Animal Crossing: Exploring Diffusive Lotka-Volterra Ecosystems

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1 Introduction

Modeling ecosystems accurately was a struggle in biomathematics since the inception of the field. In 1910, the equations were initially proposed by Alfred J. Lotka, to explain autocatalytic chemical reactions[1]. In 1926, the same equations were independently developed by the Italian mathematician and physicist Vito Volterra to help the work of marine biologist Umberto D'Ancona, who was examining fish populations in the Adriatic Sea during World War 1[2]. Volterra's insights into these interactions would never come to light had it not been for Ancona courting Volterra's daughter.

The original equations described by Alfred J. Lotka and Vito Volterra are as follows:

$$x_{i+1} = x_i + \Delta t \frac{dx}{dt}_i$$

$$\frac{dx}{dt}_i = \alpha x_i - \beta x_i y_i$$

$$y_{i+1} = y_i + \Delta t \frac{dy}{dt}_i$$

$$\frac{dy}{dt}_i = \delta x_i y_i - \gamma y_i$$

where x corresponds to prey, y corresponds to predators, α corresponds to the reproductive rate of prey, β corresponds to the rate at which the prey are eaten, and δ corresponds to the death rate of the predators.

In the diffusive predator-prey model, the animals are allowed to move along an island. Here, a one dimensional island was used. To accomplish this, a few modifications are made to the original equations:

$$x_{i+1} = x_i + \Delta t \frac{dx}{dt}_i$$

$$\frac{dx}{dt}_i = k \frac{d^2x}{dz^2}_i + x_i - x_i y_i$$

$$\frac{d^2x}{dz^2}_i = \frac{x_{i-1} + x_{i+1} - 2x_i}{\Delta z^2}$$

$$y_{i+1} = y_i + \Delta t \frac{dy}{dt}_i$$

$$\frac{dy}{dt}_i = j\frac{d^2y}{dz^2}_i + xy_i - \psi y_i$$

$$\frac{d^2y}{dz^2}_i = \frac{y_{i-1} + y_{i+1} - 2y_i}{\Delta z^2}$$

where k and j are the diffusion rates of prey and predators respectively, ψ corresponds to the death rate of the predators, z corresponds to the position on the island, and Δz corresponds to the division of gridpoints along the island. The second derivatives of position come from the finite difference approximation.

In modeling this ecosystem, only a few variables can be modified: k, j, Δz, Δt, the size of the island, and the initial placement of the animal populations. For the results presented below, the following values were used to represent a stable ecosystem.

k	j	Δz	Δt	Island size	ψ
1.6	1.5	1	0.1	500m	2

For the initial placement of the animal populations, the predators and prey were placed 10 meters apart, with 2000 prey and 200 predators.

Of these variables, k j and ψ were then modified to see how they affected the ecosystem as a whole.

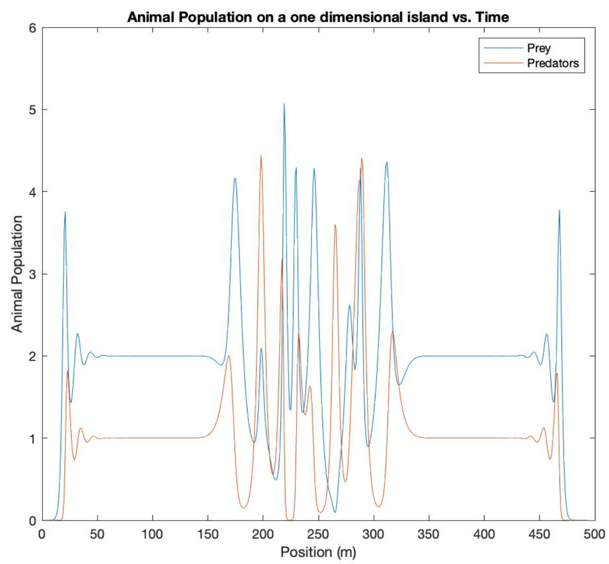
2 The Diffusive Model

2.1 Regular ecosystems

In nature, predators and prey experience a cyclical relationship where the combined population of the system is stable, only increasing slightly over time. For the values listed below, the following patterns of the ecosystem emerge.

k	j	Δz	Δt	Island size	ψ
1.6	1.5	1	0.1	500m	2

It is evident that with this ecosystem, the total population of both predators and prey is reacting to one another, and remains relatively constant through the course of the simulation. At the beginning of the simulation, the prey population skyrockets, as the predators have not reached them yet. The predator population thus gets very low. After the predators get close enough to the prey the reverse happens, as the predator population skyrockets with an enormous amount of prey suddenly available to them. This cycle continues until the amplitude of these oscillations is almost the same, and the ecosystem stabilizes. This is evident on the outer ranges of figure 1a, from 50m-150m and 350m-450m.



