#1. **(a)** 0 **(b)** Does not exist. **#2.** **(a)** $f_y(x,y) = y^3(x^4+y^4)^{-3/4}$ if $(x,y) \neq (0,0)$; $f_y(0,0)$ does not exist. **(b)** f_y is discontinuous only at $(0,0)$.

#3. **(a)** $2e^{r+s+t}/(e^r+e^s+e^t)^3$ **(b)** $-e^y/(15\cos 5z)$ **#4.** $(4/3, 2/3, -4/3)$

#5. $(e^{16}-1)/4$ **#6.** $16/3$ **#7.** 12π **#8.** $125\pi/3$

#1. *Limits.* [13.2.23], [13.2.35]

(a) Use polar coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$. Note that $(x,y) \to (0,0)$ is equivalent to $r = \sqrt{x^2+y^2} \to 0^+$. We have

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln\sqrt{x^2+y^2} = \lim_{r\to 0^+} r^2 \ln r = \lim_{r\to 0^+} \frac{\ln r}{1/r^2} \overset{(*)}{=} \lim_{r\to 0^+} \frac{1/r}{-2/r^3} = 0,$$

where in the penultimate step (*), we have used the L'Hôpital Rule.

(b) If we choose the path along the curve C: $(x,y,z) = (t,t,t^3)$ with $t \neq 0$, the function becomes $f(t,t,t^3) = t\cdot t\cdot(t^3)/(t^6+t^6+t^6) = \frac{1}{3t}$, which has no limit as $t \to 0$. Therefore, $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^6+y^6+z^2}$ does not exist.

#2. *Partial derivatives, continuity.* [13.3.108]

$f(x,y) = (x^4+y^4)^{1/4}$.

(a) For $(x,y) \neq (0,0)$, by the Chain Rule we have

$$f_y(x,y) = \frac{1}{4}(x^4+y^4)^{-3/4}\frac{\partial}{\partial y}(x^4+y^4) = \frac{1}{4}(x^4+y^4)^{-3/4}\cdot 4y^3 = \frac{y^3}{(x^4+y^4)^{3/4}}.$$

To find f_y at $(0,0)$, we cannot use the above computation; we have to resort to the definition. Note that $f(0,0) = 0$. Thus,

$$f_y(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}\frac{(y^4)^{1/4}-0}{y} = \lim_{y\to 0}\frac{|y|}{y},$$

which does not exist because the limit $\to 1$ as $y \to 0^+$, while the limit $\to -1$ as $y \to 0^-$.

(b) Since $(x^4+y^4)^{-3/4}$ is continuous everywhere in $\mathbb{R}^2$ except at $(0,0)$ and y^3 is continuous everywhere in $\mathbb{R}^2$, we see that $f_y(x,y)$ is continuous everywhere except possibly at $(0,0)$. The partial f_y is not continuous at $(0,0)$ because $f_y(0,0)$ fails to exist.

#3. *Partial derivatives, the Chain Rule.* [13.5.61], [13.5.45]

(a) We have

$$w_r = \frac{\partial w}{\partial r} = \frac{e^r}{e^r+e^s+e^t} = e^r(e^r+e^s+e^t)^{-1}.$$

Note that r, s, and t are independent variables, so

$$w_{rs} = (w_r)_s = e^r(-(e^r+e^s+e^t)^{-2}e^s) = -e^{r+s}(e^r+e^s+e^t)^{-2}.$$

Similarly,

$$w_{rst} = (w_{rs})_t = -e^{r+s}(-2(e^r+e^s+e^t)^{-3}e^t) = 2e^{r+s+t}(e^r+e^s+e^t)^{-3}.$$

(b) Regard x, y, and z as three *independent* variables and let $F(x,y,z) = xe^y + 3\sin 5z$. Then $F_x = e^y$, $F_z = 15\cos 5z$, and

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-e^y}{15\cos 5z}.$$

#4. *Lagrange multipliers.* [13.9.9]

