Calculus (I) : Midterm Exam 2 (11/30/2020, 8:15 - 11:45 AM)

** The Exam includes 7 problems with 110 points in total.*
** Please show your work for partial credits.*

Problem	Score
1. (20 pts.)	
2. (40 pts.)	
3. (10 pts.)	
4. (10 pts.)	
5. (10 pts.)	
6. (10 pts.)	
7. (10 pts.)	
Total (110 pts.)	

Name: Solution

Student ID #: ____________

Department: ____________

1. **(20 pts.)** Sketch the graph of $f(x) = \frac{\ln|x|}{x^2}$ and find the followings.

(a) Domain, range, intercepts, asymptotes, and symmetry.

(b) $f'(x)$, intervals of increasing and decreasing, and local extrema.

(c) $f''(x)$, concavity, and inflection points.

(a) Domain $= \{x \in \mathbb{R} \mid x \neq 0\}$

Range $= \{y \in \mathbb{R} \mid y \leq \frac{1}{2e}\}$

Intercepts : $(\pm 1, 0)$

Vertical asymptote : $x = 0$ $\left(\lim_{x \to 0^+} \frac{\ln|x|}{x^2} = -\infty, \lim_{x \to 0^-} \frac{\ln|x|}{x^2} = -\infty \right)$

Horizatal asymptote : $y = 0$ $\left(\lim_{x \to +\infty} \frac{\ln|x|}{x^2} = 0, \lim_{x \to -\infty} \frac{\ln|x|}{x^2} = 0 \right)$

The graph of $f(x)$ is symmetric with respect to y-axis, since $f(-x) = f(x)$.

(b) $f'(x) = \frac{d}{dx}\left(\frac{\ln|x|}{x^2}\right) = \frac{1 - 2\ln|x|}{x^3}$

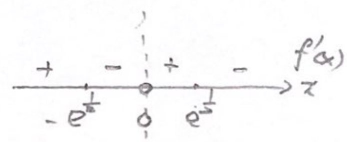
critical numbers : $x = \pm e^{\frac{1}{2}}$

increasing intervals: $(-\infty, -e^{\frac{1}{2}}), (0, e^{\frac{1}{2}})$

decreasing intervals: $(-e^{\frac{1}{2}}, 0), (e^{\frac{1}{2}}, \infty)$

local max: $f(\pm e^{\frac{1}{2}}) = \frac{1}{2e}$

local min doesn't exist.



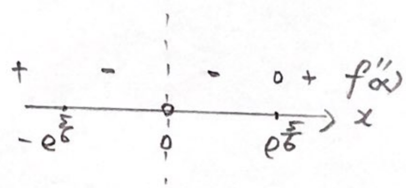
(c) $f''(x) = \frac{d}{dx}\left(\frac{1 - 2\ln|x|}{x^3}\right) = \frac{-5 + 6\ln|x|}{x^4}$

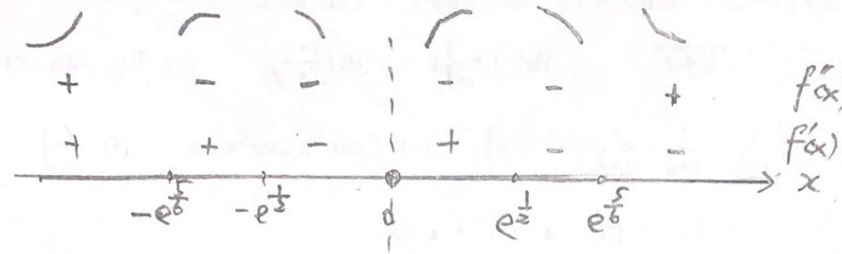
critical numbers: $x = \pm e^{\frac{5}{6}}$

