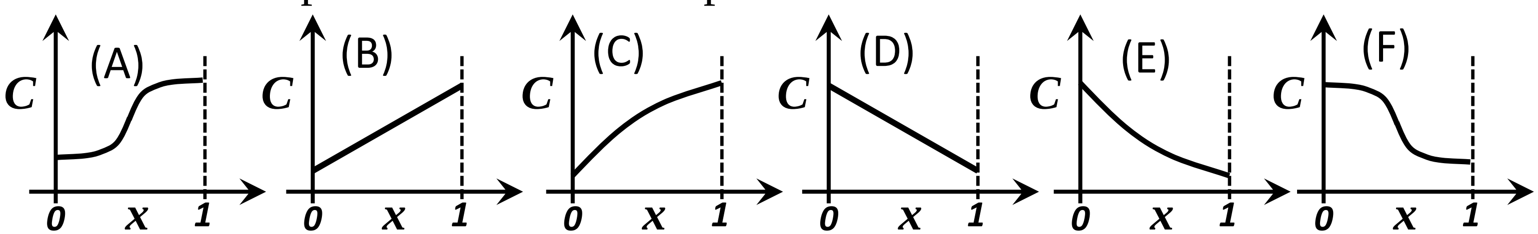
4. (20 pts) Fig. 3(a) shows a cross section of an infinitely long cylindrical conductor with radius $\boldsymbol{R}$. A total current $\boldsymbol{I_0}$ with current density $\vec{J}(r) = \hat{z}J_0[1 - (\frac{r}{R})^2]$ flows in the conductor (direction is out of page). (a) (3 pts) Find the constant $\boldsymbol{J_0}$ in terms of $\boldsymbol{I_0}$, $\boldsymbol{R}$ and other necessary constants. (b) (9 pts) Find the the magnetic fields (magnitude and direction) for $\boldsymbol{r > R}$ and $\boldsymbol{r < R}$. (c) (8 pts) We add another infinitely long plate carrying a uniform current density $\boldsymbol{J_1}$ (Fig.3(b)). The width of the plate is $\boldsymbol{R/2}$ and its center is located at $\boldsymbol{x = 4R}$. If the magnetic field at point $\boldsymbol{A}$: $(\boldsymbol{3R,0,0})$ is zero, what is the current density $\boldsymbol{J_1}$ and the direction of this current in the infinite plate?

II.選擇題(51 points)

1. (5 pts)As shown in Fig. 4(a), a capacitor consists of a dielectric slab with dielectric constant $\boldsymbol{\kappa}$ ($\boldsymbol{\kappa > 1}$) and two conducting plates of the same area $\boldsymbol{A}$. The capacitor is isolated and initially charged with the amount of charge $\boldsymbol{Q}$. Fig. 4(b) shows that the dielectric slab is then slowly pulled away from the capacitor such that $\boldsymbol{x \cdot A}$ is the area of the conductor without the slab, and $(\boldsymbol{1-x}) \cdot \boldsymbol{A}$ the area with the slab ($\boldsymbol{0 \le x \le 1}$). Which of the following show the correct relation between the capacitance $\boldsymbol{C}$ of the capacitor as a function of x?



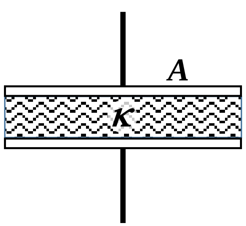


Fig. 4(a)

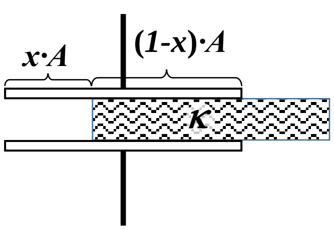


Fig. 4(b)

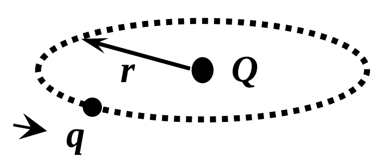


Fig. 5

2. (5pts) A student makes a short electromagnet by winding 100 turns of wire around a wooden cylinder of diameter $\boldsymbol{d}$ = 2 cm. The coil is connected to a battery producing a current of 4.0 A in the wire. At what axial distance $\boldsymbol{z}$ (>> $\boldsymbol{d}$) in unit of m will the magnetic field have the magnitude of 25 nT? (Consider the winding as an circular current loop magnetic dipole)
(A) $0.1 < z \leqq 0.3$ (B) $0.3 < z \leqq 0.5$ (C) $0.5 < z \leqq 0.7$ (D) $0.7 < z \leqq 0.9$ (E) $0.9 < z \leqq 1.1$
(F) $1.1 < z \leqq 1.3$ (G) $1.3 < z \leqq 1.5$ (H) $1.5 < z$.

