# A novel approach to the Lindelöf hypothesis

by

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**Abstract**

Lindelöf's hypothesis, one of the most important open problems in the history of mathematics, states that for large $t$, Riemann's zeta function $\zeta(\frac{1}{2}+it)$ is of order $O(t^{\varepsilon})$ for any $\varepsilon > 0$ . It is well known that for large $t$, the leading order asymptotics of the Riemann zeta function can be expressed in terms of a transcendental exponential sum. The usual approach to the Lindelöf hypothesis involves the use of ingenious techniques for the estimation of this sum. However, since such estimates can not yield an asymptotic formula for the above sum, it appears that this approach cannot lead to the proof of the Lindelöf hypothesis. Here, a completely different approach is introduced: the Riemann zeta function is embedded in a classical problem in the theory of complex analysis known as a Riemann-Hilbert problem, and then, the large $t$-asymptotics of the associated integral equation is formally computed. This yields two different results. First, the formal proof that a certain Riemann zeta-type double exponential sum satisfies the asymptotic estimate of the Lindelöf hypothesis. Second, it is formally shown that the sum of $|\zeta(1/2+it)|^2$ and of a certain sum which depends on $\epsilon$, satisfies for large $t$ the estimate of the Lindelöf hypothesis. Hence, since the above identity is valid for all $\epsilon$, this asymptotic identity suggests the validity of Lindelöf's hypothesis. The completion of the rigorous derivation of the above results will be presented in a companion paper.

# 1 Introduction

Riemann's hypothesis, which is perhaps the most celebrated open problem in the history of mathematics, can be verified numerically for $t$ up to $O(10^{13})$. This suggests that it is important to investigate the large $t$-asymptotics of the Riemann function $\zeta(s)$, $s=\sigma+it$, $0<\sigma<1$, $t\to\infty$. This investigation is closely related with Lindelöf's hypothesis. Indeed, it is well known that the leading asymptotics for large $t$ of $\zeta(s)$ can be expressed in terms of the following transcendental sum:

$$\zeta(s)\sim\sum_{m=1}^{[t]}\frac{1}{m^s},\qquad 0<\sigma<1,\quad t\to\infty, \tag{1.1}$$

where throughout this paper [A] denotes the integer part of the positive number A. Lindelöf's hypothesis, one of the most important open problems in the history of mathematics, states that for $\sigma=1/2$, this sum is of order $O(t^{\varepsilon})$ for any $\varepsilon>0$.

It is stated in [GM] that "The Lindelöf Hypothesis is implied by the Riemann Hypothesis and conversely it implies that very few zeros disobey it." Indeed, let $N(\sigma,T)$ denote the number of zeros, $\beta+it$, of $\zeta(s)$, such that $\beta>\sigma$ and $0<t\le T$. Then,

$$N(\sigma,T)\le N(T)<AT\ln T,\qquad \frac{1}{2}<\sigma<1,$$

where $N(T)$ denotes the number of zeros of $\zeta(s)$ in the domain $0\le\sigma\le1$, $t\le T$, and $A$ is a constant. According to Theorem 9.19A of [T]

$$N(\sigma,T)=O\left(T^{\frac{3}{2}-\sigma}(\ln T)^5\right),\qquad \frac{1}{2}<\sigma<1,\quad T\to\infty.$$

In [HT] it is proven that if Lindelöf's Hypothesis is valid then

$$N(\sigma,T)=O\left(T^{\epsilon}\right),\qquad \frac{3}{4}+\delta\le\sigma<1,\quad T\to\infty,$$

for $\epsilon$ and $\delta$ positive and arbitrarily small.

The first result in this direction was obtained in [IN], where under the assumption of Lindelöf's hypothesis, it was shown that

$$N(\sigma,T)=O\left(T^{2(1-\sigma)+\epsilon}\right),\qquad \frac{1}{2}\le\sigma\le1,\quad T\to\infty,$$

for $\epsilon>0$, arbitrarily small.

In [HT] it is stated that the above estimate shattered for the first time "the semblance that the Lindelöf hypothesis is much weaker than the Riemann Hypothesis and has not even an essential influence on the vertical distribution of the zeros, i.e. $N(\sigma,t)$". Furthermore, in [TU], where a slightly stronger form of the above estimate is proven by a different method, it is stated that "Lindelöf's hypothesis is much stronger than expected and even implies the estimate

$$N(\sigma,T)=O\left(T^{\epsilon}\right), \qquad \frac{1}{2}+\delta\leq\sigma<1, \quad T\to\infty,$$

for $\epsilon$ and $\delta$ positive and arbitrarily small."

The sum of the rhs of (1.1) is a particular case of an exponential sum. Pioneering results for the estimation of such sums were obtained almost 100 years ago using methods developed by Weyl [W], and Hardy and Littlewood [HL], when it was shown that $\zeta(1/2+it)=O(t^{1/6+\varepsilon})$. In the last 90 years some slight progress was made using the ingenious techniques of Vinogradov [V]. Currently, the best result is due to Bourgain [B] who has been able to reduce the exponent factor to $53/342\approx 0.155$.

It turns out that the best estimate for the growth of $\zeta(s)$ as $t\to\infty$, is based on the approximate functional equation, see page 79 of [T],

$$\zeta(s)=\sum_{n\leq x}\frac{1}{n^s}+\frac{(2\pi)^s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sum_{n\leq y}\frac{1}{n^{1-s}}+O\left(x^{-\sigma}+|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}\right),$$
$$xy=\frac{t}{2\pi}, \quad 0<\sigma<1, \quad t\to\infty, \quad (1.2)$$

where $\Gamma(s)$, $s\in\mathbb{C}$, denotes the gamma function. It is interesting that, in contrast to the usual situation in asymptotics where higher order terms in an asymptotic expansion are more complicated, the higher order terms of the asymptotic expansion of $\zeta(s)$ can be computed *explicitly*. Siegel, in his classical paper [S], presented the asymptotic expansion of $\zeta(s)$ to *all* orders in the important case of $x=y=\sqrt{t/2\pi}$. In [FL], analogous results are presented for *any* $x$ and $y$ valid to *all* orders.

An interesting relation between the Riemann hypothesis and the solution of a particular Neumann problem for the two-dimensional Laplace equation is presented in [FG].

A major obstacle in trying to prove Lindelöf's hypothesis via the estimation of the relevant exponential sum is that in estimates one "loses" something (the more powerful the technique the less the loss). Hence, it does not appear that it is possible to prove Lindelöf's hypothesis via the above technically complicated but conceptually straightforward approach. Actually, world renowned mathematicians believe that the only way of proving the Lindelöf hypothesis is by proving the Riemann hypothesis and by employing the fact that the latter implies the former [SA].

Here, a different approach to Lindelöf's hypothesis is introduced: the Riemann zeta function is embedded in a more complex mathematical structure and the asymptotics of this new structure is computed. In this approach, by using asymptotics, one avoids the loss occurring in estimates.

Of course, the main difficulty with the above approach is finding a suitable "more complex mathematical structure". After many unsuccessful attempts and with the crucial help of Jonatan Lenells and Anthony Ashton, it has been possible to embed the Riemann zeta function in the Riemann-Hilbert problem (or equivalently, singular integral equation) shown below.

## Main results

The following equation is valid:

$$\frac{t}{\pi}\fint_{-\infty}^{\infty}\Re\left\{\frac{\Gamma(it-i\tau t)}{\Gamma(\sigma+it)}\Gamma(\sigma+i\tau t)\right\}|\zeta(\sigma+i\tau t)|^2\,\mathrm{d}\tau+\mathcal{G}(\sigma,t)=0,$$
$$0<\sigma<1,\quad t>0,\quad (1.3)$$

where the principal value integral is defined with respect to $\tau=1$, and the function $\mathcal{G}(\sigma,t)$ is defined by the formulae

$$\mathcal{G}(\sigma,t)=\begin{cases}\zeta(2\sigma)+\left(\frac{\Gamma(1-\bar{s})}{\Gamma(s)}+\frac{\Gamma(1-s)}{\Gamma(\bar{s})}\right)\Gamma(2\sigma-1)\zeta(2\sigma-1)+\frac{2(\sigma-1)\zeta(2\sigma-1)}{(\sigma-1)^2+t^2}, & \sigma\neq\frac{1}{2},\\ \Re\left\{\Psi\left(\frac{1}{2}+it\right)\right\}+2\gamma-\ln 2\pi+\frac{2}{1+4t^2}, & \sigma=\frac{1}{2},\end{cases}\tag{1.4}$$

with $\Psi(z)$ denoting the digamma function, i.e.,

$$\Psi(z)=\frac{\frac{\mathrm{d}}{\mathrm{dz}}\Gamma(z)}{\Gamma(z)},\quad z\in\mathbb{C},$$

and $\gamma$ denoting the Euler constant.

The large $t$ asymptotic analysis of (1.3) yields the following asymptotic estimates:

First,

$$\sum_{m_2=\left[t^{\frac{1}{2}-\epsilon}\right]}^{[t]} \sum_{m_1=1}^{\left[\frac{m_2}{t^{\frac{1}{2}-\epsilon}}\right]} \frac{1}{(m_1+m_2)^{\frac{1}{2}+it} m_2^{\frac{1}{2}-it}} = O\left(t^{\epsilon} \ln t\right), \quad t \to \infty, \quad \text{for any } \epsilon > 0, \quad (1.5)$$

which is the analogue of the Lindelöf hypothesis for the Riemann-type double sum defined by the lhs of the above equation.

Second,

$$\left| \sum_{m=1}^{[t]} \frac{1}{m^{\frac{1}{2}+it}} \right|^2 - 2\Re\left\{S^{\epsilon}(t)\right\} = O\left(t^{\epsilon} \ln t\right), \quad t \to \infty, \quad \text{for any } \epsilon > 0, \quad (1.6)$$

with $S^{\epsilon}$ given by

$$S^{\epsilon}(t) = t^{i\left(\epsilon - \frac{1}{2}\right)t} e^{i\left(t - t^{\frac{1}{2}+\epsilon}\right)} \sum\sum_{m_1, m_2 \in M^{\epsilon}} \frac{1}{m_1^{\frac{1}{2}+it} m_2^{\frac{1}{2}-it}} e^{-i\frac{m_2}{m_1} t^{\frac{1}{2}+\epsilon}}, \quad (1.7)$$

and the set $M^{\epsilon}$ defined by

$$M^{\epsilon} = \left\{ m_j = 1, \ldots, [t], \ j = 1, 2, \ \ \frac{m_2}{m_1} \in \left(t^{-\frac{1}{2}-\epsilon}, t^{\frac{1}{2}-\epsilon} - 1\right) \right\}.$$

Taking into consideration that $S^{\epsilon}(t)$ depends on $\epsilon$, (1.6) suggests the validity of Lindelöf's hypothesis.

### Summary of the derivation of the main results

The derivation of (1.3), which is presented in section 2, is based on the so-called Plemelj formulae, which are the fundamental ingredients of the theory of Riemann-Hilbert problems. The appearance of these problems in mathematics and in applications in the last forty years has been ubiquitous, see for example [FIK, I, Z]. Furthermore, Deift and Zhou have introduced a powerful technique for the asymptotic analysis of Riemann Hilbert problems [DZ] (see also [D]).

It is shown in Corollary 2.1 that $\mathcal{G}(\sigma, t)$ satisfies

$$\mathcal{G}(\sigma, t) = \begin{cases} \zeta(2\sigma) + O\left(\frac{t^{1-2\sigma}}{1-2\sigma}\right), & \sigma \neq \frac{1}{2}, \\ \ln t + O(1), & \sigma = \frac{1}{2}, \end{cases} \qquad t \to \infty. \qquad (1.8)$$

Regarding the computation of the large $t$ asymptotics of equation (1.3), we note that the term $\Gamma(it-i\tau t)\Gamma(\sigma+i\tau t)/\Gamma(\sigma+it)$, $-\infty<\tau<\infty$, decays exponentially for large $t$, unless $\tau$ is in the interval

$$-t^{\delta_1-1}\le\tau\le 1+t^{\delta_4-1},\quad \delta_1>0,\ \delta_4>0.$$

Thus, equation (1.3) simplifies to the equation

$$\begin{aligned}&\frac{t}{\pi}\oint_{-t^{\delta_1-1}}^{1+t^{\delta_4-1}}\Re\left\{\frac{\Gamma(it-i\tau t)}{\Gamma(\sigma+it)}\Gamma(\sigma+i\tau t)\right\}|\zeta(\sigma+i\tau t)|^2\,\mathrm{d}\tau+\mathcal{G}(\sigma,t)\\&+O\left(e^{-\pi t^{\delta_{14}}}\right)=0,\qquad t\to\infty,\\&0<\sigma<1,\ \delta_1>0,\ \delta_4>0,\ \delta_{14}=\min(\delta_1,\delta_4).\end{aligned}\tag{1.9}$$

where the principal value integral is defined with respect to $\tau=1$.

It turns out that the computation of the large $t$ asymptotics of (1.9) requires splitting the interval $[-t^{\delta_1-1},1+t^{\delta_4-1}]$ into the following four subintervals:

$$\begin{aligned}&L_1=[-t^{\delta_1-1},t^{-1}],L_2=[t^{-1},t^{\delta_2-1}],L_3=[t^{\delta_2-1},1-t^{\delta_3-1}],\\&L_4=[1-t^{\delta_3-1},1+t^{\delta_4-1}],\quad \delta_2>0,\ \delta_3>0.\end{aligned}\tag{1.10}$$

Thus, the asymptotic evaluation of (1.9) reduces to the analysis of the four integrals,

$$\begin{aligned}I_j(\sigma,t)=&\frac{t}{\pi}\oint_{L_j}\Re\left\{\frac{\Gamma(it-i\tau t)}{\Gamma(\sigma+it)}\Gamma(\sigma+i\tau t)\right\}|\zeta(\sigma+i\tau t)|^2\,\mathrm{d}\tau,\\&0<\sigma<1,\quad t>0,\end{aligned}\tag{1.11}$$

where $I_1$, $I_2$, $I_3$, $I_4$ also depend on $\delta_1$, $\delta_2$, $(\delta_2,\delta_3)$, $(\delta_3,\delta_4)$, respectively, $\{L_j\}_1^4$ are defined in (1.10), and the principal value integral is needed only for $I_4$.

It is shown in section 3 that for $\delta_1$ a sufficient small positive constant, $I_1$ satisfies the estimate

$$I_1(\sigma,t,\delta_1)=\begin{cases}O\left(t^{-\sigma+\frac{4}{3}\delta_1}\right), & 0<\sigma\le\frac{1}{2},\\ O\left(t^{-\sigma+\delta_1\left(\sigma+\frac{5}{6}\right)}\right), & \frac{1}{2}<\sigma<1.\end{cases},\quad t\to\infty.\tag{1.12}$$

By employing the classical estimates of Atkinson, it is also shown in section 3 that $I_2$ satisfies the following estimate, for $0<\delta_2<1$:

$$I_2(\sigma,t,\delta_2)=\begin{cases}O\left(t^{-\sigma+2(1-\sigma)\delta_2}\zeta(2-2\sigma)\right), & 0<\sigma<\frac{1}{2},\\ O\left(t^{-\frac{1}{2}+\delta_2}\ln t\right), & \sigma=\frac{1}{2},\\ O\left(t^{-\sigma+\left(\sigma+\frac{1}{2}\right)\delta_2}\zeta(2\sigma)\right), & \frac{1}{2}<\sigma<1,\end{cases}\quad t\to\infty.\tag{1.13}$$

Let $\tilde{I}_3$ denote the integral obtained from the rhs of equation (1.11) with $j = 3$, when $|\zeta|^2$ is replaced by its leading term asymptotics:

$$\tilde{I}_3(\sigma, t, \delta_2, \delta_3) = \sum_{m_1=1}^{[t]} \sum_{m_2=1}^{[t]} \frac{1}{m_1^\sigma} \frac{1}{m_2^\sigma} \Re\left\{ J_3(\sigma, t, \delta_2, \delta_3, \lambda) \right\}, \qquad \lambda = \frac{m_2}{m_1},$$
$$0 < \sigma < 1,\ t > 0,\ 0 < \delta_2 < 1,\ 0 < \delta_3 < 1, \tag{1.14}$$

where

$$J_3(\sigma, t, \delta_2, \delta_3, \lambda) = \frac{t}{\pi} \int_{t^{\delta_2 - 1}}^{1 - t^{\delta_3 - 1}} \frac{\Gamma(it - it\tau)}{\Gamma(\sigma + it)} \Gamma(\sigma + it\tau) \lambda^{i\tau t} d\tau, \quad \lambda = \frac{m_2}{m_1},$$
$$0 < \sigma < 1, \quad t > 0; \quad 0 < \delta_2 < 1,\ 0 < \delta_3 < 1,\ m_j = 1, \dots, [t], \quad j = 1, 2. \tag{1.15}$$

The asymptotic analysis of $J_3$ as $t \to \infty$ is discussed in section 4, where it is shown that $J_3$ involves the following two terms: (i) $J_3^S$, which is due to the contribution of the stationary points, with an error due to the contribution of the lower end point $t^{\delta_2 - 1}$. (ii) $J_3^U$, which is due to the contribution of the upper end point $1 - t^{\delta_3 - 1}$. These terms yield the following representation for $\tilde{I}_3$:

$$\tilde{I}_3(\sigma, t, \delta_2, \delta_3) = \tilde{I}_3^S(\sigma, t, \delta_2, \delta_3) - \tilde{I}_3^U(\sigma, t, \delta_2, \delta_3), \tag{1.16}$$

where

$$\tilde{I}_3^S(\sigma, t, \delta_2, \delta_3) = 2\Re\left\{ \sum\sum_{m_1, m_2 \in M(\delta_2, \delta_3, t)} \frac{1}{m_2^{\bar{s}} (m_1 + m_2)^s} [1 + o(1)] \right\}$$
$$\times \left[1 + O\left(t^{-\delta_{23}}\right)\right], \qquad t \to \infty,$$
$$s = \sigma + it,\ 0 < \sigma < 1, \quad 0 < \delta_2 < 1,\ 0 < \delta_3 < 1, \delta_{23} = \min\{\delta_2, \delta_3\}, \tag{1.17}$$

with the set $M$ defined by

$$M(\delta_2, \delta_3, t) = \left\{ m_j = 1, \dots, [t],\ j = 1, 2,\ \ \frac{m_2}{m_1} \in \left( \frac{1}{t^{1-\delta_3} - 1}, t^{1-\delta_2} - 1 \right) \right\}, \tag{1.18}$$

and with $\tilde{I}_3^U$ given by

$$\tilde{I}_3^U(\sigma,t,\delta_2,\delta_3) = -\sqrt{\frac{2}{\pi}}\Re\Bigg\{ e^{\frac{i\pi}{4}} t^{-\frac{\delta_3}{2}} (1-t^{\delta_3-1})^{\sigma-\frac{1}{2}+i(t-t^{\delta_3})} t^{i(\delta_3-1)t^{\delta_3}}$$
$$\times \sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]} \frac{1}{m_1^{s-it^{\delta_3}}} \frac{1}{m_2^{\bar{s}+it^{\delta_3}}} \frac{1}{\ln\left[\frac{m_2}{m_1}\left(t^{1-\delta_3}-1\right)\right]} [1+o(1)] \Bigg\}$$
$$\times \left[1+O(t^{-\delta_{23}})\right], \qquad t\to\infty. \tag{1.19}$$