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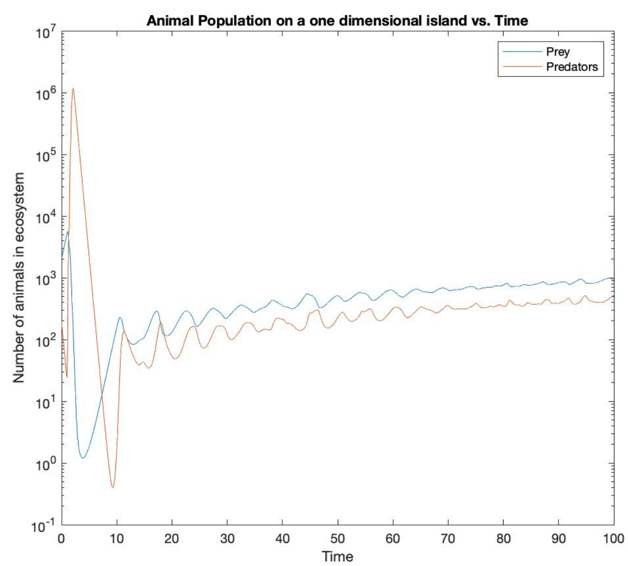

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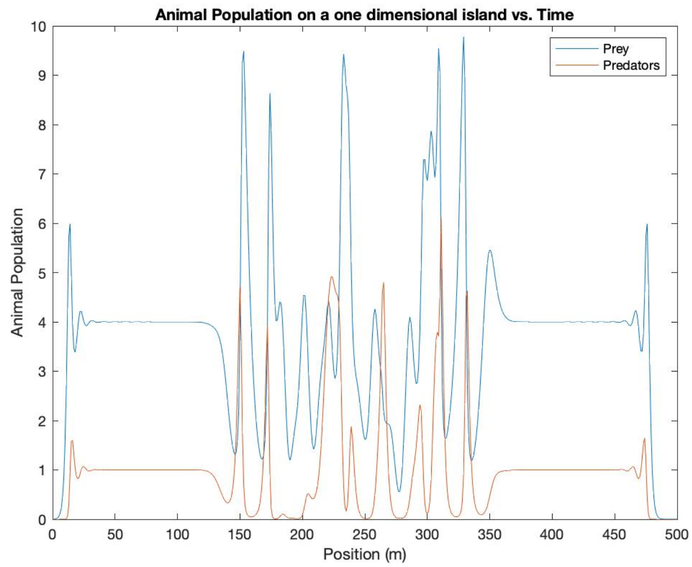
Figure 1: (a) Graph of animal population vs. position on the island at $t = 100$. (b) Graphs of the individual populations of predators and prey, showing how they vary over time.

2.2 Modifying the death rate

Now, a disease is introduced to the ecosystem that affects the predators and not the prey. This results in an increased death rate for the predators. Using the same initial population distributions, doubling the death rate of the predators yielded the following result.

k	j	Δz	Δt	Island size	ψ
1.6	1.5	1	0.1	500m	4

Modifying the death rate in this way did not impact the overall stability of the system. It still exhibits the chaotic nature in the middle of the island, but the outer edges of the island still show a stability, which is evident in the constant population counts in figure 2a over the ranges of 40-110m and 360-450m.



