

Graph States, Pivot Minor, and Universality of (X, Z) -measurements

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Abstract

The graph state formalism offers strong connections between quantum information processing and graph theory. Exploring these connections, first we show that any graph is a pivot-minor of a planar graph, and even a pivot minor of a triangular grid. Then, we prove that the application of measurements in the (X, Z) plane (i.e. one-qubit measurement according to the basis $\{\cos(\theta)|0\rangle + \sin(\theta)|1\rangle, \sin(\theta)|0\rangle - \cos(\theta)|1\rangle\}$ for some θ) over graph states represented by triangular grids is a universal measurement-based model of quantum computation. These two results are in fact two sides of the same coin, the proof of which is a combination of graph theoretical and quantum information techniques.

1 Introduction

In 2001, Raussendorf and Briegel [33, 34] introduced a model for quantum computation based on measurements where one-qubit measurements are applied on an initial entangled state, called *graph state*. This model, called the *one-way model* is not only a very promising model for the physical implementation of a quantum computer [41, 32], but it has also led to several theoretical breakthroughs in quantum information processing. For instance, the one-way model has been proved to be more favorable to the parallelization of quantum operations than the usual quantum circuits [7]; the one-way model has also given rise to the elaboration of several protocols like the blind quantum computing [5], and the quantum secret sharing with graph states [23, 19, 18].

The graph state formalism, which is used to describe the initial entangled state in the one-way model, has been broadly studied this last decade, the survey by Hein *et al.* [16] provides an excellent introduction to the domain and has more than 200 references. The graph state formalism is a powerful framework for characterizing quantum information properties in a combinatorial way, using graph theory. For instance, the ability of performing a unitary (or more generally an information preserving evolution) on a given graph state has

been characterized as the existence of a certain kind of *flow* in the corresponding graph [8, 10, 6, 24].

Another example of the graphical characterization of quantum information properties is that two graphs which are locally equivalent (i.e. equal up to local complementation, a graph transformation introduced by Kotzig [20] and investigated among others by Bouchet [11, 3]) are representing the same entanglement in the sense that the corresponding graph states are LC equivalent [39]. As a consequence, the rank-width [29] of a graph, which is invariant by local complementation, is a measure of entanglement of the corresponding quantum state [40]. Moreover, the minimal degree up to local complementation [17] and the weak odd domination [15] are other examples of graph theoretical characterization of quantum properties: the minimal distance of a quantum code for the minimal degree up to local complementation [1]; and the threshold of a graph state based quantum secret sharing schemes for weak odd domination [15].

In the present paper, we mainly prove two important results, both come from the strong connections, offered by the graph state formalism, between quantum information processing and graph theory. First we show that any graph is a pivot-minor of a planar graph (a pivot minor of a graph G is a graph one can obtain by performing pivotings – a certain combination of local complementations – and vertex deletions over the graph G). Moreover, we prove that the application of measurements in the (X, Z) plane (i.e. one-qubit measurement according to the basis $\{\cos(\theta)|0\rangle + \sin(\theta)|1\rangle, \sin(\theta)|0\rangle - \cos(\theta)|1\rangle\}$ for some θ) over graph states represented by triangular grids is a universal model of quantum computation. These two results are in fact two sides of the same coin, the proof of which is a combination of graph theoretical and quantum information techniques. In particular, the former, i.e. the proof that any graph is a pivot-minor of a planar graph, is an example of ‘classical’ (in the sense ‘non quantum’) property which is proved using quantum arguments. Other such classical results with quantum proofs are listed in this paper [12].

The latter result, the universality of (X, Z) -measurement over triangular grids, is an improvement in the quest of the minimal resources for measurement-based quantum computation [26, 30, 22, 40, 31, 5, 36], minimisation which is essential for the actual physical implementation of the model. Several regular grids (square, triangular, and hexagonal grids) are known to be universal resources for the one-way model [40], however the universality of these graph states is based on the use of single qubit measurements in the three possible axis X , Y and Z of the Bloch sphere, more precisely the performed measurements are according to Z (measurement in the standard $\{|0\rangle, |1\rangle\}$ basis) and in the (X, Y) plane (measurement in the basis $\{\frac{|0\rangle + e^{i\theta}|1\rangle}{\sqrt{2}}, \frac{|0\rangle - e^{i\theta}|1\rangle}{\sqrt{2}}\}$ for some θ). Alternatively, it has been proven in [5] that the measurements in the (X, Y) plane over ‘brickwork’ states (see Figure 1) are universal. The tradeoff to the use of the single (X, Y) plane for the measurement is that the brickwork state is not regular, so not as easy as a grid to prepare. In the present paper, we show that measurements in a single plane, namely the (X, Z) plane, over the triangular grid is universal.

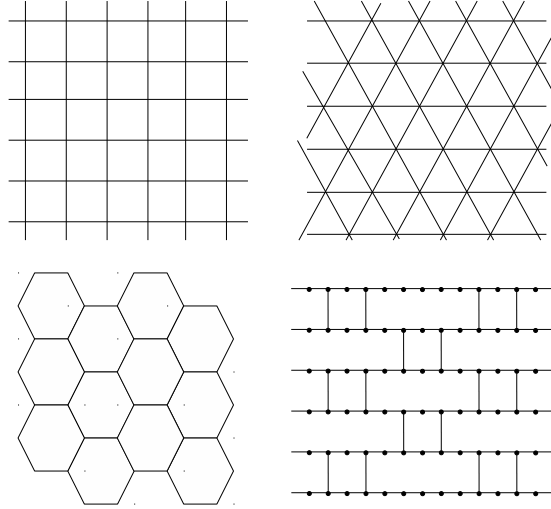


Figure 1: Top left: squarred grid (or cluster state) – Top right: triangular grid – Bottom left: hexagonal grid – Bottom right: 'brickwork' state.

