Calculus (I) : Final Exam (1/4/2021, 8:15 - 11:45 AM)

* *The Exam includes 5 problems with 114 points in total.*
* *Please show your work for partial credits.*

Problem	Score
1. (32 pts.)	
2. (8 pts.)	
3. (16 pts.)	
4. (10 pts.)	
5. (48 pts.)	
Total (114 pts.)	

Name: Solution

Student ID #: ____________________

Department: ____________________

1. **(32 pts.)** Let R be the region enclosed by curves $y = x^4$ and $y = 2 - |x|$. And, S is the solid obtained by rotating the region R about $x = 1$.

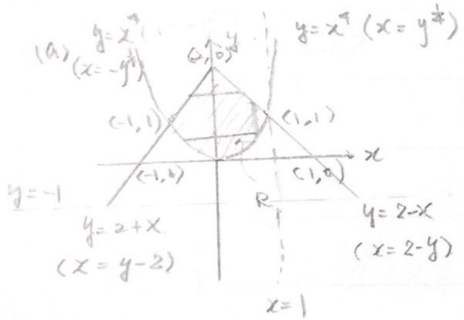
(a) Sketch the region R and find area of R.

(b) Use Washers method to find the volume of S.

(c) Use Cylindrical Shell method to find the volume of S.

(d) Find the volume of the solid obtained by rotating the region R about $y = -1$.

(a)



Area of R

$$= \int_{-1}^{0} (2 + x - x^4)\, dx + \int_{0}^{1} (2 - x - x^4)\, dx$$

$$= 2x\Big|_{-1}^{1} - \frac{x^5}{5}\Big|_{-1}^{1} + \frac{x^2}{2}\Big|_{-1}^{0} - \frac{x^2}{2}\Big|_{0}^{1}$$

$$= 4 - \frac{2}{5} - \frac{1}{2} - \frac{1}{2} = \frac{13}{5} \quad *$$

(b) Integrate w.r.t. y

$$\int_0^1 \pi\left[1 + (-y^{\frac{1}{4}})\right]^2 - \pi\left[1 - y^{\frac{1}{4}}\right]^2 dy + \int_1^2 \pi\left[1 - (y-2)\right]^2 - \pi\left[1 - (2-y)\right]^2 dy$$

$$= 4\pi \int_0^1 y^{\frac{1}{4}}\, dy + 4\pi \int_1^2 (2 - y)\, dy$$

$$= 4\pi \cdot \frac{4}{5} y^{\frac{5}{4}}\Big|_0^1 + 4\pi \cdot \left(2y - \frac{y^2}{2}\right)\Big|_1^2$$

$$= \frac{16}{5}\pi + 2\pi$$

$$= \frac{26}{5}\pi \quad *$$

(c) Integrate w.r.t. x.

$$\int_{-1}^{0} 2\pi(1-x)(2+x-x^4)\,dx + \int_0^1 2\pi(1-x)(2-x-x^4)\,dx$$

$$= 2\pi\int_{-1}^{0}(2-x-x^2-x^4+x^5)\,dx + 2\pi\int_0^1(2-3x+x^2-x^4+x^5)\,dx$$

$$= 2\pi\left[2x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^6}{6}\right]_{-1}^{0} + 2\pi\left(2x-\frac{3}{2}x^2+\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^6}{6}\right)\Big|_0^1$$

$$= 2\pi\left[2+\frac{1}{2}-\frac{1}{3}-\frac{1}{5}-\frac{1}{6}\right] + 2\pi\left[2-\frac{3}{2}+\frac{1}{3}-\frac{1}{5}+\frac{1}{6}\right]$$

$$= 2\pi\left(3-\frac{2}{5}\right)$$

$$= \frac{26}{5}\pi$$

(d) Integrate w.r.t. x (Washer's method)

Since the solid is symmetric with respect to y-axis

$$2\int_0^1\left(\pi[(2-x)-(-1)]^2 - \pi[x^4-(-1)]^2\right)dx$$

$$= 2\pi\int_0^1(8-6x+x^2-2x^4-x^8)\,dx$$

$$= 2\pi\left(8-3+\frac{1}{3}-\frac{2}{5}-\frac{1}{9}\right)$$

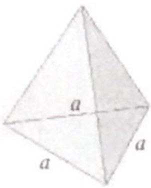
$$= \frac{434\pi}{45}$$

2. **(8 pts.)** Find the average value of the function $f(x) = 3sin^2x$ in the interval $[0, \pi]$.

