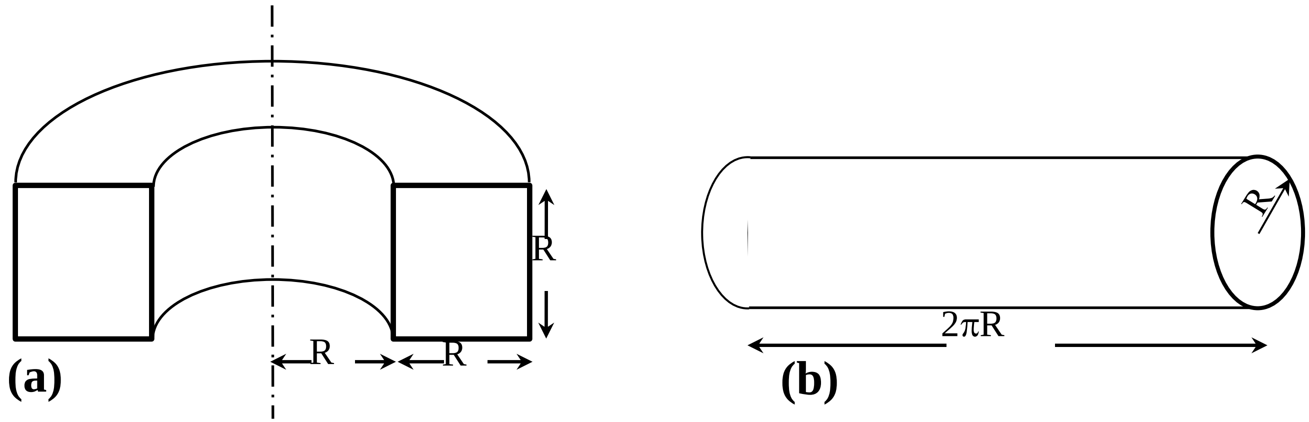
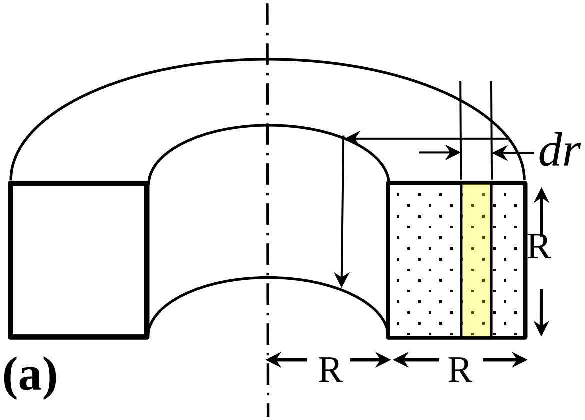
GP(II).spring.2020
HW10

HW10-1 :

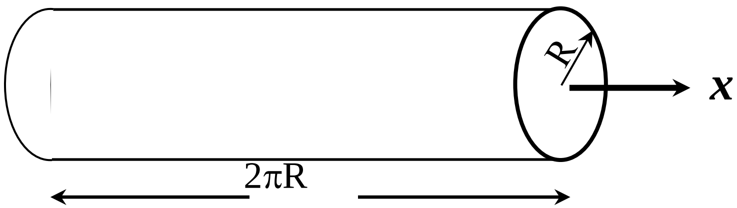
A toroidal inductor has a square cross section as show in Figure (a) on the left. There number of windings is N and its inductance is L_T. And a solenoidal inductor has the with radius R and length $2\pi R$, as shown in Figure (b), with the same number of windings and its inductance is L_S. Let $x=L_T/L_S$. Determine the value of x.



HW10-1 : A toroidal inductor has a square cross section as show in Figure (a) on the left. There number of windings is N and its inductance is L_T. And a solenoidal inductor has the with radius R and length $2\pi R$, as shown in Figure (b), with the same number of windings and its inductance is L_S. Let $x=L_T/L_S$. Determine the value of x.