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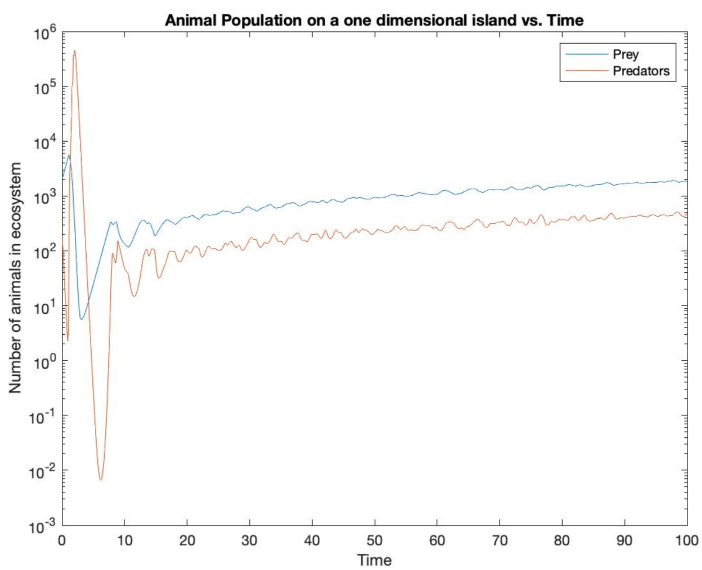

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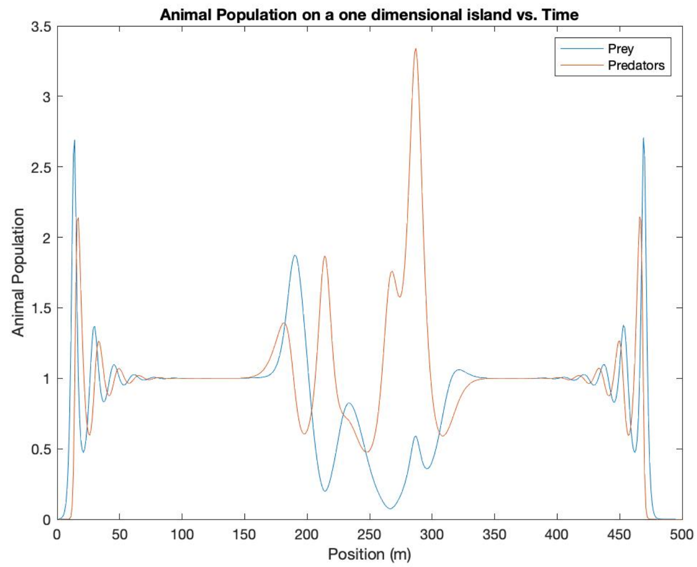
Figure 2: (a) Graph of animal population vs. position on the island at $t = 100$ after increasing the death rate. (b) Graphs of the individual populations of predators and prey, showing how they vary over time after increasing the death rate.

The predators over time, could develop a resistance to the disease. This would result in lower death rates for the predators. Using the same initial population distributions, halving the initial death rate yielded the following result.

k	j	Δz	Δt	Island size	ψ
1.6	1.5	1	0.1	500m	1

Modifying the death rate of the predators begins to yield interesting results. At the edges of the populations, the populations vary much more than the high death rate before reaching an equilibrium. It is interesting to note that over this equilibrium range, the populations of predators and prey are roughly equal, which is evident by the similar population numbers in figure 2b after $t = 20$. In the central area of the island, it still shows the chaotic competition for food that was observed in previous simulations.



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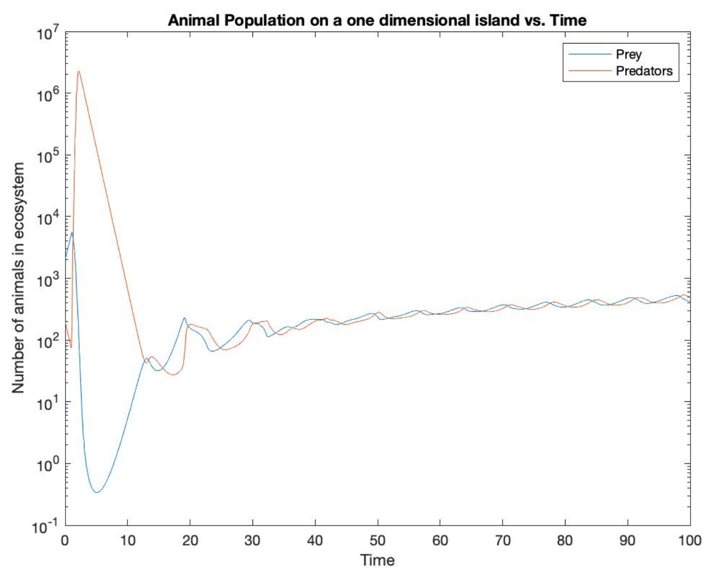

