Solutions to Practice Problems for Exam II

Note that most of these can be solved in many ways.
I have provided *a* solution to every problem, but not always the simplest one.

1. Find general solution to $\dfrac{d^4y}{dx^4} - 7\dfrac{d^2y}{dx^2} - 18y = 0$.

   Aux Polynomial: $m^4 - 7m - 18 = (m^2-9)(m^2+2) = (m-3)(m+3)(m-\sqrt{2}i)(m+\sqrt{2}i)$

   General solution: $y = c_1e^{3x} + c_2e^{-3x} + c_3\cos(\sqrt{2}x) + c_4\sin(\sqrt{2}x)$ on $\mathbb{R}$

2. Find a differential operator that annihilates $x^3(1-5x)$.

   D^5, because $D^4(x^3 - 5x^4) = 0$

3. Find general solution to $2x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + y + x = x^2$.

   $$\frac{d^2y}{dx^2} + \frac{5}{2x}\frac{dy}{dx} - + \frac{1}{2x^2}y = \frac{1}{2} - \frac{1}{2x}$$

   Homogeneous solution:

   Aux Polynomial: $2m(m-1) + 5m + 1 = 2m^2 + 3m + 1 = (2m+1)(m+1) = 0$

   $y_c = c_1x^{-1} + c_2x^{-1/2}$

   Particular Solution:

   $y_p = u_1x^{-1} + u_2x^{-1/2}$

   $$W(x^{-1}, x^{-1/2}) = \det\begin{bmatrix} x^{-1} & x^{-1/2} \\ -x^{-2} & -\frac{1}{2}x^{-3/2} \end{bmatrix} = \frac{1}{2}x^{-5/2}$$

   $$W_1 = \det\begin{bmatrix} 0 & x^{-1/2} \\ \frac{1}{2} - \frac{1}{2}x^{-1} & -\frac{1}{2}x^{-3/2} \end{bmatrix} = -\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2}$$

   $$\frac{du_1}{dx} = \frac{W_1}{W} = x - x^2 \implies u_1 = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

   $$W_2 = \det\begin{bmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{1}{2} - \frac{1}{2}x^{-1} \end{bmatrix} = \frac{1}{2}x^{-1} - \frac{1}{2}x^{-2}$$

   $$\frac{du_2}{dx} = \frac{W_2}{W} = x^{3/2} - x^{1/2} \implies u_2 = \frac{2}{5}x^{5/2} - \frac{3}{2}x^{3/2}$$

   $$y_p = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)x^{-1} + \left(\frac{2}{5}x^{5/2} - \frac{3}{2}x^{3/2}\right)x^{-1/2} = -\frac{1}{6}x + \frac{1}{15}x^2$$

   General solution: $y = y_c + y_p = \left(c_1x^{-1} + c_2x^{-1/2}\right) + \left(-\frac{1}{6}x + \frac{1}{15}x^2\right)$ on $(0,\infty)$

4. Find general solution to $6\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$ given $y_1 = e^{x/3}$.

   Proceed by reduction of order: $y_2 = ue^{x/3}$.

   ODE: $\dfrac{d^2y}{dx^2} + \dfrac{1}{6}\dfrac{dy}{dx} - \dfrac{1}{6}y = 0 \implies P = \dfrac{1}{6}$

   $y_2 = e^{x/3}\displaystyle\int \frac{e^{-\int 1/6\ dx}}{e^{2x/3}}\ dx = e^{x/3}\int e^{-x/6-2x/3}\ dx = e^{x/3}\left(-\frac{6}{5}e^{-5x/6}\right) = -\frac{6}{5}e^{-x/2}$

   General solution: $y = c_1e^{x/3} + c_2e^{-x/2}$ on $\mathbb{R}$

5. Find general solution to $\dfrac{1}{4}\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = x^2 - 2x$ via undetermined coefficients.