The paper is structured as follows: first, the standard graph theoretical notions used in the paper are presented, then are given the fundamental properties of the signed graph states, an extension of the graph state formalism. In section 5, the actions of X - and Z -measurements over signed graph states are graphically characterized. Section 6 is dedicated to the proof, based on quantum arguments, that any graph is pivot minor of a planar graph. Finally, in section 7, we prove that measurements in the (X, Z) plane on triangular grids are universal for quantum computing.

2 Pivot minor

Given a graph G and a vertex u , the local complementation on the vertex u switches edges and non-edges in the neighborhood of u : it transforms G into $G * u = G \Delta K_{\mathcal{N}(u)}$ where Δ is the symmetric difference and $K_{\mathcal{N}(u)}$ is the complete graph over the neighborhood of u . Pivoting (also called edge local complementation) [2, 27] on an edge uv is defined as a sequence of local complementations on the two vertices of the edge $G \wedge uv = G * u * v * u$ (see Figure 2).

A graph G is a pivot minor of H if and only if G can be obtained from H by a sequence of pivotings and then a sequence of vertex deletions.

It has been proven [37, 14] that for any sequence of pivotings, there exists an equivalent sequence of pivotings where each vertex is used at most once. In particular, Kwon and Oum characterized, using the rank-width, the graphs which are pivot minors of trees and paths [21] and Oum proved that any bipartite

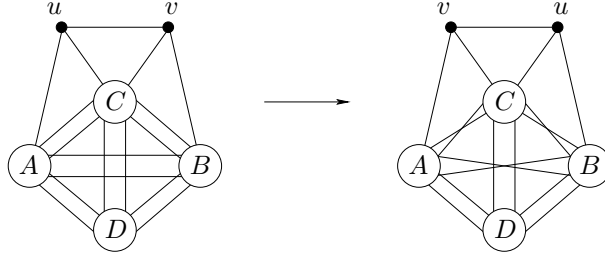


Figure 2: Pivoting on uv . $C = \mathcal{N}(u) \cap \mathcal{N}(v)$, $A = \mathcal{N}(u) \setminus C$, $B = \mathcal{N}(v) \setminus C$, and D is the rest of the vertices. Pivoting on uv exchanges vertices u and v , and for any $(x, y) \in (A \times B) \cup (B \times C) \cup (A \times C)$, the edge xy is deleted if xy was an edge, and added otherwise.

circle graph is a pivot minor of all line graphs with large rank-width [28].

3 Signed graph state formalism

In this section, we review the fundamentals of graph states [16] and signed graph states [13]. The graph state formalism consists in representing some particular quantum states using graphs. Given a graph G of order n , the corresponding quantum state is defined as the common fixpoint of n operators depending on the graph G . Each of these operators is a Pauli operator on n qubits. The group of Pauli operators acting on a set V of n qubits is generated by $\{X_u, Z_u, iI\}_{u \in V}$, where I is the identity, X_u (resp. Z_u) is an operator which acts as the identity on $V \setminus \{u\}$ and as $X : |x\rangle \mapsto |\bar{x}\rangle$ (resp. $Z : |x\rangle \mapsto (-1)^x |x\rangle$) on qubit u . More generally, for any subset $S \subseteq V$, let $X_S = \prod_{u \in S} X_u$ and $Z_S = \prod_{u \in S} Z_u$.

For a given simple undirected graph $G = (V, E)$ of order n , the *graph state* $|G\rangle \in \mathbb{C}^n$ is the unique quantum state¹ such that for any $u \in V$,

$$X_u Z_{\mathcal{N}(u)} |G\rangle = |G\rangle$$

To increase the expressive power of the graph state formalism, following [13] a *sign* is added to the graph: for a given graph $G = (V, E)$ and a subset $S \subseteq V$ of vertices, let

$$|G; S\rangle = Z_S |G\rangle$$

Proposition 1 *For any graph $G = (V, E)$, $\{|G; S\rangle\}_{S \subseteq V}$ form an orthonormal basis.*

¹In fact $|G\rangle$ is unique up to a global phase which is irrelevant in quantum computing. This global phase is chosen s.t. $\langle 0^n | G \rangle = \frac{1}{\sqrt{2^n}}$

Proof First, notice that for any $u \in V$ and any $S \subseteq V$, X_u and Z_S commute if $u \notin S$ and anticommute if $u \in S$. For any distinct $S, S' \subseteq V$, let $u \in S \Delta S'$.

$$\begin{aligned} \langle G; S | G; S' \rangle &= \langle G | Z_S Z_{S'} | G \rangle \\ &= \langle G | Z_{S \Delta S'} X_u Z_{\mathcal{N}(u)} | G \rangle \\ &= -\langle G | X_u Z_{\mathcal{N}(u)} Z_{S \Delta S'} | G \rangle \\ &= -\langle G; S | G; S' \rangle \end{aligned}$$

As a consequence, $\langle G; S | G; S' \rangle = 0$, so $\{|G; S\rangle\}_{S \subseteq V}$ is an orthonormal basis. \square

In the following lemma, it is shown that the action of any Pauli operator on a signed graph state can be captured by its sign, up to a global phase. Since quantum state are equivalent up to a global phase, for any $|\phi\rangle, |\psi\rangle$, we write $|\phi\rangle \equiv |\psi\rangle$ when there exists α such that $|\phi\rangle = e^{i\alpha} |\psi\rangle$.

