#1. $(85\sqrt{85}-8)/27$ **#2.** **(b)** $\pm\sqrt{3}$ **(c)** $\pi - 3\sqrt{3}/2$ **(d)** $2\displaystyle\int_0^{\pi} \sqrt{5+4\cos\theta}\,d\theta$

#3. **(a)** hyperboloid of one sheet **(b)** circular paraboloid **(c)** circular cylinder **(d)** horizontal plane **(e)** upper nappe of a circular cone

#4. $(\rho,\theta,\phi) = (6378, 120.5°\text{E}, 23.5°\text{N}) \approx (6378, 2.10, 1.16)$; $(x,y,z) \approx (-2968.59, 5039.68, 2543.23)$

#5. **(b)** $1/(\sqrt{2}e^t)$ **(c)** $\pi/4$

#1. *Parametric curve, arc length.* [10.1.65]

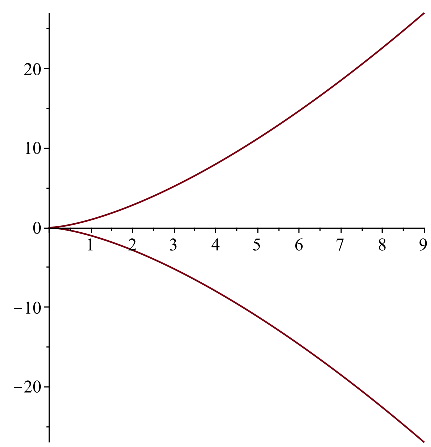
The arc length differential is

$$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

The arc length of the parametric curve $\mathbf{r}(t) = \langle t^2, t^3\rangle$, $0 \le t \le 3$, is given by

$$L = \int ds = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^3 \sqrt{(2t)^2+(3t^2)^2}\,dt = \int_0^3 t\sqrt{4+9t^2}\,dt$$
$$= \left[\frac{1}{27}(4+9t^2)^{3/2}\right]_0^3 = \frac{1}{27}(85\sqrt{85}-8).$$

Figure 1: The semicubical parabola $\mathbf{r}(t) = \langle t^2, t^3\rangle$, $-\infty < t < \infty$

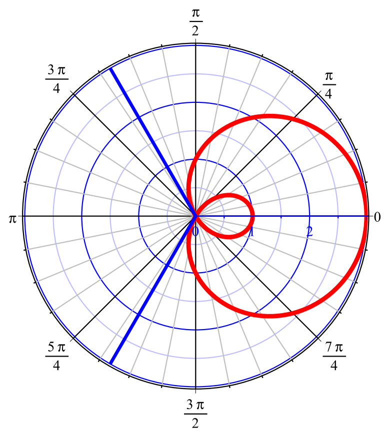


#2. *Limaçon* $C\colon r = 1 + 2\cos\theta$. [10.3.17], [10.3.33]

(a) The graph of C enjoys a symmetry about the x-axis since the polar equation is unchanged when θ is replaced by $-\theta$. [Note that the cosine is an even function: $\cos(-\theta) = \cos\theta$ for all θ.] There is an inner loop when the angle $\theta \in [2\pi/3, 4\pi/3]$, where $r \le 0$. We make a table of values of r for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	2π
$r = 1 + 2\cos\theta$	3	1	0	$1-\sqrt{3}$	-1	$1-\sqrt{3}$	0	1	3
			lower inner loop			upper inner loop			

Figure 2: The limaçon $r = 1 + 2\cos\theta$ and two tangent lines at O



(b) The slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta},$$

where $r' = dr/d\theta = -2\sin\theta$.

At the pole, we have $r = 0$, so the formula for the slope becomes

$$\left.\frac{dy}{dx}\right|_O = \frac{r'\sin\theta}{r'\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta,$$

provided that $r' \neq 0$. The pole lies on the limaçon when $r = 0$, that is, when $\cos\theta = -1/2$, or $\theta = 2\pi/3$ or $4\pi/3$. In both cases, we note that $r' = -2\sin(\pm 2\pi/3) \neq 0$. So there are two tangent lines at the pole and their slopes are given by

$$\left.\frac{dy}{dx}\right|_{(0,\pm 2\pi/3)} = \tan(\pm 2\pi/3) = \pm\sqrt{3}.$$

The polar equations for these two tangent lines are easy to write down: they are just $\theta = \pm 2\pi/3$.

