HW12
Solution

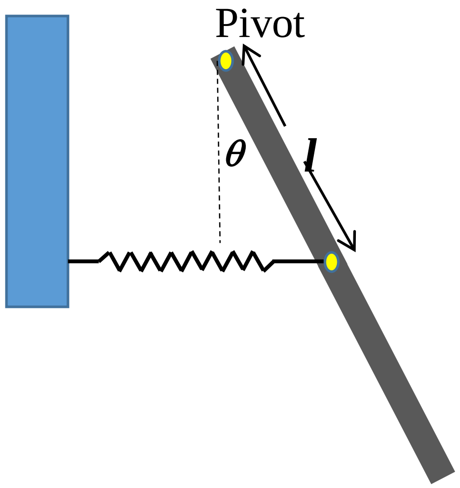
HW12-1: A particle with mass $\boldsymbol{m}$ and velocity $v = dr/dt = \dot{r}$ moves in a one-dimensional potential $U(r) = U_0\left[32\left(\frac{A}{r}\right)^{12} - \left(\frac{A}{r}\right)^{6}\right]$, where $\boldsymbol{U_0}$ and $\boldsymbol{A}$ are positive constants and $\boldsymbol{r} > 0$.

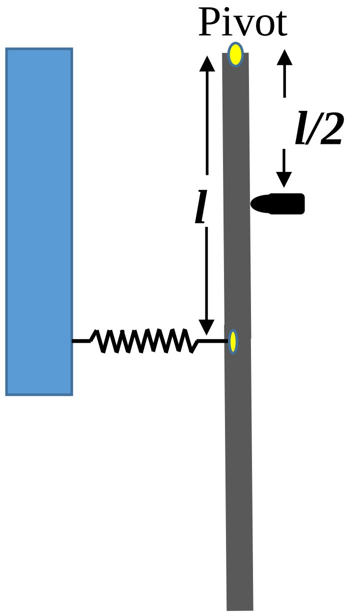
a) There is a static equilibrium point at $\boldsymbol{r = r_0}$ for this potential. Find the equilibrium point $\boldsymbol{r_0}$ and the potential at this point in terms of $\boldsymbol{A}$ and $\boldsymbol{U_0}$.
b) Find the equation of motion for this system.
c) Near the equilibrium point $\boldsymbol{r_0}$, the system can be approximated as a simple harmonic oscillator (SHO). Let $\boldsymbol{r = r_0 + x}$, rewrite the equation of motion in part b) as function of $\boldsymbol{x}$ by using the formula $(r_0 + x)^{-n} \approx r_0^{-n}(1 - nx/r_0 + \ldots)$, if $x \ll r_0$.

d) Find the period of this particle in terms of $\boldsymbol{r_0}$, $\boldsymbol{m}$, and/or $\boldsymbol{U_0}$.

HW12-2: A uniform rod with length 2$\boldsymbol{l}$, mass $\boldsymbol{M}$ hang from one end and the center of rod is attached with a horizontal massless spring with spring constant $\boldsymbol{k}$ (left figure shown below). This spring is initially at the equilibrium position (**θ = 0**). Now the rod is displaced by a small angle **θ** (left figure below) from the vertical position and is then released. Assume the angle **θ** is so small such that **sinθ ~ θ, cosθ ~ 1.**

a) Draw the free body diagram of the rod. Find the equation of the motion of the rod and its period $\boldsymbol{T}$.
b) Now assume the rod is initially at equilibrium position (θ = 0). At $\boldsymbol{t = 0}$, a bullet of mass **4*M*/3** strikes and becomes embedded inside the rod at position $\boldsymbol{l}$/2 from the pivot (right figure shown below). Assume the speed of the bullet is $\boldsymbol{v}$, Find the new period $\boldsymbol{T_N}$ of the motion of the rod, and find new **θ (t)** with initial condition given above.





