t)$ as

$$h(t) = \frac{1}{2}\widetilde{h}(t)e^{j2\pi f_0 t} + \text{c. c.} \qquad (2.310)$$

and

$$x(t) = \frac{1}{2}\tilde{x}(t)e^{j2\pi f_0 t} + \text{c. c.} \qquad (2.311)$$

respectively, where c.c. stands for the complex conjugate of the immediately preceding term.

Complex Envelope Representation of Bandpass Systems (3/5)

- Using these in (2.309), the output can be expressed as

$$y(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\tilde{h}(\lambda)e^{j2\pi f_0 \lambda} + \text{c.c.}\right]\left[\frac{1}{2}\tilde{x}(t-\lambda)e^{j2\pi f_0 (t-\lambda)} + \text{c.c.}\right] d\lambda$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \tilde{h}(\lambda)\,\tilde{x}(t-\lambda)d\lambda e^{j2\pi f_0 t} + \text{c.c.}$$

$$+ \frac{1}{4}\int_{-\infty}^{\infty} \tilde{h}(\lambda)\tilde{x}^*(t-\lambda)e^{j4\pi f_0 \lambda}d\lambda e^{-j2\pi f_0 t} + \text{c.c.} \qquad (2.312)$$

Complex Envelope Representation of Bandpass Systems (4/5)

- The second pair of terms, $\frac{1}{4}\int_{-\infty}^{\infty}\tilde{h}(\lambda)\tilde{x}^*(t-\lambda)e^{j4\pi f_0\lambda}d\lambda e^{-j2\pi f_0 t}+c.c.$,is approximately zero by virtue of the factor $e^{j4\pi f_0\lambda}=\cos(4\pi f_0\lambda)+j\sin(4\pi f_0\lambda)$ in the integrand ($\tilde{h}$ and $\tilde{x}$ are slowly varying with respect to this complex exponential, and therefore, the integrand cancels to zero, half-cycle by half-cycle).

Complex Envelope Representation of Bandpass Systems (5/5)

- Thus

$$y(t) \cong \frac{1}{4}\int_{-\infty}^{\infty} \tilde{h}(\lambda)\,\tilde{x}(t-\lambda)d\lambda e^{j2\pi f_0 t} + c.c. \qquad (2.313)$$

$$= \frac{1}{2}Re([\tilde{h}(t) * \tilde{x}(t)]e^{j2\pi f_0 t}) \triangleq \frac{1}{2}Re(\tilde{y}(t)e^{j2\pi f_0 t})$$

Where (2.314)

$$\tilde{y}(t) = \tilde{h}(t) * \tilde{x}(t) = \mathfrak{I}^{-1}[\widetilde{H}(f)\tilde{X}(f)]$$

In which $\widetilde{H}(f)$ and $\tilde{X}(f)$ are the respective Fourier transforms of $\tilde{h}(t)$ and $\tilde{x}(t)$.

Example 2.32 (1/5)

- As an example of the application of (2.313), consider the input

$$x(t) = \prod\left(\frac{t}{\tau}\right)\cos(2\pi f_0 t) \qquad (2.315)$$

To a filter with impulse response

$$h(t) = \alpha e^{-\alpha t}u(t)\cos(2\pi f_0 t) \qquad (2.316)$$

- Using the complex envelope analysis just developed with $\tilde{x}(t) = \prod({}^{t}/_{\tau})$ and $\tilde{h}(t) = \alpha e^{-\alpha t}u(t)$, we have as the complex envelope of the filter output

Example 2.32 (2/5)

$$\tilde{y}(t) = \prod\left({}^{t}/_{\tau}\right) * \alpha e^{-\alpha t} u(t)$$

$$= \left[1 - e^{-\alpha\left(t + {}^{\tau}/_{2}\right)}\right] u\left(t + \frac{\tau}{2}\right) - \left[1 - e^{-\left(t - {}^{\tau}/_{2}\right)}\right] u\left(t - \frac{\tau}{2}\right) \quad (2.317)$$

Multiplying this by $\frac{1}{2} e^{j2\pi f_0 t}$ and taking the real part results in the output of the filter in accordance with (2.313).

Example 2.32 (3/5)

- The result is

$$y(t)=\frac{1}{2}\{[1-e^{-\alpha(t+\frac{\tau}{2})}]u(t+\frac{\tau}{2})-[1-e^{-\alpha(t+\frac{\tau}{2})}]\ u(t-\frac{\tau}{2})\}\cos(2\pi f_0 t) \quad (2.318)$$

To check this result, we convolve (2.315) and (2.316) directly.

Example 2.31 (4/5)

- The superposition integral becomes

$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} \prod(\lambda/\tau) \cos(2\pi f_0 \lambda)\, \alpha e^{-\alpha(t-\lambda)} u(t - \lambda) \cos[2\pi f_0(t-\lambda)]\, d\lambda \qquad (2.319)$$

However,

$$\cos(2\pi f_0 \lambda) \cos[2\pi f_0(t-\lambda)]$$
$$= \frac{1}{2}\cos(2\pi f_0 t) + \frac{1}{2}\cos[2\pi f_0(t-2\lambda)] \qquad (2.320)$$

Example 2.31 (5/5)

- So that the superposition integral becomes

$$y(t) = \frac{1}{2}\int_{-\infty}^{\infty} \Pi(\lambda/\tau)\, \alpha e^{-\alpha(t-\lambda)} u(t-\lambda) d\lambda \cos(2\pi f_0 t) + \frac{1}{2}\int_{-\infty}^{\infty} \Pi(\lambda/\tau)\, \alpha e^{-\alpha(t-\lambda)} u(t-\lambda) \cos[2\pi f_0 (t-\lambda)] d\lambda \quad (2.321)$$

If $f_0^{-1} \ll \tau$ and $f_0^{-1} \ll \alpha^{-1}$, the second integral is approximately zero, so that we have only the first integral, which is $\Pi({}^{t}/_{\tau})$ convolved with $\alpha e^{-\alpha t} u(t)$ and the result multiplied by $\frac{1}{2}\cos(2\pi f_0 t)$, which is the same as (2.318).