(a)



(b)

(a) $$L_T = \frac{\Phi_{B,T}}{I}, \quad \Phi_{B,T} = N\int \vec{B}_T \cdot d\vec{A}$$

With Ampere's law, we obtain $\vec{B}_T = \dfrac{\mu_0 NI}{2\pi r}\hat{\phi}$

$$\Rightarrow \Phi_{B,T} = N \cdot \int_R^{2R} \frac{\mu_0 NI}{2\pi r} R dr = \frac{\mu_0 N^2 IR}{2\pi}\int_R^{2R}\frac{dr}{r}$$

$$= \frac{\mu_0 N^2 IR}{2\pi}\ln 2$$

$$\Rightarrow L_T = \frac{\Phi_{B,T}}{I} = \frac{\mu_0 N^2 R}{2\pi}\ln 2$$

(b) $$L_S = \frac{\Phi_{B,S}}{I}, \quad \Phi_{B,S} = N \cdot \vec{B}_S \cdot \vec{A}$$

With Ampere's law, we obtain $\vec{B}_S = \dfrac{\mu_0 NI}{2\pi R}\hat{x}$

$$\Rightarrow \Phi_{B,S} = N\frac{\mu_0 NI}{2\pi R}\pi R^2 = \frac{\mu_0 N^2 IR}{2}$$

$$\Rightarrow L_S = \frac{\Phi_{S,T}}{I} = \frac{\mu_0 N^2 R}{2}$$

$$\Rightarrow x = \frac{L_T}{L_S} = \left(\frac{\mu_0 N^2 R}{2\pi}\ln 2\right)\Big/\left(\frac{\mu_0 N^2 R}{2}\right)$$

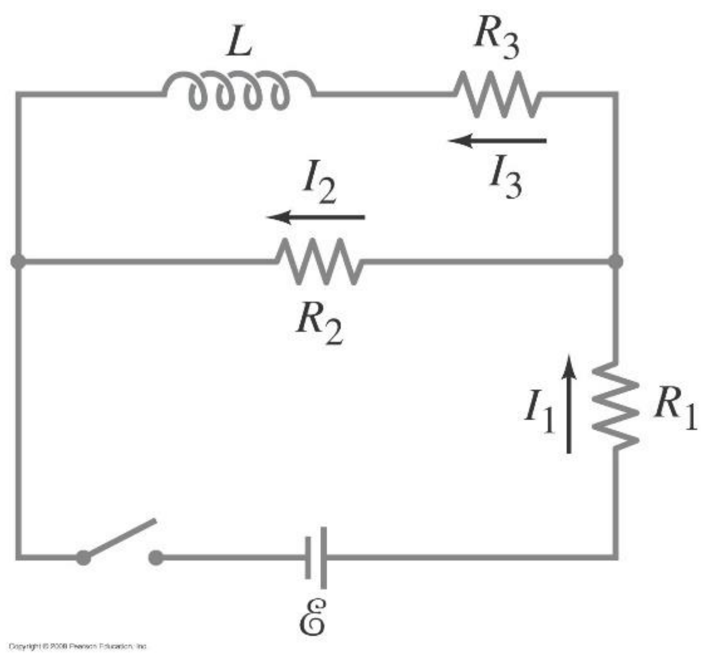
$$= \frac{\ln 2}{\pi}$$

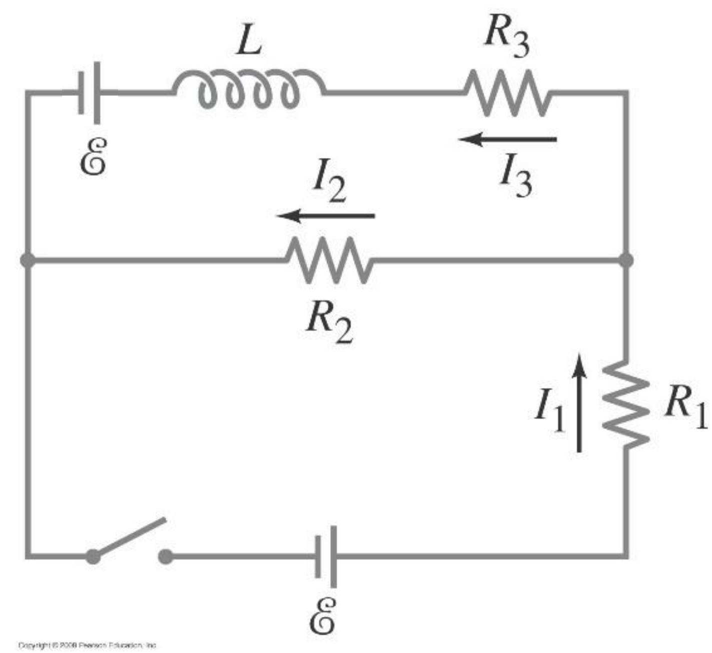
HW10-2 :

