Probability
Midterm Exam I, Mar. 28, 2017

1. (20%) An urn contains 6 balls numbered 1 to 6.
   (a) (10%) Suppose we select two balls from the urn **with** replacement. Let A be the event that "the number on the first ball is greater than 4, and the number on the second ball is greater than 5". Find $P[A]$.
   (b) (10%) Suppose we select two balls from the urn **without** replacement. Let B be the event that "the number on the first ball is greater than 4, and the number on the second ball is greater than 5". Find $P[B]$.
2. (5%) Let A and B be two events, where $P(A) > 0$ and $P(B) > 0$. Suppose A and B are mutually exclusive, i.e., $A \cap B = \{\}$. Show that A and B are always dependent using Axioms of Probability.
3. (10%) Alice has three children. Assume that all eight possible arrangements of "boy" and "girl" in the order of birth, $\{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$, are equally probable. You are given the information that at least one of Alices children is a boy, and the youngest child is not a girl. Condition on the given information, what is the probability that the Alice have two girls and one boy?
4. (25%) A box contains three coins. One of the coins is a two-headed coin, the second is a fair coin, and the third is a biased coin with probability $P(\text{head}) = p$. One of the coins is picked at random and flipped.
   (a) (10%) Find the probability of the event that a head is observed. What is its maximum value?
   (b) (15%) If a head is observed, find the probability that the picked coin is the fair coin. What are its maximum and minimum values?
5. (20%) Consider a sequential experiment in which we repeat independent Bernoulli trials until the occurrence of the first success. The probability of the Bernoulli trial is p. Let X be defined as the number of trials carried out until the occurrence of the first success.
   (a) (10%) Write down the probability mass function (PMF) of X, i.e., $p_X(k)$
   (b) (10%) What is the probability that X is an odd number?
6. (20%) Let X be an uniform random variable taking on values in $\mathcal{S}_X = \{-1, 1, 2\}$.
   (a) (5%) Write down the the probability mass function (PMF) of X, i.e., $p_X(k)$
   (b) (5%) Find the expected value of X.
   (c) (5%) Let Y be defined as $Y = X^2$. Write down the the probability mass function (PMF) of Y, i.e., $p_Y(k)$
   (d) (5%) Find the expected value of Y.