(c) The inner loop corresponds to $2\pi/3 \le \theta \le 4\pi/3$. By the symmetry about the x-axis, the area of the region enclosed by the inner loop is

$$\begin{aligned}
A &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2}r^2\,d\theta = 2\int_{2\pi/3}^{\pi} \frac{1}{2}r^2\,d\theta = \int_{2\pi/3}^{\pi} (1+2\cos\theta)^2\,d\theta \\
&= \int_{2\pi/3}^{\pi} (1 + 4\cos\theta + 4\cos^2\theta)\,d\theta = \int_{2\pi/3}^{\pi} (1 + 4\cos\theta + 4\cdot\frac{1+\cos 2\theta}{2})\,d\theta \\
&= \int_{2\pi/3}^{\pi} (3 + 4\cos\theta + 2\cos 2\theta)\,d\theta = \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{2\pi/3}^{\pi} \\
&= 3(\pi - 2\pi/3) - 4\sin(2\pi/3) - \sin(4\pi/3) = \pi - \frac{3}{2}\sqrt{3}.
\end{aligned}$$

(d) As θ ranges over $[0, \pi]$, the limaçon is traced exactly half. Therefore, by the symmetry, the total length of C is

$$L = 2\int_0^\pi \sqrt{r^2 + (r')^2}\, d\theta = 2\int_0^\pi \sqrt{(1+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta = 2\int_0^\pi \sqrt{5+4\cos\theta}\, d\theta.$$

Remark. The integral $\int \sqrt{5+4\cos\theta}\, d\theta$ is non-elementary.

#3. *Rectangular, cylindrical, or spherical coordinates.* [11.7.17], [11.8.21], [11.8.23], [11.8.30], [11.8.29 + p.835]

(a) $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{4} - (z-1)^2 = 1$ is a hyperboloid of one sheet, whose axis of symmetry is the vertical line $\mathbf{r}(t) = \langle 1, -2, 1+t\rangle$, $-\infty < t < \infty$. It is also symmetric about the horizontal plane $z = 1$ (and vertical planes $x = 1$ or $y = -2$, too).

(b) $z = r^2$ represents $z = x^2 + y^2$, which is a circular paraboloid, opening upward, above the xy-plane.

(c) $r = 4\cos\theta$, or $r^2 = 4r\cos\theta$, or $x^2 + y^2 = 4x$, or $(x-2)^2 + y^2 = 4$. It represents a circular cylinder of radius 2, extended in the direction of z-axis.

(d) $\rho\cos\phi = 2$, or $z = 2$, a horizontal plane.

(e) $\phi = \pi/4$ implies that $\tan\phi = 1$, or $\cos\phi = \sin\phi$, or $\rho\cos\phi = \rho\sin\phi$, or $z = r = \sqrt{x^2+y^2}$. This is the upper nappe of the circular cone $z^2 = x^2 + y^2$.

Figure 3: (a) A hyperboloid of one sheet (b) A circular paraboloid

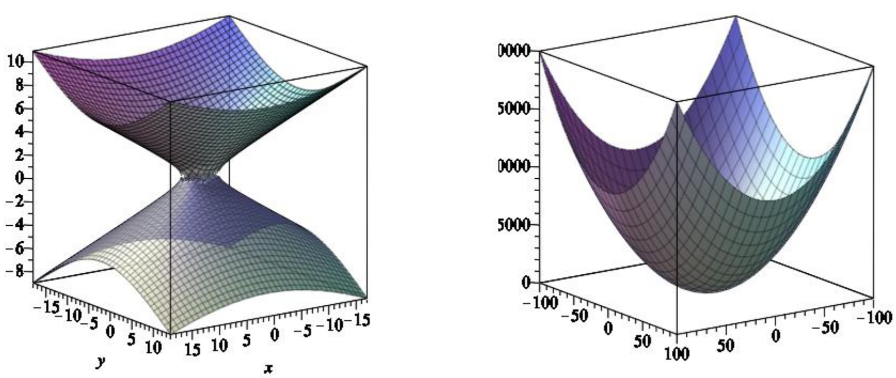
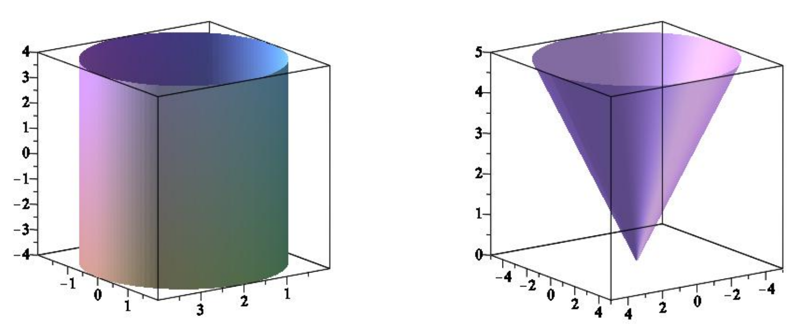


Figure 4: (c) A circular cylinder (e) The upper nappe of a circular cone



#4. *Spherical coordinates versus rectangular coordinates.* [11.8.51 + Exmp.4]

The spherical coordinates for Chiayi are $\rho = 6378$ km, $\theta = 120.5°\text{E} = 120.5\pi/180 \approx 2.10$ rad, and $\phi = 23.5°\text{N} = (90 - 23.5)\pi/180 \approx 1.16$ rad. (Note that ϕ is measured from the positive z-axis, that is, the ray from the center of the Earth toward the North Pole.)