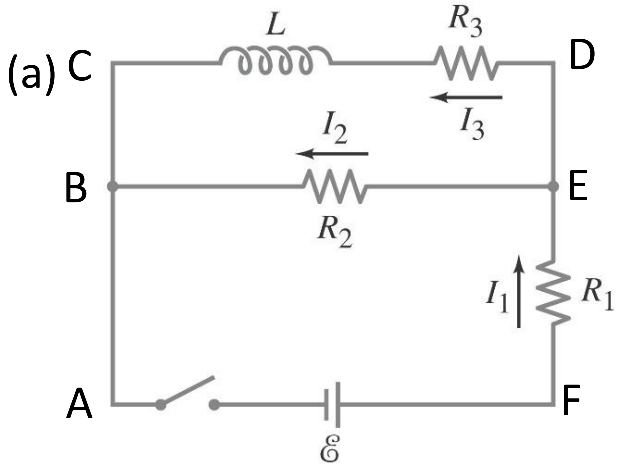
In the circuit shown on the below,

(a) Consider the circuit shown below on left, at the moment t =0, the switch is closed, determine the current $\boldsymbol{I_2}$ as function of time , the magnitude of ($\boldsymbol{I_1}$, $\boldsymbol{I_2}$, $\boldsymbol{I_3}$) right after the switch was closed, and their magnitude a long time after the switch is closed (The direction of the currents is presumed to be the direction indicated by the arrows).

(b) Now consider the circuit shown below on right, where a second battery is now inserted in the circuit in series with the inductor, and again, at the moment t =0, the switch is closed, determine the magnitude of ($\boldsymbol{I_1}$, $\boldsymbol{I_2}$, $\boldsymbol{I_3}$) right after the switch was closed, and their magnitude a long time after the switch is closed(The direction of the currents is presumed to be the direction indicated by the arrows).







Right after the switch being closed, the inductor behaves like an infinite resistor. Therefore

$$I_3 = 0,\ \ I_2 = I_1 = \frac{\varepsilon}{R_1 + R_2}$$

After the switch being closed for a long time, the inductor behaves like an conductor (zero resistance). Therefore

$$I_1 = \frac{\varepsilon}{R_1 + R_2 \parallel R_3} = \frac{\varepsilon}{R_1 + \dfrac{R_2 R_3}{R_2 + R_3}}$$

$$= \frac{\varepsilon (R_2 + R_3)}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$I_2 = \frac{\varepsilon - R_1 I_1}{R_2}$$

$$= \frac{\varepsilon - R_1 \dfrac{\varepsilon (R_2 + R_3)}{R_1 R_2 + R_1 R_3 + R_2 R_3}}{R_2}$$

$$= \frac{\varepsilon R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$I_3 = \frac{\varepsilon - R_1 I_1}{R_3}$$

$$= \frac{\varepsilon - R_1 \dfrac{\varepsilon (R_2 + R_3)}{R_1 R_2 + R_1 R_3 + R_2 R_3}}{R_3}$$

$$= \frac{\varepsilon R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

For the time dependence of I2, we need to solve the problem with Kirchhoff's circuit theorem. $I_1 = I_2 + I_3$ (1)

For Loop ABEF, $\varepsilon - I_1 R_1 - I_2 R_2 = 0$ (2)

For Loop ACDF, $\varepsilon - I_1 R_1 - I_3 R_3 - L\dfrac{dI_3}{dt} = 0$ (3)

For Loop BCDE, $I_2 R_2 - I_3 R_3 - L\dfrac{dI_3}{dt} = 0$ (4)