concave upward : $(-\infty, -e^{\frac{5}{6}}), (e^{\frac{5}{6}} \infty)$

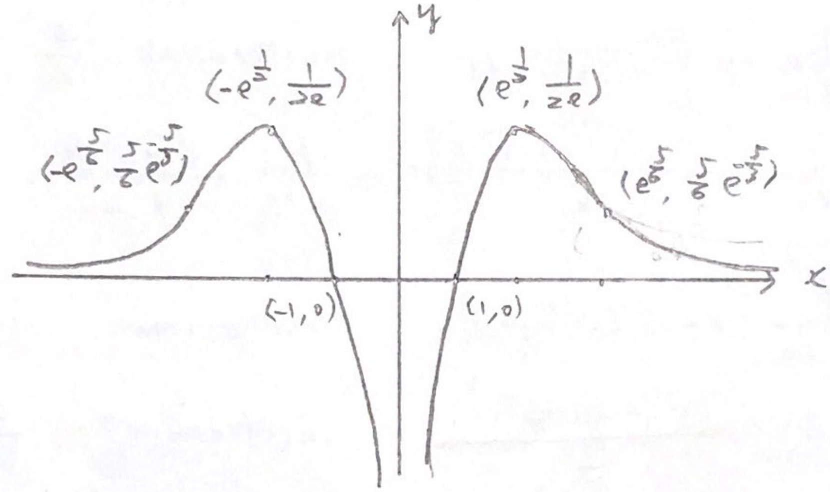
concave downward: $(-e^{-\frac{5}{6}}, 0), (0, e^{\frac{5}{6}})$

inflextion points: $\left(\pm e^{\frac{5}{6}}, \frac{5}{6} e^{-\frac{5}{3}}\right)$





f''(x)
f'(x)
x
-e^{5/6}
-e^{1/2}
0
e^{1/2}
e^{5/6}



y
x
(-e^{1/2}, 1/2e)
(e^{1/2}, 1/2e)
(-e^{5/6}, 5/6 e^{-5/3})
(e^{5/6}, 5/6 e^{-5/3})
(-1, 0)
(1, 0)

2. (40 pts.) Evaluate the followings and simply your answers.

(a) $\lim_{x\to 0}\frac{1}{x}\int_2^{2+x}\sqrt{1+t^3}\,dt$ **(b)** $\lim_{x\to\infty}\left[x-x^2\ln\left(\frac{1+x}{x}\right)\right]$ **(c)** $\lim_{x\to 0^+}(\tan 3x)^{2x}$

(d) $\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\frac{1}{1+\frac{3}{n}}+\cdots+\frac{1}{2}\right)$ **(e)** $\int \tan^3 x\sec^5 x\,dx$ **(f)** $\frac{d}{dx}\int_x^{x^2} e^{t^2}\,dt$

(g) $\int_0^1|2x^2-x|\,dx$ **(h)** $\int x^3\sqrt{x^2+1}\,dx$

(a)

$$\lim_{x\to 0}\frac{1}{x}\int_2^{2+x}\sqrt{1+t^3}\,dt \qquad \text{indeterminate: } \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\frac{d}{dx}\int_2^{2+x}\sqrt{1+t^3}\,dt}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{\sqrt{1+(2+x)^3}}{1} = 3$$

(b)

$$\lim_{x\to\infty}\left[x-x^2\ln\left(\frac{1+x}{x}\right)\right] \qquad \text{indeterminate: } \infty-\infty$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}-\ln\frac{1+x}{x}}{\frac{1}{x^2}} \qquad \text{indeterminate: } \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{x^2}-\frac{1}{1+x}+\frac{1}{x}}{-2\cdot\frac{1}{x^3}}$$

$$=\lim_{x\to 0}\frac{-\frac{1+x-x}{x^2(1+x)}}{-2\cdot\frac{1}{x^3}} = \frac{1}{2}$$

(c)

$$\lim_{x\to 0^+}(\tan 3x)^{2x} = \lim_{x\to 0^+} e^{\ln(\tan 3x)^{2x}}$$

$$= e^{\lim_{x\to 0^+} 2x\cdot\ln(\tan 3x)}$$

$$= e^{\lim_{x\to 0^+}\frac{2\ln(\tan 3x)}{\frac{1}{x}}} \qquad \text{indeterminate: } \frac{\infty}{\infty}$$

$$= e^{\lim_{x\to 0^+}\frac{2\cdot\frac{3}{\tan 3x}}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to 0^+} -\frac{3x}{\sin 3x}\cdot(2x)\cdot\cos 3x}$$

$$= e^0$$

$$= 1$$

(d)

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right)$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \ln|1+x| \Big|_0^1 = \ln 2$$

(e)

$$\int \tan^3 x \sec^5 x \, dx \qquad u = \sec x$$

$$du = \sec x \tan x \, dx$$

$$= \int \tan^2 x \sec^4 x \cdot \sec x \tan x \cdot dx$$

$$= \int (u^2-1)\, u^4 \, du = \int (u^6 - u^4)\, du = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$$

