Big Homework (Midterm II) LA 2021 Spring

Answer the questions and show all of your work clearly in this file. The maximum point is 100.

1. Determine which of the sets are subspaces of $\boldsymbol{P}_3$. (Hint: Subspace Test)

a. (10 pts) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

Ans:

$set \quad V \qquad f(x)=a_3x^3 + a_2x^2 + a_1x$

$let \ f, g \in V \qquad g(x)=b_3x^3 + b_2x^2 + b_1x$

$(f+g)(x) = (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x \ \rightarrow \ f+g \in V$

$let \ k \ \in$ R and f$\in V \quad f(x)=a_3x^3 + a_2x^2 + a_1x$

$(kf)(x) = ka_3x^3 + ka_2x^2 + ka_1x \ \rightarrow kf \in V \rightarrow V$ is subspaces of $\boldsymbol{P}_3$

b. (10 pts) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.

Ans:

Let f,g$\in$V

$f(x)=a_3x^3 + a_2x^2 + a_1x$

$g(x)=b_3x^3+b_2x^2+b_1 x$

$(f+g)(x)= (a_3+b_3)x^3+(a_2+b_2)x^2+(a_1+b_1)x+ a_0+b_0$

$= (a_3+b_3)+(a_2+b_2)+(a_1+b_1)+ (a_0+b_0)$

$=(a_3+a_2+a_1+ a_0)+(b_3+b_2+b_1+ b)$

$=0+0$

$=0$

=>f+g$\in$V

Let k$\in$R and f$\in$V

$f(x)=a_3x^3 + a_2x^2 + a_1x$

$k \cdot f(x)=k(a_3x^3 + a_2x^2 + a_1x)= ka_3x^3 + ka_2x^2 + ka_1x+ ka_0$

$=k(a_3+a_2+a_1+ a_0)=k \cdot 0=0$

=>kf$\in$V

=>V is a subspace of P_3 #

2. (10 pts) Determine whether the following polynomials span P_2.

$$p_1=1-x+2x^2 , \ p_2=3+x$$
$$p_3=5-x+4x^2, \ p_4=-2-2x+2x^2$$

Ans:

$$p_1=1-x+2x^2\ ,\ p_2=3+x$$
$$p_3=5-x+4x^2,\ p_4=-2-2x+2x^2$$

find $\alpha_1,\ \alpha_2\ \epsilon R$, $p_3=\ \alpha_1 p_1+\alpha_2 p_2$, $5-x+4x^2=\alpha_1(1-x+2x^2)+\ \alpha_2(3+x)$

$\rightarrow 5-x+4x^2=(\alpha_1+3\alpha_2)+(-\alpha_1+\alpha_2)x+2\alpha_1 x^2$, $\begin{cases} 5=\alpha 1+3\alpha 2 \\ -1=-\alpha 1+\alpha 2 \\ 4=2\alpha 1 \end{cases}$. $\rightarrow \alpha 1=2, \alpha 2=1$

Polynomials p_3 can be written as linear combination of polynomials p_1, p_2

$span\{p1,p2\}\ = span\{p1,p2,p3\}\ = span\ \ \{p1,p2,p3,p4\}$

$\dim(span\ \ \{p1,p2,p3,p4\})\ = \dim(span\ \ \{p1,p2\})\ < 3 = \dim\ (P_2)$

These four polynomials cannot span P_2.

3. In each part, determine whether the vectors are linearly independent or are linearly dependent in $\boldsymbol{R}^4$.
   a: (5 pts) (3, 8, 7, −3), (1, 5, 3, −1), (2, −1, 2, 6), (4, 2, 6, 4)
   b: (5 pts) (3, 0, −3, 6), (0, 2, 3, 1), (0, −2, −2, 0), (−2, 1, 2, 1)

Ans:

(a)

$$\begin{bmatrix} 3 & 1 & 2 & 4 & 0 \\ 8 & 5 & -1 & 2 & 0 \\ 7 & 3 & 2 & 6 & 0 \\ -3 & -1 & 6 & 4 & 0 \end{bmatrix} =>$$

$$P_{12}E_{21}\left[-\tfrac{3}{8}\right]E_{31}\left[-\tfrac{7}{8}\right]E_{41}\left[\tfrac{3}{8}\right]E_{23}\left[\tfrac{-7}{11}\right]P_{23}E_{42}\left[\tfrac{7}{11}\right]P_{34}E_{43}\left[\tfrac{-3}{41}\right]D_3[\tfrac{11}{82}]E_{23}\left[\tfrac{-23}{8}\right]E_{13}[1]D_2[\tfrac{-8}{11}]E_{12}[-5]D_1[\tfrac{1}{8}]$$

$=>\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ a=-d，b=d，c= -d => the given vectors are linearly dependent

(b)

$$\begin{bmatrix} 3 & 0 & -3 & 6 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ -2 & 1 & 2 & 1 & 0 \end{bmatrix} => D_1\left[\tfrac{1}{3}\right]D_3\left[\tfrac{-1}{2}\right]E_{41}[-2]E_{23}[-2]E_{12}[1]E_{42}[-4]E_{34}[-1]E_{23}[-1] =>$$

$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 1 & 0 & -7 & 0 \end{bmatrix}$ $=> d=0, c=0, b=0, a=0 =>$the given vectors are linearly independent

4. (10 pts) Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix},\ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

Ans:

Def: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} => \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$,

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix} = reduce\ row\ echelon\ from => \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent in $\boldsymbol{R}^4$

=> Show that the following matrices form a basis for M_{22}.

