Methods for Linear Equations with constant coefficients

$$L(y) = a_n \frac{d^n y}{dx^n} + \ldots + a_1 \frac{dy}{dx} + a_0 y = g(x), \qquad a_i \text{ constants}, \qquad f(x) = \frac{g(x)}{a_n}$$

Solutions $y(x) = y_c(x) + y_p(x)$

Technique	**Situation**	**Substitution**	**Details**
Reduction of Order	Homogeneous Order 2 $y_1(x)$ known	$y_2(x) = u(x) y_1(x)$	$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ $\implies \frac{du}{dx} = \frac{e^{-\int P\ dx}}{y_1^2(x)}$
Auxiliary Polynomial	Homogeneous Order n	$y(x) = e^{mx}$	Case I: m real, distinct $\implies y = e^{mx}$ Case II: m real, multiplicity k $\implies e^{mx}\ ,\ \ldots\ ,\ x^{k-1}e^{mx}$ Case III: $m = a \pm bi \implies e^{ax}\cos(bx)$ & $e^{ax}\sin(bx)$
Undetermined Coefficients	Nonhomogeneous Order n	y_c from Table 4.4.1	Solve Homogeneous equation for y_p Substitute y_c into the non-homogeneous and solve for c_i
Annihilators	Nonhomogeneous Order n	-	Find Annihilator L_1 so that $L_1(g(x)) = 0$ Solve $L_1(L(y)) = 0$ Identify which terms comprise of y_p Plug in $L(y_p) = g(x)$ to find constants.
Variation of Parameters	Nonhomogeneous Order n	$y_c = u_1 y_1 + \ldots + u_n y_n$	Solve Homogeneous equation for y_c $\frac{du_i}{dx} = \frac{W_i}{W}$

Wronskian: $W = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{bmatrix}$. W_i replaces ith column with $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$

Cauchy-Euler Equation

$$L(y) = a_n x^n \frac{d^n y}{dx^n} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = g(x), \qquad a_i \text{ constants}, \qquad f(x) = \frac{g(x)}{a_n}$$

Solutions $y(x) = y_c(x) + y_p(x)$

Technique	**Situation**	**Substitution**	**Details**
Auxiliary Polynomial	Homogeneous Order n	$y(x) = x^m$	Case I: m real, distinct $\implies y = x^m$ Case II: m real, multiplicity k $\implies x^m\ ,\ \ln(x)x^m\ ,\ \ldots\ ,\ \ln^{k-1}(x)x^m$ Case III: $m = a \pm bi \implies x^a \cos(b\ln(x))$ & $x^a \sin(b\ln(x))$
Variation of Parameters	Nonhomogeneous Order n	$y_c = u_1 y_1 + \ldots + u_n y_n$	Solve Homogeneous equation for y_c $\frac{du_i}{dx} = \frac{W_i}{W}$

Approximations

Technique	**Situation**	**Details**
Euler's Method	$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$	Choose step size h to increment x $x_{n+1} = x_n + h$ $y_{n+1} = y_n + hf(x_n, y_n)$
Improved Euler's Method	$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$	Choose step size h to increment x $x_{n+1} = x_n + h$ $y*_{n+1} = y_n + hf(x_n, y_n)$ $y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y*_{n+1})}{2}$
Taylor Polynomial Method	Order n $y(x_0) = y_0$ $\vdots$ $y^{n-1}(x_0) = y_{n-1}$	$y = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)(x-x_0)^2}{2} + \ldots + \frac{y^k(x_0)(x-x_0)^k}{k!}$ Find later $y^k(x_0)$ by plugging into the ODE