Lemma 2 *For any graph $G = (V, E)$, any subset $S \subseteq V$, and any Pauli operator P , there exists $S' \subseteq V$ such that $P|G; S\rangle \equiv |G; S'\rangle$.*

Proof P can be decomposed, up to a global phase, into a product of X and Z operators: $\exists S_1, S_2 \subseteq V$, and $d \in [0, 3]$ such that $P = i^d X_{S_1} Z_{S_2}$. Thus $P|G; S\rangle = i^d X_{S_1} Z_{S_2 \Delta S} |G\rangle = i^d X_{S_1} Z_{S_2 \Delta S} (\prod_{u \in S_1} X_u Z_{\mathcal{N}(u)} |G\rangle) = i^d Z_{S_2 \Delta S} \prod_{u \in S_1} Z_{\mathcal{N}(u)} |G\rangle$, where Δ is the symmetric difference. \square

All signed graph states represent distinct quantum states:

Lemma 3 *For any graphs G_1, G_2 and any signs S_1, S_2 ,*

$$|G_1; S_1\rangle \equiv |G_2; S_2\rangle \implies G_1 = G_2 \text{ and } S_1 = S_2$$

Proof Let $S = S_1 \Delta S_2$. $|G_1; S_1\rangle \equiv |G_2; S_2\rangle$ implies $|G_1; S\rangle \equiv |G_2\rangle$. For any vertex u , since $Z_S X_u Z_{\mathcal{N}_{G_1}(u)} |G_1\rangle \equiv X_u Z_{\mathcal{N}_{G_2}(u)} |G_2\rangle$, $|G_1; S\rangle \equiv |G_2; \mathcal{N}_{G_1}(u) \Delta \mathcal{N}_{G_2}(u)\rangle$. Thus, $|G_2\rangle \equiv |G_2; \mathcal{N}_{G_1}(u) \Delta \mathcal{N}_{G_2}(u)\rangle$. By proposition 1, $\mathcal{N}_{G_1}(u) \Delta \mathcal{N}_{G_2}(u) = \emptyset$. Thus, $G_1 = G_2$ since for any u , $\mathcal{N}_{G_1}(u) = \mathcal{N}_{G_2}(u)$. As a consequence, $|G_1; S\rangle \equiv |G_1\rangle$, so by proposition 1, $S = \emptyset$ so $S_1 = S_2$. \square

4 Combinatorial properties of Graph States

The success of the graph state formalism is mainly due to the ability to characterize quantum properties by means of graph theoretic ones. For instance, local Clifford equivalence has been characterized by local complementation [39]. Here, we recall that the action of local Clifford transformations can be characterized by local complementations, and we prove that the action of local real Clifford transformations can be characterized by pivoting.

A Clifford transformation C is a map which transforms Pauli operators to Pauli operators: for any Pauli P , CPC^\dagger is a Pauli operator. A local Clifford is a Clifford that can be decomposed into the tensor product of one-qubit unitaries:

$C_1 \otimes \dots \otimes C_n$. The group of local Clifford on V is generated by $\{\sqrt{Z}_u, \sqrt{X}_u\}_{u \in V}$ where $\sqrt{P}_u := \frac{e^{i\pi/4}}{\sqrt{2}}(I - iP_u)$. For any subset S of qubits, let $\sqrt{P}_S := \prod_{u \in S} \sqrt{P}_u$.

A particularly important Clifford operation is the so-called Hadamard transformation: for any $u \in V$ H acts as the identity on $V \setminus \{u\}$ and as $H : |x\rangle \mapsto \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}$ on u . Moreover, for any $S \subseteq V$, let $H_S = \prod_{u \in S} H_u$.

H together with Z generates a subgroup of the Clifford operations, which corresponds exactly to the local *real* Clifford operations, i.e. those that can be represented by matrices with real entries in the computational basis $\{|0\rangle, |1\rangle\}$.

Some quantum transformations on graph states can be interpreted in terms of graph transformations. In particular, the application of some Clifford transformations can be interpreted in terms of local complementations [39]:

Proposition 4 *For any graph $G = (V, E)$, for any vertex $u \in V$,*

$$\sqrt{X}_u^\dagger \sqrt{Z}_{N(u)} |G\rangle = |G * u\rangle$$

Such Clifford operators act on signed graph states as follows:

$$\sqrt{X}_u^\dagger \sqrt{Z}_{N(u)} |G; S\rangle = \begin{cases} |G * u; S\rangle & \text{if } u \notin S \\ |G * u; S \Delta N(u)\rangle & \text{if } u \in S \end{cases}$$

Moreover Van den Nest proved that the action of local Clifford transformations is characterized by local complementation:

Lemma 5 ([39]) *For any graphs G and G' , there exists a local Clifford transformation C such that $C|G\rangle = |G'\rangle$ if and only if G and G' are locally equivalent, i.e. there exists a sequence of local complementation transforming G in G' .*

In [38], a combinatorial characterization of the action of Hadamard transformations on two neighbor qubits has been introduced².

