SOLUTIONS

(25 pts) 1. Find a general solution the ODE $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 8e^x$ by any technique.

Homogeneous solution:

Aux Polynomial: $m^2 - 2m + 5 = (m-1)^2 + 4 = 0 \implies m = 1 \pm 2i$

$y_c = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$

Indetermined coefficients:

$y_p = Ae^x$

$\dfrac{dy_p}{dx} = Ae^x$

$\dfrac{d^2y_p}{dx^2} = Ae^x$

$(Ae^x) - 2(Ae^x) + 5(Ae^x) = e^x$

$4Ae^x = 8e^x \implies A = 2,$

$y_p = 2e^x$

General solution: $y = y_c + y_p = \Big(c_1 e^x \cos(2x) + c_2 e^x \sin(2x)\Big) + \Big(2e^x\Big)$ on $\mathbb{R}$

OR

Annihilators:

$(D^2 - 2D + 5)y = 2e^x \implies (D-1)(D^2 - 2D + 5)y = 0$

$m = 1 \pm 2i,\ 1.$

$y = c_1 e^x \cos(2x) + c_2 e^x \sin(2x) + c_3 e^x$

$y_p = c_3 e^x$

$\dfrac{dy_p}{dx} = c_3 e^x$

$\dfrac{d^2y_p}{dx^2} = c_3 e^x$

$(c_3 e^x) - 2(c_3 e^x) + 5(c_3 e^x) = e^x$

$4c_3 e^x = 8e^x \implies c_3 = 2,$

General solution: $y = y_c + y_p = \Big(c_1 e^x \cos(2x) + c_2 e^x \sin(2x)\Big) + \Big(2e^x\Big)$ on $\mathbb{R}$

OR

Variation of Parameters:

$$y_p = u_1 e^x \cos(2x) + u_2 e^x \sin(2x).$$

$$W = \det \begin{bmatrix} e^x \cos(2x) & e^x \sin(2x) \\ e^x \cos(2x) - 2e^x \sin(2x)) & e^x \sin(2x) + 2e^x \cos(2x) \end{bmatrix}$$

$$= \cancel{e^{2x} \sin(2x) \cos(2x)} + 2e^{2x} \cos^2(2x) - \cancel{e^{2x} \sin(2x) \cos(2x)} + 2e^{2x} \sin^2(2x) = 2e^{2x}(\cos^2(2x) + \sin^2(2x))$$

$$= 2e^{2x}$$

$$W_1 = \det \begin{bmatrix} 0 & e^x \sin(2x) \\ 8e^x & e^x \sin(2x) + 2e^x \cos(2x) \end{bmatrix}$$

$$= -8e^{2x} \sin(2x)$$

$$\frac{du_1}{dx} = \frac{W_1}{W} = \frac{-8e^{2x} \sin(2x)}{2e^{2x}} = -4 \sin(2x) \implies u_1 = 2 \cos(2x)$$

$$W_2 = \det \begin{bmatrix} e^x \cos(2x) & 0 \\ e^x \cos(2x) - 2e^x \sin(2x) & 8e^x \end{bmatrix}$$

$$= 8e^{2x} \cos(2x)$$

$$\frac{du_2}{dx} = \frac{W_2}{W} = \frac{8e^{2x} \cos(2x)}{2e^{2x}} = 4 \cos(2x) \implies u_2 = 2 \sin(2x)$$

$$y_p = (2 \cos(2x))\, e^x \cos(2x) + (2 \sin(2x) e^x \sin(2x) = 2e^x(\cos^2(2x) + \sin^2(2x)) = 2e^x.$$

General solution: $y = y_c + y_p = \Big(c_1 e^x \cos(2x) + c_2 e^x \sin(2x)\Big) + \Big(2e^x\Big)$ on $\mathbb{R}$

(25 pts) 2. The statements below are all ***false***. Correct them to create true statements.
(some may be corrected in multiple ways)

(a) A set of functions on a shared domain is linearly independent if the Wrońskian is zero at an initial condition.

A set of functions is linearly independent if the Wrońskian is ***never zero on the shared domain***.

A set of functions is linearly ***dependent*** if the Wrońskian is zero at ***some value in the shared domain***.

(b) If we know $y_1(x)$ is a solution to a second order non-homogeneous linear ODE with constant coefficients, we can use variation of parameters to find a linearly independent solution $y_2(x)$.

If we know $y_1(x)$ is a solution to a second order ***homogeneous*** linear ODE with constant coefficients, we can use ***reduction of order*** to find a linearly independent solution $y_2(x)$.

If we know $y_1(x)$ ***and*** $\mathbf{y_2(x)}$ ***are linearly independent*** solutions to a second order ***homogeneous*** linear ODE with constant coefficients, we can use variation of parameters to find ***particular solution*** $\mathbf{y_p}$.

