General Physics II: Hw 1

Problem 1:

Do the integral of the following

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{3-x}}$ (ii) $\displaystyle\int_3^6 \frac{x\,dx}{(9x^2-25)^{1/2}}$ (iii) $\displaystyle\int_0^1 \frac{2dx}{12-3x}$ (iv) $\displaystyle\int_0^{\pi/3} \sin 2x \cos 2x\,dx$

(v) $\displaystyle\int_0^{\pi} \sin^2 3x\,dx$ (vi) $\displaystyle\int_0^1 \frac{dx}{\left(25x^2+4\right)^{3/2}}$ (vii) $\displaystyle\int_2^3 \frac{dx}{\sqrt{9x^2-25}}$

Problem 2:

Solve q(t) of the following equations

(i) $\displaystyle\frac{dq}{dt} = 2q - 3 \quad ; \; t = 0 \;\; q = 5$

(ii) $\displaystyle\varepsilon + 3\frac{q(t)}{C} + 4R\frac{dq(t)}{dt} = 0 \; ; \; t = 0 \;\; q = 2 \; ; \; where\ \varepsilon, R, C\ are$ constant

Problem 3:

There are 9 point charges as shown in Fig. 1 and the position of the charges are given by

$$(r_n;\theta_n) = (R;\frac{n\pi}{8}) \qquad for \quad n = 0,1,2,3,4,5,6,7,8$$

Determine the electric field at the point (0, 0, z) for the following cases

(a) $q_n = q$

(b) $q_n = q$ for n = 0,2,4,6,8 ; $q_n = -q$ for n = 1,3,5,7

(Express your answers in terms of R, q, z, and any other constants)

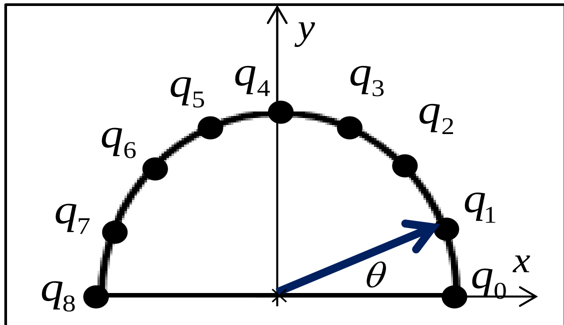


Fig. 1

Problem 1

Do the integral of the following

(i) $\int_0^1 \frac{dx}{\sqrt{3-x}}$

$Sol:$

$let\ u = 3-x$

$du = -dx$

$Then$

$$\int \frac{dx}{\sqrt{3-x}} = \int \frac{-du}{\sqrt{u}} = -2\sqrt{u} + c$$

$Hence$

$$\int_0^1 \frac{dx}{\sqrt{3-x}} = -2\sqrt{3-x}\Big|_0^1$$

$$= 2\sqrt{3} - 2\sqrt{2}$$

(ii) $\int_3^6 \frac{xdx}{(9x^2-25)^{1/2}}$

$Sol:$

$let\ u = 9x^2 - 25$

$du = 18xdx$

$Then$

$$\int \frac{xdx}{(9x^2-25)^{1/2}} = \int \frac{1}{u^{1/2}} \frac{du}{18}$$

$$= \frac{1}{18}\int u^{-\frac{1}{2}} du = \frac{1}{9}(2u^{\frac{1}{2}}) + c$$

$Hence$

$$\int_3^6 \frac{xdx}{(9x^2-25)^{1/2}} = \frac{1}{9}\sqrt{9x^2-25}\Big|_3^6$$

$$= \frac{1}{9}(\sqrt{299} - 2\sqrt{14})$$

(iii) $\int_0^1 \frac{2dx}{12-3x}$

$Sol:$

$let\ u = 12 - 3x$

$du = -3dx$

$Then$

$$\int \frac{2dx}{12-3x} = \int \frac{2}{u}\left(-\frac{1}{3}du\right)$$

$$= -\frac{2}{3}\int \frac{1}{u} du = -\frac{2}{3}\ln|u| + c$$

$Hence$

$$\int_0^1 \frac{2dx}{12-3x} = -\frac{2}{3}\ln|12-3x|\Big|_0^1$$

$$= -\frac{2}{3}\ln\left|\frac{3}{4}\right|$$

(iv) $\int_0^{\pi/3} \sin 2x \cos 2x dx$

Sol :

let $u = \sin 2x$

$du = 2\cos 2x dx$

Then

$\int \sin 2x \cos 2x dx = \int u du$

$= \frac{1}{2}\int u du = \frac{1}{2}(\frac{1}{2}u^2) + c = \frac{1}{4}u^2 + c$

Hence

$$\int_0^{\pi/3} \sin 2x \cos 2x dx = \frac{1}{4}(\sin 2x)^2 \Big|_0^{\frac{\pi}{3}}$$