3. (5 pts) As shown in Fig. 5, a negative charge $\boldsymbol{q}$ with mass $\boldsymbol{m}$ is traveling in a circular orbit around a fixed positive charge $\boldsymbol{Q}$ due to the Coulomb force between them. If the radius of the orbit increase by a factor of 2 ($\boldsymbol{r} \rightarrow \boldsymbol{2r}$), then the magnetic dipole moment resulted from $\boldsymbol{q}$ orbiting $\boldsymbol{Q}$ increases by a factor $\boldsymbol{x}$ ($\boldsymbol{\mu} \rightarrow \boldsymbol{x \cdot \mu}$). Which of the following is correct?
(A) $x < 1/5$ (B) $1/5 \leq x < 1/3$ (C) $1/3 \leq x < ½$ (D) $1/2 \leq x < 1$
(E) $1 \leq x < 2$ (F) $2 \leq x < 3$ (G) $3 \leq x < 5$ (H) $5 \leq x$

4. (5 pts) Shown in Fig. 6, the switch is open at first and the currents reach steady state. Then the switch is closed. How much does the charge $\boldsymbol{\Delta Q}$ flow out of capacitor $\mathbf{C_1}$ after a long time ?
($\boldsymbol{\Delta Q}$ in μC) (A) $0 < \Delta Q \leq 20$ (B) $20 < \Delta Q \leq 40$ (C) $40 < \Delta Q \leq 60$
(D) $60 < \Delta Q \leq 80$ (E) $80 < \Delta Q \leq 100$ (F) $100 < \Delta Q \leq 120$

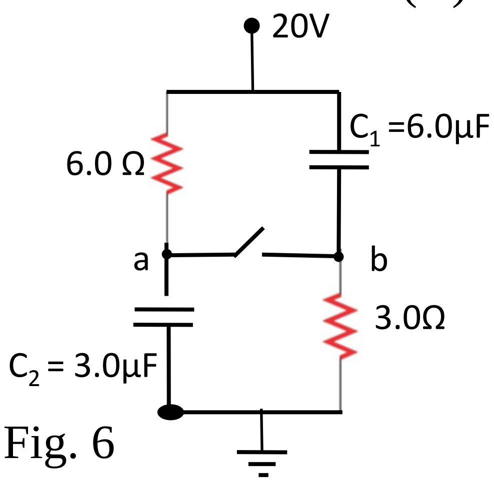


Fig. 6

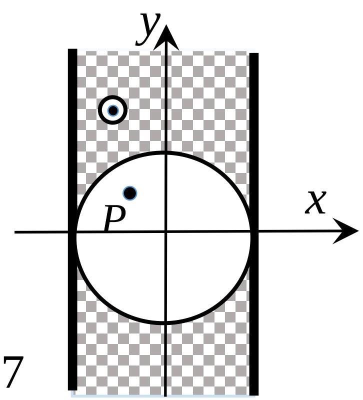


Fig. 7

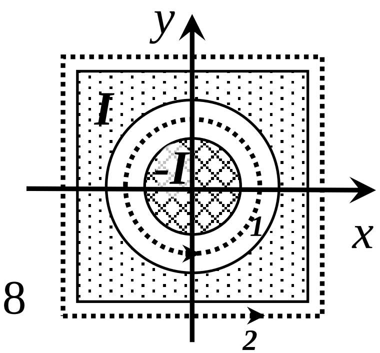
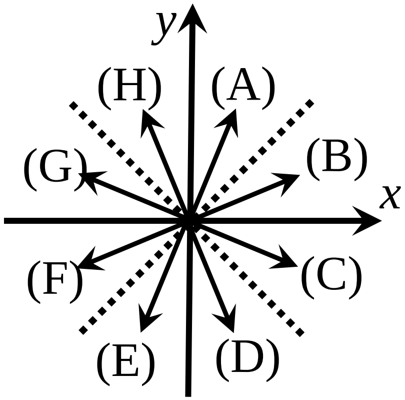


Fig. 8

5. (5pts)As shown in Fig. 7, an infinite conducting plate with thickness **2*d*** carries a uniform current density ***J*** in **+z** axis, in the middle of the plate there is an infinitely long hollow cylindrical region with radius ***d*** and its axis coincides with the ***z***-axis. Which of the following could be the direction of the magnetic field at point ***P*** =(***-d/2,d/2,0***)?