Proposition 6 *For any graph $G = (V, E)$ and any edge $(u, v) \in E$,*

$$H_{u,v} |G\rangle = Z_{N(u) \cap N(v)} |G \wedge uv\rangle$$

Proof Since the original statement in [38] has been introduced without the Pauli factor $Z_{N(u) \cap N(v)}$, a proof of the proposition is given. The proof is based on the facts that $\sqrt{X}_u^\dagger \sqrt{Z}_u \sqrt{X}_u^\dagger = e^{i\pi/4} X_u H_u X_u$ and $\sqrt{Z}_u \sqrt{X}_u^\dagger \sqrt{X}_u = e^{-i\pi/4} Z_u H_u Z_u$.

$$\begin{aligned} |G \wedge uv\rangle &= |G * u * v * u\rangle \\ &= Z_{(N(u) \cup N(v)) \Delta \{u,v\}} \sqrt{X}_u^\dagger \sqrt{Z}_u \sqrt{X}_u^\dagger \sqrt{Z}_v \sqrt{X}_v^\dagger \sqrt{X}_v |G\rangle \\ &= Z_{(N(u) \cup N(v)) \Delta \{u,v\}} X_u Z_v H_u H_v X_u Z_v |G\rangle \\ &= X_u Z_{(N(u) \cup N(v)) \Delta \{u\}} H_{u,v} Z_{N(u) \Delta \{v\}} |G\rangle \end{aligned}$$

Moreover $|G \wedge uv\rangle = X_u Z_{N_{G \wedge uv}(u)} |G \wedge uv\rangle = X_u Z_{N_G(v) \Delta \{u,v\}} |G \wedge uv\rangle$, so $|G \wedge uv\rangle = H_{u,v} Z_{N_G(u) \cap N_G(v)} |G\rangle$. \square

²In [38] the characterisation has been given up to Pauli operation.

The action of Hadamard transformations is extended to signed graph states: Given a graph $G = (V, E)$ and an edge $(u, v) \in E$, for any $S \subseteq V \setminus \{u, v\}$:

$$\begin{aligned} H_{u,v}|G; S\rangle &= |G \wedge uv; S\Delta(\mathcal{N}(u) \cap \mathcal{N}(v))\rangle \\ H_{u,v}|G; S \cup \{u\}\rangle &= |G \wedge uv; S\Delta(\mathcal{N}(u) \cap \overline{\mathcal{N}(v)})\rangle \\ H_{u,v}|G; S \cup \{v\}\rangle &= |G \wedge uv; S\Delta(\overline{\mathcal{N}(u)} \cap \mathcal{N}(v))\rangle \\ H_{u,v}|G; S \cup \{u, v\}\rangle &= -|G \wedge uv; S\Delta(\mathcal{N}(u) \cup \mathcal{N}(v))\rangle \end{aligned}$$

Now we prove that the action of the local real Clifford transformations on graph states is captured by the graphical transformation of pivoting.

Lemma 7 *For any real local Clifford transformation R , and for any graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, if $R|G_1, S_1\rangle \equiv |G_2, S_2\rangle$ then $\exists A, S'_1, S'_2 \subseteq V_1$ s.t. $H_A|G_1, S'_1\rangle \equiv |G_2, S'_2\rangle$.*

Proof Any real local Clifford can be decomposed (up to a global phase) into a product of H , X and Z . More precisely, for any real local Clifford R , $\exists A, B, C$ s.t. $R \equiv H_A X_B Z_C$. So $R|G_1, S_1\rangle \equiv H_A X_B Z_C|G_1, S_1\rangle \equiv H_A X_B|G_1, S_1\Delta C\rangle \equiv H_A|G_1, S_1\Delta C\Delta \text{Odd}(B)\rangle \equiv |G_2, S_2\rangle$ where $\text{Odd}(B) := \{v \text{ s.t. } |N(v) \cap B| = 1 \pmod{2}\}^3$. \square

Lemma 8 *For any graphs G and G' , any signs S, S' , and any real local Clifford transformation R , if $R|G; S\rangle = |G'; S'\rangle$ then G and G' are pivot equivalent, i.e. there exists a sequence of pivotings transforming G in G' .*

Proof Thanks to Lemma 7, there exist A, S_1, S_2 s.t. $H_A|G, S_1\rangle \equiv |G', S_2\rangle$. For any $u \in A$, notice that $A \cap \mathcal{N}(u) \neq \emptyset$. Otherwise, if $A \cap N_G(u) = \emptyset$ then $|G'; S_2\rangle \equiv H_A Z_{S_1} X_u Z_{\mathcal{N}(u)}|G\rangle \equiv Z_u Z_{\mathcal{N}(u)} H_A Z_{S_1}|G\rangle \equiv Z_{\{u\} \cup \mathcal{N}(u)}|G'; S_2\rangle$ which contradicts proposition 1. Thus, for any $u \in A$, there exists $v \in A \cap \mathcal{N}(u)$ and a sign S_3 such that $H_A|G; S_1\rangle \equiv H_{A \setminus \{u, v\}} H_{u, v}|G; S_1\rangle \equiv H_{A \setminus \{u, v\}}|G \wedge uv; S_3\rangle \equiv |G'; S_2\rangle$. As a consequence, by induction on the size of A , G is a pivot equivalent to G' . \square

5 Local measurements and Measurement-based quantum computing

In this section, the action of local measurements over the qubits of a graph states is considered. We give graphical interpretation of such measurements, but also a computational interpretation since the application of local measurement over a graph state is the key ingredient of the one-way model.

