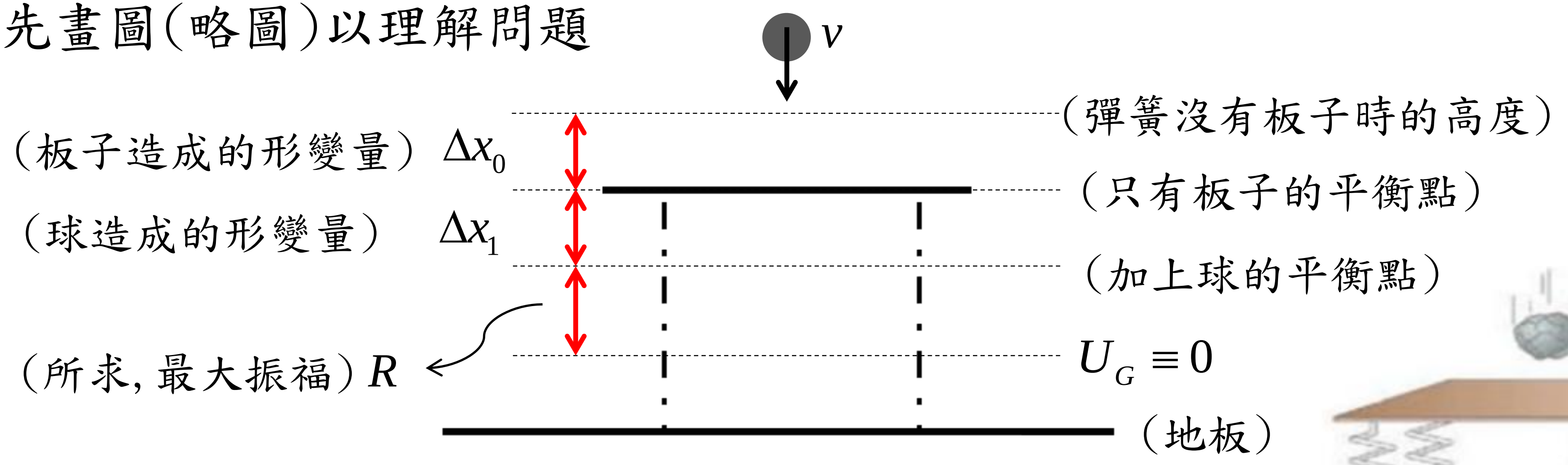
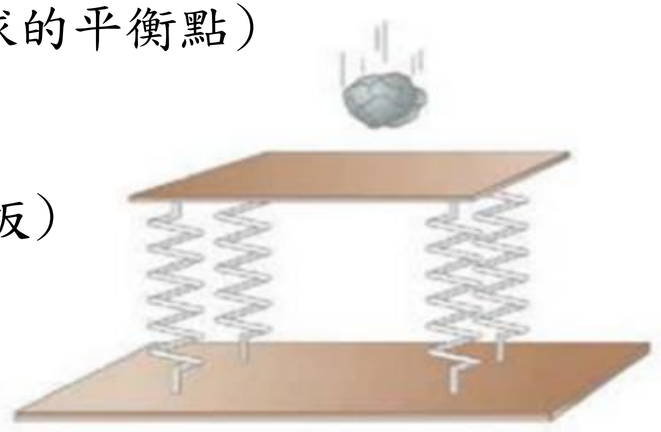
GP HW11