In analogy with $\tilde{I}_3$, the integral $\tilde{I}_4$ is obtained from $I_4$ by replacing $|\zeta|^2$ with its large $t$ asymptotics. The analysis of $\tilde{I}_4$ as $t\to\infty$ is discussed in section 5, where using novel analytical and asymptotic techniques implemented in the complex plane, it is shown that $\tilde{I}_4$ also involves two terms: (i) One term can be computed exactly in terms of the leading asymptotics of $|\zeta(s)|^2$. (ii) The second term can be expressed in terms of the sum $2\Re\left\{S_4^P\right\}$ which can be evaluated exactly in terms of a certain residue computation, and the sum $\Re\left\{S_4^{SD}\right\}$ which involves a steepest descent computation. These terms imply the following result for $\tilde{I}_4$:

$$\tilde{I}_4(\sigma,t,\delta_3,\delta_4) = \Bigg[ -\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]} \frac{1}{m_1^s m_2^{\bar{s}}}$$
$$+ 2\Re\left\{S_4^P(\sigma,t,\delta_3)\right\} - \Re\left\{S_4^{SD}(\sigma,t,\delta_3)\right\}\Bigg]\left[1+O(t^{2\delta_{34}-1})\right],\ t\to\infty,$$
$$0<\sigma<1,\ 0<\delta_3<\frac{1}{2},\ 0<\delta_4<\frac{1}{2},\ \delta_{34}=\min(\delta_3,\delta_4), \tag{1.20}$$

where $S_4^P$ is defined by

$$S_4^P(\sigma,t,\delta_3) = \sum\sum_{m_1,m_2\in M_4(\delta_3,t)} \frac{1}{m_1^s m_2^{\bar{s}}} e^{-\frac{im_2}{m_1}t}, \tag{1.21}$$

with the set $M_4$ defined by

$$M_4(\delta_3,t) = \left\{ m_j = 1,\ldots,[t],\ j=1,2,\quad \frac{m_1}{m_2}\in(t^{1-\delta_3},t)\right\}, \tag{1.22}$$

whereas $S_4^{SD}$ is defined by

$$S_4^{SD}(\sigma,t,\delta_3) = \frac{1}{\pi}\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]} \frac{1}{m_1^s m_2^{\bar{s}}} E_4^{SD}(t,\delta_3,M), \tag{1.23}$$

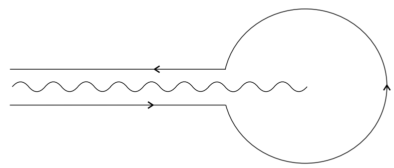

Figure 1: The Hankel contour.

with $E_4^{SD}(t,\delta_3,M)$ defined by

$$E_4^{SD} = \int_{H_1} \frac{e^{t^{\delta_3}[\omega-\frac{\pi}{2}+i\ln(Mt^{\delta_3}\omega)]}}{\omega[\frac{\pi}{2}-i\ln(Mt^{\delta_3}\omega)]}d\omega, \quad M=\frac{m_1}{m_2 t}, \tag{1.24}$$

and $H_1$ denoting the Hankel contour, with a branch cut along the negative real axis, see figure 1,

$$H_1 = \left\{re^{-i\pi}|1<r<\infty\right\} \cup \left\{e^{i\theta}|-\pi<\theta<\pi\right\} \cup \left\{re^{i\pi}|1<r<\infty\right\}.$$

It turns out that the sum of the rhs of (1.17) can be related to the large $t$-asymptotics of $|\zeta(s)|^2$. This is a consequence of the following exact identities:

$$\begin{aligned}2\Re\left\{\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_2^{\bar{s}}(m_1+m_2)^s}\right\} - \left(\sum_{m=1}^{[t]}\frac{1}{m^s}\right)\left(\sum_{m=1}^{[t]}\frac{1}{m^{\bar{s}}}\right) \\ = -\sum_{m=1}^{[t]}\frac{1}{m^{2\sigma}} + 2\Re\left\{\sum_{m=1}^{[t]}\sum_{n=[t]+1}^{[t]+m}\frac{1}{m^{\bar{s}}n^s}\right\}, \qquad s=\sigma+it\in\mathbb{C},\end{aligned} \tag{1.25}$$

and

$$\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_2^{\bar{s}}(m_1+m_2)^s} = \sum\sum_{(m_1,m_2)\in M}\frac{1}{m_2^{\bar{s}}(m_1+m_2)^s}+S_2(\sigma,t,\delta_2)+S_3(\sigma,t,\delta_3), \tag{1.26}$$

with $S_2$ and $S_3$ defined below,

$$\begin{aligned}S_2(\sigma,t,\delta_2) &= \sum\sum_{(m_1,m_2)\in M_2}\frac{1}{m_2^{\bar{s}}(m_1+m_2)^s}, \\ M_2(\delta_2,t) &= \left\{m_j=1,\ldots,[t],\ j=1,2,\quad \frac{m_2}{m_1}>t^{1-\delta_2}\right\},\end{aligned} \tag{1.27}$$

and

$$S_3(\sigma,t,\delta_3) = \sum\sum_{(m_1,m_2)\in M_3} \frac{1}{m_2^{\bar{s}}(m_1+m_2)^s}$$

$$M_3(\delta_3,t) = \left\{ m_j = 1,\ldots,[t],\ j=1,2,\quad \frac{m_2}{m_1} < \frac{1}{t^{1-\delta_3}-1} \right\}. \tag{1.28}$$

Furthermore the following estimates are derived in [FKL] and [KF], respectively (see also section 6):

$$S_3(\sigma,t,\delta_3) = S_4^P(\sigma,t,\delta_3)\left[1 + O\left(t^{2\delta_3-1}\right)\right], \qquad t\to\infty, \tag{1.29}$$

and

$$\sum_{m=1}^{[t]}\sum_{n=[t]+1}^{[t]+m} \frac{1}{m^{\bar{s}}n^s} = \begin{cases} O\left(t^{\frac{1}{2}-\frac{5}{3}\sigma}\ln t\right), & 0\le\sigma\le\frac{1}{2}, \\ O\left(t^{\frac{1}{3}-\frac{4}{3}\sigma}\ln t\right), & \frac{1}{2}<\sigma<1, \end{cases} \qquad t\to\infty. \tag{1.30}$$

**The case $\sigma = \frac{1}{2}$**

We restrict our consideration to the case $\sigma = \frac{1}{2}$, with $\delta_1, \delta_3, \delta_4$ arbitrarily small positive constants and $0 < \delta_2 < 1$.

Adding the expressions for $\tilde{I}_3$ and $\tilde{I}_4$ given by equations (1.16) and (1.20), and then using equations (1.25) and (1.26) to simplify the resulting expression, we find

$$\tilde{I}_3(\sigma,t,\delta_2,\delta_3) + \tilde{I}_4(\sigma,t,\delta_3,\delta_4) \sim$$

$$\Re\left\{S_4^{SD}(\sigma,t,\delta_3)\right\} - \tilde{I}_3^U(\sigma,t,\delta_3) - 2\Re\left\{S_2(\sigma,t,\delta_2)\right\} - \sum_{m=1}^{[t]}\frac{1}{m^{2\sigma}}, \quad t\to\infty,$$

$$\sigma = \frac{1}{2},\ 0<\delta_2<1,\ \delta_3,\ \delta_4 \text{ small positive constants.} \tag{1.31}$$

Comparing this equation with (1.9), namely

$$\tilde{I}_3 + \tilde{I}_4 \sim -I_2 - \mathcal{G}(\sigma,t), \qquad t\to\infty, \tag{1.32}$$

evaluated at $\sigma = \frac{1}{2}$, where the asymptotic behaviour of $\mathcal{G}(\sigma,t)$ ad $t\to\infty$ is given by (1.8), we find the following: first, the leading asymptotics of $\Re\left\{S_4^{SD}\right\}$ is given by the rhs of (1.19) (this result can be verified explicitly using the steepest descent computation in remark 5.1). Second,

$$\Re\left\{S_2\left(\frac{1}{2},t,\delta_2\right)\right\} = I_2\left(\frac{1}{2},t,\delta_3\right) + O(1), \qquad t\to\infty. \tag{1.33}$$

Using (1.13) and (1.27) evaluated at $\sigma = \frac{1}{2}$, this equation yields

$$\sum_{m_2=\left[t^{1-\delta_2}\right]}^{[t]} \sum_{m_1=1}^{\left[\frac{m_2}{t^{1-\delta_2}}\right]} \frac{1}{(m_1+m_2)^{\frac{1}{2}+it} m_2^{\frac{1}{2}-it}} = O\left(t^{-\frac{1}{2}+\delta_2} \ln t\right) + O(1), \quad t \to \infty. \tag{1.34}$$

For the particular case of $\delta_2 = 1/2 + \epsilon$, the above equation becomes

$$\sum_{m_2=\left[t^{\frac{1}{2}-\epsilon}\right]}^{[t]} \sum_{m_1=1}^{\left[\frac{m_2}{t^{\frac{1}{2}-\epsilon}}\right]} \frac{1}{(m_1+m_2)^{\frac{1}{2}+it} m_2^{\frac{1}{2}-it}} = O\left(t^{\epsilon} \ln t\right), \qquad t \to \infty, \tag{1.35}$$

for any $\epsilon > 0$.

The extensive cancellations occurring in the asymptotic evaluation of $\tilde{I}_3 + \tilde{I}_4$ motivate the direct asymptotic evaluation of this integral: let $\tilde{I}_{34}$ denote the integral obtained by replacing $|\zeta|^2$ with its leading asymptotics. It is shown in section 7 that

$$\begin{aligned}\tilde{I}_{34}\left(\frac{1}{2}, t, \delta_2, \delta_4\right) = \Bigg[ &- \left|\sum_{m=1}^{[t]} \frac{1}{m^{\frac{1}{2}+it}}\right|^2 + 2\Re\left\{S_{34}(\sigma, t, \delta_2)\right\} \\ &+ O\left(t^{\frac{1}{2}-2\delta_2} \ln t\right)\Bigg]\left[1 + O(t^{-\delta_2})\right], \quad 0 < \delta_2 < 1, \quad t \to \infty,\end{aligned} \tag{1.36}$$

where $S_{34}$ is defined by

$$S_{34}(\sigma, t, \delta_2) = t^{i(\delta_2-1)t} e^{i(t-t^{\delta_2})} \sum\sum_{m_1, m_2 \in M_{34}(\delta_2, t)} \frac{1}{m_1^{\frac{1}{2}+it} m_2^{\frac{1}{2}-it}} e^{-i\frac{m_2}{m_1} t^{\delta_2}}, \tag{1.37}$$

and the set $M_{34}$ is defined by

$$M_{34} = \left\{ m_j = 1, \dots, [t],\ j = 1, 2,\ \ \frac{m_2}{m_1} \in (t^{-\delta_2}, t^{1-\delta_2} - 1) \right\}.$$

Comparing (1.36) with the equation (1.32) evaluated at $\sigma = \frac{1}{2}$, we conclude that

$$\begin{aligned}&\left|\sum_{m=1}^{[t]} \frac{1}{m^{\frac{1}{2}+it}}\right|^2 - 2\Re\left\{S_{34}\left(\frac{1}{2}, t, \delta_2\right)\right\} = -\mathcal{G}\left(\frac{1}{2}, t\right) + I_2\left(\frac{1}{2}, t, \delta_2\right) + O\left(t^{\frac{1}{2}-2\delta_2} \ln t\right) \\ &= -\ln t + O(1) + O\left(t^{-\frac{1}{2}+\delta_2} \ln t\right) + O\left(t^{\frac{1}{2}-2\delta_2} \ln t\right), \quad 0 < \delta_2 < 1, \quad t \to \infty.\end{aligned} \tag{1.38}$$

For the particular case of $\delta_2 = 1/2 + \epsilon$, the above equation becomes

$$\left| \sum_{m=1}^{[t]} \frac{1}{m^{\frac{1}{2}+it}} \right|^2 - 2\Re\left\{ S_{34}\left(\frac{1}{2}, t, \frac{1}{2}+\epsilon\right)\right\} = O\left(t^{\epsilon}\ln t\right), \quad \text{for any } \epsilon > 0, \tag{1.39}$$

with $S_{34}$ given in (1.37).

## 2 A Singular Integral Equation for $|\zeta(s)|^2$ and its asymptotic form

In this section we derive equations (1.3) and (1.9).

**Proposition 2.1** *Let $u = \sigma + it$, $\sigma > 1$, $\omega \in \mathbb{R}$, $|\omega| < 1$. The following identity is valid:*

$$\Gamma(u)(1+\omega)^{-u} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \Gamma(u+z)\Gamma(-z)\omega^z dz, \quad -\sigma < c < 0. \tag{2.1}$$

**Proof** Consider a clockwise contour $C$ enclosing $0, 1, 2, \ldots$, but not $-1, -2, \ldots$. Then $\Gamma(u+z)$ does not have any poles in the domain enclosed by $C$, whereas $\Gamma(-z)$ has poles at $z = n$ for $n = 0, 1, 2, \ldots$, with residues given by

$$\mathrm{Res}_n\Gamma(-z) = \frac{(-1)^{n+1}}{n!}, \quad n = 0, 1, \ldots.$$

Hence

$$\frac{1}{2i\pi}\int_C \Gamma(u+z)\Gamma(-z)\omega^z dz = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\Gamma(u+n)\omega^n = (1+\omega)^{-u}\Gamma(u),$$

where in the last step we have used the binomial theorem and the fact that $|w| < 1$.

By Stirling's approximation, we have the formula

$$\Gamma(z) = z^{-z}z^{z}\left(\frac{2\pi}{z}\right)^{\frac{1}{2}}(1 + O(z^{-1})), \quad z \to \infty, \quad |\arg z| \leq \pi - \delta.$$

Thus,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{1-b}, \quad z \to \infty.$$

This estimate, together with the identity

$$\Gamma(-z) = -\frac{\pi}{\sin(\pi z)\Gamma(1+z)},$$

yield

$$\Gamma(-z)\Gamma(z+u) \sim z^{u-1}\frac{\pi}{2i(e^{-i\pi z} - e^{i\pi z})}.$$

Thus, the product of the above gamma functions decays, and since $\omega^z = e^{z\ln\omega}$ also decays for $|\omega| < 1$, it follows that we can replace the contour $C$ by the straight line from $c - i\infty$ to $c + i\infty$. **QED**

**Proposition 2.2** *Let $s = \sigma + it$, $\sigma > 1$, $t \geq 0$. Define the modified Hurwitz function by the expression*

$$\zeta_1(s,\alpha) = \sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^s}, \quad \Re s > 1, \quad \alpha \geq 0, \tag{2.2}$$

*and via analytic continuation for all $s \in \mathbb{C}$. The following identity is valid:*

$$|\zeta_1(s,\alpha)|^2 = \zeta_1(2\sigma,\alpha) + \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{\Gamma(s+z)}{\Gamma(s)} + \frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right) \times \Gamma(-z)\zeta(-z)\zeta_1(2\sigma+z,\alpha)dz, \tag{2.3}$$

*where $\bar{s} = \sigma - it$ and*

$$\max(-\sigma, 1-2\sigma) < c < -1, \quad \sigma > 1, \quad t > 0.$$

**Proof** Define the function $f(u,v,\alpha)$ by

$$f(u,v,\alpha) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(n+\alpha)^{-u}(n+m+\alpha)^{-v}, \qquad \alpha \geq 0,\ \Re u > 1,\ \Re v > 1. \tag{2.4}$$

Letting in (2.1) $\omega = \frac{m}{n+\alpha}$, we find

$$\Gamma(u)(n+\alpha)^u(m+n+\alpha)^{-u} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\Gamma(u+z)\Gamma(-z)m^z(n+\alpha)^{-z}dz, \ \ -\sigma < c < 0.$$

Hence, multiplying this equation by $\Gamma(u)^{-1}(n+\alpha)^{-u}(n+\alpha)^{-v}$, we find

$$(n+\alpha)^{-v}(m+n+\alpha)^{-u} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(u+z)}{\Gamma(u)}\Gamma(-z)m^z(n+\alpha)^{-(z+u+v)}dz. \tag{2.5}$$

Summing over $m,\ n$ and using the definition of $f$ we obtain

$$f(u,v,\alpha)=\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(u+z)}{\Gamma(u)}\Gamma(-z)\zeta(-z)\zeta_1(u+v+z,\alpha)dz. \qquad (2.6)$$

The definitions of $\zeta_1(u,\alpha)$ and $f(u,v,\alpha)$ yield the identity

$$\zeta_1(u,\alpha)\zeta_1(v,\alpha)=\zeta_1(u+v,\alpha)+f(u,v,\alpha)+f(v,u,\alpha), \qquad (2.7)$$

through straightforward calculations.

Replacing in (2.7) the functions $f$ via the representations (2.6), and then replacing in the resulting equation $u$ and $v$ by $s$ and $\bar{s}$ we obtain equation (2.3).

**QED**

**Lemma 2.1** *The Riemann zeta function $\zeta(s)$ satisfies the identity*

$$|\zeta(s)|^2=\mathcal{G}(\sigma,t)+\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z)dz,$$

$$s=\sigma+it,\quad \sigma>0,\quad t>0,\quad \max(-1,-\sigma)<c<\min(0,1-2\sigma), \qquad (2.8)$$

*where the function $\mathcal{G}(\sigma,t)$ is defined by the following expressions:*

$$\mathcal{G}(\sigma,t)=\zeta(2\sigma)+\left(\frac{\Gamma(1-\bar{s})}{\Gamma(s)}+\frac{\Gamma(1-s)}{\Gamma(\bar{s})}\right)\Gamma(2\sigma-1)\zeta(2\sigma-1)+\frac{2(\sigma-1)\zeta(2\sigma-1)}{(\sigma-1)^2+t^2},\qquad \sigma\neq\frac{1}{2}, \qquad (2.9)$$

$$\mathcal{G}\left(\frac{1}{2},t\right)=\Re\left(\Psi\left(\frac{1}{2}+it\right)\right)+2\gamma-\ln 2\pi+\frac{2}{1+4t^2}, \qquad (2.10)$$

*with $\Psi(z)$ denoting the digamma function, i.e.,*

$$\Psi(z)=\frac{\frac{\mathrm{d}}{\mathrm{dz}}\Gamma(z)}{\Gamma(z)},\quad z\in\mathbb{C}, \qquad (2.11)$$

*and $\gamma$ denoting the Euler constant.*

**Proof**

Letting $\alpha=0$ in equation (2.3) we find

$$|\zeta(s)|^2=\zeta(2\sigma)$$

$$+\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z)\mathrm{d}z,$$

$$s=\sigma+it,\quad \sigma>1,\quad t>0,\quad \max(-\sigma,1-2\sigma)<c<-1. \tag{2.12}$$

We first assume that $\sigma\neq\frac{1}{2}$ and deform the contour occurring in equation (2.12) past the poles at $z=1-2\sigma$ and $z=-1$. This yields

$$|\zeta(s)|^2=\zeta(2\sigma)+R_1+R_2$$

$$+\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z)\mathrm{d}z,$$

$$s=\sigma+it,\quad \sigma>0,\quad t>0,\quad \max(-1,-\sigma)<c<\min(0,1-2\sigma), \tag{2.13}$$

where $R_1$ and $R_2$ are defined by the following expressions:

$$R_1=\operatorname*{Res}_{z=1-2\sigma}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z),$$

$$R_2=-\operatorname*{Res}_{z=-1}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z).$$

The above residues can be computed as follows:

$$R_1=\left(\frac{\Gamma(1-\bar{s})}{\Gamma(s)}+\frac{\Gamma(1-s)}{\Gamma(\bar{s})}\right)\Gamma(2\sigma-1)\zeta(2\sigma-1)\operatorname*{Res}_{z=1-2\sigma}\zeta(2\sigma+z), \tag{2.14}$$

$$R_2=-\left(\frac{\Gamma(s-1)}{\Gamma(s)}+\frac{\Gamma(\bar{s}-1)}{\Gamma(\bar{s})}\right)\Gamma(1)\zeta(2\sigma-1)\operatorname*{Res}_{z=-1}\zeta(-z). \tag{2.15}$$

Using the identities

$$\Gamma(u)=u\Gamma(u-1),\ u\in\mathbb{C};\quad \Gamma(1)=1, \tag{2.16}$$

and

$$\zeta(1+\varepsilon)=\frac{1}{\varepsilon}+\gamma+O(\varepsilon),\quad \varepsilon\to 0, \tag{2.17}$$

equations (2.14) and (2.15) become

$$R_1=\left(\frac{\Gamma(1-\bar{s})}{\Gamma(s)}+\frac{\Gamma(1-s)}{\Gamma(\bar{s})}\right)\Gamma(2\sigma-1)\zeta(2\sigma-1), \tag{2.18}$$

$$R_2=\frac{2(\sigma-1)}{(\sigma-1)^2+t^2}\ \zeta(2\sigma-1). \tag{2.19}$$

Substituting in (2.13) the expressions for $R_1$ and $R_2$ given by equations (2.18) and (2.19) we find equation (2.9).