(J) The magnetic field is zero.

6. (5 pts) Fig. 8 shows the cross section of infinitely long co-axial (同軸) conductors along the **z**-axis (out of page), The outer conductor carries a uniform current ***I*** (***I*** > 0) in the **+z**-direction, and the inner cylindrical conductor carries a uniform current **–*I***. Consider two closed loops labelled as ***1*** and ***2*** on the x-y plane with their centers at the origin. Let $\boldsymbol{B_1}(x,y,z)$ and $\boldsymbol{B_2}(x,y,z)$ be the B-field generated by the currents at each point along loop ***1*** and loop ***2***, respectively, and $K_1 = \oint_{Loop1} \vec{B}_1 \cdot d\vec{\ell}$, *and* $K_2 = \oint_{Loop2} \vec{B}_2 \cdot d\vec{\ell}$

(A) $|\boldsymbol{B_1}|$ = ***constant***, $K_1 = 0$, $|\boldsymbol{B_2}| = 0$, $K_2 = 0$

(B) $|\boldsymbol{B_1}|$ = ***constant***, $K_1 \neq 0$, $|\boldsymbol{B_2}| = 0$, $K_2 = 0$

(C) $|\boldsymbol{B_1}|$ = ***constant***, $K_1 \neq 0$, $|\boldsymbol{B_2}| \neq 0$, $K_2 \neq 0$

(D) $|\boldsymbol{B_1}| \neq$ ***constant***, $K_1 \neq 0$, $|\boldsymbol{B_2}| \neq 0$, $K_2 = 0$

(E) $|\boldsymbol{B_1}|$ = ***constant***, $K_1 \neq 0$, $|\boldsymbol{B_2}| \neq 0$, $K_2 = 0$

(F) $|\boldsymbol{B_1}| \neq$ ***constant***, $K_1 \neq 0$, $|\boldsymbol{B_2}| = 0$, $K_2 = 0$

(G) $|\boldsymbol{B_1}| \neq$ ***constant***, $K_1 \neq 0$, $|\boldsymbol{B_2}| \neq 0$, $K_2 \neq 0$

Integration Formula for reference

$$\mu_0 = 4\pi \times 10^{-7} (T \cdot m / \mathrm{A})$$

$$\int \frac{dx}{\sqrt{x^2 \pm b^2}} = \ln\left(x + \sqrt{x^2 \pm b^2}\right)$$

$$\int \frac{x^2 dx}{\sqrt{x^2 \pm b^2}} = \frac{x\sqrt{x^2 \pm b^2}}{2} - \frac{b^2}{2} \ln\left(x + \sqrt{x^2 \pm b^2}\right)$$

$$\int \frac{dx}{\left(x^2 \pm b^2\right)^{3/2}} = \frac{\pm x}{b^2 \sqrt{x^2 \pm b^2}}$$

$$\int \frac{x^2 dx}{\left(x^2 \pm b^2\right)^{3/2}} = \frac{-x}{\sqrt{x^2 \pm b^2}} + \ln\left(x + \sqrt{x^2 \pm b^2}\right)$$