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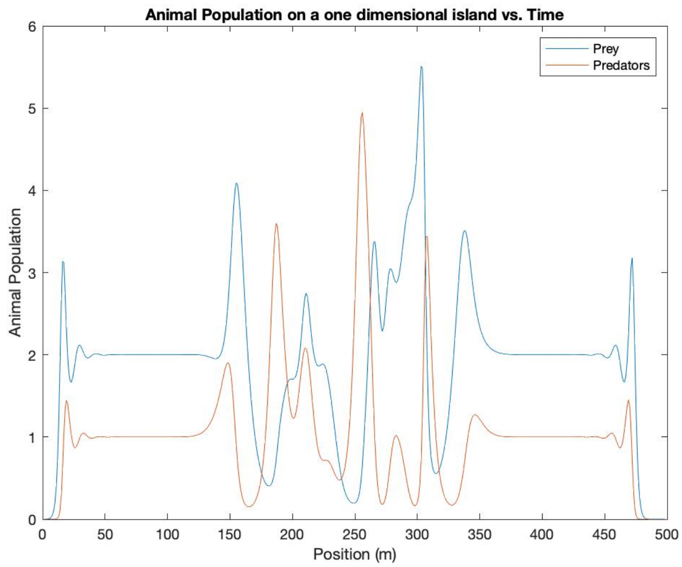
Figure 3: (a) Graph of animal population vs. position on the island at $t = 100$ after decreasing the death rate of predators. (b) Graphs of the individual populations of predators and prey, showing how they vary over time after decreasing the death rate of predators.

2.3 Modifying the predator diffusion rate

How fast the predators diffuse along the island is an important factor in understanding the relationship between the predators and prey. Using the same initial population distributions, doubling the original predator diffusion rate yielded the following result.

k	j	Δz	Δt	Island size	ψ
1.6	3	1	0.1	500m	2

Modifying the predator diffusion rate in this way yields an intricate system, where the predators move faster than the prey, and can subsequently compete with the pray to eat them as they diffuse, instead of lagging behind in previous simulations of their position over time. They still show an equilibrium over and outer range, and have more regions of competition in the middle of the island, where they can overtake and eat the prey easier. This is evident by figure 4b, where the total population of the system increases over time, but only slightly when compared to figure 1b.



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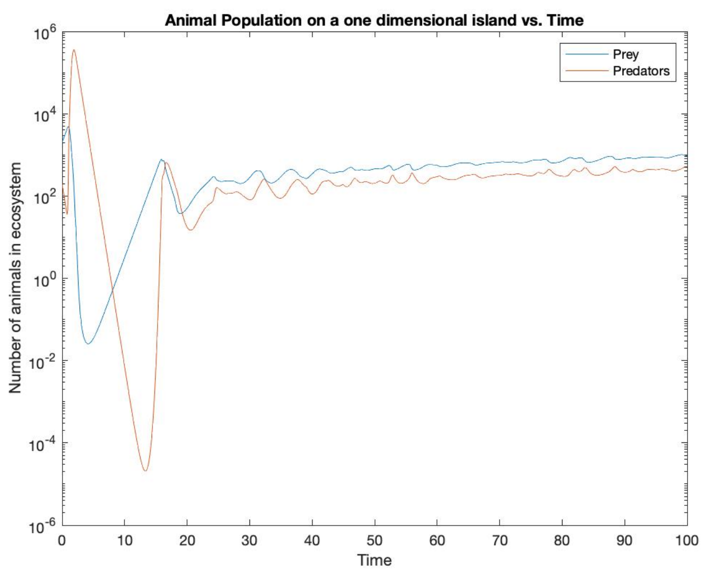

