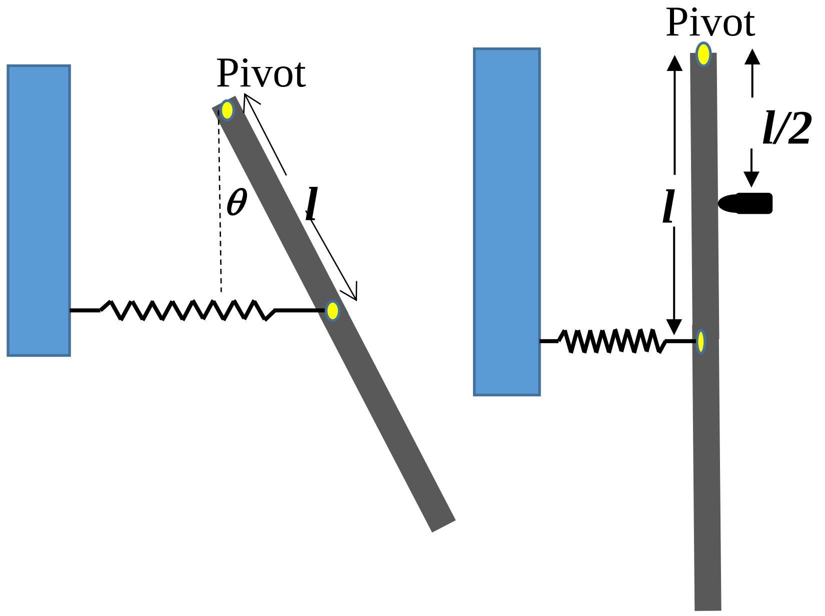
HW12-1: A particle with mass $\boldsymbol{m}$ and velocity $v = dr/dt = \dot{r}$ moves in a one-dimensional potential $U(r) = U_0\left[32\left(\frac{A}{r}\right)^{12} - \left(\frac{A}{r}\right)^{6}\right]$, where $\boldsymbol{U_0}$ and $\boldsymbol{A}$ are positive constants and $\boldsymbol{r} > 0$.

a) There is a static equilibrium point at $\boldsymbol{r = r_0}$ for this potential. Find the equilibrium point $\boldsymbol{r_0}$ and the potential at this point in terms of $\boldsymbol{A}$ and $\boldsymbol{U_0}$.
b) Find the equation of motion for this system.
c) Near the equilibrium point $\boldsymbol{r_0}$, the system can be approximated as a simple harmonic oscillator (SHO). Let $\boldsymbol{r = r_0 + x}$, rewrite the equation of motion in part b) as function of $\boldsymbol{x}$ by using the formula $(r_0 + x)^{-n} \approx r_0^{-n}(1 - nx/r_0 + \ldots)$, if $x << r_0$.

d) Find the period of this particle in terms of $\boldsymbol{r_0}$, $\boldsymbol{m}$, and/or $\boldsymbol{U_0}$.

HW12-2: A uniform rod with length 2$\boldsymbol{l}$, mass $\boldsymbol{M}$ hang from one end and the center of rod is attached with a horizontal massless spring with spring constant $\boldsymbol{k}$ (left figure shown below). This spring is initially at the equilibrium position (**θ = 0**). Now the rod is displaced by a small angle **θ** (left figure below) from the vertical position and is then released. Assume the angle **θ** is so small such that **sinθ ~ θ, cosθ ~ 1.**



a) Draw the free body diagram of the rod. Find the equation of the motion of the rod and its period $\boldsymbol{T}$.
b) Now assume the rod is initially at equilibrium position (θ = 0). At $\boldsymbol{t}$ **= 0**, a bullet of mass **4*M*/3** strikes and becomes embedded inside the rod at position $\boldsymbol{l}$/2 from the pivot (right figure shown below). Assume the speed of the bullet is $\boldsymbol{v}$, Find the new period $\boldsymbol{T_N}$ of the motion of the rod, and find new **θ (t)** with initial condition given above.

