(c) $x_3(t) = \cos(4\pi t) + 9\sin(21\pi t)$

(d) $x_4(t) = 3\sin(4\pi t) + 5\cos(7\pi t) + 6\sin(11\pi t)$

(e) $x_5(t) = \cos(17\pi t) + 5\cos(18\pi t)$

(f) $x_6(t) = \cos(2\pi t) + 7\sin(3\pi t)$

(g) $x_7(t) = 4\cos(7\pi t) + 5\cos(11\pi t)$

(h) $x_8(t) = \cos(120\pi t) + 3\cos(377t)$

(i) $x_9(t) = \cos(19\pi t) + 2\sin(21\pi t)$

(j) $x_{10}(t) = 5\cos(6\pi t) + 6\sin(7\pi t)$

2.4 Sketch the single-sided and double-sided amplitude and phase spectra of

(a) $x_1(t) = 5\cos(12\pi t - \pi/6)$

(b) $x_2(t) = 3\sin(12\pi t) + 4\cos(16\pi t)$

(c) $x_3(t) = 4\cos(8\pi t)\cos(12\pi t)$

(*Hint:* Use an appropriate trigonometric identity.)

(d) $x_4(t) = 8\sin(2\pi t)\cos^2(5\pi t)$

(*Hint:* Use appropriate trigonometric identities.)

(e) $x_5(t) = \cos(6\pi t) + 7\cos(30\pi t)$

(f) $x_6(t) = \cos(4\pi t) + 9\sin(21\pi t)$

(g) $x_7(t) = 2\cos(4\pi t) + \cos(6\pi t) + 6\sin(17\pi t)$

2.5

(a) Show that the function $\delta_\epsilon(t)$ sketched in Figure 2.4(b) has unity area.

(b) Show that

$$\delta_\epsilon(t) = \epsilon^{-1}e^{-t/\epsilon}u(t)$$

has unity area. Sketch this function for $\epsilon = 1, \frac{1}{2}$, and $\frac{1}{4}$. Comment on its suitability as an approximation for the unit impulse function.

(c) Show that a suitable approximation for the unit impulse function as $\epsilon \to 0$ is given by

$$\delta_\epsilon(t) = \begin{cases} \epsilon^{-1}\left(1 - |t|/\epsilon\right), & |t| \le \epsilon \\ 0, & \text{otherwise} \end{cases}$$

2.6 Use the properties of the unit impulse function given after (2.14) to evaluate the following relations.

(a) $\int_{-\infty}^{\infty} [t^2 + \exp(-2t)]\delta(2t-5)\,dt$

(b) $\int_{-10^-}^{10^+} (t^2+1)\left[\sum_{n=-\infty}^{\infty} \delta(t-5n)\right] dt$ (Note: 10^+ means just to the right of 10; -10^- means just to the left of -10)

(c) $10\delta(t) + A\frac{d\delta(t)}{dt} + 3\frac{d^2\delta(t)}{dt^2} = B\delta(t) + 5\frac{d\delta(t)}{dt} + C\frac{d^2\delta(t)}{dt^2}$; find A, B, and C

(d) $\int_{-2}^{11} [e^{-4\pi t} + \tan(10\pi t)]\delta(4t+3)\,dt$

(e) $\int_{-\infty}^{\infty} [\cos(5\pi t) + e^{-3t}]\frac{d\delta^2(t-2)}{dt^2}\,dt$

2.7 Which of the following signals are periodic and which are aperiodic? Find the periods of those that are periodic. Sketch all signals.

(a) $x_a(t) = \cos(5\pi t) + \sin(7\pi t)$

(b) $x_b(t) = \sum_{n=0}^{\infty} \Lambda(t-2n)$

(c) $x_c(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$

(d) $x_d(t) = \sin(3t) + \cos(2\pi t)$

(e) $x_e(t) = \sum_{n=-\infty}^{\infty} \Pi(t-3n)$

(f) $x_f(t) = \sum_{n=0}^{\infty} \Pi(t-3n)$

2.8 Write the signal $x(t) = \cos(6\pi t) + 2\sin(10\pi t)$ as

(a) The real part of a sum of rotating phasors.

(b) A sum of rotating phasors plus their complex conjugates.

(c) From your results in parts (a) and (b), sketch the single-sided and double-sided amplitude and phase spectra of $x(t)$.

Section 2.2

2.9 Find the normalized power for each signal below that is a power signal and the normalized energy for each signal that is an energy signal. If a signal is neither a power signal nor an energy signal, so designate it. Sketch each signal (α is a positive constant).

(a) $x_1(t) = 2\cos(4\pi t + 2\pi/3)$

(b) $x_2(t) = e^{-\alpha t}u(t)$

(c) $x_3(t) = e^{\alpha t}u(-t)$

(d) $x_4(t) = \left(\alpha^2 + t^2\right)^{-1/2}$

(e) $x_5(t) = e^{-\alpha|t|}$

(f) $x_6(t) = e^{-\alpha t}u(t) - e^{-\alpha(t-1)}u(t-1)$

2.10 Classify each of the following signals as an energy signal or as a power signal by calculating E, the energy, or P, the power (A, B, θ, ω, and τ are positive constants).

(a) $x_1(t) = A|\sin(\omega t + \theta)|$

(b) $x_2(t) = A\tau/\sqrt{\tau + jt},\ j = \sqrt{-1}$

(c) $x_3(t) = Ate^{-t/\tau}u(t)$

(d) $x_4(t) = \Pi(t/\tau) + \Pi(t/2\tau)$

(e) $x_5(t) = \Pi(t/2) + \Lambda(t)$

(f) $x_6(t) = A\cos(\omega t) + B\sin(2\omega t)$

2.11 Find the powers of the following periodic signals. In each case provide a sketch of the signal and give its period.

(a) $x_1(t) = 2\cos(4\pi t - \pi/3)$

(b) $x_2(t) = \sum_{n=-\infty}^{\infty} 3\Pi\left(\frac{t-4n}{2}\right)$

(c) $x_3(t) = \sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t-6n}{2}\right)$

(d) $x_4(t) = \sum_{n=-\infty}^{\infty} \left[\Lambda(t-4n) + \Pi\left(\frac{t-4n}{2}\right)\right]$

2.12 For each of the following signals, determine both the normalized energy and power. Tell which are power signals, which are energy signals, and which are neither. (Note: 0 and ∞ are possible answers.)

(a) $x_1(t) = 6e^{(-3+j4\pi)t}u(t)$

(b) $x_2(t) = \Pi[(t-3)/2] + \Pi(\frac{t-3}{6})$

(c) $x_3(t) = 7e^{j6\pi t}u(t)$

(d) $x_4(t) = 2\cos(4\pi t)$

(e) $x_5(t) = |t|$

(f) $x_6(t) = t^{-1/2}u(t-1)$

2.13 Show that the following are energy signals. Sketch each signal.

(a) $x_1(t) = \Pi(t/12)\cos(6\pi t)$

(b) $x_2(t) = e^{-|t|/3}$

(c) $x_3(t) = 2u(t) - 2u(t-8)$

(d) $x_4(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda - 2\int_{-\infty}^{t-10} u(\lambda)\,d\lambda + \int_{-\infty}^{t-20} u(\lambda)\,d\lambda$

(*Hint:* Consider what the indefinite integral of a step function is first.)

2.14 Find the energies and powers of the following signals (note that 0 and ∞ are possible answers). Tell which are energy signals and which are power signals.

(a) $x_1(t) = \cos(10\pi t)u(t)\,u(2-t)$

(b) $x_2(t) = \sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t-3n}{2}\right)$

(c) $x_3(t) = e^{-|t|}\cos(2\pi t)$

(d) $x_4(t) = \Pi\left(\frac{t}{2}\right) + \Lambda(t)$

Section 2.3

2.15 Using the uniqueness property of the Fourier series, find exponential Fourier series for the following signals (f_0 is an arbitrary frequency):

(a) $x_1(t) = \sin^2(2\pi f_0 t)$

(b) $x_2(t) = \cos(2\pi f_0 t) + \sin(4\pi f_0 t)$

(c) $x_3(t) = \sin(4\pi f_0 t)\cos(4\pi f_0 t)$

(d) $x_4(t) = \cos^3(2\pi f_0 t)$

(e) $x_5(t) = \sin(2\pi f_0 t)\cos^2(4\pi f_0 t)$

(f) $x_6(t) = \sin^2(3\pi f_0 t)\cos(5\pi f_0 t)$

(*Hint*: Use appropriate trigonometric identities and Euler's theorem.)

2.16 Expand the signal $x(t) = 2t^2$ in a complex exponential Fourier series over the interval $|t| \le 2$. Sketch the signal to which the Fourier series converges for all t.

2.17 If $X_n = |X_n|\exp[j\underline{/X_n}]$ are the Fourier coefficients of a real signal, $x(t)$, fill in all the steps to show that:

(a) $|X_n| = |X_{-n}|$ and $\underline{/X_n} = -\underline{/X_{-n}}$.

(b) X_n is a real, even function of n for $x(t)$ even.

(c) X_n is imaginary and an odd function of n for $x(t)$ odd.

(d) $x(t) = -x(t + T_0/2)$ (halfwave odd symmetry) implies that $X_n = 0$, n even.

2.18 Obtain the complex exponential Fourier series coefficients for the (a) pulse train, (b) half-rectified sinewave, (c) full-rectified sinewave, and (d) triangular waveform as given in Table 2.1.

2.19 Find the ratio of the power contained in a rectangular pulse train for $|nf_0| \le \tau^{-1}$ to the total power for each of the following cases:

(a) $\tau/T_0 = \frac{1}{2}$ (c) $\tau/T_0 = \frac{1}{10}$

(b) $\tau/T_0 = \frac{1}{5}$ (d) $\tau/T_0 = \frac{1}{20}$

(*Hint:* You can save work by noting the spectra are even about $f = 0$.)

2.20

(a) If $x(t)$ has the Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

and $y(t) = x(t - t_0)$, show that

$$Y_n = X_n e^{-j2\pi n f_0 t_0}$$

where the Y_n's are the Fourier coefficients for $y(t)$.

(b) Verify the theorem proved in part (a) by examining the Fourier coefficients for $x(t) = \cos(\omega_0 t)$ and $y(t) = \sin(\omega_0 t)$.

