Chapter 3

Basic Modulation Techniques

3.1 Problems

Problem 3.1
The demodulated output, in general, is

$$y_D(t) = \text{Lp}\{x_c(t)\, 2\cos[\omega_c t + \theta(t)]\}$$

where Lp $\{\bullet\}$ denotes the lowpass portion of the argument. With

$$x_c(t) = A_c m(t) \cos[\omega_c t + \phi_0]$$

the demodulated output becomes

$$y_D(t) = \text{Lp}\{2A_c m(t) \cos[\omega_c t + \phi_0] \cos[\omega_c t + \theta(t)]\}$$

Performing the indicated multiplication and taking the lowpass portion yields

$$y_D(t) = A_c m(t) \cos[\theta(t) - \phi_0]$$

If $\theta(t) = \theta_0$ (a constant), the demodulated output becomes

$$y_D(t) = A_c m(t) \cos[\theta_0 - \phi_0]$$

Letting $A_c = 1$ gives the error

$$\varepsilon(t) = m(t)[1 - \cos(\theta_0 - \phi_0)]$$

The mean-square error is

$$\left\langle \varepsilon^2(t) \right\rangle = \left\langle m^2(t)[1 - \cos(\theta_0 - \phi_0)]^2 \right\rangle$$

where $\langle\cdot\rangle$ denotes the time-average value. Since the term $[1-\cos(\theta_0-\phi_0)]$ is a constant, we have

$$\left\langle \varepsilon^2(t)\right\rangle = \left\langle m^2(t)\right\rangle [1-\cos(\theta_0-\phi_0)]^2$$

Note that for $\theta_0=\phi_0$, the demodulation carrier is phase coherent with the original modulation carrier, and the error is zero. For $\theta(t)=\omega_0 t$ we have the demodulated output

$$y_D(t) = A_c m(t)\cos(\omega_0 t-\phi_0)$$

Letting $A_c=1$, for convenience, gives the error

$$\varepsilon(t) = m(t)[1-\cos(\omega_0 t-\phi_0)]$$

giving the mean-square error

$$\left\langle \varepsilon^2(t)\right\rangle = \left\langle m^2(t)[1-\cos(\omega_0 t-\phi_0)]^2\right\rangle$$

In many cases, the average of a product is the product of the averages. (We will say more about this in Chapters 4 and 5). For this case

$$\left\langle \varepsilon^2(t)\right\rangle = \left\langle m^2(t)\right\rangle \left\langle [1-\cos(\omega_0 t-\phi_0)]^2\right\rangle$$

Note that $1-\cos(\omega_0 t-\phi_0)$ is periodic. Taking the average over an integer number of periods yields

$$\begin{aligned}\left\langle [1-\cos(\omega_0 t-\phi_0)]^2\right\rangle &= \left\langle 1-2\cos(\omega_0 t-\phi_0)+\cos^2(\omega_0 t-\phi_0)\right\rangle \\ &= 1+\frac{1}{2}=\frac{3}{2}\end{aligned}$$

Thus

$$\left\langle \varepsilon^2(t)\right\rangle = \frac{3}{2}\left\langle m^2(t)\right\rangle$$

Problem 3.2

We are given that

$$m(t)=\sum_{k=1}^{5}\frac{10}{k}\sin(2\pi k f_m t)=\sum_{k=1}^{5}\frac{10}{k}\cos\left(2\pi k f_m t-\frac{\pi}{2}\right)$$

With a carrier of

$$c(t)=100\cos(200\pi t)$$

we have

$$x_c(t)=\sum_{k=1}^{5}\frac{500}{k}\left\{\cos\left[(200+2kf_m)\,\pi t-\frac{\pi}{2}\right]+\cos\left[(200-2kf_m)\,\pi t+\frac{\pi}{2}\right]\right\}$$

The complex exponential Fourier series is

$$\begin{aligned}x_c(t) &= \sum_{k=1}^{5}\frac{250}{k}\left\{\exp\left[(200+2kf_m)\,\pi t-\frac{\pi}{2}\right]+\exp\left[(200-2kf_mt)\,\pi t+\frac{\pi}{2}\right]\right\}\\ &\quad+\sum_{k=1}^{5}\frac{250}{k}\left\{\exp\left[(-200-2kf_m)\,\pi t+\frac{\pi}{2}\right]+\exp\left[(-200+2kf_mt)\,\pi t-\frac{\pi}{2}\right]\right\}\end{aligned}$$

The top row represents the positive frequency terms and the second row represents the negative frequency terms. The transmitted power is

$$P_T=\sum_{k=1}^{5}2\left(\frac{250}{k}\right)^2=125000\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}\right)=182.960\,\mathrm{kW}$$

Problem 3.3

A full-wave rectifier takes the form shown in Figure 3.1. The waveforms are shown in Figure 3.2, with the half-wave rectifier on top and the full-wave rectifier on the bottom. The message signal is the envelopes. Decreasing exponentials can be drawn from the peaks of the waveform as depicted in Figure 3.3(b) in the text. It is clear that the full-wave rectified $x_c\,(t)$ defines the message better than the half-wave rectified $x_c\,(t)$ since the carrier frequency is effectively doubled.

Problem 3.4

Using

$$E_{ff}=\frac{a^2\left\langle m_n^2(t)\right\rangle}{1+a^2\left\langle m_n^2(t)\right\rangle}$$

yields the table below.

Part	$\left\langle m_n^2\,(t)\right\rangle$	$a=0.2$	$a=0.3$	$a=0.4$	$a=7$	$a=1$
a	1/3	$E_{ff}=1.32\%$	$E_{ff}=2.91\%$	$E_{ff}=5.06\%$	$E_{ff}=14.04\%$	$E_{ff}=25\%$
b	1/3	$E_{ff}=1.32\%$	$E_{ff}=2.91\%$	$E_{ff}=5.06\%$	$E_{ff}=14.04\%$	$E_{ff}=25\%$
c	1	$E_{ff}=3.85\%$	$E_{ff}=8.26\%$	$E_{ff}=13.79\%$	$E_{ff}=32.89\%$	$E_{ff}=50\%$

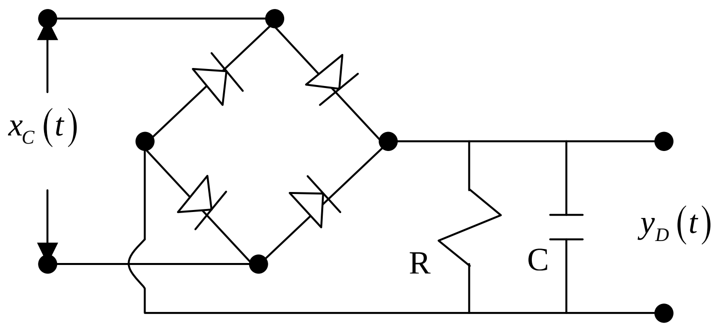

Figure 3.1: Full wave rectifier.

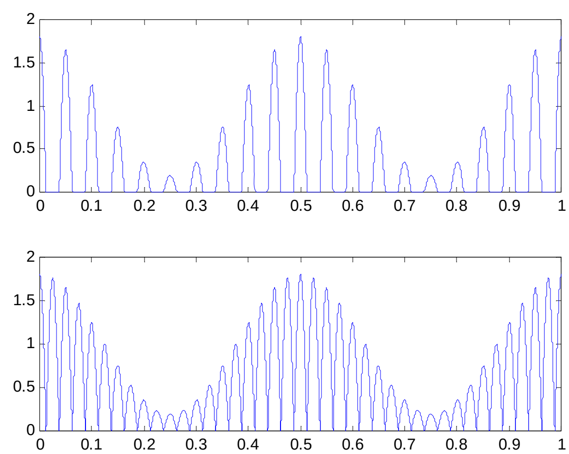

Figure 3.2: Output waveforms for a half-wave rectifier (top) and a full-wave rectifier (bottom).

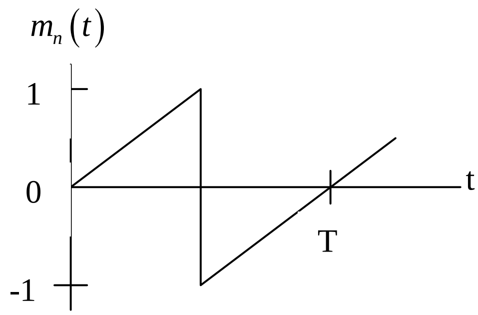

Figure 3.3: Normalized message signal for Problem 3.5.