We consider local measurements according to the basis $\{|0^{(\alpha)}\rangle, |1^{(\alpha)}\rangle\}$ parametrised by an angle $\alpha \in [0, 2\pi)$, where

$$\begin{aligned} |0^{(\alpha)}\rangle &:= \cos\left(\frac{\alpha}{2}\right)|0\rangle + \sin\left(\frac{\alpha}{2}\right)|1\rangle \\ |1^{(\alpha)}\rangle &:= \sin\left(\frac{\alpha}{2}\right)|0\rangle - \cos\left(\frac{\alpha}{2}\right)|1\rangle \end{aligned}$$

³ $\text{Odd}(B)$ can be alternatively defined inductively as follows: $\text{Odd}_G(\emptyset) := \emptyset$, and $\text{Odd}_G(\{u_0, \dots, u_n\}) = \text{Odd}_G(\{u_0, \dots, u_{n-1}\}) \Delta N_G(u_n)$

This family of measurements is said to be in the (X, Z) -plane since all these basis states are actually in the (X, Z) -plane in the Bloch sphere representation. Moreover, for any α , the observable associated with the measurement in the basis $\{|0^{(\alpha)}\rangle, |1^{(\alpha)}\rangle\}$ is $\cos(\alpha)Z + \sin(\alpha)X$. Notice that when $\alpha = 0$ it corresponds to the standard basis measurement $\{|0\rangle, |1\rangle\}$ also called Z-measurement and when $\alpha = \pi/2$ to the diagonal basis measurement $\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\}$ or X-measurement.

One-way quantum computation is generally made upon (X, Y) -measurements together with Z-measurements. We prove that using (X, Z) measurements instead also give rise to a universal model of quantum computing.

A measurement has non deterministic evolution which consists in projecting the state of the measured qubit onto one of the two basis states: $|s^{(\alpha)}\rangle$, $s \in \{0, 1\}$ is the classical outcome of the measurement. Thus the action of such a measurement, up a re-normalisation is $\langle s^{(\alpha)}|$.

The action of a Z-measurement corresponds to a vertex deletion: for any graph $G = (V, E)$, any vertex $u \in V$, any $s \in \{0, 1\}$,

$$\sqrt{2} \langle s_u^{(0)} | G \rangle = Z_{\mathcal{N}(u)}^s | G \setminus u \rangle$$

The action of Z-measurement is extended to signed graph states: for any graph $G = (V, E)$, any sign $S \subseteq V$, any vertex $u \in V$, and any $s \in \{0, 1\}$,

$$\sqrt{2} \langle s_u^{(0)} | G; S \rangle = | G \setminus u; (S \setminus u) \Delta (\mathcal{N}(u))^s \rangle$$

A single X-measurement cannot be interpreted as a graph transformation, since for any connected graph G and any vertex u of G , an X-measurement of qubit u of $|G\rangle$ leads to a quantum state which is no more a graph state. However, the application of a pair of X-measurements on adjacent qubits can be interpreted as a pivoting followed by vertex deletions:

Proposition 9 *For any graph G , any edge uv , and any $r, s \in \{0, 1\}$,*

$$2 \langle s_u^{(\pi/2)} r_v^{(\pi/2)} | G \rangle = Z_S | G \wedge uv \setminus u \setminus v \rangle$$

where $S = (\mathcal{N}(u) \cap \mathcal{N}(v)) \Delta (\mathcal{N}(u) \setminus v)^r \Delta (\mathcal{N}(v) \setminus u)^s$

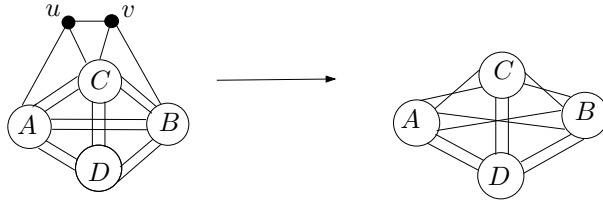


Figure 3: Evolution of a graph state by X measuring u and v

Proof Since $\langle s_u^{(\frac{\pi}{2})} | = \langle s_u^{(0)} | H_u$, $2\langle s_u^{(\frac{\pi}{2})} r_v^{(\frac{\pi}{2})} | G \rangle = 2\langle s_u^{(0)} r_v^{(0)} | H_{u,v} | G \rangle$
 $= 2\langle s_u^{(0)} r_v^{(0)} | Z_{N_G(u) \cap N_G(v)} | G \wedge uv \rangle$
 $= Z_{(N_G(u) \cap N_G(v)) \Delta (N_{G \wedge uv}(v) \setminus u)^r \Delta (N_{G \wedge uv}(v) \setminus v)^r} | G \wedge uv \setminus u \setminus v \rangle$
 $= Z_{(N_G(u) \cap N_G(v)) \Delta (N_G(u) \setminus v)^r \Delta (N_G(v) \setminus u)^r} | G \wedge uv \setminus u \setminus v \rangle \quad \square$

The action of the X -measurement of two neighbors extends to the signed graphs: for any graph $G = (V, E)$, any sign $S \subseteq V$, any edge $uv \in E$ and any $r, s \in \{0, 1\}$:

$$2\langle s_u^{(\pi/2)} r_v^{(\pi/2)} | G; S \rangle = | G \wedge uv \setminus u \setminus v; S' \rangle$$

where $S' = (S \setminus u \setminus v) \Delta (\mathcal{N}(u) \cap \mathcal{N}(v)) \Delta (\mathcal{N}(u) \setminus v)^{r \oplus |v \cap S|} \Delta (\mathcal{N}(v) \setminus u)^{s \oplus |u \cap S|}$

Lemma 10 *For any graphs G, G' , and any signs S, S' , if a sequence of X - and Z -measurements transforms $|G; S\rangle$ into $|G'; S'\rangle$ then G' is a pivot minor of G .*

