Solutions to Practice Problems for Exam I

1. $x\dfrac{dy}{dx} + 4y = x^3 - x$

First order Linear.

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 - 1 \quad \Longrightarrow P(x) = \frac{4}{x} \quad \Longrightarrow \mu(x) = e^{\int (4/x)\ dx} = e^{\ln|x^4|} = x^4$$

$$x^4\frac{dy}{dx} + 4x^3y = x^6 - x^4 \quad \Longrightarrow \frac{d}{dx}\left(x^4y\right) = x^6 - x^4 \quad \Longrightarrow x^4y = \frac{x^7}{7} - \frac{x^5}{5} + C \quad \Longrightarrow y = \frac{x^3}{7} - \frac{x}{5} + \frac{C}{x^4} \text{ on } (0,\infty)$$

Explicit Solution

2. $(x)\ dx + (x^2y + 4y)\ dy = 0$

First order Separable.

$$y\ dy = -\frac{x}{x^2+4}\ dx \quad \Longrightarrow \frac{y^2}{2} = -\frac{1}{2}\ln|x^2+4| \quad \Longrightarrow y^2 = \ln\left(\frac{1}{x^2+4}\right) \text{ on } (-\infty,\infty)$$

Implicit Solution

3. $(xe^x - 2y)\ dx = x\ dy$

First order Linear.

$$\frac{dy}{dx} + \frac{2}{x}y = e^x \quad \Longrightarrow P(x) = \frac{2}{x} \quad \Longrightarrow \mu(x) = e^{\int (2/x)\ dx} = e^{\ln|x^2|} = x^2$$

$$x^2\frac{dy}{dx} + 2xy = x^2e^x \quad \Longrightarrow \frac{d}{dx}\left(x^2y\right) = x^2e^x \quad \Longrightarrow x^2y = x^2e^x - 2xe^x + 2e^x + C$$

$$\Longrightarrow y = e^x - \tfrac{2e^x}{x} + \tfrac{2e^x}{x^2} + \tfrac{C}{x^2} \text{ on } (0,\infty)$$

Explicit Solution

4. $\dfrac{dy}{dx} = 1 + e^{y-x+5}$

First order, not anything we care about

$$u = y - x + 5 \quad \Longrightarrow y = u + x - 5 \quad \Longrightarrow \frac{dy}{dx} = \frac{du}{dx} + 1$$

$$\frac{du}{dx} + 1 = 1 + e^u \quad \Longrightarrow \frac{du}{dx} = e^u$$

New ODE is first order separable

$$e^{-u}\ du = 1\ dx \quad \Longrightarrow -e^u = x + c \quad \Longrightarrow C = e^{y-x+5} + x \text{ on } (-\infty,\infty)$$

Implicit Solution

5. $\dfrac{dy}{dx} = y(xy^3 - 1)$

$\dfrac{dy}{dx} + y = xy^4$

First order, Bernoulli with $n = 4$

$u = y^{-3} \quad \Longrightarrow \; y = u^{-1/3} \quad \Longrightarrow \; \dfrac{dy}{dx} = -\dfrac{1}{3u^{4/3}}\dfrac{du}{dx}$

$-\dfrac{1}{3u^{4/3}}\dfrac{du}{dx} + u^{-1/3} = xu^{-4/3} \quad \Longrightarrow \; \dfrac{du}{dx} - 3u = -3x$

New ODE is first order Linear

$\Longrightarrow \; P(x) = -3 \quad \Longrightarrow \; \mu(x) = e^{\int -3 \; dx} = e^{-3x}$

$e^{-3x}\dfrac{du}{dx} - 3e^{-3x}u = -3xe^{-3x} \quad \Longrightarrow \; \dfrac{d}{dx}\left(e^{-3x}u\right) = -3xe^{-3x} \quad \Longrightarrow \; e^{-3x}u = xe^{-3x} + \dfrac{1}{3}e^{-3x} + C$

$\Longrightarrow \; u = x + \dfrac{1}{3} + Ce^{3x} \quad \Longrightarrow \; \dfrac{1}{y^3} = x + \dfrac{1}{3} + Ce^{3x} \quad \Longrightarrow \; y = \dfrac{1}{\sqrt[3]{x + \frac{1}{3} + Ce^{3x}}}$

Explicit Solution defined wherever $x + 1/3 + Ce^{3x} > 0$

6. $\dfrac{dy}{dx} + 2xy^2 = 0$

First order Separable.