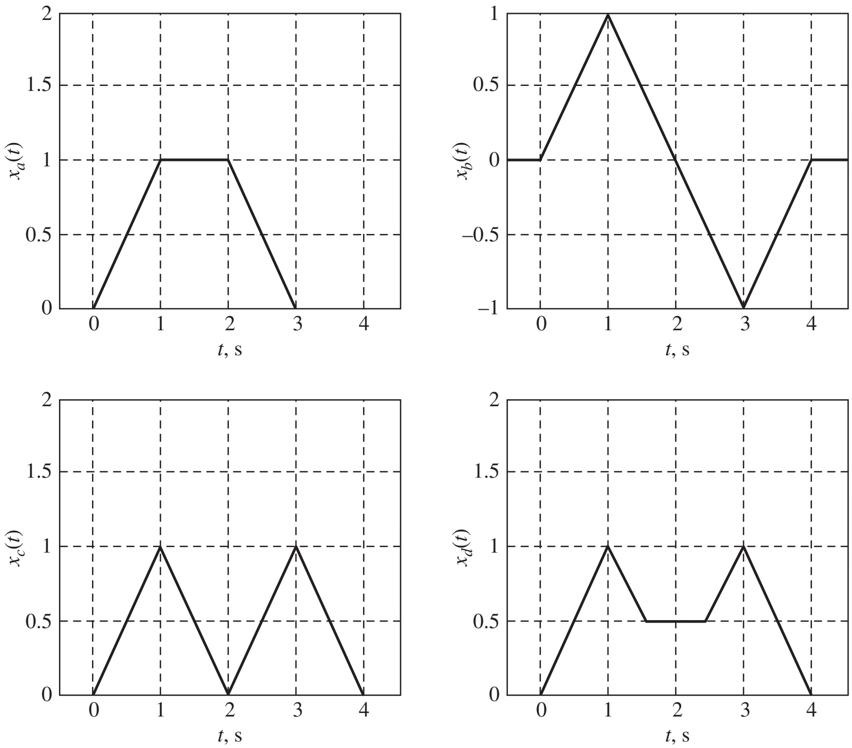

Figure 2.36

(b) Use the result above and the relation $u(t) = \frac{1}{2}\left[\operatorname{sgn}(t) + 1\right]$ to find the Fourier transform of the unit step.

(c) Use the integration theorem and the Fourier transform of the unit impulse function to find the Fourier transform of the unit step. Compare the result with part (b).

2.26 Using only the Fourier transform of the unit impulse function and the differentiation theorem, find the Fourier transforms of the signals shown in Figure 2.36.

2.27

(a) Write the signals of Figure 2.37 as the linear combination of two delayed triangular functions. That is, write $x_a(t) = a_1\Lambda\left(\left(t - t_1\right)/T_1\right) + a_2\Lambda\left(\left(t - t_2\right)/T_2\right)$ by finding appropriate values for a_1, a_2, t_1, t_2, T_1, and T_2. Do similar expressions for all four signals shown in Figure 2.36.

(b) Given the Fourier-transform pair $\Lambda(t) \longleftrightarrow \operatorname{sinc}^2(f)$, find their Fourier transforms using the superposition, scale-change, and time-delay theorems. Compare your results with the answers obtained in Problem 2.26.

2.28

(a) Given $\Pi(t) \longleftrightarrow \operatorname{sinc}(f)$, find the Fourier transforms of the following signals using the frequency-translation followed by the time-delay theorem.

(i) $x_1(t) = \Pi(t-1)\exp[j4\pi(t-1)]$

(ii) $x_2(t) = \Pi(t+1)\exp[j4\pi(t+1)]$

(b) Repeat the above, but now applying the time-delay theorem followed by the frequency-translation theorem.

2.29 By applying appropriate theorems and using the signals defined in Problem 2.28, find Fourier transforms of the following signals:

(a) $x_a(t) = \frac{1}{2}x_1(t) + \frac{1}{2}x_1(-t)$

(b) $x_b(t) = \frac{1}{2}x_2(t) + \frac{1}{2}x_2(-t)$

Provide MATLAB plots of $y(t)$ and $Y(f)$ [note that $Y(f)$ is real]. Compare with part (a).

(c) Use Equation (2.134) with the result of part (a) to find the Fourier transform of the half-rectified cosine wave.

2.39 Provide plots of the following functions of time and find their Fourier transforms. Tell which ones should be real and even functions of f and which ones should be imaginary and odd functions of f. Do your results bear this out?

(a) $x_1(t) = \Lambda\left(\frac{t}{2}\right) + \Pi\left(\frac{t}{2}\right)$

(b) $x_2(t) = \Pi\left(\frac{t}{2}\right) - \Lambda(t)$

(c) $x_3(t) = \Pi\left(t + \frac{1}{2}\right) - \Pi\left(t - \frac{1}{2}\right)$

(d) $x_4(t) = \Lambda(t-1) - \Lambda(t+1)$

(e) $x_5(t) = \Lambda(t)\mathrm{sgn}(t)$

(f) $x_6(t) = \Lambda(t)\cos(2\pi t)$

Section 2.5

2.40

(a) Obtain the time-average autocorrelation function of $x(t) = 3 + 6\cos(20\pi t) + 3\sin(20\pi t)$.

(*Hint*: Combine the cosine and sine terms into a single cosine with a phase angle.)

(b) Obtain the power spectral density of the signal of part (a). What is its total average power?

2.41 Find the power spectral densities and average powers of the following signals.

(a) $x_1(t) = 2\cos(20\pi t + \pi/3)$

(b) $x_2(t) = 3\sin(30\pi t)$

(c) $x_3(t) = 5\sin(10\pi t - \pi/6)$

(d) $x_4(t) = 3\sin(30\pi t) + 5\sin(10\pi t - \pi/6)$

2.42 Find the autocorrelation functions of the signals having the following power spectral densities. Also give their average powers.

(a) $S_1(f) = 4\delta(f-15) + 4\delta(f+15)$

(b) $S_2(f) = 9\delta(f-20) + 9\delta(f+20)$

(c) $S_3(f) = 16\delta(f-5) + 16\delta(f+5)$

(d) $S_4(f) = 9\delta(f-20) + 9\delta(f+20) + 16\delta(f-5) + 16\delta(f+5)$

2.43 By applying the properties of the autocorrelation function, determine whether the following are acceptable for autocorrelation functions. In each case, tell why or why not.

(a) $R_1(\tau) = 2\cos(10\pi\tau) + \cos(30\pi\tau)$

(b) $R_2(\tau) = 1 + 3\cos(30\pi\tau)$

(c) $R_3(\tau) = 3\cos(20\pi\tau + \pi/3)$

(d) $R_4(\tau) = 4\Lambda(\tau/2)$

(e) $R_5(\tau) = 3\Pi(\tau/6)$

(f) $R_6(\tau) = 2\sin(10\pi\tau)$

2.44 Find the autocorrelation functions corresponding to the following signals.

(a) $x_1(t) = 2\cos(10\pi t + \pi/3)$

(b) $x_2(t) = 2\sin(10\pi t + \pi/3)$

(c) $x_3(t) = \mathrm{Re}\left[3\exp(j10\pi t) + 4j\exp(j10\pi t)\right]$

(d) $x_4(t) = x_1(t) + x_2(t)$

2.45 Show that the $R(\tau)$ of Example 2.20 has the Fourier transform $S(f)$ given there. Plot the power spectral density.

Section 2.6

2.46 A system is governed by the differential equation (a, b, and c are nonnegative constants)

$$\frac{dy(t)}{dt} + ay(t) = b\frac{dx(t)}{dt} + cx(t)$$

(a) Find $H(f)$.

(b) Find and plot $|H(f)|$ and $\underline{/H(f)}$ for $c = 0$.

(c) Find and plot $|H(f)|$ and $\underline{/H(f)}$ for $b = 0$.

2.47 For each of the following transfer functions, determine the unit impulse response of the system.

(a) $H_1(f) = \dfrac{1}{7 + j2\pi f}$

(b) $H_2(f) = \dfrac{j2\pi f}{7 + j2\pi f}$

(*Hint*: Use long division first.)

(c) $H_3(f) = \dfrac{e^{-j6\pi f}}{7 + j2\pi f}$

(d) $H_4(f) = \dfrac{1 - e^{-j6\pi f}}{7 + j2\pi f}$

2.48 A filter has frequency response function $H(f) = \Pi(f/2B)$ and input $x(t) = 2W\,\mathrm{sinc}(2Wt)$.

(a) Find the output $y(t)$ for $W < B$.

(b) Find the output $y(t)$ for $W > B$.

(c) In which case does the output suffer distortion? What influenced your answer?

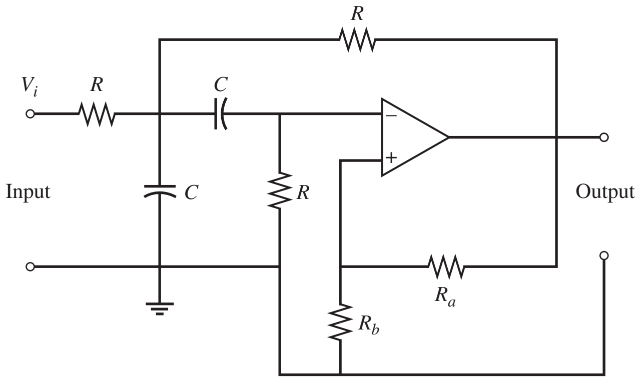

Figure 2.37

2.49 A second-order active bandpass filter (BPF), known as a bandpass Sallen–Key circuit, is shown in Figure 2.37.

(a) Show that the frequency response function of this filter is given by

$$H(j\omega) = \frac{\left(K\omega_0/\sqrt{2}\right)(j\omega)}{-\omega^2 + \left(\omega_0/Q\right)(j\omega) + \omega_0^2}, \quad \omega = 2\pi f$$

where

$$\omega_0 = \sqrt{2}(RC)^{-1}$$

$$Q = \frac{\sqrt{2}}{4 - K}$$

$$K = 1 + \frac{R_a}{R_b}$$

(b) Plot $|H(f)|$.

(c) Show that the 3-dB bandwidth of the filter can be expressed as $B = f_0/Q$, where $f_0 = \omega_0/2\pi$.

(d) Design a BPF using this circuit with center frequency $f_0 = 1000$ Hz and 3-dB bandwidth of 300 Hz. Find values of R_a, R_b, R, and C that will give these desired specifications.

2.50 For the two circuits shown in Figure 2.38, determine $H(f)$ and $h(t)$. Sketch accurately the amplitude and phase responses. Plot the amplitude response in decibels. Use a logarithmic frequency axis.

2.51 Using the Paley-Wiener criterion, show that

$$|H(f)| = \exp(-\beta f^2)$$

is not a suitable amplitude response for a causal, linear time-invariant filter.

2.52 Determine whether or not the filters with impulse responses given below are BIBO stable. α and f_0 are postive constants.

