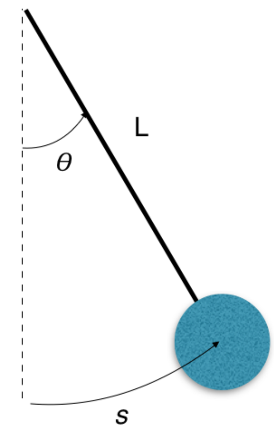
Scientific Computing Assignment 4:
Basic Time-Stepping and the Pendulum



The Physical System

Newton's second law can be applied to a mass hanging from a rigid pivoting rod, leading to the following equations:

$$\theta = s/L$$
$$dv/dt = -g \ \sin\theta$$
$$ds/dt = v$$

Here, θ is the angle the pendulum makes with the vertical, $s(t)$ is the distance the pendulum bob has swung from equilibrium, and $v(t)$ is the velocity of the pendulum bob.
Let's solve this system with the following parameters:

g = 9.8 m/s^2
L = 1 meter
0 < t < 10 seconds
Initial v (t = 0) = 0
Initial s (t = 0) = s_0 = 0.1 meter

There are two important things to know about this system from Intro Physics:

1. The system cannot be solved analytically, the sin θ term is a problem. But if the pendulum bob isn't moving very far (we can make the approximation $s << L$), then a solution can be found:
   $$s(t) \ = s_0 \ \cos(2\,\pi\, t\,/\,T) \tag{1}$$
   where the period (repeat time) of the oscillation is
   $$T \ = \ 2\,\pi\,\sqrt{L/g} \tag{2}$$
2. The total energy of the system (kinetic plus gravitational potential) is conserved and does not change over time. For this system, the total energy per unit mass is:
   $$E \ = \ \tfrac{1}{2}v^2 \ + \ g\,L\,(1 \ - \ \cos\theta) \tag{3}$$

Part 1: The Euler Forward Method

In general, the Euler Forward method solves one or more differential equations of the form

$$du/dt \ = f(u, v, \ ... \ t) \tag{4}$$
$$dv/dt \ = g(u, v, \ ... \ t) \tag{5}$$
… etc…

at timesteps $t_i = i\,\Delta t$ to get the values of *u, v, etc.* at those timesteps as follows:

$$u_{i+1} = u_i + f(u_i, v_i, t_i)\ \Delta t$$
$$v_{i+1} = v_i + g(u_i, v_i, t_i)\ \Delta t$$

Crucially, the right-hand-side “tendency functions” *f(u,v,...,t)* and *g(u,v,...,t)* are evaluated using the values from the *previous timestep*.

Task 1. Write some code that uses Euler forward to solve the pendulum problem as described above.

Question 1. Generate a numerical solution using Δt=0.05 seconds. Plot the analytical solution (using equation 1) and your numerical solution on the same graph. Describe how they differ. Also plot the energy of the system over time: is it conserved?

You should find that your numerical system doesn’t obey the law of conservation of energy, and gradually gains energy over time. Let’s see if this problem can be solved with a smaller \timestep.

Question 2: Generate numerical solutions using various choices of Δt between 10^{-4} and 10^{-1} seconds. Plot the difference in energy between the start and end of the simulation for each of your values. (You can do it by hand if you like.)

Question 3: The position *s* (*t*) behaves strangely when *s* gets bigger than π. Can you explain physically what’s happening here?

Part 2: The Symplectic Euler Method

Symplectic integration is a fancy name, but the idea is simple: having calculated a new up-to-date u_{i+1} in equation (4), we should use it, not the old u_i, to update v_{i+1} in equation (5). Which is to say:

$$u_{i+1} = u_i + f(u_i, v_i, t_i)\ \Delta t$$
$$v_{i+1} = v_i + g(u_{i+1}, v_i, t_i)\ \Delta t$$

Does such a tiny change make a difference?

Task 2: Write a new code to implement this symplectic method. This should be just changing one term on one line.

Question 4: Repeat Questions 1 and 2 using this method. Is it better? Is it perfect?

Question 5: Now that we’ve got a numerical method that isn’t terrible, let’s look at hopefully-real changes in the pendulum’s behavior as we increase the initial displacement s_0. Using your symplectic method with Δt = 0.001 seconds, measure the period (time to repeat) with different initial conditions: s_0 = 0.4, 1, 2, and 3.12 meters. Describe the results, and any differences in behavior from the analytical solution (Equation 1 and 2).