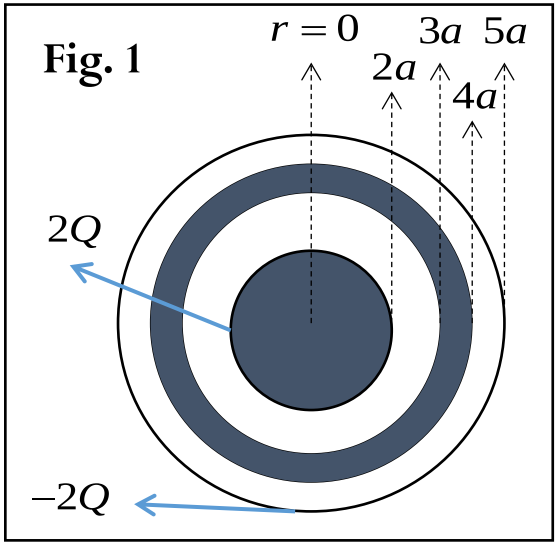
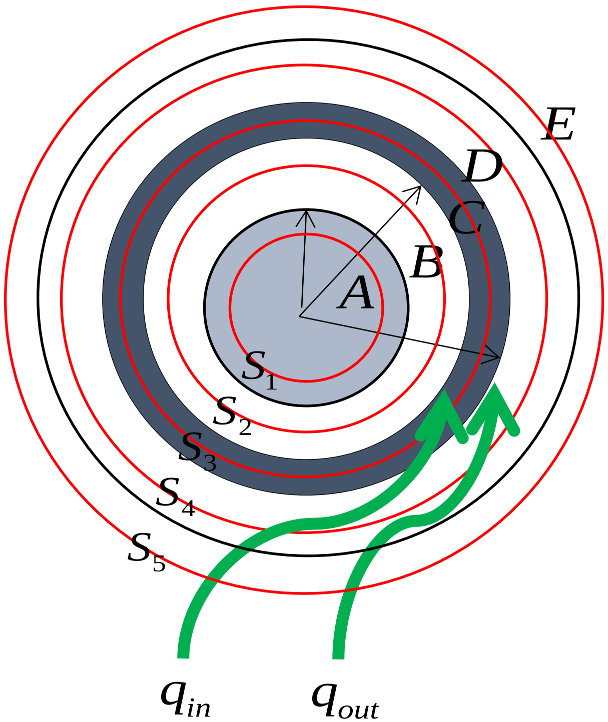
HW4-1: A spherical structure consists of four concentric spherical shells have radii ***2a, 3a,4a,*** and ***5a*** (Fig. below). The most inner spherical shell (***r =2a***) carries total charge **+2*Q*** and the outer most shell (***r =5a***) carries total charge **-2*Q***. Divide the system into regions: **A:** ***r < 2a***, **B:** ***2a<r<3a***, **C:*3a<r<4a***, **D:** ***4a<r<5a***, **E:** ***r >5a*** (infinitely thin shell). The regions ***A, C***, and ***E*** are conductors and other regions are empty.

(a) Draw the Guassian surfaces and calculate the electric fields (both the direction and magnitude) and electric potential in each region. (let ***V = 0*** at ***r =*** ∞)

(b) Now, we connect region **C** and **E** with a wire. Once the system equilibrated, find the electric field and potential for each region in this case.



Solution HW4-1:



Charge problems:

$$S_3 : \oint_{S_3} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0}(2Q + q_{in}) = 0$$

$$q_{in} = -2Q$$

Charge conservation :

$$Q_{tot} = \text{constant} = 0$$
$$= q_{in} + q_{out} = 0$$
$$\Rightarrow \quad q_{out} = 2Q$$

Region A: S_1 : Interior of conductor

$\Rightarrow$ no electric field, $\vec{E}(r) = \hat{0}$

Region B: $S_2 : \oint_{S_2} \vec{E} \cdot d\vec{a} = E \cdot 4\pi r^2 \ , \ \frac{1}{\varepsilon_0} Q_{enc} = \frac{2Q}{\varepsilon_0}$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{r^2} \hat{r}$$

Region C: S_3 : Interior of conductor

$\Rightarrow$ no electric field, $\vec{E}(r) = \hat{0}$

Region D: $S_4 : \oint_{S_4} \vec{E} \cdot d\vec{a} = E \cdot 4\pi r^2$

$$\frac{1}{\varepsilon_0} Q_{enc} = \frac{1}{\varepsilon_0}(2Q + q_{in} + q_{out}) = \frac{2Q}{\varepsilon_0}$$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{r^2} \hat{r}$$

Region E: $S_5 : \oint_{S_4} \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = 0 \quad \Rightarrow \vec{E}(r) = 0$