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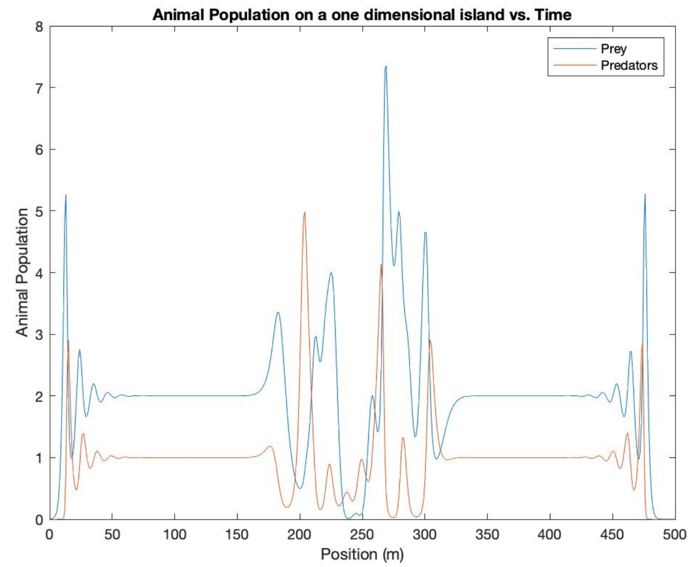
Figure 4: (a) Graph of animal population vs. position on the island at $t = 100$ after increasing the predator diffusion rate. (b) Graphs of the individual populations of predators and prey, showing how they vary over time after increasing the predator diffusion rate.

Decreasing the diffusion rate of the predators would also have an affect on the ecosystem, allowing for higher population rates of prey. Using the same initial population distributions, halving the original predator diffusion rate yielded the following result.

k	j	Δz	Δt	Island size	ψ
1.6	0.75	1	0.1	500m	2

Modifying the diffusion rate in this way produces similar results to decreasing the death rate of the predators. In figure 5a, it shows more oscillations on the fringes of the populations before it reaches stable counts of each population. Figure 5b shows the total counts of each population, which oscillate with respect to each other at lower amplitudes. The total population of the system can also be seen to be increasing steadily over time.



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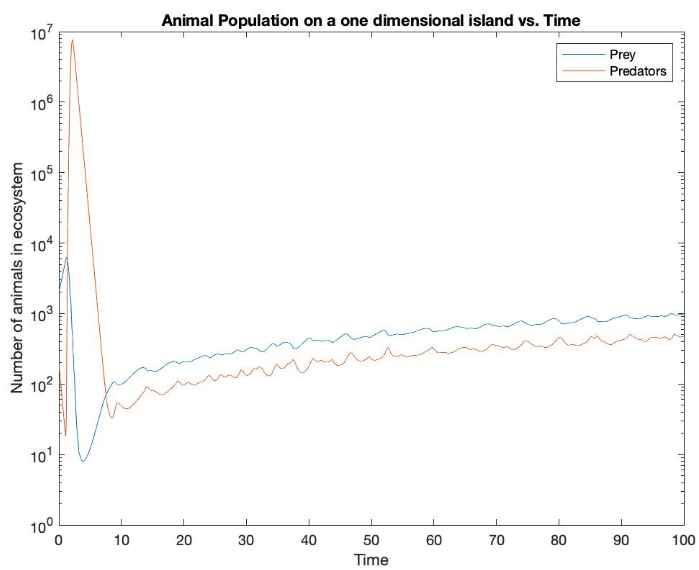


(b)

Figure 5: (a) Graph of animal population vs. position on the island at $t = 100$ after decreasing the predator diffusion rate. (b) Graphs of the individual populations of predators and prey, showing how they vary over time after decreasing the predator diffusion rate.

2.4 Modifying the prey diffusion rate

How fast the prey diffuse along the island is an important factor in understanding the relationship between the predators and prey. Using the same initial population distributions, doubling the original prey diffusion rate yielded the following result.

k	j	Δz	Δt	Island size	ψ
3.2	1.5	1	0.1	500m	2

In increasing the diffusion rate of the prey, it is important to understand the type of island we are dealing with. This one dimensional island is surrounded by lava, and in the event that a predator chases prey off of the island, it is considered to have died as it is outside the bounds of the simulation. As a result of this, both the edges of the island experience the same effect as seen at the edges of the competitive zone, near the middle of the island. The effect isn't as pronounced, as only cycle is ever present at a time. This does result in a much larger region of equilibrium populations, which is visible in figure 6a. Figure 6b shows the total population counts, where something interesting emerges. As time increases, the change in the total population of the system decreases as the total population seems to approach a single value. This would imply that if the simulation continued forever, a carrying capacity of the island would be reached. The death rate of the predators appears to be proportional to the equilibrium populations also, as the ratio between the populations in the equilibrium zone is observed to be