We wish to find a point $P(x, y, z)$ lying on the circle $C\colon x^2 + y^2 + z^2 = 4$ and P has smallest distance to the plane $E\colon 2x + y - 2z = 9$. The plane E has a normal vector $\mathbf{n} = \langle 2, 1, -2\rangle$ and contains the point $A(4, 1, 0)$. If θ is the angle between $\overrightarrow{AP}$ and $\mathbf{n}$, then the distance $d(P, E) = \|\overrightarrow{AP}\||\cos\theta| = |\overrightarrow{AP}\cdot\mathbf{n}|/\|\mathbf{n}\|$.

In other words, we wish to optimize $d(x, y, z) = \overrightarrow{AP}\cdot\mathbf{n} = \langle x-4, y-1, z\rangle\cdot\langle 2, 1, -2\rangle = 2(x-4)+(y-1)-2z = 2x+y-2z-9$ subject to the constraint $g(x, y, z) = x^2+y^2+z^2 = 4$. By the method of Lagrange multipliers, we need to solve $\lambda\nabla d = \nabla g$, or $\lambda\langle 2, 1, -2\rangle = \langle 2x, 2y, 2z\rangle$. Altogether, we have four equations and four unknowns x, y, z, and λ:

$$\begin{cases} 2x = 2\lambda, \\ 2y = \lambda, \\ 2z = -2\lambda, \\ x^2 + y^2 + z^2 = 4. \end{cases}$$

Since $\lambda \neq 0$ (why?), we see that $x = -z = 2y$. Substituting these back to the fourth equation gives $(2y)^2 + y^2 + (-2y)^2 = 4$, or $y^2 = 4/9$, or $y = \pm 2/3$. There are only two possible solutions for P, namely, $\pm\frac{2}{3}(2, 1, -2)$.

Now, we check: $d(\frac{2}{3}(2, 1, -2)) = -3$ and $d(\frac{2}{3}(-2, -1, 2)) = -15$. Since $\|\mathbf{n}\| = 3$, the minimum distance is $|d(\frac{2}{3}(2, 1, -2))|/\|\mathbf{n}\| = 1$, and $P = \frac{2}{3}(2, 1, -2)$ is the point on C that is closest to E.

#5. *Double integral, change of the order of integration.* [14.2.55]

Since the integral $\int e^{x^4}\,dx$ is non-elementary, we need to reverse the order of integration. The region R was written as a Type II: $0 \le y \le 8$, $y^{1/3} \le x \le 2$. We rewrite it as a Type I region, $R\colon 0 \le x \le 2$, $0 \le y \le x^3$. Then

$$\begin{aligned} \int_0^8 \int_{y^{1/3}}^2 e^{x^4}\,dx\,dy = \iint_R e^{x^4}\,dA = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^2 \Big[e^{x^4} y\Big]_{y=0}^{x^3}\,dx \\ = \int_0^2 e^{x^4} x^3\,dx = \frac{1}{4}e^{x^4}\Big|_0^2 = \frac{1}{4}(e^{16} - 1). \end{aligned}$$

#6. *Double integral, volume.* [14.2.38]

The projection of the solid G to the xy-plane is a quarter-circular region $R\colon x^2+y^2 \le 4$, $x \ge 0$, $y \ge 0$. We regard G as the solid below the surface $S\colon z = \sqrt{4-x^2}$ over R. The volume of the solid is

$$\begin{aligned} V &= \iint_R z\,dA = \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx = \int_0^2 \Big[y\sqrt{4-x^2}\Big]_{y=0}^{\sqrt{4-x^2}}\,dx \\ &= \int_0^2 (4-x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}. \end{aligned}$$

#7. *Surface area, parametric surfaces.* [14.4.23]