$= \frac{1}{4}(\frac{3}{4} - 0) = \frac{3}{16}$

(v) $\int_0^{\pi} \sin^2 3x dx$

Sol :

※ $\cos 2\theta = 1 - 2\sin^2\theta$

$\rightarrow \sin^2 3x = \frac{1-\cos 6x}{2}$

Then

$\int \sin^2 3x dx = \int \frac{1-\cos 6x}{2} dx$

$= \frac{1}{2}x - \frac{1}{12}\sin 6x + c$

Hence

$$\int_0^{\pi} \sin^2 3x dx = \frac{1}{2}x - \frac{1}{12}\sin 6x \Big|_0^{\pi}$$

$= \frac{1}{2}(\pi - 0) - \frac{1}{12}(0-0) = \frac{1}{2}\pi$

(vi) $\int_0^1 \frac{dx}{(25x^2+4)^{3/2}} = \int_0^1 \frac{\frac{1}{8}dx}{\left(\left(\frac{5x}{2}\right)^2 + 1\right)^{3/2}}$

Sol :

let $\frac{5x}{2} = \tan\theta$

$\frac{5}{2}dx = \sec^2\theta d\theta$

$\tan^2\theta + 1 = \sec^2\theta$

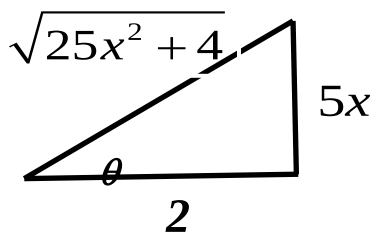
Then $\int_0^1 \frac{\frac{1}{8}dx}{\left(\left(\frac{5x}{2}\right)^2 + 1\right)^{3/2}} = \int \frac{\frac{1}{20}\sec^2\theta d\theta}{(\tan^2\theta + 1)^{3/2}}$

$= \frac{1}{20}\int \frac{d\theta}{\sec\theta} = \frac{1}{20}\sin\theta + c = \frac{1}{20}\frac{5x}{\sqrt{25x^2+4}} + c$

Hence $\int_0^1 \frac{dx}{(25+4x^2)^{3/2}} = \frac{1}{20}\frac{5x}{\sqrt{25x^2+4}}\Big|_0^1$

$= \frac{1}{20}\frac{5}{\sqrt{29}}$

$\tan\theta = \frac{5}{2}x$



(vii) $\displaystyle\int_2^3 \frac{dx}{\sqrt{9x^2-25}} = \int_2^3 \frac{\frac{1}{5}dx}{\sqrt{\left(\frac{3x}{5}\right)^2 - 1}}$

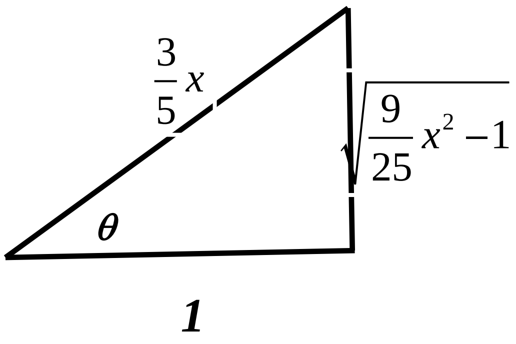
Sol :

let $\displaystyle \frac{3}{5}x = \sec\theta \rightarrow \frac{3}{5}dx = \tan\theta\sec\theta d\theta$ $\boxed{\tan^2\theta + 1 = \sec^2\theta}$

