1、 $f(x)=y'=5y$, $h=0.2$

$$\begin{cases} y_{n+1}=y_n+h\dfrac{f(x_n,y_n)+f(x_{n+1},y^*_{n+1})}{2} \\ y^*_{n+1}=y_n+hf(x_n,y_n) \end{cases}$$

$$y_1^*=1+0.2(5\cdot1)=2$$

$$y_1=1+0.2\left(\frac{5+(5\cdot2)}{2}\right)=1+0.1(15)=2.5$$

$$y_2^*=2.5+0.2(5\cdot2.5)=5$$

$$y_2=2.5+0.2\left(\frac{(5\cdot2.5)+(5\cdot5)}{2}\right)=2.5+0.1(37.5)=6.25 \quad \#$$

2、

$$X=\begin{pmatrix}2e^t & -e^{-2t}\\ -e^t & e^{-2t}\end{pmatrix},\quad X'=\begin{pmatrix}2e^t & 2e^{-2t}\\ -e^t & -2e^{-2t}\end{pmatrix}$$

$$右式=\begin{pmatrix}4 & 6\\ -3 & -5\end{pmatrix}X=\begin{pmatrix}4 & 6\\ -3 & -5\end{pmatrix}\begin{pmatrix}2e^{-t} & -e^{-2t}\\ -e^t & e^{-2t}\end{pmatrix}$$

$$=\begin{pmatrix}2e^t & 2e^{-2t}\\ -e^t & -2e^{-2t}\end{pmatrix}=X',\ 左式=右式$$

$$W(x_1,x_2)=\begin{vmatrix}2e^t & -e^{-2t}\\ -e^t & e^{-2t}\end{vmatrix}=2e^{-t}-e^{-t}=e^{-t}\neq0$$

$\Rightarrow x_1, x_2$ linearly independent

3、 $x' = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

$\det(A - \lambda I) = \begin{pmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$

$\lambda = 1 . 1$

$\lambda = 1 ,\ \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} k = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,\ k = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} P = \begin{pmatrix} -1 \\ 2 \end{pmatrix} . P = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$x = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} t e^t + \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^t \right]$

$x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ -3c_2 \end{pmatrix}$

$\Rightarrow c_1 = -1 , c_2 = -1$

$x = -1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t - \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} t e^t + \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^t \right]$

$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -2 \end{pmatrix} t e^t + \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^t$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -2 \end{pmatrix} t e^t = \begin{pmatrix} t \\ 1-2t \end{pmatrix} e^t$ #

4、 $X' = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} X + \begin{pmatrix} -3 \\ 4e^{t} \end{pmatrix}, \ X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\det(AI - \lambda) = \begin{vmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix} = 8 - 2\lambda - 4\lambda + \lambda^2 - 3$$

$$= (\lambda - 5)(\lambda - 1), \quad \lambda = 5, 1$$

① $\lambda = 1$, $\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} k = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

② $\lambda = 5$, $\begin{pmatrix} -3 & -1 \\ -3 & -1 \end{pmatrix} p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ p = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$$\Phi(t) = \begin{pmatrix} e^{t} & e^{5t} \\ e^{t} & -3e^{5t} \end{pmatrix}$$

$$\Phi^{-1}(t) = \frac{1}{-4e^{6t}} \begin{pmatrix} -3e^{5t} & -e^{5t} \\ -e^{t} & e^{t} \end{pmatrix} = \begin{pmatrix} \frac{3}{4}e^{-t} & \frac{1}{4}e^{-t} \\ \frac{1}{4}e^{-5t} & -\frac{1}{4}e^{-5t} \end{pmatrix}$$