(c) For order two and above, there do not exist analytic solutions to non-linear ODEs and hence solutions to IVP and BVPs must be numerically approximated.

For order two and above, ***there are non-linear ODEs for which*** there do not exist analytic solutions and hence solutions to IVP and BVPs ***often*** must be numerically approximated.

(d) An IVP where the ODE is non-homogeneous 3rd order linear with non-constant coefficients does not necessarily have solution.

An IVP where the ODE is non-homogeneous 3rd order linear with non-constant coefficients ***has unique solutions***.

A ***BVP*** where the ODE is non-homogeneous 3rd order linear with non-constant coefficients does not necessarily have solution.

(e) Euler's method can be used to approximate the solution to IVPs with ODE n-th order linear with constant coefficients.

Euler's method can be used to approximate the solution to IVPs with ODE ***first order linear***.

A Taylor polynomial can be used to ***locally*** approximate the solution to ***any*** IVP.

(25 pts) 3. Solve the following System of linear ODEs:

$$\frac{dx}{dt} = x - y$$

$$\frac{dy}{dt} = 2x + 4y$$

The system is given as $Dx = x - y$ and $Dy = 2x + 4y$.
We can rewrite this as

$$(D-1)x + y = 0$$
$$-2x + (D-4)y = 0$$

Eliminate y:

Apply $(D-4)$ to the the top equation and multiply the bottom equation by -1 to get

$$(D-4)(D-1)x + (D-4)y = 0$$
$$2x - (D-4)y = 0$$

Adding the equations yields $(D^2 - 5D + 6)x = (D-3)(D-2)y = 0$

Hence $x(t) = c_1e^{2x} + c_2e^{3x}$

Eliminate x:

Multiply the top equation by 2 and apply $D-1$ to the bottom equation to get

$$2(D-1)x + 2y = 0$$
$$-2(D-1)x + (D-1)(D-4)y = 0$$

Adding the equations yields $(D^2 - 5D + 6)y = (D-3)(D-2)y = 0$

Hence $y(t) = c_3e^{2x} + c_4e^{3x}$

Solve for constants:

Plugging into the first equation yields:

$$\left(2c_1e^{2x} + 3c_2e^{3x}\right) - \left(c_1e^{2x} + c_2e^{3x}\right) + \left(c_3e^{2x} + c_4e^{3x}\right) = 0$$

$$e^{2x}\left(2c_1 - c_1 + c_3\right) + e^{3x}\left(3c_2 - c_2 + c_4\right) = 0$$

$c_3 = -c_1$ and $c_4 = -2c_2$

$$x(t) = c_1e^{2x} + c_2e^{3x}$$
$$y(t) = -c_1e^{2x} - 2c_2e^{3x} \qquad \text{on } \mathbb{R}$$

(25 pts) 4. Find a general solution the ODE $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 2x^4 + x^2$ by any technique.

Homogeneous solution:

Aux Polynomial: $m(m-1) - 4m + 6 = m^2 - 5m + 6 = (m-2)(m-3) = 0 \implies m = 2, 3$

$y_c = c_1x^2 + c_2x^3$

Variation of Parameters:

$$\frac{d^2y}{dx^2} - \frac{4}{x}\frac{dy}{dx} + 6y = 2x^2 + 1$$

$y_p = u_1x^2 + u_2x^3.$

$$W = \det\begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix} = 3x^4 - 2x^4 = x^4$$

$$W_1 = \det\begin{bmatrix} 0 & x^3 \\ 2x^2+1 & 3x^2 \end{bmatrix} = -2x^5 - x^3$$

$$\frac{du_1}{dx} = \frac{W_1}{W} = \frac{-2x^5 - x^3}{x^4} = -2x - \frac{1}{x} \implies u_1 = -x^2 - \ln(x)$$

$$W_2 = \det\begin{bmatrix} x^2 & 0 \\ 2x & 2x^2+1 \end{bmatrix} = 2x^4 + x^2$$

$$\frac{du_2}{dx} = \frac{W_2}{W} = \frac{2x^4 + x^2}{x^4} = 2 + \frac{1}{x^2} \implies u_2 = 2x - \frac{1}{x}$$

$$y_p = \left(-x^2 - \ln(x)\right)x^2 + \left(2x - \frac{1}{x}\right)x^3 = -x^4 - x^2\ln(x) + 2x^4 - x^2 = x^4 - x^2\ln(x) - x^2.$$

General solution: $y = y_c + y_p = \left(c_1x^2 + c_2x^3\right) + \left(x^4 - x^2\ln(x) - x^2\right) = c_1x^2 + c_2x^3 + x^4 - x^2\ln(x)$ on $(0, \infty)$