3. **(16 pts.)** Determine whether the improper integral converges or diverges, and if converges, find its value.

(a) $\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$ **(b)** $\int_2^\infty \frac{1}{2x-\ln x}\,dx$

4. **(10 pts.)** Apply an appropriate definite integral to find the volume of a pyramid with height 6 and base an equilateral triangle with side $a = 8$.



2. $$\frac{1}{\pi-0}\int_0^{\pi} 3\sin^2 x\,dx = \frac{3}{\pi}\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx$$

$$= \frac{3}{\pi}\left[\frac{x}{2}-\frac{\sin 2x}{4}\right]\Big|_0^{\pi} = \frac{3}{2}$$ ※

3. (a) $$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}(1+x)}\,dx$$

$u = \sqrt{x} \Rightarrow du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$$\int \frac{1}{\sqrt{x}(1+x)}\,dx = 2\int \frac{1}{1+u^2}\,du = 2\tan^{-1}\sqrt{x} + C$$

$$\lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{t\to 0^+}\left[2\tan^{-1}1 - 2\tan^{-1}\sqrt{t}\right] = \frac{\pi}{2}$$

$$\lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\tan^{-1}1\right] = \frac{\pi}{2}.$$

Hence, $\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi$. is convergent ※

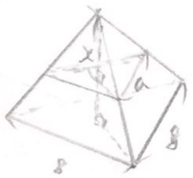
(b) $2x - \ln x < 2x$ for $x \geq 2 \Rightarrow \frac{1}{2x - \ln x} > \frac{1}{2x}$ for $x \geq 2$

Hence, $\int_2^\infty \frac{1}{2x - \ln x}\,dx > \int_2^\infty \frac{1}{2x}\,dx$

Since $\int_2^\infty \frac{1}{2x}\,dx = \lim_{t\to\infty} \left[\frac{1}{2}\ln t - \frac{1}{2}\ln 2\right] = \infty$,

By Comparison Thm, $\underline{\underline{\int_2^\infty \frac{1}{2x - \ln x}\,dx \text{ is divergent.}}}$ #

4.



Let x be the height for the pyramid and the base with side a, then $0 \leq x \leq 6$ and $0 \leq a \leq 8$.

The area of the cross-section with side a is $\frac{1}{2} \times a \times \frac{\sqrt{3}}{2} a = \frac{\sqrt{3}}{4} a^2$

And, $\frac{x}{6} = \frac{a}{8} \Rightarrow a = \frac{4}{3}x$

Thus, the volume of the pyramid is

$$\int_0^6 \frac{\sqrt{3}}{4} a^2\,dx = \int_0^6 \frac{\sqrt{3}}{4}\left(\frac{4}{3}x\right)^2 dx$$

$$= \frac{4\sqrt{3}}{9} \cdot \frac{x^3}{3}\Big|_0^6$$

$$= \underline{\underline{32\sqrt{3}}}$$ #

5. (48 pts.) Evaluate the integrals.

(a) $\int e^{4x}\sin 2x\,dx$ **(b)** $\int x\ln(x+1)\,dx$ **(c)** $\int \frac{1}{x^2\sqrt{x^2+36}}\,dx$

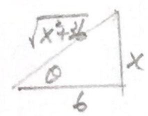
(d) $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$ **(e)** $\int_0^{\pi/2} cos^6 x\,dx$ **(f)** $\int_{\pi/6}^{\pi/3} \frac{3\ln(\tan x)}{7\sin x\cos x}\,dx$

(g) $\int \frac{x^3-3x^2+6x-2}{x^3-2x^2+x}\,dx$ **(h)** $\int \frac{1}{x\sqrt{1-x}}dx$

(a)
$$\int e^{4x}\sin 2x\,dx = \frac{1}{4}e^{4x}\sin 2x - \int \frac{1}{4}e^{4x}\cdot\cos 2x\cdot 2\,dx$$
$$= \frac{1}{4}e^{4x}\sin 2x - \left[\frac{1}{2}\cdot\frac{1}{4}e^{4x}\cos 2x - \int \frac{1}{2}\cdot\frac{1}{4}e^{4x}(-\sin 2x)\cdot 2\,dx\right]$$
$$\frac{5}{4}\int e^{4x}\sin 2x\,dx = \frac{1}{4}e^{4x}\sin 2x - \frac{1}{2}\cdot\frac{1}{4}e^{4x}\cos 2x$$
$$\int e^{4x}\sin 2x\,dx = \underline{\frac{1}{5}e^{4x}\sin 2x - \frac{1}{10}e^{4x}\cos 2x + C}\ \#$$