(f)

$$\frac{d}{dx}\int_x^{x^2} e^{t^2}\,dt = \frac{d}{dx}\left(\int_0^{x^2} e^{t^2}\,dt - \int_0^{x} e^{t^2}\,dt\right)$$

$$= 2xe^{x^4} - e^{x^2}$$

(g)

$$\int_0^1 |2x^2 - x| \, dx = -\int_0^{\frac{1}{2}} (2x^2 - x)\,dx + \int_{\frac{1}{2}}^1 (2x^2 - x)\,dx$$

$$= -\left[\frac{2}{3}x^3 - \frac{1}{2}x^2\right]\Big|_0^{\frac{1}{2}} + \left[\frac{2}{3}x^3 - \frac{1}{2}x^2\right]\Big|_{\frac{1}{2}}^1$$

$$= +\frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{6} + \frac{1}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{4}$$

(h)

$$\int x^3\sqrt{x^2+1}\,dx \qquad u = x^2+1 \Rightarrow x^2 = u-1$$

$$du = 2x\,dx$$

$$= \frac{1}{2}\int x^2 \cdot \sqrt{x^2+1} \cdot 2x\,dx$$

$$= \frac{1}{2}\int (u-1)\,u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du = \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

3. (10 pts.) **(a)** State the Mean Value Theorem.

(b) Show the inequality: $|\cos a - \cos b| \leq |a - b|$ for all a and b.

4. (10 pts.) **(a)** Let $f(x) = \begin{cases} \frac{\sin x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$, find the values of x where the function

$\mathrm{Si}(x) = \int_0^x f(t)\,dt$ has local maximum values.

(b) Draw the graph of Si(x).

5. (10 pts.) **(a)** Explain the idea of Newton's Method and derive the formula.

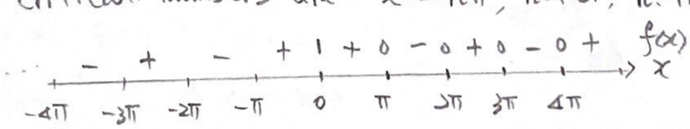
(b) Perform two iterations of Newton's Method to approximate $\sqrt{3}$ staring with $x_1 = 2$.

3. (a) If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number $c \in (a, b)$ so that $f'(c) = \frac{f(b) - f(a)}{b - a}$

(b) Consider $f(x) = \cos x$, then f is continuous on $[a, b]$ and differentiable on (a, b) for any $a, b \in \mathbb{R}$.
And, $f'(x) = -\sin x$.
Hence, there is a number $c \in (a, b)$. so that $-\sin c = \frac{\cos b - \cos a}{b - a}$
Then $-\sin c = \frac{\cos a - \cos b}{a - b}$
Hence, $\left|\frac{\cos a - \cos b}{a - b}\right| = |-\sin c| \leq 1$
Therefore, $|\cos a - \cos b| < |a - b|$ for all a, b

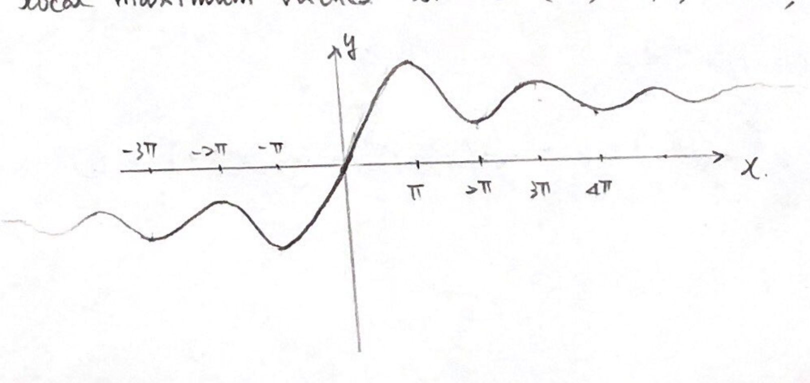
4. (a) $\frac{d}{dx} \mathrm{Si}(x) = \frac{d}{dx} \int_0^x f(t)\,dt = f(x) = \begin{cases} \frac{\sin x}{x} & , x \neq 0. \\ 1 & , x = 0 \end{cases}$

critical numbers are $x = n\pi$, $n \neq 0$, n: integers.