Problem 3.5
By inspection, the normalized message signal is as shown in Figure 3.3. Thus

$$m_n(t) = \frac{2}{T}t, \qquad 0 \le t \le \frac{T}{2}$$

and

$$\left\langle m_n^2(t) \right\rangle = \frac{2}{T}\int_0^{T/2} \left(\frac{2}{T}t\right)^2 dt = \frac{2}{T}\left(\frac{2}{T}\right)^2 \frac{1}{3}\left(\frac{T}{2}\right)^3 = \frac{1}{3}$$

Also

$$\begin{aligned} A_c[1+a] &= 40 \\ A_c[1-a] &= 10 \end{aligned}$$

This yields

$$\frac{1+a}{1-a} = \frac{40}{10} = 4$$

or

$$\begin{aligned} 1+a &= 4-4a \\ 5a &= 3 \end{aligned}$$

Thus

$$a = 0.6$$

Since the index is 0.6, we can write

$$A_c[1+0.6] = 40$$

This gives

$$A_c = \frac{40}{1.6} = 25$$

This carrier power is

$$P_c = \frac{1}{2}A_c^2 = \frac{1}{2}(25)^2 = 312.5 \quad \text{Watts}$$

The efficiency is

$$E_{ff} = \frac{(0.6)^2\left(\frac{1}{3}\right)}{1+(0.6)^2\left(\frac{1}{3}\right)} = \frac{0.36}{3.36} = 0.107 = 10.7\%$$

Thus

$$\frac{P_{sb}}{P_c + P_{sb}} = 0.107$$

where P_{sb} represents the power in the sidebands and P_c represents the power in the carrier. The above expression can be written

$$P_{sb} = 0.107 + 0.107 P_{sb}$$

This gives

$$P_{sb} = \frac{0.107}{1.0 - 0.107} P_c = 97.48 \quad \text{Watts}$$

Problem 3.6

Using (3.14) with $A_c = 100$, $a = 0.7$ and $\left\langle m_n^2(t)\right\rangle = 1$ gives

$$\left\langle x_c^2(t)\right\rangle = 0.5(100)^2 + 0.5(100)^2(0.7)^2$$

The power in the sidebands is

$$P_{sb} = 2.45\,\text{kW}$$

The total power is 7.45 kW. Thus the efficiency is

$$E_{ff} = \frac{2.45}{7.45} = 0.3289 = 32.89\%$$

This may be checked by using

$$E_{ff} = \frac{a^2\left\langle m_n^2(t)\right\rangle}{1 + a^2\left\langle m_n^2(t)\right\rangle} = \frac{(0.7)^2}{1+(0.7)^2} = 0.3289$$

The modulation trapezoid is shown below.

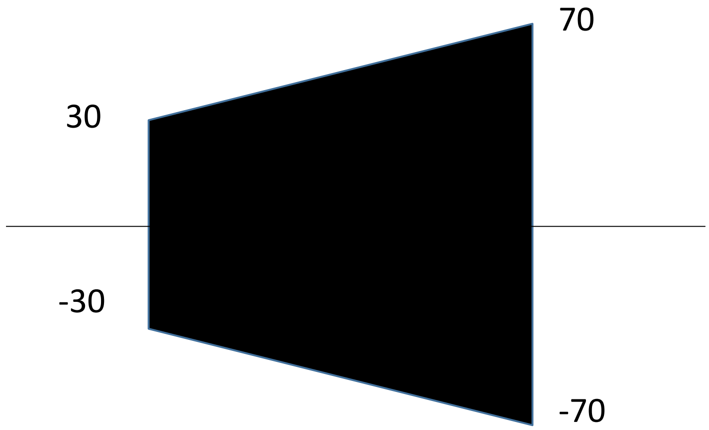

Figure 3.4: Modulation trapezoid for Problem 3.6.

Problem 3.7
For the first signal

$$-\tau + 5(T - \tau) = 0$$

so that

$$\tau = \frac{5T}{6}, \quad m_n(t) = m(t)$$

Also

$$\left\langle m_n^2 \right\rangle = \frac{1}{T}\left[(-1)^2 \frac{5T}{6} + (5)^2 \frac{T}{6}\right] = \frac{30T}{6T} = 5$$

For $a = 0.7$ the efficiency is

$$E_{ff} = \frac{a^2 \left\langle m_n^2(t) \right\rangle}{1 + a^2 \left\langle m_n^2(t) \right\rangle} = \frac{(0.7)^2(5)}{1 + (0.7)^2(5)} = 0.7101 = 71.01\%$$

and for $a = 1$ the efficiency is

$$E_{ff} = \frac{a^2 \left\langle m_n^2(t) \right\rangle}{1 + a^2 \left\langle m_n^2(t) \right\rangle} = \frac{(1)^2(5)}{1 + (1)^2(5)} = 0.83.33 = 83.33\%$$

For the second signal

$$-5\tau + (T - \tau) = 0$$

so that

$$\tau = \frac{T}{6}, \quad m_n(t) = \frac{1}{5}m(t)$$

Also

$$\left\langle m_n^2 \right\rangle = \frac{1}{T}\left(\frac{1}{5}\right)^2 \left[(1)^2\frac{T}{6} + (-5)^2\frac{5T}{6}\right] = \frac{5T}{25T} = 0.2$$

For $a = 0.7$ the efficiency is

$$E_{ff} = \frac{a^2 \left\langle m_n^2(t) \right\rangle}{1 + a^2 \left\langle m_n^2(t) \right\rangle} = \frac{(0.7)^2(0.2)}{1 + (0.7)^2(0.2)} = 0.0893 = 8.93\%$$

and for $a = 1$ the efficiency is

$$E_{ff} = \frac{a^2 \left\langle m_n^2(t) \right\rangle}{1 + a^2 \left\langle m_n^2(t) \right\rangle} = \frac{(1)^2(0.2)}{1 + (1)^2(0.2)} = 0.1667 = 16.67\%$$

Problem 3.8

(a) With $m(t) = 9\cos(20\pi t) - 8\cos(60\pi t)$ we can use MATLAB and either plot the waveform or use a root finding algorithm to find the most negative value of $m(t)$. Plotting the waveform shows that it varies between -12.9 and $+12.9$. Therefore $m_n(t) = \frac{1}{12.9}\left[9\cos(20\pi t) - 8\cos(60\pi t)\right]$

(b) $\left\langle m_n^2(t) \right\rangle = \left(\frac{1}{12.9}\right)^2 \left(\frac{1}{2}\right) \left[(9)^2 + (8)^2\right] = 0.4357$

(c) $E = \left[(0.8)^2(0.4357)\right] / \left[1 + (0.8)^2(0.4357)\right] = 0.2180 = 21.80\%$

(d) The expression for $x_c(t)$ is

$$\begin{aligned} x_c(t) &= 110\left[1 + \frac{1}{12.9}\left(9\cos(20\pi t) - 8\cos(60\pi t)\right)\right]\cos 200\pi t \\ &= -34.1085\cos(140\pi t) + 38.3721\cos(180\pi t) \\ &\quad +110\cos(200\pi t) \\ &\quad +38.3721\cos(220\pi t) - 34.1085\cos(260\pi t) \end{aligned}$$

The spectrum is illustrated below.

The positive and negative terms dedined in the table below so that the magnitude spectrum is even an the phase spectrum is odd.

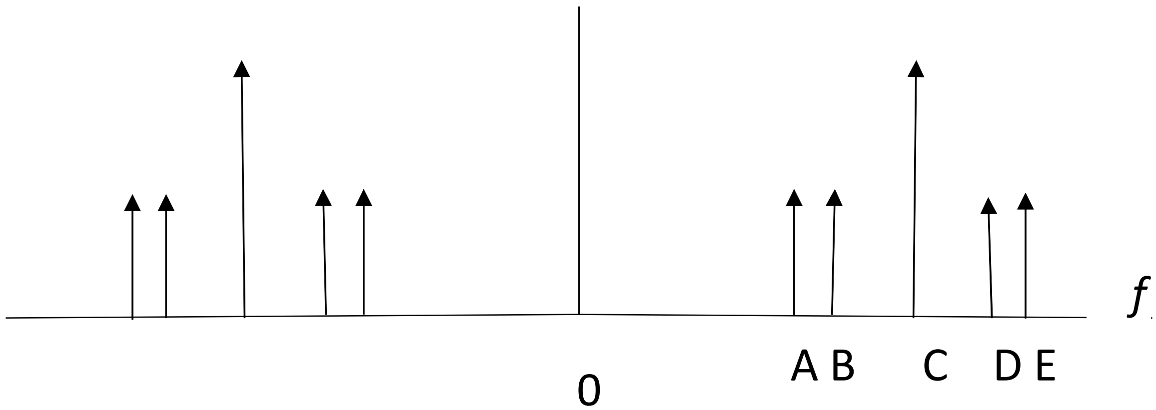

Figure 3.5: Spectrum of transmitted signal using AM.

Component	Frequency (Hz)	Magnitude	Phase
A	70	34.1085	$-\pi$
B	90	38.3721	0
C	100	110	0
D	110	38.3721	0
E	130	34.1085	π
-A	-70	34.1085	π
-B	-90	38.3721	0
-C	-100	110	0
-D	-110	38.3721	0
-E	-130	34.1085	$-\pi$

Problem 3.9

(a) With $m(t) = 9\cos(20\pi t) + 8\cos(60\pi t)$, the most negative value of $m(t) = -17$ and falls at $t = 0.05$. Therefore $m_n(t) = \frac{1}{17}\left[9\cos(20\pi t) - 8\cos(60\pi t)\right]$

(b) $\left\langle m_n^2(t)\right\rangle = \left(\frac{1}{17}\right)^2\left(\frac{1}{2}\right)\left[(9)^2 + (8)^2\right] = 0.2509$

(c) $E = \left[(0.8)^2(0.2509)\right] / \left[1 + (0.8)^2(0.2509)\right] = 0.1384 = 13.84\%$

(d) The expression for $x_c(t)$ is

$$\begin{aligned} x_c(t) &= 110\left[1+\frac{1}{17}\left(9\cos(20\pi t)+8\cos(60\pi t)\right)\right]\cos 200\pi t \\ &= 25.8824\cos(140\pi t)+29.1176\cos(180\pi t) \\ &\quad +110\cos(200\pi t) \\ &\quad +29.1176\cos(220\pi t)+25.8824\cos(260\pi t) \end{aligned}$$

The spectrum is as in the previous problem. Components are identified below. Note that the change in sign makes phase spectrum everywhere zero.