$$X_p = \Phi(t) \int \Phi^{-1}(t) F(t)\, dt = \begin{pmatrix} e^{t} & e^{5t} \\ e^{t} & -3e^{5t} \end{pmatrix} \int \begin{pmatrix} \frac{3}{4}e^{-t} & \frac{1}{4}e^{-t} \\ \frac{1}{4}e^{-5t} & -\frac{1}{4}e^{-5t} \end{pmatrix} \begin{pmatrix} -3 \\ 4e^{t} \end{pmatrix} dt$$

$$= \begin{pmatrix} e^{t} & e^{5t} \\ e^{t} & -3e^{5t} \end{pmatrix} \begin{pmatrix} \frac{9}{4}e^{-t} + t \\ \frac{3}{20}e^{-5t} + \frac{1}{4}e^{-4t} \end{pmatrix} = \begin{pmatrix} \frac{12}{5} + te^{t} + \frac{1}{4}e^{t} \\ \frac{9}{5} + te^{t} - \frac{3}{4}e^{t} \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{t} & e^{5t} \\ e^{t} & -3e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{12}{5} + te^{t} + \frac{1}{4}e^{t} \\ \frac{9}{5} + te^{t} - \frac{3}{4}e^{t} \end{pmatrix}, \ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{4} \\ -\frac{2}{5} \end{pmatrix}$$

$$X = X_c + X_p = \begin{pmatrix} e^{t} & e^{5t} \\ e^{t} & -3e^{5t} \end{pmatrix} \begin{pmatrix} -\frac{5}{4} \\ -\frac{2}{5} \end{pmatrix} + \begin{pmatrix} \frac{12}{5} + te^{t} + \frac{1}{4}e^{t} \\ \frac{9}{5} + te^{t} - \frac{3}{4}e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{5}e^{5t} - e^{t} + te^{t} + \frac{12}{5} \\ \frac{6}{5}e^{5t} - 2e^{t} + te^{t} + \frac{9}{5} \end{pmatrix}$$

#

5、 $A = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{pmatrix}$

$$(sI-A) = \begin{pmatrix} s-3 & -4 & -5 \\ 0 & s-5 & -4 \\ 0 & 0 & s-3 \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s-3)^2(s-5)} \begin{pmatrix} (s-5)(s-3) & 4(s-3) & 5s-9 \\ 0 & (s-3)^2 & 4(s-3) \\ 0 & 0 & (s-3(s-5) \end{pmatrix}$$

$$e^{At} = \mathcal{L}\{(sI-A)^{-1}\} = \begin{pmatrix} \frac{1}{s-3} & \frac{4}{(s-3)(s-5)} & \frac{5s-9}{(s-3)^2(s-5)} \\ 0 & \frac{1}{s-5} & \frac{4}{(s-3)(s-5)} \\ 0 & 0 & \frac{1}{s-3} \end{pmatrix}$$

$$\frac{5s-9}{(s-3)^2(s-5)} = \frac{-3}{(s-3)^2} + \frac{A}{(s-3)} + \frac{4}{(s-5)}$$

$$= \frac{-3(s-5)}{(s-3)^2(s-5)} + \frac{A(s-3)(s-5)}{(s-3)^2(s-5)} + \frac{4(s-3)^2}{(s-3)^2(s-5)}$$

$$\Rightarrow (-3s+15) + A(s^2-8s+15) + 4(s^2-6s+9) = 5s-9$$

$\Rightarrow A+4=0$

$\Rightarrow A=-4$ 代回

$$\therefore x = e^{At}c = \begin{pmatrix} e^{3t} & -2e^{3t}+2e^{5t} & 4e^{5t}-(3t+4)e^{3t} \\ 0 & e^{5t} & -2e^{3t}+2e^{5t} \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