$\dfrac{1}{y^2}dx = -2xdx \quad \Longrightarrow \; -\dfrac{1}{y} = -x^2 + C \quad \Longrightarrow \; y = \dfrac{1}{x^2 + C}$ on $(-\infty, \infty)$

Explicit Solution

7. $(\tan(x) - \sin(x)\sin(y)) \; dx + (\cos(x)\cos(y)) \; dy = 0$

$\dfrac{\partial}{\partial y}\Big(\tan(x) - \sin(x)\sin(y)\Big) = -\sin(x)\cos(y)$

$\dfrac{\partial}{\partial x}\Big(\cos(x)\cos(y)\Big) = -\sin(x)\cos(y)$

First order Exact.

$F(x, y) = \displaystyle\int \cos(x)\cos(y) \; dy + g(x) = \cos(x)\sin(y) + g(x)$

$\dfrac{\partial}{\partial x}\Big(F(x, y)\Big) = -\sin(x)\sin(y) + g'(x) = \tan(x) - \sin(x)\sin(y) \quad \Longrightarrow \; g'(x) = \tan(x) \quad \Longrightarrow \; g(x) = \ln|\sec(x)|$

$\Longrightarrow \; d\Big(\cos(x)\sin(y) + \ln|\sec(x)|\Big) = 0 \quad \Longrightarrow \; \cos(x)\sin(y) + \ln|\sec(x)| = C$

Implicit Solution

8. $x^2 \dfrac{dy}{dx} + y^2 = ty$

$\dfrac{dy}{dx} - \dfrac{t}{x^2} y = -\dfrac{1}{x^2} y^2$

First order, Bernoulli with $n = 2$

$u = y^{-1} \quad \implies y = u^{-1} \quad \implies \dfrac{dy}{dx} = -\dfrac{1}{u^2}\dfrac{du}{dx}$

$-\dfrac{1}{u^2}\dfrac{du}{dx} - \dfrac{t}{x^2}u^{-1} = -\dfrac{1}{x^2}u^{-2} \quad \implies \dfrac{du}{dx} + \dfrac{t}{x^2}u = \dfrac{1}{x^2}$

New ODE is first order Linear

$\implies P(x) = \dfrac{t}{x^2} \quad \implies \mu(x) = e^{\int t/(x^2)\ dx} = e^{-t/x}$

$e^{-t/x}\dfrac{du}{dx} + \dfrac{t}{x^2}e^{-t/x}u = \dfrac{1}{x^2}e^{-t/x} \quad \implies \dfrac{d}{dx}\left(e^{-t/x}u\right) = \dfrac{1}{x^2}e^{-t/x} \quad \implies e^{-t/x}u = \dfrac{1}{t}e^{-t/x} + C$

$u = \dfrac{1}{t} + Ce^{t/x} \quad \implies \dfrac{1}{y} = \dfrac{1}{t} + Ce^{t/x} \quad \implies y = \dfrac{1}{\frac{1}{t} + Ce^{t/x}} \quad \implies y = \dfrac{t}{1 + tCe^{t/x}}$ on $(0, \infty)$

9. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

First order Separable.

$\dfrac{1}{\sqrt{1-y^2}}dx = xdx \quad \implies \sin^{-1}(y) = -\dfrac{1}{2}x^2 + C \quad \implies y = \sin\left(C - \dfrac{1}{2}x^2\right)$ on $(-\infty, \infty)$

Explicit Solution

10. $(5y - 2x)\ dy - (2y)\ dx = 0$

$\dfrac{\partial}{\partial y}\Big(-2y\Big) = -2$

$\dfrac{\partial}{\partial x}\Big(5y - 2x\Big) = -2$

First order Exact. (it is also 1-homogeneous)

$F(x, y) = \displaystyle\int -2y\ dx + g(y) = -2xy + g(y)$

$\dfrac{\partial}{\partial y}\Big(F(x, y)\Big) = -2x + g'(y) \quad \implies g'(y) = 5y \quad \implies g(x) = \dfrac{5y^2}{2}$

$\implies d\Big(\dfrac{5y^2}{2} - 2xy\Big) = 0 \quad \implies \dfrac{5y^2}{2} - 2xy = C$

Implicit Solution

11. $(x)\ dx + (y - 2x)\ dy = 0$

First order, 1-homogeneous

$$x = vy \implies dx = v\ dy + y\ dv \implies (vy)(v\ dy + y\ dv) + (y - 2vy)\ dy \implies (vy^2)\ dv + (v^2y + y - 2vy)\ dy = 0$$

New ODE is first order Separable

$$vy^2\ dv = -y(v^2 - 2v + 1)\ dy \implies -\frac{v}{(v-1)^2}\ dv = \frac{1}{y}\ dy \implies \frac{1}{v-1} - \ln|v-1| + C = \ln|y|$$

$$\implies y = \frac{Ce^{1/(v-1)}}{v-1} \implies y = \frac{Ce^{1/((x/y)-1)}}{(x/y)-1} = \frac{Cye^{y/(x-y)}}{x-y}$$

