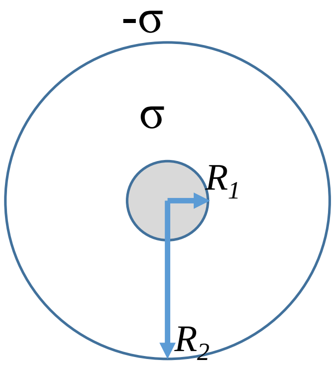
HW5-1: Consider a capacitor is made of two concentric metal spherical shells with radia $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ respectively. The inner shell has a surface charge density $+\boldsymbol{\sigma}$, the outer shell $-\boldsymbol{\sigma}$.

(a) Find the electric field and electric potential difference between the two shells?

(b) What is the capacitance of this system?

(c) What is the total energy stored in this system? What is the energy density between the plates?

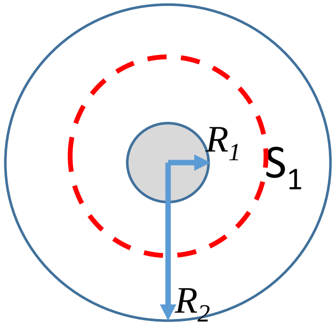


HW5-1: solution

(a) By using the Gauss's law, calculate the electric field between the two spherical shell:

Choose the concentria spherical Gaussian surface (三維球殼) S_1:

$$4\pi r^2 E_r = \frac{Q}{\varepsilon_0} \Rightarrow \vec{E} = k\frac{Q}{r^2}\hat{r}$$



Calculate the electric potential difference between the two spherical shells:

$$V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{S} = -\int_a^b E_r dr$$

$$= -\int_a^b k\frac{Q}{r^2}dr = -kQ\left(-\frac{1}{r}\right)\Bigg|_a^b = kQ\left(\frac{1}{b}-\frac{1}{a}\right)$$

(b) By using the definition,calculate the capacitance:

$$C = \frac{Q}{\Delta V} = \frac{ab}{k(b-a)}$$

(c) Energy stored in the capacitor: $$U = \frac{Q^2}{2C} = \frac{k(b-a)Q^2}{2ab}$$ (depends on *a*, *b*, *Q*.)

Energy density: $$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{Q^2}{32\pi^2\varepsilon_0}\cdot\frac{1}{r^4}$$ (depends on r.)

HW5-2:

(a) The solid lines in Fig.2 shows two conducting coaxial cylinders with inner and outer radius $\boldsymbol{R_a=a, R_b=4a}$ and length $\boldsymbol{L}$. Calculate the capacitance of this device. Ignoring the end effects.

(b) A cylindrical metallic tube is inserted into this system with radius $\boldsymbol{R_c= 2a}$ and $\boldsymbol{R_d = 3a}$, shown in dashed lines in the figure. What is the capacitance for this new structure? (hint: you may use result in part (a))

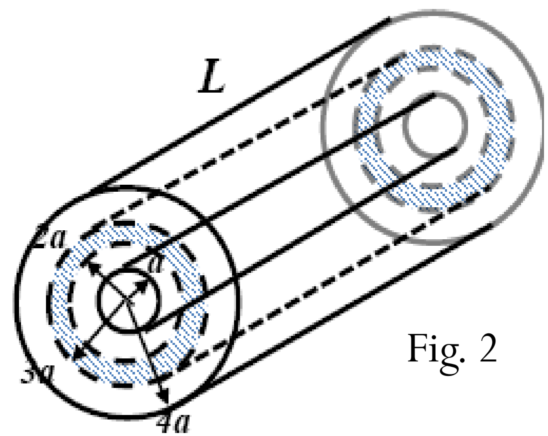
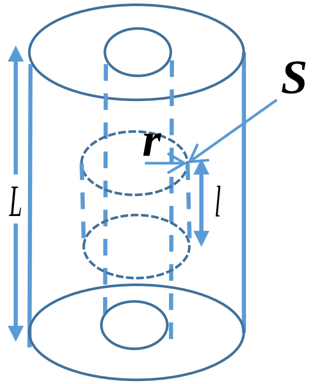


Fig. 2

HW5-2: solution

(a)



Choose cylindrical surface S as the Gaussian surface.

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0} \qquad \Rightarrow E\,2\pi r l = \frac{\left(\dfrac{Q}{2\pi R_a L}\right)\cdot 2\pi R_a l}{\varepsilon_0}$$

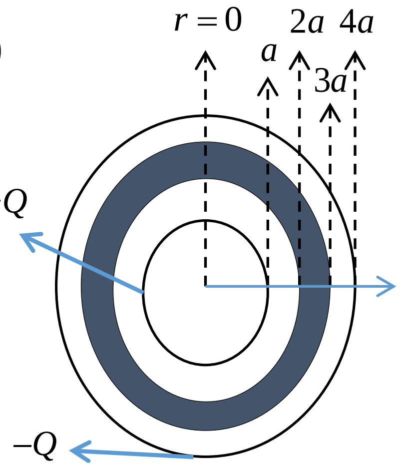
$$\text{set } \lambda = \frac{Q}{L} \qquad \Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

$$\Delta V = V(a) - V(4a) = -\int_{4a}^{a} \vec{E}\cdot d\vec{l}$$

$$= -\int_{4a}^{a} \frac{\lambda}{2\pi\varepsilon_0 r} dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{a}{4a}\right) = \frac{Q/L}{2\pi\varepsilon_0}\ln 4$$

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{|V(4a)-V(a)|} = \frac{Q}{\dfrac{Q/L}{2\pi\varepsilon_0}\ln 4} = \frac{2\pi\varepsilon L}{\ln 4} = \frac{\pi\varepsilon L}{\ln 2}$$

(b)



since $E=0$ for $2a \le r \le 3a$

Take spherical Gaussian surface near r = 2a:

$\oint \vec{E}\cdot d\vec{A}=0$ ➡ No charge within r $\le$2a

➡ There are -Q on surface r=2a and Q on surface r=3a.

Then using the Gauss theorem in those three region, we obtain:

$$\vec{E}_1=\frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} \quad \text{for a} \le \text{r} \le \text{2a}$$

$$\vec{E}_2=0 \quad \text{for 2a} \le \text{r} \le \text{3a}$$

$$\vec{E}_3=\frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} \quad \text{for 3a} \le \text{r} \le \text{4a}$$

$$V(a)-V(4a)=-\int_{4a}^{a}\vec{E}\cdot d\vec{l}=-\left(\int_{4a}^{3a}\vec{E}_1\cdot d\vec{l}+\int_{3a}^{2a}\cancel{\vec{E}_2}\cdot d\vec{l}+\int_{2a}^{a}\vec{E}_3\cdot d\vec{l}\right)$$

$$=-\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{3a}{4a}\right)-\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{a}{2a}\right)$$

$$C=\frac{Q}{|V(4a)-V(a)|}=\frac{2\pi\varepsilon L}{\ln(8/3)}$$