Then

$$\int_2^3 \frac{\frac{1}{5}dx}{\sqrt{\left(\frac{3x}{5}\right)^2 - 1}} = \frac{1}{3}\int \frac{\tan\theta\sec\theta d\theta}{\sqrt{\sec^2\theta - 1}} = \frac{1}{3}\int \sec\theta d\theta$$

$$\sec\theta = \frac{3}{5}x$$



$$= \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| \Bigg|_{\theta_1}^{\theta_2} = \frac{1}{3}\ln\left|\frac{3}{5}x + \sqrt{\frac{9}{25}x^2 - 1}\right| \Bigg|_{x=2}^{x=3}$$

$$= \frac{1}{3}\ln\left|\frac{\frac{9}{5} + \frac{\sqrt{56}}{5}}{\frac{6}{5} + \frac{\sqrt{11}}{5}}\right| = \frac{1}{3}\ln\left|\frac{9+\sqrt{56}}{6+\sqrt{11}}\right|$$

Problem 2

Solve q(t) of the following equations

(i) $$\frac{dq}{dt} = 2q - 3 \quad ; \; t = 0 \;\; q = 5$$

$$\frac{dq}{dt} = 2q - 3$$

$$\frac{dq}{q - \frac{3}{2}} = 2dt$$

$$\int_{q_0}^{q(t)} \frac{dq}{q - \frac{3}{2}} = 2\int_0^t dt$$

$$\ln\left(\frac{q(t) - \frac{3}{2}}{q_0 - \frac{3}{2}}\right) = 2t$$

$$q(t) - \tfrac{3}{2} = \left(q_0 - \tfrac{3}{2}\right)e^{2t}$$

$$q(t) = \left(q_0 - \tfrac{3}{2}\right)e^{2t} + \tfrac{3}{2}$$

$$at\ t = 0, \quad q(t = 0) = q_0 = 5$$

$$q(0) = \left(q_0 - \frac{3}{2}\right) + \frac{3}{2} = 5 \Rightarrow q_0 = 5$$

$$\Rightarrow q(t) = \frac{7}{2}e^{2t} + \frac{3}{2}$$

(ii) $\varepsilon + 3\dfrac{q(t)}{C} + 4R\dfrac{dq(t)}{dt} = 0$; $t = 0 \;\; q = 2$; $\textit{where } \varepsilon, R, C \textit{ are}$ constant

$$Sol: \;\; \frac{dq(t)}{dt} = -\frac{3\left(q(t) + \varepsilon C/3\right)}{4RC} \Rightarrow \frac{dq(t)}{q(t) + \varepsilon C/3} = -\frac{3dt}{4RC}$$

$$\int_{q(0)}^{q(t)} \frac{dq(t)}{q(t) + \varepsilon C/3} = \int_{t=0}^{t} -\frac{3dt'}{4RC}$$

$$\Rightarrow \ln\left|q(t) + \frac{\varepsilon C}{3}\right| \Big|_{q(0)}^{q(t)} = -\frac{3t}{4RC}$$

$$\Rightarrow \ln\left|\frac{q(t) + \frac{\varepsilon C}{3}}{q(0) + \frac{\varepsilon C}{3}}\right| = -\frac{3t}{4RC} \;\; , q(0) = 2$$

$$\Rightarrow q(t) + \frac{\varepsilon C}{3} = (2 + \frac{\varepsilon C}{3})e^{-\frac{3t}{4RC}}$$

$$q(t) = (\frac{C\varepsilon}{3} + 2)e^{-\frac{3t}{4RC}} - \frac{C\varepsilon}{3}$$

Problem 3

There are 9 point charges as shown in Fig. 1 and the position of the charges are given by

$$(r_n;\theta_n)=(R;\frac{n\pi}{8}) \qquad for \quad n=0,1,2,3,4,5,6,7,8$$

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(Express your answers in terms of R, q, z, and any other constants)

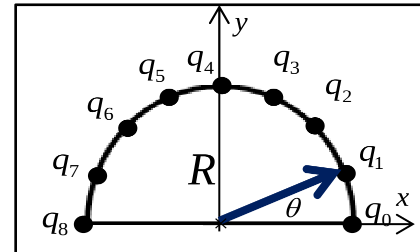
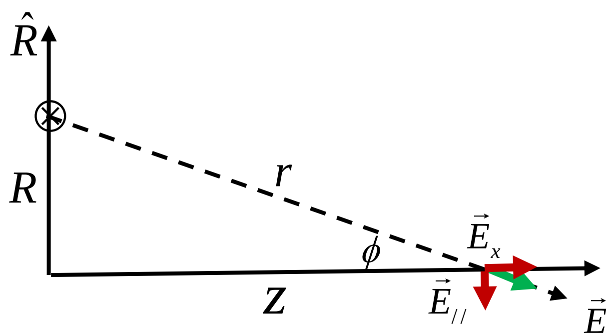
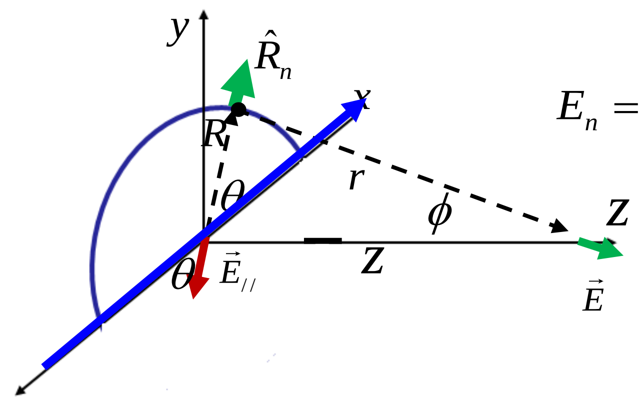