   $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 4x^2 - 8x$

   Homogeneous solution:

   Aux Polynomial: $m^2 + 4m + 4 = (m+2)^2 = 0$

   $y_c = c_1e^{-2x} + xc_2e^{-2x}$

   Particular Solution:

   $y_p = Ax^2 + Bx + C$

   $\dfrac{dy_p}{dx} = 2Ax + B$

   $\dfrac{d^2y_p}{dx^2} = 2A$

   $(2A) + 4(2Ax + B) + 4(Ax^2 + Bx + C) = 4x^2 - 8x$

   $= x^2(4A) + x(8A + 4B) + (2A + 4B + 4C) = 4x^2 - 8x$

   $\implies A = 1, B = -4, C = 7/2$

   $y_p = x^2 - 4x + \dfrac{7}{2}$

   General solution: $y = y_c + y_p = \left(c_1e^{-2x} + xc_2e^{-2x}\right) + \left(x^2 - 4x + \frac{7}{2}\right)$ on $(0, \infty)$

6. Find general solution to $\dfrac{d^4y}{dx^4} - 4\dfrac{d^2y}{dx^2} = x + e^x$ given $y_c = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x}$.

   This can be done a few ways, I chose annihilators

   Particular Solution:

   $x + e^x$ is annihilated by $D(D-1)$ and the homogeneous solution has linear term already, hence

   $y_p = c_1x^3 + c_2x^2 + c_3e^x$

   $\dfrac{dy_p}{dx} = 3c_1x^2 + 2c_2x + c_3e^x$

   $\dfrac{d^2y_p}{dx^2} = 6c_1x + 2c_2 + c_3e^x$

   $\dfrac{d^3y_p}{dx^3} = 6c_1 + c_3e^x$

$$\frac{d^4y_p}{dx^4} = c_3e^x$$

$$(c_3e^x) - 4(6c_1x + 2c_2 + c_3e^x) = x + e^x$$
$$x(-24c_1) + (-8c_2) + e^x(c_3 - 4c_3) = x + e^x$$
$$\implies c_1 = -1/24,\ c_2 = 0,\ c_3 = -1/3$$
$$y_p = -\frac{1}{24}x^3 - \frac{1}{3}e^x$$

General solution: $y = y_c + y_p = \left(c_1 + c_2x + c_3e^{2x} + c_4e^{-2x}\right) + \left(-\frac{1}{24}x^3 - \frac{1}{3}e^x\right)$ on $\mathbb{R}$

7. Consider the IVP $\frac{dy}{dx} = y^2 + x^2$, $y(0) = 1$. For the following, write the expressions, and then use a calculator to compute.

(a) Approximate $y(0.2)$ via Euler's method with step size 0.1.
$y_0 = 1$
$y_1 = 1 + (0.1)(0^2 + 1^2) = 1.1.$
$y_2 = 1.1 + (0.1)(0.1^2 + 1.1^2) = 1.1 + (0.1)(0.01 + 1.21) = 1.1 + 0.122 = 1.222.$

(b) Approximate $y(0.2)$ via improved Euler's method with step size 0.1.
$y_0 = 1$
$y_1^* = 1 + (0.1)(0^2 + 1^2) = 1.1.$

$$y_1 = 1 + (0.1)\left(\frac{(0^2 + 1^2) + (0.1^2 + 1.1^2)}{2}\right) = 1 + (0.1)\left(\frac{1 + 0.122}{2}\right) = 1 + 0.111 = 1.111$$

$y_2^* = 1.111 + (0.1)(0.1^2 + 1.111^2) = 1.1 + (0.01 + 1.234321) = 1.12354321.$

$$y_2 = 1.111 + (0.1)\left(\frac{(0.1^2 + 1.111^2) + (0.2^2 + 1.12354321^2)}{2}\right) = 1.23833$$

(this one had an arithmetic error previously)

(c) The true solution is $y(0.2) = 1.25302$. What are the absolute and relative errors?
Euler's absolute error: $|1.25302 - 1.222| = 0.03102$
Euler's relative error: $0.03102/1.25302 = 0.02476$, 2.476%.

Improved Euler's absolute error: $|1.25302 - 1.25153| = 0.00149$
Improved Euler's relative error: $0.00149/1.25302 = 0.001189$, 0.1189%.