1	2	3	4	5	6				
D	E	E	B	X*	D				

*此題無正確答案, 送分

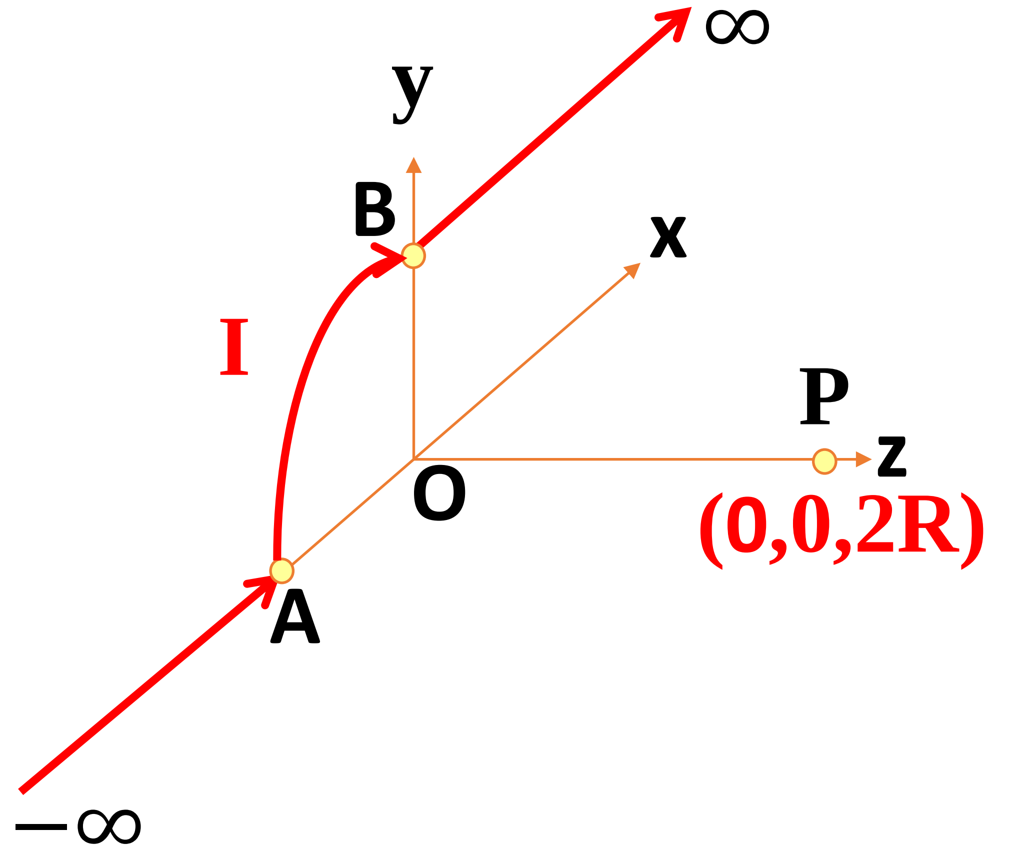
1. (20pts) Figure shows a three-section conducting wire on x-y plane with current I. The first section is from ∞ to A on the x-axis. The section is from A to B is a quarter of a circle with radius ***R***. The last section is from B to -∞ and parallel to x-axis. Find the *x*-, *y*-, *z*-components of the magnetic field at point ***P*** on the *z*-axis due to

(a) (7pts) current in the section from -∞ to A,

(b) (5pts) current in the section from B to ∞, and

(c) (8pts) current in the section from A to B.

The coordinates of ***A***, ***B***, and ***P*** are **(-*R*, *0*, *0*)**, **(*0*, *R*, *0*),** and **(0, *0*, *2R*)**, respectively.



(a) Line segment on x-axis:

$$\vec{r}\,' = (x,0,0);\quad \vec{r} = (-x,0,2R);\quad d\vec{l} = (dx,0,0);\quad d\vec{l}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & 0 & 0 \\ -x & 0 & 2R \end{vmatrix} = -2Rdx\hat{j}$$

1 1 1

$$\vec{B}_1 = \int_{-\infty}^{-R} \frac{\mu_0 I}{4\pi} \frac{-2Rdx\hat{j}}{(4R^2+x^2)^{3/2}} = \frac{\mu_0 I(-2R)}{4\pi(4R^2)}\hat{j}[\frac{-1}{\sqrt{5}} - (-1)] = \boxed{\frac{\mu_0 I}{8\pi R}(\frac{1}{\sqrt{5}} - 1)\hat{j}}$$