Fig. 1

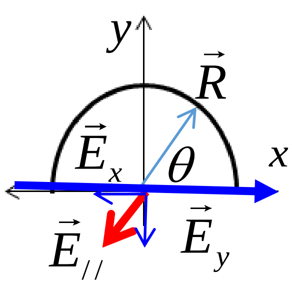
Solution

Method 1





$$E_n = k\frac{q_n}{r^2} = k\frac{q_n}{R^2+z^2}$$



$$E_{nz} = E_n\cos\phi \quad ; \quad E_{n//} = E_n\sin\phi$$

$$E_{nx} = -E_{n//}\cos\theta = -E_n\sin\phi\cos\theta_n$$

$$E_{ny} = -E_{n//}\sin\theta_n = -E_n\sin\phi\sin\theta_n$$

$$E_x = \sum_{n=0}^{8} E_{nx} = -\sum_{n=0}^{8} E_n \sin\phi \cos\theta_n = -\sum_{n=0}^{8} \frac{q_n R \cos\theta_n}{\left(R^2 + z^2\right)^{3/2}}$$

$$E_y = \sum_{n=0}^{8} E_{ny} = -\sum_{n=0}^{8} E_n \sin\phi \sin\theta_n = -\sum_{n=0}^{8} \frac{q_n R \sin\theta_n}{\left(R^2 + z^2\right)^{3/2}}$$

$$E_z = \sum_{n=0}^{6} E_{nz} = \sum_{n=0}^{8} E_n \cos\phi = \sum_{n=0}^{8} \frac{q_n z}{\left(R^2 + z^2\right)^{3/2}}$$

$$(a)\vec{E}_{total} = \frac{-kq}{(R^2+z^2)^{3/2}} R(\cos 0 + \cos\frac{\pi}{8} + \cos\frac{2\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{4\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{6\pi}{8} + \cos\frac{7\pi}{8} + \cos\frac{8\pi}{8})\hat{x}$$

$$+ \frac{-kq}{(R^2+z^2)^{3/2}} R(\sin 0 + \sin\frac{\pi}{8} + \sin\frac{2\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{4\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{6\pi}{8} + \sin\frac{7\pi}{8} + \sin\frac{8\pi}{8})\hat{y} + \frac{kq}{(R^2+z^2)^{3/2}}(9z)\hat{z}$$

$$= \frac{kq}{(R^2+z^2)^{3/2}}\left[0\hat{x} - (1+\sqrt{2}+\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}})R\hat{y} + 9z\hat{z}\right]$$

(b) $q_n = q$ for n = 0,2,4,6,8 ; $q_n = -q$ for n = 1,3,5,7

$$\vec{E}_{total} = \frac{-kq}{(R^2+z^2)^{3/2}} R(\cos 0 - \cos\frac{\pi}{8} + \cos\frac{2\pi}{8} - \cos\frac{3\pi}{8} + \cos\frac{4\pi}{8} - \cos\frac{5\pi}{8} + \cos\frac{6\pi}{8} - \cos\frac{7\pi}{8} + \cos\frac{8\pi}{8})\hat{x}$$

$$+ \frac{-kq}{(R^2+z^2)^{3/2}} R(\sin 0 - \sin\frac{\pi}{8} + \sin\frac{2\pi}{8} - \sin\frac{3\pi}{8} + \sin\frac{4\pi}{8} - \sin\frac{5\pi}{8} + \sin\frac{6\pi}{8} - \sin\frac{7\pi}{8} + \sin\frac{8\pi}{8})\hat{y} + \frac{kqz}{(R^2+z^2)^{3/2}}\hat{z}$$