Implicit solution

12. $\dfrac{dy}{dx} = \sqrt{y}$

First order autonomous.

$$y^{-1/2}\ dy = 1\ dx \implies 2y^{1/2} = x + C \implies y = \left(\frac{x}{2} + C\right)^2 \text{ on } (-\infty, \infty)$$

Explicit Solution

13. $(y^2 + yx)\ dx + (x^2)\ dy = 0$

First order, 2-homogeneous

$$y = ux \implies dy = u\ dx + x\ du \implies (u^2x^2 + ux^2)\ dx + (x^2)(u\ dx + x\ du) = 0 \implies (u^2x^2 + 2ux^2)\ dx + (x^3)\ du = 0$$

New ODE is first order Separable

$$\frac{1}{u^2 + 2u}\ du = -\frac{1}{x}\ dx \implies \left(\frac{1}{2u} - \frac{1}{2(u+2)}\right) = -\frac{1}{x}\ dx \implies \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| = -\ln|x| + C$$

$$\implies \ln\left|\frac{u}{u+2}\right| = \ln\left|\frac{1}{x^2}\right| + C \implies \frac{u}{u+2} = \frac{C}{x^2} \implies \frac{y/x}{y/x+2} = \frac{C}{x^2} \implies \frac{y}{y+2x} = \frac{C}{x^2}$$

Implicit solution

14. $(3+3x^2)\dfrac{dy}{dx} = 2xy(y^3-1)$

$$\frac{dy}{dx} + \left(\frac{2x}{3+3x^2}\right)y = \left(\frac{2x}{3+3x^2}\right)y^4$$

First order, Bernoulli with $n = 4$

$$u = y^{-3} \quad \Longrightarrow y = u^{-1/3} \quad \Longrightarrow \frac{dy}{dx} = -\frac{1}{3u^{4/3}}\frac{du}{dx}$$

$$-\frac{1}{3u^{4/3}}\frac{du}{dx} + \left(\frac{2x}{3+3x^2}\right)u^{-1/3} = \left(\frac{2x}{3+3x^2}\right)u^{-4/3} \quad \Longrightarrow \frac{du}{dx} + \left(-\frac{2x}{1+x^2}\right)u = \left(-\frac{2x}{1+x^2}\right)$$

New ODE is first order Linear

$$\Longrightarrow P(x) = \left(-\frac{2x}{1+x^2}\right) \quad \Longrightarrow \mu(x) = e^{\int\left(-\frac{2x}{1+x^2}\right)\,dx} = e^{-\ln|1+x^2|} = \frac{1}{1+x^2}$$

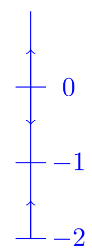
$$\frac{1}{1+x^2}\frac{du}{dx} + \left(-\frac{2x}{(1+x^2)^2}\right)u = \left(-\frac{2x}{(1+x^2)^2}\right) \quad \Longrightarrow \frac{d}{dx}\left(\frac{1}{1+x^2}u\right) = -\frac{2x}{(1+x^2)^2}$$

$$\Longrightarrow u = (x^2+1)\left(\frac{1}{1+x^2}\right) + C(x^2+1) \quad \Longrightarrow u = Cx^2+C+1 \quad \Longrightarrow y = \frac{1}{\sqrt[3]{Cx^2+C+1}}$$

Explicit Solution defined wherever $Cx^2+C+1 \neq 0$

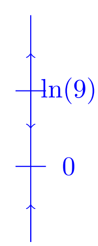
15. $\dfrac{dy}{dx} = y\ln(y+2)$

First order autonomous
Critical points are $y=-1$ and $y=0$
$y=-1$ is an attractor and $y=0$ is a repeller.



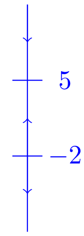
16. $\dfrac{dz}{dx} = \dfrac{ze^z - 9z}{e^z}$

First order autonomous
Critical points are $z=0$ and $z=\ln(9)$
$z=0$ is an attractor and $z=\ln(9)$ is a repeller.



17. $\dfrac{dy}{dx} = 10 + 3y - y^2$

First order autonomous
Critical points are $y = -2$ and $y = 5$
$z = 5$ is an attractor and $y = -2$ is a repeller.