HW12-3: Problem 19-32 in Giancoli (pp. 607) ; Problem 19-32 in Giancoli (pp. 523)

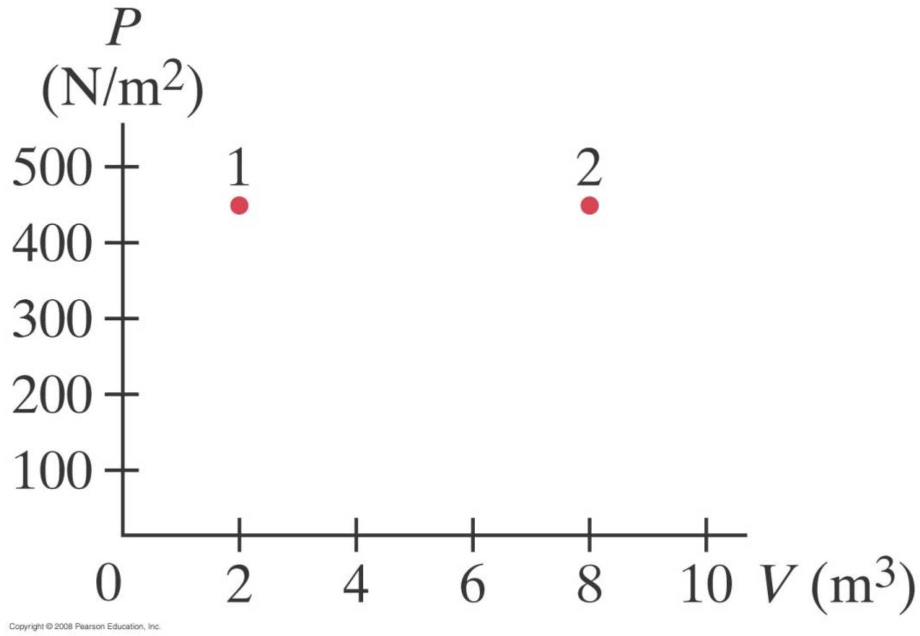
The *PV* diagram in Fig. 19–31 shows two possible states of a system containing 1.55 moles of a monatomic ideal gas. ($P_1=P_2=455$ N/m^2, $V_1=2.00$ m^3, $V_2=8.00$ m^3.)

(*a*) Draw the process which depicts an isobaric expansion from state 1 to state 2, and label this process A.

(*b*) Find the work done by the gas and the change in internal energy of the gas in process A.

(*c*) Draw the two-step process which depicts an isothermal expansion from state 1 to the volume V_2 followed by an isovolumetric increase in temperature to state 2, and label this process B.

(*d*) Find the change in internal energy of the gas for the two-step process B.



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d) Find the period of this particle in terms of $\boldsymbol{r_0}$, $\boldsymbol{m}$, and/or $\boldsymbol{U_0}$.

Sol:

(a) 求極值➔一次微分=0

$$\frac{dU}{dr} = U_0\left[-12\cdot 32\frac{A^{12}}{r^{13}} - (-6)\frac{A^6}{r^7}\right] = 0 \Rightarrow r = r_0 = 2A$$

$$U(r_0) = U_0\left[2^{-7}\left(\frac{r_0}{r}\right)^{12} - 2^{-6}\left(\frac{r_0}{r}\right)^{6}\right]_{r=r_0} = U_0\left[2^{-7} - 2^{-6}\right] = -\frac{U_0}{128}$$

(b)

$$E_{tot} = KE + PE = \frac{1}{2}m\dot{r}^2 + U(r) = \frac{1}{2}m\dot{r}^2 + U_0\left[2^{-7}\left(\frac{r_0}{r}\right)^{12} - 2^{-6}\left(\frac{r_0}{r}\right)^{6}\right]$$

只有保守力做功➔力學能守恆

$$\frac{dE_{tot}}{dt} = 0 \Rightarrow 0 = m\dot{r}\frac{d\dot{r}}{dt} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right]\dot{r} = \dot{r}\left(m\ddot{r} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right]\right)$$

Equation of motion: $m\ddot{r} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right] = 0$

Or from $F = ma$ → $F = m\ddot{r} = -\frac{dU}{dr} = -\frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right]$

➔ $m\ddot{r} + \frac{3U_0}{32r_0}\left[\left(\frac{r_0}{r}\right)^7 - \left(\frac{r_0}{r}\right)^{13}\right] = 0$

(c) Equation of motion: $m\ddot{r}+\dfrac{3U_0}{32r_0}\left[\left(\dfrac{r_0}{r}\right)^7-\left(\dfrac{r_0}{r}\right)^{13}\right]=0$

$$x\equiv r-r_0 \rightarrow \begin{array}{c} r=x+r_0 \\ \dot{x}=\dot{r} \end{array}$$

$$\left(\frac{r_0}{r}\right)^7=\frac{1}{(1+x/r_0)^7}\simeq 1-7\frac{x}{r_0} \qquad \left(\frac{r_0}{r}\right)^{13}=\frac{1}{(1+x/r_0)^{13}}\simeq 1-13\frac{x}{r_0}$$

Equation of motion becomes: $m\ddot{x}+\dfrac{3U_0}{32r_0}\left[1-\dfrac{7x}{r_0}-1+\dfrac{13x}{r_0}\right]=m\ddot{x}+\dfrac{9U_0}{16r_0^2}x=0$

