SOLUTIONS

(20 pts) 1. Consider the ODE $\frac{dy}{dx} = y^4 - 5y^3 - 14y^2$.

(a) Classify the ODE as specifically as you can.

(b) Find all critical points of the ODE.

(c) Construct a one-dimensional phase portrait.

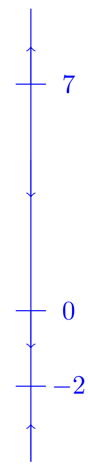
(d) Classify the critical points.

(e) If we also know that $y(-1) = 6$, what is the long term behavior of the system?

(a) This is first order Autonomous ODE

(b) $\frac{dy}{dx} = y^2(y-7)(y+2)$. Critical points occur at $y = -2, 0, 7$

(c) Portrait on the right:



(d) $y = -2$ is an attractor

$y = 7$ is a repeller

$y = 0$ is semi-stable

(e) $y \to 0$

(20 pts) 2. Find an implicit solution to the initial value problem: $\dfrac{dy}{dx} + y = xy^4$, y(0) = 1.
Do not find an interval.

Bernoulli with $n = 4$

$$u = y^{-3} \quad \Longrightarrow y = u^{-1/3} \quad \Longrightarrow \frac{dy}{dx} = -\frac{1}{3u^{4/3}}\frac{du}{dx}$$

$$-\frac{1}{3u^{4/3}}\frac{du}{dx} + u^{-1/3} = xu^{-4/3} \quad \Longrightarrow \frac{du}{dx} - 3u = -3x$$

New ODE is first order Linear

$$\Longrightarrow P(x) = -3 \quad \Longrightarrow \mu(x) = e^{\int -3\ dx} = e^{-3x}$$

$$e^{-3x}\frac{du}{dx} - 3e^{-3x}u = -3xe^{-3x} \quad \Longrightarrow \frac{d}{dx}\left(e^{-3x}u\right) = -3xe^{-3x} \ * \quad \Longrightarrow e^{-3x}u = xe^{-3x} + \frac{1}{3}e^{-3x} + C$$

$$\Longrightarrow u = x + \frac{1}{3} + Ce^{3x} \quad \Longrightarrow \frac{1}{y^3} = x + \frac{1}{3} + Ce^{3x} \quad \Longrightarrow y^3 = \frac{1}{x + \frac{1}{3} + Ce^{3x}} = \frac{3}{3x + 1 + Ce^{3x}}$$

$$y(0) = 1 \Longrightarrow 1 = \frac{3}{0 + 1 + C} \quad \Longrightarrow C + 1 = 3 \quad \Longrightarrow C = 2$$

$$y^3 = \frac{3}{3x + 1 + 2e^{3x}}$$

(there are a TON of ways to leave the implicit expression)

$*$ integration by parts

$$u = x \quad du = \ dx \qquad dv = -3e^{-3x}\ dx \quad v = e^{-3x}$$

$$\int -3xe^{-3x} = (x)(e^{-3x}) - \int \left(e^{-3x}\right)\ (dx) = xe^{-3x} + \frac{1}{3}e^{-3x} + C$$

(20 pts) 3. Solve the ODE: $\left(2x\sin(y)+\dfrac{1}{x}\right)\,dx+\left(x^2\cos(y)-\dfrac{1}{2\sqrt{y}}\right)\,dy=0.$

Classify what kind of solution you find and give restrictions on x and y that must hold.

$$\frac{\partial}{\partial y}\left(2x\sin(y)+\frac{1}{x}\right)=2x\cos(y)$$

$$\frac{\partial}{\partial x}\left(x^2\cos(y)-\frac{1}{2\sqrt{y}}\right)=2x\cos(y)$$

Therefore the ODE is exact.

Method One:

$$F(x,y)=\int 2x\sin(y)+\frac{1}{x}\;dx+g(y)=x^2\sin(y)+\ln|x|+g(y)$$

$$\frac{\partial}{\partial y}\Big(F(x,y)\Big)=x^2\cos(y)+g'(y)=x^2\cos(y)-\frac{1}{2\sqrt{y}}$$

$$\Longrightarrow\;F(x,y)=x^2\sin(y)+\ln|x|-\sqrt{y}$$

Method Two:

$$F(x,y)=\int x^2\cos(y)-\frac{1}{2\sqrt{y}}\;dy+g(x)=x^2\sin(y)-\sqrt{y}+g(x)$$

$$\frac{\partial}{\partial x}\Big(F(x,y)\Big)=2x\sin(y)+g'(x)=2x\sin(y)+\frac{1}{x}$$

$$\Longrightarrow\;F(x,y)=x^2\sin(y)-\sqrt{y}+\ln|x|$$

$$d\Big(x^2\sin(y)+\ln|x|-\sqrt{y}\Big)=0$$

$$x^2\sin(y)+\ln|x|-\sqrt{y}=C$$

This is a one-parameter family of implicit solution that have the restriction that $x\neq 0$ and $y>0$

(20 pts) 4. The statements below are all ***false***. Correct them to create true statements.
(some may be corrected in multiple ways)

(a) The order of an ODE is the highest power to which y is raised.

The order of an ODE is the **order of the highest order derivative taken**.

