

Please show all work. Answers without justification may not receive full credits.
Do all problems. Points total 130.

1. (10 pts) Find the arc length of the parametric curve $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $0 \leq t \leq 3$.
2. (40 pts) Let C be the limaçon given by the polar equation $r = 1 + 2 \cos \theta$.
 - (a) Sketch C . [Hint: There is an inner loop corresponding to $r \leq 0$.]
 - (b) Find the slopes of the tangent lines to C at the pole.
 - (c) Find the area of the region enclosed by the inner loop.
 - (d) Set up, but do not evaluate, the integral for the total length of C .
3. (40 pts) Identify and sketch the following **surfaces** whose equations are given in rectangular, cylindrical, or spherical coordinates.
 - (a) $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} - (z-1)^2 = 1$
 - (b) $z = r^2$
 - (c) $r = 4 \cos \theta$
 - (d) $\rho \cos \phi = 2$
 - (e) $\phi = \pi/4$
4. (10 pts) Assume that the Earth is a sphere of radius $\rho = 6378$ km and adopt the rectangular coordinate system with its origin at the center of the Earth, its positive z -axis passing through the North Pole, and its positive x -axis passing through the prime meridian. The city of Chiayi is located at 120.5° east longitude and 23.5° north latitude. Find its spherical $\rho\theta\phi$ -coordinates and rectangular xyz -coordinates.
5. (30 pts) Recall that the curvature of a smooth curve $\mathbf{r}(t)$ is given by

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

- (a) Use this to prove that the curvature of a plane parametric curve $x = f(t)$, $y = g(t)$ is

$$\kappa(t) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}},$$

where primes denote differentiation with respect to t .

- (b) Find the curvature of the logarithmic spiral

$$x = e^t \cos t, \quad y = e^t \sin t.$$

- (c) Show that the angle between \mathbf{v} and \mathbf{a} is constant for the logarithmic spiral in (b). Find the angle.
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