(a) $h_1(t) = \exp(-\alpha|t|)\cos\left(2\pi f_0 t\right)$

(b) $h_2(t) = \cos\left(2\pi f_0 t\right) u(t)$

(c) $h_3(t) = t^{-1}u(t-1)$

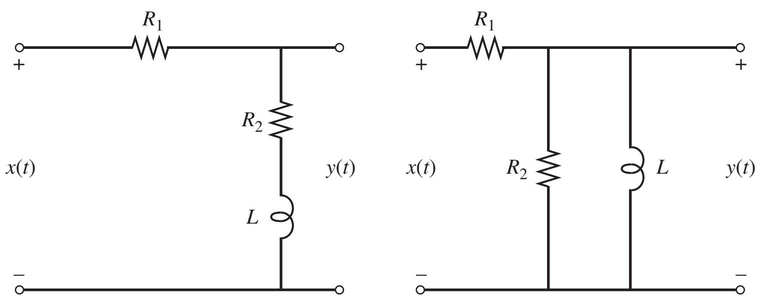

Figure 2.38

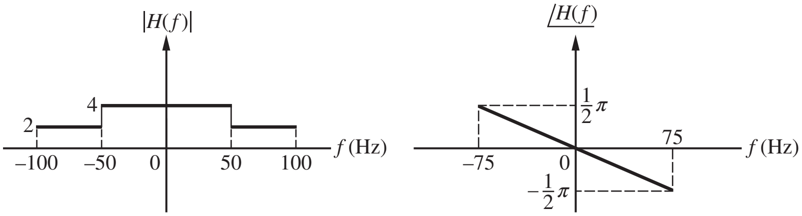

Figure 2.39

(d) $h_4(t) = e^{-t}u(t) - e^{-(t-1)}u(t-1)$

(e) $h_5(t) = t^{-2}u(t-1)$

(f) $h_6(t) = \text{sinc}(2t)$

2.53 Given a filter with frequency response function

$$H(f) = \frac{5}{4 + j(2\pi f)}$$

and input $x(t) = e^{-3t}u(t)$, obtain and plot accurately the energy spectral densities of the input and output.

2.54 A filter with frequency response function

$$H(f) = 3\Pi\left(\frac{f}{62}\right)$$

has as an input a half-rectified cosine waveform of fundamental frequency 10 Hz. Determine an analytical expression for the output of the filter. Plot the output using MATLAB.

2.55 Another definition of bandwidth for a signal is the 90% energy containment bandwidth. For a signal with energy spectral density $G(f) = |X(f)|^2$, it is given by B_{90} in the relation

$$0.9E_{\text{Total}} = \int_{-B_{90}}^{B_{90}} G(f)\,df = 2\int_0^{B_{90}} G(f)\,df;$$

$$E_{\text{Total}} = \int_{-\infty}^{\infty} G(f)\,df = 2\int_0^{\infty} G(f)\,df$$

Obtain B_{90} for the following signals if it is defined. If it is not defined for a particular signal, state why it is not.

(a) $x_1(t) = e^{-\alpha t}u(t)$, where α is a positive constant

(b) $x_2(t) = 2W\,\text{sinc}(2Wt)$ where W is a positive constant

(c) $x_3(t) = \Pi(t/\tau)$ (requires numerical integration)

(d) $x_4(t) = \Lambda(t/\tau)$ (requires numerical integration)

(e) $x_5(t) = e^{-\alpha|t|}$

2.56 An ideal quadrature phase shifter has frequency response function

$$H(f) = \begin{cases} e^{-j\pi/2}, & f > 0 \\ e^{+j\pi/2}, & f < 0 \end{cases}$$

Find the outputs for the following inputs:

(a) $x_1(t) = \exp(j100\pi t)$

(b) $x_2(t) = \cos(100\pi t)$

(c) $x_3(t) = \sin(100\pi t)$

(d) $x_4(t) = \Pi(t/2)$

2.57 A filter has amplitude response and phase shift shown in Figure 2.39. Find the output for each of the inputs given below. For which cases is the transmission distortionless? Tell what type of distortion is imposed for the others.

(a) $\cos(48\pi t) + 5\cos(126\pi t)$

(b) $\cos(126\pi t) + 0.5\cos(170\pi t)$

(c) $\cos(126\pi t) + 3\cos(144\pi t)$

(d) $\cos(10\pi t) + 4\cos(50\pi t)$

2.58 Determine and accurately plot, on the same set of axes, the group delay and the phase delay for the systems with unit impulse responses:

(a) $h_1(t) = 3e^{-5t}u(t)$

(b) $h_2(t) = 5e^{-3t}u(t) - 2e^{-5t}u(t)$

(c) $h_3(t) = \text{sinc}\left[2B\left(t - t_0\right)\right]$ where B and t_0 are positive constants

(d) $h_4(t) = 5e^{-3t}u(t) - 2e^{-3(t-t_0)}u\left(t - t_0\right)$ where t_0 is a positive constant

2.59 A system has the frequency response function

$$H(f) = \frac{j2\pi f}{(8 + j2\pi f)(3 + j2\pi f)}$$

Determine and accurately plot the following: (a) The amplitude response; (b) The phase response; (c) The phase delay; (d) The group delay.

2.60 The nonlinear system defined by

$$y(t) = x(t) + 0.1x^2(t)$$

has an input signal with the bandpass spectrum

$$X(f) = 2\Pi\left(\frac{f-10}{4}\right) + 2\Pi\left(\frac{f+10}{4}\right)$$

Sketch the spectrum of the output, labeling all important frequencies and amplitudes.

2.61 Given a filter with frequency response function

$$H(f) = \frac{j2\pi f}{\left(9 - 4\pi^2 f^2\right) + j0.3\pi f}$$

Determine and accurately plot the following: (a) The amplitude response; (b) The phase response; (c) The phase delay; (d) The group delay.

2.62 Given a nonlinear, zero-memory device with transfer characteristic

$$y(t) = x^3(t),$$

find its output due to the input

$$x(t) = \cos(2\pi t) + \cos(6\pi t)$$

List all frequency components and tell whether thay are due to harmonic generation or intermodulation terms.

2.63 Find the impulse response of an ideal highpass filter with the frequency response function

$$H_{\mathrm{HP}}(f) = H_0\left[1 - \Pi\left(\frac{f}{2W}\right)\right] e^{-j2\pi f t_0}$$

2.64 Verify the pulsewidth-bandwidth relationship of Equation (2.234) for the following signals. Sketch each signal and its spectrum.

- **(a)** $x(t) = A\exp(-t^2/2\tau^2)$ (Gaussian pulse)
- **(b)** $x(t) = A\exp(-\alpha|t|)$, $\alpha > 0$ (double-sided exponential)

2.65

- **(a)** Show that the frequency response function of a second-order Butterworth filter is

$$H(f) = \frac{f_3^2}{f_3^2 + j\sqrt{2}f_3 f - f^2}$$

where f_3 is the 3-dB frequency in hertz.

- **(b)** Find an expression for the group delay of this filter. Plot the group delay as a function of f/f_3.
- **(c)** Given that the step response for a second-order Butterworth filter is

$$y_s(t) = \left[1 - \exp\left(-\frac{2\pi f_3 t}{\sqrt{2}}\right) \times \left(\cos\frac{2\pi f_3 t}{\sqrt{2}} + \sin\frac{2\pi f_3 t}{\sqrt{2}}\right)\right] u(t)$$

where $u(t)$ is the unit step function, find the 10% to 90% risetime in terms of f_3.

Section 2.7

2.66 A sinusoidal signal of frequency 1 Hz is to be sampled periodically.

- **(a)** Find the maximum allowable time interval between samples.
- **(b)** Samples are taken at $\frac{1}{3}$-s intervals (i.e., at a rate of $f_s = 3$ sps). Construct a plot of the sampled signal spectrum that illustrates that this is an acceptable sampling rate to allow recovery of the original sinusoid.
- **(c)** The samples are spaced $\frac{2}{3}$ s apart. Construct a plot of the sampled signal spectrum that shows what the recovered signal will be if the samples are passed through a lowpass filter such that only the lowest frequency spectral lines are passed.

2.67 A flat-top sampler can be represented as the block diagram of Figure 2.40.

- **(a)** Assuming $\tau \ll T_s$, sketch the output for a typical $x(t)$.
- **(b)** Find the spectrum of the output, $Y(f)$, in terms of the spectrum of the input, $X(f)$. Determine the relationship between τ and T_s required to minimize distortion in the recovered waveform?

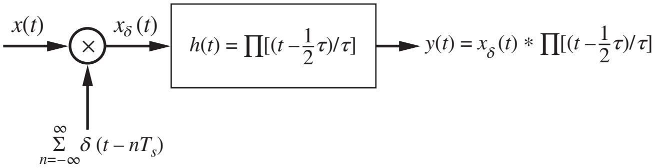

Figure 2.40

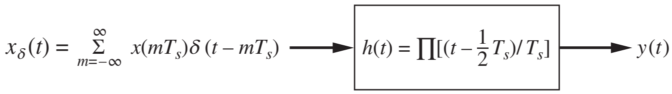

Figure 2.41

2.68 Figure 2.41 illustrates so-called *zero-order-hold reconstruction*.

(a) Sketch $y(t)$ for a typical $x(t)$. Under what conditions is $y(t)$ a good approximation to $x(t)$?

(b) Find the spectrum of $y(t)$ in terms of the spectrum of $x(t)$. Discuss the approximation of $y(t)$ to $x(t)$ in terms of frequency-domain arguments.

2.69 Determine the range of permissible cutoff frequencies for the ideal lowpass filter used to reconstruct the signal

$$x(t) = 10\cos^2(600\pi t)\cos(2400\pi t)$$

which is sampled at 4500 samples per second. Sketch $X(f)$ and $X_\delta(f)$. Find the minimum allowable sampling frequency.

2.70 Given the bandpass signal spectrum shown in Figure 2.42, sketch spectra for the following sampling rates f_s and indicate which ones are suitable.

(a) $2B$ (b) $2.5B$ (c) $3B$ (d) $4B$ (e) $5B$ (f) $6B$

Section 2.8

2.71 Using appropriate Fourier-transform theorems and pairs, express the spectrum $Y(f)$ of

$$y(t) = x(t)\cos(\omega_0 t) + \hat{x}(t)\sin(\omega_0 t)$$

in terms of the spectrum $X(f)$ of $x(t)$, where $X(f)$ is lowpass with bandwidth

$$B < f_0 = \frac{\omega_0}{2\pi}$$

Sketch $Y(f)$ for a typical $X(f)$.