To parametrize the portion σ of the sphere $x^2+y^2+z^2 = 9$ that is on or above the plane $z = 1$, we use polar coordinates for (x, y). So let $x = r\cos\theta$, $y = r\sin\theta$, then $z = \sqrt{9-(x^2+y^2)} = \sqrt{9-r^2}$. The condition $z \ge 1$ yields $r^2 \le 8$, or $0 \le r \le 2\sqrt{2}$. A parametrization with parameters r and θ:

$$\sigma\colon x = r\cos\theta, \quad y = r\sin\theta, \quad z = \sqrt{9-r^2}, \quad (0 \le r \le 2\sqrt{2},\ \ 0 \le \theta \le 2\pi).$$

We note that the projection of σ onto xy-plane is a disk $D\colon x^2+y^2 \le 8$. Also, σ is the surface $z=\sqrt{9-x^2-y^2}$ directly above D. Thus,

$$\begin{aligned}
S = \operatorname{area}(\sigma) &= \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \\
&= \iint_D \sqrt{1+\Big(\frac{-x}{\sqrt{9-x^2-y^2}}\Big)^2+\Big(\frac{-y}{\sqrt{9-x^2-y^2}}\Big)^2}\,dA \\
&= \iint_D \sqrt{\frac{9}{9-x^2-y^2}}\,dA = 3\int_0^{2\pi}\int_0^{2\sqrt{2}} \frac{1}{\sqrt{9-r^2}}\,r\,dr\,d\theta \\
&= 3\int_0^{2\pi} d\theta \int_0^{2\sqrt{2}} \frac{r}{\sqrt{9-r^2}}\,dr = 3(2\pi)\left[-\sqrt{9-r^2}\right]_0^{2\sqrt{2}} = 3(2\pi)(3-1) = 12\pi.
\end{aligned}$$

Solution 2. Using the parametrization given above, the surface σ takes the vector form:

$$\mathbf{r}(r,\theta) = \langle x(r,\theta), y(r,\theta), z(r,\theta)\rangle = \langle r\cos\theta,\ r\sin\theta,\ \sqrt{9-r^2}\rangle,\quad 0\le r\le 2\sqrt{2},\quad 0\le\theta\le 2\pi.$$

Note that the domain of the vector-valued function $\mathbf{r}$ is a rectangle

$$R\colon 0\le r\le 2\sqrt{2},\quad 0\le\theta\le 2\pi$$

in the $r\theta$-plane. We compute the surface area differential $dS = \|\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial \theta}\|\,dA$.

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial \theta} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \frac{-r}{\sqrt{9-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\
&= \left\langle \begin{vmatrix} \sin\theta & \frac{-r}{\sqrt{9-r^2}} \\ r\cos\theta & 0 \end{vmatrix}, -\begin{vmatrix} \cos\theta & \frac{-r}{\sqrt{9-r^2}} \\ -r\sin\theta & 0 \end{vmatrix}, \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \right\rangle \\
&= r\left\langle \frac{r\cos\theta}{\sqrt{9-r^2}}, \frac{r\sin\theta}{\sqrt{9-r^2}}, 1\right\rangle.
\end{aligned}$$

Hence,

$$\|\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial \theta}\| = r\sqrt{\Big(\frac{r\cos\theta}{\sqrt{9-r^2}}\Big)^2+\Big(\frac{r\sin\theta}{\sqrt{9-r^2}}\Big)^2+1} = \frac{3r}{\sqrt{9-r^2}}.$$

Now, the computation proceeds as above.

#8. *Triple integrals, spherical coordinates.* [14.6.13]

The given solid G is bounded above by the sphere $\rho=5$ and below by the cone $\phi=\pi/3$. It is best to describe it by spherical coordinates.

$$G\colon 0\le\theta\le 2\pi,\quad 0\le\phi\le\pi/3,\quad 0\le\rho\le 5.$$

A spherical box! Since the volume element in spherical coordinates is $dV=\rho^2\sin\phi\,d\rho d\phi d\theta$, the volume of G is

$$\begin{aligned}
V &= \iiint_G dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^5 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/3}\sin\phi\,d\phi\int_0^5\rho^2\,d\rho \\
&= 2\pi\Big[-\cos\phi\Big]_0^{\pi/3}\Big[\frac{\rho^3}{3}\Big]_0^5 = 2\pi\Big(1-\frac{1}{2}\Big)\frac{125}{3} = \frac{125\pi}{3}.
\end{aligned}$$