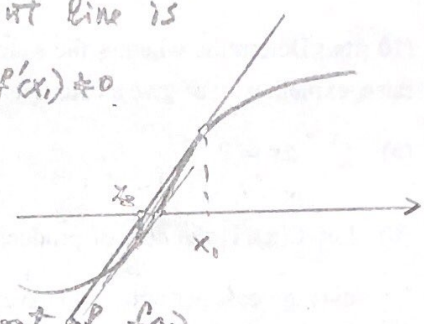
local maximum values at $x = (-1)^{n+1} n\pi$, $n \neq 0$, n: integer

(b)



5. (a) If x_1 is close enough to the root of $f(x)$, we use the tangent line of $f(x)$ at $(x_1, f(x_1))$ to approximate the curve $y = f(x)$, so the tangent line is

$$y - f(x_1) = f'(x_1)(x - x_1) \quad \text{if } f'(x_1) \neq 0$$



Let $y = 0$, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

is an approximate for the root of $f(x)$.

We apply the same idea successively to obtain a sequence $x_1, x_2, \cdots, x_n, \cdots$ approaching to root of $f(x)$.

(b) Let $x = \sqrt{3}$, then $x^2 = 3$

Choose $f(x) = x^2 - 3$, then $\sqrt{3}$ is a root of $f(x)$

and $f'(x) = 2x$.

By Newton's Method, $x_1 = 2$

$$x_2 = 2 - \frac{2^2 - 3}{2 \cdot 2} = \frac{7}{4}$$

$$x_3 = \frac{7}{4} - \frac{\left(\frac{7}{4}\right)^2 - 3}{2 \cdot \frac{7}{4}} = \frac{97}{56} \approx 1.732$$

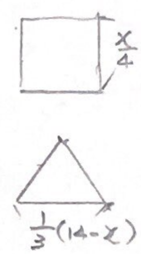
6. **(10 pts.)** A piece of wire 14 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut for the squares so that the total area enclosed is minimum? ($\sqrt{3} \approx 1.732$)

7. **(10 pts.)** Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give a counterexample.

   (a) $\int_{-1}^{1} \frac{1}{x} dx = 0.$

   (b) Let C(x) is the cost of producing x units of a commodity, and $AC(x) = \frac{C(x)}{x}$ is the average cost per unit. If the average cost is a minimum, then the marginal cost equals the average cost.

6. Suppose we cut x m. long for the square, then $0<x<14$
and the length for the triangle is $(14-x)$ m.



The total area $A(x) = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{14-x}{3}\right)^2$

$$A'(x) = 2\left(\frac{x}{4}\right)\cdot\frac{1}{4} + \frac{\sqrt{3}}{4}\cdot 2\cdot\frac{(14-x)}{3}\cdot\left(-\frac{1}{3}\right)$$

$$= \left(\frac{1}{8}+\frac{\sqrt{3}}{18}\right)x - \frac{7\sqrt{3}}{9}$$

$$A'(x)=0 \Rightarrow x = \frac{56\sqrt{3}}{9+4\sqrt{3}} \approx 6.09 \text{ (m)}$$

Since $A'(x)>0$ when $14>x>\frac{56\sqrt{3}}{9+4\sqrt{3}}$ and $A'(x)<0$ when $0<x<\frac{56\sqrt{3}}{9+4\sqrt{3}}$,
by the 1st derivative test, $A\left(\frac{56\sqrt{3}}{9+4\sqrt{3}}\right)$ is the minimum area.
We should cut 6.09 m. long for the square

7. (a) False.

$$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

$$\int_{-1}^{0} \frac{1}{x}\,dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x}\,dx = \lim_{b\to 0^-} \ln|x| \Big|_{-1}^{b} = \lim_{b\to 0^-} \ln|b| = -\infty$$

$$\int_{0}^{1} \frac{1}{x}\,dx = \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x}\,dx = \lim_{a\to 0^+} \ln|x| \Big|_{a}^{1} = \lim_{a\to 0^+} -\ln|a| = +\infty$$

$\int_{-1}^{0} \frac{1}{x}\,dx$ and $\int_{0}^{1} \frac{1}{x}\,dx$ are diverges,

hence $\int_{-1}^{1} \frac{1}{x}\,dx$ diverges.

(b) True.

$$AC(x) = \frac{C(x)}{x} \Rightarrow AC'(x) = \frac{C'(x)\cdot x - C(x)}{x^2}$$

$$AC'(x) = 0 \Rightarrow C'(x)\cdot x - C(x) = 0 \Rightarrow C'(x) = \frac{C(x)}{x}$$

$AC'(x) < 0$ when $0 < x < \frac{C(x)}{C'(x)}$

$AC'(x) > 0$ when $x > \frac{C(x)}{C'(x)}$

By the 1st Derivative Test, $AC(x)$ has minimum value at $x = \frac{C(x)}{C'(x)}$

i.e. $C'(x) = \frac{C(x)}{x}$.