2 2

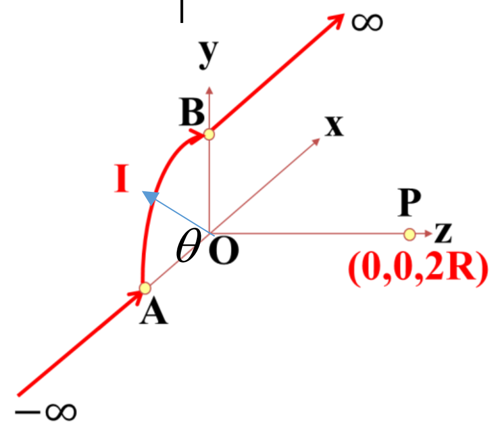
(b) Line segment parallel to x-axis:

$$\vec{r}\,' = (x,R,0);\ \vec{r} = (-x,-R,2R);\ d\vec{l} = (dx,0,0);\ d\vec{l}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & 0 & 0 \\ -x & -R & 2R \end{vmatrix} = -2Rdx\hat{j} - Rdx\hat{k}$$

1 1 1

$$\vec{B}_2 = \int_0^{\infty} \frac{\mu_0 I}{4\pi} \frac{-Rdx}{(5R^2+x^2)^{3/2}}(2\hat{j}+\hat{k}) = -\frac{\mu_0 I}{20\pi R}(2\hat{j}+\hat{k})$$

1 1



(c) quarter of a circle on x-y plane:

$$\vec{r}\,' = (-R\cos\theta, R\sin\theta, 0);\quad \vec{r} = (R\cos\theta, -R\sin\theta, 2R);$$

1

$$d\vec{l} = \frac{d\vec{r}\,'}{d\theta}d\theta = (R\sin\theta d\theta, R\cos\theta d\theta, 0);$$

2

$$d\vec{l}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ R\sin\theta d\theta & R\cos\theta d\theta & 0 \\ R\cos\theta & -R\sin\theta & 2R \end{vmatrix}$$

2

$$= 2R^2\cos\theta d\theta\hat{i} - 2R^2\sin\theta d\theta\hat{j} - R^2 d\theta\hat{k}$$

$$\boxed{\vec{B}_3 = \int_0^{\pi/2} \frac{\mu_0 I}{4\pi} \frac{(2R^2\cos\theta d\theta, -2R^2\sin\theta d\theta, -R^2 d\theta)}{5\sqrt{5}R^3}}$$

1

$$= \frac{\mu_0 I}{20\sqrt{5}\pi R}(2\hat{i} - 2\hat{j} - \frac{\pi}{2}\hat{k})$$

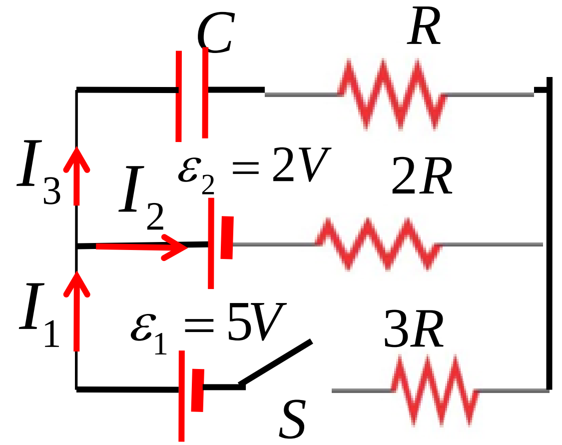
2

3. (15 points) Consider the circuit with uncharged capacitor shown in Fig. 1. Before $t = 0$, the switch is open for a long time. And then the switch is closed at $t = 0$.

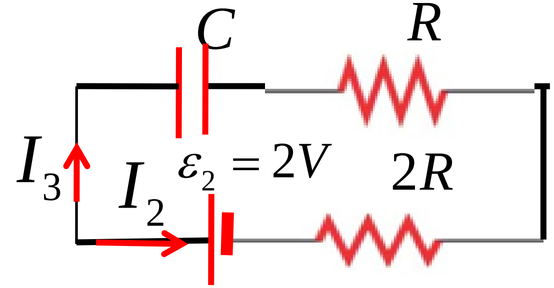
(A) (3 pts) Find the values of I_1, I_2 I_3 and $Q(t=0^-)= Q_i$.

(B) (3 pts) Find the values of I_1, I_2 I_3 and $Q(t=\infty)= Q_0$.

(C) (9 pts) Find the time constant τ and the charge on the capacitor Q(t) after the switch is closed.



(A) Find the values of I_1, I_2 I_3 and Q before $t = 0$.