Component	Frequency (Hz)	Magnitude	Phase
A	70	25.8824	0
B	90	29.1176	0
C	100	110	0
D	110	29.1176	0
E	130	25.8824	0
-A	-70	25.8824	0
-B	-90	29.1176	0
-C	-100	110	0
-D	-110	29.1176	0
-E	-130	25.8824	0

Problem 3.10
The modulator output

$$x_c(t) = 40\cos 2\pi(200)t + 5\cos 2\pi(180)t + 5\cos 2\pi(220)t$$

can be written

$$x_c(t) = [40 + 10\cos 2\pi(20)t]\cos 2\pi(200)t$$

or

$$x_c(t) = 40\left[1+\frac{10}{40}\cos 2\pi(20)t\right]\cos 2\pi(200)t$$

By inspection, the modulation index is

$$a = \frac{10}{40} = 0.25$$

Since the component at 200 Hertz represents the carrier, the carrier power is

$$P_c = \frac{1}{2}(40)^2 = 800 \quad \text{Watts}$$

The components at 180 and 220 Hertz are sideband terms. Thus the sideband power is

$$P_{sb} = \frac{1}{2}(5)^2 + \frac{1}{2}(5)^2 = 25 \quad \text{Watts}$$

Thus, the efficiency is

$$E_{ff} = \frac{P_{sb}}{P_c + P_{sb}} = \frac{25}{800 + 25} = 0.3030 = 3.30\%$$

Problem 3.11

The sideband power is

$$P_{sb} = \frac{B^2}{2} + \frac{B^2}{2} = B^2$$

The efficiency is

$$E_{ff} = \frac{P_{sb}}{P_c + P_{sb}} = \frac{B^2}{(A^2/2) + B^2} = \frac{2B^2}{A^2 + 2B^2}$$

The carrier power is

$$P_c = \frac{A^2}{2} = 200$$

Thus

$$A = \sqrt{400} = 20$$

This gives

$$E_{ff} = \frac{P_{sb}}{P_c + P_{sb}} = \frac{B^2}{200 + B^2} = 0.3$$

Therefore

$$60 = B^2 - 0.3B^2 = 0.7B^2$$

and

$$B = \sqrt{60/0.7} = 9.2582$$

We determine the modulation index by writing

$$x_c(t) = A\left[1 + \frac{2B}{A}\cos(2\pi(20)\pi t\right]\cos(2\pi(200)\pi t)$$

The modulation index is

$$a = \frac{2B}{A} = \frac{2(9.2582)}{20} = 0.9285$$

Summarizing

$$A = 20 \qquad B = 9.2582 \qquad a = 0.9258$$

Problem 3.12
The modulator output

$$x_c(t) = 25\cos 2\pi(150)t + 5\cos 2\pi(160)t + 5\cos 2\pi(140)t$$

is

$$x_c(t) = 25\left[1 + \frac{10}{25}\cos 2\pi(10)t\right]\cos 2\pi(150)t$$

Thus, the modulation index, a, is

$$a = \frac{10}{25} = 0.4$$

The carrier power is

$$P_c = \frac{1}{2}(25)^2 = 312.5 \quad \text{Watts}$$

and the sideband power is

$$P_{sb} = \frac{1}{2}(5)^2 + \frac{1}{2}(5)^2 = 25 \quad \text{Watts}$$

Thus, the efficiency is

$$E_{ff} = \frac{25}{312.5 + 25} = 0.0741$$

Problem 3.13
(a) By plotting $m(t)$ or by using a root-finding algorithm we see that the minimum value of $m(t)$ is $M = -3.432$. Thus

$$m_n(t) = 0.5828\cos(2\pi f_m t) + 0.2914\cos(4\pi f_m t) + 0.5828\cos(10\pi f_m t)$$

The AM signal is

$$\begin{aligned} x_c(t) &= A_c[1 + 0.8m_n(t)]\cos 2\pi f_c t \\ &= 0.2331A_c\cos 2\pi(f_c - 5f_m)t \\ &\quad +0.1166A_c\cos 2\pi(f_c - 2f_m)t \\ &\quad +0.2331A_c\cos 2\pi(f_c - f_m)t \\ &\quad +A_c\cos 2\pi f_c t \\ &\quad +0.2331A_c\cos 2\pi(f_c + f_m)t \\ &\quad +0.1166A_c\cos 2\pi(f_c + 2f_m)t \\ &\quad +0.2331A_c\cos 2\pi(f_c + 5f_m)t \end{aligned}$$

The spectrum is drawn from the expression for $x_c(t)$. It contains 14 discrete components as shown.

Comp	Freq	Amp	Comp	Freq	Amp
1	$-f_c - 5f_m$	$0.1166A_c$	8	$f_c - 5f_m$	$0.1166A_c$
2	$-f_c - 2f_m$	$0.0583A_c$	9	$f_c - 2f_m$	$0.0583A_c$
3	$-f_c - f_m$	$0.1166A_c$	10	$f_c - f_m$	$0.1166A_c$
4	$-f_c$	$0.5A_c$	11	f_c	$0.5A_c$
5	$-f_c + f_m$	$0.1166A_c$	12	$f_c + f_m$	$0.1166A_c$
6	$-f_c + 2f_m$	$0.0583A_c$	13	$f_c + 2f_m$	$0.0583A_c$
7	$-f_c + 5f_m$	$0.1166A_c$	14	$f_c + 5f_m$	$0.1166A_c$

(b) Since

$$\left\langle m_n^2(t)\right\rangle = \frac{1}{2}\left[(0.5838)^2 + (0.2914)^2 + (0.5838)^2\right] = 0.3833$$

the efficiency is

$$E_{ff} = \frac{a^2\left\langle m_n^2(t)\right\rangle}{1 + a^2\left\langle m_n^2(t)\right\rangle} = \frac{(0.8)^2(0.3833)}{1 + (0.8)^2(0.3833)} = 0.1970$$

Problem 3.14

(a) From Figure 3.35

$$x(t) = m(t) + \cos\omega_c t$$

With the given relationship between $x(t)$ and $y(t)$ we can write

$$y(t) = 4\{m(t) + \cos\omega_c t\} + 2\{m(t) + \cos\omega_c t\}^2$$

which can be written

$$y(t) = 4m(t) + 4\cos\omega_c t + 2m^2(t) + 4m(t)\cos\omega_c t + 1 + 1\cos 2\omega_c t$$

The equation for $y(t)$ is more conveniently expressed

$$y(t) = 1 + 4m(t) + 2m^2(t) + 4[1 + m(t)]\cos\omega_c t + \cos 2\omega_c t$$

(b) The spectrum illustrating the terms involved in $y(t)$ is shown in Figure 3.6. The center frequency of the filter is f_c and the bandwidth must be greater than or equal to $2W$. In addition, $f_c - W > 2W$ or $f_c > 3W$, and $f_c + W < 2f_c$. The last inequality states that $f_c > W$, which is redundant since we know that $f_c > 3W$.

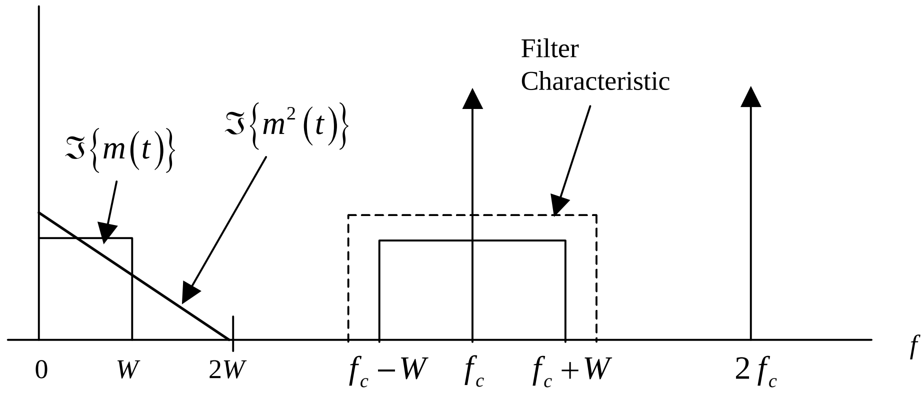

Figure 3.6: Generation of an AM signal using a nonlinear operation.

(c) From the definition of $m(t)$ we have

$$m(t) = M m_n(t)$$

so that

$$g(t) = 4[1 + M m_n(t)] \cos \omega_c t$$

It follows that

$$a = 0.1 = M$$

Thus

$$M = 0.1$$

(d) This method of forming a DSB signal avoids the need for an analog multiplier.

Problem 3.15
With

$$m(t) = 4\cos(2\pi f_m t) + \cos(4\pi f_m t)$$

we have

$$\widehat{m}(t) = 4\sin(2\pi f_m t) + \sin(4\pi f_m t)$$

Therefore

$$x_c(t) = 5[4\cos(2\pi f_m t) + \cos(4\pi f_m t)]\cos(2\pi f_c t) \pm 5[4\sin(2\pi f_m t) + \sin(4\pi f_m t)]\sin(2\pi f_c t)$$

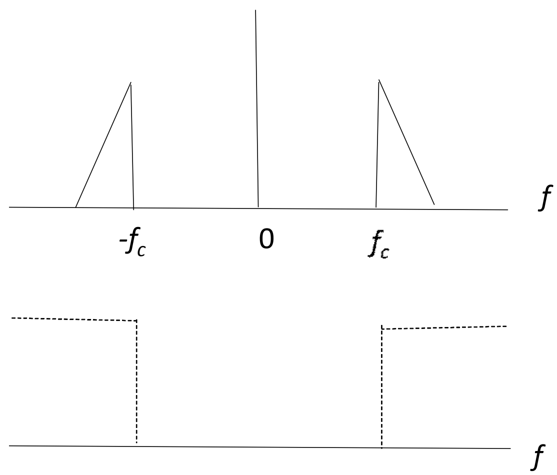

Figure 3.7: Generation of upper-sideband SSB using sideband filtering.