Solution 2. It is also possible to describe G in cylindrical coordinates, albeit a little cumbersome. The solid, when projected onto the xy-plane, gives a disk: $D\colon 0\le r\le 5\sqrt{3}/2$. The upper surface, the sphere $\rho = 5$, takes the form $r^2 + z^2 = 25$, or $z_{\text{top}} = \sqrt{25 - r^2}$. The lower surface, the cone $\phi = \pi/3$, becomes $z_{\text{bot}} = r \cot \phi = r/\sqrt{3}$.

$$G\colon 0 \le \theta \le 2\pi, \quad 0 \le r \le 5\sqrt{3}/2, \quad r/\sqrt{3} \le z \le \sqrt{25 - r^2}.$$

The volume element in cylindrical coordinates is $dV = r dr d\theta dz$. Therefore, G has volume

$$\begin{aligned}
V &= \iiint_G dV = \int_0^{2\pi} \int_0^{5\sqrt{3}/2} \int_{r/\sqrt{3}}^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{5\sqrt{3}/2} r\Big(\sqrt{25 - r^2} - \frac{r}{\sqrt{3}}\Big)\, dr \\
&= 2\pi \Big[-\frac{1}{3}(25 - r^2)^{3/2} - \frac{r^3}{3\sqrt{3}}\Big]_0^{5\sqrt{3}/2} \\
&= 2\pi \cdot \frac{1}{3}\Big(25^{3/2} - (25 - \frac{75}{4})^{3/2} - \frac{1}{\sqrt{3}} \cdot \frac{125}{8} \cdot 3\sqrt{3}\Big) \\
&= 2\pi \cdot \frac{125}{3}(1 - \frac{1}{8} - \frac{3}{8}) = \frac{125\pi}{3}.
\end{aligned}$$

Remark. One can also start with

$$V = \iiint_G dV = \iint_D (z_{\text{top}} - z_{\text{bot}})\, dA_{xy},$$

and then transform D and the area differential dA_{xy} in terms of polar coordinates, so that $D\colon 0 \le r \le 5\sqrt{3}/2,\ \ 0 \le \theta \le 2\pi$ and $dA_{xy} = r\, dr d\theta$. The rest of the computation goes the same way as above.

Solution 3. (Not recommended.) If one insists on doing the problem through rectangular coordinates, here is the way. The Cartesian equation for the upper semisphere is $z = \sqrt{25 - x^2 - y^2}$. As for the cone, note that $z = r \cot \phi$, so for the cone $\phi = \pi/3$, its Cartesian equation is $z^2 = r^2 \cot^2(\pi/3) = r^2/3$, or $z = r/\sqrt{3} = \sqrt{(x^2 + y^2)/3}$. The projection of the solid G onto the xy-plane is the disk $D\colon x^2 + y^2 \le (5\sqrt{3}/2)^2 = 75/4$. In terms of Cartesian coordinates, the volume takes the form:

$$\begin{aligned}
V &= \iiint_G dV = \int_{-5\sqrt{3}/2}^{5\sqrt{3}/2} \int_{-\sqrt{75/4 - x^2}}^{\sqrt{75/4 - x^2}} \int_{\sqrt{(x^2+y^2)/3}}^{\sqrt{25 - x^2 - y^2}} dz\, dy\, dx \\
&= 4 \int_0^{5\sqrt{3}/2} \int_0^{\sqrt{75/4 - x^2}} \int_{\sqrt{(x^2+y^2)/3}}^{\sqrt{25 - x^2 - y^2}} dz\, dy\, dx.
\end{aligned}$$

The computation is much more involved.