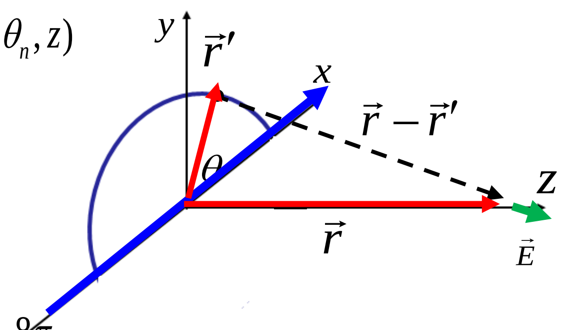
$$= \frac{kq}{(R^2+z^2)^{3/2}}\left[0\hat{x} - (1+\sqrt{2}-\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}})R\hat{y} + z\hat{z}\right]$$

Method 2

$Sol:$ (a) $\vec{r} = (0,0,z)$ $\vec{r}\,' = (R\cos\theta, R\sin\theta, 0)$ $\vec{r} - \vec{r}\,' = (-R\cos\theta, -R\sin\theta, z)$

$$\vec{E}_n = \frac{kq_n}{|\vec{r}-\vec{r}\,'|^2}\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} = \frac{kq_n}{R^2\cos^2\theta + R^2\sin^2\theta + z^2}\frac{1}{\sqrt{R^2\cos^2\theta + R^2\sin^2\theta + z^2}}\cdot(-R\cos\theta_n, -R\sin\theta_n, z)$$

$$= \frac{kq}{(R^2+z^2)^{3/2}}\left(-R\cos\frac{n\pi}{8}\hat{x} - R\sin\frac{n\pi}{8}\hat{y} + z\hat{z}\right)$$



$$\vec{E}_{total} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 + \vec{E}_5 + \vec{E}_6 + \vec{E}_7 + \vec{E}_8$$

$$= \frac{-kq}{(R^2+z^2)^{3/2}}R\left(\cos 0 + \cos\frac{\pi}{8} + \cos\frac{2\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{4\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{6\pi}{8} + \cos\frac{7\pi}{8} + \cos\frac{8\pi}{8}\right)\hat{x}$$

$$+ \frac{-kq}{(R^2+z^2)^{3/2}}R\left(\sin 0 + \sin\frac{\pi}{8} + \sin\frac{2\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{4\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{6\pi}{8} + \sin\frac{7\pi}{8} + \sin\frac{8\pi}{8}\right)\hat{y} + \frac{kq}{(R^2+z^2)^{3/2}}(9z)\hat{z}$$

$$= \frac{kq}{(R^2+z^2)^{3/2}}\left[0\hat{x} - \left(1 + \sqrt{2} + \sqrt{2-\sqrt{2}} + \sqrt{2+\sqrt{2}}\right)R\hat{y} + 9z\hat{z}\right]$$

(b) $q_n = q$ for $n = 0,2,4,6,8$; $q_n = -q$ for $n = 1,3,5,7$

$$\vec{E}_{total} = \frac{-kq}{(R^2+z^2)^{3/2}}R\left(\cos 0 - \cos\frac{\pi}{8} + \cos\frac{2\pi}{8} - \cos\frac{3\pi}{8} + \cos\frac{4\pi}{8} - \cos\frac{5\pi}{8} + \cos\frac{6\pi}{8} - \cos\frac{7\pi}{8} + \cos\frac{8\pi}{8}\right)\hat{x}$$

$$+ \frac{-kq}{(R^2+z^2)^{3/2}}R\left(\sin 0 - \sin\frac{\pi}{8} + \sin\frac{2\pi}{8} - \sin\frac{3\pi}{8} + \sin\frac{4\pi}{8} - \sin\frac{5\pi}{8} + \sin\frac{6\pi}{8} - \sin\frac{7\pi}{8} + \sin\frac{8\pi}{8}\right)\hat{y} + \frac{kqz}{(R^2+z^2)^{3/2}}\hat{z}$$

$$= \frac{kq}{(R^2+z^2)^{3/2}}\left[0\hat{x} - \left(1 + \sqrt{2} - \sqrt{2-\sqrt{2}} - \sqrt{2+\sqrt{2}}\right)R\hat{y} + z\hat{z}\right]$$