We next compute the limit of the rhs of equation (2.9) as $\sigma \to \frac{1}{2}$. For this purpose, in addition to equation (2.17) we will also use the following identities:

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon), \quad \varepsilon \to 0, \tag{2.20a}$$

and

$$\zeta(\varepsilon) = -\frac{1}{2}\left(1 + \varepsilon \ln 2\pi + O(\varepsilon^2)\right), \quad \varepsilon \to 0. \tag{2.20b}$$

Letting $\sigma = \frac{1}{2} + \frac{\varepsilon}{2}$, we find

$$\frac{\Gamma(1-\bar{s})}{\Gamma(s)} = \frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)} = \frac{\Gamma\left(\frac{1}{2}+it-\frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{1}{2}+it+\frac{\varepsilon}{2}\right)}$$

$$= \frac{1 - \frac{\varepsilon}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right) + O(\varepsilon^2)}{1 + \frac{\varepsilon}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right) + O(\varepsilon^2)}.$$

Thus,

$$\frac{\Gamma(1-\bar{s})}{\Gamma(s)} = 1 - \varepsilon\Psi\left(\frac{1}{2}+it\right) + O(\varepsilon^2). \tag{2.21}$$

Equations (2.20) imply

$$\Gamma(2\sigma-1)\zeta(2\sigma-1) = \Gamma(\varepsilon)\zeta(\varepsilon) = -\frac{1}{2}\left(\frac{1}{\varepsilon} + \ln 2\pi - \gamma + O(\varepsilon)\right),$$

$$\varepsilon \to 0. \tag{2.22}$$

Using in equation (2.9), equations (2.17) and (2.20) we find

$$\mathcal{G}\left(\frac{1}{2}+\frac{\varepsilon}{2},t\right) = \frac{1}{\varepsilon} + \gamma + \frac{2}{1+4t^2} + O(\varepsilon)$$

$$-\left[1 - \varepsilon\Re\left(\Psi\left(\frac{1}{2}+it\right)\right) + O(\varepsilon^2)\right]\left[\frac{1}{\varepsilon} + \ln 2\pi - \gamma + O(\varepsilon)\right], \quad \varepsilon \to 0.$$

Hence,

$$\mathcal{G}\left(\frac{1}{2}+\frac{\varepsilon}{2},t\right) = \Re\left(\Psi\left(\frac{1}{2}+it\right)\right) + 2\gamma - \ln 2\pi + \frac{2}{1+4t^2} + O(\varepsilon), \quad \varepsilon \to 0,$$

and equation (2.10) follows. **QED**

Using equation (2.8) and employing the Plemelj formulae, the following basic equation can be derived.

**Theorem 2.1** *The Riemann zeta function $\zeta(s)$ satisfies the integral equation*

$$\frac{t}{\pi}\oint_{-\infty}^{\infty}\Re\left\{\frac{\Gamma(it-i\tau t)}{\Gamma(\sigma+it)}\Gamma(\sigma+i\tau t)\right\}|\zeta(\sigma+i\tau t)|^2\,d\tau+\mathcal{G}(\sigma,t)=0,$$

$$0<\sigma<1,\quad t>0, \tag{2.23}$$

*where $\mathcal{G}(\sigma,t)$ is defined by equations (2.9) and (2.10), and the principal value integral in equation (2.23) is defined with respect to $\tau=1$.*

**Proof** We take the limit of equation (2.8) as $c\downarrow-\sigma$. In this limit the poles $z=-s$ and $z=-\bar{s}$ approach the contour of integration from the left. Using Plemelj's formulae we find the following equation:

$$|\zeta(s)|^2=\mathcal{G}(\sigma,t)+P_1+P_2$$

$$+\frac{1}{2i\pi}\oint_{-\sigma-i\infty}^{-\sigma+i\infty}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z)\mathrm{d}z, \tag{2.24}$$

where

$$P_1=\frac{i\pi}{2i\pi}\operatorname*{Res}_{z=-s}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z), \tag{2.25}$$

$$P_2=\frac{i\pi}{2i\pi}\operatorname*{Res}_{z=-\bar{s}}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z). \tag{2.26}$$

Employing equation (2.20a) it follows that

$$P_1=P_2=\frac{1}{2}|\zeta(s)|^2. \tag{2.27}$$

Thus, equation (2.24) simplifies to the equation

$$\frac{1}{2i\pi}\oint_{-\sigma-i\infty}^{-\sigma+i\infty}\left(\frac{\Gamma(s+z)}{\Gamma(s)}+\frac{\Gamma(\bar{s}+z)}{\Gamma(\bar{s})}\right)\Gamma(-z)\zeta(-z)\zeta(2\sigma+z)\mathrm{d}z+\mathcal{G}(\sigma,t)=0,$$

$$s=\sigma+it,\quad t>0,\quad 0<\sigma<1. \tag{2.28}$$

Letting $z=-\sigma+i\tau t$, equation (2.28) becomes

$$\frac{t}{2\pi}\oint_{-\infty}^{\infty}\left(\Gamma(it+i\tau t)\frac{\Gamma(\sigma-i\tau t)}{\Gamma(\sigma+it)}+\Gamma(-it+i\tau t)\frac{\Gamma(\sigma-i\tau t)}{\Gamma(\sigma-it)}\right)|\zeta(\sigma+i\tau t)|^2\,\mathrm{d}\tau$$

$$+\mathcal{G}(\sigma,t)=0, \tag{2.29}$$

where now the principal value integral is defined with respect to $\tau = 1$ and $\tau = -1$.

Letting $\tau \to -\tau$ in the first part of the integral in the lhs of equation (2.29) we find

$$\frac{t}{2\pi} \oint_{-\infty}^{\infty} \Gamma(it + i\tau t) \frac{\Gamma(\sigma - i\tau t)}{\Gamma(\sigma + it)} \left|\zeta(\sigma + i\tau t)\right|^2 \mathrm{d}\tau$$
$$= \frac{t}{2\pi} \oint_{-\infty}^{\infty} K(\sigma, t, \tau) \left|\zeta(\sigma + i\tau t)\right|^2 \mathrm{d}\tau,$$

where the kernel $K$ is defined by

$$K(\sigma, t, \tau) = \Gamma(it - i\tau t) \frac{\Gamma(\sigma + i\tau t)}{\Gamma(\sigma + it)}, \quad 0 < \sigma < 1, \quad t > 0, \quad \tau \in (-\infty, \infty). \tag{2.30}$$

Hence, equation (2.29) can be rewritten in the form of equation (2.23).

**QED**

**Remark 2.1** *Equation* (2.20a) *implies the following estimate for the singularity of* $\Gamma(it - i\tau t)$ *at* $\tau = 1$*:*

$$K(\sigma, t, \tau) = -\frac{1}{it(\tau - 1)} + O(1), \quad \tau \to 1. \tag{2.31}$$

*The integral equation* (2.23) *involves the real part of* $K$ *which according to equation* (2.31) *is non-singular at* $\tau = 1$*, hence no principal value integral is needed in* (2.23)*. However, in the analysis that follows, instead of the real part of* $K$ *we will first compute* $K$*, and for this reason it is useful to retain the principal value.*

In order to analyse the large $t$ behaviour of equation (2.23) we first use the fact that for $-\infty < \tau < +\infty$ the gamma functions occurring in the lhs of (2.23) decay exponentially, unless $-t^{\delta_1 - 1} < \tau < 1 + t^{\delta_4 - 1}$, where $\delta_1$ and $\delta_4$ are positive constants.

**Lemma 2.2** *The Riemann zeta function* $\zeta(s)$ *satisfies the integral equation*

$$\frac{t}{\pi} \oint_{-t^{\delta_1 - 1}}^{1 + t^{\delta_4 - 1}} \Re\left\{ \frac{\Gamma(it - i\tau t)}{\Gamma(\sigma + it)} \Gamma(\sigma + i\tau t) \right\} \left|\zeta(\sigma + i\tau t)\right|^2 d\tau + \mathcal{G}(\sigma, t)$$
$$+ O\left(e^{-\pi t^{\delta_{14}}}\right) = 0, \quad 0 < \sigma < 1, \; \delta_1 > 0, \; \delta_4 > 0, \; \delta_{14} = \min(\delta_1, \delta_4), \quad t \to \infty, \tag{2.32}$$

*where $\Gamma(z)$, $z \in \mathbb{C}$, denotes the gamma function, $\mathcal{G}(\sigma,t)$ is defined by equations* (2.9) *and* (2.10)*, and the principal value integral is defined with respect to $\tau = 1$.*

**Proof** Starting with Stirling's formula, the following formulae are derived in the appendix of [FL]:

$$\Gamma(\sigma + i\xi) = \sqrt{2\pi}\xi^{\sigma-\frac{1}{2}}e^{-\frac{\pi\xi}{2}}e^{-\frac{i\pi}{4}}e^{-i\xi}\xi^{i\xi}e^{\frac{i\pi\sigma}{2}}\left[1 + O\left(\frac{1}{\xi}\right)\right], \quad \xi \to \infty, \tag{2.33a}$$

and

$$\Gamma(\sigma - i\xi) = \sqrt{2\pi}\xi^{\sigma-\frac{1}{2}}e^{-\frac{\pi\xi}{2}}e^{\frac{i\pi}{4}}e^{i\xi}\xi^{-i\xi}e^{-\frac{i\pi\sigma}{2}}\left[1 + O\left(\frac{1}{\xi}\right)\right], \quad \xi \to \infty. \tag{2.33b}$$

These formulae imply that the kernel $K$ defined in (2.30) satisfies the following estimate:

$$\begin{aligned} K(\sigma,t,\tau) &= O\left((|1-\tau|t)^{-\frac{1}{2}}e^{-\frac{\pi}{2}|1-\tau|t}\ \frac{(|\tau|t)^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|\tau|t}}{t^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}t}}\right) \\ &= O\left((|1-\tau|t)^{-\frac{1}{2}}|\tau|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}t(|1-\tau|+|\tau|-1)}\right), \end{aligned}$$

provided that as $t \to \infty$,

$$|\tau t| \to \infty, \quad |1-\tau|t \to \infty. \tag{2.34}$$

In order to ensure the validity of (2.34), the boundaries $\tau = 0$ and $\tau = 1$ must be analysed carefully. In this connection, we decompose the infinite line as the union of the following three subintervals:

$$-\infty < \tau \leq -t^{\delta_1 - 1}, \qquad -t^{\delta_1-1} \leq \tau \leq 1 + t^{\delta_4 - 1}, \qquad 1 + t^{\delta_4-1} < \tau < \infty,$$

with $\delta_1 > 0$, $\delta_4 > 0$.

For $\tau$ in the first interval we find

$$e^{-\frac{\pi}{2}t(|1-\tau|+|\tau|-1)} = e^{-\frac{\pi}{2}t(1-\tau-\tau-1)} = e^{\pi\tau t}.$$

Hence,

$$\left| t\int_{-\infty}^{-\frac{t^{\delta_1}}{t}} K(\sigma,t,\tau)|\zeta(\sigma+it\tau)|^2 d\tau \right| \leq t\int_{-\infty}^{-\frac{t^{\delta_1}}{t}} \frac{1}{\sqrt{t}}|\tau|^{\sigma-\frac{1}{2}}e^{\pi\tau t}|\zeta(\sigma+it\tau)|^2 d\tau$$

$$= t^{-\sigma} \int_{t^{\delta_1}}^{\infty} x^{\sigma-\frac{1}{2}} e^{-\pi x} |\zeta(\sigma+ix)|^2 dx \le e^{-\pi t^{\delta_1}}.$$

Similar considerations apply for $\tau$ in the third interval, where

$$e^{-\frac{\pi}{2}t(|1-\tau|+|\tau|-1)} = e^{-\frac{\pi}{2}t(\tau-1+\tau-1)} = e^{-\pi t(\tau-1)}.$$

Thus, for large $t$ the kernel $K$ is exponentially small for $\tau$ outside the interval $-t^{\delta_1-1} \le \tau \le 1+t^{\delta_4-1}$, and hence equation (2.23) becomes equation (2.32).

**QED**

Theorem 2.1 implies the following result.

**Corollary 2.1** *The Riemann zeta function $\zeta(s)$ satisfies the integral equation*

$$\frac{t}{\pi} \fint_{-t^{\delta_1-1}}^{1+t^{\delta_4-1}} \Re\left\{ \frac{\Gamma(it-i\tau t)}{\Gamma(\sigma+it)} \Gamma(\sigma+i\tau t) \right\} |\zeta(\sigma+i\tau t)|^2 \, d\tau$$
$$+ \begin{cases} \zeta(2\sigma) + 2\Gamma(2\sigma-1)\zeta(2\sigma-1)\sin(\pi\sigma) t^{1-2\sigma} \left(1 + O\left(\frac{1}{t}\right)\right), & 0<\sigma<1, \quad \sigma \neq \frac{1}{2}, \\ \ln t + 2\gamma - \ln 2\pi, & \sigma = \frac{1}{2}, \end{cases}$$
$$+ O\left(e^{-\pi t^{\delta_{14}}}\right) + O\left(\frac{1}{t^2}\right) = 0, \quad \delta_1 > 0, \ \delta_4 > 0, \ \delta_{14} = \min(\delta_1, \delta_4), \quad t \to \infty, \tag{2.35}$$

*where $\Gamma(z)$, $z \in \mathbb{C}$, denotes the gamma function, the principal value integral is defined with respect to $\tau = 1$.*

**Proof** Employing (2.33) with $\xi = t$ in the definition of $\mathcal{G}(\sigma, t)$ we find

$$\mathcal{G}(\sigma,t) = 2\Gamma(2\sigma-1)\zeta(2\sigma-1)\sin(\pi\sigma)t^{1-2\sigma}\left[1 + O\left(\frac{1}{t}\right)\right] + \zeta(2\sigma) + O\left(\frac{1}{t^2}\right),$$
$$0<\sigma<1, \ t \to \infty. \tag{2.36}$$

Furthermore, the estimate

$$\Psi\left(\frac{1}{2}+it\right) = \ln t + \frac{i\pi}{2} + O\left(\frac{1}{t^2}\right), \quad t \to \infty, \tag{2.37}$$

implies

$$\mathcal{G}\left(\frac{1}{2}, t\right) = \ln t + 2\gamma - \ln 2\pi + O\left(\frac{1}{t^2}\right), \quad t \to \infty. \tag{2.38}$$

Replacing in equation (2.32) the function $\mathcal{G}(\sigma, t)$ by equations (2.36) and (2.38), equation (2.32) becomes equation (2.35).

**QED**

# 3 The Rigorous Estimation of $I_1$ and $I_2$

**Lemma 3.1** *Let $I_1(\sigma, t, \delta_1)$ be defined by*

$$I_1(\sigma, t, \delta_1) = \frac{t}{\pi} \int_{-t^{\delta_1 - 1}}^{\frac{1}{t}} \Re\left\{ \frac{\Gamma(it - it\tau)}{\Gamma(\sigma + it)} \Gamma(\sigma + it\tau) \right\} |\zeta(\sigma + it\tau)|^2 d\tau, \quad (3.1)$$

$$0 < \sigma < 1, \quad t > 0, \quad 0 < \delta_1 < 1.$$

*Then, for sufficiently small $\delta_1$,*

$$I_1(\sigma, t, \delta_1) = \begin{cases} O\left( \frac{t^{\delta_1\left(\sigma + \frac{5}{6}\right)}}{t^\sigma} \right), & \frac{1}{2} < \sigma < 1, \\ O\left( \frac{t^{\frac{4}{3}\delta_1}}{t^\sigma} \right), & 0 < \sigma \le \frac{1}{2}. \end{cases}, \quad t \to \infty. \quad (3.2)$$

**Proof** In the interval of integration, we have

$$-t^{\delta_1} \le t\tau \le 1. \quad (3.3)$$

Thus,

$$t - 1 \le t - \tau t \le t + t^{\delta_1}.$$

Hence, $t - \tau t \to \infty$ as $t \to \infty$, and therefore we can use the asymptotic formula (2.33a) to compute both $\Gamma(\sigma + it)$ and $\Gamma(it - it\tau)$:

$$\Gamma(\sigma + it) = \sqrt{2\pi} t^{\sigma - \frac{1}{2}} e^{-\frac{\pi t}{2}} e^{-\frac{i\pi}{4}} e^{\frac{i\pi\sigma}{2}} e^{-it} t^{it} \left[ 1 + O\left(\frac{1}{t}\right) \right], \quad t \to \infty, \quad (3.4a)$$

$$\Gamma(it - it\tau) = \sqrt{2\pi} t^{-\frac{1}{2}} (1-\tau)^{-\frac{1}{2}} e^{-\frac{\pi t}{2}(1-\tau)} e^{-\frac{i\pi}{4}} e^{-it(1-\tau)} t^{it(1-\tau)} (1-\tau)^{it(1-\tau)}$$
$$\times \left[ 1 + O\left( \frac{1}{t - t\tau} \right) \right], \quad t \to \infty. \quad (3.4b)$$

Equations (3.4) together with the inequality

$$\frac{1}{t - \tau t} \le \frac{1}{t-1},$$

imply that

$$\frac{\Gamma(it - it\tau)}{\Gamma(\sigma + it)} = t^{-\sigma} (1-\tau)^{-\frac{1}{2}} e^{\frac{\pi t\tau}{2}} e^{i\varphi(t,\tau)} \left[ 1 + O\left(\frac{1}{t}\right) \right], \quad t \to \infty, \quad (3.5)$$

where $\varphi$ is a real function.

Equation (3.3) implies that $\tau t$ is either O(1), or $\tau t = O(t^{\Delta})$, $\Delta < \delta_1$.