Proof Since all measurements are commuting because they are local, we assume w.l.o.g. that all the Z -measurements are performed first, leading to a signed graph state $|G \setminus A, S_1\rangle$, where A is the set of the qubits which are Z -measured. Then, inductively, if a pair of 2 neighbors uv are X -measured then by proposition 9, these two measurements lead to the sign graph state $|((G \setminus A) \wedge uv) \setminus \{u, v\}, S_2\rangle$. We repeat this step inductively until there is no two neighbors which are X -measured. So it leads to a graph state $|\tilde{G}, S_3\rangle$ such that \tilde{G} is a pivot minor of G and such that the remaining X -measurements have to be performed on an independent set B of \tilde{G} . If B is empty then $G' = \tilde{G}$ so G' is a pivot minor of G . Otherwise, let $u \in B$. The state $|\phi\rangle$ after the X -measurement of u is $\sqrt{2} \cdot \langle s_u^{(\pi/2)} | \tilde{G}; S_3 \rangle$ with $s \in \{0, 1\}$. $|\phi\rangle$ is an eigenvector of $Z_{\mathcal{N}_{\tilde{G}}(u)}$, indeed $Z_{\mathcal{N}_{\tilde{G}}(u)}(u) \langle s_u^{(\pi/2)} | \tilde{G}; S_3 \rangle = (-1)^s Z_{\mathcal{N}_{\tilde{G}}(u)}(u) \langle s_u^{(\pi/2)} | X_u Z_{\mathcal{N}_{\tilde{G}}(u)} | \tilde{G}; S_3 \rangle = (-1)^s \langle s_u^{(\pi/2)} | \tilde{G}; S_3 \rangle$. Moreover, since none of the neighbors of u are measured, the final state $|G', S'\rangle$ is also a eigenvector of $Z_{\mathcal{N}_{\tilde{G}}(u)}$, so according to lemma 3 $\mathcal{N}_{\tilde{G}}(u) = \emptyset$ and u is isolated. For an isolated vertex u , one can easily show that a X -measurement of u produces the state $|\tilde{G} \setminus u, S_3\rangle$. By induction on the independent set B , it comes that G' is a pivot minor of G . \square

6 From Quantum computation to graph theory

6.1 Embedding in a planar graph.

The main result of this section is that any graph is pivot minor of a planar graph. The proof consists in encoding the preparation of $|G\rangle$, for any G , with a measurement-calculus pattern [9, 10] (the language used for formal description of one-way quantum computation) composed of X -measurements only.

First we recall some properties of the measurement-based model:

An open graph is a triplet (G, I, O) where $G = (V, E)$ is a graph and $I, O \subseteq V$ are representing respectively the input and the output qubits of the computation. Intuitively, the input qubits are initialised in a state which is the state of the input of the computation and then they are entangled with the rest of the qubits. Then, the non output qubits are measured and finally, at the end of the computation, the output of the computation is located on the remaining qubits, i.e. the output qubits. For a given open graph (G, I, O) the corresponding entangling operation $M_{(G, I, O)}$ is:

$$M_{(G, I, O)} : |x\rangle \mapsto \frac{1}{\sqrt{2^{|I|}}} \sum_{S \subseteq I} (-1)^{x \bullet \mathbb{I}_S} |G; S\rangle$$

where \mathbb{I}_S is a binary n -bit vector indexed by V such that $\mathbb{I}_S(u) = 1 \iff u \in S$, and $\forall x, y \in \{0, 1\}^V$, $x \bullet y = \sum_{u \in V} x_u y_u$.

When the input state is the uniform superposition $|\phi\rangle = \frac{1}{\sqrt{2^{|I|}}} \sum_{x \in \{0, 1\}^I} |x\rangle$, the graph state $|G\rangle$ is prepared, indeed $\langle G | M_{(G, I, O)} | \phi \rangle = \frac{1}{2^{|I|}} \sum_{S \subseteq I, x \in \{0, 1\}^I} (-1)^{x \bullet \mathbb{I}_S} \langle G | G; S \rangle = \frac{1}{2^{|I|}} \sum_{x \in \{0, 1\}^I} (-1)^{x \bullet \mathbb{I}_\emptyset} = 1$.

Open graphs can be composed: the composition $(G_2 = (V_2, E_2), A, O) \circ (G_1 = (V_1, E_1), I, A) = (G' = (V_1 \cup V_2, E_1 \Delta E_2), I, O)$, is well defined if $V_1 \cap V_2 = A$.

In the following, we consider that the input qubits are always X -measured. The other qubits are measured in the (X, Z) -plane.

In order to simulate a unitary tranformation using measurements a correction strategy needs to be given to guarantee that the result of the simulation does not depend on the various classical outcomes obtained during the simulation. This strategy consists in applying the unitary maps Pauli X or Z on unmeasured qubits when the outcome of a measurement is 1 (see [25] for a detailed correction strategy). These corrections are applied in such a way that the state of the system after the correction is the same state as the state of the system if the classical outcome 0 would have occurred. One of the simplest example of MBQC consist in the graph P_2 composed of two vertices 1 and 2 connected by an edge. Notice that the pivoting according to the edge $(1, 2)$ does not change the state: $P_2 \wedge 12 = P_2$, so $H_{1,2}|P_2\rangle = |P_2\rangle$.