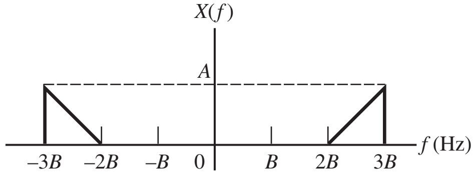

Figure 2.42

2.72 Show that $x(t)$ and $\hat{x}(t)$ are orthogonal for the following signals:

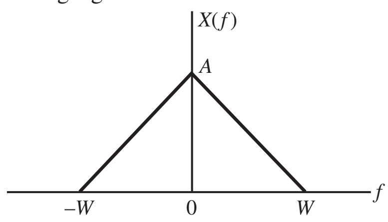

Figure 2.43

(a) $x_1(t) = \sin(\omega_0 t)$

(b) $x_2(t) = 2\cos(\omega_0 t) + \sin(\omega_0 t)\cos(2\omega_0 t)$

(c) $x_3(t) = A\exp(j\omega_0 t)$

2.73 Assume that the Fourier transform of $x(t)$ is real and has the shape shown in Figure 2.43. Determine and plot the spectrum of each of the following signals:

(a) $x_1(t) = \frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)$

(b) $x_2(t) = \left[\frac{3}{4}x(t) + \frac{3}{4}j\hat{x}(t)\right]e^{j2\pi f_0 t},\ f_0 \gg W$

(c) $x_3(t) = \left[\frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)\right]e^{j2\pi W t}$

(d) $x_4(t) = \left[\frac{2}{3}x(t) - \frac{1}{3}j\hat{x}(t)\right]e^{j\pi W t}$

2.74 Following Example 2.30, consider

$$x(t) = 2\cos(52\pi t)$$

Find $\hat{x}(t)$, $x_p(t)$, $\tilde{x}(t)$, $x_R(t)$, and $x_I(t)$ for the following cases: (a) $f_0 = 25$ Hz; (b) $f_0 = 27$ Hz; (c) $f_0 = 10$ Hz; (d) $f_0 = 15$ Hz; (e) $f_0 = 30$ Hz; (f) $f_0 = 20$ Hz.

2.75 Consider the input

$$x(t) = \Pi(t/\tau)\cos[2\pi(f_0 + \Delta f)t],\quad \Delta f \ll f_0$$

to a filter with impulse response

$$h(t) = \alpha e^{-\alpha t}\cos(2\pi f_0 t)u(t)$$

Find the output using complex envelope techniques.

Also $A_c = 20$ V and $f_c = 300$ Hz. Determine the expression for the upper-sideband SSB signal and the lower-sideband SSB signal. Write these in a way that shows the amplitude and frequency of all transmitted components.

3.7 Equation (3.63) gives the amplitude and phase for the VSB signal components centered about $f = +f_c$. Give the amplitude and phase of the signal comonents centered about $f = -f_c$. Using these values show that the VSB signal is real.

3.8 An AM radio uses the standard IF frequency of 455 kHz and is tuned to receive a signal having a carrier frequency of 1020 kHz. Determine the frequency of the local oscillator for both low-side tuning and high-side tuning. Give the image frequencies for each.

3.9 The input to an AM receiver input consists of both modulated carrier (the message signal is a single tone) and interference terms. Assuming that $A_i = 100$ V, $A_m = 0.2$ V, $A_c = 1$ V, $f_m = 10$ Hz, $f_c = 300$ Hz, and $f_i = 320$ Hz, approximate the envelope detector output by giving the amplitudes and frequencies of all components at the envelope detector output.

3.10 A PAM signal is formed by sampling an analog signal at 5 kHz. The duty cycle of the generated PAM pulses is to be 5%. Define the transfer function of the holding circuit by giving the value of τ in (3.92). Define the transfer function of the equalizing filter.

3.11 Rewrite (3.100) to show that relationship between δ_0/A and $T_s f_1$. A signal defined by

$$m(t) = A\cos(40\pi t)$$

is sampled at 1000 Hz to form a DM signal. Give the minium value of δ_0/A to prevent slope overload.

3.12 A TDM signal consists of four signals having bandwidths of 1000, 2000, 4000, and 6000 Hz. What is the total bandwidth of the composite TDM signal. What is the lowest possible sampling frequency for the TDM signal?

Problems

Section 3.1

3.1 Assume that a DSB signal

$$x_c(t) = A_c m(t)\cos(2\pi f_c t + \phi_0)$$

is demodulated using the demodulation carrier $2\cos[2\pi f_c t + \theta(t)]$. Determine, in general, the demodulated output $y_D(t)$. Let $A_c = 1$ and $\theta(t) = \theta_0$, where θ_0 is a constant, and determine the mean-square error between $m(t)$ and the demodulated output as a function of ϕ_0 and θ_0. Now let $\theta_0 = 2\pi f_0 t$ and compute the mean-square error between $m(t)$ and the demodulated output.

3.2 A message signal is given by

$$m(t) = \sum_{k=1}^{5} \frac{10}{k}\sin(2\pi k f_m t)$$

and the carrier is given by

$$c(t) = 100\cos(200\pi t)$$

Write the transmitted signal as a Fourier series and determine the transmitted power.

Section 3.2

3.3 Design an envelope detector that uses a full-wave rectifier rather than the half-wave rectifier shown in Figure 3.3. Sketch the resulting waveforms, as was done in for a half-wave rectifier. What are the advantages of the full-wave rectifier?

3.4 Three message signals are periodic with period T, as shown in Figure 3.32. Each of the three message signals is applied to an AM modulator. For each message signal, determine the modulation efficiency for $a = 0.2$, $a = 0.3$, $a = 0.4$, $a = 0.7$, and $a = 1$.

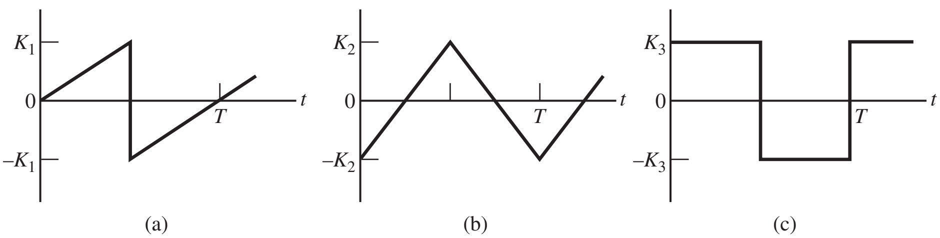

Figure 3.32

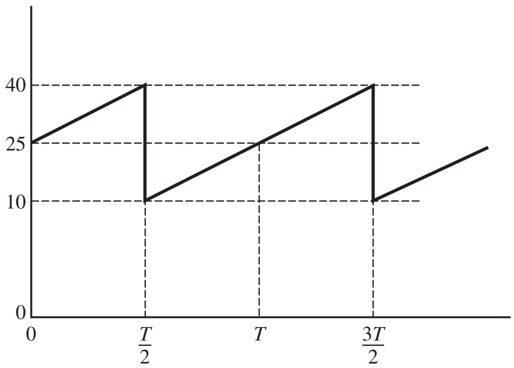

Figure 3.33

3.5 The positive portion of the envelope of the output of an AM modulator is shown in Figure 3.33. The message signal is a waveform having zero DC value. Determine the modulation index, the carrier power, the efficiency, and the power in the sidebands.

3.6 A message signal is a square wave with maximum and minimum values of 8 and −8 V, respectively. The modulation index $a = 0.7$ and the carrier amplitude $A_c = 100$ V. Determine the power in the sidebands and the efficiency. Sketch the modulation trapezoid.

3.7 In this problem we examine the efficiency of AM for the case in which the message signal does not have symmetrical maximum and minimum values. Two message signals are shown in Figure 3.34. Each is periodic with period T, and τ is chosen such that the DC value of $m(t)$ is zero. Calculate the efficiency for each $m(t)$ for $a = 0.7$ and $a = 1$.

3.8 An AM modulator operates with the message signal

$$m(t) = 9\cos(20\pi t) - 8\cos(60\pi t)$$

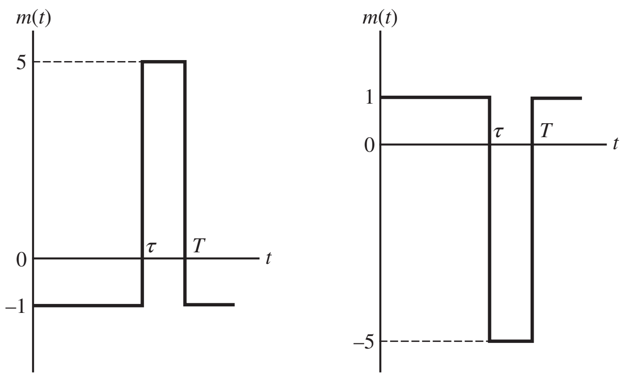

Figure 3.34

The unmodulated carrier is given by $110\cos(200\pi t)$, and the system operates with an index of 0.8.

(a) Write the equation for $m_n(t)$, the normalized signal with a minimum value of −1.

(b) Determine $\langle m_n^2(t)\rangle$, the power in $m_n(t)$.

(c) Determine the efficiency of the modulator.

(d) Sketch the double-sided spectrum of $x_c(t)$, the modulator output, giving the weights and frequencies of all components.

3.9 Rework Problem 3.8 for the message signal

$$m(t) = 9\cos(20\pi t) + 8\cos(60\pi t)$$

3.10 An AM modulator has output

$$x_c(t) = 40\cos[2\pi(200)t] + 5\cos[2\pi(180)t] + 5\cos[2\pi(220)t]$$

Determine the modulation index and the efficiency.

3.11 An AM modulator has output

$$x_c(t) = A\cos[2\pi(200)t] + B\cos[2\pi(180)t] + B\cos[2\pi(220)t]$$

The carrier power is P_0 and the efficiency is E_{ff}. Derive an expression for E_{ff} in terms of P_0, A, and B. Determine A, B, and the modulation index for $P_0 = 200\ W$ and $E_{ff} = 30\%$.

3.12 An AM modulator has output

$$x_c(t) = 25\cos[2\pi(150)t] + 5\cos[2\pi(160)t] + 5\cos[2\pi(140)t]$$

Determine the modulation index and the efficiency.