If $\tau t = O(1)$, then

$$\Gamma(\sigma + i\tau t) = O(1), \quad t \to \infty.$$

If $\tau t \to -\infty$, then using (2.33b) with $\xi = \tau t$, we find

$$e^{\frac{\pi\tau t}{2}}\Gamma(\sigma + i\tau t) = O\left(|\tau t|^{\sigma-\frac{1}{2}}\right), \quad \tau t \to -\infty.$$

Thus, recalling that $\tau t = O\left(t^{\delta_1}\right)$, we find

$$e^{\frac{\pi\tau t}{2}}\Gamma(\sigma + i\tau t) = \begin{cases} O\left(t^{\delta_1\left(\sigma-\frac{1}{2}\right)}\right), & \frac{1}{2} < \sigma < 1, \\ O\left(1\right), & 0 < \sigma \leq \frac{1}{2}. \end{cases}$$

Combining the above equations with (3.5), we find

$$\Re\left\{\frac{\Gamma(it - it\tau)}{\Gamma(\sigma + it)}\Gamma(\sigma + it\tau)\right\} = O(t^{-\sigma})\left(1 - \frac{O(1)}{t}\right)^{-\frac{1}{2}}\left[1 + O\left(\frac{1}{t}\right)\right] \\ \times \begin{cases} O\left(t^{\delta_1\left(\sigma-\frac{1}{2}\right)}\right), & \frac{1}{2} < \sigma < 1, \\ O\left(1\right), & 0 < \sigma \leq \frac{1}{2}. \end{cases}, \quad t \to \infty.$$

Using the estimate (see Theorems 5.12 and 5.18 in [T])

$$|\zeta(\sigma + it^{\Delta})| = O\left(t^{\frac{\Delta}{6}}\right), \quad t \to \infty,$$

where $\Delta$ is a positive constant, we conclude that

$$\Re\left\{\frac{\Gamma(it - it\tau)}{\Gamma(\sigma + it)}\Gamma(\sigma + it\tau)\right\}|\zeta(\sigma + it\tau)|^2 = O\left(t^{\frac{\delta_1}{3}-\sigma}\right)\left[1 + O\left(\frac{1}{t}\right)\right] \\ \times \begin{cases} O\left(t^{\delta_1\left(\sigma-\frac{1}{2}\right)}\right), & \frac{1}{2} < \sigma < 1, \\ O\left(1\right), & 0 < \sigma \leq \frac{1}{2}. \end{cases}, \quad t \to \infty.$$

The above equation together with the mean value theorem, imply

$$I_1 = \frac{t}{\pi}\left(\frac{1}{t} + \frac{t^{\delta_1}}{t}\right)O\left(t^{\frac{\delta_1}{3}-\sigma}\right) \times \begin{cases} O\left(t^{\delta_1\left(\sigma-\frac{1}{2}\right)}\right), & \frac{1}{2} < \sigma < 1, \\ O\left(1\right), & 0 < \sigma \leq \frac{1}{2}. \end{cases}, \quad t \to \infty,$$

which yields (3.2). **QED**

**Lemma 3.2** *Let $I_2(\sigma, t, \delta_2)$ be defined by*

$$I_2(\sigma, t, \delta_2) = \frac{t}{\pi} \int_{\frac{1}{t}}^{t^{\delta_2 - 1}} \Re\left\{ \frac{\Gamma(it - it\tau)}{\Gamma(\sigma + it)} \Gamma(\sigma + it\tau) \right\} |\zeta(\sigma + it\tau)|^2 d\tau, \quad (3.6)$$

$$0 < \sigma < 1, \quad t > 0, \quad 0 < \delta_2 < 1.$$

*Then,*

$$I_2\left(\frac{1}{2}, t, \delta_2\right) = O\left(t^{-\frac{1}{2}+\delta_2} \ln t\right), \quad \sigma = \frac{1}{2}, \quad t \to \infty, \quad (3.7)$$

*and*

$$I_2(\sigma, t, \delta_2) = \begin{cases} O\left(t^{-\sigma + \left(\sigma + \frac{1}{2}\right)\delta_2} \zeta(2\sigma)\right), & \frac{1}{2} < \sigma < 1, \quad t \to \infty, \\ O\left(t^{-\sigma + 2(1-\sigma)\delta_2} \zeta(2 - 2\sigma)\right), & 0 < \sigma < \frac{1}{2}, \quad t \to \infty. \end{cases} \quad (3.8)$$

**Proof** In the interval of integration, we have

$$1 \leq t\tau \leq t^{\delta_2}.$$

Thus,

$$t - t^{\delta_2} \leq t - t\tau \leq t - 1.$$

Hence, $t - t\tau \to \infty$ as $\tau \to \infty$, and therefore we can use (2.33a) to compute $|\Gamma(it - it\tau)$. Thus, taking into consideration the inequality

$$\frac{1}{t - t\tau} \leq \frac{1}{t - t^{\delta_2}} = \frac{1}{t(1 - t^{\delta_2 - 1})} = \frac{1}{t} + O(t^{\delta_2 - 2}), \quad t \to \infty,$$

we find that equation (3.5) is still valid. Inserting the expression (3.6) in the definition of $I_2$ and using the change of variables $\rho = t\tau$, we find

$$I_2 = \frac{1}{\pi t^\sigma} \int_1^{t^{\delta_2}} \left(1 - \frac{\rho}{t}\right)^{-\frac{1}{2}} \Re\left\{ \left(e^{\frac{\pi\rho}{2}} \Gamma(\sigma + i\rho)\right) e^{i\varphi(t,\rho)} \right\} |\zeta(\sigma + i\rho)|^2 d\rho$$
$$\times \left[1 + O\left(\frac{1}{t}\right)\right], \quad t \to \infty. \quad (3.9)$$

We will estimate the integral (3.9) by employing the first mean value theorem for integrals: since $|\zeta(\sigma + i\rho)|^2$ does not change sign and it is integrable for $\rho \in [1, t^{\delta_2}]$, it follows that there exists a $c(t)$,

$$1 < c(t) < t^{\delta_2},$$

such that

$$I_2 = \frac{1}{\pi t^\sigma}\left(1-\frac{c(t)}{t}\right)^{-\frac{1}{2}} \Re\Big\{\left(e^{\frac{\pi c(t)}{2}}\Gamma(\sigma+ic(t))\right)e^{i\varphi(t,c(t))}\Big\}\int_1^{t^{\delta_2}}|\zeta(\sigma+i\rho)|^2 d\rho$$
$$\times\left[1+O\left(\frac{1}{t}\right)\right],\quad t\to\infty. \quad (3.10)$$

If $c(t)=O(1)$, then $e^{\frac{\pi c(t)}{2}}\Gamma(\sigma+ic(t))=O(1)$. If $c(t)\to\infty$, as $t\to\infty$, then equation (2.33a) with $\xi=c(t)$ yields

$$e^{\frac{\pi c(t)}{2}}\Gamma(\sigma+ic(t))=O\left(c(t)^{\left(\sigma-\frac{1}{2}\right)}\right),\quad t\to\infty.$$

Thus, since $c(t)<t^{\delta_2}$, we find

$$e^{\frac{\pi c(t)}{2}}\Gamma(\sigma+ic(t))=\begin{cases} O\left(t^{\delta_2\left(\sigma-\frac{1}{2}\right)}\right), & \frac{1}{2}<\sigma<1,\\ O(1), & 0<\sigma\le\frac{1}{2}.\end{cases} \quad (3.11)$$

We recall Atkinson's asymptotic formula (theorem 7.4 of [T]):

$$\int_1^{t^{\delta_2}}\left|\zeta\left(\frac{1}{2}+i\rho\right)\right|^2 \mathrm{d}\rho = t^{\delta_2}\ln t^{\delta_2}+(2\gamma-1-\ln 2\pi)t^{\delta_2}+O\left(t^{\delta_2\left(\frac{1}{2}+\varepsilon\right)}\right),$$
$$\varepsilon>0,\quad t\to\infty. \quad (3.12)$$

Similarly, for $\sigma\neq\frac{1}{2}$,

$$\int_1^{t^{\delta_2}}|\zeta(\sigma+i\rho)|^2\,\mathrm{d}\rho=\begin{cases} O\left(t^{\delta_2}\zeta(2\sigma)\right), & \frac{1}{2}<\sigma\le 1,\quad t\to\infty,\\ O\left(t^{2(1-\sigma)\delta_2}\zeta(2-2\sigma)\right), & 0<\sigma<\frac{1}{2},\quad t\to\infty.\end{cases} \quad (3.13)$$

Using in equation (3.10) the expressions obtained from equations (3.11)-(3.13), we find equation (3.8). **QED**

Combining Theorem 2.2 and Lemmas 3.1 and 3.2 we obtain the following result.

**Theorem 3.1** *Let $0<\sigma<1$, $t>0$, and let $\delta_1$ be a sufficiently small positive constant, whereas the constants $\delta_j$ satisfy $0<\delta_j<1$, $j=2,3,4$. Let the integrals $I_3(\sigma,t,\delta_2,\delta_3)$ and $I_4(\sigma,t,\delta_3,\delta_4)$ be defined by (1.11) with $j=3$ and $j=4$. The Riemann zeta function $\zeta(s)$, $s\in\mathbb{C}$ satisfies the following equation:*

$$I_3(\sigma,t,\delta_2,\delta_3)+I_4(\sigma,t,\delta_3,\delta_4)+\ln t+2\gamma-\ln 2\pi+O\left(t^{\delta_2-\frac{1}{2}}\ln t\right)+O\left(t^{\frac{4}{3}\delta_1-\frac{1}{2}}\right)$$
$$+O\left(e^{-\pi t^{\delta_{14}}}\right)=0,\quad \delta_{14}=\min(\delta_1,\delta_4),\quad \sigma=\frac{1}{2},\quad t\to\infty, \quad (3.14)$$

*as well as*

$$I_3(\sigma,t,\delta_2,\delta_3)+I_4(\sigma,t,\delta_3,\delta_4)+\zeta(2\sigma)+2\Gamma(2\sigma-1)\zeta(2\sigma-1)\sin(\pi\sigma)t^{1-2\sigma}\left(1+O\left(\frac{1}{t}\right)\right)$$
$$+\begin{cases} O\left(t^{-\sigma+\left(\frac{3}{2}-\sigma\right)\delta_2}\ln t\right)+O\left(t^{-\sigma+\frac{4}{3}\delta_1}\right), & 0<\sigma<\frac{1}{2} \\ O\left(t^{-\sigma+\left(\sigma+\frac{1}{2}\right)\delta_2}\zeta(2\sigma)\right)+O\left(t^{-\sigma+\left(\frac{5}{6}+\sigma\right)\delta_1}\right), & \frac{1}{2}<\sigma<1 \end{cases}$$
$$+O\left(e^{-\pi t^{\delta_{14}}}\right)=0, \quad t\to\infty. \quad (3.15)$$

**Proof** Decomposing the interval of integration of the lhs of equation (2.35) into the four subintervals defined in (1.10), we find that the lhs of equation (2.35) equals the sum of the four integrals $\{I_j\}_1^4$ defined by equations (1.11). Replacing the integrals $I_1$ and $I_2$ via equations (3.1) and (3.6) respectively, equation (2.35) yields equations (3.14) and (3.15). **QED**

# 4 The Asymptotics of $\tilde{I}_3$

Equation (1.14) expresses $\tilde{I}_3$ in terms of the integral $J_3$, which we analyse below.

**Proposition 4.1** *Let* $J_3$ *be defined by*

$$J_3(\sigma,t,\delta_2,\delta_3,\lambda)=\frac{t}{\pi}\int_{t^{\delta_2-1}}^{1-t^{\delta_3-1}}\frac{\Gamma(it-it\tau)}{\Gamma(\sigma+it)}\Gamma(\sigma+it\tau)\lambda^{i\tau t}d\tau, \quad \lambda=\frac{m_2}{m_1},$$
$$0<\sigma<1, \quad t>0, \quad 0<\delta_2<1,\ 0<\delta_3<1, \quad m_j=1,2,\dots,[t], \quad j=1,2. \quad (4.1)$$

*Then,*

$$J_3(\sigma,t,\delta_2,\delta_3,\lambda)=\sqrt{\frac{2t}{\pi}}e^{-\frac{i\pi}{4}}\tilde{J}_3(\sigma,t,\delta_2,\delta_3,\lambda)\left[1+O(t^{-\delta_{23}})\right],$$
$$\delta_{23}=\min\{\delta_2,\delta_3\}, \quad t\to\infty, \quad (4.2)$$

*where* $\tilde{J}_3$ *is defined by*

$$\tilde{J}_3(\sigma,t,\delta_2,\delta_3,\lambda)=\int_{t^{\delta_2-1}}^{1-t^{\delta_3-1}}G(\sigma,\tau)e^{iF(\tau,\lambda)}d\tau, \quad (4.3)$$

*with*

$$G(\sigma,\tau)=(1-\tau)^{-\frac{1}{2}}\tau^{\sigma-\frac{1}{2}}, \quad F(\tau,\lambda)=(1-\tau)\ln(1-\tau)+\tau\ln\tau+\tau\ln\lambda. \quad (4.4)$$

**Proof** In the interval of integration, we have

$$t^{\delta_2} \le t\tau \le t - t^{\delta_3}. \tag{4.5}$$

Thus,

$$t^{\delta_3} \le t - t\tau \le t - t^{\delta_2}. \tag{4.6}$$

Hence, $t\tau \to \infty$ and $t - t\tau \to \infty$ as $t \to \infty$, therefore we can employ the asymptotic formula (2.33a) with $\xi = t$, $t\tau$, $t-\tau$, to compute the ratio of the gamma functions appearing in the rhs of (4.1). Expressions for $\Gamma(\sigma + it)$ and the $\Gamma(it - it\tau)$ are given in (3.4). Similarly,

$$\Gamma(\sigma+it\tau) = \sqrt{2\pi}(t\tau)^{\sigma-\frac{1}{2}}e^{-\frac{\pi\tau t}{2}}e^{-\frac{i\pi}{4}}e^{\frac{i\pi\sigma}{2}}e^{-i\tau t}(\tau t)^{i\tau t}\left[1+O\left(\frac{1}{\tau t}\right)\right], \quad t\to\infty. \tag{4.7}$$

Equations (4.5) and (4.6) imply the inequalities

$$\frac{1}{\tau t} \le t^{-\delta_2}, \quad \frac{1}{t-t\tau} \le t^{-\delta_3}. \tag{4.8}$$

Equations (3.4), (4.7) and (4.8) yield

$$\frac{\Gamma(it-it\tau)}{\Gamma(\sigma+it)}\Gamma(\sigma+it\tau) = \sqrt{\frac{2\pi}{t}}e^{-\frac{i\pi}{4}}G(\sigma,\tau)e^{it[(1-\tau)\ln(1-\tau)+\tau\ln\tau]} \times\left[1+O(t^{-\delta_{23}})\right], \quad t\to\infty. \tag{4.9}$$

Substituting equation (4.9) in the rhs of (4.1) we obtain (4.2). **QED**

It is well known that the main contributions to the asymptotic analysis of integrals such as $\tilde{J}_3$ come from possible singularities, from possible stationary points, and from the end points of the interval of integration [AF]. The integral $\tilde{J}_3$ possesses a stationary point at $\tau = 1/(1+\lambda)$. Thus, for this integral there exist two contributions, one from the associated stationary point and one from the end points of the interval of integration. Assuming that the stationary point does not approach the end points, the latter contributions can be computed using integration by parts: the contribution to $\tilde{J}_3$ from the upper and lower end points are of the following order respectively:

$$\frac{G(\sigma,1-t^{\delta_3-1})}{t} \sim \frac{t^{-\frac{\delta_3-1}{2}}}{t} = \frac{t^{-\frac{\delta_3}{2}}}{\sqrt{t}}, \quad t\to\infty, \tag{4.10}$$

and

$$\frac{G(\sigma,t^{\delta_2-1})}{t} \sim \frac{t^{(\delta_2-1)(\sigma-\frac{1}{2})}}{t} = \frac{t^{\delta_2(\sigma-\frac{1}{2})}}{t^{\sigma+\frac{1}{2}}}, \quad t\to\infty. \tag{4.11}$$

We first compute the contribution from the stationary point, where we also include the error term arising from the contributions of the lower end point.

**Proposition 4.2** *Let $\tilde{J}_3$ be defined by* (4.3). *Then,*

$$\tilde{J}_3 = \tilde{J}_3^S - \tilde{J}_3^U, \tag{4.12}$$

*where $\tilde{J}_3^S$ and $\tilde{J}_3^U$ are defined as follows:*

$$\tilde{J}_3^S(\sigma, t, \delta_2, \lambda) = \int_{L(\delta_2)} G(\sigma, \tau) e^{iF(\tau,\lambda)} d\tau, \tag{4.13}$$

*and*

$$\tilde{J}_3^U(\sigma, t, \delta_3, \lambda) = \int_{1-t^{\delta_3}}^{\infty e^{i\varphi}} G(\sigma, \tau) e^{iF(\tau,\lambda)} d\tau, \quad 0 < \varphi < \arctan \frac{\pi}{|\ln \lambda|}, \tag{4.14}$$

*$\sigma$, $t$, $\delta_2$, $\delta_3$, $\lambda$ are as in* (4.1), *whereas $L(\delta_2)$ denotes the contour in the complex $\tau$-plane, starting at the point $t^{\delta_2 - 1}$, going down into the lower half complex plane, up through the point $\tau = 1/(1+\lambda)$ and continuing to $\infty e^{i\varphi}$.*

*$\tilde{J}_3^S$ is given by*

$$\tilde{J}_3^S(\sigma, t, \delta_2, \lambda) = \sqrt{\frac{2\pi}{t}} e^{\frac{i\pi}{4}} \frac{\lambda^{it}}{(1+\lambda)^{\sigma+it}} [1 + o(1)], \quad t \to \infty, \tag{4.15a}$$

*where the first term in* (4.15a) *occurs iff*

$$\frac{1}{t^{1-\delta_3} - 1} < \lambda < t^{1-\delta_2} - 1, \tag{4.15b}$$

*whereas $\tilde{J}_3^U$ is given by*

$$\tilde{J}_3^U(\sigma, t, \delta_3, \lambda) = \frac{it^{-\frac{\delta_3}{2}}}{\sqrt{t}} t^{i(\delta_3 - 1)t^{\delta_3}} (1 - t^{\delta_3 - 1})^{\sigma - \frac{1}{2} + i(t - t^{\delta_3})} \frac{\lambda^{i(t - t^{\delta_3})}}{\ln(\lambda(t^{1-\delta_3} - 1))} \times [1 + o(1)], \quad t \to \infty. \tag{4.16}$$

**Proof** For the function $F(\tau, \lambda)$, $\tau \in \mathbb{C}$, defined by the second of equations (4.4), we chose the branch cuts $[-\infty, 0] \cup [1, \infty]$. The function $F$ satisfies the equation

$$\frac{\partial F}{\partial \tau} = -\ln(1-\tau) + \ln \tau + \ln \lambda. \tag{4.17}$$

Thus, the integral $\tilde{J}_3$ possesses a stationary point at $\tau = \tau_1$, provided that $1 - \tau_1 = \tau_1 \lambda$, i.e.,

$$\tau_1 = \frac{1}{1+\lambda}. \tag{4.18}$$

This occurs if and only if

$$\frac{t^{\delta_2}}{t} < \tau_1 < 1 - \frac{t^{\delta_3}}{t},$$

i.e., if and only if $\lambda$ satisfies the inequality (4.15b).

We deform the contour of integration to the contour $L(\delta_2)$ plus the contour from $\infty e^{i\varphi}$ back to the point $1 - t^{\delta_3 - 1}$.

We claim that if $\varphi$ is sufficiently small, namely if $\varphi$ satisfies the second of equations in (4.14), then the integral $\tilde{J}_3^U$ converges. Indeed, employing the change of variables

$$\tau = \Delta(t) + \rho e^{i\varphi}, \quad \Delta(t) = 1 - \frac{t^{\delta_3}}{t},$$

we find that $F$ becomes

$$F = (1 - \Delta - e^{i\varphi}\rho)\ln(1 - \Delta - e^{i\varphi}\rho) + (\Delta + e^{i\varphi}\rho)[\ln\lambda + \ln(\Delta + e^{i\varphi}\rho)].$$

For fixed $t$ and large $\rho$, we have

$$F \sim \rho e^{i\varphi}[\ln\lambda + \ln(\rho e^{i\varphi}) - \ln(-\rho e^{i\varphi})], \quad \rho \to \infty.$$

Using

$$\ln(\rho e^{i\varphi}) - \ln(-\rho e^{i\varphi}) = i\pi,$$

it follows that

$$\Im F \sim \rho[\ln\lambda \sin\varphi + \pi\cos\varphi], \quad \rho \to \infty.$$

For the convergence of $\tilde{J}_3^U$ we require $\Im F > 0$ as $\rho \to \infty$. Thus, if $\lambda \geq 1$ then we have convergence for all $\varphi \in (0, \pi/2)$, whereas if $\lambda \in (0,1)$ we require that $\varphi$ satisfies the condition displayed in (4.14).