Lemma 11 *The open graph $(P_n, \{1\}, \{n\})$ where the qubits $1 \dots n-1$ are X -measured simulates the Hadamard transformation if n is even and the identity if n is odd.*

Proof

$$\begin{aligned}
\sqrt{2} \langle r^{(\pi/2)} | \mathbf{M}_{(P_2, \{1\}, \{2\})} | x \rangle &= \sqrt{2} \langle r^{(\pi/2)} | \frac{1}{\sqrt{2|\{1\}|}} \sum_{S \subseteq \{1\}} (-1)^{x \bullet \mathbf{1}_S} | P_2; S \rangle \\
&= \langle r_1^{(\pi/2)} | P_2 \rangle + (-1)^x \langle r_1^{(\pi/2)} | Z_1 | P_2 \rangle \\
&= \langle r_1^{(\pi/2)} | P_2 \rangle + (-1)^x \langle r_1^{(\pi/2)} | X_2 | P_2 \rangle \\
&= (I + (-1)^x X_2) \langle r_1^{(\pi/2)} | P_2 \rangle \\
&= (I + (-1)^x X_2) \langle r_1^{(\pi/2)} | H_{1,2} | P_2 \rangle \\
&= (I + (-1)^x X_2) H_2 \langle r_1^{(0)} | P_2 \rangle \\
&= H_2 (I + (-1)^x Z_2) Z_2^r \frac{|0_2\rangle + |1_2\rangle}{2} \\
&= X_2^r H_2 | x \rangle
\end{aligned}$$

So it simulates H up to a Pauli operator X which depends on the classical outcome of the measurement.

By composing this elementary open graph state, it comes that any path of size n where all but the output qubit is X -measured, implements H if the number of edges is odd and the identity otherwise, since $H^2 = I$. \square

Lemma 12 *The open graph $(P_2, \{1, 2\}, \{1, 2\})$ with no measurement, simulates the two-qubit unitary transformation $\Lambda Z: |x, y\rangle \mapsto (-1)^{xy} |x, y\rangle$*

Proof The simulation of ΛZ is obtained by considering a the open graph $(P_2, \{1, 2\}, \{1, 2\})$. Thus both qubits are inputs and outputs. The simulation is done without measurement. For any $x, y \in \{0, 1\}^{\{1, 2\}}$,

$$\begin{aligned}
\langle y | \mathbf{M}_{(P_2, \{1, 2\}, \{1, 2\})} | x \rangle &= \frac{1}{2} \sum_{S \subseteq \{1, 2\}} (-1)^{x \bullet \mathbf{1}_S} \langle y | P_2; S \rangle \\
&= \frac{1}{2} \sum_{S \subseteq \{1, 2\}} (-1)^{x \bullet \mathbf{1}_S} \langle y | Z_S | P_2 \rangle \\
&= \frac{1}{2} \sum_{S \subseteq \{1, 2\}} (-1)^{x \bullet \mathbf{1}_S} (-1)^{y \bullet \mathbf{1}_S} \langle y | P_2 \rangle \\
&= \frac{1}{2} \left(\sum_{S \subseteq \{1, 2\}} (-1)^{(x \oplus y) \bullet \mathbf{1}_S} \right) \langle y | P_2 \rangle
\end{aligned}$$

If $x \neq y$, then $\sum_{S \subseteq \{1, 2\}} (-1)^{(x \oplus y) \bullet \mathbf{1}_S} = 0$, so $\langle y | \mathbf{M}_{(P_2, \{1, 2\}, \{1, 2\})} | x \rangle = 0$. When $x = y$, $\sum_{S \subseteq \{1, 2\}} (-1)^{(x \oplus y) \bullet \mathbf{1}_S} = 4$. As a consequence, For any $x \in$

$$\{0, 1\}^{\{1,2\}},$$

$$\begin{aligned} \langle x | M_{(P_2, \{1,2\}, \{1,2\})} | x \rangle &= 2 \langle x | P_2 \rangle \\ &= 2 \langle x_1^{(0)} | \langle x_2^0 | P_2 \rangle \\ &= \langle x_1^{(0)} | (|0_1\rangle + (-1)^{x_2} |1_1\rangle) \\ &= (-1)^{x_1 \cdot x_2} \end{aligned}$$

So $M_{(P_2, \{1,2\}, \{1,1\})}$ implements the map $|x_1 x_2\rangle \mapsto (-1)^{x_1 \cdot x_2} |x_1 x_2\rangle$, i.e. ΛZ . \square

An other property required for the proof of the main theorem is the existence of a planar circuit preparing the state $|G\rangle$.

Lemma 13 *For any graph $G = (V, E)$ of order $n = |V|$, there exists a n -qubit planar circuit C_G of size $O(n^3)$ composed of ΛZ and H to prepare $|G\rangle$.*

Proof There exists a circuit of size $O(n^2)$ composed of ΛZ only for preparing the graph state $|G\rangle$ [16]. This circuit can be made planar by interspersing at most n SWAP gates between every ΛZ . Moreover every SWAP gate can be decomposed using a constant number of ΛZ et H : $\forall u, v$,

$$\text{SWAP}_{u,v} = H_u \Lambda Z_{u,v} H_{u,v} \Lambda Z_{u,v} H_{u,v} \Lambda Z_{u,v} H_u$$

As a consequence $|G\rangle$ can be prepared by a planar circuit of size $O(n^3)$ acting on n qubits. \square

Theorem 14 *Any graph on n vertices is pivot minor of a planar graph of $O(n^3)$ vertices.*