<u>HW12-3:</u> Problem 19-32 in Giancoli (pp. 523)

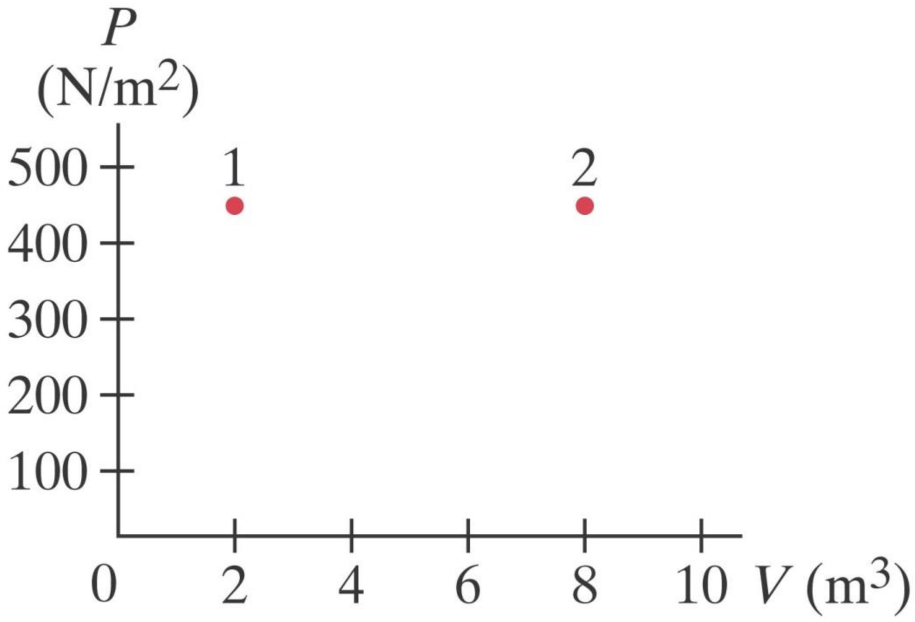
The *PV* diagram in Fig. 19–31 shows two possible states of a system containing 1.55 moles of a monatomic ideal gas. (P_1=P_2=455 N/m^2, V_1=2.00 m^3, V_2=8.00 m^3.)

(*a*) Draw the process which depicts an isobaric expansion from state 1 to state 2, and label this process A.

(*b*) Find the work done by the gas and the change in internal energy of the gas in process A.

(*c*) Draw the two-step process which depicts an isothermal expansion from state 1 to the volume V_2 followed by an isovolumetric increase in temperature to state 2, and label this process B.

(*d*) Find the change in internal energy of the gas for the two-step process B.



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c) Near the equilibrium point $\boldsymbol{r_0}$, the system can be approximated as a simple harmonic oscillator (SHO). Let $\boldsymbol{r = r_0 + x}$, rewrite the equation of motion in part b) as function of $\boldsymbol{x}$ by using the formula $(r_0 + x)^{-n} \approx r_0^{-n}(1 - nx/r_0 + \ldots)$, if $x \ll r_0$.

d) Find the period of this particle in terms of $\boldsymbol{r_0}$, $\boldsymbol{m}$, and/or $\boldsymbol{U_0}$.

Sol:

(a) 求極值➔一次微分=0

$$\frac{dU}{dr} = U_0\left[-12\cdot 32\frac{A^{12}}{r^{13}} - (-6)\frac{A^6}{r^7}\right] = 0 \Rightarrow r = r_0 = 2A$$

$$U(r_0) = U_0\left[2^{-7}\left(\frac{r_0}{r}\right)^{12} - 2^{-6}\left(\frac{r_0}{r}\right)^{6}\right]_{r=r_0} = U_0\left[2^{-7} - 2^{-6}\right] = -\frac{U_0}{128}$$

(b)

$$E_{tot} = KE + PE = \frac{1}{2}m\dot{r}^2 + U(r) = \frac{1}{2}m\dot{r}^2 + U_0\left[2^{-7}\left(\frac{r_0}{r}\right)^{12} - 2^{-6}\left(\frac{r_0}{r}\right)^{6}\right]$$

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$$\frac{dE_{tot}}{dt} = 0 \Rightarrow 0 = m\dot{r}\frac{d\dot{r}}{dt} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right]\dot{r} = \dot{r}\left(m\ddot{r} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right]\right)$$