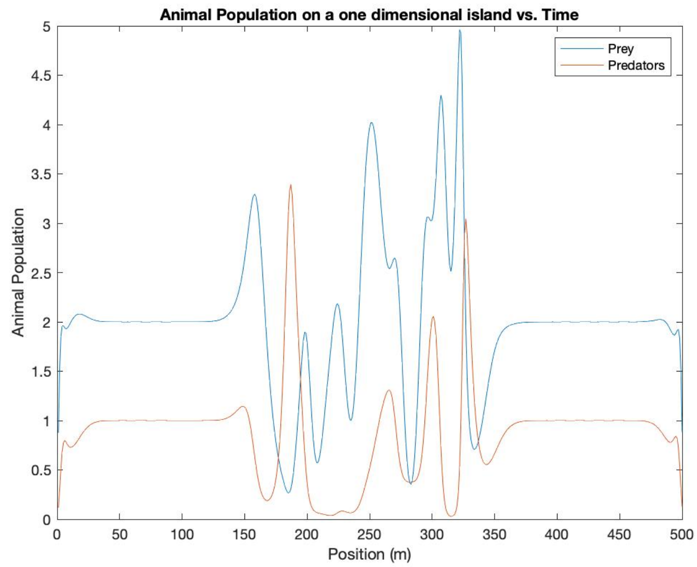
$$\frac{P_{predator}}{P_{prey}} = \frac{1}{\psi}$$

This conclusion can also be drawn from figures 2a, 3a, and 4a, where the equilibrium populations follow this rule. From this, the carrying capacity populations of the predators and prey can be determined from the product of the total number of grid points and the population of these points. Thus, the carrying capacity populations are derived as

$$P_{predator} = L\Delta z$$

$$P_{prey} = \psi L\Delta z$$

The capacities of this system can then be shown as $P_{predator} = 500$ and $P_{prey} = 1000$.



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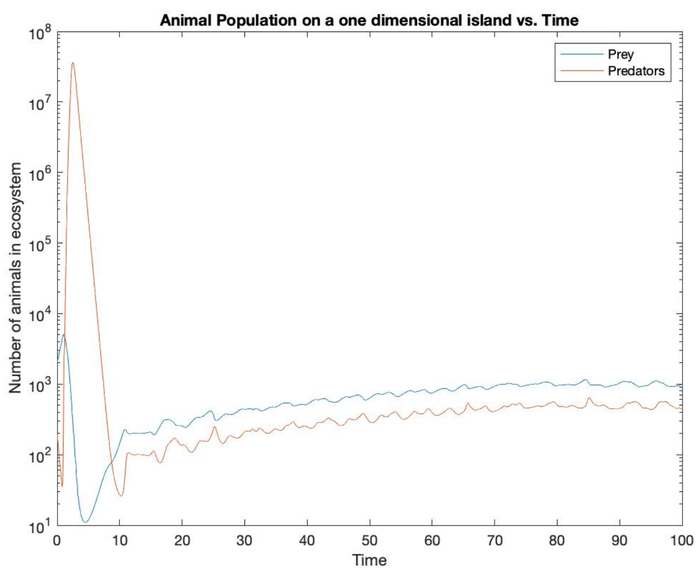

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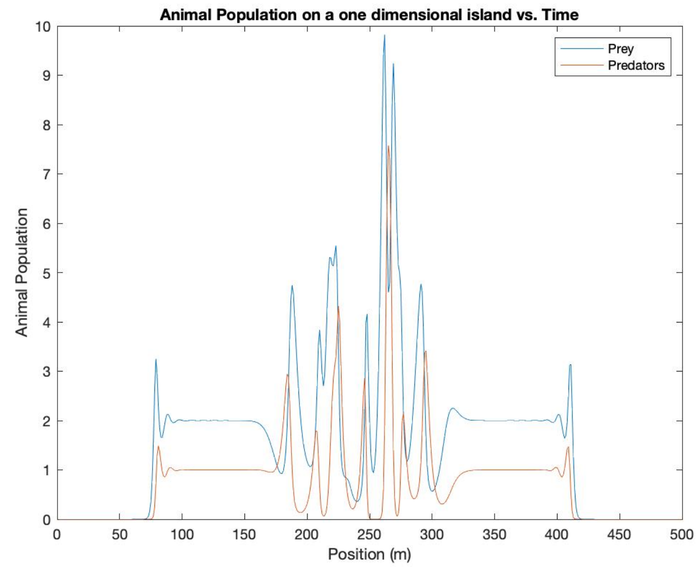
Figure 6: (a) Graph of animal population vs. position on the island at $t = 100$ after increasing the prey diffusion rate. (b) Graphs of the individual populations of predators and prey, showing how they vary over time after increasing the prey diffusion rate.