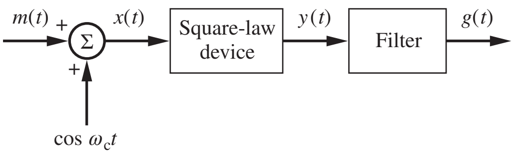

Figure 3.35

3.13 An AM modulator is operating with an index of 0.8. The modulating signal is

$$m(t) = 2\cos(2\pi f_m t) + \cos(4\pi f_m t) + 2\cos(10\pi f_m t)$$

(a) Sketch the spectrum of the modulator output showing the weights of all impulse functions.

(b) What is the efficiency of the modulation process?

3.14 Consider the system shown in Figure 3.35. Assume that the average value of $m(t)$ is zero and that the maximum value of $|m(t)|$ is M. Also assume that the square-law device is defined by $y(t) = 4x(t) + 2x^2(t)$.

(a) Write the equation for $y(t)$.

(b) Describe the filter that yields an AM signal for $g(t)$. Give the necessary filter type and the frequencies of interest.

(c) What value of M yields a modulation index of 0.1?

(d) What is an advantage of this method of modulation?

Section 3.3

3.15 Assume that a message signal is given by

$$m(t) = 4\cos(2\pi f_m t) + \cos(4\pi f_m t)$$

Calculate an expression for

$$x_c(t) = \frac{1}{2}A_c m(t)\cos(2\pi f_c t) \pm \frac{1}{2}A_c \widehat{m}(t)\sin(2\pi f_c t)$$

for $A_c = 10$. Show, by sketching the spectra, that the result is upper-sideband or lower-sideband SSB depending upon the choice of the algebraic sign.

3.16 Redraw Figure 3.10 to illustrate the generation of upper-sideband SSB. Give the equation defining the upper-sideband filter. Complete the analysis by deriving the expression for the output of an upper-sideband SSB modulator.

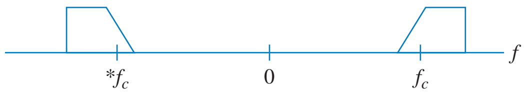

Figure 3.36

3.17 Squaring a DSB or AM signal generates a frequency component at twice the carrier frequency. Is this also true for SSB signals? Show that it is or is not.

Section 3.4

3.18 Prove analytically that carrier reinsertion with envelope detection can be used for demodulation of VSB.

3.19 Figure 3.36 shows the spectrum of a VSB signal. The amplitude and phase characteristics are the same as described in Example 3.3. Show that upon coherent demodulation, the output of the demodulator is real.

Section 3.5

3.20 Sketch Figure 3.20 for the case where $f_{\mathrm{LO}} = f_c - f_{\mathrm{IF}}$.

3.21 A mixer is used in a short-wave superheterodyne receiver. The receiver is designed to receive transmitted signals between 10 and 30 MHz. High-side tuning is to be used. Determine an acceptable IF frequency and the tuning range of the local oscillator. Strive to generate a design that yields the minimum tuning range.

3.22 A superheterodyne receiver uses an IF frequency of 455 kHz. The receiver is tuned to a transmitter having a carrier frequency of 1100 kHz. Give two permissible frequencies of the local oscillator and the image frequency for each. Repeat assuming that the IF frequency is 2500 kHz.

Section 3.6

3.23 A DSB signal is squared to generate a carrier component that may then be used for demodulation. (A technique for doing this, namely the phase-locked loop, will be studied in the next chapter.) Derive an expression that illustrates the impact of interference on this technique.

Section 3.7

3.24 A continuous-time signal is sampled and input to a holding circuit. The product of the holding time and the sampling frequency is τf_s. Plot the amplitude response of the required equalizer as a function of τf_s. What problem, or problems, arise if a large value of τ is used while the sampling frequency is held constant?

Section 3.8

3.25 A continuous data signal is quantized and transmitted using a PCM system. If each data sample at the receiving end of the system must be known to within $\pm 0.25\%$ of the peak-to-peak full-scale value, how many binary symbols must each transmitted digital word contain? Assume that the message signal is speech and has a bandwidth of 4 kHz. Estimate the bandwidth of the resulting PCM signal (choose k).

3.26 A delta modulator has the message signal

$$m(t) = 3 \sin 2\pi(10)t + 4 \sin 2\pi(20)t$$

Determine the minimum sampling frequency required to prevent slope overload, assuming that the impulse weights δ_0 are 0.05π.

3.27 Five messages bandlimited to $W, W, 2W, 4W$, and $4W$ Hz, respectively, are to be time-division multiplexed. Devise a commutator configuration such that each signal is periodically sampled at its own minimum rate and the samples are properly interlaced. What is the minimum transmission bandwidth required for this TDM signal?

3.28 Repeat the preceding problem assuming that the commutator is run at twice the minimum rate. What are the advantages and disadvantages of doing this?

3.29 Five messages bandlimited to $W, W, 2W, 5W$, and $7W$ Hz, respectively, are to be time-division multiplexed. Devise a sampling scheme requiring the minimum sampling frequency.

3.30 In an FDM communication system, the transmitted baseband signal is

$$x(t) = m_1(t) \cos(2\pi f_1 t) + m_2(t) \cos(2\pi f_2 t)$$

This system has a second-order nonlinearity between transmitter output and receiver input. Thus, the received baseband signal $y(t)$ can be expressed as

$$y(t) = a_1 x(t) + a_2 x^2(t)$$

Assuming that the two message signals, $m_1(t)$ and $m_2(t)$, have the spectra

$$M_1(f) = M_2(f) = \Pi\left(\frac{f}{W}\right)$$

sketch the spectrum of $y(t)$. Discuss the difficulties encountered in demodulating the received baseband signal. In many FDM systems, the subcarrier frequencies f_1 and f_2 are harmonically related. Describe any additional problems this presents.

Computer Exercises

3.1 In Example 3.1 we determined the minimum value of $m(t)$ using MATLAB. Write a MATLAB program that provides a complete solution for Example 3.1. Use the FFT for finding the amplitude and phase spectra of the transmitted signal $x_c(t)$.

3.2 The purpose of this exercise is to demonstrate the properties of SSB modulation. Develop a computer program to generate both upper-sideband and lower-sideband SSB signals and display both the time-domain signals and the amplitude spectra of these signals. Assume the message signal

$$m(t) = 2\cos(2\pi f_m t) + \cos(4\pi f_m t)$$

Select both f_m and f_c so that both the time and frequency axes can be easily calibrated. Plot the envelope of the SSB signals, and show that both the upper-sideband and the lower-sideband SSB signals have the same envelope. Use the FFT algorithm to generate the amplitude spectrum for both the upper-sideband and the lower-sideband SSB signal.

3.3 Using the same message signal and value for f_m used in the preceding computer exercise, show that carrier reinsertion can be used to demodulate an SSB signal. Illustrate the effect of using a demodulation carrier with insufficient amplitude when using the carrier reinsertion technique.

3.4 In this computer exercise we investigate the properties of VSB modulation. Develop a computer program (using MATLAB) to generate and plot a VSB signal and the corresponding amplitude spectrum. Using the program, show that VSB can be demodulated using carrier reinsertion.

3.5 Using MATLAB simulate delta modulation. Generate a signal, using a sum of sinusoids, so that the bandwidth is known. Sample at an appropriate sampling frequency (no slope overload). Show the stairstep approximation. Now reduce the sampling frequency so that slope overload occurs. Once again, show the stairstep approximation.

3.6 Using a sum of sinusoids as the sampling frequency, sample and generate a PAM signal. Experiment with various values of τf_s. Show that the message signal is recovered by lowpass filtering. A third-order Butterworth filter is suggested.

4.5 A signal, which is treated as narrowband angle modulation has a modulation index $\beta = 0.2$. Determine the ratio of sideband power to carrier power. Describe the spectrum of the transmitted signal.

4.6 An angle-modulated signal, with sinusoidal $m(t)$ has a modulation index $\beta = 5$. Determine the ratio of sideband power to carrier power assuming that 5 sidebands are transmitted each side of the carrier.

4.7 An FM signal is formed by narrowband-to-wideband conversion. The peak frequency deviation of the narrowband signal is 40 Hz and the bandwidth of the message signal is 200 Hz. The wideband (transmitted) signal is to have a deviation ratio of 6 and a carrier frequency of 1 MHz. Determine the multiplying factor n, the carrier frequency of the narrowband signal, and, using Carson's rule, estimate the bandwidth of the wideband signal.

4.8 A first-order PLL has a total loop gain of 10. Determine the lock range.

4.9 A second-order loop filter, operating in the tracking mode, has a loop gain of 10 and a loop filter transfer function of $(s + a)/s$. Determine the value of a so that the loop damping factor is 0.8. With this choice of a, what is the loop natural frequency?

4.10 A first-order PLL has a loop gain of 300. The input to the loop instantaneously changes frequency by 40 Hz. Determine the steady-state phase error due to this step change in frequency.

4.11 An FDM system is capable of transmitting a baseband signal having a bandwidth of 100 kHz. One channel is input to the system without modulation (about $f = 0$). Assume that all message signals have a lowpass spectrum with a bandwidth of 2 kHz and that the guardband between channels is 1 kHz. How many channels can be multiplexed together to form the baseband?

4.12 A QM system has two message signals defined by

$$m_1(t) = 5\cos(8\pi t)$$

and

$$m_2(t) = 8\sin(12\pi t)$$

Due to a calibration error, the demodulation carriers have a phase error of 10 degrees. Determine the two demodulated message signals.

Problems

Section 4.1

4.1 Let the input to a phase modulator be $m(t) = u(t - t_0)$, as shown in Figure 4.1(a). Assume that the unmodulated carrier is $A_c \cos(2\pi f_c t)$ and that $f_c t_0 = n$, where n is an integer. Sketch accurately the phase modulator output for $k_p = \pi$ and $-\frac{3}{8}\pi$ as was done in Figure 4.1(c) for $k_p = \frac{1}{2}\pi$.

4.2 Repeat the preceding problem for $k_p = -\frac{1}{2}\pi$ and $\frac{3}{8}\pi$.

4.3 Redraw Figure 4.4 assuming $m(t) = A\sin\left(2\pi f_m t + \frac{\pi}{6}\right)$.

4.4 We previously computed the spectrum of the FM signal defined by

$$x_{c1}(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

Now assume that the modulated signal is given by

$$x_{c2}(t) = A_c \cos[2\pi f_c t + \beta \cos(2\pi f_m t)]$$

Show that the amplitude spectrum of $x_{c1}(t)$ and $x_{c2}(t)$ are identical. Compute the phase spectrum of $x_{c2}(t)$ and compare with the phase spectrum of $x_{c1}(t)$.

4.5 Compute the single-sided amplitude and phase spectra of

$$x_{c3}(t) = A\sin[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

and

$$x_{c4}(t) = A_c \sin[2\pi f_c t + \beta \cos(2\pi f_m t)]$$

Compare the results with Figure 4.5.