This gives

$$\begin{aligned} x_c(t) &= 10\cos\left[2\pi\left(f_c+f_m\right)t\right]+2.5\cos\left[2\pi\left(f_c+2f_m\right)t\right] \\ &\quad +10\cos\left[2\pi\left(f_c-f_m\right)t\right]+2.5\cos\left[2\pi\left(f_c-2f_m\right)t\right] \\ &\quad \mp10\cos\left[2\pi\left(f_c+f_m\right)t\right]+2.5\cos\left[2\pi\left(f_c+2f_m\right)t\right] \\ &\quad \pm10\cos\left[2\pi\left(f_c-f_m\right)t\right]+2.5\cos\left[2\pi\left(f_c-2f_m\right)t\right] \end{aligned}$$

Note that by using the top, +, sign we have

$$x_c(t)=20\cos\left[2\pi\left(f_c-f_m\right)t\right]+5\cos\left[2\pi\left(f_c-2f_m\right)t\right]$$

The upper-sideband (frquency components above f_c) cancel and we are left with only lower-sideband terms (frquency components below f_c). Note that by using the bottom, +, sign we have

$$x_c(t)=20\cos\left[2\pi\left(f_c+f_m\right)t\right]+5\cos\left[2\pi\left(f_c+2f_m\right)t\right]$$

The lower-sideband (frquency components below f_c) cancel) and we are left with only upper-sideband terms (frquency components above f_c). The spectrum in each case is obvious.

Problem 3.16

Consider the filter transfer function shown below

From the above figure, and comparing it with Figure 3.10 in the text, it is clear that the uppersideband filter is given by

$$H_U(f) = \frac{1}{2}\left[1 + \mathrm{sgn}\left(f - f_c\right)\right] + \frac{1}{2}\left[1 - \mathrm{sgn}\left(f + f_c\right)\right]$$

Going through the same steps as in the text (pp 125-127) gives

$$x_{USB}(t) = \frac{1}{2}A_c m(t)\cos\omega_c t - \frac{1}{2}A_c \widehat{m}(t)\sin\omega_c t$$

Problem 3.17

We have

$$x_{ssb}^2(t) = \left[\frac{1}{2}A_c m(t)\cos\omega_c t \pm \frac{1}{2}A_c \widehat{m}(t)\sin\omega_c t\right]^2$$

which is

$$x_{ssb}^2(t) = \frac{1}{4}A_c^2 m^2(t)\left[1 + \cos(2\omega_c t)\right] \pm \frac{1}{2}A_c^2 m(t)\widehat{m}(t)\cos(\omega_c t)\sin(\omega_c t) + \frac{1}{4}A_c^2 \widehat{m}^2(t)\left[1 - \cos(2\omega_c t\right]$$

or

$$x_{ssb}^2(t) = \frac{1}{4}A_c^2\left[m^2(t) + \widehat{m}^2(t) + m^2(t)\cos(2\omega_c t) \pm m(t)\widehat{m}(t)\sin(\omega_c t) - \widehat{m}^2(t)\cos(2\omega_c t)\right]$$

Therefore, yes, components are created at twice the carrier frequency.

Problem 3.18

We assume that the VSB waveform is given by

$$\begin{aligned} x_c(t) &= \frac{1}{2}A\varepsilon\cos(\omega_c - \omega_1)t \\ &\quad + \frac{1}{2}A(1-\varepsilon)\cos(\omega_c + \omega_1)t \\ &\quad + \frac{1}{2}B\cos(\omega_c + \omega_2)t \end{aligned}$$

We let $y(t)$ be $x_c(t)$ plus a carrier. Thus

$$y(t) = x_c(t) + K\cos\omega_c t$$

It can be shown that $y(t)$ can be written

$$y(t) = y_1(t)\cos\omega_c(t) + y_2(t)\sin\omega_c t$$

where

$$\begin{aligned} y_1(t) &= \frac{A}{2}\cos\omega_1 t + \frac{B}{2}\cos\omega_2 t + K \\ y_2(t) &= \left(A\varepsilon + \frac{A}{2}\right)\sin\omega_1 t - \frac{B}{2}\sin\omega_2 t \end{aligned}$$

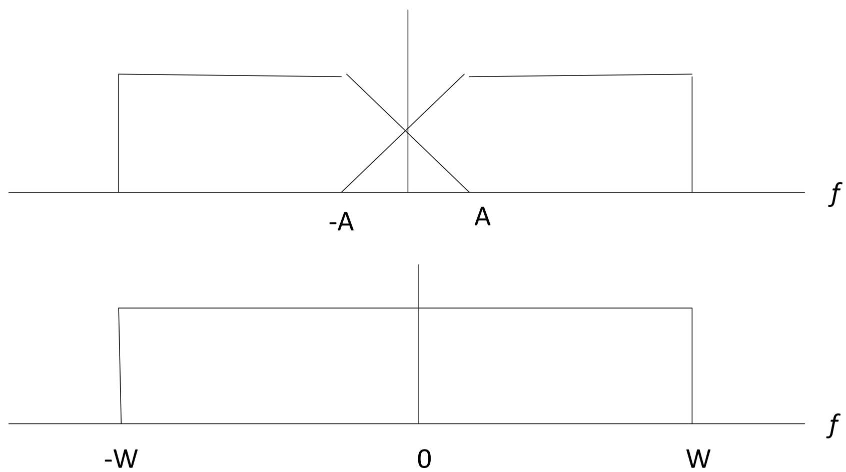

Figure 3.8: Coherent demodulation of VSB.

Also

$$y(t) = R(t)\cos(\omega_c t + \theta)$$

where $R(t)$ is the envelope and is therefore the output of an envelope detector. It follows that

$$R(t) = \sqrt{y_1^2(t) + y_2^2(t)}$$

For K large, $y_1(t) \gg y_2(t)$ for all t so that $y_2(t)$ can be neglected. Therefore, $R(t) = |y_1(t)|$, which is $\frac{1}{2}m(t) + K$, where K is a dc bias. Thus if the detector is ac coupled, K is removed and the output $y(t)$ is $m(t)$ scaled by $\frac{1}{2}$.

Problem 3.19

Coherent demodulation translates the positive-frequency portion of the spectrum to $f = 0$ and to $f = 2f_c$. The negative-frequency portion of the spectrum is translated to $f = 0$ and to $f = -2f_c$. The following figure illustrates the portion of the spectrum about $f = 0$.

The portion of the spectrum is the portion between $-$A and A. Using the required symmetry properties for a VSB filter, the two portions of the translated spectra are conjugate symmetric. Thus they add resulting in the spectrum shown in the bottom pane.

Problem 3.20

The result is shown in the following figure.