Equation of motion: $m\ddot{r} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right] = 0$

Or from $F = ma$ → $F = m\ddot{r} = -\frac{dU}{dr} = -\frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right]$

➔ $m\ddot{r} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right] = 0$

(c) Equation of motion: $m\ddot{r} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right] = 0$

$$x \equiv r - r_0 \rightarrow \begin{array}{c} r = x + r_0 \\ \dot{x} = \dot{r} \end{array}$$

$$\left(\frac{r_0}{r}\right)^7 = \frac{1}{(1+x/r_0)^7} \simeq 1 - 7\frac{x}{r_0} \qquad \left(\frac{r_0}{r}\right)^{13} = \frac{1}{(1+x/r_0)^{13}} \simeq 1 - 13\frac{x}{r_0}$$

Equation of motion becomes: $m\ddot{x} + \frac{3U_0}{32r_0}\left[1 - \frac{7x}{r_0} - 1 + \frac{13x}{r_0}\right] = m\ddot{x} + \frac{9U_0}{16r_0^2}x = 0$

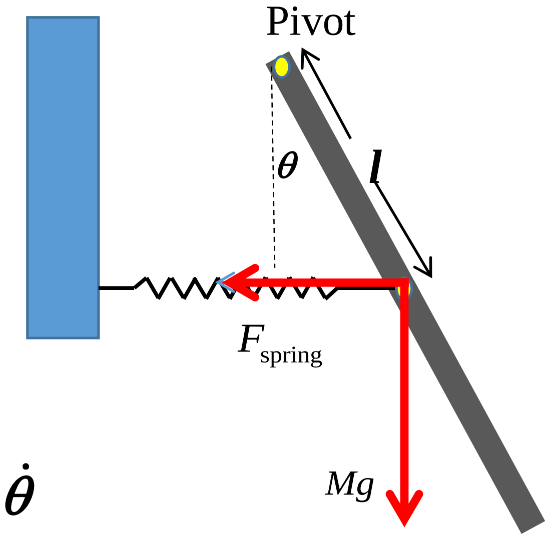
(d)

$$\omega^2 = \frac{9U_0}{16mr_0^2} = \frac{9U_0}{64mA^2} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\left(\frac{16mr_0^2}{9U_0}\right)^{1/2}$$

HW12-2: A uniform rod with length $2l$, mass M hang from one end and the center of rod is attached with a horizontal massless spring with spring constant k (left figure shown below). This spring is initially at the equilibrium position ($\theta = 0$). Now the rod is displaced by a small angle θ (left figure below) from the vertical position and is then released. Assume the angle θ is so small such that $\sin\theta \sim \theta$, $\cos\theta \sim 1$.

a) Draw the free body diagram of the rod. Find the equation of the motion of the rod and its period T.

b) Now assume the rod is initially at equilibrium position ($\theta = 0$). At $t = 0$, a bullet of mass $4M/3$ strikes and becomes embedded inside the rod at position $l/2$ from the pivot (right figure shown below). Assume the speed of the bullet is v, Find the new period T_N of the motion of the rod, and find new $\theta(t)$ with initial condition given above.



Sol:

a) **solving by the conservation of the mechanical energy**

$$KE = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\frac{M(2\ell)^2}{3}\dot{\theta}^2$$

$$PE = \frac{1}{2}k(\Delta\ell)^2 + Mgy_{CM}$$

$$\Delta\ell = x_{CM} = \ell\sin\theta$$

$$y_{CM} = \ell(1-\cos\theta)$$

$$E_{tot} = \frac{1}{2}\frac{M(2\ell)^2}{3}\dot{\theta}^2 + \frac{1}{2}k(\Delta\ell)^2 + Mgy_{CM}$$

$$= \frac{2M\ell^2}{3}\dot{\theta}^2 + \frac{1}{2}k\ell^2\sin^2\theta + Mg\ell(1-\cos\theta)$$