4.6 The power of an unmodulated carrier signal is 50 W, and the carrier frequency is $f_c = 40$ Hz. A sinusoidal message signal is used to FM modulate it with index $\beta = 10$. The sinusoidal message signal has a frequency of 5 Hz. Determine the average value of $x_c(t)$. By drawing appropriate spectra, explain this apparent contradiction.

4.7 Given that $J_0(5) = -0.178$ and that $J_1(5) = -0.328$, determine $J_3(5)$ and $J_4(5)$.

4.8 Determine and sketch the spectrum (amplitude and phase) of an angle-modulated signal assuming that the instantaneous phase deviation is $\phi(t) = \beta\sin(2\pi f_m t)$. Also assume $\beta = 10$, $f_m = 30$ Hz, and $f_c = 2000$ Hz.

4.9 A transmitter uses a carrier frequency of 1000 Hz so that the unmodulated carrier is $A_c\cos(2\pi f_c t)$. Determine both the phase and frequency deviation for each of the following transmitter outputs:

(a) $x_c(t) = \cos[2\pi(1000)t + 40\sin\left(5t^2\right)]$

(b) $x_c(t) = \cos[2\pi(600)t]$

4.10 Repeat the preceding problem assuming that the transmitter outputs are defined by

(a) $x_c(t) = \cos[2\pi(1200)t^2]$

(b) $x_c(t) = \cos[2\pi(900)t + 10\sqrt{t}]$

4.11 An FM modulator has output

$$x_c(t) = 100\cos\left[2\pi f_c t + 2\pi f_d \int^t m(\alpha)d\alpha\right]$$

where $f_d = 20$ Hz/V. Assume that $m(t)$ is the rectangular pulse $m(t) = 4\Pi\left[\frac{1}{8}(t-4)\right]$

(a) Sketch the phase deviation in radians.

(b) Sketch the frequency deviation in hertz.

(c) Determine the peak frequency deviation in hertz.

(d) Determine the peak phase deviation in radians.

(e) Determine the power at the modulator output.

4.12 Repeat the preceding problem assuming that $m(t)$ is the triangular pulse $4\Lambda\left[\frac{1}{3}(t-6)\right]$.

4.13 An FM modulator with $f_d = 10$ Hz/V. Plot the frequency deviation in Hz and the phase deviation in radians for the three message signals shown in Figure 4.37.

4.14 An FM modulator has $f_c = 2000$ Hz and $f_d = 20$ Hz/V. The modulator has input $m(t) = 5\cos[2\pi(10)t]$.

(a) What is the modulation index?

(b) Sketch, approximately to scale, the magnitude spectrum of the modulator output. Show all frequencies of interest.

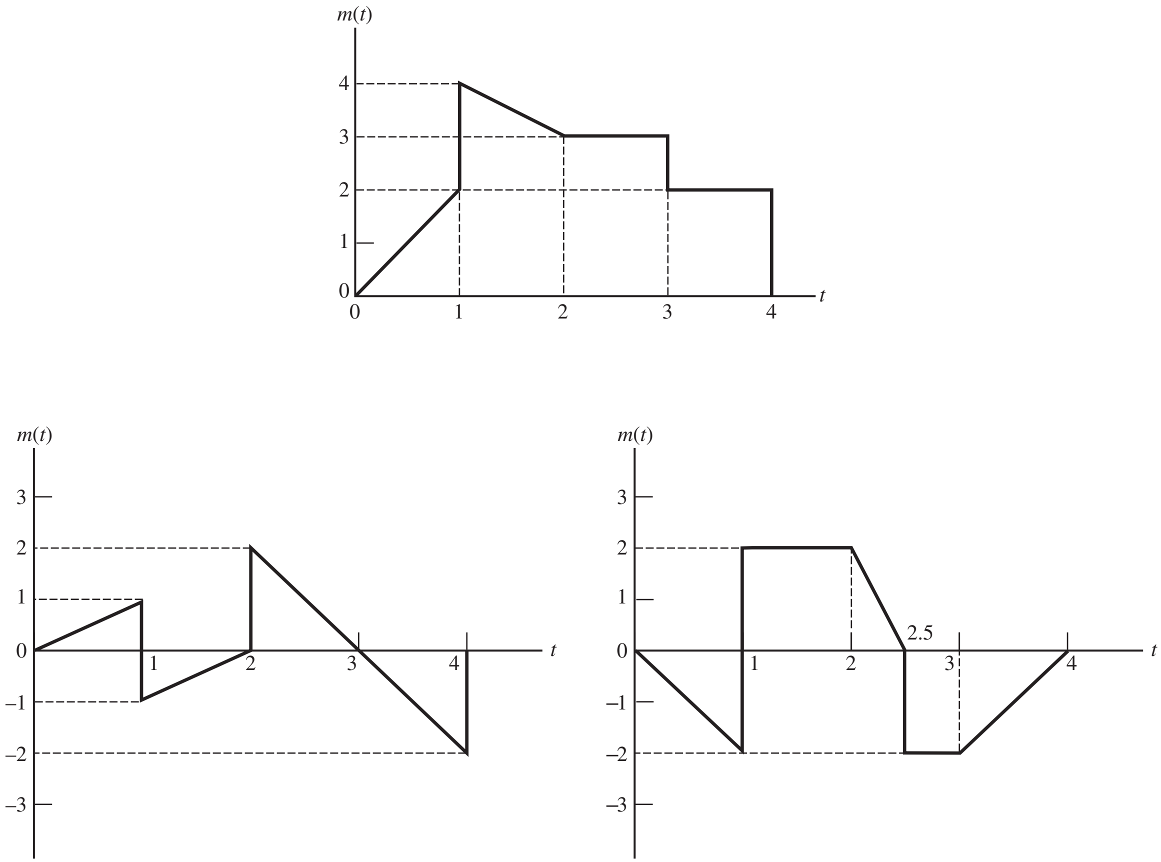

Figure 4.37

(c) $f(x) = Ce^{-\gamma x} u(x-1)$

(d) $f(x) = C[u(x) - u(x-\tau)]$

6.12 Test X and Y for independence if

(a) $f_{XY}(x, y) = Ae^{-|x|-2|y|}$

(b) $f_{XY}(x, y) = C(1-x-y), 0 \le x \le 1- \quad y$ and $0 \le y \le 1$

Prove your answers.

6.13 The joint pdf of two random variables is

$$f_{XY}(x, y) = \begin{cases} C(1+xy), & 0 \le x \le 4,\ 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the following:

(a) The constant C

(b) $f_{XY}(1, 1.5)$

(c) $f_{XY}(x, 3)$

(d) $f_{X|Y}(x \mid 3)$

6.14 The joint pdf of the random variables X and Y is

$$f_{XY}(x, y) = Axye^{-(x+y)},\ x \ge 0 \text{ and } y \ge 0$$

(a) Find the constant A.

(b) Find the marginal pdfs of X and Y, $f_X(x)$ and $f_Y(y)$.

(c) Are X and Y statistically independent? Justify your answer.

6.15

(a) For what value of $\alpha > 0$ is the function

$$f(x) = \alpha x^{-2} u(x-\alpha)$$

a probability-density function? Use a sketch to illustrate your reasoning and recall that a pdf has to integrate to one. [$u(x)$ is the unit step function.]

(b) Find the corresponding cumulative-distribution function.

(c) Compute $P(X \ge 10)$.

6.16 Given the Gaussian random variable with the pdf

$$f_X(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$

where $\sigma > 0$ is the standard deviation. If $Y = X^2$, find the pdf of Y. *Hint*: Note that $Y = X^2$ is symmetrical about $X = 0$ and that it is impossible for Y to be less than zero.

6.17 A nonlinear system has input X and output Y. The pdf for the input is Gaussian as given in Problem 6.16. Determine the pdf of the output, assuming that the nonlinear system has the following input/output relationship:

(a) $Y = \begin{cases} aX, & X \ge 0 \\ 0, & X < 0 \end{cases}$

Hint: When $X < 0$, what is Y? How is this manifested in the pdf for Y?

(b) $Y = |X|$;

(c) $Y = X - X^3/3$.

Section 6.3

6.18 Let $f_X(x) = A\exp(-bx)u(x-2)$ for all x where A and b are positive constants.

(a) Find the relationship between A and b such that this function is a pdf.

(b) Calculate $E(X)$ for this random variable.

(c) Calculate $E(X^2)$ for this random variable.

(d) What is the variance of this random variable?

6.19

(a) Consider a random variable uniformly distributed between 0 and 2. Show that $E(X^2) > E^2(X)$.

(b) Consider a random variable uniformly distributed between 0 and 4. Show that $E(X^2) > E^2(X)$.

(c) Can you show in general that for any random variable it is true that $E(X^2) > E^2(X)$ unless the random variable is zero almost always? (*Hint:* Expand $E\left\{[X - E(X)]^2 \ge 0\right\}$ and note that it is 0 only if $X = 0$ with probability 1.)

6.20 Verify the entries in Table 6.5 for the mean and variance of the following probability distributions:

(a) Rayleigh;

(b) One-sided exponential;

(c) Hyperbolic;

(d) Poisson;

(e) Geometric.

6.21 A random variable X has the pdf

$$f_X(x) = Ae^{-bx}[u(x) - u(x-B)]$$

where $u(x)$ is the unit step function and A, B, and b are positive constants.

(a) Find the proper relationship between the constants A, b, and B. Express b in terms of A and B.

(b) Determine and plot the cdf.

(c) Compute $E(X)$.

(d) Determine $E(X^2)$.

(e) What is the variance of X?

6.22 If

$$f_X(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

show that

(a) $E[X^{2n}] = 1 \cdot 3 \cdot 5 \cdots (2n-1)\sigma^{2n}$, for $n = 1, 2, \ldots$

(b) $E[X^{2n-1}] = 0$ for $n = 1, 2, \ldots$

6.23 The random variable has pdf

$$f_X(x) = \frac{1}{2}\delta(x-5) + \frac{1}{8}[u(x-4) - u(x-8)]$$

where $u(x)$ is the unit step. Determine the mean and the variance of the random variable thus defined.