Decreasing the diffusion rate of the prey would also have an affect on the ecosystem, allowing for higher population rates of predators as they can move to the prey quicker than originally intended. Using the same initial population distributions, halving the original prey diffusion rate yielded the following result.

k	j	Δz	Δt	Island size	ψ
0.8	1.5	1	0.1	500m	2

Modifying the diffusion rates in this way had one immediate change on the system, where the right side of the island had a higher population of prey than the left side of the island. This is significant due to the fact that the prey were initially placed on the left side of the island. It also took more time for the equilibrium zone to appear, which can be explained by the slower diffusion rate of prey. This resulted in much less diffusion over the island, which can be observed in figure 7a. In figure 7b, the total population is increasing slowly and starting to show characteristics of a carrying capacity in the ecosystem, and the rate of change in the populations begins to decrease as time increases.



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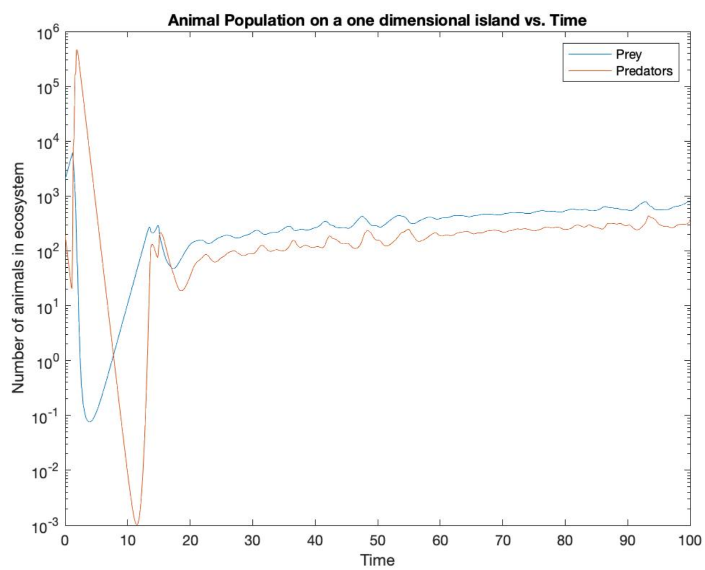

(b)

Figure 7: (a) Graph of animal population vs. position on the island at $t = 100$ after decreasing the prey diffusion rate. (b) Graphs of the individual populations of predators and prey, showing how they vary over time after decreasing the prey diffusion rate.

3 Conclusion

The model designed responds as expected, with predators and prey proportionally increasing and decreasing in response to each others proximity. It shows two main areas on the island. The first is a central zone of competition, where the predator and prey populations vary with respect to one another. The second are is on the outer ranges of the populations, where an equilibrium population of both predators and prey can be observed. The ratio between the two populations is proportional to the death rate of the predators, ψ. Other notable inferences on the system include a carrying capacity emerging when the animals diffuse entirely along the island, and the proportionality to the predator death rate ψ. Improvements and future work with this model could expand to a two dimensional island, with matrices updating both total counts and x, y positions. Additionally, a three animal ecosystem could also be explored to investigate how the dynamics of such a system would behave.

List of Figures

References

[1] A. J. Lotka, "Contribution to the theory of periodic reactions," *The Journal of Physical Chemistry*, vol. 14, no. 3, p. 271–274, 1909.

[2] V. Volterra, "Variazioni e fluttuazioni del numero d'individui in specie animali conviventi," *Accademia dei Lincei*, vol. 2, p. 31–113, 1926.