In order to compute the contribution from the stationary point $\tau = \tau_1$ we employ the well known formula [M]

$$\int_{b_1}^{b_2} g(\tau)e^{itF(\tau)}d\tau = \sqrt{\frac{2\pi}{t|F''(\tau_1)|}}\, g(\tau_1)e^{itF(\tau_1) + \frac{i\pi}{4}sgnF''(\tau_1)} + O\left(\frac{g(b_1)}{t}\right) + O\left(\frac{g(b_2)}{t}\right), \quad t \to \infty. \tag{4.19}$$

Using (4.17) we obtain

$$\frac{\partial^2 F(\tau_1,\lambda)}{\partial\tau^2} = \frac{1}{\tau_1(1-\tau_1)} = \frac{(1+\lambda)^2}{\lambda}.$$

Evaluating $F(\tau,\lambda)$ at $\tau=\tau_1$ we find

$$\begin{aligned} F(\tau_1,\lambda) &= \frac{\lambda}{1+\lambda}\ln\left(\frac{\lambda}{1+\lambda}\right) + \frac{1}{1+\lambda}\ln\left(\frac{1}{1+\lambda}\right) + \frac{1}{1+\lambda}\ln\lambda \\ &= \frac{\lambda}{1+\lambda}\ln\left(\frac{\lambda}{1+\lambda}\right) + \frac{1}{1+\lambda}\ln\left(\frac{1}{1+\lambda}\right) = \ln\frac{\lambda}{1+\lambda} = -\ln\left(1+\frac{1}{\lambda}\right). \end{aligned}$$

Thus,

$$\sqrt{\frac{2\pi}{t|F''(\tau_1)|}}e^{itF(\tau_1)+\frac{i\pi}{4}sgnF''(\tau_1)} = \sqrt{\frac{2\pi}{t}}e^{\frac{i\pi}{4}}\frac{\lambda^{\frac12}}{(1+\lambda)}(1+\lambda)^{-it} = \sqrt{\frac{2\pi}{t}}e^{\frac{i\pi}{4}}\frac{\lambda^{\frac12+it}}{(1+\lambda)^{1+it}}. \tag{4.20}$$

The definition of $G(\sigma,\tau)$ in the first of equations (4.4) implies

$$g(\tau_1) = G(\sigma,\tau_1) = \big(1-\frac{1}{1+\lambda}\big)^{-\frac12}\frac{1}{(1+\lambda)^{\sigma-\frac12}} = \frac{\lambda^{-\frac12}}{(1+\lambda)^{\sigma-1}}. \tag{4.21}$$

Substituting equations (4.11), (4.20) and (4.21) into equation (4.19), we find (4.15a).

Assuming that $1/(1+\lambda)$ does not approach $1-t^{\delta_3-1}$, it is straightforward to compute the large $t$-asymptotics of $\tilde{J}_3^U$ via integration by parts:

$$\begin{aligned} \tilde{J}_3^U &= \frac{1}{it}\int_{1-t^{\delta_3-1}}^{\infty e^{i\varphi}} \frac{G}{\partial F/\partial\tau}\left(\frac{\partial}{\partial\tau}e^{itF}\right)d\tau \\ &= -\frac{Ge^{itF}}{it\partial F/\partial\tau}\Bigg|_{\tau=1-t^{\delta_3-1}} - \frac{1}{it}\int_{1-t^{\delta_3-1}}^{\infty e^{i\varphi}}\left[\frac{\partial}{\partial\tau}\left(\frac{G}{\partial F/\partial\tau}\right)\right]e^{itF}d\tau. \end{aligned} \tag{4.22}$$

Using the identities

$$F(1-t^{\delta_3-1},\lambda) = t^{\delta_3-1}\ln t^{\delta_3-1} + (1-t^{\delta_3-1})\ln(1-t^{\delta_3-1}) + (1-t^{\delta_3-1})\ln\lambda,$$

$$\frac{\partial F}{\partial\tau}(1-t^{\delta_3-1},\lambda) = \ln\left(\frac{\lambda(1-t^{\delta_3-1})}{t^{\delta_3-1}}\right),$$

$$G(\sigma,1-t^{\delta_3-1}) = t^{-\frac{\delta_3-1}{2}}(1-t^{\delta_3-1})^{\sigma-\frac12},$$

we find that the first term of the rhs of (4.22) yields the leading term of the rhs of (4.16). The rigorous derivation of the relevant error term, as well as the analysis of the case that the stationary point approaches $1-t^{\delta_3-1}$, is presented in [FSF]. **QED**

**Remark 4.1** *The contour $L(\delta_2)$ can be deformed to a contour which can be written in the form*

$$L(\delta_2)=\left[t^{\delta_2-1},-\infty e^{i\Phi}\right]\cup\left[-\infty e^{i\Phi},\infty e^{i\varphi}\right],$$

*where $\Phi$ is appropriately constrained so that the associated integral converges. Hence, $J_3^S$ can be written*

$$J_3^S=J_3^{SD}+J_3^L$$

*The leading behaviour of $J_3^{SD}$ is given by the rhs of* (4.15a) *and the leading behaviour of $J_3^L$ is given via integration by parts:*

$$\tilde{J}_3^L(\sigma,t,\delta_2,\lambda)=-\frac{1}{it}\frac{Ge^{itF}}{\frac{\partial F}{\partial\tau}}\Bigg|_{\tau=t^{\delta_2-1}}. \tag{4.23}$$

*Using the identities*

$$F(t^{\delta_2-1},\lambda)=(1-t^{\delta_2-1})\ln(1-t^{\delta_2-1})+t^{\delta_2-1}\ln t^{\delta_2-1}+t^{\delta_2-1}\ln\lambda,$$

$$\frac{\partial F}{\partial\tau}(t^{\delta_2-1},\lambda)=\ln\left(\frac{\lambda}{t^{1-\delta_2}-1}\right),$$

$$G(\sigma,t^{\delta_2-1})=t^{(\delta_2-1)\left(\sigma-\frac{1}{2}\right)}(1-t^{\delta_2-1})^{-\frac{1}{2}},$$

*we find*

$$\tilde{J}_3^L(\sigma,t,\delta_2,\lambda)=\frac{it^{\delta_2\left(\sigma-\frac{1}{2}\right)}}{t^{\sigma+\frac{1}{2}}}\frac{(1-t^{\delta_2-1})^{-\frac{1}{2}+i(t-t^{\delta_2})}}{\ln\left(\frac{\lambda}{t^{1-\delta_2}-1}\right)}\lambda^{it^{\delta_2}}t^{i(\delta_2-1)t^{\delta_2}}. \tag{4.24}$$

*Employing the above equation in (1.14) yields a contribution to $\tilde{I}_3$ similar to the $E_{34}^{SD}$ defined in (7.25). Thus, for the estimate of the relevant expression we refer to remark 7.1, where we indicate that for $\sigma=1/2$ and $\delta>1/4$ the relevant contribution is negligible.*

*The rigorous derivation of* (4.23) *as well as the analysis of the case that the stationary point approaches the lower end point $t^{\delta_2-1}$ is similar with the analysis presented in [FSF]; details are given in [FKL].*

**Theorem 4.1** *Let $\tilde{I}_3$ be defined in (1.14). Then,*

$$\tilde{I}_3(\sigma, t, \delta_2, \delta_3) =$$
$$2\Re\left\{\sum\sum_{m_1, m_2 \in M(\delta_2,\delta_3)} \frac{1}{m_2^{\bar{s}}(m_1+m_2)^s}[1+o(1)]\right\}\left[1+O(t^{-\delta_{23}})\right]$$
$$-\sqrt{\frac{2}{\pi}}\Re\left\{e^{\frac{i\pi}{4}}t^{-\frac{\delta_3}{2}}(1-t^{\delta_3-1})^{\sigma-\frac{1}{2}+i(t-t^{\delta_3})}t^{i(\delta_3-1)t^{\delta_3}}\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_1^{s-it^{\delta_3}}}\frac{1}{m_2^{\bar{s}+it^{\delta_3}}}\right.$$
$$\left.\times\frac{1}{\ln\left[\frac{m_2}{m_1}\left(t^{1-\delta_3}-1\right)\right]}[1+o(1)]\right\}\left[1+O(t^{-\delta_{23}})\right], \quad t\to\infty, \quad (4.25)$$

*where $\sigma, \delta_2, \delta_3, \delta_{23}$, as well as the set $M$ are as in (1.17) and (1.18).*

**Proof** Expressing in (1.14) $\tilde{J}_3$ in terms of $\tilde{J}_3 = \tilde{J}_3^S - \tilde{J}_3^U$, using equations (4.15a) and (4.16) for $\tilde{J}_3^S$ and $\tilde{J}_3^U$ respectively, and simplifying the resulting formulae, we find (4.25). **QED**

# 5 The Leading asymptotics of $I_4$

Let $I_4$ be defined by

$$I_4(\sigma,t,\delta_2,\delta_3) = \frac{t}{\pi}\oint_{1-t^{\delta_3-1}}^{1+t^{\delta_4-1}}\Re\left\{\frac{\Gamma(it-it\tau)}{\Gamma(\sigma+it)}\Gamma(\sigma+it\tau)\right\}|\zeta(\sigma+it\tau)|^2 \mathrm{d}\tau,$$
$$0<\sigma<1, \quad t>0, \quad 0<\delta_2<1, \quad 0<\delta_3<1, \quad (5.1)$$

where the principal value integral is with respect to $\tau = 1$. Using the change of variables $1-\tau=\rho$, $I_4$ becomes

$$I_4 = \frac{t}{\pi}\oint_{-t^{\delta_4-1}}^{t^{\delta_3-1}}\Re\left\{\Gamma(it\rho)\frac{\Gamma(\sigma+it-it\rho)}{\Gamma(\sigma+it)}\right\}|\zeta(\sigma+it-it\rho)|^2\,\mathrm{d}\rho,$$

where now the principal value integral is defined with respect to $\rho = 0$. The change of variables $t\rho = x$ yields

$$I_4 = \frac{1}{\pi}\oint_{-t^{\delta_4}}^{t^{\delta_3}}\Re\left\{\Gamma(ix)\frac{\Gamma(\sigma+it-ix)}{\Gamma(\sigma+it)}\right\}|\zeta(\sigma+it-ix)|^2\,\mathrm{d}x. \quad (5.2)$$

In the interval of integration we have

$$-t^{\delta_4} \le x \le t^{\delta_3}.$$

Thus,

$$t - t^{\delta_3} \leq t - x \leq t + t^{\delta_4}. \tag{5.3}$$

Replacing in (5.2), $|\zeta|^2$ by its leading order asymptotics we find

$$\zeta\left(\sigma + i(t-x)\right) \sim \sum_{m=1}^{[\frac{\eta}{2\pi}]} \frac{1}{m^{\sigma+i(t-x)}}, \quad \eta > t - x. \tag{5.4}$$

Since $t - x \leq t + t^{\delta 4}$, we take $\eta = 2\pi t > t + t^{\delta_4}$. Thus,

$$|\zeta(\sigma + i(t-x))|^2 \sim \sum_{m_1=1}^{[t]} \sum_{m_2=1}^{[t]} \frac{1}{m_1^s m_2^{\bar{s}}} \left(\frac{m_1}{m_2}\right)^{ix}. \tag{5.5}$$

Let $\tilde{I}_4$ denote the expression obtained from $I_4$ by replacing $|\zeta|^2$ with the rhs of (5.5), i.e,

$$\tilde{I}_4(\sigma, t, \delta_3, \delta_4) = \Re\left\{\sum_{m_1=1}^{[t]} \sum_{m_2=1}^{[t]} \frac{1}{m_1^s} \frac{1}{m_2^{\bar{s}}} J_4(\sigma, t, \delta_2, \delta_3, \frac{m_1}{m_2})\right\}, \tag{5.6}$$

where $J_4$ is defined by

$$J_4(\sigma, t, \delta_3, \delta_4, \frac{m_1}{m_2}) = \frac{1}{\pi} \fint_{-t^{\delta_4}}^{t^{\delta_3}} \Gamma(ix) \frac{\Gamma(\sigma + it - ix)}{\Gamma(\sigma + it)} \left(\frac{m_1}{m_2}\right)^{ix} \mathrm{d}x,$$
$$0 < \sigma < 1, \quad t > 0, \quad 0 < \delta_3 < 1, \quad 0 < \delta_4 < 1, \quad m_j = 1, 2, \ldots, [t], \tag{5.7}$$

with the principal value integral defined with respect to $x = 0$.

**Proposition 5.1** *Let $J_4$ be defined by (5.7). Let $H_1$ denote the Hankel contour with a branch cut along the negative real axis, see figure 1, defined by*

$$H_1 = \left\{re^{-i\pi} | 1 < r < \infty\right\} \cup \left\{e^{i\theta} | -\pi < \theta < \pi\right\} \cup \left\{re^{i\pi} | 1 < r < \infty\right\}. \tag{5.8}$$

*Then,*

$$J_4(\sigma, t, \delta_3, \delta_4, \frac{m_1}{m_2}) = \frac{1}{\pi} \int_{H_1} \frac{e^z}{z} \tilde{J}_4(\sigma, t, \delta_3, \delta_4, A) dz \left[1 + O\left(\frac{1}{t}\right)\right], \quad t \to \infty,$$
$$0 < \sigma < 1, \; 0 < \delta_3 < \frac{1}{2}, \; 0 < \delta_4 < \frac{1}{2}, \; A = \frac{m_1}{m_2} \frac{z}{t}, \tag{5.9}$$

*where*

$$\tilde{J}_4(\sigma,t,\delta_3,\delta_4,A) = \oint_{-t^{\delta_4}}^{t^{\delta_3}} \frac{e^{\frac{\pi x}{2}} A^{ix}}{e^{-\pi x} - e^{\pi x}} \left(1-\frac{x}{t}\right)^{\sigma-\frac{1}{2}} e^{ix} \left(1-\frac{x}{t}\right)^{i(t-x)} dx, \tag{5.10}$$

*with the principal value integral defined with respect to $x=0$.*

**Proof**. Equation (2.33a) together with the inequality

$$\frac{1}{t-x} \le \frac{1}{t-t^{\delta_3}} = \frac{1}{t} + O\left(t^{\delta_3-2}\right), \quad t\to\infty,$$

imply

$$\Gamma(\sigma+i(t-x)) = \sqrt{2\pi}(t-x)^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}(t-x)} e^{-\frac{i\pi}{4}} e^{\frac{i\pi\sigma}{2}} e^{-i(t-x)} (t-x)^{i(t-x)} \left(1+O\left(\frac{1}{t}\right)\right), \ t\to\infty. \tag{5.11}$$

The above equation together with (3.4a) yield

$$\frac{\Gamma\left(\sigma+i(t-x)\right)}{\Gamma(\sigma+it)} = \left(1-\frac{x}{t}\right)^{\sigma-\frac{1}{2}} e^{\frac{\pi x}{2}} t^{-ix} e^{ix} \left(1-\frac{x}{t}\right)^{i(t-x)} \left[1+O\left(\frac{1}{t}\right)\right], \ t\to\infty. \tag{5.12}$$

Replacing in equation (5.7), $\Gamma(\sigma+i(t-x))/\Gamma(\sigma+it)$ by the rhs of (5.12), as well as employing the formula

$$\Gamma(ix) = \frac{1}{e^{-\pi x}-e^{\pi x}} \int_{H_1} \frac{e^z}{z} z^{ix} \mathrm{d}z, \tag{5.13}$$

equation (5.7) becomes equation (5.9). **QED**

**Proposition 5.2** *Let $\tilde{J}_4$ be defined in (5.10). Then,*

$$\tilde{J}_4(\sigma,t,\delta_3,\delta_4,A) = \left[\frac{i}{2}(-1+\frac{2}{1-iA}) + \frac{e^{it^{\delta_3}\ln A} e^{-\frac{\pi t^{\delta_3}}{2}}}{\frac{\pi}{2} - i\ln A}\right] \left[1+O\left(t^{-\delta_{34}}\right)\right], \quad t\to\infty,$$

$$0<\sigma<1, \ 0<\delta_3<\frac{1}{2}, \ 0<\delta_4<\frac{1}{2}, \ \delta_{34}=\min\{\delta_3,\delta_4\}, \ A=\frac{m_1 z}{m_1 t}. \tag{5.14}$$

**Proof** Using the identity

$$\begin{aligned} e^{ix}\left(1-\frac{x}{t}\right)^{i(t-x)} &= e^{ix} e^{i(t-x)\ln\left(1-\frac{x}{t}\right)} = e^{ix} e^{i(t-x)\left(-\frac{x}{t}+O\left(\frac{x^2}{t^2}\right)\right)} \\ &= e^{iO\left(\frac{x^2}{t}\right)} = 1+O\left(\frac{x^2}{t}\right), \quad \frac{x}{t}\to 0, \end{aligned}$$

we find

$$\left(1-\frac{x}{t}\right)^{\sigma-\frac{1}{2}} e^{ix}\left(1-\frac{x}{t}\right)^{i(t-x)} = \left[1+O\left(t^{2\delta_{34}-1}\right)\right],\ t\to\infty,$$

where we have used that $|x|<t^{\delta_{34}}$. Then, equation (5.10) becomes

$$\tilde{J}_4(\sigma,t,\delta_3,\delta_4,A) = \oint_{-t^{\delta_4}}^{t^{\delta_3}} \frac{e^{\frac{\pi x}{2}}A^{ix}}{e^{-\pi x}-e^{\pi x}}\mathrm{d}x\left[1+O\left(t^{2\delta_{34}-1}\right)\right],\quad t\to\infty. \quad (5.15)$$

It is remarkable that the leading order term of the above integral can be computed in closed form within an error which is exponentially small as $t\to\infty$. Indeed,

$$\lim_{\varepsilon\to0}\left\{\int_{-t^{\delta_4}}^{-\varepsilon}\frac{e^{\frac{\pi x}{2}}A^{ix}}{e^{-\pi x}-e^{\pi x}}\mathrm{d}x+\int_{\varepsilon}^{t^{\delta_3}}\frac{e^{\frac{\pi x}{2}}A^{ix}}{e^{-\pi x}-e^{\pi x}}\mathrm{d}x\right\}$$

$$=\lim_{\varepsilon\to0}\left\{\int_{\varepsilon}^{t^{\delta_4}}\frac{e^{-\frac{3\pi x}{2}}A^{-ix}}{1-e^{-2\pi x}}\mathrm{d}x-\int_{\varepsilon}^{t^{\delta_3}}\frac{e^{-\frac{\pi x}{2}}A^{ix}}{1-e^{-2\pi x}}\mathrm{d}x\right\}$$

$$=\lim_{\varepsilon\to0}\left(\int_{\varepsilon}^{t^{\delta_4}}e^{-\frac{3\pi x}{2}}A^{-ix}\left(\sum_{k=0}^{\infty}e^{-2\pi kx}\right)\mathrm{d}x-\int_{\varepsilon}^{t^{\delta_3}}e^{-\frac{\pi x}{2}}A^{ix}\left(\sum_{k=0}^{\infty}e^{-2\pi kx}\right)\mathrm{d}x\right)$$

$$=\lim_{\varepsilon\to0}\sum_{k=0}^{\infty}\left\{\int_{\varepsilon}^{t^{\delta_4}}e^{-x\left(2\pi k+\frac{3\pi}{2}+i\ln A\right)}\mathrm{d}x-\int_{\varepsilon}^{t^{\delta_3}}e^{-x\left(2\pi k+\frac{\pi}{2}-i\ln A\right)}\mathrm{d}x\right\}$$

$$=-\sum_{k=0}^{\infty}\left\{\frac{e^{-t^{\delta_4}\left(2\pi k+\frac{3\pi}{2}+i\ln A\right)}-1}{2\pi k+\frac{3\pi}{2}+i\ln A}-\frac{e^{-t^{\delta_3}\left(2\pi k+\frac{\pi}{2}-i\ln A\right)}-1}{2\pi k+\frac{\pi}{2}-i\ln A}\right\}.$$