(b)
$$\int x\ln(x+1)\,dx = \frac{x^2}{2}\cdot\ln(x+1) - \int \frac{x^2}{2}\cdot\frac{1}{(x+1)}\,dx$$
$$\int \frac{x^2}{x+1}\,dx = \int \left(x-1+\frac{1}{x+1}\right)dx = \frac{x^2}{2} - x + \ln|x+1| + C$$
$$\int x\ln(x+1)\,dx = \underline{\frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1) + C}\ \#$$

(c) $x = 6\tan\theta \Rightarrow dx = 6\sec^2\theta\,d\theta$
$$\int \frac{1}{x^2\sqrt{x^2+36}}\,dx = \int \frac{6\sec^2\theta\,d\theta}{36\tan^2\theta\cdot 6\sec\theta}$$



$$= \frac{1}{36}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$
$$= -\frac{1}{36}\cdot\frac{1}{\sin\theta} + C = \underline{-\frac{1}{36}\cdot\frac{\sqrt{x^2+36}}{x} + C}\ \#$$

(d) $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\,d\theta$
$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta$$
$$= 2\int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\,d\theta$$
$$= 2\cdot\left[x - \frac{\sin 2\theta}{2}\right]\Big|_0^{\frac{\pi}{4}} = \underline{\frac{\pi}{2} - 1}\ \#$$

(e) $\int_0^{\frac{\pi}{2}} \cos^6 x \, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right)^3 dx = \frac{1}{8}\int_0^{\frac{\pi}{2}} (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\, dx$

Let $u = 2x$, $\therefore dx = \frac{1}{2} du$. $\quad = \frac{1}{8}\left[\frac{\pi}{2} + \frac{3\pi}{4}\right] = \underline{\underline{\frac{5\pi}{32}}}$ #

$\int_0^{\frac{\pi}{2}} \cos 2x \, dx = \frac{1}{2}\int_0^{\pi} \cos u \, du = 0$

$\int_0^{\frac{\pi}{2}} \cos^2 2x \, dx = \frac{1}{2}\int_0^{\pi} \cos^2 u \, du = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$

$\int_0^{\frac{\pi}{2}} \cos^3 2x \, dx = \frac{1}{2}\int_0^{\pi} \cos^3 u \, du = 0.$

(f) $u = \ln(\tan x) \Rightarrow du = \frac{\sec^2 x}{\tan x} dx = \frac{1}{\sin x \cos x} dx$

$\int_{\pi/6}^{\pi/3} \frac{3\ln(\tan x)}{\pi \sin x \cos x} dx = \int_{-\ln\sqrt{3}}^{\ln\sqrt{3}} \frac{3}{\pi} u \, du = \underline{\underline{0}}$ #

(g) $\frac{x^3 - 3x^2 + 6x - 2}{x^3 - 2x^2 + x} = 1 + \frac{-x^2 + 5x - 2}{x(x-1)^2} = 1 + \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$-x^2 + 5x - 2 = A(x-1)^2 + Bx(x-1) + Cx$

$x = 0 \Rightarrow A = -2, \quad x = 1 \Rightarrow C = 2, \quad A + B = -1 \Rightarrow B = 1$

$\int \frac{x^3 - 3x^2 + 6x - 2}{x^3 - 2x^2 + x} dx = \int \left(1 + \frac{-2}{x} + \frac{1}{x-1} + \frac{2}{(x-1)^2}\right) dx$

$= \underline{\underline{x - 2\ln|x| + \ln|x-1| - \frac{2}{(x-1)} + C}}$ #

(h) $u = \sqrt{1-x} \Rightarrow du = \frac{1}{2} \cdot \frac{-1}{\sqrt{1-x}} dx, \quad x = 1 - u^2$

$\int \frac{1}{x\sqrt{1-x}} dx = -2\int \frac{1}{1-u^2} du = -\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du$

$= -\left[-\ln|1-u| + \ln|1+u|\right] + C$

$= \underline{\underline{\ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right| + C}}$ #