Problem 3.21

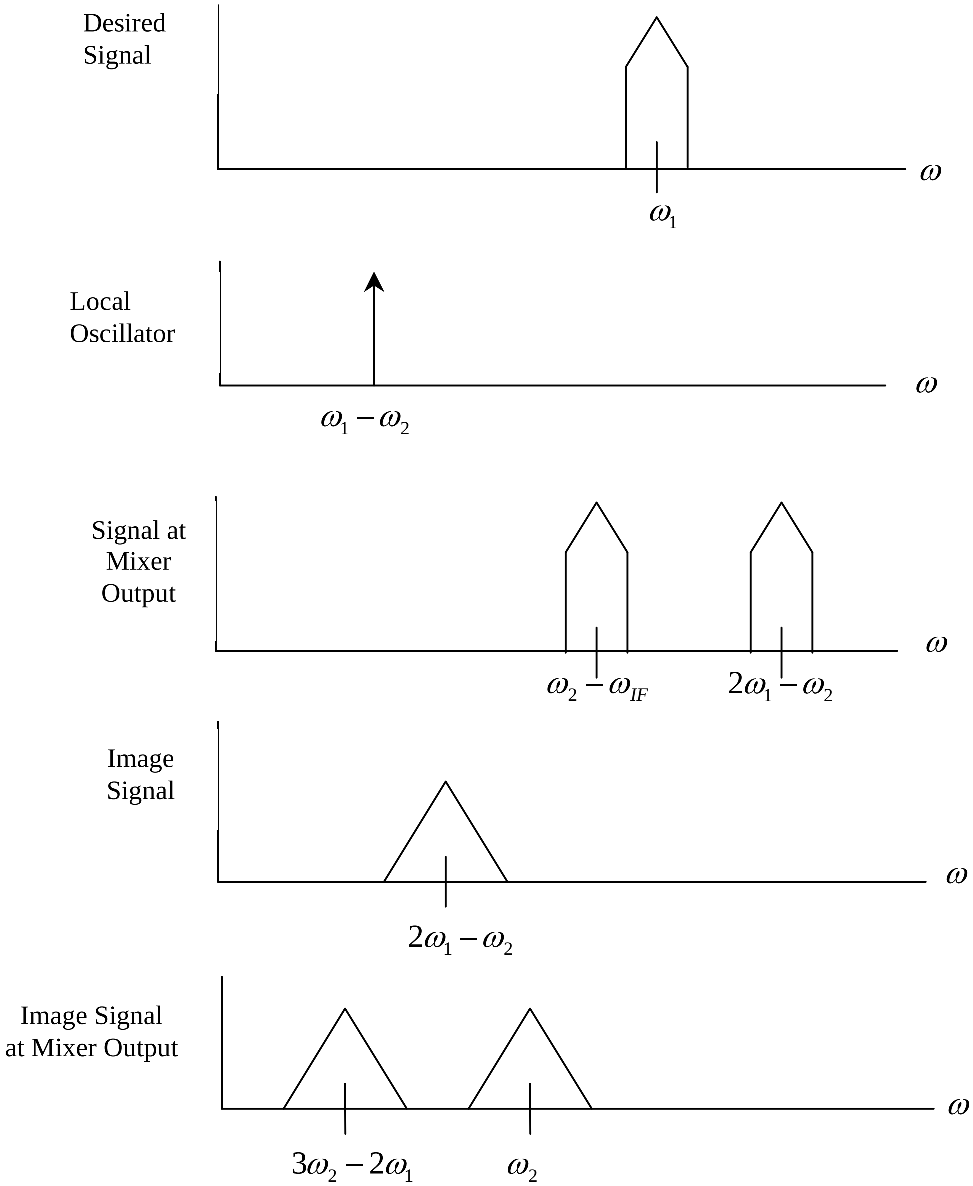

Figure 3.9: Result for Problem 3.20.

Since high-side tuning is used, the local oscillator frequency is

$$f_{LO} = f_i + f_{IF}$$

where f_i, the carrier frequency of the input signal, varies between 10 and 30 MHz. The ratio is

$$R = \frac{f_{IF} + 30}{f_{IF} + 10}$$

where f_{IF} is the IF frequency expressed in MHz. We make the following table

f_{IF},MHz	R
0.4	2.9231
0.6	2.8868
0.8	2.8519
1.0	2.8182
5.0	2.3333
10.0	2.0000

A plot of R as a function of f_{IF} is the required plot. However, we see that R is nearly constant and decreases as f_{IF} increases. Therefore, in this case, a large IF frequency should be used consistant with hardware requirements and availability.

Problem 3.22
With $f_{IF} = 455$ kHz we have for high-side tuning

$$\begin{aligned} f_{LO} &= f_i + f_{IF} = 1100 + 455 = 1555 \quad \text{kHz} \\ f_{IMAGE} &= f_i + 2f_{IF} = 1100 + 910 = 2010 \quad \text{kHz} \end{aligned}$$

and for low-side tuning we have

$$\begin{aligned} f_{LO} &= f_i - f_{IF} = 1100 - 455 = 665 \quad \text{kHz} \\ f_{IMAGE} &= f_i - 2f_{IF} = 1100 - 910 = 210 \quad \text{kHz} \end{aligned}$$

With $f_{IF} = 2500$ we have for high-side tuning

$$\begin{aligned} f_{LO} &= f_i + f_{IF} = 1100 + 2500 = 3600 \quad \text{kHz} \\ f_{IMAGE} &= f_i + 2f_{IF} = 1100 + 5000 = 6100 \quad \text{kHz} \end{aligned}$$

and for low-side tuning we have

$$\begin{aligned} f_{LO} &= f_i - f_{IF} = 1100 - 2500 = -1400 \quad \text{kHz} \\ f_{LO} &= 1400 \quad \text{kHz} \\ f_{IMAGE} &= f_i - 2f_{IF} = 1100 - 5000 = -3900 \quad \text{kHz} \\ f_{IMAGE} &= 3900 \quad \text{kHz} \end{aligned}$$

In the preceding development for low-side tuning, recall that the spectra are symmetrical about $f = 0$.

Problem 3.23

We assume single-tone interference as in the body of the text. Therefore we assume

$$x_c(t) = Am(t)\cos(2\pi f_c t) + A_i \cos(2\pi f_i t)$$

Therefore

$$x_c^2(t) = \frac{A^2}{2}m^2(t)\left[1 + \cos(4\pi f_c t)\right] + 2AA_i\cos(2\pi f_c t)\cos(2\pi f_i t) + \frac{A_i^2}{2}\left[1 + \cos(4\pi f_i t)\right]$$

Ignoring the dc terms, we have

$$x_c^2(t) = \frac{A^2}{2}m^2(t)\cos(4\pi f_c t) + AA_i\left[\cos(2\pi(f_c + f_i)t\right] + AA_i\left[\cos(2\pi(f_c - f_i)t\right] + \frac{A_i^2}{2}\cos(4\pi f_i t)$$

We therefore see that the squaring operation generates a component at twice the carrier frequency and a component at twice the interference frequency. A pair of sidebands are also generated about the carrier frequency, above and below the carrier frequency, and separated from the carrier frequency by the interference frequency.

Problem 3.24

From (3.96) the transfer function of the holding operation is

$$H(f) = \tau\frac{\sin(2\pi f\tau)}{2\pi f\tau}\exp(-j\pi f\tau)$$

Sampling reproduces this about $f = 0$ and all harmonics of the sampling frequency. We consider the terms about $f = 0$ and $f = f_s$. Ignoring the phase, which is linear and therefore only induces a delay, we have

$$H_1(f) = \tau\frac{\sin(2\pi f\tau)}{2\pi f\tau} * f_s\left[\delta(f) + \delta(f - f_s)\right]$$

Finally we have

$$H_1(f) = \tau f_s\frac{\sin(2\pi f\tau)}{2\pi f\tau} + \tau f_s\frac{\sin(2\pi(f - f_s)\tau)}{2\pi(f - f_s)\tau}$$

The equalizer will have the transfer function

$$H_{eq}(f) = \frac{1}{\tau f_s}\frac{2\pi f\tau}{\sin(2\pi f\tau)}$$

In order for the frequency response to be bounded we must have $2f\tau < 1$ or $\tau < \frac{1}{2f}$ where f is the highest frequency in the message signal. We must also have $2f\tau - 2f_s\tau > -1$. To plot this requires that f be set as a parameter.

Problem 3.25

Let A be the peak-to-peak value of the data signal. The peak error is 0.5% ($\pm$0.25%) and the peak-to-peak error is $0.01\,A$. The required number of quantizating levels is

$$\frac{A}{0.01A} = 100 \leq 2^n = q$$

so we choose $q = 128$ and $n = 7$. The bandwidth is

$$B = 2Wk\log_2 q = 2Wk(7)$$

The value of k is estimated by assuming that the speech is sampled at the Nyquist rate. Then the sampling frequency is $f_s = 2W = 8\,\text{kHz}$. Each sample is encoded into $n = 7$ pulses. Let each pulse be τ with corresponding bandwidth $\frac{1}{\tau}$. For our case

$$\tau = \frac{1}{nf_s} = \frac{1}{2Wn}$$

Thus the bandwidth is

$$\frac{1}{\tau} = 2Wn = 2W\log_2 q$$

and so $k = 1$. For $k = 1$

$$B = 2\,(8,000)\,(7) = 112 \text{ kHz}$$

Problem 3.26

The message signal is

$$m(t) = 3\sin[2\pi(10)t] + 4\sin[2\pi(20)t]$$

The derivative of the message signal is

$$\frac{dm\,(t)}{dt} = 60\pi\cos[2\pi\,(10)\,t] + 160\pi\cos[2\pi\,(20t)]$$

The maximum value of $dm\,(t)\,/dt$ is obviously 280π and the maximum occurs at $t = 0$. Thus

$$\frac{\delta_0}{T_s} \geq 220\pi$$

or

$$f_s \geq \frac{220\pi}{\delta_0} = \frac{220\pi}{0.05\pi} = 4400$$

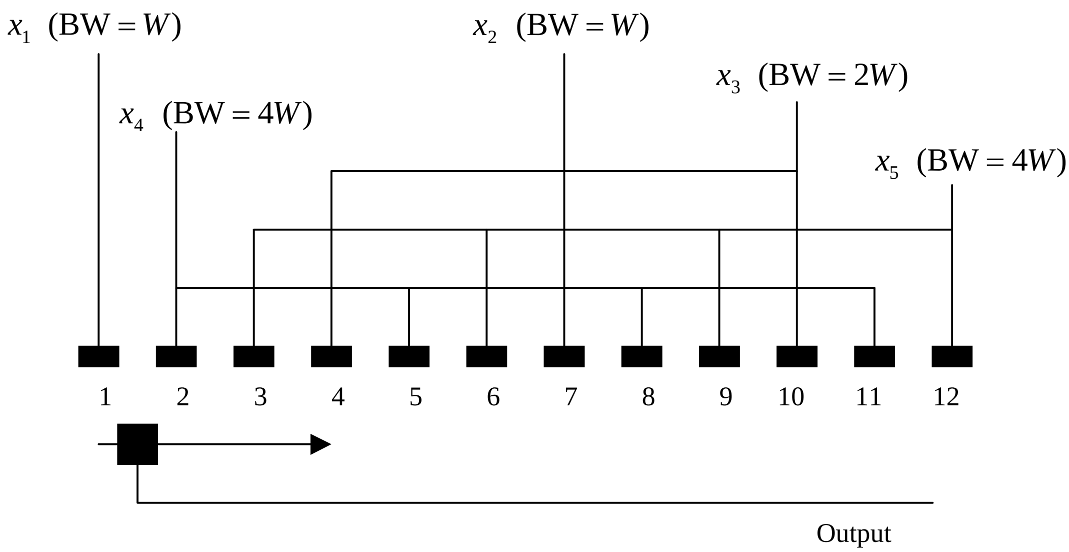

Figure 3.10: Commutator configuration for Problem 3.27.