Taking into consistent that $\arg z\in[-\pi,\pi]$, it follows that the terms involving $t^{\delta_3}$ and $t^{\delta_4}$ decay exponentially expect for the term involving $t^{\delta_3}$ and $k=0$. Hence,

$$\oint_{-t^{\delta_4}}^{t^{\delta_3}}\frac{e^{\frac{\pi x}{2}}A^{ix}}{e^{-\pi x}-e^{\pi x}}\mathrm{d}x=\sum_{k=0}^{\infty}\left\{\frac{1}{2\pi k+\frac{3\pi}{2}+i\ln A}-\frac{1}{2\pi k+\frac{\pi}{2}-i\ln A}\right\}+\frac{e^{it^{\delta_3}\ln A}e^{-\frac{\pi t^{\delta_3}}{2}}}{\frac{\pi}{2}-i\ln A}$$

$$+O\left(e^{-t^{\delta_{34}}}\right),\quad t\to\infty. \quad (5.16)$$

Let $S$ denote the first term of the rhs of (5.16). Then,

$$S=\frac{1}{2\pi}\left(\frac{1}{k+1-b}-\frac{1}{k+b}\right), \quad (5.17)$$

with
$$b = \frac{1}{4} - \frac{i}{2\pi} \ln A. \tag{5.18}$$

Let $\Psi(z)$, $z \in \mathbb{C}$, denotes the digamma function defined in (2.11), then (5.17) takes the form

$$S = -\frac{1}{2\pi} \left( \Psi(1-b) - \Psi(b) \right). \tag{5.19}$$

Employing in (5.19) the reflection formula for $\Psi(z)$, namely

$$\Psi(1-z) - \Psi(z) = \pi \cot(\pi z), \tag{5.20}$$

we find
$$S = -\frac{1}{2} \tan\left( \frac{\pi}{4} + \frac{i}{2} \ln A \right). \tag{5.21}$$

This formula can be further simplified as follows:

$$\tan\left( \frac{\pi}{4} + \frac{i}{2} \ln A \right) = \frac{1}{i} \frac{e^{i\left(\frac{\pi}{4} + \frac{i}{2}\ln A\right)} - e^{-i\left(\frac{\pi}{4} + \frac{i}{2}\ln A\right)}}{e^{i\left(\frac{\pi}{4} + \frac{i}{2}\ln A\right)} + e^{-i\left(\frac{\pi}{4} + \frac{i}{2}\ln A\right)}}$$

$$= \frac{1}{i} \frac{e^{\frac{i\pi}{4}} A^{-\frac{1}{2}} - e^{\frac{-i\pi}{4}} A^{\frac{1}{2}}}{e^{\frac{i\pi}{4}} A^{-\frac{1}{2}} + e^{\frac{-i\pi}{4}} A^{\frac{1}{2}}} = \frac{1+iA}{i(1-iA)} = \frac{1}{i}\left( -1 + \frac{2}{1-iA} \right).$$

Hence
$$S = \frac{i}{2}\left( -1 + \frac{2}{1-iA} \right). \tag{5.22}$$

Replacing in (5.15) the leading order term with the rhs of (5.16), where $S$ is given by (5.22), we find (5.14). **QED**

**Proposition 5.3** *Let $J_4$ be defined by (5.7). Then,*

$$J_4(\sigma, t, \delta_3, \delta_4) = \left[ -1 + E_4(t, \delta_3, M) \right] \left[ 1 + O(t^{2\delta_{34}-1}) \right], \quad t \to \infty, \tag{5.23}$$

*where*

$$E_4(t, \delta_3, M) = \frac{1}{\pi} \int_{H_1} \frac{e^z}{z} \frac{e^{it^{\delta_3} \ln(Mz) - \frac{\pi t^{\delta_3}}{2}}}{\frac{\pi}{2} - i\ln(Mz)} dz, \quad M = \frac{m_1}{m_2 t}. \tag{5.24}$$

**Proof** Equation (5.23) follows from equations (5.9) and (5.14) with the aid of the following identity:

$$\frac{i}{2} \int_{H_1} \frac{e^z}{z} \left( -1 + \frac{2}{1-iA} \right) dz = -\pi, \quad A = \frac{m_1}{m_2} \frac{z}{t}. \tag{5.25}$$

In order to derive (5.25) we will employ the following residue formulae:

$$\int_{H_1} \frac{e^z}{z} \mathrm{d}\tau = 2\pi i,$$

and

$$\int_{H_1} \frac{e^z}{z(z+ic)} \mathrm{d}z = 2\pi i \operatorname*{Res}_{z=0} \frac{e^z}{z(z+ic)} = \frac{2\pi}{c}, \quad c \neq 0.$$

These formulae imply the following identities for the two terms occurring in the lhs of (5.25):

$$-\frac{i}{2} \int_{H_1} \frac{e^z}{z} \mathrm{d}z = \pi,$$

and

$$i \int_{H_1} \frac{e^z}{z(1-iA)} \mathrm{d}z = \frac{i}{-i\left(\frac{m_1}{m_2 t}\right)} \int_{H_1} \frac{e^z}{z\left(z + \frac{im_2 t}{m_1}\right)} \mathrm{d}z = -2\pi.$$

Hence, (5.24) follows.

The pole of the integrand of $E_4$ occurs on the contour $H_1$ iff $m_1/m_2 = t$. Letting

$$m_2 = 1 + \epsilon_1, \quad m_1 = t - \epsilon_2, \qquad \epsilon_1 > 0, \; \epsilon_2 > 0,$$

it follows that

$$M \sim 1 - \epsilon, \qquad \epsilon = \epsilon_1 + \frac{\epsilon_2}{t}, \quad \epsilon \to 0.$$

By employing the Plemelj formula it is straightforward to compute the limit $E_4$ as $\epsilon \to 0$. Details will be presented in [FKL]. **QED**

**Proposition 5.4** *Let $E_4$ be defined in (5.24). Then,*

$$E_4(t, \delta_3, M) = 2e^{-\frac{i}{M}} + E_4^{SD}(t, \delta_3, M), \quad M = \frac{m_1}{m_2 t}, \tag{5.26}$$

*where the first term occurs iff*

$$\frac{m_1}{m_2} \in (t^{1-\delta_3}, t), \tag{5.27}$$

*and $E_4^{SD}$ is defined by*

$$E_4^{SD} = \frac{1}{\pi} \int_{H_1} \frac{e^{t^{\delta_3}[\omega - \frac{\pi}{2} + i\ln(Mt^{\delta_3}\omega)]}}{\omega[\frac{\pi}{2} - i\ln(Mt^{\delta_3}\omega)]} d\omega. \tag{5.28}$$

**Proof** In order to estimate $E_4$ we let $z = t^\delta \omega$. Then,

$$E_4 = \frac{1}{\pi} \int_{H_{t^{-\delta_3}}} \frac{e^{t^{\delta_3}[\omega - \frac{\pi}{2} + i\ln(Mt^{\delta_3}\omega)]}}{\omega[\frac{\pi}{2} - i\ln(Mt^{\delta_3}\omega)]} d\omega, \tag{5.29}$$

where $H_{t^{-\delta_3}}$ is the Hankel contour involving a circle of radius $t^{-\delta_3}$. The above integral has a stationary point at

$$\omega_{sp} = -i. \tag{5.30}$$

Thus, in order to estimate $E_4$, we deform the above circle in the $\omega$-complex plane to a circle of radius 1:

$$E_4 = \frac{1}{\pi} \int_{H_1} \frac{e^{t^{\delta_3}[\omega - \frac{\pi}{2} + i\ln(Mt^{\delta_3}\omega)]}}{\omega[\frac{\pi}{2} - i\ln(Mt^{\delta_3}\omega)]} d\omega + E_4^P,$$

where $E_4^P$ is the contribution of the pole

$$\omega_p = -\frac{i}{t^{\delta_3} M}. \tag{5.31}$$

This pole contribution occurs, iff

$$\frac{1}{t^{\delta_3} M} \in (t^{-\delta_3}, 1),$$

or

$$\frac{m_2 t}{m_1 t^{\delta_3}} \in (t^{-\delta_3}, 1),$$

which implies equation (5.27). The residue of the integral of the rhs of (5.29) associate with $\omega_p$ is given by $ie^{-i/M}$. Cauchy's theorem yields

$$\frac{1}{\pi}\left(\int_{H_1} - \int_{H_{t^{-\delta_3}}}\right) \frac{e^{t^{\delta_3}[\omega - \frac{\pi}{2} + i\ln(Mt^{\delta_3}\omega)]}}{\omega[\frac{\pi}{2} - i\ln(Mt^{\delta_3}\omega)]} d\omega = -2e^{-i/M}, \tag{5.32}$$

and then equation (5.26) follows.

The case that the pole $\omega_p$ approaches the stationary point $\omega_{sp}$ can be analysed by employing the Plemelj formulae, as well as a slight modification of the steepest descent computation [FKL]. **QED**

**Theorem 5.1** *Let $\tilde{I}_4$ denote the integral obtained from $I_4$ defined in (1.11) with $j=4$, with $|\zeta|^2$ replaced by its leading term asymptotics. Then,*

$$\tilde{I}_4(\sigma,t,\delta_3,\delta_4) = -\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]} \frac{1}{m_1^s m_2^{\bar{s}}}\left[1+O(t^{2\delta_{34}-1})\right]$$
$$+2\Re\left\{\sum\sum_{m_1,m_2\in M_4(\delta_3,t)} \frac{1}{m_1^s m_2^{\bar{s}}} e^{-\frac{im_2}{m_1}t}\right\}\left[1+O(t^{2\delta_{34}-1})\right]$$
$$+\frac{\Re}{\pi}\left\{\frac{1}{\pi}\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]} \frac{1}{m_1^s m_2^{\bar{s}}} E_4^{SD}(t,\delta_3,M)\right\}\left[1+O(t^{2\delta_{34}-1})\right],\ t\to\infty$$
$$0<\sigma<1,\ 0<\delta_3<\frac{1}{2},\ 0<\delta_4<\frac{1}{2},\ M=\frac{m_1}{m_2 t}, \tag{5.33}$$

*where the set $M_4$ is defined by*

$$M_4(\delta_3,t) = \left\{m_j = 1,\ldots,[t],\ j=1,2,\quad \frac{m_1}{m_2}\in(t^{1-\delta_3},t)\right\},$$

*and $E_4^{SD}$ is defined by (5.28) with $M=\frac{m_1}{m_2 t}$.*

**Proof** $\tilde{I}_4$ can be expressed in terms of $J_4$ by equation (5.6) and $J_4$ is given by (5.23). Replacing in the latter equation $E_4$ by the rhs of (5.26) we find (5.33). **QED**

**Remark 5.1** *A steepest descent computation implies that the leading order term of $E_4^{SD}$ is as follows:*

$$E_4^{SD} \sim -\sqrt{\frac{2}{\pi}} e^{\frac{i\pi}{4}} t^{-\frac{\delta_3}{2}} e^{-it^{\delta_3}} t^{i(\delta_3-1)t^{\delta_3}} \frac{1}{\ln\left(\frac{m_2}{m_1}t^{1-\delta_3}\right)} \left(\frac{m_1}{m_2}\right)^{it^{\delta_3}},\quad t\to\infty. \tag{5.34}$$

**Remark 5.2** *Recall that*

$$\left(1-\frac{x}{t}\right)^{\sigma-\frac{1}{2}} e^{ix}\left(1-\frac{x}{t}\right)^{i(t-x)} = \left[1+O\left(\frac{x}{t}\right)\right]\left[1+O\left(\frac{x^2}{t}\right)\right],\quad t\to\infty. \tag{5.35}$$

*This estimate, together with the fact that $|x|<\delta_{34}$ implies the restriction $x^2/t<t^{2\delta_{34}-1}$, which then imposes that both $\delta_3$ and $\delta_4$ must be less than $1/2$. However, the constraint $x^2/t<t^{2\delta_{34}-1}$ can be eliminated and hence $\delta_3$*

*and $\delta_4$ only need to be constrained to be less than 1: one can estimate the integral $\tilde{J}_4$ defined in (5.10) directly without eliminating the term defined by the lhs of (5.35). The relevant approach is very similar to the one used in section 7, where it is no possible to eliminate the lhs of (5.35) since in that case $x^2/t$ may grow.*

# 6 The Analysis of the Double Sums Arising from $I_4$ and from the Stationary Points of $I_3$

It turns out that the double sum occurring in $\tilde{I}_3$ which involves $m_2^{-\bar{s}}(m_1 + m_2)^{-s}$, can be related for large $t$ with $|\zeta(s)|^2$ which occurs in $I_4$. In this connection the following exact result will be useful.

**Lemma 6.1** *Define the functions $f(u,v)$ and $g(u,v)$ by*

$$f(u,v) = \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \frac{1}{m_2^u} \frac{1}{(m_1+m_2)^v}, \tag{6.1}$$

$$g(u,v) = \sum_{m=1}^{N} \sum_{n=N+1}^{N+m} \frac{1}{m^u n^v}, \tag{6.2}$$

*where $N$ is an arbitrary finite positive integer and $u \in \mathbb{C}$, $v \in \mathbb{C}$. These functions satisfy the identity*

$$f(u,v)+f(v,u)+\sum_{m=1}^{N} \frac{1}{m^{u+v}} = (\sum_{m=1}^{N} \frac{1}{m^u})(\sum_{n=1}^{N} \frac{1}{n^v})+g(u,v)+g(v,u). \tag{6.3}$$

**Proof** Letting $m_2 = m$, $m_1 + m_2 = n$ in $f(u,v)$ and in $f(v,u)$, and then exchanging $m$ and $n$ in the expression of $f(v,u)$, we find the following:

$$f(u,v) + f(v,u) = \Big(\sum_{m=1}^{N} \sum_{n=m+1}^{m+N} + \sum_{n=1}^{N} \sum_{m=N+1}^{N+n}\Big) \frac{1}{m^u n^v}$$

$$= \Big(\sum_{m=1}^{N} \sum_{n=m+1}^{N} + \sum_{m=1}^{N} \sum_{n=N+1}^{N+m} + \sum_{n=1}^{N} \sum_{m=n+1}^{N} + \sum_{n=1}^{N} \sum_{m=N+1}^{N+n}\Big) \frac{1}{m^u n^v}.$$

The second sum above equals $g(u,v)$, and by exchanging $m$ and $n$ in the last sum it follows that the latter sum equals $g(v,u)$. Thus, the above identity becomes

$$f(u,v)+f(v,u) = \Big(\sum_{m=1}^{N}\sum_{n=m+1}^{N} + \sum_{n=1}^{N}\sum_{m=n+1}^{N}\Big)\frac{1}{m^u n^v} + g(u,v)+g(v,u). \quad (6.4)$$

But

$$\sum_{n=1}^{N}\sum_{m=n+1}^{N}\frac{1}{m^u n^v} = \sum_{m=1}^{N}\sum_{n=1}^{m-1}\frac{1}{m^u n^v}. \quad (6.5)$$

Using the identity (6.5) in (6.4), adding to both sides of (6.4) the term

$$\sum_{m=1}^{N}\frac{1}{m^u m^v},$$

and noting that

$$\Big(\sum_{m=1}^{N}\sum_{n=m+1}^{N} + \sum_{m=1}^{N}\sum_{n=1}^{m-1}\Big)\frac{1}{m^u n^v} + \sum_{m=1}^{N}\frac{1}{m^u m^v} = \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{1}{m^u n^v},$$

equation (6.4) becomes (6.3).

**QED**

**Corollary 6.1** *The following identity is valid:*

$$2\Re\left\{\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_2^{\bar{s}}(m_1+m_2)^s}\right\} - \left(\sum_{m=1}^{[t]}\frac{1}{m^s}\right)\left(\sum_{m=1}^{[t]}\frac{1}{m^{\bar{s}}}\right)$$
$$= -\sum_{m=1}^{[t]}\frac{1}{m^{2\sigma}} + 2\Re\left\{\sum_{m=1}^{[t]}\sum_{n=[t]+1}^{[t]+m}\frac{1}{m^{\bar{s}}n^s}\right\}, \qquad s=\sigma+it\in\mathbb{C}. \quad (6.6)$$

**Proof** Letting $u=\bar{s}$, $v=s$, $N=[t]$, equation (6.3) becomes(6.6).

**QED**

**Lemma 6.2** *The following identity is valid:*

$$\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]} = \sum_{(m_1,m_2)\in M} + \sum_{m_1=1}^{\left[\frac{t}{t^{1-\delta_3}-1}\right]-1}\ \sum_{m_2=\left[(t^{1-\delta_3}-1)m_1\right]+1}^{[t]} + \sum_{m_1=\left[t^{1-\delta_2}\right]}^{[t]}\ \sum_{m_2=1}^{\left[\frac{m_1}{t^{1-\delta_2}-1}\right]-1}. \quad (6.7)$$

**Proof**

$$\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]} = \left( \sum_{m_1=1}^{\left[\frac{t}{t^{1-\delta_3}-1}\right]-1} + \sum_{m_1=\left[\frac{t}{t^{1-\delta_3}-1}\right]}^{\left[t^{1-\delta_2}\right]-1} + \sum_{m_1=\left[t^{1-\delta_2}\right]}^{[t]} \right) \sum_{m_2=1}^{[t]}$$

$$= \sum_{m_1=1}^{\left[\frac{t}{t^{1-\delta_3}-1}\right]-1} \sum_{m_2=1}^{[t]} + \sum_{m_1=\left[\frac{t}{t^{1-\delta_3}-1}\right]}^{\left[t^{1-\delta_2}\right]-1} \sum_{m_2=1}^{[t]} + \sum_{m_1=\left[t^{1-\delta_2}\right]}^{[t]} \sum_{m_2=1}^{[t]} . \tag{6.8}$$

We subdivide the sum over $m_2$ occurring in the first and third double sums in (6.8) as follows:

$$\sum_{m_2=1}^{[t]} = \sum_{m_2=1}^{\left[(t^{1-\delta_3}-1)m_1\right]} + \sum_{m_2=\left[(t^{1-\delta_3}-1)m_1\right]+1}^{[t]} ,$$

and

$$\sum_{m_2=1}^{[t]} = \sum_{m_2=1}^{\left[\frac{m_1}{t^{1-\delta_2}-1}\right]-1} + \sum_{m_2=\left[\frac{m_1}{t^{1-\delta_2}-1}\right]}^{[t]} .$$

Substituting the above expressions in (6.8) we find

$$\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]} = \sum_{m_1=1}^{\left[\frac{t}{t^{1-\delta_3}-1}\right]-1} \left( \sum_{m_2=1}^{\left[(t^{1-\delta_3}-1)m_1\right]} + \sum_{m_2=\left[(t^{1-\delta_3}-1)m_1\right]+1}^{[t]} \right)$$

$$+ \sum_{m_1=\left[\frac{t}{t^{1-\delta_3}-1}\right]}^{\left[t^{1-\delta_2}\right]-1} \sum_{m_2=1}^{[t]} + \sum_{m_1=\left[t^{1-\delta_2}\right]}^{[t]} \left( \sum_{m_2=1}^{\left[\frac{m_1}{t^{1-\delta_2}-1}\right]-1} + \sum_{m_2=\left[\frac{m_1}{t^{1-\delta_2}-1}\right]}^{[t]} \right). \tag{6.9}$$

The sum of the first, third, and fifth double sums in (6.9) equals the first term of the rhs of (6.7), whereas the second and fourth double sums in (6.9) are the second and third terms of the rhs of (6.7).