To convert them into the rectangular coordinates, we use the formulas: $r = \rho\sin\phi$ and

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta \approx -2968.59, \\ y &= r\sin\theta = \rho\sin\phi\sin\theta \approx 5039.68, \\ z &= \rho\cos\phi \approx 2543.23. \end{aligned}$$

The rectangular coordinates for Chiayi are $(x, y, z) \approx (-2968.59, 5039.68, 2543.23)$.

#5. *Curvature, logarithmic spiral.* [12.5.23a], [12.6.34], [12.6.22]

(a) The (parametric) plane curve $x = f(t)$, $y = g(t)$ has the vector equation $\mathbf{r}(t) = \langle x, y, 0\rangle$. (Here and below, we will drop the variable t and use the prime notation for derivatives with respect to t, if no confusion arises.) We have $\mathbf{v} = \mathbf{r}' = \langle x', y', 0\rangle$, $\mathbf{a} = \mathbf{r}'' = \langle x'', y'', 0\rangle$, and so $v = \|\mathbf{v}\| = \|\mathbf{r}'\| = \sqrt{x'^2 + y'^2}$ and

$$\mathbf{v}\times\mathbf{a} = \mathbf{r}'\times\mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & 0 \\ x'' & y'' & 0 \end{vmatrix} = \langle 0, 0, x'y'' - x''y'\rangle.$$

Thus, $\|\mathbf{r}'\times\mathbf{r}''\| = \|\langle 0, 0, x'y'' - x''y'\rangle\| = |x'y'' - x''y'|$ and the curvature is therefore

$$\kappa(t) = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{3/2}}.$$

(b) For the logarithmic spiral $x = e^t\cos t$, $y = e^t\sin t$, we have

$$\begin{aligned} x' &= e^t(\cos t - \sin t), \quad y' = e^t(\cos t + \sin t), \\ x'' &= (e^t\cos t)' - (e^t\sin t)' = x' - y' = -2e^t\sin t, \end{aligned}$$

and

$$y'' = (e^t\cos t)' + (e^t\sin t)' = x' + y' = 2e^t\cos t.$$

Hence, we have

$$\begin{aligned} \mathbf{v} &= \langle e^t(\cos t - \sin t),\ e^t(\cos t + \sin t)\rangle, \\ \mathbf{a} &= \langle -2e^t\sin t,\ 2e^t\cos t\rangle, \end{aligned}$$

$v = \|\mathbf{v}\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} = \sqrt{2}e^t$, and

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t(\cos t - \sin t) & e^t(\cos t + \sin t) & 0 \\ -2e^t\sin t & 2e^t\cos t & 0 \end{vmatrix} = \langle 0, 0, 2e^{2t}\rangle,$$

So, $\|\mathbf{v}\times\mathbf{a}\| = \|\langle 0, 0, 2e^{2t}\rangle\| = 2e^{2t} = v^2$ and

$$\kappa(t) = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{v^2}{v^3} = \frac{1}{v} = \frac{1}{\sqrt{2}e^t}.$$

(c) From the computation in (b), we see that

$$\|\mathbf{a}\| = \sqrt{(-2e^t\sin t)^2 + (2e^t\cos t)^2} = 2e^t.$$

The dot product $\mathbf{v}\cdot\mathbf{a} = e^t(\cos t - \sin t)(-2e^t \sin t) + e^t(\cos t + \sin t)\cdot 2e^t \cos t = 2e^{2t}$. If the angle between $\mathbf{v}(t)$ and $\mathbf{a}(t)$ is $\theta = \theta(t)$, then

$$\cos\theta(t) = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|\,\|\mathbf{a}\|} = \frac{2e^{2t}}{\sqrt{2}e^t\cdot 2e^t} = \frac{1}{\sqrt{2}}.$$

Hence, $\theta(t) = \pi/4$ for all t, that is, $\mathbf{v}$ and $\mathbf{a}$ always keep a constant angle of $\pi/4$.

Remark. It's easy to see that $\mathbf{r}\cdot\mathbf{a} = 0$, so $\mathbf{r}(t) \perp \mathbf{a}(t)$ for all t. Hence, $\mathbf{r}$ and $\mathbf{v}$ also keep a constant angle (of $\pi/4$). This is why the logarithmic spiral is also called an *equiangular spiral.*

Figure 5: The equiangular spiral with polar equation $r = e^\theta$ for $\theta \in [-2\pi, 0], [0, \pi/2], [\pi/2, 2\pi]$