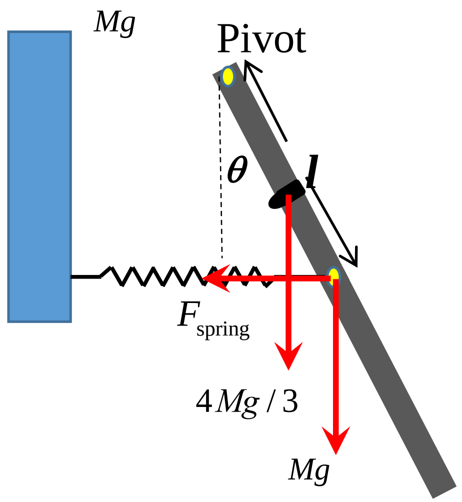
$$\frac{dE_{tot}}{dt} = 0 = \frac{4M\ell^2}{3}\dot{\theta}\ddot{\theta} + k\ell^2\sin\theta\cos\theta\dot{\theta} + (Mg\ell\sin\theta)\dot{\theta}$$

$$\Rightarrow \ddot{\theta} + \frac{(k\ell + Mg)}{\frac{4M\ell}{3}}\theta = 0 \qquad \theta << 1 \Rightarrow \quad \cos\theta \approx 1 \quad , \quad \sin\theta \approx \theta \quad ,$$

$$\Rightarrow \omega^2 = \frac{3(k\ell + Mg)}{4M\ell} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4M\ell}{3(Mg + k\ell)}}$$

A bullet with mass **4*M*/3** embedded at ***l*/2** the KE and PE become

(b)



$$KE = \frac{1}{2} I\dot{\theta}^2 = \frac{1}{2}\left(\frac{M(2\ell)^2}{3} + \frac{4}{3}M\left(\frac{\ell}{2}\right)^2\right)\dot{\theta}^2$$

$$PE = \frac{1}{2}k(\ell\sin\theta)^2 + Mg\ell(1-\cos\theta) + \frac{4}{3}Mg\frac{\ell}{2}(1-\cos\theta)$$

$$E_{tot} = \frac{5M\ell^2}{6}\dot{\theta}^2 + \frac{1}{2}k\ell^2\sin^2\theta + \frac{5}{3}Mg\ell(1-\cos\theta)$$

$$\frac{dE_{tot}}{dt} = 0 = \frac{5M\ell^2}{3}\dot{\theta}\ddot{\theta} + k\ell^2\sin\theta\cos\theta\dot{\theta} + \left(\frac{5}{3}Mg\ell\sin\theta\right)\dot{\theta}$$

$$\Rightarrow \ddot{\theta} + \frac{(k\ell + \frac{5}{3}Mg)}{\frac{5M\ell}{3}}\theta = 0 \qquad \theta \ll 1 \Rightarrow \quad \cos\theta \approx 1 \quad , \quad \sin\theta \approx \theta \quad ,$$

$$\Rightarrow \omega^2 = \frac{3k\ell + 5Mg}{5M\ell} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5M\ell}{5Mg + 3k\ell}}$$

Initial condition: $\theta(0)=0, \quad \dot{\theta}(0)=\dot{\theta}_0$

Set $\theta(t)=A\cos(\omega t+\phi)$ $\Rightarrow \theta(t)=A\cos\left(\sqrt{\dfrac{3k\ell+5Mg}{5M\ell}}\cdot t+\phi\right)$ $\because \theta(0)=0\Rightarrow \phi=\dfrac{\pi}{2}$

相對於轉軸，合力舉為零 ⟶ 角動量守恆 $\vec{L}_i=\vec{L}_f$

$$L_i=-\frac{\ell}{2}\frac{4M}{3}v,\quad L_f=I\dot{\theta}_0=\frac{5}{3}M\ell^2\dot{\theta}_0, \quad \Rightarrow \dot{\theta}_0=-\frac{2v}{5\ell}$$

$$\dot{\theta}(0)=\frac{d\theta}{dt}(t=0)=-A\omega\sin(\phi)=-A\sqrt{\frac{3k\ell+5Mg}{5M\ell}}\sin(\phi)=\dot{\theta}_0=-\frac{2v}{5l}$$