Solution

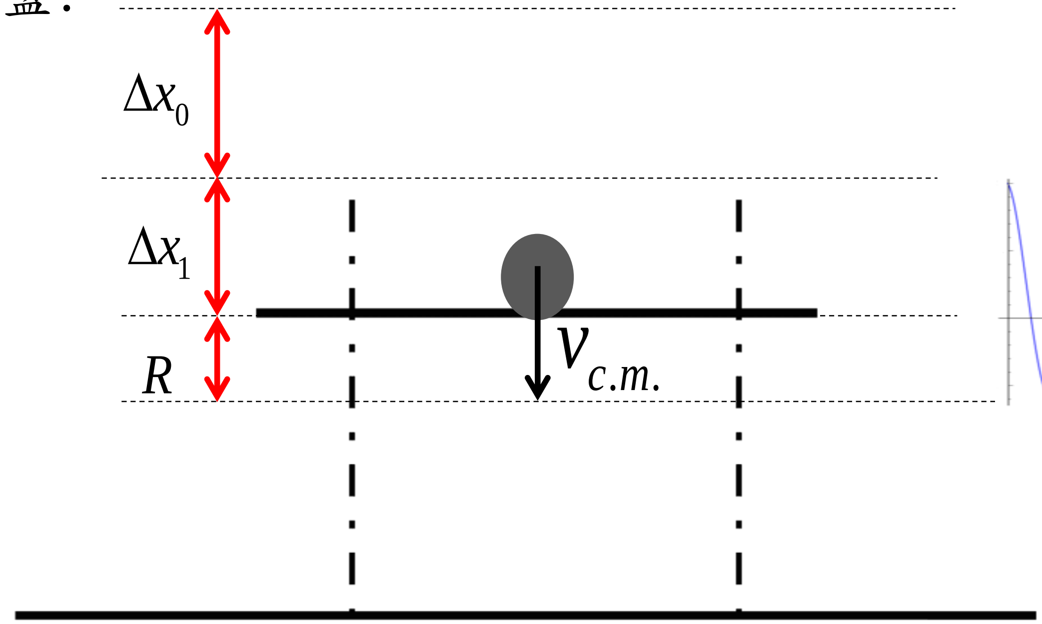
HW11-1: Problem 14-83 in Giancoli (pp. 393)

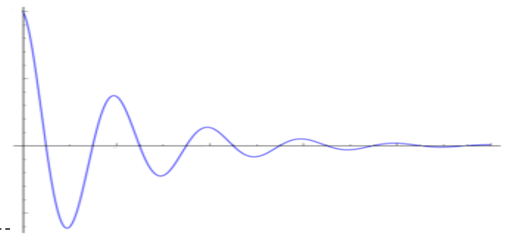
先畫圖(略圖)以理解問題





黏在一起後的震盪:





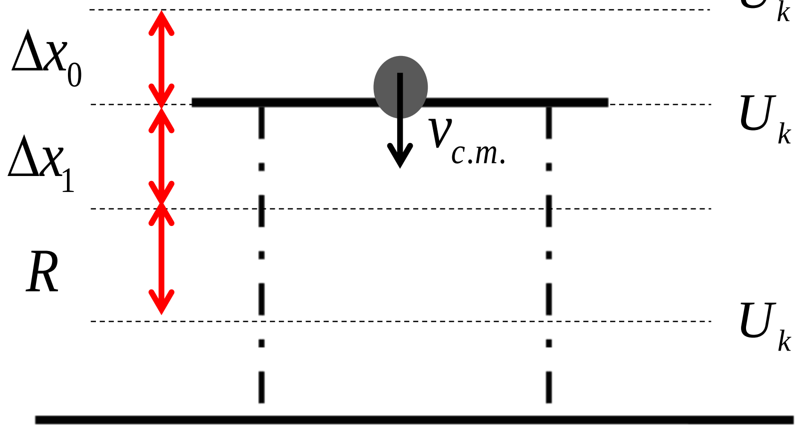
(a):

$$F=-k\Delta x \Rightarrow \begin{cases} -Mg=-k_{eff}\Delta x_0 \\ -(M+m)g=-k_{eff}(\Delta x_0+\Delta x_1) \end{cases}$$

$$\Rightarrow mg=k_{eff}\Delta x_1\ ,\ k_{eff}=\frac{mg}{\Delta x_1}\approx 130.67(\mathrm{kg/s^2})$$

(b):