The **degree** of an ODE is the highest power to which **the highest order derivative** is raised

(b) A first order α-homogeneous equation can always be reduced to an exact ODE via $u = y^{1-n}$ for some n.

A first order α-homogeneous equation can always be reduced to a **separable** ODE via **one of either** $u = y/x$ **or** $v = x/y$.

A first order **Bernoulli** equation can always be reduced to a **linear** ODE via $u = y^{1-n}$ for some n.

(c) A F.O. exact ODE can always be expressed as $\frac{d}{dx}\Big(\mu(x)y\Big) = 0$ for some differentiable function $\mu(x)$

A F.O. exact ODE can always be expressed as $d\big(F(x, y)y\big) = 0$ for some **twice** differentiable $f(x, y)$

A F.O. **linear homogeneous** ODE can always be expressed as $\frac{d}{dx}\Big(\mu(x)y\Big) = 0$ forsome differentiable $\mu(x)$

A F.O. **linear** ODE can always be expressed as $\frac{d}{dx}\Big(\mu(x)y\Big) = \mu(x)g(x)$ for some differentiable $\mu(x)$

(d) An integration factor is a dependent variable that can be substituted into a F.O. ODE to produce an exact equation from an inexact one.

An integration factor is **a function** μ that can be **be multiplied by** a F.O. ODE to produce an exact equation from an inexact one.

An integration factor is **a function** $\mu(x)$ that can be **multiplied by** a F.O. **linear** ODE to produce **an explicit solution to the ODE**.

(e) A first order initial value problem given by $\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$ has a unique solution on an interval when y is continuous near (x_0, y_0).

A first order initial value problem given by $\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$ has a unique solution on an interval when **both** $f(x, y)$ **and** $\frac{\partial f}{\partial y}$ are continuous near (x_0, y_0).

(20 pts) 5. Find an explicit solution to $x^2\dfrac{dy}{dx} - 3xy + y = -x^3.$

$$\frac{dy}{dx} + \left(\frac{1}{x^2} - \frac{3}{x}\right) y = -x$$

$$P(x) = \frac{1}{x^2} - \frac{3}{x} \quad \implies \mu(x) = e^{\int 1/x^2 - 3/x \ dx} = e^{-1/x + \ln|x^{-3}|} = x^{-3}e^{-1/x}$$

$$\implies \frac{1}{x^3}e^{-1/x}\frac{dy}{dx} + \left(\frac{1}{x^5} - \frac{3}{x^4}\right)e^{-1/x}y = -\cancel{x}\frac{1}{x^{\cancel{3}\,2}}e^{-1/x} \quad \implies \frac{d}{dx}\left(\frac{1}{x^3}e^{-1/x}y\right) = -\frac{1}{x^2}e^{-1/x} \ *$$

$$\frac{1}{x^3}e^{-1/x}y = -e^{-1/x} + C$$

$$y = -x^3 + Cx^3e^{1/x} \text{ on } (0,\infty)$$

This is explicit because y is a function of x here

$*$ integration by substitution

$$u = -\frac{1}{x} \quad du = \frac{1}{x^2}dx$$

$$\int -\frac{1}{x^2}e^{-1/x} \ dx = -\int e^u \ du = -e^u + C = -e^{-1/x} + C$$

(+15 pts) 6. In mathematical biology, predator-prey systems can be modeled continuously deterministically with ODEs. Let's consider Capybaras (world's largest rodents), which are hunted by Jaguars (big cats).
Let $x(t) \geq 0$ be the number of Capybaras in an ecosystem at time t, and $y(t) \geq 0$ be the number of Jaguars. Consider the following facts:

- If there are no Jaguars, the Capybara population will grow proportionally to the size of the population of Capybaras, at a constant rate $\alpha > 0$.
- If there are no Capybaras, the Jaguar population will starve to death proportionally to the size of the population of Jaguars, at a constant rate $\gamma > 0$.
- The rate at which Capybaras are hunted is proportional to the product of the number of Capybaras and the number of Jaguars, at a constant rate $\beta > 0$.
- The rate at which Jaguars are born is proportional to the product of the number of Capybaras and the number of Jaguars, at a constant rate $\delta > 0$.

(a) Write a system of ODEs that describes this system. (called *Lotka-Volterra equations*).

(b) Find the critical point(s) (x, y) of the system of ODEs by solving $\dfrac{dx}{dt} = 0 = \dfrac{dy}{dt}$.

(c) Give me your best drawing of a Capybara and a Jaguar (ONLY ATTEMPT IF EXAM COMPLETE)

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

$$0 = \alpha x - \beta xy \quad \Longrightarrow \quad \alpha = \beta y \ \text{ OR } \ x = 0 \quad \Longrightarrow \quad y = \frac{\alpha}{\beta} \text{ OR } x = 0$$
$$\text{Case I: if } x = 0 \quad \Longrightarrow 0 = 0 - \gamma y \quad \Longrightarrow \quad y = 0$$
$$\text{Case II: if } y = \frac{\alpha}{\beta} \quad \Longrightarrow 0 = \frac{\alpha\delta}{\beta}x - \frac{\alpha\gamma}{\beta} \quad \Longrightarrow \quad x = \frac{\gamma}{\delta}$$
$$\text{Critical points are } \left(\frac{\gamma}{\delta}, \frac{\alpha}{\delta}\right) \text{ and } (0, 0)$$