18. $x^2 \dfrac{dy}{dx} + xy = y$ y(-1) = -1

First order Separable.

$$\frac{1}{y}dy = \frac{(1-x)}{x^2}dx \implies \ln|y| = -\frac{1}{x} - \ln(x) + C \implies y = C\left(\frac{1}{xe^{1/x}}\right) \text{ on } (0,\infty)$$

Explicit Solution

$$y(-1) = -1 = C(-e) \implies C = \frac{1}{e} \implies y = \frac{1}{xe^{1/x-1}}$$

$$f(x,y) = \frac{y - xy}{x^2} \quad \frac{\partial f}{\partial y} = \frac{1}{x^2} - \frac{1}{x}$$

Both are continuous when $x \neq 0$, so the solution to our IVP is unique.

19. $x\dfrac{dy}{dx} + y = 4x + 1$ y(1) = 8

First order Linear.

$$\frac{dy}{dx} + \frac{1}{x}y = 4 + \frac{1}{x} \implies P(x) = \frac{1}{x} \implies \mu(x) = e^{\int (1/x)\ dx} = e^{\ln|x|} = x$$

$$x\frac{dy}{dx} + y = 4x + 1 \implies \frac{d}{dx}(xy) = 4x + 1 \implies xy = 2x^2 + x + C \implies y = 2x + 1 + \frac{C}{x} \text{ on } (0,\infty)$$

Explicit Solution

$$y(1) = 8 = 3 + C \implies C = 5 \implies y = 2x + 1 + \frac{5}{x}$$

$$f(x,y) = 4 + \frac{1}{x} - \frac{y}{x} \quad \frac{\partial f}{\partial y} = -\frac{1}{x}$$

Both are continuous when $x \neq 0$, so the solution to our IVP is unique.

20. Removed due to difficulty

21. $\sin(x)\ dx + y\ dy = 0$ y(0) = -1

First order Separable.

$ydy = -\sin(x)dx \quad \Longrightarrow \frac{1}{2}y^2 = \cos(x) + C \quad \Longrightarrow y^2 = 2\cos(x) + C$

Implicit Solution

From the initial condition, we know we need $(0, -1)$ to be a point on our curve, hence $y = -\sqrt{C - 2\cos(x)}$

$y(0) = (-1) = \sqrt{C + 2\cos(0)} \quad \Longrightarrow C = -1 \quad \Longrightarrow y = \sqrt{2\cos(x) - 1}$ on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$

$f(x, y) = \frac{-\sin(x)}{y} \quad \frac{\partial f}{\partial y} = \frac{\sin(x)}{y^2}$

Both are continuous when $y \neq 0$, so the solution we found to our IVP is unique.

22. $(x^2 + y^2 - 5)\ dx + (y + xy)\ dy = 0 \qquad$ y(0) = 1

$$\frac{\partial}{\partial y}\Big(x^2 + y^2 - 5\Big) = 2y$$

$$\frac{\partial}{\partial x}\Big(y + xy\Big) = y$$

First order not Exact, not linear, not homogeneous, not separable, not autonomous. Seems 'almost' exact

Integrating Factor: $\dfrac{2y - y}{y + xy} = \dfrac{1}{1 + x} \qquad \Longrightarrow\ \mu(x) = e^{\int 1/(x+1)\ dx} = e^{\ln|x+1|} = x + 1$

$$\Longrightarrow\ (x+1)(x^2 + y^2 - 5)\ dx + (x+1)(y + xy)\ dy = 0$$

$$\frac{\partial}{\partial y}\Big((x+1)(x^2 + y^2 - 5)\Big) = 2xy + 2y$$

$$\frac{\partial}{\partial x}\Big((x+1)(y + xy)\Big) = y + yx + xy + y = 2xy + 2y$$

New ODE is first order exact.

$$F(x,y) = \int x^2y + 2xy + y\ dy + g(x) = \frac{1}{2}x^2y^2 + xy^2 + \frac{1}{2}y^2 + g(x)$$

$$\frac{\partial}{\partial x}\Big(F(x,y)\Big) = xy^2 + y^2 + g'(y) = x^3 + xy^2 - 5x + x^2 + y^2 - 5 \qquad \Longrightarrow\ g'(y) = x^3 - 5x + x^2 - 5$$

$$\Longrightarrow\ g(x) = \frac{1}{4}x^4 - \frac{5}{2}x^2 + \frac{1}{3}x^3 - 5x \qquad \Longrightarrow\ d\Big(xy^2 + y^2\frac{1}{4}x^4 - \frac{5}{2}x^2 + \frac{1}{3}x^3 - 5x\Big) = 0$$

$$xy^2 + y^2 + \frac{1}{4}x^4 - \frac{5}{2}x^2 + \frac{1}{3}x^3 - 5x = C \qquad \Longrightarrow\ 12xy^2 + 12y^2 + 3x^4 - 30x^2 + 4x^3 - 60x = C$$