6.24 Two random variables X and Y have means and variances given below:

$$m_x = 1 \quad \sigma_x^2 = 4 \quad m_y = 3 \quad \sigma_y^2 = 7$$

A new random variable Z is defined as

$$Z = 3X - 4Y$$

Determine the mean and variance of Z for each of the following cases of correlation between the random variables X and Y:

(a) $\rho_{XY} = 0$

(b) $\rho_{XY} = 0.2$

(c) $\rho_{XY} = 0.7$

(d) $\rho_{XY} = 1.0$

6.25 Two Gaussian random variables X and Y, with zero means and variances σ^2, between which there is a correlation coefficient ρ, have a joint probability-density function given by

$$f(x, y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2\sigma^2\left(1-\rho^2\right)}\right]$$

The marginal pdf of Y can be shown to be

$$f_Y(y) = \frac{\exp\left(-y^2/\left(2\sigma^2\right)\right)}{\sqrt{2\pi\sigma^2}}$$

Find the conditional pdf $f_{X|Y}(x \mid y)$.

6.26 Using the definition of a conditional pdf given by Equation (6.62) and the expressions for the marginal and joint Gaussian pdfs, show that for two jointly Gaussian random variables X and Y, the conditional density function of X given Y has the form of a Gaussian density with conditional mean and the conditional variance given by

$$E(X|Y) = m_x + \frac{\rho\sigma_x}{\sigma_y}(Y - m_y)$$

and

$$\operatorname{var}(X|Y) = \sigma_x^2(1-\rho^2)$$

respectively.

6.27 The random variable X has a probability-density function uniform in the range $0 \le x \le 2$ and zero elsewhere. The independent variable Y has a density uniform in the range $1 \le y \le 5$ and zero elsewhere. Find and plot the density of $Z = X + Y$.

6.28 A random variable X is defined by

$$f_X(x) = 4e^{-8|x|}$$

The random variable Y is related to X by $Y = 4 + 5X$.

(a) Determine $E[X]$, $E[X^2]$, and σ_x^2.

(b) Determine $f_Y(y)$.

(c) Determine $E[Y]$, $E[Y^2]$, and σ_y^2. (*Hint:* The result of part (b) is not necessary to do this part, although it may be used.)

(d) If you used $f_Y(y)$ in part (c), repeat that part using only $f_X(x)$.

6.29 A random variable X has the probability-density function

$$f_X(x) = \begin{cases} ae^{-ax}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

where a is an arbitrary positive constant.

(a) Determine the characteristic function $M_x(jv)$.

(b) Use the characteristic function to determine $E[X]$ and $E[X^2]$.

(c) Check your results by computing

$$\int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

for $n = 1$ and 2.

(d) Compute σ_x^2.

Section 6.4

6.30 Compare the binomial, Laplace, and Poisson distributions for

(a) $n = 3$ and $p = \frac{1}{5}$

(b) $n = 3$ and $p = \frac{1}{10}$

(c) $n = 10$ and $p = \frac{1}{5}$

(d) $n = 10$ and $p = \frac{1}{10}$

6.31 An honest coin is flipped 10 times.

(a) Determine the probability of the occurrence of either 5 or 6 heads.

(b) Determine the probability of the first head occurring at toss number 5.

(c) Repeat parts (a) and (b) for flipping 100 times and the probability of the occurrence of 50 to 60 heads inclusive and the probability of the first head occurring at toss number 50.

6.32 Passwords in a computer installation take the form $X_1X_2X_3X_4$, where each character X_i is one of the 26 letters of the alphabet. Determine the maximum possible number of different passwords available for assignment for each of the two following conditions:

(a) A given letter of the alphabet can be used only once in a password.

(b) Letters can be repeated if desired, so that each X_i is completely arbitrary.

(c) If selection of letters for a given password is completely random, what is the probability that your competitor could access, on a single try, your computer in part (a)? part (b)?

6.33 Assume that 20 honest coins are tossed.

(a) By applying the binomial distribution, find the probability that there will be fewer than 3 heads.

(b) Do the same computation using the Laplace approximation.

(c) Compare the results of parts (a) and (b) by computing the percent error of the Laplace approximation.

6.34 A digital data transmission system has an error probability of 10^{-5} per digit.

(a) Find the probability of exactly 1 error in 10^5 digits.

(b) Find the probability of exactly 2 errors in 10^5 digits.

(c) Find the probability of more than 5 errors in 10^5 digits.

6.35 Assume that two random variables X and Y are jointly Gaussian with $m_x = m_y = 1$, $\sigma_x^2 = \sigma_y^2 = 4$.

(a) Making use of (6.194), write down an expression for the margininal pdfs of X and of Y.

(b) Write down an expression for the conditional pdf $f_{X\mid Y}(x \mid y)$ by using the result of (a) and an expression for $f_{XY}(x, y)$ written down from (6.189). Deduce that $f_{Y\mid X}(y \mid x)$ has the same form with y replacing x.

(c) Put $f_{X\mid Y}(x \mid y)$ into the form of a marginal Gaussian pdf. What is its mean and variance? (The mean will be a function of y.)

6.36 Consider the Cauchy density function

$$f_X(x) = \frac{K}{1 + x^2}, \quad -\infty \le x \le \infty$$

(a) Find K.

(b) Show that $\mathrm{var}\{X\}$ is not finite.

(c) Show that the characteristic function of a Cauchy random variable is $M_x(jv) = \pi K e^{-|v|}$.

(d) Now consider $Z = X_1 + \cdots + X_N$ where the X_i's are independent Cauchy random variables. Thus, their characteristic function is

$$M_Z(jv) = (\pi K)^N \exp(-N|v|)$$

Show that $f_Z(z)$ is Cauchy. (*Comment*: $f_Z(z)$ is not Gaussian as $N \to \infty$ because $\mathrm{var}\{X_i\}$ is not finite and the conditions of the central-limit theorem are violated.)

6.37 (Chi-squared pdf) Consider the random variable $Y = \sum_{i=1}^N X_i^2$ where the X_i's are independent Gaussian random variables with pdfs $n(0, \sigma)$.

(a) Show that the characteristic function of X_i^2 is

$$M_{X_i^2}(jv) = \left(1 - 2jv\sigma^2\right)^{-1/2}$$

(b) Show that the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{y^{N/2-1}e^{-y/2\sigma^2}}{2^{N/2}\sigma^N\Gamma(N/2)}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$

where $\Gamma(x)$ is the gamma function, which, for $x = n$, an integer is $\Gamma(n) = (n-1)!$. This pdf is known as the χ^2 (chi-squared) pdf with N degrees of freedom. *Hint*: Use the Fourier-transform pair

$$\frac{y^{N/2-1}e^{-y/\alpha}}{\alpha^{N/2}\Gamma(N/2)} \leftrightarrow (1-j\alpha v)^{-N/2}$$

(c) Show that for N large, the χ^2 pdf can be approximated as

$$f_Y(y) = \frac{\exp\left[-\frac{1}{2}\left(\frac{y-N\sigma^2}{\sqrt{4N\sigma^4}}\right)^2\right]}{\sqrt{4N\pi\sigma^4}}, \quad N \gg 1$$

Hint: Use the central-limit theorem. Since the x_i's are independent,

$$\bar{Y} = \sum_{i=1}^{N} \overline{X_i^2} = N\sigma^2$$

and

$$\text{var}(Y) = \sum_{i=1}^{N} \text{var}\,(X_i^2) = N\,\text{var}\,(X_i^2)$$

(d) Compare the approximation obtained in part (c) with $f_Y(y)$ for $N = 2, 4, 8$.

(e) Let $R^2 = Y$. Show that the pdf of R for $N = 2$ is Rayleigh.

6.38 Compare the Q-function and the approximation to it for large arguments given by (6.202) by plotting both expressions on a log-log graph. (*Note*: MATLAB is handy for this problem.)

6.39 Determine the cdf for a Gaussian random variable of mean m and variance σ^2. Express in terms of the Q-function. Plot the resulting cdf for $m = 0$, and $\sigma = 0.5$, 1, and 2.

6.40 Prove that the Q-function may also be represented as $Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\phi}\right) d\phi$.

6.41 A random variable X has the probability-density function

$$f_X(x) = \frac{e^{-(x-10)^2/50}}{\sqrt{50\pi}}$$

Express the following probabilities in terms of the Q-function and calculate numerical answers for each:

(a) $P(|X| \le 15)$;

(b) $P(10 < X \le 20)$;

(c) $P(5 < X \le 25)$;

(d) $P(20 < X \le 30)$.

6.42

(a) Prove Chebyshev's inequality. *Hint*: Let $Y = (X - m_x)/\sigma_x$ and find a bound for $P(|Y| < k)$ in terms of k.

(b) Let X be uniformly distributed over $|x| \le 1$. Plot $P(|X| \le k\sigma_x)$ versus k and the corresponding bound given by Chebyshev's inequality.

6.43 If the random variable X is Gaussian, with zero mean and variance σ^2, obtain numerical values for the following probabilities:

(a) $P(|X| > \sigma)$;

(b) $P(|X| > 2\sigma)$;

(c) $P(|X| > 3\sigma)$.

6.44 Speech is sometimes idealized as having a Laplacian-amplitude pdf. That is, the amplitude is distributed according to

$$f_X(x) = \left(\frac{a}{2}\right)\exp(-a\,|x|)$$

(a) Express the variance of X, σ^2, in terms of a. Show your derivation; don't just simply copy the result given in Table 6.4.

(b) Compute the following probabilities: $P(|X| > \sigma)$; $P(|X| > 2\sigma)$; $P(|X| > 3\sigma)$.

6.45 Two jointly Gaussian zero-mean random variables, X and Y, have respective variances of 3 and 4 and correlation coefficient $\rho_{XY} = -0.4$. A new random variable is defined as $Z = X + 2Y$. Write down an expression for the pdf of Z.

6.46 Two jointly Gaussian random variables, X and Y, have means of 1 and 2, and variances of 3 and 2, respectively. Their correlation coefficient is $\rho_{XY} = 0.2$. A new random variable is defined as $Z = 3X + Y$. Write down an expression for the pdf of Z.

6.47 Two Gaussian random variables, X and Y, are independent. Their respective means are 5 and 3, and their respective variances are 1 and 2.

(a) Write down expressions for their marginal pdfs.

(b) Write down an expression for their joint pdf.

(c) What is the mean of $Z_1 = X + Y$? $Z_2 = X - Y$?

(d) What is the variance of $Z_1 = X + Y$? $Z_2 = X - Y$?

(e) Write down an expression for the pdf of $Z_1 = X + Y$.

(f) Write down an expression for the pdf of $Z_2 = X - Y$.

6.48 Two Gaussian random variables, X and Y, are independent. Their respective means are 4 and 2, and their respective variances are 3 and 5.

(a) Write down expressions for their marginal pdfs.

(b) Write down an expression for their joint pdf.

(c) What is the mean of $Z_1 = 3X + Y$? $Z_2 = 3X - Y$?