Region E: $E(r)=0, \quad V(r)=-\int_{\infty}^{r}\vec{E}\cdot d\vec{r}=0$

Region D: $V(r)=-\int_{\infty}^{r}\vec{E}\cdot d\vec{r}=-\int_{5a}^{r}\vec{E}\cdot d\vec{r}=\frac{1}{4\pi\varepsilon_0}\frac{2Q}{r'}\Big|_{5a}^{r}=\frac{2Q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{5a}\right)$

Region C: $V(r)=-\int_{\infty}^{r}\vec{E}\cdot d\vec{r}=-(\int_{5a}^{4a}\vec{E}\cdot d\vec{r}+\int_{4a}^{r}\vec{E}\cdot d\vec{r})=\frac{2Q}{4\pi\varepsilon_0}\left(\frac{1}{4a}-\frac{1}{5a}\right)$

Region B: $V(r)=-\int_{\infty}^{r}\vec{E}\cdot d\vec{r}=-(\int_{5a}^{4a}\vec{E}\cdot d\vec{r}+\int_{4a}^{3a}\vec{E}\cdot d\vec{r}+\int_{3a}^{r}\vec{E}\cdot d\vec{r})$

$$=\frac{2Q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{3a}+\frac{1}{4a}-\frac{1}{5a}\right)$$

Region A: $V(r)=-\int_{\infty}^{r}\vec{E}\cdot d\vec{r}=-(\int_{5a}^{4a}\vec{E}\cdot d\vec{r}+\int_{4a}^{3a}\vec{E}\cdot d\vec{r}+\int_{3a}^{2a}\vec{E}\cdot d\vec{r})$

$$=\frac{2Q}{4\pi\varepsilon_0}\left(\frac{1}{2a}-\frac{1}{3a}+\frac{1}{4a}-\frac{1}{5a}\right)$$

(b) Connect region C and E with a wire.

The electric field:

Region A: $\vec{E}(r) = \hat{0}$, unchanged.

Region B: $\vec{E}(r) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r^2}\hat{r}$

,unchanged.

Region C: $\vec{E}(r) = \hat{0}$, unchanged.

Region D: $\Delta V = V(4a) - V(3a) = 0$

$\Rightarrow \vec{E}(r) = 0$, changed.

Region E: $\vec{E}(r) = 0$, unchanged.

The electric potential:

Region E: $E(r) = 0, \quad V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r} = 0$

Region D: $V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r} = -\int_{5a}^{r} \vec{E}\cdot d\vec{r} = 0$

Region C: $V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r}$

$$= -\left(\int_{5a}^{4a} \vec{E}\cdot d\vec{r} + \int_{4a}^{r} \vec{E}\cdot d\vec{r}\right) = 0$$

Region B: $V(r) = -\left(\int_{5a}^{4a} \vec{E}\cdot d\vec{r} + \int_{4a}^{3a} \vec{E}\cdot d\vec{r} + \int_{3a}^{r} \vec{E}\cdot d\vec{r}\right)$

$$= \frac{2Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{3a}\right)$$

Region A: $V(r) = -\left(\int_{5a}^{4a} \vec{E}\cdot d\vec{r} + \int_{4a}^{3a} \vec{E}\cdot d\vec{r} + \int_{3a}^{2a} \vec{E}\cdot d\vec{r}\right)$

$$= \frac{2Q}{4\pi\varepsilon_0}\left(\frac{1}{2a} - \frac{1}{3a}\right)$$

HW4-2: There is charged sphere with charge density $\rho(r) = Ar^{5/2}$, and radius ***R***, as shown in Fig. 3.

(A) What is total charge of the sphere?

(B) Find the electric field, magnitude and direction, for ***r* > *R*** and ***r* < *R***.

(C) Find the electric potential for ***r* > *R*** and ***r* < *R***. (set ***V* = 0** at ***r*** $\rightarrow \infty$)

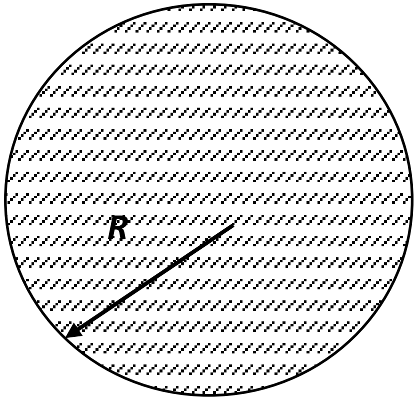
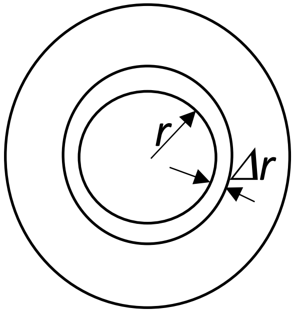


Fig. 3

HW4.2. There is charged sphere with charge density $\rho(r) = Ar^{5/2}$, and radius ***R***, as shown in Fig. 3.