From (1) and (2), we obtain

$$I_2 = \frac{\varepsilon}{R_1 + R_2} - \frac{R_1}{R_1 + R_2} I_3 \quad (5)$$

By inserting (5) into (4) we obtain

$$\left(\frac{\varepsilon}{R_1 + R_2} - \frac{R_1}{R_1 + R_2} I_3\right) R_2 - I_3 R_3 - L\frac{dI_3}{dt} = 0$$

$$\Rightarrow L\frac{dI_3}{dt} + \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 + R_2} I_3 = \frac{\varepsilon R_2}{R_1 + R_2}$$

$$\Rightarrow \frac{dI_3}{dt} = -\frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{(R_1 + R_2)L} \cdot$$

$$\left(I_3 - \frac{\varepsilon R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}\right)$$

$$\Rightarrow \frac{dI_3}{I_3 - \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3}} = -\frac{R_1R_2 + R_1R_3 + R_2R_3}{(R_1 + R_2)L} dt$$

$$\Rightarrow \int_{I_3(0)}^{I_3(t)} \frac{dI_3}{I_3 - \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3}} = -\int_0^t \frac{R_1R_2 + R_1R_3 + R_2R_3}{(R_1 + R_2)L} dt$$

$$\Rightarrow \ln \frac{I_3(t) - \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3}}{I_3(0) - \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3}} = -\frac{R_1R_2 + R_1R_3 + R_2R_3}{(R_1 + R_2)L} t$$

$$\Rightarrow I_3(t) - \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3} = (I_3(0) - \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3}) e^{-\frac{R_1R_2 + R_1R_3 + R_2R_3}{(R_1 + R_2)L} t}$$

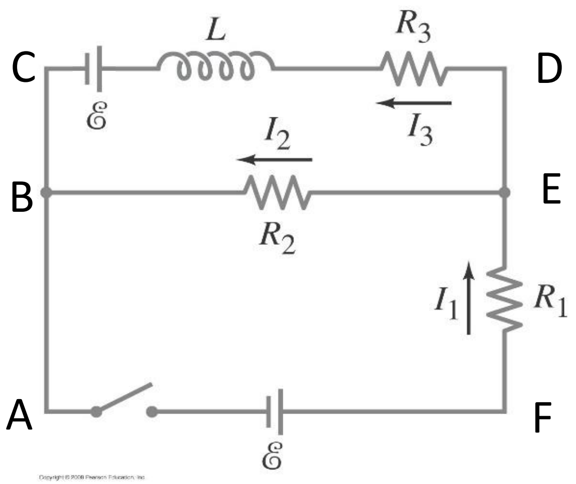
For I_3(0)=0, we get

$$I_3(t) = \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3} (1 - e^{-\frac{R_1R_2 + R_1R_3 + R_2R_3}{(R_1 + R_2)L} t}) \quad (6)$$

From (5) and (6), we get

$$I_2 = \frac{\varepsilon}{R_1 + R_2} - \frac{R_1}{R_1 + R_2} \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3} (1 - e^{-\frac{R_1R_2 + R_1R_3 + R_2R_3}{(R_1 + R_2)L} t})$$

(b)



When the switch is open, the inductor behaves as a conductor (zero resistance)

$$I_3 = \frac{-\varepsilon}{R_3 + R_2} = -I_2 \quad (1)$$

Right after the switch being closed (t=0), the inductor $\boldsymbol{L}$ reacts to the change according to the Faraday and Lentz law such that the current $\boldsymbol{I_3}$ through itself does not change immediately. Therefore $\boldsymbol{I_3}$ remains the same as before the switch being closed, i.e.

$$I_3 = \frac{-\varepsilon}{R_3 + R_2} \quad (2)$$

The voltage across CD may change to keep (2) true momentary.

For Loop ABEF,

$$\varepsilon - I_1R_1 - I_2R_2 = 0 \quad (3)$$

At the nodal point E,

$$I_1 = I_2 + I_3 \Rightarrow I_1 = I_2 + \frac{-\varepsilon}{R_3 + R_2} \quad (4)$$

Solve (3) and (4) for I1 and I2, we get

$$\Rightarrow I_1 = \frac{\varepsilon R_3}{(R_1 + R_2)(R_3 + R_2)}$$

$$\Rightarrow I_2 = \frac{(R_1 + R_2 + R_3)\varepsilon}{(R_1 + R_2)(R_3 + R_2)}$$

Note that $\boldsymbol{V_{CD}}$ will change from

$$R_2 I_{2,\text{Before Switch being closed}} = \frac{\varepsilon R_2}{R_3 + R_2}$$

to

$$R_2 I_{2,\text{Right After Switch closed}} = \frac{\varepsilon R_2(R_1 + R_2 + R_3)}{(R_1 + R_2)(R_3 + R_2)}$$

$\boldsymbol{V_L}$ is changed to maintain the same I_3.