**QED**

**Lemma 6.3**

$$2\Re\left\{\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_2^{\bar{s}}(m_1+m_2)^s}\right\}-\left|\sum_{m=1}^{[t]}\frac{1}{m^s}\right|^2$$
$$=-\begin{cases}\ln t+O(1), & \sigma=\frac{1}{2},\\ \dfrac{t^{1-2\sigma}}{1-2\sigma}+O(1), & 0<\sigma<1,\quad \sigma\neq\frac{1}{2},\end{cases}$$
$$+\begin{cases}O\left(t^{\frac{1}{2}-\frac{5}{3}\sigma}\ln t\right), & 0<\sigma\leq\frac{1}{2},\\ O\left(t^{\frac{1}{3}-\frac{4}{3}\sigma}\ln t\right), & \frac{1}{2}<\sigma<1,\end{cases}\qquad t\to\infty. \tag{6.10}$$

**Proof** In order to prove (6.10), we need to estimate the two terms of the rhs of (6.6). The first term yields

$$\sum_{m=1}^{[t]}\frac{1}{m^{2\sigma}}=\int_0^{[t]}\frac{1}{x^{2\sigma}}dx+O(1),$$

which gives rise to the first term of the rhs of (6.10). The second term of (6.6) is estimated in lemma 3.2 of [KF], and then (6.10) follows.
**QED**

# 7 The Analysis of $I_3+I_4$

The interval of integration of $I_4$ can be as rewritten as follows:

$$\left[1-t^{\delta_3-1},1+t^{\delta_4-1}\right]=\left[1-t^{\delta_3-1},t^{\delta_2-1}\right]+\left[t^{\delta_2-1},1+t^{\delta_4-1}\right].$$

Thus,

$$I_{34}\doteqdot I_3+I_4=\frac{t}{\pi}\oint_{t^{\delta_2-1}}^{1+t^{\delta_4-1}}\Re\left\{\frac{\Gamma(it-it\tau)}{\Gamma(\sigma+it)}\Gamma(\sigma+it\tau)\right\}|\zeta(\sigma+it\tau)|^2\mathrm{d}\tau,$$
$$0<\sigma<1,\quad t>0,\quad 0<\delta_2<1,\quad 0<\delta_4<1, \tag{7.1}$$

where the principal value is with respect to $\tau=1$. Making the change of variables $t\tau=t-x$, equation (7.1) becomes

$$I_{34}=\frac{1}{\pi}\oint_{-t^{\delta_4}}^{t-t^{\delta_2}}\Re\left\{\frac{\Gamma(ix)\Gamma(\sigma+it-ix)}{\Gamma(\sigma+it)}\right\}|\zeta(\sigma+it-ix)|^2\mathrm{d}x, \tag{7.2}$$

where the principle value is now with respect to $x = 0$.

Since, $-t^{\delta_4} \leq x \leq t - t^{\delta_2}$, we have $t - x \geq t^{\delta_2}$, hence, $t - x \to \infty$ as $t \to \infty$, and we can use the large $t - x$ asymptotics for both $|\zeta(\sigma + it - ix)|$ and $\Gamma(\sigma + it - ix)$.

Let $\tilde{I}_{34}$ denote the expression obtained from $I_{34}$ by replacing $|\Gamma(\sigma + it - ix)|^2$ with its large $t - x$ asymptotics, namely by the rhs of (5.5) (note that $t - x \leq t + t^{\delta_4}$):

$$\tilde{I}_{34}(\sigma, t, \delta_2, \delta_4) = \Re \left\{ \sum_{m_1=1}^{[t]} \sum_{m_2=1}^{[t]} \frac{1}{m_1^s m_2^{\bar{s}}} J_{34}(\sigma, t, \delta_2, \delta_4, \frac{m_1}{m_2}) \right\}, \tag{7.3}$$

where $J_{34}$ is defined by

$$J_{34}(\sigma, t, \delta_2, \delta_4, \frac{m_1}{m_2}) = \frac{1}{\pi} \int_{-t^{\delta_4}}^{t - t^{\delta_2}} \frac{\Gamma(ix)\Gamma(\sigma + it - ix)}{\Gamma(\sigma + it)} \left( \frac{m_1}{m_2} \right)^{ix} \mathrm{d}x, \tag{7.4}$$

$$0 < \sigma < 1, \quad t > 0, \quad 0 < \delta_2 < 1, \quad 0 < \delta_4 < 1, \quad m_j = 1, 2, \ldots, [t], \quad j = 1, 2.$$

**Proposition 7.1** *Let $J_{34}$ be defined by (7.4). Let $H_1$ denote the Hankel contour defined in (5.8) and depicted in figure 1. Then,*

$$J_{34}(\sigma, t, \delta_2, \delta_4, \frac{m_1}{m_2}) = \frac{1}{\pi} \int_{H_1} \frac{e^z}{z} \tilde{J}_{34}(\sigma, t, \delta_2, \delta_4, A) dz \left[ 1 + O\left( t^{-\delta_2} \right) \right], \quad t \to \infty,$$

$$0 < \sigma < 1, \quad 0 < \delta_2 < 1, \quad 0 < \delta_4 < 1, \quad A = \frac{m_1 z}{m_2 t}, \tag{7.5}$$

*where $\tilde{J}_{34}$ is defined by*

$$\tilde{J}_{34}(\sigma, t, \delta_2, \delta_4, A) = \oint_{-t^{\delta_4}}^{t - t^{\delta_2}} \frac{e^{\frac{\pi x}{2}} A^{ix}}{e^{-\pi x} - e^{\pi x}} \left( 1 - \frac{x}{t} \right)^{\sigma - \frac{1}{2}} e^{ix} \left( 1 - \frac{x}{t} \right)^{i(t - x)} dx, \tag{7.6}$$

*and the principle value is with respect to $x = 0$.*

**Proof.** The proof is identical with the proof of proposition 5.1 expect that now

$$t^{\delta_2} \leq t - x \leq t + t^{\delta_4},$$

thus,

$$\frac{1}{t - x} \leq t^{-\delta_2},$$

hence the error term of $\Gamma(\sigma + i(t - x))$ is of order $O(t^{-\delta_2})$. **QED**

**Proposition 7.2** *Let $\tilde{J}_{34}$ be defined in (7.6). Then,*

$$\tilde{J}_{34}(\sigma,t,\delta_2,\delta_4,A)=\frac{i}{2}(-1+\frac{2}{1-iA})+\tilde{E}_{34}(\sigma,t,\delta_2,A)\left(1+O\left(\frac{\sigma-\frac{1}{2}}{t^{\delta_2}}\frac{1}{\check{A}},\frac{1}{t^{\delta_2}}\frac{1}{\check{A}^2}\right)\right)$$

$$+O\left(\frac{\sigma-\frac{1}{2}}{t}\Psi'(\tilde{A}),\frac{1}{t}\Psi''(\tilde{A}),\frac{\sigma-\frac{1}{2}}{t}\Psi'(\hat{A}),\frac{1}{t}\Psi''(\hat{A})\right),\quad t\to\infty,$$

$0<\sigma<1,\quad 0<\delta_2<1,\quad 0<\delta_4<1,$

$$\tilde{A}=\frac{\pi}{2}-i\ln A,\quad \hat{A}=\frac{3\pi}{2}+i\ln A,\quad \check{A}=\frac{\pi}{2}-i\ln A+i(\delta_2-1)\ln t, \tag{7.7}$$

*where $\Psi$ denotes the digamma function $\Gamma'/\Gamma$, prime denotes differentiation, and $\tilde{E}_{34}$ is defined by*

$$\tilde{E}_{34}(\sigma,t,\delta_2,A)=t^{(\delta_2-1)(\sigma-\frac{1}{2}+it^{\delta_2})}e^{i(t-t^{\delta_2})}\frac{e^{(t-t^{\delta_2})(i\ln A-\frac{\pi}{2})}}{\frac{\pi}{2}-i\ln A+i(\delta_2-1)\ln t}. \tag{7.8}$$

**Proof** In the relevant interval of integration $\frac{x^2}{t}$ may grow, thus we cannot use the identity

$$\left(1-\frac{x}{t}\right)^{\sigma-\frac{1}{2}}e^{ix}\left(1-\frac{x}{t}\right)^{i(t-x)}=1+O\left(\frac{x^2}{t}\right),\quad \frac{x^2}{t}\to 0.$$

We split the interval of integration into

$$\left[-t^{\delta_4},-\epsilon\right]\bigcup\left[\epsilon,t-t^{\delta_2},\right].$$

In the integral corresponding to the first interval we replace $x$ by $-x$, and then in the resulting integral, as well as in the integral corresponding to the second interval above, we expand the term $(1-e^{-2\pi x})^{-1}$. In this way, we find

$$\tilde{J}_{34}=\sum_{k=0}^{\infty}\left[J_1(\sigma,t,\delta_4,k,A)+J_2(\sigma,t,\delta_2,k,A)\right], \tag{7.9}$$

where $J_1$ is defined by

$$J_1=\lim_{\epsilon\to 0}\int_{\epsilon}^{t^{\delta_4}}(1+\frac{x}{t})^{\sigma-\frac{1}{2}}e^{-F_1(x,t,k,A)}\mathrm{d}x, \tag{7.10}$$

with

$$F_1 = x\left(\frac{3\pi}{2} + 2\pi k + i\ln A\right) + i\left[x - (t+x)\ln(1+\frac{x}{t})\right], \tag{7.11}$$

whereas $J_2$ is defined by

$$J_2 = \lim_{\epsilon\to 0}\int_{t-t^{\delta_2}}^{\epsilon}\left(1-\frac{x}{t}\right)^{\sigma-\frac{1}{2}} e^{F_2(x,t,k,A)}\mathrm{d}x, \tag{7.12}$$

with

$$F_2 = -x\left(2\pi k + \frac{\pi}{2} - i\ln A\right) + i\left[x + (t-x)\ln(1-\frac{x}{t})\right]. \tag{7.13}$$

We can estimate $J_1$ and $J_2$ using integration by parts. In this respect we note that

$$\frac{\partial F_1}{\partial x} = \frac{3\pi}{2} + 2\pi k + i\ln A - i\ln(1+\frac{x}{t}),$$

and

$$\frac{\partial F_2}{\partial x} = -\left[\frac{\pi}{2} + 2\pi k - i\ln A + i\ln(1-\frac{x}{t})\right].$$

Thus,

$$\begin{aligned}
J_1 &= \lim_{\epsilon\to 0}\left\{\left.\frac{(1+\frac{x}{t})^{\sigma-\frac{1}{2}}e^{-F_1}}{-\partial F_1/\partial x}\right|_{x=\epsilon}^{x=t^{\delta_4}} - \int_{\epsilon}^{t^{\delta_4}}\frac{\partial}{\partial x}\left(\frac{(1+\frac{x}{t})^{\sigma-\frac{1}{2}}}{-\partial F_1/\partial x}\right)e^{-F_1}dx\right\} \\
&= \lim_{\epsilon\to 0}\left\{\left.\frac{(1+\frac{x}{t})^{\sigma-\frac{1}{2}}e^{-F_1}}{-\partial F_1/\partial x}\right|_{x=\epsilon}^{x=t^{\delta_4}}\right. \\
&\quad \left. - \int_{\epsilon}^{t^{\delta_4}}\left[\frac{(\sigma-\frac{1}{2})(1+\frac{x}{t})^{\sigma-\frac{3}{2}}}{t(\partial F_1/\partial x)^2} + i\frac{(1+\frac{x}{t})^{\sigma-\frac{1}{2}}}{(\partial F_1/\partial x)^3}\frac{1}{(t+x)}\right]\left(\frac{\partial}{\partial x}e^{-F_1}\right)dx\right\} \\
&= \frac{1}{\frac{3}{2}\pi + 2\pi k + i\ln A} + O\left(\frac{\sigma-\frac{1}{2}}{t(\frac{3}{2}\pi+2\pi k+i\ln A)^2}, \frac{1}{t}\frac{1}{(\frac{3}{2}\pi+2\pi k+i\ln A)^3}\right), \\
&\qquad t\to\infty,
\end{aligned} \tag{7.14}$$

where we have used the fact that $\Re\{3\pi/2 + 2\pi k + i\ln A\} > 0$ for all $k$, hence the contribution from the end point $t^{\delta_4}$ yields an exponentially small term.

Similarly, using integration by parts to estimate $J_2$ we find the following:

$$
\begin{aligned}
J_2 &= \lim_{\epsilon\to 0}\left\{ \left. \frac{(1-\frac{x}{t})^{\sigma-\frac{1}{2}} e^{F_2}}{\partial F_2/\partial x} \right|_{x=t-t^{\delta_2}}^{\epsilon} - \int_{t-t^{\delta_2}}^{\epsilon} \frac{\partial}{\partial x}\left( \frac{(1-\frac{x}{t})^{\sigma-\frac{1}{2}}}{\partial F_2/\partial x}\right) e^{F_2} dx \right\} \\
&= \lim_{\epsilon\to 0}\left\{ \left. \frac{(1-\frac{x}{t})^{\sigma-\frac{1}{2}} e^{F_2}}{\partial F_2/\partial x} \right|_{x=t-t^{\delta_2}}^{\epsilon} \right. \\
&\quad \left. + \int_{t-t^{\delta_2}}^{\epsilon} \left[ \frac{(\sigma-\frac{1}{2})(1-\frac{x}{t})^{\sigma-\frac{3}{2}}}{t(\partial F_2/\partial x)^2} + i\frac{(1-\frac{x}{t})^{\sigma-\frac{1}{2}}}{(\partial F_2/\partial x)^3}\frac{1}{(t-x)}\right] \left(\frac{\partial}{\partial x} e^{F_2}\right) dx \right\} \\
&= -\frac{1}{\frac{\pi}{2}+2\pi k - i\ln A} + \frac{t^{(\delta_2-1)(\sigma-\frac{1}{2}+it^{\delta_2})} e^{i(t-t^{\delta_2})} e^{-(t-t^{\delta_2})(\frac{\pi}{2}+2\pi k - i\ln A)}}{\frac{\pi}{2}+2\pi k - i\ln A + i(\delta_2-1)\ln t} \\
&\quad + O\left( \frac{\sigma-\frac{1}{2}}{t(\frac{\pi}{2}+2\pi k + i\ln A)^2}, \frac{1}{t}\frac{1}{(\frac{\pi}{2}+2\pi k + i\ln A)^3}\right) \\
&\quad + O\left( \frac{(\sigma-\frac{1}{2})}{t}\frac{t^{(\delta_2-1)(\sigma-\frac{3}{2}+it^{\delta_2})} e^{i(t-t^{\delta_2})} e^{-(t-t^{\delta_2})(\frac{\pi}{2}+2\pi k - i\ln A)}}{(\frac{\pi}{2}+2\pi k - i\ln A + i(\delta_2-1)\ln t)^2}, \right. \\
&\quad \left. \frac{1}{t^{\delta_2}}\frac{t^{(\delta_2-1)(\sigma-\frac{1}{2}+it^{\delta_2})} e^{i(t-t^{\delta_2})} e^{-(t-t^{\delta_2})(\frac{\pi}{2}+2\pi k - i\ln A)}}{(\frac{\pi}{2}+2\pi k - i\ln A + i(\delta_2-1)\ln t)^3}\right), \quad t\to\infty. \qquad (7.15)
\end{aligned}
$$

Taking into consideration that $\arg z \in [-\pi,\pi]$, it follows that the second term, as well as the last two terms, in the rhs of (7.15) decay exponentially, unless $k=0$. Thus,

$$
J_2 = -\frac{1}{2\pi k + \frac{\pi}{2} - i\ln A} + \tilde{E}_{34}(\sigma,t,\delta_2,A) + O(A_1,A_2,A_3,A_4), \qquad (7.16)
$$

where $\tilde{E}_{34}$ is given in (7.8) and $A_1, A_2, A_3, A_4$ are defined as follows:

$$
\begin{aligned}
A_1 &= \frac{\sigma-\frac{1}{2}}{t(\frac{\pi}{2}+2\pi k + i\ln A)^2}, \qquad A_2 = \frac{1}{t}\frac{1}{(\frac{\pi}{2}+2\pi k + i\ln A)^3}, \\
A_3 &= \frac{(\sigma-\frac{1}{2})}{t^{\delta_2}}\frac{t^{(\delta_2-1)(\sigma-\frac{1}{2}+it^{\delta_2})} e^{i(t-t^{\delta_2})} e^{-(t-t^{\delta_2})(\frac{\pi}{2}-i\ln A)}}{(\frac{\pi}{2}-i\ln A + i(\delta_2-1)\ln t)^2}, \\
A_4 &= \frac{1}{t^{\delta_2}}\frac{t^{(\delta_2-1)(\sigma-\frac{1}{2}+it^{\delta_2})} e^{i(t-t^{\delta_2})} e^{-(t-t^{\delta_2})(\frac{\pi}{2}-i\ln A)}}{(\frac{\pi}{2}-i\ln A + i(\delta_2-1)\ln t)^3}.
\end{aligned}
$$

Regarding the summation over all $k$ of this error term, as well as the error term of $J_1$ in the rhs of (7.14), we recall the following identities:

$$
\sum_{k=0}^{\infty}\frac{1}{(k+a)^2} = \Psi'(a), \quad \sum_{k=0}^{\infty}\frac{1}{(k+a)^3} = -\frac{1}{2}\Psi''(a), \qquad (7.17)
$$

where $\Psi$ denotes the digamma function $\Gamma'/\Gamma$ and prime denotes differentiation.

