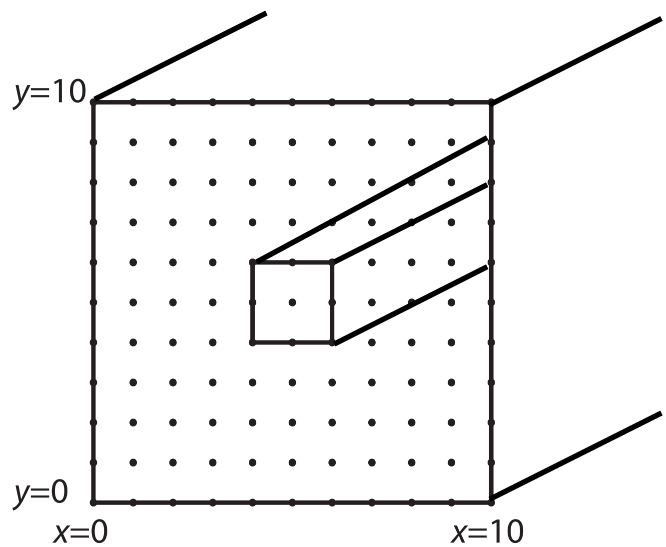
For Problem Set 6:
Relaxation Method for Laplace and Poisson Problems

Option A) Do this part of the problem if you haven't taken Scientific Computing. If you have, you've already done this, so do Option B) instead. In many real-world situations, the geometry of charged objects is complicated enough that Coulomb's Law and Gauss's Law are impractical. Often, the only workable solution is a computer-assisted *numerical solution*. One of the simplest techniques for solving electric potential problems using a computer is the "relaxation method".

As discussed in class, if the charge density $\rho = 0$ in a given region of space, then $\nabla^2\phi = 0$ there. $\nabla^2\phi$ is a measure of the *curvature* of the potential function: if it's positive at some point, ϕ there is lower than its surroundings; if it's negative, ϕ is higher than its surroundings. If $\nabla^2\phi = 0$, one can prove that ϕ at that point is equal to the **average value** of its surroundings.

Consider the problem at right: two long, concentric hollow tubes with square cross-sections. The outer one is grounded (zero potential), the inner one has a potential of 1 volt. Compute the potential at the grid points indicated by dots in the figure, using the following procedure:



a) Set the starting value of each of the grid points: set grid points on the outer boundary to 0 volts, set points on the inner boundary to 1 volt. Pick a starting "guess" for the points in between: the value doesn't matter much.

b) Set the potential at each of the points in the interior to the **average** of the four values above, below, to the left, and to the right of it. Do not change the value of the points on the boundary.

c) **Repeat step b over and over**. With each successive repetition, all the interior values change, so each point is never *quite* equal to the average of its neighbors, but as you do more and more repetitions, the error becomes less and less: the numerical solution converges toward the correct answer.

You may use any computer tool you like to perform this procedure. Most programming languages can do it quite easily and quickly; I've done it in MATLAB, Mathematica and Python; Maple should also be able to do it.

If you don't know any programming languages, you can do it using a spreadsheet. Create a new 11x11 table for each repetition, with cell formulas that refer to the previous table.

Repeat the procedure until the solution converges (if you're using a spreadsheet, do at least 10 repetitions. If you're using a programming language, do at least 50). Plot the potential along the line x = 4.

Option B) Do this version if you've taken Scientific Computing. You might want to read Option A as a review.

Let's figure out how to do Gauss-Seidel relaxation on a Poisson problem – that is, let's try to find the voltage in a situation where there is charge present. Let V(i,j) refer to the voltage at row i, column j of a rectangular grid, with Δx and Δy the spacing between the rows and columns.

As discussed in class, the differential equation for voltage as a function of charge density is:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi k \rho$$

a) Discretize this equation using the centered second difference and solve for V(i,j). Show your work: you should find

$$V(i,j) = \frac{V(i+1,j) + V(i-1,j) + V(i,j+1) + V(i,j-1)}{4} + \pi k \rho(i,j)\Delta x \Delta y$$

That is, if charge is present, you can calculate the voltage by averaging the four nearest neighbors and then adding a value proportional to the charge density. As always, you must repeat this process over and over until the solution converges.

Use this equation to find the voltage everywhere inside a square domain 1 meter across in both dimensions, containing a uniform charge density $\rho = 10^{-9}$ coulombs/m^3. Assume the square domain is surrounded by a grounded conducting box, so V = 0 at the edges. I'll leave the numerical details (number of gridpoints, number of iterations) up to you. Plot the voltage in the box, and tell me the voltage at the center of the box.