#

6、 $f_1(x)=e^x$, $f_2(x)=xe^{-x}-e^{-x}$

$$\int_a^b f_1(x)f_2(x)dx=\int_0^2 e^x(xe^{-x}-e^{-x})dx$$

$$=\int_0^2(x-1)dx=\frac{x^2}{2}-x\Big|_0^2=(2-2)-0=0$$

$\Rightarrow$ orthogonal #

7、 $f(x)=\begin{cases}1, & -1\le x<0\\ x, & 0\le x<1\end{cases}$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\pi x)+b_n\sin(n\pi x))$$

$$a_0=\int_{-1}^{1}f(x)dx=\int_{-1}^{0}1\,dx+\int_0^1 x\,dx=x\Big|_{-1}^{0}+\frac{1}{2}x^2\Big|_0^1=\frac{3}{2}$$

$$a_n=\int_{-1}^{1}f(x)\cos n\pi x\,dx=\int_{-1}^{0}\cos n\pi x\,dx+\int_0^1 x\cdot\cos n\pi x\,dx$$

$$=\frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0}+\frac{x}{n\pi}\sin(n\pi x)\Big|_0^1-\int_0^1\frac{1}{n\pi}\sin(n\pi x)dx$$

$$=0+\frac{1}{n\pi}\sin(n\pi)+\frac{1}{(n\pi)^2}\cos(n\pi x)\Big|_0^1=\frac{(-1)^n}{(n\pi)^2}-\frac{1}{(n\pi)^2}$$

$$b_n=\int_{-1}^{1}f(x)\sin(n\pi x)dx=\int_{-1}^{0}\sin(n\pi x)dx+\int_0^1 x\sin(n\pi x)dx$$

$$=\frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0}+\left[-\frac{x}{n\pi}\cos(n\pi x)+\frac{1}{(n\pi)^2}\sin(n\pi x)\right]\Big|_0^1$$

$$=\frac{-1}{n\pi}+\frac{(-1)^n}{n\pi}-\frac{(-1)^n}{n\pi}+0$$

$$=\frac{-1}{n\pi}$$

$$\Rightarrow \frac{3}{4}+\sum_{n=1}^{\infty}\frac{-1+(-1)^n}{n^2\pi^2}\cos(n\pi x)-\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(n\pi x)$$

#

8、 $y''+4y=[1-\mu(t-\pi)]\sin 2t$, $y(0)=1$, $y'(0)=2$

$$s^2Y(s)-s-2+4Y(s)=\frac{2}{(s^2+4)}-\frac{2e^{-\pi s}}{s^2+4}$$

$$Y(s)=\frac{2}{(s^2+4)^2}-\frac{2e^{-\pi s}}{(s^2+4)^2}+\frac{s}{s^2+4}+\frac{2}{s^2+4}$$

$$\Rightarrow \frac{1}{8}(\sin 2t-2t\cos 2t)-\frac{1}{8}(\sin 2t+2t\cos 2t)\mu(t-\pi)+\cos 2t+\sin 2t$$

$$=\frac{9}{8}\sin 2t-\frac{t}{4}\cos 2t+\cos 2t-\left(\frac{\sin 2t}{8}-\frac{t}{4}\cos 2t\right)u(t-\pi)$$

#

9、$Y(s)=\dfrac{s}{(s+1)(s^2+4)}$, $\mathcal{L}\{f*g\}=F(s)G(s)$

$\mathcal{L}\{F(s)G(s)\}^{-1}=f*g$

$$f*g=\int_0^t e^{-\tau}\cos2(t-\tau)\,d\tau$$

$u=e^{-\tau}$, $du=-e^{-\tau}d\tau$

$V=\dfrac{\sin2(t-\tau)}{-2}$, $dv=\cos2(t-\tau)\,d\tau$

$$\Rightarrow e^{-\tau}\frac{\sin2(t-\tau)}{-2}\Bigg|_0^t-\frac{1}{2}\int_0^t e^{-\tau}\sin2(t-\tau)\,d\tau$$

$$\Rightarrow \frac{1}{2}\sin2t-\frac{1}{2}\left[e^{-\tau}\left(\frac{\cos2(t-\tau)}{2}\right)\Bigg|_0^t+\int_0^t e^{-\tau}\frac{\cos2(t-\tau)}{2}\,d\tau\right]$$