Adding the expressions (7.14), (7.16) and employing (7.17), equation (7.9) yields (7.7). In this respect, we note that following the same steps used in the proof of proposition 5.2, we find that the first terms of the rhs of (7.14) and (7.16) yield

$$\frac{i}{2}(-1+\frac{2}{1-iA}).$$

**QED**

**Proposition 7.3** *Let $J_{34}$ be defined by (7.4). Then,*

$$J_{34}(\sigma,t,\delta_2,\delta_3) = [-1+E_{34}(\sigma,t,\delta_2,M)+o(1)]\,[1+O(t^{-\delta_2})], \quad t\to\infty,$$
$$0<\sigma<1,\ 0<\delta_2<1,\ 0<\delta_4<1,\ M=\frac{m_1}{m_2 t}, \tag{7.18}$$

*where $E_{34}$ is defined by*

$$E_{34}(\sigma,t,\delta_2,M) = \frac{1}{\pi}t^{(\delta_2-1)(\sigma-\frac{1}{2}+it^{\delta_2})}e^{i(t-t^{\delta_2})}\int_{H_1}\frac{e^z}{z}\frac{e^{(t-t^{\delta_2})[i\ln(Mz)-\frac{\pi}{2}]}[1+O(t^{-\delta_2})]dz}{\left[\frac{\pi}{2}-i\ln(Mz)+i(\delta_2-1)\ln t\right]}, \tag{7.19}$$

*with $H_1$ denoting the Hankel contour depicted in figure 1.*

**Proof** Equation (7.5) shows that $J_{34}$ involves multiplying $\tilde{J}_{34}$ by $e^z/z$ and integrating along the Hankel contour. Thus, following precisely the same steps used in the proof of proposition 5.3, the first term of the rhs of (7.7) yields $-1$. The second term of the rhs of (7.7), namely $\tilde{E}_{34}$, yields $E_{34}$. It is shown in [FKL] that the first two error terms in the rhs of (7.7) yield the error term $O(t^{-\delta_2})$ of the rhs of (7.19), and the last four error terms in the rhs of (7.7) yield the error term $o(1)$ of the rhs of (7.18). **QED**

**Proposition 7.4** *Let $E_{34}$ be defined by (7.19). Then,*

$$E_{34}(\sigma,t,\delta_2,M) = 2t^{(\delta_2-1)(\sigma-\frac{1}{2}+it)}e^{i(t-t^{\delta_2})}e^{-\frac{im_2}{m_1}t^{\delta_2}} + E_{34}^{SD}(\sigma,t,\delta_2,M),$$
$$0<\sigma<1,\ 0<\delta_2<1,\ t>0,\ M=\frac{m_1}{m_2 t}, \tag{7.20}$$

*where the first term of the rhs of (7.20) occurs iff*

$$\frac{m_2}{m_1}\in\left(t^{-\delta_2},t^{1-\delta_2}-1\right), \tag{7.21}$$

*and $E_{34}^{SD}$ is defined by*

$$E_{34}^{SD}(\sigma,t,\delta_2,M)=\frac{t^{(\delta_2-1)(\sigma-\frac{1}{2}+it^{\delta_2})}}{\pi}e^{i(t-t^{\delta_2})}\int_{H_1}\frac{e^{(t-t^{\delta_2})[\omega+i\ln(M(t-t^{\delta_2})\omega)-\frac{\pi}{2}]}[1+O(t^{-\delta_2})]d\omega}{\frac{\pi}{2}-i\ln(M(t-t^{\delta_2})\omega)+i(\delta_2-1)\ln t}, \tag{7.22}$$

*with $H_1$ denoting the Hankel contour defined in figure 1.*

**Proof** The derivation of the above result is very similar with the proof of proposition 5.4: we make in the integral $E_{34}$ the change of variables $z=T\omega, T=t-t^{\delta_2}$, and then we obtain an integral similar with the rhs of (7.22), but with $H_1$ replaced by $H_{T^{-1}}$ which denotes a Hankel contour whose associated circle has radius $T^{-1}$ instead of 1. The integrand of this integral has a stationary point at

$$\omega_{sp}=-i. \tag{7.23}$$

This motivates the deformation of the contour $H_{T^{-1}}$ to the contour $H_1$. Taking into consideration that the above integrand has a pole at

$$\omega_p=-\frac{i}{MTt^{1-\delta_2}}, \tag{7.24}$$

it follows that deformation from $H_{T^{-1}}$ to $H_1$ yields a contribution from the pole $\omega_p$, iff

$$\frac{1}{MTt^{1-\delta_2}}\in\left(\frac{1}{T},1\right),$$

i.e., iff

$$\frac{m_2t}{m_1(t-t^{\delta_2})t^{1-\delta_2}}\in\left(\frac{1}{t-t^{\delta_2}},1\right),$$

which implies equation (7.21). The pole contribution equals the negative of the first term of the rhs of (7.20), and then (7.20) follows via Cauchy's theorem, in analogy with (5.32).

The case that the pole $\omega_p$ approaches the stationary point $\omega_{sp}$ can be analysed by employing the Plemelj formulae [FKL], as well as a slight modification of the steepest descent computation. **QED**

**Remark 7.1** *A steepest descent computation implies that the leading order term of $E_{34}^{SD}$ is as follows:*

$$E_{34}^{SD}\sim-\sqrt{\frac{2}{\pi}}e^{\frac{i\pi}{4}}t^{(\delta_2-1)(\sigma-\frac{1}{2}+it^{\delta_2})}\frac{[M(t-t^{\delta_2})]^{i(t-t^{\delta_2})}}{\sqrt{t-t^{\delta_2}}\ln[M(t-t^{\delta_2})t^{1-\delta_2}]},\quad t\to\infty. \tag{7.25}$$

**Theorem 7.1** *Let $\tilde{I}_{34}$ denote the integral obtained from $I_3+I_4$ by replacing $|\zeta|^2$ with its leading term asymptotics. Then,*

$$\begin{aligned}\tilde{I}_{34}\left(\frac{1}{2},t,\delta_2,\delta_4\right) &= \Bigg[-\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_1^{\frac{1}{2}+it}m_2^{\frac{1}{2}-it}} \\ &+2\Re\Bigg\{t^{i(\delta_2-1)t}e^{i(t-t^{\delta_2})}\sum\sum_{m_1,m_2\in M_{34}(\delta_2,t)}\frac{1}{m_1^{\frac{1}{2}+it}m_2^{\frac{1}{2}-it}}e^{-i\frac{m_2}{m_1}t^{\delta_2}} \\ &-\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t-t^{\delta_2}}}e^{\frac{i\pi}{4}}t^{i(\delta_2-1)t^{\delta_2}}\left(1-t^{\delta_2-1}\right)^{i(t-t^{\delta_2})} \\ &\times\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_1^{\frac{1}{2}+it^{\delta_2}}m_2^{\frac{1}{2}-it^{\delta_2}}}\frac{[1+O(t^{-\delta_2})]}{\ln\left[\frac{m_1}{m_2}\left(t^{1-\delta_2}-1\right)\right]}\Bigg\}\Bigg] \\ &\times\left[1+O(t^{-\delta_2})\right],\qquad 0<\delta_2<1,\quad t\to\infty,\end{aligned}\tag{7.26}$$

*where the set $M_{34}$ is defined by*

$$M_{34}=\left\{m_j=1,\ldots,[t],\ j=1,2,\ \ \frac{m_2}{m_1}\in\left(t^{-\delta_2},t^{1-\delta_2}-1\right)\right\}.$$

**Proof** Replacing in (7.3) the expression for $J_{34}$ given by (7.18), we find that the first term of the rhs of (7.18) and the first term of the rhs of $E_{34}$, defined in (7.20), yield the first two terms of the rhs of (7.26). The third term of (7.26) comes from the steepest descent contribution given in (7.25). The neighbourhoods of the points $\frac{1}{tM}=t^{-\delta_2}$, $\frac{1}{tM}=t^{1-\delta_2}-1$ can be evaluated in the same way as the neighbourhoods of the points $M=1$ and $M=t^{-\delta_3}$ in propositions 5.3 and 5.4. Estimates of $E_{34}^{SD}$ and of the error term in the rhs of (7.18) presented in [FKL] yield the error term in the rhs of (7.26). **QED**

**Remark 7.2** *Rigorous estimates of the steepest descent contribution are presented in [FKL]. However, the following heuristic argument implies that for the interesting case of $\delta_2=1/2$, the contribution of the error term is negligible: employing (1.3) of [FL] with $\eta\to 2\pi t$, $t\to t^{\delta}$, $\sigma=1/2$, $\delta\in(0,1)$, we obtain*

$$\zeta\left(\frac{1}{2}+it^{\delta}\right)=\sum_{n=1}^{[t]}\frac{1}{n^{\frac{1}{2}+it^{\delta}}}+O\left(t^{\frac{1}{2}-\delta}\right),\quad t\to\infty,$$

*thus*

$$\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_1^{\frac{1}{2}+it^\delta}m_2^{\frac{1}{2}-it^\delta}}=\left|\zeta\left(\frac{1}{2}+it^\delta\right)\right|^2+O\left(t^{\frac{1}{2}-\delta}\left|\zeta\left(\frac{1}{2}+it^\delta\right)\right|\right)+O\left(t^{1-2\delta}\right),\quad t\to\infty.$$

*Using the estimate* $\zeta\left(\frac{1}{2}+it^\delta\right)=O\left(t^{\frac{\delta}{6}}\right)$, *the above expression yields*

$$\frac{1}{\sqrt{t}}\sum_{m_1=1}^{[t]}\sum_{m_2=1}^{[t]}\frac{1}{m_1^{\frac{1}{2}+it^\delta}m_2^{\frac{1}{2}-it^\delta}}=O\left(t^{\frac{\delta}{3}-\frac{1}{2}}\right)+O\left(t^{-\frac{5}{6}\delta}\right)+O\left(t^{\frac{1}{2}-2\delta}\right),\quad t\to\infty.$$

*The last equation suggests that for* $\delta_2>1/4$ *the steepest descent contribution of (7.26) is bounded by a decreasing function of* $t$*, which vanishes as* $t\to\infty$.

# 8 Conclusions

The main results presented here are the following:

**The rigorous derivation of the exact integral equation** (1.3) **satisfied by** $|\zeta(s)|^2$

This result is based on a certain identity relating the Riemann and Hurwitz zeta functions derived in [ASF], and on the use of the Plemelj formulae, which provide the main tools for the analysis of the Riemann-Hilbert problem.

**The derivation of the asymptotic identity** (1.5)

The derivation of this result is based on the large $t-$analysis of the integrals $\{I_j\}_1^4$ defined in (1.11). In more details the derivation of (1.5) is based on the following:

- The rigorous estimation of $I_1$ is straightforward.
- The rigorous estimation of $I_2$ is based on Atkinson's classical estimates.
- The asymptotics of the integral $\tilde{I}_3$, which denotes the integral obtained from $I_3$ by replacing $|\zeta|^2$ with its large $t$ asymptotics, can be obtained via standard asymptotic techniques for integrals. Indeed, the main contributions of $I_3$ arise from the associated stationary points (the relevant rigorous computation is straightforward), as well as from the

end points. The contribution from the upper end point is rigorously computed in [FSF], where the analysis of the non-generic case that the stationary point approaches the upper end point is also presented. The rigorous computation of the analogous contribution of the lower end point can be obtained in a very similar manner.

- The asymptotics of $\tilde{I}_4$, which denotes the integral obtained by replacing $|\zeta|^2$ with its large $t$- asymptotics , can be obtained via novel asymptotic techniques. Indeed, it turns out that the relevant analysis give rise to an integral along the Hankel contour $H_1$ whose integrand involves two terms. Remarkably, the Hankel integral of the first term can be computed analytically, and thus one is left with the computation of the Hankel integral of the second term, denoted by $E_4$. By deforming the Hankel contour to pass over the relevant stationary point, and by employing Cauchy's theorem, it follows that $E_4$ yields a steepest descent contribution, plus a contribution due to the associated residue.

- The rigorous analysis of the relation between two different Riemann zeta-type double sums arising in the asymptotics of $\tilde{I}_3$ and $\tilde{I}_4$ (see section 6).

In order to complete the rigorous derivation of (1.5) the following tasks are required:

- The analysis of the contribution to $\tilde{I}_3$ from the lower end point of integration (which is very similar to the analysis presented in [FSF]).

- The investigation of the non-generic case that the stationary point and the pole associated with $E_4$ coincide (which can be analysed by employing the Plemelj formulae, as well as a slight modification of the steepest descent computation).

- The analysis of the error terms arising from replacing in $\{I_j\}_1^4$, $|\zeta|^2$ by its leading asymptotics.

Clearly, the challenging task is the last one, which fortunately *can* be avoided: in analogy with equation (1.3), there exists an equation satisfied by the double sum expressing the leading asymptotics of $|\zeta|^2$, thus one only needs to analyse $\{\tilde{I}_j\}_1^4$, instead of $\{I_j\}_1^4$. In this connection we note that it is straightforward to estimate $\tilde{I}_1$, instead of $I_1$. Furthermore, Atkinson's estimates which are used for the estimation of $I_2$, are also valid for the double sum expressing the leading asymptotics of $|\zeta|^2$ (actually the proof of Atkinson is based on this sum instead of $|\zeta|^2$).

**The derivation of the asymptotic identity** (1.6)

The derivation of this result is based on the rigorous analysis of $I_1$ and $I_2$ described in point 2., as well as on the large $t$-asymptotics of the integral $\tilde{I}_{34} = \tilde{I}_3 + \tilde{I}_4$.

The analysis of this integral is similar to the analysis of $\tilde{I}_4$. However, before obtaining the two Hankel integrals described in point 2., it is now necessary to use integration by parts to simplify an integral which appears in the integrand of the Hankel integral. Thus, the completion of the rigorous justification of equation (1.6) requires the following:

- The rigorous investigation of the asymptotics employing integration by parts, mentioned above.
- The investigation of the non-generic case that the stationary point and the pole associated with $E_{34}$ coincide (which can be analysed by employing the Plemelj formulae, as well as a slight modification of the steepest descent computation).

The derivation of an equation analogous to (1.3) satisfied by the double sum expressing the leading large $t$ asymptotics of $|\zeta|^2$, as well as the rigorous analysis of the remaining issues discussed above, will be pursued in [FKL].

## Appendix A Numerical verification of (5.14).

Let $t = 10^7$, $\delta_3 = \delta_4 = \delta = \frac{1}{4}$. Let $\tilde{J}_4$ be defined by (5.10). We compute $\tilde{J}_4$ and the leading term of the rhs of (5.14), at the following four different values of $A$: $\{2+3i, -2+3i, -2-3i, 2-3i\}$. These points are in the four different quadrants of the complex $z$-plane. The results are shown below:

$$
\begin{array}{lll}
A = 2+3i & \text{lhs}= -0.1 - i0.3 & \text{rhs}= -0.1 - i0.3 \\
A = -2+3i & \text{lhs}= 0.1 - i0.3 & \text{rhs}= 0.1 - i0.3 \\
A = -2-3i & \text{lhs}= 4.68\times 10^{13} - i1.56\times 10^{14} & \text{rhs}= 4.68\times 10^{13} - i1.56\times 10^{14} \\
A = 2-3i & \text{lhs}= -0.25 - i0.75 & \text{rhs}= -0.25 - i0.75\ .
\end{array}
$$

The relative errors are

$$
\begin{array}{ll}
A = 2+3i & \text{re}=\left|\frac{\text{rhs}-\text{lhs}}{\text{rhs}}\right| = 1.42\times 10^{-8} \\
A = -2+3i & \text{re}=\left|\frac{\text{rhs}-\text{lhs}}{\text{rhs}}\right| = 1.42\times 10^{-8} \\
A = -2-3i & \text{re}=\left|\frac{\text{rhs}-\text{lhs}}{\text{rhs}}\right| = 1.5\times 10^{-9} \\
A = 2-3i & \text{re}=\left|\frac{\text{rhs}-\text{lhs}}{\text{rhs}}\right| = 1.26\times 10^{-8}.
\end{array}
$$

The first term of the rhs of (5.14) is dominant in all cases except for the third case where $A$ is in the third quadrant. In this case, as expected, the dominant term is the second term of the rhs of (5.14) with the relevant contribution growing like $e^{t^{\delta}\left(-\frac{\pi}{2}-\arg A\right)}$, with $\left(-\frac{\pi}{2}-\arg A\right) \approx 0.588$.

## Appendix B Numerical verification of (5.26).

Let $t = 6\times 10^7 + 0.45$ and $\delta_3 = \frac{1}{4}$. Let $E_4$ be defined by (5.24). Figure 2 depicts the relative error $\left|\frac{\text{rhs}-\text{lhs}}{\text{lhs}}\right|$ of (5.26) with $E_4^{SD}$ computed via (5.34).

Recall the constraint (5.27), and that $M = \frac{m_1}{m_2 t}$: if $M < t^{-\delta_3}$, then the leading asymptotic behaviour of the rhs of (5.26) is obtained by considering only the steepest descent contribution, whereas if $M > t^{-\delta_3}$ one has to consider the additional pole contribution.

Figure 2 depicts the relative errors for different values of $M = at^{-\delta_3}$: for the left figure $a \in (0, 2/3)$ and the right figure $a \in (4/3, 2)$.

The error is small provided that the pole does not approach the stationary point, namely, $a$ does not approach the value 1.

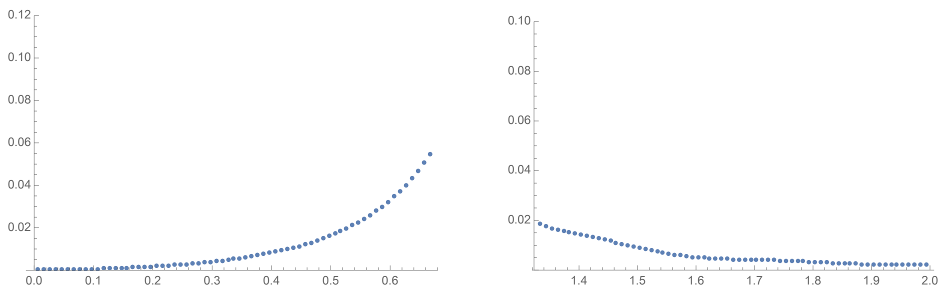


Figure 2: The relative errors between the lhs and rhs of (5.26). For the left figure, the pole contribution in the rhs of (5.26) is *not* taken into account, whereas for the right figure, both terms of the rhs of (5.26) are taken into consideration.

# Appendix C Numerical verification of (7.20).

The mathematical analysis of $E_{34}$ is similar with $E_4$. However, it is more difficult to obtain numerical verifications for $E_{34}$ because now the interval of integration is large. Thus, we only consider a relatively small value of $t$, namely we let $t = 39.17$ and $\delta_2 = \frac{1}{4}$. Let $E_{34}$ be given by (7.19). We compute the relative error $\left|\frac{\text{rhs}-\text{lhs}}{\text{lhs}}\right|$ of (7.20) with $E_{34}^{SD}$ given by (7.25), for all points $m_j = 1, \ldots, [t]$, $j = 1, 2$, namely for $39 \times 39 = 1521$ points. The red dots correspond to the points $(m_1, m_2)$ which satisfy the constraint (7.21), thus the computation of the rhs of (7.20) involves the contribution from pole $\omega_p$, whereas for the blue dots only the contribution from the stationary point $\omega_{sp}$ is taken into account.

Again the relative error is small, unless the pole approaches the stationary point (this feature is more striking for small values of $m_1$).

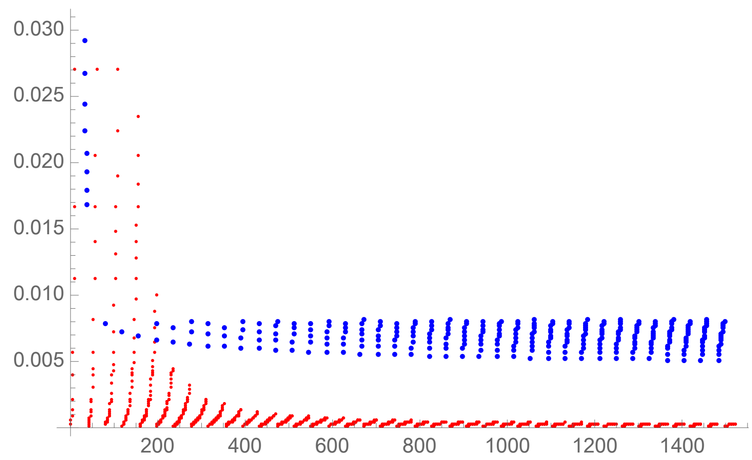


Figure 3: The relative errors between the lhs and rhs of (7.20). The blue dots correspond to the steepest descent contribution, whereas the red dots correspond to both the steepest descent and the pole contributions.

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The rigorous estimates of the integral $J_3^U$ is presented in our joint paper with Arran Fernandez and Euan Spence.

The starting point of the approach developed here is equation (2.2) which is derived in our joint paper with Anthony Ashton [ASF].

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