Proof For any graph G , the planar circuit which implements G on input $\frac{1}{\sqrt{2^{|V|}}} \sum_{x \in \{0,1\}^{|V|}} |x\rangle$ can be simulated using an open graph G' on with all but output qubits are X -measured: this open graph is obtained by composing the open graphs simulating H and ΛZ accordingly to the circuit. Since only X measurements occur, G is a pivot minor of H (lemma 8). Therefore any graph is a pivot minor of a planar graph. \square

6.2 Some particular planar graphs

Theorem 15 *Any graph of n vertices is pivot minor of a triangular grid on $O(n^4)$ vertices.*

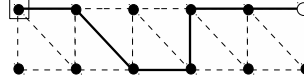
Proof The simulation of the circuit C_G that prepares $|G\rangle$ can be embedded into a triangular grid of size $4n * 4d$ where $n = |V|$ and d is the depth C_G :

The simulation with X and Z measurement of any circuit using the ΛZ and H gates is explained by the following figures where the non-output vertices on the bold line are X -measured and the other non-output vertices are Z -measured.

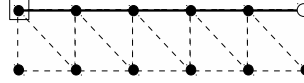
Note that it is important to be able to simulate the identity in this model in order to compose the different simulations.

The output qubits (represented with a white circle) state corresponds to the application of the simulated unitary transformation on the input qubits. (represented by a square).

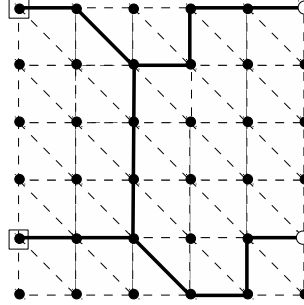
- Simulation of Id :



- Simulation of H :



- Simulation of ΛZ :



As a consequence any graph is a pivot minor of a triangular grid of size at most $O(n^4)$ ($=O(n * d) = O(n * n^2 * n)$) \square

Other regular graph states, like rectangular⁴ or hexagonal grids, are commonly used in quantum computation. But, contrary to triangular grids, we show that some graphs are pivot minor of no rectangular nor hexagonal grid:

Lemma 16 *There exists a graph which is a pivot minor of no rectangular nor hexagonal grid.*

Proof Pivot minors of bipartite graphs are bipartite [4], thus, since any rectangular or hexagonal grid is bipartite, all pivot minors of these kind of grids are bipartite. As a consequence, any graph which is not bipartite (e.g. a triangle) is a pivot minor of no rectangular nor hexagonal grid. \square

⁴Graph states described by rectangular grids are also called *cluster states*.

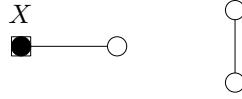
Theorem 16 implies that some graph states, like the triangle, cannot be reached from the traditional cluster state (graph state described by a rectangular grid) by a sequence of X - and Z -measurements. Triangular grids turn out to be more adapted to one-way quantum computation than rectangular or hexagonal ones, as it is underlined in the next section, by proving that measurements in the (X, Z) -plane on a triangular graph state is a universal model of quantum computation.

7 Universality of (X, Z) -measurements

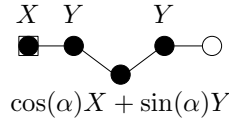
Briegel and Raussendorf [33] have proved that any unitary transformation can be simulated by applying on a rectangular grid, a sequence of one-qubit measurements, such that each measurement is either according to Z , or in the (X, Y) -plane, i.e. according to an observable $\cos(\alpha)X + \sin(\alpha)Y$ for some α . We prove that any (real) unitary transformation can be simulated by a sequences of measurements in the (X, Z) -plane, on triangular grids.

Theorem 17 *(X, Z) -measurements are universal for one-way quantum computation.*

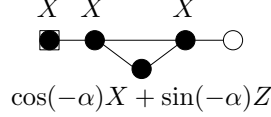
Proof The set of unitary transformations $\{H, \Lambda Z, P(\alpha), \alpha \in [0, 2\pi)\}$ where $P(\alpha) = \cos(\alpha/2)X + \sin(\alpha/2)Z$ is universal for (real) quantum computation [35]. ΛZ and H can be implemented in the one-way model using X -measurements as follows (see lemmas 11 and 12):



Regarding $P(\alpha)$, it is implemented by the following one-way quantum computation using (X, Y) -measurements [34]:



This computation is equivalent to a one-way quantum computation involving measurement in the (X, Z) plane only:



The equivalency between the two computations can be proved using the local complementation. Indeed, the two underlying graphs are locally equivalent by applying a local complementation on the central vertex. Thus, if one takes the latter graph (with a triangle), applies \sqrt{X}^\dagger on the central vertex and \sqrt{Z} on its neighbors and then measures the central vertex according to $\cos(\alpha)X + \sin(\alpha)Y$ and its neighbors according to Y then it leads to the simulation of $P(\alpha)$. A unitary U followed by a measurement according to an observable A is equivalent to a measurement according to the rotated observable UAU^\dagger , thus the application of a measurement according to $\sqrt{X}^\dagger(\cos(\alpha)X + \sin(\alpha)Y)\sqrt{X} = \cos(-\alpha)X + \sin(-\alpha)Z$ on the central vertex and according to $\sqrt{Z}Y\sqrt{Z}^\dagger = X$ on its neighbors lead to the simulation of $P(\alpha)$. \square

Theorem 18 *Triangular graph states are universal resources for one-way quantum computation based on (X, Z) -measurements.*

Proof Any real unitary can be implemented by composing the one-way computations using (X, Z) measurements only (lemma 13). The graph state corresponding to this composition can be obtained from the triangular grid by means of X and Z measurements only (theorem 15). \square

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