黏在一起後的瞬間:



$$U_k=\frac{1}{2}k\Delta x^2=0$$

$$U_k=\frac{1}{2}k\Delta x_0^{\,2},\ U_G=(m+M)g(\Delta x_1+R),\ K_{c.m.}$$

$$U_k=\frac{1}{2}k(\Delta x_0+\Delta x_1+R)^2,\ U_G\equiv 0$$

最大震幅,
即考慮動能、重力位能轉換成彈力位能 $\Rightarrow$ 最低點動能為零

先考慮碰撞(完全非彈性碰撞)時間甚短,所以動量變化量可忽略,即動量守恆:

$$\vec{p} = const \;\Rightarrow\; mv = (m+M)v_{c.m}\,,\quad v_{c.m} = \frac{mv}{m+M}$$

$$\Rightarrow\; (K_{c.m})_i = \frac{1}{2}M_{c.m}v^2{}_{c.m} = \frac{1}{2}\frac{m^2}{m+M}v^2$$

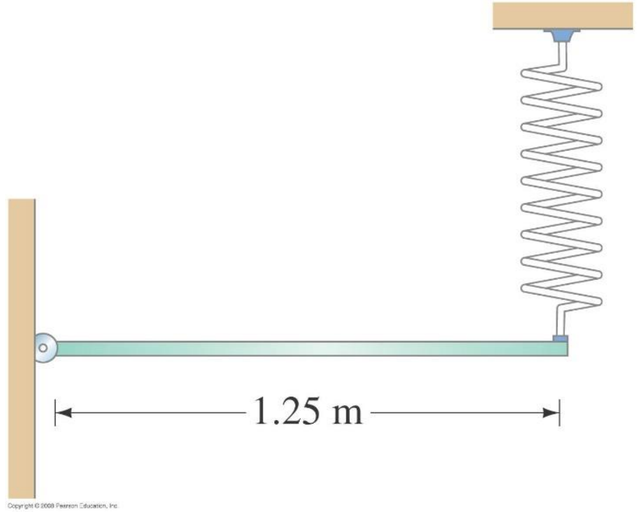
力學能守恆: $E_{tot} = K + U_k + U_G = const$

$$(K_{c.m})_i + \frac{1}{2}k\Delta x_0{}^2 + M_{c.m.}g(\Delta x_1 + R) = 0 + \frac{1}{2}k(\Delta x_0 + \Delta x_1 + R)^2 + 0$$

所求為R ,將等式整理,並用一元二次公式解,得 $R \approx 9.56(cm)$

HW11-2: Problem 14-11 in Giancoli (pp. 389)

A uniform meter stick of mass M is pivoted on a hinge at one end and held horizontal by a spring with spring constant k attached at the other end (Fig. 14–28). If the stick oscillates up and down slightly, what is its frequency?
[*Hint*: Write a torque equation about the hinge.]



達平衡時伸長量為y_0，淨力矩：$\sum \tau = Mg\left(\frac{1}{2}\ell\right) - F_s\ell = \frac{1}{2}Mg\ell - ky_0\ell = 0$

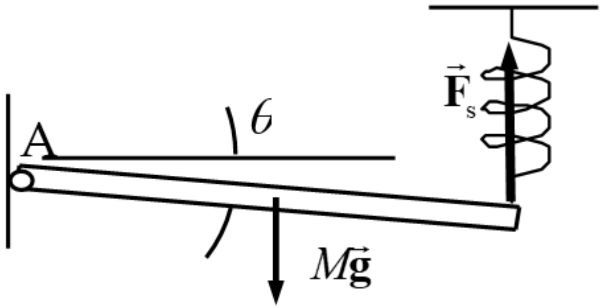
伸長y時力矩：$\sum \tau = Mg\left(\frac{1}{2}\ell\right) - F_s\ell = \frac{1}{2}Mg\ell - k\left(y + y_0\right)\ell = I\alpha = \frac{1}{3}M\ell^2 \dfrac{d^2\theta}{dt^2}$