Implicit Solution

$$y(0) = 1 \qquad \Longrightarrow\ 0 + 12 + 0 - 0 + 0 - 0 = C$$

$$12xy^2 + 12y^2 + 3x^4 - 30x^2 + 4x^3 - 60x = 12$$

$$f(x,y) = \frac{5 - x^2 - y^2}{y + xy} \quad \frac{\partial f}{\partial y} = \frac{(-2y)(y + xy) - (5 - x^2 - y^2)(1 + x)}{(y + xy)^2}$$

Both are continuous when $y + xy = (y+1)x \neq 0$,
so the solution we found to our IVP is unique when when this occurs .

23. $xy^2 \dfrac{dy}{dx} = y^3 - x^3 \qquad \text{y}(1) = 2$

$xy^2\ dy = (y^3 - x^3)\ dx$

First order, 3-homogeneous

$y = ux \quad \Longrightarrow\ dy = u\,dx + x\,du \quad \Longrightarrow\ (u^2x^3)(u\,dx + x\,du) = (u^3x^3 - x^3)\,dx \quad \Longrightarrow\ (u^2x^4)\,du = (u^3x^3 - x^3 - u^3x^3)\,dx$

New ODE is first order Separable

$u^2\ du = -\dfrac{1}{x}\ dx \quad \Longrightarrow\ \dfrac{u^3}{3} = \ln\left|\dfrac{1}{x}\right| + C \quad \Longrightarrow\ u^3 = \ln\left|\dfrac{1}{x^3}\right| + C$

$\dfrac{y^3}{x^3} = \ln\left|\dfrac{1}{x^3}\right| + C \quad \Longrightarrow\ y = \sqrt[3]{x^3 \ln\left|\dfrac{1}{x^3}\right| + Cx^3}$

Explicit solution

$y(1) = 2 \quad \Longrightarrow\ \frac{2^3}{1} = \ln|1| + C \quad \Longrightarrow\ C = 8$

$y = \sqrt[3]{x^3 \ln\left|\dfrac{1}{x^3}\right| + Cx^3}$ on $(0, \infty)$

$f(x, y) = \dfrac{y^3 - x^3}{xy^2} \quad \dfrac{\partial f}{\partial y} = \dfrac{(3y^2)(xy^2) - (2xy)(y^3 - x^3}{(xy^2)^2}$

Both are continuous when $x, y \neq 0$, so the solution we found to our IVP is unique

24. $\sqrt{y}\dfrac{dy}{dx} + y^{3/2} = 1 \qquad \text{y}(1) = 1/2$

$\dfrac{dy}{dx} + y = y^{-1/2}$

First order, Bernoulli with $n = -1/2$

$u = y^{3/2} \quad \Longrightarrow \ y = u^{2/3} \quad \Longrightarrow \ \dfrac{dy}{dx} = \dfrac{2}{3u^{1/3}}\dfrac{du}{dx}$

$\dfrac{2}{3u^{1/3}}\dfrac{du}{dx} + u^{2/3} = \dfrac{1}{u^{1/3}} \quad \Longrightarrow \ \dfrac{du}{dx} + \dfrac{3}{2}u = \dfrac{3}{2}$

New ODE is first order Linear

$\Longrightarrow \ P(x) = \dfrac{3}{2} \quad \Longrightarrow \ \mu(x) = e^{\int (3/2)\ dx} = e^{3x/2}$

$e^{3x/2}\dfrac{du}{dx} + \dfrac{3}{2}e^{3x/2}u = \dfrac{3}{2}e^{3x/2} \quad \Longrightarrow \ \dfrac{d}{dx}\left(e^{3x/2}u\right) = \dfrac{3}{2}e^{3x/2} \quad \Longrightarrow \ e^{3x/2}u = e^{3x/2} + C$

$\Longrightarrow \ u = 1 + Ce^{-3x/2} \quad \Longrightarrow \ y^{3/2} = 1 + Ce^{-3x/2} \quad \Longrightarrow \ y = \left(1 + Ce^{-3x/2}\right)^{2/3}$ on $(0, \infty)$

Explicit solution

$y(1) = \dfrac{1}{2} = (1 + C)^{2/3} \ \Longrightarrow \ C = \dfrac{1 - \sqrt{8}}{\sqrt{8}}$

$f(x, y) = \dfrac{1}{\sqrt{y}} + y \quad \dfrac{\partial f}{\partial y} = -\dfrac{1}{2y^{3/2}} + 1$

Both are continuous when $y \neq 0$, so the solution we found to our IVP is unique.