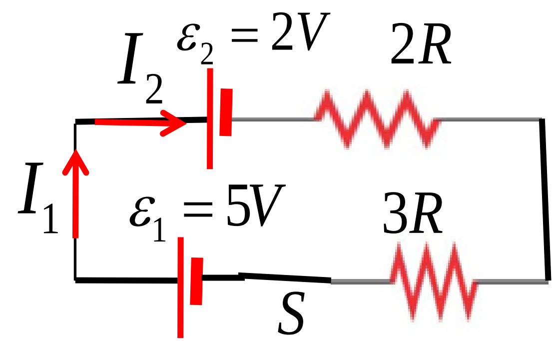


Before t = 0, the switch is open, then the circuit should be like the figure on the right. Now the capacitor reaches fully charged, there is no current in the circuit, so

$\Rightarrow$ $I_1 = I_2 = I_3 = 0$ 2

$$\frac{Q_i}{C} = \varepsilon_2 \Rightarrow Q_i = 2C$$ 1

(B) Find the values of I_1, I_2 I_3 and $Q(t=\infty) = Q_0$.



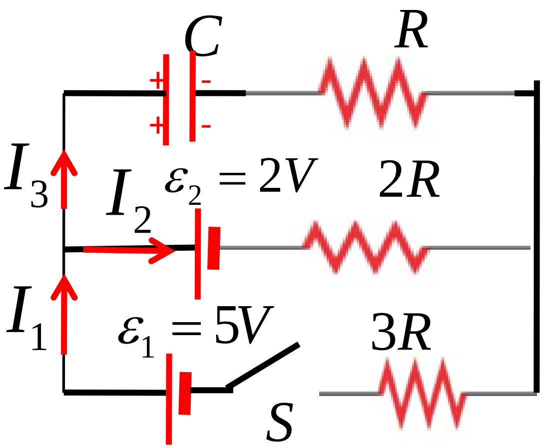
After the switch is closed for a long time, the capacitor reaches fully charged, and acts like a open circuit. Then the circuit become the figure on the right.

$I_1 = I_2$ and $\varepsilon_1 - \varepsilon_2 - I_1 \cdot (2R) - I_1 \cdot (3R) = 0$

$$\Rightarrow I_1 = I_2 = \frac{\varepsilon_1 - \varepsilon_2}{5R} = \frac{3}{5R}$$ 1 and $I_3 = 0$ 1

$$\frac{Q_0}{C} = \varepsilon_2 + I_2 \cdot 2R = \frac{16}{5}(volt) \Rightarrow Q_0 = \frac{16C}{5}$$ 1

(C) Find the time constant τ and the charge on the capacitor Q(t) after the switch is closed.



$I_1 = I_2 + I_3$ 1

$\varepsilon_1 - \varepsilon_2 - I_2 \cdot (2R) - I_1 \cdot (3R) = 0$ 1

$-\frac{Q}{C} - I_3 \cdot (R) + I_2 \cdot (2R) + \varepsilon_2 = 0$ 1

and $I_3 = \frac{dQ}{dt}$ 1

Rearrange the first three equations:

$$\begin{aligned} I_1 \quad - I_2 \quad - I_3 &= 0 \\ 3RI_1 + 2RI_2 \qquad &= (\varepsilon_1 - \varepsilon_2) \\ 2RI_2 - RI_3 &= \frac{Q}{C} - \varepsilon_2 \end{aligned}$$

$$I_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 3R & 2R & \varepsilon_1 - \varepsilon_2 \\ 0 & 2R & \frac{Q}{C} - \varepsilon_2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 \\ 3R & 2R & 0 \\ 0 & 2R & -R \end{vmatrix}} = \frac{\left(\frac{Q}{C} - 2\right) \cdot (2R + 3R) - 6R}{-11R^2}$$