偏移角度極小時，由泰勒展開式可得： $y = \ell \sin\theta \approx \ell\theta$

將y代回可得： $\frac{1}{2}Mg\ell - ky\ell - ky_0\ell = \frac{1}{3}M\ell^2 \dfrac{d^2\theta}{dt^2} \rightarrow \frac{1}{2}Mg\ell - ky\ell - \frac{1}{2}Mg\ell = \frac{1}{3}M\ell^2 \dfrac{d^2\theta}{dt^2}$

$$\rightarrow -k\ell^2\theta = \frac{1}{3}M\ell^2 \frac{d^2\theta}{dt^2} \rightarrow \frac{d^2\theta}{dt^2} + \frac{3k}{M}\theta = 0$$

又 $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = 0$ 故可得知 $\Rightarrow \omega^2 = \dfrac{3k}{M} = 4\pi^2 f^2 \rightarrow f = \dfrac{1}{2\pi}\sqrt{\dfrac{3k}{M}}$



HW11-3:

A particle is confined to move in *x*-direction between x=0 and x= ∞, and it experiences an conservative force $F(x)$ such that its potential energy $U(x) = -bx^2 \cdot e^{-ax}$, where $a,b>0$,

(a) (6pts) Determine this conservative force $F(x)$ as a function of x,

(b) (4pts) At the equilibrium point $x = S$, $F(S) = 0$, determine the value of S.

(c) (4pts) If the particle is moving around S, and if we define $z = x - S$, write down the equation of motion of the particle in terms of z,

(d) (3pts) For the case if $z/S \ll 1$, the particle executes a simple harmonic oscillation around S, determine the period of the oscillation of the particle near S.

Useful formula: $(1+z)^n \approx 1+nz, \quad e^{az} \approx 1+az, \qquad for\ |z| \ll 1, |az| \ll 1$

(a) $$U(x) = -bx^2 e^{-ax}$$

$$F(x) = -\frac{dU(x)}{dx} = -\frac{d(-bx^2 e^{-ax})}{dx}$$

$$= 2bxe^{-ax} - abx^2 e^{-ax} = -(ax-2)bxe^{-ax}$$

(b) $$for\ F(S) = 0, \Rightarrow 2bSe^{-aS} - abS^2 e^{-aS} = 0$$

$$\Rightarrow S = \frac{2}{a}$$

(c) $$\sum \vec{F} = m\vec{a}, \quad F(x) = -(ax-2)bxe^{-ax} = m\frac{d^2x}{dt^2}$$

$$-(ax-2)bxe^{-ax} = m\frac{d^2x}{dt^2}$$

$$z \equiv x - S \Rightarrow x = z + S = z + 2/a$$

$$\Rightarrow -(a(z+2/a)-2)(z+2/a)\mathrm{b}e^{-a(z+2/a)} = m\frac{d^2(z+2/a)}{dt^2}$$

$$\Rightarrow -az(z+2/a)be^{-a(z+2/a)} = m\frac{d^2z}{dt^2}$$

$$\Rightarrow m\frac{d^2z}{dt^2} + az(z+S)be^{-az-2} = 0$$

(d) $$for\ |z/S| \ll 1, |az| \ll 1$$

$$az(z+S)be^{-a(z+S)} = azS(1+\frac{z}{S})be^{-az} \cdot e^{-aS}$$

$$\approx azS(1+\frac{z}{S})(1-az) \cdot be^{-aS}$$

$$= aS(z + (\frac{1}{S} - a)z^2 - \frac{a}{S}z^3) \cdot be^{-aS} \approx aSz \cdot be^{-aS} = 2be^{-2} \cdot z$$

$$\Rightarrow m\frac{d^2z}{dt^2} + 2be^{-2} \cdot z = 0$$

$$\Rightarrow \omega = \sqrt{\frac{2be^{-2}}{m}} = \sqrt{\frac{2b}{m}}e^{-1}$$

$$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi e\sqrt{\frac{m}{2b}}$$