$$A=\frac{2v}{5l}\sqrt{\frac{5M\ell}{3k\ell+5Mg}}$$

$$\Rightarrow \theta(t)=\frac{2v}{5l}\sqrt{\frac{5M\ell}{3k\ell+5Mg}}\cos\left(\sqrt{\frac{3k\ell+5Mg}{5M\ell}}t+\frac{\pi}{2}\right)$$

$$\text{or}\quad \theta(t)=\frac{2v}{5l}\sqrt{\frac{5M\ell}{3k\ell+5Mg}}\sin\left(\sqrt{\frac{3k\ell+5Mg}{5M\ell}}t\right)$$

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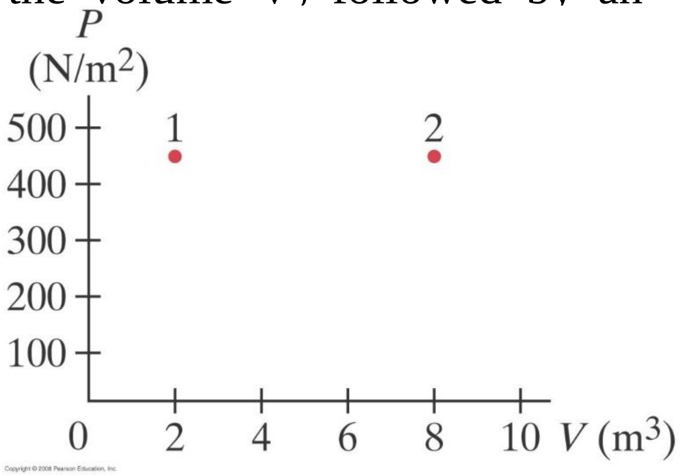


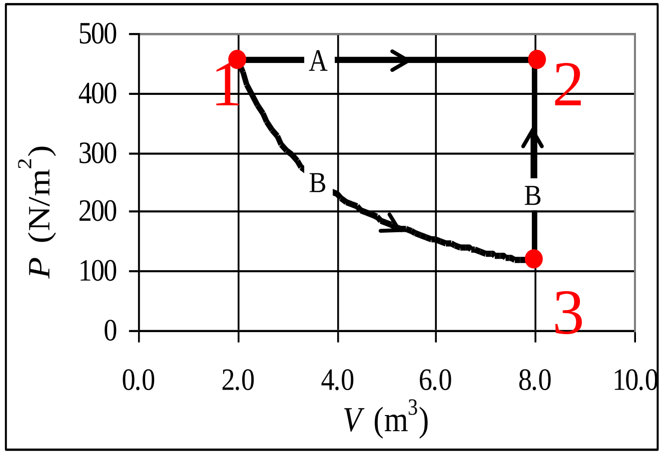
Solution:

(a) isobaric : no change in pressure (1→2)

(c) isothermal : no change in temperature $\Rightarrow$ $\boldsymbol{PV} = \textbf{constant}$ (1→3)

isovolumetric : no change in volume (3→2)

(isochoric)



(b)

$$W=\int_i^f PdV=P\int_i^f dV=P(V_f-V_i)=P\Delta V$$

$$W=P(V_2-V_1)=(455\,\mathrm{N/m^2})(8m^3-2m^3)=2730J$$

$$\Delta E_{\mathrm{int}}=n\frac{f}{2}R\Delta T \quad \text{for ideal monatomic gas : } f=3$$

$$=\frac{3}{2}(nRT_2-nRT_1)=\frac{3}{2}(P_2V_2-P_1V_1)=\frac{3}{2}P(V_2-V_1)=\frac{3}{2}W$$

$$=4.10\times10^3\,J$$

(d)

$$\Delta E_{\mathrm{int}}=\frac{3}{2}nR\Delta T=\frac{3}{2}(nRT_2-nRT_1)=\frac{3}{2}(P_2V_2-P_1V_1)$$

$$=\frac{3}{2}P(V_2-V_1)=\frac{3}{2}W=4.10\times10^3\,J$$