Thus, the minimum sampling frequency is 4400 Hz.

Problem 3.27

One possible commutator configuration is illustrated in Figure 3.10. The signal at the point labeled "output" is the baseband signal. The minimum commutator speed is $2W$ revolutions per second. For simplicity the commutator is laid out in a straight line. Thus, the illustration should be viewed as it would appear wrapped around a cylinder. After taking the sample at point 12 the commutator moves to point 1. On each revolution, the commutator collects 4 samples of x_4 and x_5, 2 samples of x_3, and one sample of x_1 and x_2. The minimum transmission bandwidth is

$$\begin{aligned} B &= \sum_i W_i = W + W + 2W + 4W + 4W \\ &= 12W \end{aligned}$$

Problem 3.28

If the speed of the comutator is doubled from 2W to 4W, the sampling frequency is obviously doubled. The sampling frequencies are 4W, 4W, 8W, 16W, and 16W, respectively. The required transmission freqency is doubled. However, the guard frequencies are doubled.

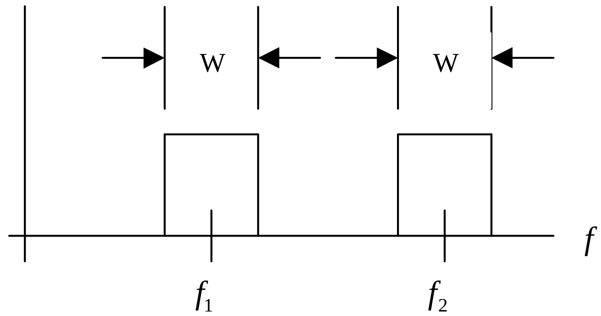

Figure 3.11: Single-sided spectrum for Problem 3.30.

Problem 3.29
The minimum baseband bandwidth is

$$B = W + W + 2W + 5W + 7W = 16W$$

A suitable scheme for this takes a bit of thought. However, the problem is not much different than when we consider the requirement that a sum of sinusoids be periodic.To be periodic, the frequencies must be commensurable (measurable by the same standard). In this case all bandwidths must be factors of a common bandwidth, which is 70W. Thus,the communator must have 70 distinct psitions.There are many possibilities one is shown in the Table below.

Single Bandwidth	Communator Positions	Note
$7W$	10, 20, 30, 40, 50, 60, 70	Spacing $= \frac{70}{7} = 10$
$5W$	1, 15, 29, 43, 57	Spacing $= \frac{70}{5} = 14$
$2W$	2, 37	Spacing $= \frac{70}{2} = 35$
W	3	Spacing $= \frac{70}{1} = 70$
W	4	Spacing $= \frac{70}{1} = 70$

The positions are arbitrary but the spacings are not. The communator rotates at a minimum rate of 2W. Note that there are many unused positions. These unused positions can be used for other signals as long as the commensurable conditions hold.klmmmmmmmmmmmmmmmmmmmmmmmm

Problem 3.30
The single-sided spectrum for $x(t)$ is shown in Figure 3.11.
From the definition of $y(t)$ we have

$$Y(s) = a_1 X(f) + a_2 X(f) * X(f)$$

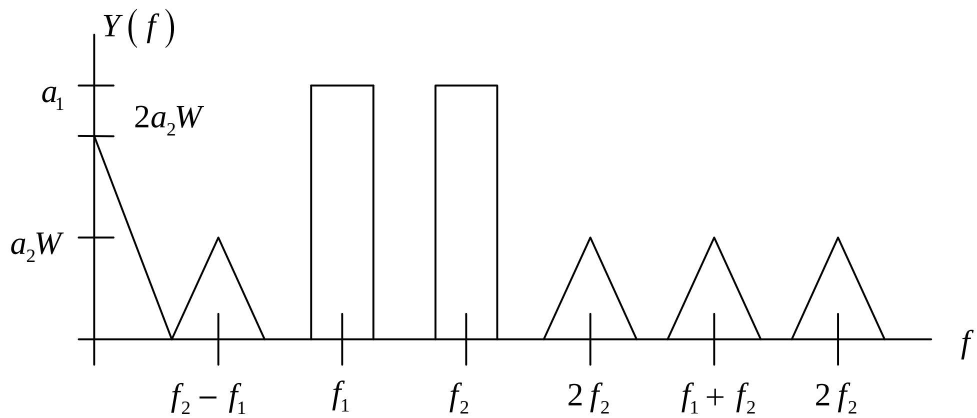

Figure 3.12: Output spectrum for Problem 3.54.

The spectrum for $Y(f)$ is given in Figure 3.12. Demodulation can be a problem since it may be difficult to filter the desired signals from the harmonic and intermodulation distortion caused by the nonlinearity. As more signals are included in $x(t)$, the problem becomes more difficult. The difficulty with harmonically related carriers is that portions of the spectrum of $Y(f)$ are sure to overlap. For example, assume that $f_2 = 2f_1$. For this case, the harmonic distortion arising from the spectrum centered about f_1 falls exactly on top of the spectrum centered about f_2.

3.2 Computer Exercises

Computer Exercise 3.1
The MATLAB program is

```
% File:  ce3_1.m
t = 0:0.001:1;
fm = 1;
fc =10;
m = 4*cos(2*pi*fm*t-pi/9) + 2*sin(4*pi*fm*t);
[minmessage,index] = min(m);
mncoefs = [4 2]/abs(minmessage)
mn = m/abs(minmessage);
```

```
plot(fm*t,mn,'k'),
grid, xlabel('Normalized Time'), ylabel('Amplitude')
mintime = 0.001*(index-1);
dispa = ['The minimum of m(t) is ', num2str(minmessage,'%15.5f'), ...
     'and falls at ',num2str(mintime,'%15.5f'), ' s.']; disp(dispa)
%
% Now we compute the efficiency.
%
mnsqave = mean(mn.*mn);      % mean-square value
a = 0.5;                     % modulation index
asq = a^2;
efficiency = asq*mnsqave/(1+asq*mnsqave);
dispb = ['The efficiency is ', num2str(efficency,'%15.5f'),' .']; disp(dispb)
%
% Now compute the carrier amplitude.
%
cpower = 50;                 % carrier power
camplitude = sqrt(2*cpower); % carrier amplitude
%
% Compute the magnitude and phase spectra.
%
xct = camplitude*(1+a*mn).*cos(2*pi*fc*t);
Npts = 1000;
fftxct = fft(xct,Npts)/Npts;
s1 = [conj(fliplr(fftxct(2:25))) fftxct(1:25)];
mags1 = abs(s1);
angles1 = angle(s1);
for k=1:length(s1)
   if mags1(k) <= 0.01
      angles1(k)=0;
   end
end
%
% Plot the magnitude and phase spectra.
%
figure
subplot(2,1,1), stem(-24:24,mags1,'.')
xlabel('frequency'); ylabel('magnitude')
subplot(2,1,2), stem(-24:24,angles1,'.')
xlabel('frequency'); ylabel('phase')
```

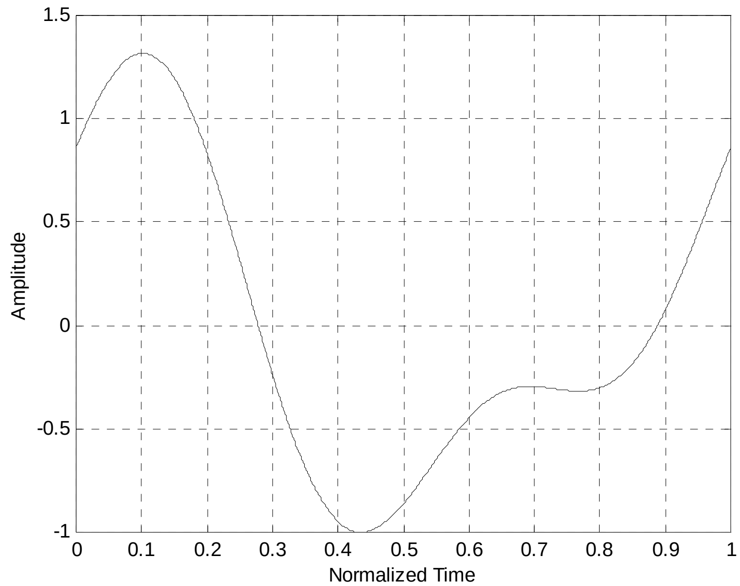

Figure 3.13: Message signal.

```
% End of script file.
```

Executing the program gives:

```
» ce3_1
mncoefs =
 0.9165 0.4583
The minimum of m(t) is -4.36424 and falls at 0.43500 s.
The efficiency is 0.11607 .
```

The program also generates Figures 3.13 and **??**.

