2.2 a) $u_{tt} - c^2 u_{xx} = 0$ where $\Box u = 0$ and $\Box = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$ is the wave operator

since $\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)$,

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u$$

$$= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)(u_t + c u_x) \quad \text{let } v = u_t + c u_x$$

$$= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)(v)$$

$$= v_t - c v_x$$

thus $u_{tt} - c^2 u_{xx}$ is equivalent to $v = u_t + c u_x$
$v_t - c v_x = 0$

b)

$v_t - c v_x = 0$ gives $v(t,x) = h(x+ct)$

$u_t + c u_x = v$ becomes $u_t + c u_x = h(x+ct)$

now solve by looking for solutions of the form:

$u = u_{hom} + u_p$ where u_p is a particular solution

and u_{hom} solves the homogenous equation $u_t + c u_x = 0$

The general solution to $u_t + c u_x = 0$ gives

$u_{hom} = g(x-ct)$

Then guess a particular solution of the form

$u_p = f(x+ct)$ and plugging it in gives $f'(z) = \frac{h(z)}{2c}$

so the whole solution $u = u_{hom} + u_p$ has the general form $u(t,x) = g(x-ct) + f(x+ct)$

2.3 a) $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(0,x) = x^2$, $u_t(0,x) = e^x$

$\phi(x) = x^2 \quad \psi(x) = e^x$

$\Box u = 0$ where $\Box = \frac{\partial^2}{\partial x^2} - 3\frac{\partial}{\partial x}\frac{\partial}{\partial t} - 4\frac{\partial^2}{\partial t^2}$

$\Rightarrow \left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u = \left(\frac{\partial}{\partial x} - \frac{4\partial}{\partial t}\right)(u_x + u_t)$ let $v = u_x + u_t$

$= \left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)v$

$= v_x - 4v_t = 0$

thus $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ becomes $v_x - 4v_t = 0$
$v = u_x + u_t$

Then $u(x,t) = f(x+4t) + g(x-t)$

① $u(0,x) = F(x) + G(x) = \phi(x)$, ② $u_t(0,x) = -4F'(x) - G'(x) = \psi(x)$

integrate ↑

$\int_0^x -4F'(z) - G'(z)dz$

③ $= -4F(x) - G(x) = \int_0^x \psi(z)dz + A$

Add ① and ③

$-3F(x) = \phi(x) - \int_0^x \psi(x)dx - A$

④ $\Rightarrow F(x) + \frac{1}{4}G(x) = -\frac{1}{4}\int_0^x \psi(x)dx + A$

Subtract ① and ④

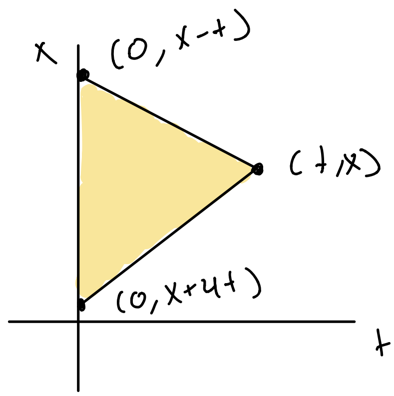
$-\frac{3}{4}G(x) = \phi(x) + \frac{1}{4}\int_0^x \psi(x)dx - A$

$F(x+4t) + G(x-t) = -\frac{1}{3}\phi(x+4t) + \frac{1}{3}\int_0^{x+4t}\psi(x)dx - \frac{4}{3}\phi(x-t) - 3\int_0^{x-t}\psi(x)dx$

✓ $= -\frac{1}{3}(x+4t)^2 + \frac{1}{3}\int_0^{x+4t} e^x dx - \frac{4}{3}(x-t)^2 - 3\int_0^{x-t} e^x dx$

2.1

b)



the shaded triangle shows the domain of dependence

$x+4t < x_0 < x-t$

2. 4 a) $$E'(t) = \int_{-\infty}^{\infty} [u_t u_{tt} + c^2 u_x u_{xt}]\,dx$$

$$= \int_{-\infty}^{\infty} [u_t u_{tt} - c^2 u_{xx} u_t]\,dx + c^2 u_x u_t \Big|_{x\to -\infty}^{x\to\infty}$$

$$= \int_{-\infty}^{\infty} u_t [u_{tt} - c^2 u_{xx}]\,dx$$

$$= -\int_{-\infty}^{\infty} a u_t^2\,dx \quad \leq 0$$

b) suppose v and w are solutions. Let $u = v - w$

$$E_u(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2)\,dx$$ which implies

$$E_u(0) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t^2(x,0) + c^2 u_t^2(x,0))\,dx = 0$$

by part a, since $E'_u(t) \leq 0$, $E_u(t) \leq 0$ for all $t \geq 0$.
However, by def. $E_u(t) \geq 0$ therefore $E_u(t) \equiv 0$ for all $t \geq 0$
we can conclude that $u_t = 0 = u_x$ and since $u(t,x) \equiv C$

for some constant C and by assumption, $u(0,x)=0$,
$C=0$. Then $u(t,x)\equiv 0$ and thus $v(x,t)=u(x,t)$

2.5 a) $u_{tt}-u_{xx}=0$, $u(0,x)=\sigma(x)$ and $u_t(0,x)=0$

$$u(t,x)=\frac{1}{2}\left[\phi(x-ct)+\phi(x+ct)\right]$$

$$=\frac{1}{2}\left[\sigma(x-t)+\sigma(x+t)\right]$$

See mathematica for plots

b) $u_{tt}-u_{xx}=0$, $u(0,x)=0$, $u_t(0,x)=\sigma(x)$

$$u(t,x)=\frac{1}{2}\int_{x-t}^{x+t}\sigma(z)\,dz \quad \text{when} \quad x<0$$

$$u(t,x)=\frac{1}{2}\int_{x-t}^{x+t}0\,dz=0 \quad \text{when} \quad x<0$$

ramp function

$$u(t,x)=\frac{1}{2}\int_{x-t}^{x+t}\sigma(z)\,dz \quad \text{when} \quad x>0$$

$$=\frac{1}{2}\int_{x-t}^{x+t}1\,dz=\frac{1}{2}(x+t-x+t)$$

$$=\frac{1}{2}(2t)=t$$

See mathematica

c) $u_{tt}-u_{xx}=\sigma(x)$, $u(0,x)$, $u_t(0,x)=0$

Index of comments

1.1 PS2: 33/39

1.2 2.1: 6/6

1.3 +5/5

1.4 +5/5

2.1 +3/5: Your function F should be a function of x + (1/4) t instead of x + 4t. I'm not completely following your algebra to obtain the solution from the initial conditions. I would take a second look at your solution to the system of equations. You should evaluate the integrals involved in the solution.

3.1 +2/2

3.2 +5/5

4.1 +4/5: Can you justify why we have zero initial conditions for u?

4.2 +2/2

4.3 +1/2: Missing Mathematica plot

4.4 +0/2