$$\Rightarrow \frac{1}{2}\sin2t-\frac{1}{4}e^{-t}+\frac{\cos2t}{4}-\frac{1}{4}\int_0^t e^{-\tau}\cos2(t-\tau)\,d\tau$$

$$\therefore \int_0^t e^{-\tau}\cos2(t-\tau)\,d\tau=-\frac{1}{4}\int_0^t e^{-\tau}\cos2(t-\tau)\,d\tau+\frac{1}{2}\sin2t-\frac{1}{4}e^{-t}+\frac{\cos2t}{4}$$

$$\frac{5}{4}\int_0^t e^{-\tau}\cos2(t-\tau)\,d\tau=\frac{1}{2}\sin2t-\frac{1}{4}e^{-t}+\frac{\cos2t}{4}$$

$$\int_0^t e^{-\tau}\cos2(t-\tau)\,d\tau=\frac{2}{5}\sin2t-\frac{1}{5}e^{-t}+\frac{1}{5}\cos2t$$

#

9、 $Y(s)=\dfrac{s}{(s+1)(s^2+4)}$

$$=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$A(s^2+4)+Bs^2+(B+C)s+C$$

$$=As^2+4A+Bs^2+(B+C)s+C$$

$$=(A+B)s^2+(B+C)s+4A+C$$

$$\begin{cases}A+B=0\\B+C=1\\4A+C=0\end{cases}\Rightarrow\begin{cases}A=-\frac{1}{5}\\B=\frac{1}{5}\\C=\frac{4}{5}\end{cases}$$

$$\Rightarrow\frac{-\frac{1}{5}}{s+1}+\frac{\frac{1}{5}s+\frac{4}{5}}{s^2+4}$$

$$\Rightarrow\frac{-1}{5}e^{-t}+\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t$$

#

10、 $y''-2y'=1+\delta(t-2),\ y(0)=0,\ y'(0)=1$

$$s^2Y(s)-\cancel{sy(0)}^{0}-y'(0)-2\left(sY(s)-\cancel{y(0)}^{0}\right)=\frac{1}{s}+e^{-2s}$$

$$\Rightarrow(s^2-2s)Y(s)=\frac{1}{s}+e^{-2s}+1$$

$$Y(s)=\frac{1}{s^2(s-2)}+\frac{e^{-2s}}{s(s-2)}+\frac{1}{s(s-2)}$$

$$=\frac{-\frac{1}{4}s-\frac{1}{2}}{s^2}+\frac{\frac{1}{4}}{s-2}+\frac{-\frac{1}{2}}{s}e^{-2s}+\frac{\frac{1}{2}}{s-2}e^{-2s}+\frac{-\frac{1}{2}}{s}+\frac{\frac{1}{2}}{s-2}$$

$$=-\frac{1}{4}\frac{1}{s}-\frac{1}{2}\frac{1}{s^2}+\frac{1}{4}\frac{1}{s-2}-\frac{1}{2}\frac{1}{s}e^{-2s}+\frac{1}{2}\frac{1}{s-2}e^{-2s}+\left(-\frac{1}{2}\right)\frac{1}{s}+\frac{1}{2}\left(\frac{1}{s-2}\right)$$

$$\mathcal{L}^{-1}\Rightarrow-\frac{1}{4}-\frac{1}{2}t+\frac{1}{4}e^{2t}-\frac{1}{2}\mathcal{U}(t-2)+\frac{1}{2}e^{2(t-2)}\mathcal{U}(t-2)-\frac{1}{2}+\frac{1}{2}e^{2t}$$

$$y(t)\Rightarrow-\frac{3}{4}-\frac{1}{2}t+\frac{3}{4}e^{2t}-\frac{1}{2}\mathcal{U}(t-2)+\frac{1}{2}e^{2(t-2)}\mathcal{U}(t-2)$$

#