Computer Exercise 3.2
The MATLAB code written for Computer Exercise 3.2 follows.

```
% File:  ce3_2.m
t = 0:0.001:1;
fm = 1;
```

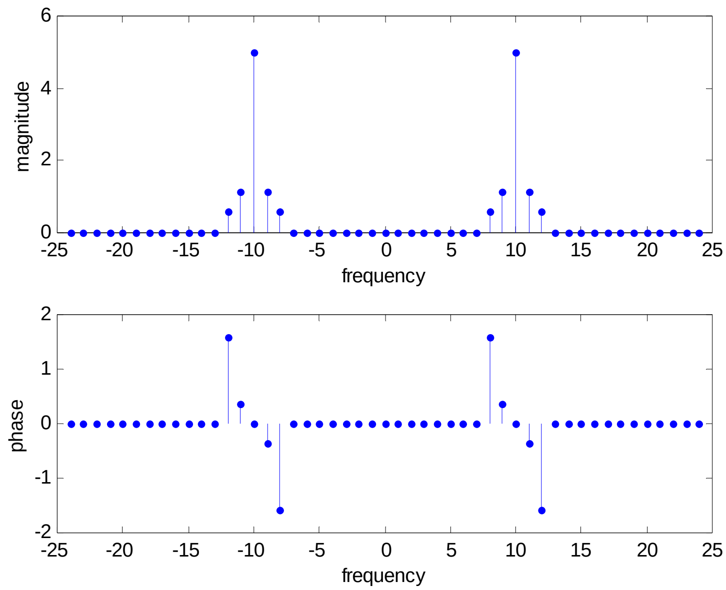

Figure 3.14: Spectra of AM signal.

```
fc =10;
m = 2*cos(2*pi*fm*t) + cos(4*pi*fm*t);
mhil = 2*sin(2*pi*fm*t) + sin(4*pi*fm*t);
xctusb = 0.5*m.*cos(2*pi*fc*t)-0.5*mhil.*sin(2*pi*fc*t);
xctlsb = 0.5*m.*cos(2*pi*fc*t)+0.5*mhil.*sin(2*pi*fc*t);
%
% Plot time-domain signals.
%
subplot(2,1,1), plot(t,xctusb)
xlabel('time'); ylabel('USB signal')
subplot(2,1,2), plot(t,xctlsb)
xlabel('time'); ylabel('LSB signal')
%
% Determine and plot spectra.
%
Npts = 1000;
fftxctu = fft(xctusb,Npts)/Npts;
fftxctl = fft(xctlsb,Npts)/Npts;
sxctusb = [conj(fliplr(fftxctu(2:25))) fftxctu(1:25)];
magsusb = abs(sxctusb);
sxctlsb = [conj(fliplr(fftxctl(2:25))) fftxctl(1:25)];
magslsb = abs(sxctlsb);
figure
subplot(2,1,1), stem(-24:24,magsusb)
xlabel('frequency'); ylabel('USB magnitude')
subplot(2,1,2), stem(-24:24,magslsb)
xlabel('frequency'); ylabel('LSB magnitude')
% End of script file.
```

Executing the code gives Figures 3.15 and 3.16.

Computer Exercise 3.3
In this computer example we investigate the demodulation of SSB using carrier resertion for several values of the constant k.for an assumed message signal.

```
% File:  ce3_3.m
t = 0:0.001:1;
fm = 1;
m = 2*cos(2*pi*fm*t) + cos(4*pi*fm*t);
mhil = 2*sin(2*pi*fm*t) + sin(4*pi*fm*t);
```

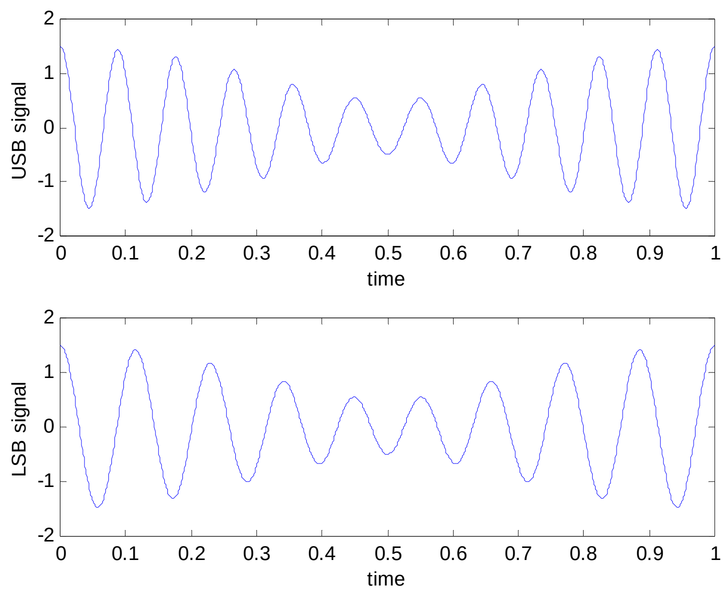

Figure 3.15: Time-domain waveforms.

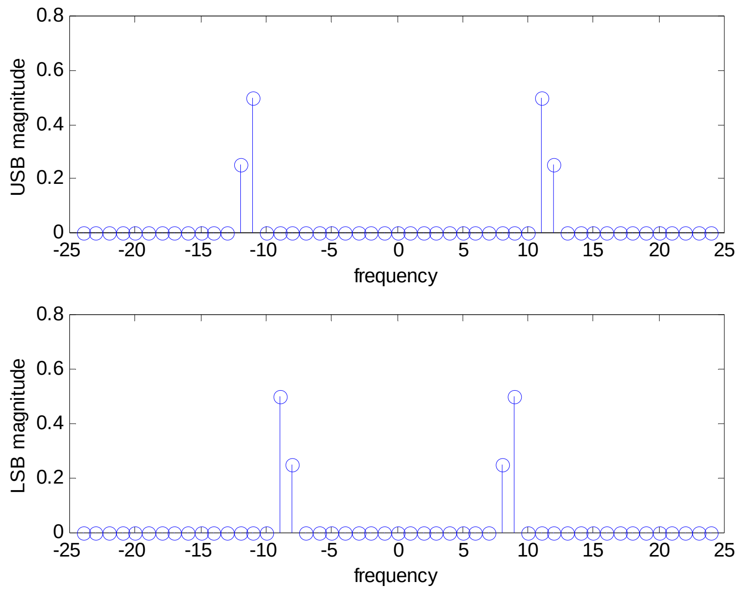

Figure 3.16: Amplitude spectra.

```
k = 1;
ydt = sqrt((k+m).*(k+m)+mhil.*mhil);
ydt1 = ydt-k;
k = 10;
ydt = sqrt((k+m).*(k+m)+mhil.*mhil);
ydt2 = ydt-k;
k = 100;
ydt = sqrt((k+m).*(k+m)+mhil.*mhil);
ydt3 = ydt-k;
subplot(3,1,1), plot(t,m,'k',t,ydt1,'k--')
ylabel('k=1')
subplot(3,1,2), plot(t,m,'k',t,ydt2,'k--')
ylabel('k=10')
subplot(3,1,3), plot(t,m,'k',t,ydt3,'k--')
ylabel('k=100')
% End of script file.
```

Executing the preceding MATLAB program gives the results illustrated in Figure 3.17. In the three plots the message signal is the solid line and the dashed line represents the output resulting from carrier reinsertion. It can be seen that the demodulated output improves for increasing k and that the distortion is negligible for $k = 100$.

Computer Exercise 3.4

Two MATLAB programs are generated for this Computer Exercise, `ce3_4a` ancd `ce3_4b`. The first of these two programs illustrates the message signal, the VSB signal and the VSB signal after carrier reinsertion. The second program illustrates the spectrum of a VSB signal The MATLAB code for the first program is

```
% File:  ce3_4a.m
 clear all
 t = 0:0.001:1;
 m = cos(2*pi*t)-cos(4*pi*t)+cos(6*pi*t);
 e1 = 0.64; e2 = 0.78; e3 = 0.92;
 fc = 25; A = 100;
 ct = A*cos(2*pi*fc*t);
 xct = e1*cos(2*pi*(fc+1)*t)+(1-e1)*cos(2*pi*(fc-1)*t)-...
 e2*cos(2*pi*(fc+2)*t)-(1-e2)*cos(2*pi*(fc-2)*t)+...
 e3*cos(2*pi*(fc+3)*t)+(1-e3)*cos(2*pi*(fc-3)*t)+...
 cos(2*pi*(fc+4)*t)+cos(2*pi*(fc+5)*t)+cos(2*pi*(fc+6)*t);
```

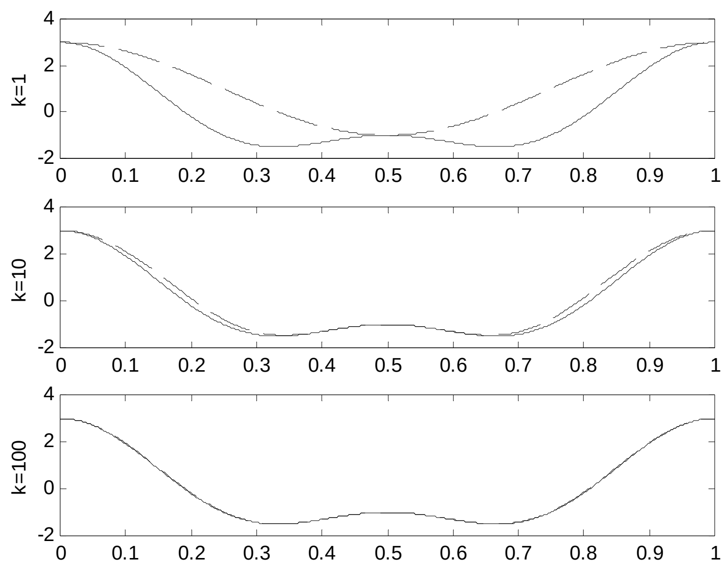

Figure 3.17: Illustration of demodulation resertion for $k = 1$, 10, and 100.