4 Appendix

4.1 Main Function

```
    clear

% The program below shows how an ecosystem with 2 predators

L=500; % The size of the island
dx = 1; %
tend = 100;
```

```
del_t = 0.1;
timevec = 0:del_t:tend;
distvec = 0:dx:L; % initial values, these produce the most interesting
    results! (if you ask me)

predposvec = zeros(tend/del_t,L/dx);
preyposvec = zeros(tend/del_t,L/dx);
predcount = zeros(tend/del_t,1);
preycount = zeros(tend/del_t,1); % initializing all the matrices

d_rate = 2; % death rate of foxes
pred_diff = 1.5; % diffusion rate of predators
prey_diff = 1.6; % diffusion rate of prey

preyposvec(1,245) = 2000; % initial conditions for matrices
predposvec(1,255) = 200; % Feel free to modify these two lines to mess
    around with the initial conditions for this program!
preycount(1) = counter(preyposvec(1,:),L/dx);
predcount(1) = counter(predposvec(1,:),L/dx);

for i = 2:tend/del_t+1 % time loop, increasing in time as you go down the
     rows
    for j = 1:L/dx % position loop, increasing in position as you move to
         the right
        if j~=1 && j~=L/dx
            ddprey_dzz = (preyposvec(i-1,j+1)+preyposvec(i-1,j-1)-2*
                preyposvec(i-1,j))/dx^2 ; % second derivative of prey
                position
            ddpred_dzz = (predposvec(i-1,j+1)+predposvec(i-1,j-1)-2*
                predposvec(i-1,j))/dx^2 ; % second derivative of predator
                 position
        else
            if j==1
                ddprey_dzz = (preyposvec(i-1,j+1)-2*preyposvec(i-1,j))/dx
                    ^2 ;
                ddpred_dzz = (predposvec(i-1,j+1)-2*predposvec(i-1,j))/dx
                    ^2 ;
            else
                ddprey_dzz = (preyposvec(i-1,j-1)-2*preyposvec(i-1,j))/dx
                    ^2 ;
                ddpred_dzz = (predposvec(i-1,j-1)-2*predposvec(i-1,j))/dx
                    ^2 ;
            end
        end
        preyposvec(i,j) = preyposvec(i-1,j) + del_t*(prey_diff*ddprey_dzz
            +preyposvec(i-1,j)*(1-predposvec(i-1,j))); % calculating the
            predator and prey at each location
        predposvec(i,j) = predposvec(i-1,j) + del_t*(pred_diff*ddpred_dzz
            +predposvec(i-1,j)*(preyposvec(i-1,j)-d_rate));
        if preyposvec(i,j)<0 % some error checking here
```

```
            preyposvec(i,j) = 0;
        end
        if predposvec(i,j)<0
            predposvec(i,j) = 0;
        end
    end
    preycount(i) = counter(preyposvec(i,:),L/dx); % counting the total
         population in
    predcount(i) = counter(predposvec(i,:),L/dx);
end

% ALL PLOTS BELOW!

semilogy(timevec,predcount) % Plot of the amount of predators in the
     ecosystem and how it changes over time
xlabel ('Time')
ylabel ('Number of predators in ecosystem')
title('Predator Population on a one dimensional island vs. Time')

semilogy(timevec,preycount) % Plot of the amount of prey in the ecosystem
      and how it changes over time
xlabel ('Time')
ylabel ('Number of prey in ecosystem')
title('Prey Population on a one dimensional island vs. Time')

for i=1:tend/del_t % A loop that shows how the ecosystem evolves over
     time
    plot(1:L/dx,preyposvec(i,:),1:L/dx,predposvec(i,:))
    xlabel ('Position (m)')
    ylabel ('Animal Population')
    legend('Prey','Predators')
    title('Animal Population on a one dimensional island vs. Time')
    drawnow
    pause(0.02)
end

semilogy(timevec,preycount,timevec,predcount) % Plot of the amount of
     animals (both pred and prey) in the ecosystem and how it changes
     over time
xlabel ('Time')
ylabel ('Number of animals in ecosystem')
legend('Prey','Predators')
title('Animal Population on a one dimensional island vs. Time')
```

4.2 Counter function

```
    function [retval] = counter(array,length)
retval = 0;
for i = 1:length
    retval = retval + array(i);
```

```
end
end
```