(d) What is the variance of $Z_1 = 3X + Y$? $Z_2 = 3X - Y$?

(e) Write down an expression for the pdf of $Z_1 = 3X + Y$.

(f) Write down an expression for the pdf of $Z_2 = 3X - Y$.

6.49 Find the probabilities of the following random variables, with pdfs as given in Table 6.4, exceeding their means. That is, in each case, find the probability that $X \geq m_X$, where X is the respective random variable and m_X is its mean.

(a) Uniform;

(b) Rayleigh;

(b) One-sided exponential.

Computer Exercises

6.1 In this exercise we examine a useful technique for generating a set of samples having a given pdf.

(a) First, prove the following theorem: If X is a continuous random variable with cdf $F_X(x)$, the random variable

$$Y = F_X(X)$$

is a uniformly distributed random variable in the interval $(0, 1)$.

(b) Using this theorem, design a random number generator to generate a sequence of exponentially distributed random variables having the pdf

$$f_X(x) = \alpha e^{-\alpha x} u(x)$$

where $u(x)$ is the unit step. Plot histograms of the random numbers generated to check the validity of the random number generator you designed.

6.2 An algorithm for generating a Gaussian random variable from two independent uniform random variables is easily derived.

(a) Let U and V be two statistically independent random numbers uniformly distributed in $[0, 1]$. Show that the following transformation generates two statistically independent Gaussian random numbers with unit variance and zero mean:

$$X = R\cos(2\pi U)$$
$$Y = R\sin(2\pi U)$$

where

$$R = \sqrt{-2\ln(V)}$$

Hint: First show that R is Rayleigh.

(b) Generate 1000 random variable pairs according to the above algorithm. Plot histograms for each set (i.e., X and Y), and compare with Gaussian pdfs after properly scaling the histograms (i.e., divide each cell by the total number of counts times the cell width so that the histogram approximates a probability-density function). *Hint*: Use the hist function of MATLAB.

6.3 Using the results of Problem 6.26 and the Gaussian random number generator designed in Computer Exercise 6.2, design a Gaussian random number generator that will provide a specified correlation between adjacent samples. Let

$$P(\tau) = e^{-\alpha|\tau|}$$

and plot sequences of Gaussian random numbers for various choices of α. Show how stronger correlation between adjacent samples affects the variation from sample to sample. (*Note*: To get memory over more than adjacent samples, a digital filter should be used with independent Gaussian samples at the input.)

6.4 Check the validity of the central-limit theorem by repeatedly generating n independent uniformly distributed random variables in the interval $(-0.5, 0.5)$, forming the sum given by (6.187), and plotting the histogram. Do this for $N = 5, 10$, and 20. Can you say anything qualitatively and quantitatively about the approach of the sums to Gaussian random numbers? Repeat for exponentially distributed component random variables (do Computer Exercise 6.1 first). Can you think of a drawback to the approach of summing uniformly distributed random variables to generating Gaussian random variables (*Hint:* Consider the probability of the sum of uniform random variables being greater than $0.5N$ or less than $-0.5N$. What are the same probabilities for a Gaussian random variable?

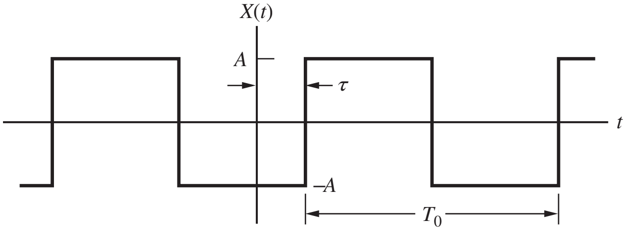

Figure 7.15

7.4 Let the sample functions of a random process be given by

$$X(t) = A\cos 2\pi f_0 t$$

where ω_0 is fixed and A has the pdf

$$f_A(a) = \frac{e^{-\alpha^2/2\sigma_a^2}}{\sqrt{2\pi}\sigma_a}$$

This random process is passed through an ideal integrator to give a random process $Y(t)$.

(a) Find an expression for the sample functions of the output process $Y(t)$.

(b) Write down an expression for the pdf of $Y(t)$ at time t_0. *Hint*: Note that $\sin 2\pi f_0 t_0$ is just a constant.

(c) Is $Y(t)$ stationary? Is it ergodic?

7.5 Consider the random process of Problem 7.3.

(a) Find the time-average mean and the autocorrelation function.

(b) Find the ensemble-average mean and the autocorrelation function.

(c) Is this process wide-sense stationary? Why or why not?

7.6 Consider the random process of Example 7.1 with the pdf of θ given by

$$p(\theta) = \begin{cases} 2/\pi, & \pi/2 \le \theta \le \pi \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the statistical-average and time-average mean and variance.

(b) Find the statistical-average and time-average autocorrelation functions.

(c) Is this process ergodic?

7.7 Consider the random process of Problem 7.4.

(a) Find the time-average mean and the autocorrelation function.

(b) Find the ensemble-average mean and the autocorrelation function.

(c) Is this process wide-sense stationary? Why or why not?

7.8 The voltage of the output of a noise generator whose statistics are known to be closely Gaussian and stationary is measured with a dc voltmeter and a true root-mean-square (rms) voltmeter that is ac coupled. The dc meter reads 6 V, and the true rms meter reads 7 V. Write down an expression for the first-order pdf of the voltage at any time $t = t_0$. Sketch and dimension the pdf.

Section 7.3

7.9 Which of the following functions are suitable autocorrelation functions? Tell why or why not. (ω_0, τ_0, τ_1, A, B, C, and f_0 are positive constants.)

(a) $A\cos\omega_0\tau$

(b) $A\Lambda\left(\tau/\tau_0\right)$, where $\Lambda(x)$ is the unit-area triangular function defined in Chapter 2

(c) $A\Pi\left(\tau/\tau_0\right)$, where $\Pi(x)$ is the unit-area pulse function defined in Chapter 2

(d) $A\exp\left(-\tau/\tau_0\right)u(\tau)$ where $u(x)$ is the unit-step function

(e) $A\exp\left(-|\tau|/\tau_0\right)$

(f) $A\,\mathrm{sinc}\left(f_0\tau\right) = \frac{\sin\left(\pi f_0\tau\right)}{\pi f_0\tau}$

7.10 A bandlimited white-noise process has a double-sided power spectral density of 2×10^{-5} W/Hz in the frequency range $|f| \le 1$ kHz. Find the autocorrelation function of the noise process. Sketch and fully dimension the resulting autocorrelation function.

7.11 Consider a random binary pulse waveform as analyzed in Example 7.6, but with half-cosine pulses given by $p(t) = \cos(2\pi t/2T)\Pi(t/T)$. Obtain and sketch the autocorrelation function for the two cases considered in Example 7.6, namely,

(a) $a_k = \pm A$ for all k, where A is a constant, with $R_m = A^2$, $m = 0$, and $R_m = 0$ otherwise.

(b) $a_k = A_k + A_{k-1}$ with $A_k = \pm A$ and $E[A_k A_{k+m}] = A^2, m = 0$, and zero otherwise.

(c) Find and sketch the power spectral density for each preceding case.

7.12 Two random processes are given by

$$X(t) = n(t) + A\cos(2\pi f_0 t + \theta)$$

and

$$Y(t) = n(t) + A\sin(2\pi f_0 t + \theta)$$

where A and f_0 are constants and θ is a random variable uniformly distributed in the interval $[-\pi, \pi)$. The first term, $n(t)$, represents a stationary random noise process with autocorrelation function $R_n(\tau) = B\Lambda(\tau/\tau_0)$, where B and τ_0 are nonnegative constants.

(a) Find and sketch their autocorrelation functions. Assume values for the various constants involved.

(b) Find and sketch the cross-correlation function of these two random processes.

7.13 Given two independent, wide-sense stationary random processes $X(t)$ and $Y(t)$ with autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$, respectively.

(a) Show that the autocorrelation function $R_Z(\tau)$ of their product $Z(t) = X(t)Y(t)$ is given by

$$R_Z(\tau) = R_X(\tau)R_Y(\tau)$$

(b) Express the power spectral density of $Z(t)$ in terms of the power spectral densities of $X(t)$ and $Y(t)$, denoted as $S_X(f)$ and $S_Y(f)$, respectively.

(c) Let $X(t)$ be a bandlimited stationary noise process with power spectral density $S_x(f) = 10\Pi(f/200)$, and let $Y(t)$ be the process defined by sample functions of the form

$$Y(t) = 5\cos(50\pi t + \theta)$$

where θ is a uniformly distributed random variable in the interval $(0, 2\pi)$. Using the results derived in parts (a) and (b), obtain the autocorrelation function and power spectral density of $Z(t) = X(t)Y(t)$.

7.14 A random signal has the autocorrelation function

$$R(\tau) = 9 + 3\Lambda(\tau/5)$$

where $\Lambda(x)$ is the unit-area triangular function defined in Chapter 2. Determine the following:

(a) The ac power.

(b) The dc power.

(c) The total power.

(d) The power spectral density. Sketch it and label carefully.

7.15 A random process is defined as $Y(t) = X(t) + X(t-T)$, where $X(t)$ is a wide-sense stationary random process with autocorrelation function $R_X(T)$ and power spectral density $S_x(f)$.

(a) Show that $R_Y(\tau) = 2R_X(\tau) + R_X(\tau + T) + R_X(\tau - T)$.

(b) Show that $S_Y(f) = 4S_X(f)\cos^2(\pi f T)$.

(c) If $X(t)$ has autocorrelation function $R_X(\tau) = 5\Lambda(\tau)$, where $\Lambda(\tau)$ is the unit-area triangular function, and $T = 0.5$, find and sketch the power spectral density of $Y(t)$ as defined in the problem statement.

7.16 The power spectral density of a wide-sense stationary random process is given by

$$S_X(f) = 10\delta(f) + 25\mathrm{sinc}^2(5f) + 5\delta(f - 10) + 5\delta(f + 10)$$

(a) Sketch and fully dimension this power spectral density function.

(b) Find the power in the dc component of the random process.

(c) Find the total power.

(d) Given that the area under the main lobe of the sinc-squared function is approximately 0.9 of the total area, which is unity if it has unity amplitude, find the fraction of the total power contained in this process for frequencies between 0 and 0.2 Hz.

7.17 Given the following functions of τ:

$$R_{X_1}(\tau) = 4\exp(-\alpha|\tau|)\cos 2\pi\tau$$

$$R_{X_2}(\tau) = 2\exp(-\alpha|\tau|) + 4\cos 2\pi b\tau$$

$$R_{X_3}(f) = 5\exp(-4\tau^2)$$