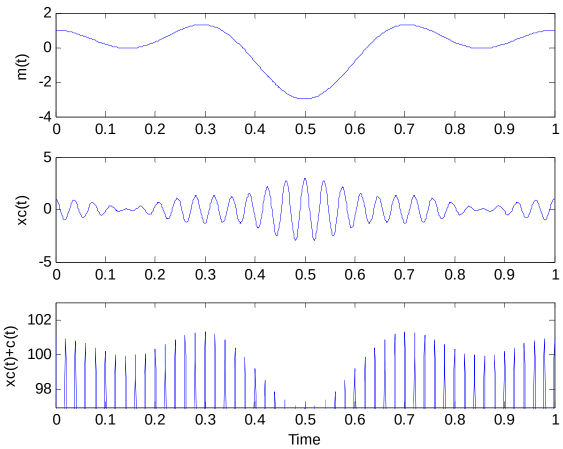

Figure 3.18: Illustration of VSB (message signal, VSB signal, and VSB signal plus carrier reinsertion).

```
xdem = abs(xct+ct);
subplot(3,1,1), plot(t,m), ylabel('m(t)')
subplot(3,1,2), plot(t,xct), ylabel('xc(t)')
subplot(3,1,3), plot(t,xdem), ylabel('xc(t)+c(t)')
xlabel('Time'), axis([0 1 97 103])
% End of script file.
```

The result of demodulation using carrier reinsertion if shown in the bottom pane. The message signal can be recovered using a dc-coupled envelope detection.

We now consider the spectrum of a VSB signal. We change the message signal to include a carrier component and to include six components above the carrier. In addition, all components in the message signal are set equal in order to illustrate the action of the VSB filter. The MATLAB program is

```
% File:  ce3_4b.m
clear all
t = 0:0.001:1;
e1 = 0.64; e2 = 0.78; e3 = 0.92;
fc = 25;
ct = 0.5*cos(2*pi*fc*t);
```

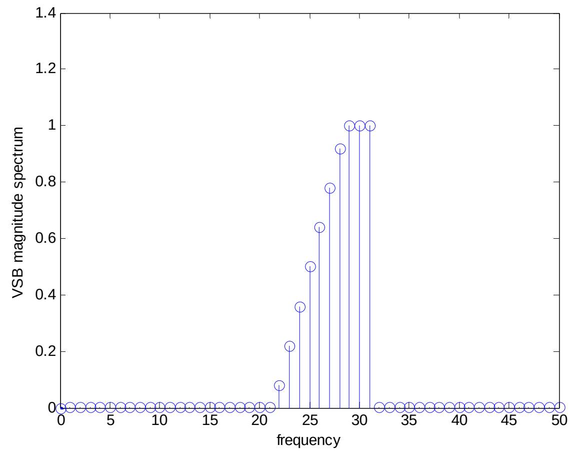

Figure 3.19: VSB spectrum.

```
xct = ct+e1*cos(2*pi*(fc+1)*t)+(1-e1)*cos(2*pi*(fc-1)*t)-...
e2*cos(2*pi*(fc+2)*t)-(1-e2)*cos(2*pi*(fc-2)*t)+...
e3*cos(2*pi*(fc+3)*t)+(1-e3)*cos(2*pi*(fc-3)*t)+...
cos(2*pi*(fc+4)*t)+cos(2*pi*(fc+5)*t)+cos(2*pi*(fc+6)*t);
Npts = 1000;
fftxctu = fft(xct,Npts)/Npts;
magvsb=2*abs(fftxctu(1:51));
freq=0:50;
stem(freq, magvsb)
xlabel('frequency'); ylabel('VSB magnitude spectrum')
% End of script file.
```

The resulting spectrum follows. Note that there is a carrier component at $f = 25$. There are three attenuated components above the carrier and three non-attenuated components. Below the carrier there are the complements of the three attenuated components above the carrier.

Computer Exercise 3.5

The MATLAB program used appears below. Given the annotations, the code should be understandable. (Thanks to Dr. John Tranter of the University of Minnesota for his help on this problem.)

```
% File CE3.5.m
clear all; close all;
f1 = 70; f2 = 300; % sine frequencies
fmin = min(f1,f2); % use lowest freq.  for 1 cycle on time axis
A1 = 3; A2 = 1.5; % sine amplitudes
samp = [15 60]; % sample period (in MATLAB increments)
N = 1500; % discretize signal with N points
hl = 1; % staircase amplitude
t = 1/N/fmin:1/N/fmin:1/fmin; % x-axis in seconds
x = A1*sin(2*pi*f1*t)+A2*sin(2*pi*f2*t); % build our signal
for kk = 1:2
    d(1:samp(kk))=0;

    for ii=samp(kk)+1:samp(kk):N
        dd(ii) = sign(x(ii) - d(ii-samp(kk))); % this is our delta
        d(ii:ii+samp(kk)-1) = d(ii-samp(kk)) + hl*dd(ii);
    end
    d = d(1:N); % trim/pad extra samples (if sample
    dd = [dd zeros(1,N-numel(dd))]; % period does not exactly divide N)

    subplot(3,2,[kk kk+2]),plot(t,x,'k'),...
           axis([t(1) t(N) min([x d]) max([x d])])
    hold on,stairs(t,d,'r'),xlabel('Time (s)')
    if (kk == 1)
        str = 'no slope overload)';
    else
        str = 'slope overload)';
    end
    title(strcat('Stairstep Approximation (',str));
    subplot(3,2,kk+4),stem(t(1:samp:end),dd(1:samp:end),'marker','none')
    axis([t(1) t(N) -1.2 1.2]),xlabel('Time (s)')
   title(strcat('Delta Modulation (',str));
    clear d; clear dd; % diff.  sample rate = diff.  dimensions...
end
% This is a neat trick that automatically maximizes your figure
screen_size = get(0, 'ScreenSize');
set(figure(1), 'Position', [0 0 screen_size(3) screen_size(4) ] );
% End of script file.
```

Executing the program yields the following output.

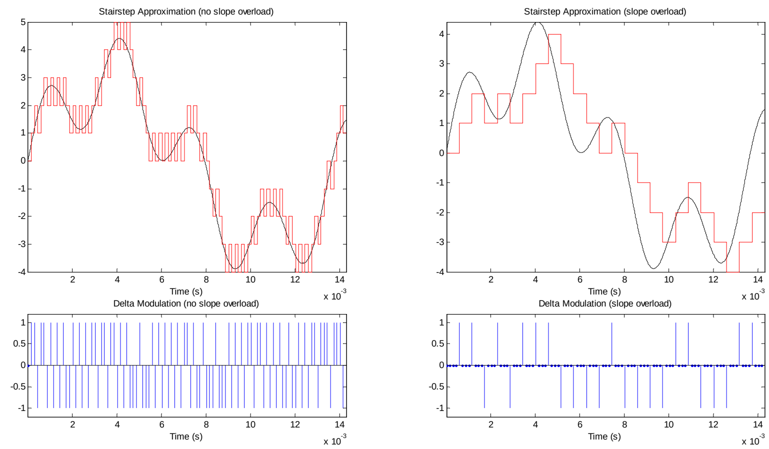

Figure 3.20: Delta modulation without slope overload (left two panes) and with slope overload (right two panes).

Computer Exercise 3.6

The MATLAB code follows.

```
% File ce3_6.m
clear all
t=0:0.001:(2-0.001);
m=zeros(1,2000);
x=zeros(1,2000);
m=cos(2*pi*t)-cos(4*pi*t)+cos(6*pi*t);
tau=5;
for k=1:2000
    if rem(k,10)==0
        x(k)=1;
    end
end
xsamp=m.*x;
h=zeros(1,10);
for k=1:tau
    h(k)=1;
end
xsh=conv(xsamp,h);
```

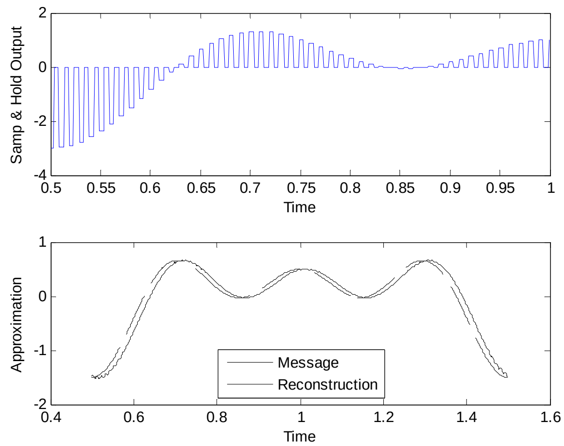

Figure 3.21: PAM and reconstruction. (Top pane - Sample and hold output. Bottom pane - original message signal and reconstructured signal.)

```
[B,A]=butter(3,0.05);
y=filter(B,A,xsh);
subplot(2,1,1), plot(t(500:1000),xsh(500:1000))
axis([0.5 1 -4 2])
xlabel('Time'); ylabel('Samp & Hold Output')
subplot(2,1,2), plot(t(500:1500),y(500:1500),'k',...
 t(500:1500),0.5*m(500:1500),'k--')
xlabel('Time'); ylabel('Approximation')
legend('Message','Reconstruction','Location','South')
% End of script file.
```

Note that for this example, the simulation sampling frequency is 1000 Hz. The message signal is sampled every 10 simulation steps. Thus $f_s = 100$ Hz. The width of the PAM pulse is 5 simulation steps so $\tau = 0.005$ s. Thus $\tau f_s = 0.5$. The 3 dB frequency of the Butterworth filter is 0.5(1000/2)=250 Hz. Other values for τf_s are easily tried.