8. Solve the system given by the equations $\frac{dx}{dt} - 4y = 1$ and $\frac{dy}{dt} + x = 2$.

$Dx - 4y = 1$
$x + Dy = 2$

Eliminate y:

$$D^2x - 4Dy = 0$$
$$4x + 4Dy = 8$$
$$\implies D^2x + 4x = (D^2 + 4)x = 8$$
$$\implies x = c_1\cos(t) + c_2\sin(t) + 2$$

Eliminate x:

$-Dx + 4y = -1$

$Dx + D^2 y = 0$

$\implies D^2 y + 4y = (D^2 + 4)y = -1$

$\implies y = c_3 \cos(t) + c_4 \sin(t) - \frac{1}{4}$

Solve for constants:

$Dx - 4y = \Big(-2c_1 \sin(2t) + 2c_2 \cos(2t)\Big) - 4\Big(c_3 \cos(t) + c_4 \sin(t) - \frac{1}{4}\Big) = 1$

$\cos(2t)\Big(2c_2 - 4c_3\Big) + \sin(2t)\Big(-2c_1 - 4c_4\Big) + 1 = 1$

$\implies c_3 = 2c_2 \quad c_4 = -2c_1$

$x = c_1 \cos(2t) + c_2 \sin(2t) + 2$

$y = 2c_2 \cos(2t) - 2c_1 \sin(2t) - \dfrac{1}{4}$

9. Find general solution to $y^2 \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}$

Since there are no x's in the equation, let $u = \dfrac{dy}{dx}$ so that $\dfrac{du}{dx} = \dfrac{d^2y}{dx^2}$

$\implies y^2 \dfrac{du}{dx} = u$

Now use $\dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx} = u\dfrac{du}{dy}$

$\implies y^2 u \dfrac{du}{dy} = u \quad \implies \quad du = y^{-2}\, dy$

$\implies \quad \dfrac{dy}{dx} = u = -\dfrac{1}{y} + c_1 = \dfrac{c_1 y - 1}{y}$ First order separable

$\implies dx = \dfrac{y}{Cy - 1}\, dy = \dfrac{1}{c_1}\left(1 + \dfrac{1}{c_1 y - 1}\right)\, dy$

$\implies x + c_2 = \dfrac{1}{c_1} y + \dfrac{1}{c_1^2} \ln|y - 1|$, implicit solution.

10. Find general solution to $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 8x^2$ via annihilators.

Annihilation:

ODE: $D^2(D+1)y = 8x^2 \quad \implies \quad D^5(D+1)y = 0$

$y = c_1 + c_2 x + c_3 e^{-x} + c_4 x^2 + c_5 x^3 + c_6 x^4$

First three terms come from $D^2(D+1)$, hence $y_c = c_1 + c_2 x + c_3 e^{-x}$

Particular Solution:

$y_p = c_4 x^2 + c_5 x^3 + c_6 x^4$

$\dfrac{dy_p}{dx} = 2c_4 x + 3c_5 x^2 + 4c_6 x^3$

$\dfrac{d^2y_p}{dx^2} = 2c_4 + 6c_5 x + 12c_6 x^2$

$$\frac{d^3y_p}{dx^3} = 6c_5 + 24c_6x$$

$$(6c_5 + 24c_6x) + (2c_4 + 6c_5x + 12c_6x^2) = 8x^2$$
$$x^2(12c_6) + x(24c_6 + 6c_5) + (6x_5 + 2c_4) = 8x^2$$
$$\implies c_6 = 2/3,\ c_5 = -8/3,\ c_4 = 8$$
$$y_p = \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2$$

General solution: $y = y_c + y_p = \Big(c_1 + c_2x + c_3e^{-x}\Big) + \Big(\frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2\Big)$ on $\mathbb{R}$

11. Find a differential operator that annihilates $13x + 9x^2 - \sin(4x)$.

$D^3(D^2 + 16)$ because $D^3(13x + 9x^2) = 0$ and $D^2\sin(4x) = 16\sin(4x)$.

12. Find general solution to $\dfrac{d^2y}{dx^2} + y = \sec^2(x)$ via variation of parameters given $y_c = c_1\cos(x) + c_2\sin(x)$.