1

$$= -\frac{5\frac{Q}{C} - 10 - 6}{11R}$$

$$= -\frac{5}{11RC}(Q - 16C/5)$$

$$= -\frac{Q - Q_0}{\tau}$$

$\tau = \frac{11}{5}RC$ 2 and $Q_0 = \frac{16C}{5}$

Then

$$\frac{dQ}{dt} = -\frac{Q - Q_0}{\tau} \Rightarrow \int_{Q_i}^{Q} \frac{dQ}{Q - Q_0} = -\int_0^t \frac{dt}{\tau}$$ 1

$$\Rightarrow \ln\left(\left|\frac{Q - Q_0}{Q_i - Q_0}\right|\right) = -\frac{t}{\tau}$$

$$\Rightarrow Q(t) = Q_0 + (Q_i - Q_0)e^{-\frac{t}{\tau}}$$

$$\Rightarrow Q(t) = \frac{16C}{5} - \frac{6C}{5}e^{-\frac{t}{\tau}}$$ 1

4. (20 pts) In Fig. XXa, a cross section of an infinite long cylindrical conductor with radius R. A total current I_0 with current density $\vec{J}(r) = \hat{z}J_0[1 - (\frac{r}{R})^2]$ flows in the conductor (direction is out of page). (a) (3 pts) Find the constant J_0 in terms of I_0, R and other necessary constants. (b) (9 pts) Find the the magnetic fields (magnitude and direction) for $r > R$ and $r < R$. (c) (8 pts) We add another infinite long plane carrying a uniform current density J_1 (FigXXb). The width of the plane is $R/2$ and its center is located at $x = 4R$. If the magnetic field at point A : $(3R,0,0)$ is zero, what is the current density J_1 and the direction of this current in the infinite plane?

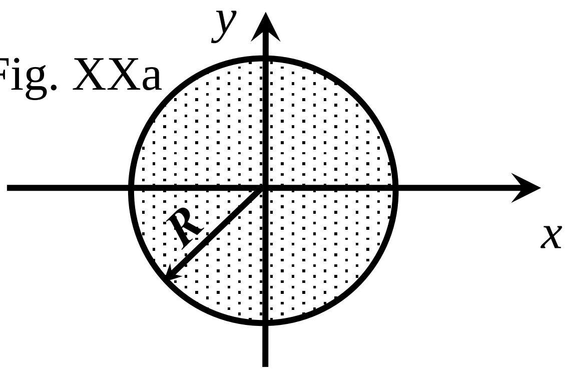


Fig. XXa

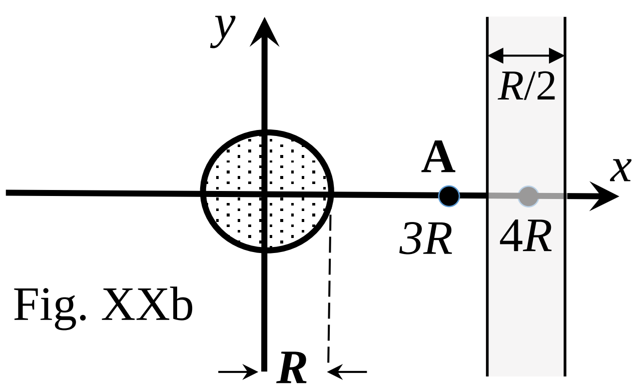
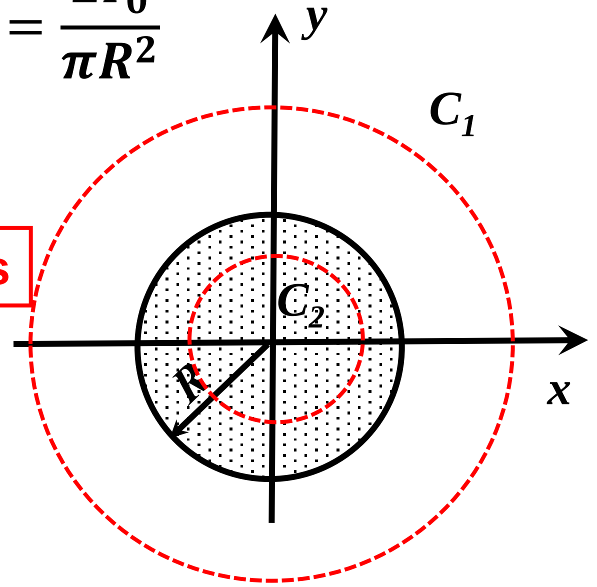


Fig. XXb

(a) $$I_0 = \int_0^R J_0\left[1-\left(\frac{r}{R}\right)^2\right]2\pi r dr = \frac{\pi}{2}J_0R^2 \rightarrow J_0 = \frac{2I_0}{\pi R^2}$$