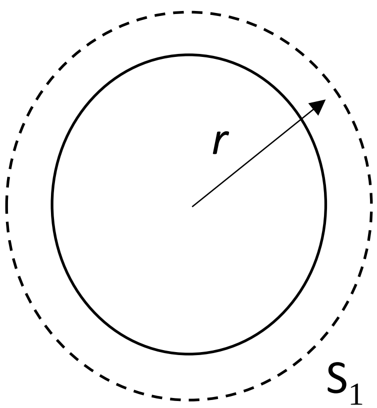
(a)

$$Q_{total} = \int \rho(r)dV = \int Ar^{5/2} \cdot 4\pi r^2 dr = A \cdot 4\pi \cdot \frac{2}{11} r^{11/2}\Big|_0^R$$

$$= \frac{8\pi A}{11} R^{11/2}$$

(b) (i) r > R case:

Choose Guass's surface S_1 with radius r which is larger than R.



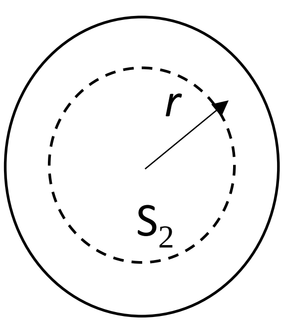
$$\oint \vec{E} \cdot d\vec{A} = E(r) \cdot 4\pi r^2$$ and $$q_{in} = Q_{total} = \frac{8\pi A}{11} R^{11/2}$$

then $$\vec{E} = \frac{Q_{total}}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} \cdot \hat{r}$$

(b) (ii) r < R case:

Choose Guass's surface S_2 with radius r which is smaller than R.

$$\oint \vec{E} \cdot d\vec{A} = E(r) \cdot 4\pi r^2$$ and $$q_{in} = \int_0^r Ar^{5/2} \cdot 4\pi r^2 dr = \frac{8\pi A}{11} r^{11/2} = Q_{total} \left(\frac{r}{R}\right)^{\frac{11}{2}}$$



then $$\vec{E} = \frac{Q_{total}}{4\pi\varepsilon_0} \cdot \frac{r^{\frac{7}{2}}}{R^{\frac{11}{2}}} \cdot \hat{r}$$

r > R : $V(r) = -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0} \frac{Q_{total}}{r'^2} \hat{r} \cdot dr'\hat{r} = -\frac{Q_{total}}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{dr'}{r'^2}$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q_{total}}{r'} \Bigg|_{r'=\infty}^{r} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{total}}{r}$$

R > r : $V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r} = -(\int_{\infty}^{R} \vec{E} \cdot d\vec{r} + \int_{R}^{r} \vec{E} \cdot d\vec{r})$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q_{total}}{r} \Bigg|_{\infty}^{r=R} - \frac{Q_{total}}{4\pi\varepsilon_0} \int_{R}^{r} \frac{r^{\frac{7}{2}}}{R^{\frac{11}{2}}} \cdot d\vec{r}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q_{total}}{R} + \frac{Q_{total}}{4\pi\varepsilon_0 R^{\frac{11}{2}}} \cdot \frac{2}{9} \left(r^{\frac{9}{2}} - R^{\frac{9}{2}} \right)$$

HW4-3. Fig. 3 shows the electric potential built by three charged infinite plates. One plate at x = *-1* m, one at x = *1* m, and the other is at the x = *2* m. What is the surface charge densities of the plates at x = *-1* m, x = *1* m and x = *2* m ?