Long after the switch being closed (t→0), the inductor $\boldsymbol{L}$ becomes a conductor (zero resistance),i.e. we can obtain $\boldsymbol{I_1, I_2, I_3}$ with Kirchhoff's theorem.

$$I_1 = I_2 + I_3 \quad (1)$$

For Loop ABEF, $\varepsilon - I_1R_1 - I_2R_2 = 0 \quad (2)$

For Loop ACDF, $\varepsilon - I_1R_1 - I_3R_3 - \varepsilon = 0 \quad (3)$

From (3), we obtain

$$I_1 = -\frac{R_3}{R_1} I_3 \quad (4)$$

From (1)&(4), we obtain

$$-\frac{R_3}{R_1} I_3 = I_2 + I_3 \;\Rightarrow I_2 = -(\frac{R_1 + R_3}{R_1}) I_3 \quad (5)$$

From (2)&(5), we obtain

$$\varepsilon - (-\frac{R_3}{R_1} I_3) R_1 - (-(\frac{R_1 + R_3}{R_1}) I_3) R_2 = 0$$

$$\Rightarrow \varepsilon + R_3 I_3 + (\frac{R_1R_2 + R_2R_3}{R_1}) I_3 = 0$$

$$\Rightarrow I_3 = -\frac{\varepsilon R_1}{R_1R_2 + R_2R_3 + R_1R_3} \quad (6)$$

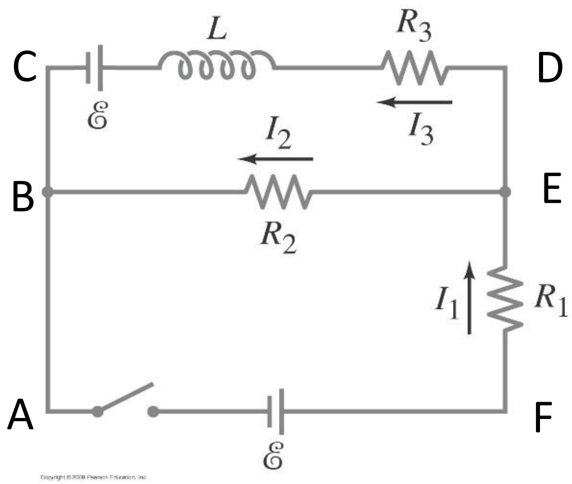
From (4)&(6), we obtain

$$I_1 = -\frac{R_3}{R_1} I_3 = \frac{\varepsilon R_3}{R_1R_2 + R_2R_3 + R_1R_3}$$

From (5)&(6), we obtain

$$I_2 = -(\frac{R_1 + R_3}{R_1}) I_3 = \frac{\varepsilon (R_1 + R_3)}{R_1R_2 + R_2R_3 + R_1R_3}$$

Support information: solution of $\boldsymbol{I_3}$ as a of time,
(This is not required by the assignment)



For the time dependence of $\boldsymbol{I_2}$, we need to solve the problem with Kirchhoff's circuit theorem. $I_1 = I_2 + I_3$ (1)

For Loop ABEF, $\varepsilon - I_1R_1 - I_2R_2 = 0$ (2)

For Loop ACDF, $\varepsilon - I_1R_1 - I_3R_3 - L\dfrac{dI_3}{dt} - \varepsilon = 0$ (3)

For Loop BCDE, $I_2R_2 - I_3R_3 - L\dfrac{dI_3}{dt} - \varepsilon = 0$(4)