3 pts



2 pts

(b) $r > R$

9 pts

$$\oint_{C_1}\vec{B}\cdot d\vec{r} = \int_0^{2\pi}(B\hat{\varphi})\cdot(\hat{\varphi}rd\varphi) = 2\pi rB$$

$$= \mu_0 I_0$$

$$\rightarrow \vec{B} = \frac{\mu_0 I_0}{2\pi r}\hat{\varphi}$$ 3 pts

$r < R$

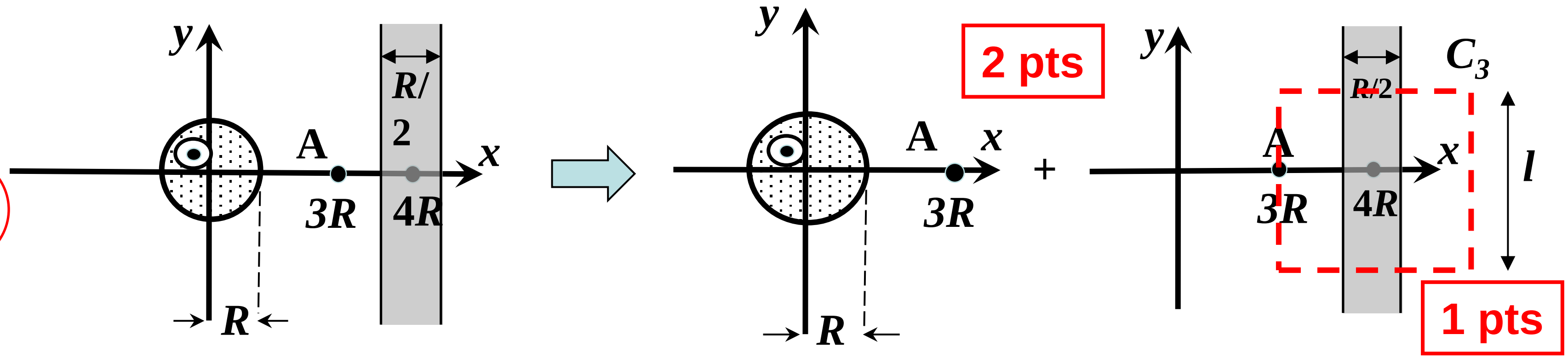
$$\oint_{C_2}\vec{B}\cdot d\vec{r} = \int_0^{2\pi}(B\hat{\varphi})\cdot(\hat{\varphi}rd\varphi) = 2\pi rB$$

$$= \mu_0 I_{enc} = \mu_0\int_0^r\left[1-\left(\frac{r\prime}{R}\right)^2\right]2\pi r' dr' = 2\pi\mu_0 J_0\left(\frac{r^2}{2}-\frac{r^4}{4R^2}\right)$$

$$\rightarrow \vec{B} = \frac{\mu_0 J_0}{r}\left(\frac{r^2}{2}-\frac{r^4}{4R^2}\right)\hat{\varphi} = \frac{\mu_0 I_0}{\pi r}\left(\frac{r^2}{R^2}-\frac{r^4}{2R^4}\right)\hat{\varphi}$$ 4 pts

(c)

8 pts



The magnetic field at point A is the vector sum of the cylindrical current and the infinite plane current :

$$\vec{B}_A = \frac{\mu_0 I_0}{6\pi R}\hat{y} + \vec{B_1}, \qquad \vec{B}_1(x=3R) = -B_1\hat{y}$$

$$\vec{B}_1(x=5R) = +B_1\hat{y}$$

$$\oint_{C_3} \vec{B_1}\cdot d\vec{r}\bigg|_{c.c.w.} = 2B_1 l = \mu_0 \int \vec{J}\cdot d\vec{A} = \mu_0(J_1\hat{z})\cdot\left(\frac{R}{2}l\,\hat{z}\right) = \mu_0\frac{RlJ_1}{2}$$

$$\rightarrow \vec{B}_1(x=3R) = -\frac{\mu_0 R J_1}{4}\hat{y}$$

2 pts

$$\vec{B}_A = \frac{\mu_0 I_0}{6\pi R}\hat{y} + \vec{B_1} = 0 \rightarrow \vec{J_1} = \frac{2}{3}\frac{I_0}{\pi R^2}\hat{z}$$

3 pts