5. (10 pts) Find a basis for the subspace of $\boldsymbol{R}^3$ that is spanned by the vectors (Show details for the correct one)

$\mathrm{v}_1 = (1,0,0), \mathrm{v}_2 = (1,0,1), \mathrm{v}_3 = (3,0,1), \mathrm{v}_4 = (0,0,-2).$

a: v_1 and v_2 form a basis for span $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$.

Ans:

$v_3 = 2v_1 + v_2$ 且 $v_4 = 2v_1 - 2v_2$ $\rightarrow$ v_3 and v_4 can be removed from the set without changing the span

$\rightarrow$ Let a,b $\in \mathbb{R}$ and $av_1 + bv_2 = 0 \rightarrow a(1,0,0) + b(1,0,1) = (0,0,0) \rightarrow a = b = 0 \rightarrow$ v_1 and v_2 are linearly independent

$\rightarrow$ Because a basis is just a linearly independent spanning set, so $\{\mathrm{v}_1, \mathrm{v}_2\}$ forms a basis for the subspace span $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$ ——**<ans>**

b: v_2 and v_3 form a basis for span $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$.

c: v_3 and v_4 form a basis for span $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$.

d: v_1 and v_3 form a basis for span $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$.

e: v_2 and v_4 form a basis for span $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$.

f: v_1 and v_4 form a basis for span $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$.

6. (10 pts) The matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 4 \\ 0 & 3 & 3 \end{bmatrix}$ is the transition matrix from <u>what basis</u> $B = \{v_1, v_2, v_3\}$ to the basis $\{(1,1,1), (1,1,0), (1,0,0)\}$ for $\boldsymbol{R}^3$?

Ans:

$$\begin{bmatrix}1&0&0\\0&7&4\\0&3&3\end{bmatrix}B=\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0&1&0&0\\0&1&0&0&7&4\\0&0&1&0&3&3\end{bmatrix}\to\begin{bmatrix}0&0&1&0&3&3\\0&1&0&0&7&4\\1&0&0&1&0&0\end{bmatrix}\to$$

$$\begin{bmatrix}1&0&1&1&3&3\\0&1&0&0&7&4\\1&0&0&1&0&0\end{bmatrix}\to \quad \begin{bmatrix}1&1&1&1&10&7\\0&1&0&0&7&4\\1&0&0&1&0&0\end{bmatrix}\to\begin{bmatrix}1&1&1&1&10&7\\1&1&0&1&7&4\\1&0&0&1&0&0\end{bmatrix}$$

Hence $B=\{v_1,v_2,v_3\}=\{(1,1,1),(10,7,0),(7,4,0)\}$

7. $A=\begin{bmatrix}1&4&5&6&7\\2&-7&-5&-3&-16\\-1&0&-1&2&1\\3&7&10&13&11\end{bmatrix}$

a: (10 pts) What is <u>the basis for</u> null space of ***A***?

Let $Ax=0=A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}$

By Gaussian Elimination:

$E_{23}[2]E_{43}[3]E_{31}[1]E_{31}[1]E_{42}[1]D_3[\frac{1}{4}]D_4[\frac{1}{20}]E_{23}[7]E_{24}[-15]E_{13}[-4]E_{14}[2]E_{34}[-2]A$

we can get reduced row echelon form of $A=\begin{bmatrix}1&0&1&0&-1\\0&0&0&0&0\\0&1&1&0&2\\0&0&0&1&0\end{bmatrix}$

then we set $x_3=s\ ,x_5=t\ ,x_1=-s+t\ ,x_2=-s-2t\ ,x_4=0$

$x=s\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}1\\-2\\0\\0\\1\end{bmatrix}\quad\therefore\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\-2\\0\\0\\1\end{bmatrix}$ is the basis#

b: (10 pts) What is <u>the basis for</u> row space of ***A***?

$A=\begin{bmatrix}1&4&5&6&7\\2&-7&-5&-3&-16\\-1&0&-1&2&1\\3&7&10&13&11\end{bmatrix}\to \mathrm{R}=\begin{bmatrix}1&0&1&0&-1\\0&1&1&0&2\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}$

$r_1=[1,0,1,0,-1]$ $r_2=[0,1,1,0,2]$ $r_3=[0,0,0,1,0]$

8. Find the largest possible value for the rank of $\boldsymbol{A}$ and the smallest possible value for the nullity of $\boldsymbol{A}$ where $\boldsymbol{A}$ is 7×11.

   a: (5 pts) Find the largest possible value for the rank of A.

   b: (5 pts) Find the smallest possible value for the nullity.

   Ans:

   rank(A)<=min(7,11) so the largest possible value for the rank of A is 7.

   rank(A) +nullity(A)=11 so the smallest possible value for the nullity is 4.

9. (*5 pts) Bonus: Give me any information about your case (date, problems or answers)