Particular Solution:

$$y_p = u_1\cos(x) + u_2\sin(x)$$
$$W(\cos(x), \sin(x)) = \det\begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} = 1$$
$$W_1 = \det\begin{bmatrix} 0 & \sin(x) \\ \sec^2(x) & \cos(x) \end{bmatrix} = \sin(x)\sec^2(x) = -\sec(x)\tan(x)$$
$$\frac{du_1}{dx} = \frac{W_1}{W} = -\sec(x)\tan(x) \implies u_1 = -\sec(x)$$
$$W(\cos(x), \sin(x)) = \det\begin{bmatrix} \cos(x) & 0 \\ -\sin(x) & \sec^2(x) \end{bmatrix} = \sec(x)$$
$$\frac{du_2}{dx} = \frac{W_2}{W} = \sec(x) \implies u_2 = \ln|\sec(x) + \tan(x)|$$
$$y_p = \cos(x)\sec(x) + \sin(x)\ln|\sec(x) + \tan(x)|$$

General solution: $y = y_c + y_p = \Big(c_1\cos(x) + c_2\sin(x)\Big) + \Big(-1 + \sin(x)\ln|\sec(x) + \tan(x)|\Big)$ on $(-\pi/4, \pi/4)$

13. Find general solution to $x^3\dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} - y = 0$.

Aux Polynomial: $m(m-1)(m-2) + m - 1 = m^3 - 3m^2 - 3m + 1 = (m-1)^3 = 0$

$y = c_1x + c_2x\ln(x) + c_3x\ln^2(x)$

14. Find general solution to $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = x\sin(x)$ via annihilators.

Annihilation:

ODE: $(D^2 + D + 1)y = x\sin(x) \implies (D^2+1)^2(D^2 + D + 1)y = 0$

$$y = c_1e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) + c_2e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right) + c_3\cos(x) + c_4\sin(x) + c_5x\cos(x) + c_6x\sin(x)$$

First three terms come from $(D^2 + D + 1)$, hence $y_c = c_1e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) + c_2e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)$

Particular Solution:

$$y_p = c_3\cos(x) + c_4\sin(x) + c_5 x\cos(x) + c_6 x\sin(x)$$
$$\frac{dy_p}{dx} = (c_4+c_5)\cos(x) + (c_6-c_3)\sin(x) + c_6 x\cos(x) - c_5 x\sin(x)$$
$$\frac{d^2y_p}{dx^2} = (-c_3+2c_6)\cos(x) + (-c_4-2c_5)\sin(x) - c_5 x\cos(x) - c_6 x\sin(x)$$
$$(\cancel{c_3} + c_4 + c_5 - \cancel{c_3} + 2c_6)\cos(x) + (\cancel{c_4} + c_6 - c_3 - \cancel{c_4} - 2c_5)\sin(x) + (c_6)x\cos(x) + (-c_5)x\sin(x) = x\sin(x)$$

$$\implies c_3 = 2,\ c_4 = 1,\ c_5 = -1,\ c_6 = 0$$
$$y_p = 2\cos(x) + \sin(x) - x\cos(x)$$

General solution: $y = y_c + y_p = \left(c_1 e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)\right) + \left(2\cos(x) + \sin(x) - x\cos(x)\right)$ on $\mathbb{R}$

15. Find general solution to $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 18y = 0$.

Aux Polynomial: $m^3 - m^2 - 18 = (m-3)(m^2+2m+6) = (m-3)\left(m+1-\sqrt{5}\right)\left(m+1+\sqrt{5}\right)$

General solution: $y = c_1 e^{3x} + c_2 e^{-x}\cos(\sqrt{5}) + c_3 e^{3x}\sin(\sqrt{5})$ on $\mathbb{R}$

16. Find general solution to $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = e^x - x + 16$.

This can be done using any method, but I chose annihilators.

Annihilation:

ODE: $(D-1)^3 y = e^x - x + 16 \quad \implies \quad D^2(D-1)^4 y = 0$

$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 x^3 e^x + c_5 + c_6 x$

First three terms come from $(D-1)^3$, hence $y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