(d)

$$\omega^2=\frac{9U_0}{16mr_0^2}=\frac{9U_0}{64mA^2}\Rightarrow T=\frac{2\pi}{\omega}=2\pi\left(\frac{16mr_0^2}{9U_0}\right)^{1/2}$$

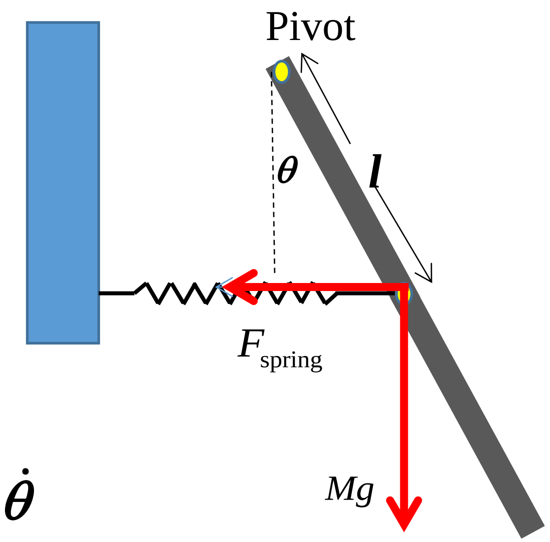
HW12-2: A uniform rod with length 2***l***, mass ***M*** hang from one end and the center of rod is attached with a horizontal massless spring with spring constant ***k*** (left figure shown below). This spring is initially at the equilibrium position (**θ = 0**). Now the rod is displaced by a small angle **θ** (left figure below) from the vertical position and is then released. Assume the angle **θ** is so small such that **sinθ ~ θ, cosθ ~ 1.**

a) Draw the free body diagram of the rod. Find the equation of the motion of the rod and its period ***T***.

b) Now assume the rod is initially at equilibrium position (θ = 0). At ***t*** **= 0**, a bullet of mass **4*M*/3** strikes and becomes embedded inside the rod at position ***l***/2 from the pivot (right figure shown below). Assume the speed of the bullet is ***v***, Find the new period T_N of the motion of the rod, and find new **θ (t)** with initial condition given above.

Sol:

a) **solving by the conservation of the mechanical energy**



$$KE = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\frac{M(2\ell)^2}{3}\dot{\theta}^2$$

$$PE = \frac{1}{2}k(\Delta\ell)^2 + Mgy_{CM}$$

$$\Delta\ell = x_{CM} = \ell\sin\theta$$

$$y_{CM} = \ell(1-\cos\theta)$$

$$E_{tot} = \frac{1}{2}\frac{M(2\ell)^2}{3}\dot{\theta}^2 + \frac{1}{2}k(\Delta\ell)^2 + Mgy_{CM}$$

$$= \frac{2M\ell^2}{3}\dot{\theta}^2 + \frac{1}{2}k\ell^2\sin^2\theta + Mg\ell(1-\cos\theta)$$

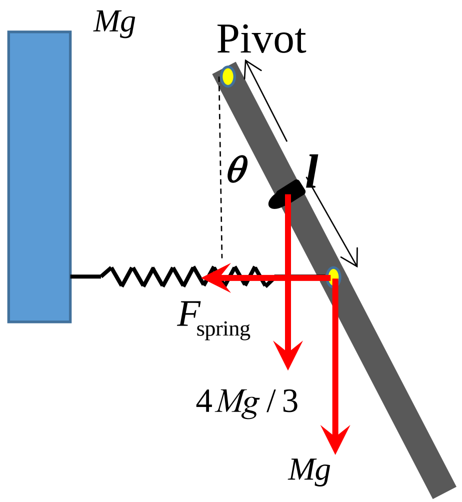
$$\frac{dE_{tot}}{dt} = 0 = \frac{4M\ell^2}{3}\dot{\theta}\ddot{\theta} + k\ell^2\sin\theta\cos\theta\dot{\theta} + (Mg\ell\sin\theta)\dot{\theta}$$

$$\Rightarrow \ddot{\theta} + \frac{(k\ell + Mg)}{\frac{4M\ell}{3}}\theta = 0 \qquad \theta << 1 \Rightarrow \quad \cos\theta \approx 1 \quad , \quad \sin\theta \approx \theta \quad ,$$

$$\Rightarrow \omega^2 = \frac{3(k\ell + Mg)}{4M\ell} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4M\ell}{3(Mg + k\ell)}}$$