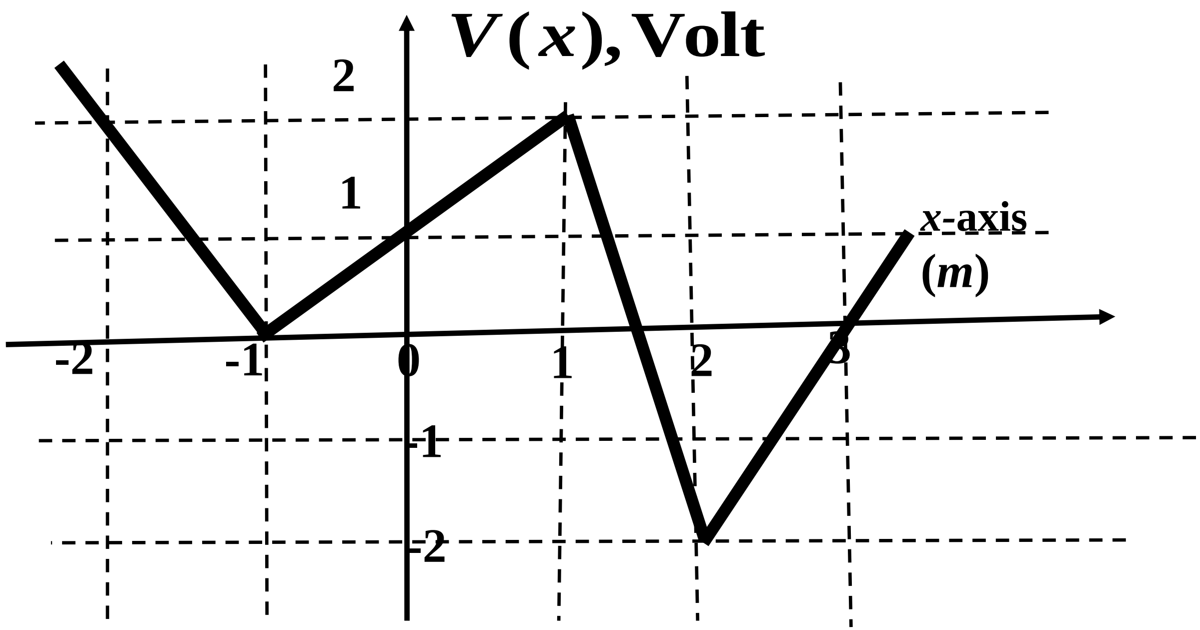
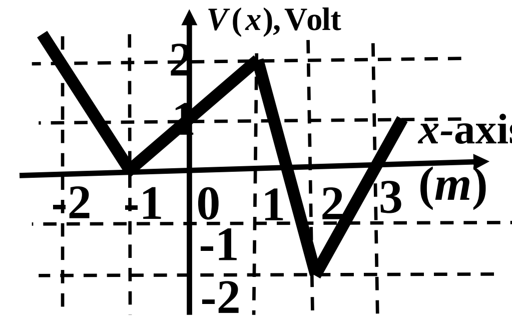


Fig. 3

Solution HW4-3 : $\vec{E} = -\dfrac{dV}{dx}\hat{i} = -\dfrac{\Delta V}{\Delta x}\hat{i}$

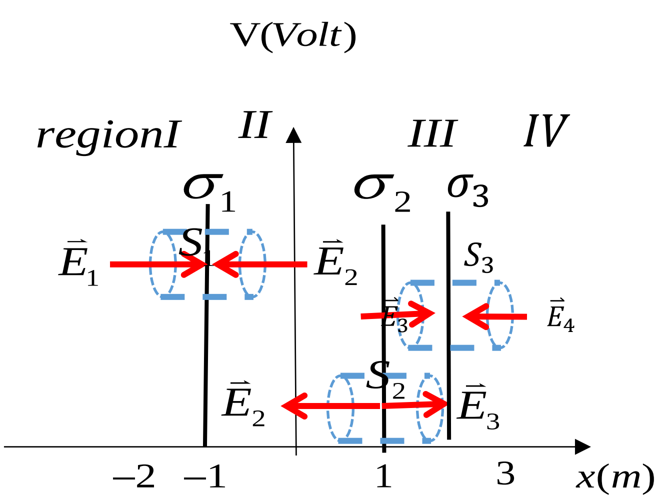


(i) x<-1 : $\vec{E} = -\dfrac{0-(-2)}{(-1)-(-2)}\hat{i} = 2\hat{i}\,(N/C)$

(ii) -1<x<1 : $\vec{E} = -\dfrac{2-0}{1-(-1)}\hat{i} = -1\hat{i}\,(N/C)$

(iii) 1< x<2 : $\vec{E} = -\dfrac{-2-2}{2-1}\hat{i} = 4\hat{i}\,(N/C)$

(iv) 2< x : $\vec{E} = -\dfrac{0-2}{3-2}\hat{i} = -2\hat{i}\,(N/C)$



(E_i are known, σ_i are unkonw)

For Gauss's surface S_1: $-E_1\cdot\pi r^2 - E_2\cdot\pi r^2 == \dfrac{\sigma_1\cdot\pi r^2}{\varepsilon_0}$

$$\Rightarrow \sigma_1 = -\varepsilon_0\left(E_1 + E_2\right) = -3\varepsilon_0$$

For Gauss's surface S_2: $+E_2\cdot\pi r^2 + E_3\cdot\pi r^2 == \dfrac{\sigma_2\cdot\pi r^2}{\varepsilon_0}$

$$\Rightarrow \sigma_2 = \varepsilon_0(E_2 + E_3) = 5\varepsilon_0$$

For Gauss's surface S_3: $-E_3\cdot\pi r^2 - E_4\cdot\pi r^2 == \dfrac{\sigma_3\cdot\pi r^2}{\varepsilon_0}$

Gauss' law: $\displaystyle\oint_{\partial\Omega}\vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc}$

$$\Rightarrow \sigma_3 = -\varepsilon_0(E_3 + E_4) = -6\varepsilon_0$$