Particular Solution:

$$y_p = c_4 x^3 e^x + c_5 + c_6 x$$
$$\frac{dy_p}{dx} = c_4 x^3 e^x + 3c_4 x^2 e^x + c_6$$
$$\frac{d^2y_p}{dx^2} = c_4 x^3 e^x + 6c_4 x^2 e^x + 6c_4 x e^x$$
$$\frac{d^3y_p}{dx^3} = c_4 x^3 e^x + 9c_4 x^2 e^x + 18c_4 x e^x + 6c_4 e^x$$

$$(c_4 x^3 e^x + 9c_4 x^2 e^x + 18c_4 x e^x + 6c_4 e^x) - 3(c_4 x^3 e^x + 6c_4 x^2 e^x + 6c_4 x e^x)$$
$$+3(c_4 x^3 e^x + 3c_4 x^2 e^x + c_6) - (c_4 x^3 e^x + c_5 + c_6 x) = e^x - x + 16$$

$$x^3 e^x(\cancel{c_4 - 3c_4 + 3c_4 - c_4}) + x^2 e^x(\cancel{9c_4 - 18c_4 + 9c_4}) + x e^x(\cancel{18c_4 - 18c_4})$$
$$+ e^x(6c_4) + x(-c_6) + (3c_6 - c_5) = e^x - x + 16$$

$$\implies c_4 = 1/6,\ c_5 = -13,\ c_6 = 1$$
$$y_p = \frac{1}{6}x^3 e^x - 13 + x$$

General solution: $y = y_c + y_p = \left(c_1 e^x + c_2 x e^x + c_3 x^2 e^x\right) + \left(\frac{1}{6}x^3 e^x - 13 + x\right)$ on $\mathbb{R}$

17. Solve the system given by the equations $\dfrac{dx}{dt} + \dfrac{dy}{dt} = e^t$ and $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - x - y = 0$.

$Dx + Dy = e^t$
$(D^2 - D - 1)x - y = 0$

Eliminate y:

$Dx + Dy = e^t$

$D(D^2 - D - 1)x - Dy = 0$

$\implies D^2(D-1)x = e^t$

$\implies x = c_1 + c_2 t + c_3 e^t + te^t$

Use second equation:

Equation two tells us $y = (D^2 - D - 1)x$

$\implies y = \Big((c_3+2)e^t + te^t\Big) - \Big(c_2 + (c_3+1)e^t + te^t\Big) - \Big(c_1 + c_2 t + c_3 e^t + te^t\Big)$

$\implies y = -c_1 - c_2 - c_2 t + (2 - c_3)e^t - te^t$

Constants are already good!

18. Find general solution to $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = x - 4e^x$ via undetermined coefficients.

$\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 4x^2 - 8x$

Homogeneous solution:

Aux Polynomial: $m^3 - 3m^2 + 3m - 1 = (m-1)^3$

$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

Particular Solution:

$y_p = Ax + B + Cx^3 e^x$

$\dfrac{dy_p}{dx} = A + Cx^3e^x + 3Cx^2e^x$

$\dfrac{d^2y_p}{dx^2} = Cx^3e^x + 6Cx^2e^x + 6Cxe^x$

$\dfrac{d^3y_p}{dx^3} = Cx^3e^x + 9Cx^2e^x + 18Cxe^x + 6Ce^x$

$(Cx^3e^x + 9Cx^2e^x + 18Cxe^x + 6Ce^x) - 3(Cx^3e^x + 6Cx^2e^x + 6Cxe^x) + 3(A + Cx^3e^x + 3Cx^2e^x) - (Ax + B + Cx^3e^x)$
$= x^3e^x\cancel{(C - 3C + 3C - C)} + x^2e^x\cancel{(9C - 18C + 9C)} + xe^x\cancel{(18C - 18C)} + e^x(6C) + x(-A) + (3A - B) = x - 4e^x$

$\implies A = -1, B = -3, C = -2/3$

$y_p = -x^2 - 3 - \dfrac{7}{2}x^3e^x$

General solution: $y = y_c + y_p = \Big(c_1e^x + c_2xe^x + c_3x^2e^x\Big) + \Big(-x^2 - 3 - \frac{7}{2}x^3e^x\Big)$ on $(0, \infty)$

19. Find general solution to $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sec(2x)$ via variation of parameters.

Homogeneous solution:

Aux Polynomial: $m^3 + 4m = m(m^2+4) = m(m-2i)(m+2i)$

$y_c = c_1 + c_2\cos(2x) + c_3\sin(2x)$

Particular Solution:

$y_p = u_1 + u_2\cos(2x) + u_3\sin(2x)$

$$W(1,\cos(2x),\sin(2x)) = \det\begin{bmatrix} 1 & \cos(2x) & \sin(2x) \\ 0 & -2\sin(2x) & 2\cos(2x) \\ 0 & -4\cos(2x) & 4\sin(2x) \end{bmatrix} = 8$$

$$W_1 = \det\begin{bmatrix} 0 & \cos(2x) & \sin(2x) \\ 0 & -2\sin(2x) & 2\cos(2x) \\ \sec(2x) & -4\cos(2x) & 4\sin(2x) \end{bmatrix} = 2\sec(2x)$$

$$\frac{du_1}{dx} = \frac{W_1}{W} = -\frac{2\sec(2x)}{8} \implies u_1 = \frac{1}{8}\ln|\sec(2x)+\tan(2x)|$$

$$W_2 = \det\begin{bmatrix} 1 & 0 & \sin(2x) \\ 0 & 0 & 2\cos(2x) \\ 0 & \sec(2x) & 4\sin(2x) \end{bmatrix} = -2$$

$$\frac{du_2}{dx} = \frac{W_2}{W} = -\frac{2}{8} \implies u_1 = -\frac{1}{4}x$$

$$W_3 = \det\begin{bmatrix} 1 & \cos(2x) & 0 \\ 0 & -2\sin(2x) & 0 \\ 0 & -4\cos(2x) & \sec(2x) \end{bmatrix} = -2\tan(2x)$$

$$\frac{du_3}{dx} = \frac{W_3}{W} = -\frac{2\tan(2x)}{8} \implies u_1 = \frac{1}{8}\ln|\cos(2x)|$$

$$y_p = \frac{1}{8}\ln|\sec(2x)+\tan(2x)| - \frac{1}{4}x\cos(2x) - \frac{1}{8}\sin(2x)\ln|\cos(2x)|$$

General solution:
$y = y_c + y_p = \Big(u_1 + u_2\cos(2x) + u_3\sin(2x)\Big) + \Big(\frac{1}{8}\ln|\sec(2x)+\tan(2x)| - \frac{1}{4}x\cos(2x) - \frac{1}{8}\sin(2x)\ln|\cos(2x)|\Big)$
on $(-\pi/4, \pi/4)$

20. Find general solution to $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = 0$ given $y_1 = x\sin(\ln(x))$.

Proceed by reduction of order: $y_2 = ux\sin(\ln(x))$.

ODE: $\dfrac{d^2y}{dx^2} - \dfrac{1}{x}\dfrac{dy}{dx} + \dfrac{2}{x^2}y = 0 \implies P = -\dfrac{1}{x}$

$$y_2 = x\sin(\ln(x))\int \frac{e^{-\int -1/x\ dx}}{x\sin(\ln(x))}\,dx = x\sin(\ln(x))\int \frac{e^{\ln|x|}}{x\sin(\ln(x))}\,dx = x\sin(\ln(x))\int \frac{x}{x\sin(\ln(x))}\,dx$$

$= x\sin(\ln(x))\int \frac{\csc^2(\ln(x))}{x}\,dx = x\sin(\ln(x))\Big(-\cot(\ln(x))\Big) = -x\cos(\ln(x))$

General solution: $y = c_1x\sin(\ln(x)) + c_2x\cos(\ln(x))$ on $(0,\infty)$

21. Use a degree five Taylor polynomial to approximate the solution to $\dfrac{d^2y}{dx^2} + y^2 = 1$, $y(0) = 2$, $\dfrac{dy}{dx}(0) = 3$.

$y(0) = 2$

$y'(0) = 3$

$y''(x) = 1 - y^2$
$y''(0) = 1 - (2)^2 = -3$

$y'''(x) = -2yy'$
$y'''(0) = -2(2)(3) = -12$

$y^{IV}(x) = -2(y')^2 - 2yy''$
$y^{IV}(0) = -2(3)^2 - 2(2)(-3) = -6$

$y^{V}(x) = -2yy''' - 6y'y''$
$y^{V}(0) = -2(2)(-12) - 6(3)(-3) = 102$

$y \approx 2 + 3x - \frac{3}{2}x^2 - 2x^3 - \frac{1}{4}x^4 + \frac{17}{20}x^5$ near $x = 0$