A bullet with mass **4*M*/3** embedded at ***l*/2** the KE and PE become

(b)



$$KE = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2}\left(\frac{M(2\ell)^2}{3} + \frac{4}{3} M \left(\frac{\ell}{2}\right)^2 \right)\dot{\theta}^2$$

$$PE = \frac{1}{2} k (\ell \sin\theta)^2 + Mg\ell(1-\cos\theta) + \frac{4}{3} Mg \frac{\ell}{2}(1-\cos\theta)$$

$$E_{tot} = \frac{5M\ell^2}{6}\dot{\theta}^2 + \frac{1}{2} k\ell^2 \sin^2\theta + \frac{5}{3} Mg\ell(1-\cos\theta)$$

$$\frac{dE_{tot}}{dt} = 0 = \frac{5M\ell^2}{3}\dot{\theta}\ddot{\theta} + k\ell^2 \sin\theta\cos\theta\dot{\theta} + \left(\frac{5}{3} Mg\ell\sin\theta\right)\dot{\theta}$$

$$\Rightarrow \ddot{\theta} + \frac{(k\ell + \frac{5}{3}Mg)}{\frac{5M\ell}{3}}\theta = 0 \qquad \theta \ll 1 \Rightarrow \quad \cos\theta \approx 1 \quad , \quad \sin\theta \approx \theta \quad ,$$

$$\Rightarrow \omega^2 = \frac{3k\ell + 5Mg}{5M\ell} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5M\ell}{5Mg + 3k\ell}}$$

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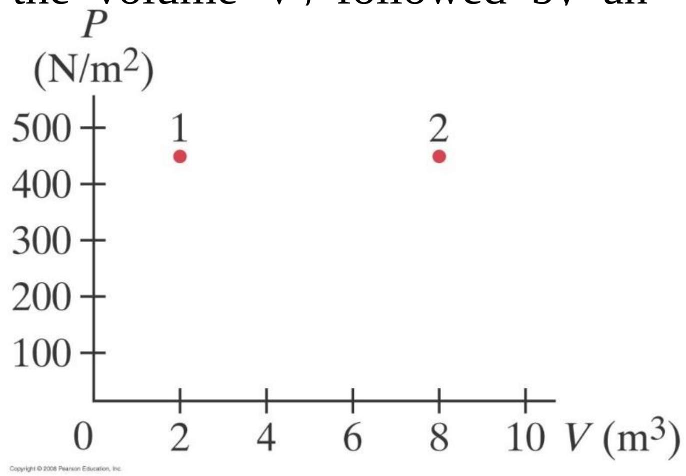


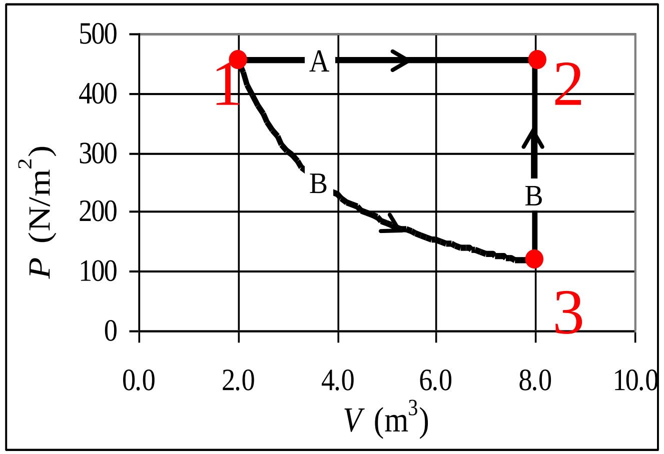
Solution:

(a) isobaric : no change in pressure (1→2)

(c) isothermal : no change in temperature $\Rightarrow$ $\boldsymbol{PV = \text{constant}}$ (1→3)

isovolumetric : no change in volume (3→2)

(isochoric)



(b)

$$W = \int_i^f P dV = P\int_i^f dV = P(V_f - V_i) = P\Delta V$$

$$W = P(V_2 - V_1) = (455\,\mathrm{N/m^2})(8m^3 - 2m^3) = 2730J$$

$$\Delta E_{\mathrm{int}} = n\frac{f}{2}R\Delta T \quad \text{for ideal monatomic gas : } f = 3$$

$$= \frac{3}{2}(nRT_2 - nRT_1) = \frac{3}{2}(P_2V_2 - P_1V_1) = \frac{3}{2}P(V_2 - V_1) = \frac{3}{2}W$$

$$= 4.10\times 10^3 J$$

(d)

$$\Delta E_{\mathrm{int}} = \frac{3}{2}nR\Delta T = \frac{3}{2}(nRT_2 - nRT_1) = \frac{3}{2}(P_2V_2 - P_1V_1)$$

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