From (1) and (2), we obtain

$$I_1 = \frac{\varepsilon}{R_1 + R_2} + \frac{R_2}{R_1 + R_2} I_3 \quad (5)$$

$$I_2 = \frac{\varepsilon}{R_1 + R_2} - \frac{R_1}{R_1 + R_2} I_3 \quad (6)$$

Inset (5) into (3), we obtain

$$-\left(\frac{\varepsilon}{R_1 + R_2} + \frac{R_2}{R_1 + R_2} I_3\right)R_1 - I_3R_3 - L\frac{dI_3}{dt} = 0$$

$$\Rightarrow \frac{R_1\varepsilon}{R_1 + R_2} + \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1 + R_2} I_3 + L\frac{dI_3}{dt} = 0$$

$$\Rightarrow \frac{dI_3}{dt} = -\frac{R_1R_2 + R_1R_3 + R_2R_3}{L(R_1 + R_2)}\left(I_3 + \frac{R_1\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}\right)$$

$$\Rightarrow \frac{dI_3}{I_3 + \dfrac{R_1\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}} = -\frac{R_1R_2 + R_1R_3 + R_2R_3}{L(R_1 + R_2)}dt$$

$$\Rightarrow \int_{I_3(0)}^{I_3(t)} \frac{dI_3}{I_3 + \dfrac{R_1\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}} = -\int_0^t \frac{R_1R_2 + R_1R_3 + R_2R_3}{L(R_1 + R_2)}dt$$

$$\Rightarrow \ln\left(\frac{I_3(t) + \dfrac{R_1\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}}{I_3(0) + \dfrac{R_1\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}}\right) = -\frac{R_1R_2 + R_1R_3 + R_2R_3}{L(R_1 + R_2)}t$$

$$\Rightarrow I_3(t)+\frac{R_1\varepsilon}{R_1R_2+R_1R_3+R_2R_3}=(I_3(0)+\frac{R_1\varepsilon}{R_1R_2+R_1R_3+R_2R_3})e^{-\frac{R_1R_2+R_1R_3+R_2R_3}{L(R_1+R_2)}t}$$

$$\Rightarrow I_3(t)=-\frac{R_1\varepsilon}{R_1R_2+R_1R_3+R_2R_3}(1-e^{-\frac{R_1R_2+R_1R_3+R_2R_3}{L(R_1+R_2)}t})+I_3(0)\cdot e^{-\frac{R_1R_2+R_1R_3+R_2R_3}{L(R_1+R_2)}t} \quad (7)$$

In this circuit, $I_3(0)=-\frac{\varepsilon}{R_2+R_3}$ (8)

Insert (8) into (7), we get

$$I_3(t)=-\frac{R_1\varepsilon}{R_1R_2+R_1R_3+R_2R_3}(1-e^{-\frac{R_1R_2+R_1R_3+R_2R_3}{L(R_1+R_2)}t})-\frac{\varepsilon}{R_2+R_3}\cdot e^{-\frac{R_1R_2+R_1R_3+R_2R_3}{L(R_1+R_2)}t} \quad (9)$$

For t→ ∞ (long after the switch being closed), from (9) we get

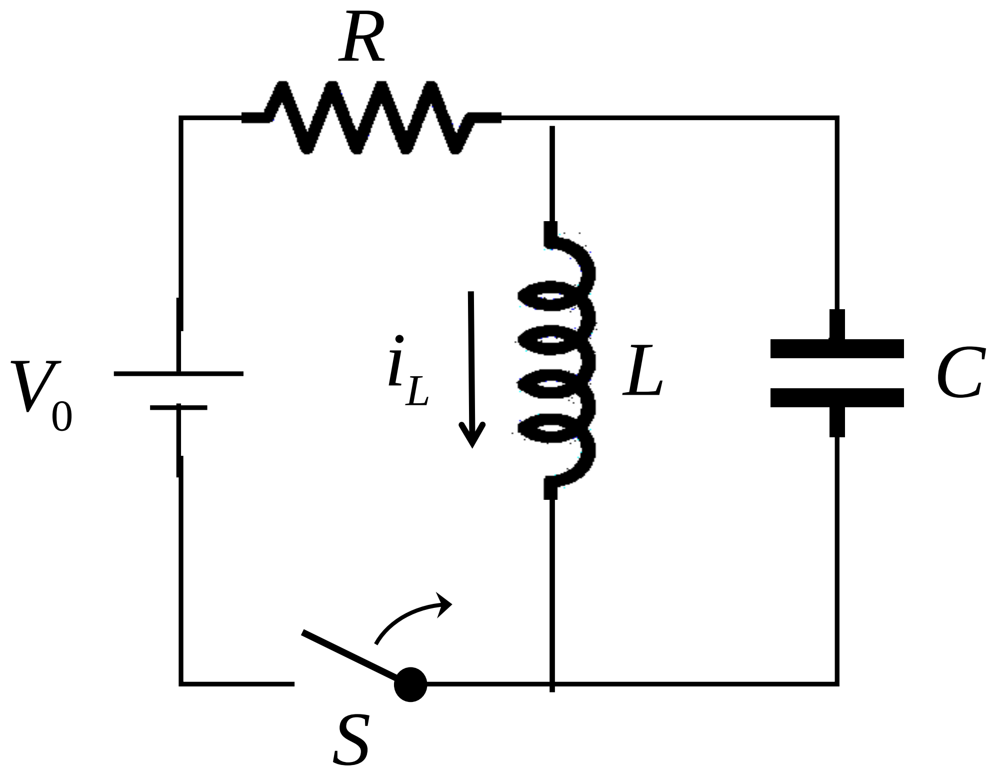
$$I_3(t)=-\frac{R_1\varepsilon}{R_1R_2+R_1R_3+R_2R_3}$$

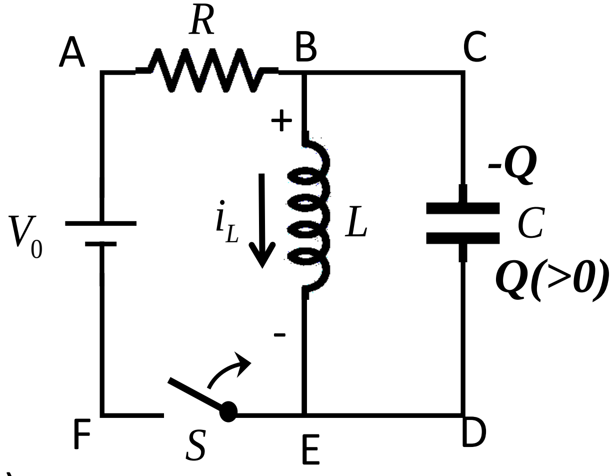
HW10-3 :

The figure on the right shows a DC circuit with the switch S being closed for t<0, and at t=0, the switch is opened,

(a) Determine the voltage across and the current through each device in the circuit, i.e. the resister, the inductor, and the capacitor for t<0, and immediately right after S being opened.

(b) Determine the current i_L through the inductor as a function of time for t ≥ 0.





(a)

For t<0, the indictor behaves like a conductor (zero resistance) and the capacitor like a open circuit, therefore

$$V_R = V_0, \quad V_L = 0 = V_C,$$

Right after S being opened, $\boldsymbol{i_L}$ remains the same momentarily, and there will be no current through the resister, therefore

$$V_R = 0, \quad i_L = \frac{V_0}{R},$$

From the loop BCDE, and with the choice of the direction of $\boldsymbol{i_L}$ and the polarity of the charge Q on the capacitor, We have

$$L\frac{di_L}{dt} + \frac{Q}{C} = 0, \quad \text{and} \quad i_L = \frac{dQ}{dt}$$

We then get

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0,$$

$$\Rightarrow Q = Q_0\cos(\omega t + \phi), \qquad \omega = \frac{1}{\sqrt{LC}}$$

$$i_L = \frac{dQ}{dt} = -Q_0\omega\sin(\omega t + \phi)$$

Right after S being opened $\boldsymbol{i_L = V_0/R}$, and Q =0, We have

$$0 = Q_0\cos(\phi), \qquad \frac{V_0}{R} = -Q_0\omega\sin(\phi)$$

$$\Rightarrow \phi = \frac{\pi}{2}, \text{ and } \; Q_0 = \frac{-V_0}{\omega R}$$

$$\Rightarrow Q(t) = \frac{-\sqrt{LC}V_0}{R}\cos\left(\frac{t}{\sqrt{LC}} + \frac{\pi}{2}\right)$$

$$= \frac{\sqrt{LC}V_0}{R}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow i_L = \frac{dQ}{dt} = \frac{V_0}{R}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

Right after S being opened

$$V_{\mathrm{C}} = \frac{Q(0)}{C} = 0$$

$$\Rightarrow V_L = L\frac{di_L}{dt}\bigg|_{t=0} = -\frac{LV_0}{\sqrt{LC}R}\sin\left(\frac{t}{\sqrt{LC}}\right)\bigg|_{t=0} = 0$$