Scientific Computing

Assignment 5: Advanced Time-Stepping and the Chaotic Lorenz Attractor

In 1963, Ed Lorenz published a remarkable paper in the Journal of the Atmospheric Sciences. He was investigating atmospheric convection: when the ground becomes warmer than the air above it, warm air near the ground rises while cold air sinks. This overturning acts to mix the lower atmosphere. There's an interesting interaction here: the overturning becomes stronger when the ground-level air is very warm, but when overturning takes place, it brings cool air down to ground level, reducing ground-level temperature. This is a negative feedback, but a rather complicated one.

Lorenz came up with a highly-simplified set of equations which capture the physics of this phenomenon. Unforunately his choice of variable names isn't very good, but we're stuck with it:

X	Strength of overturning circulation
Y	Warmth of updrafts / cooldness of downdrafts
Z	Warmth of surface air

I'm being a little vague with these definitions, because the actual equations involve Fourier decomposition and other math concepts beyond the scope of this class. Starting from the basic equations of fluid flow and heat transport, Lorenz was able to reduce them down to a set of three coupled differential equations:

$$\frac{dX}{dt} = \sigma(Y - X)$$

$$\frac{dY}{dt} = -XZ + rX - Y$$

$$\frac{dZ}{dt} = XY - bZ$$

(the greek letter in the first equation is "sigma"). Without going into too much detail, there's a three-way interaction between the speed of updrafts and downdrafts, their temperature, and the vertical differences in temperature which drive the updrafts in the first place. There are three tuneable parameters in this system, σ, r, and b, which are related to the amount of friction, the strength of ground heating, and the height-to-width ratio of the circulation, respectively.

This is a more complicated system of equations than we've looked at before, but it's vastly simplified compared to the original equations of fluid flow.

Lorenz made a remarkable discovery about these equations: they are “chaotic”. Mathematically, a chaotic system is defined as one in which infinitesmal changes in the initial state of a deterministic system eventually lead to very large changes in its behavior.

Lorenz speculated that not just convection, but the entire system of global weather as a whole is also chaotic in the same way, and as it turns out, he was right. Our weather forecasting models are limited by this chaotic behavior. Since our instruments are not perfect, we do not know precisely what initial conditions we should give to our weather models. Because weather is chaotic, that means that even the slightest error in measurement eventually leads to a giant error in our forecast. The chaotic nature of weather means that it’s impossible to generate an accurate weather forecast more than a week or two in advance.

In practical terms, this phenomenon is known as the “butterfly effect”: as Lorenz once remarked, “the flap of a butterfly’s wings in Brazil [might] set off a tornado in Texas.”

In this lab, we’ll see this in action, using the Runge Kutta scheme to get an accurate simulation. But first, let's take a look at the Runge-Kutta scheme in the context of the pendulum problem we considered last week.

Runge-Kutta for the Pendulum

Recall the equations of motion for the pendulum from last week:

$$\dot{s} = v$$

$$\dot{v} = -g \sin(s/l)$$

where dots are used to indicate the time derivative.

Task 1: Write a function of the form:

[sdot, vdot] = pendulum(s,v,g,l)

that calculates the "tendency" (time rate of change) for the position and velocity of the pendulum.

Task 2: Write a time-stepping function which performs the "4th-order Runge Kutta" time-stepping scheme on your pendulum:

$$[\dot{s}_A, \dot{v}_A] = \text{pendulum}(s_i, v_i, g, l)$$

$$[\dot{s}_B, \dot{v}_B] = \text{pendulum}(s_i + \dot{s}_A \Delta t/2, v_i + \dot{v}_A \Delta t/2, g, l)$$

$$[\dot{s}_C, \dot{v}_C] = \text{pendulum}(s_i + \dot{s}_B \Delta t/2, v_i + \dot{v}_B \Delta t/2, g, l)$$

$$[\dot{s}_D, \dot{v}_D] = \text{pendulum}(s_i + \dot{s}_C \Delta t, v_i + \dot{v}_C \Delta t, g, l)$$

$$s_{i+1} = s_i + \frac{\dot{s}_A + 2\dot{s}_B + 2\dot{s}_C + \dot{s}_D}{6} \Delta t$$

$$v_{i+1} = v_i + \frac{\dot{v}_A + 2\dot{v}_B + 2\dot{v}_C + \dot{v}_D}{6} \Delta t$$

Question 1: Perform a controlled experiment to test the numerical accuracy of this scheme against the ordinary Euler Forward scheme (non-symplectic) and discuss your results.

Runge-Kutta for the Lorenz System

Task 3: Write a function of the form:

[Xdot, Ydot, Zdot] = lorenzfunction(X, Y, Z, sigma, r, b)

which computes the rate of change of X, Y, and Z over time (their "tendency"), given X, Y, Z, σ, r, and b.

Task 4: Write a time-stepping function which performs the "4th-order Runge Kutta" time-stepping scheme on your Lorenz function:

$$[\dot{X}_A, \dot{Y}_A, \dot{Z}_A] = \text{lorenzfunction}(X_i, Y_i, Z_i, \sigma, r, b)$$
$$[\dot{X}_B, \dot{Y}_B, \dot{Z}_B] = \text{lorenzfunction}(X_i + \dot{X}_A\Delta t/2, Y_i + \dot{Y}_A\Delta t/2, Z_i + \dot{Z}_A\Delta t/2, \sigma, r, b)$$
$$[\dot{X}_C, \dot{Y}_C, \dot{Z}_C] = \text{lorenzfunction}(X_i + \dot{X}_B\Delta t/2, Y_i + \dot{Y}_B\Delta t/2, Z_i + \dot{Z}_B\Delta t/2, \sigma, r, b)$$
$$[\dot{X}_D, \dot{Y}_D, \dot{Z}_D] = \text{lorenzfunction}(X_i + \dot{X}_C\Delta t, Y_i + \dot{Y}_C\Delta t, Z_i + \dot{Z}_C\Delta t, \sigma, r, b)$$

$$X_{i+1} = X_i + \frac{\dot{X}_A + 2\dot{X}_B + 2\dot{X}_C + \dot{X}_D}{6}\Delta t$$
$$Y_{i+1} = Y_i + \frac{\dot{Y}_A + 2\dot{Y}_B + 2\dot{Y}_C + \dot{Y}_D}{6}\Delta t$$
$$Z_{i+1} = Z_i + \frac{\dot{Z}_A + 2\dot{Z}_B + 2\dot{Z}_C + \dot{Z}_D}{6}\Delta t$$

Question 2: Show and describe how the Lorenz system evolves in time, given the following parameters. These parameter values match what Lorenz chose in 1963:

σ = 10
b = 8/3
r = 30

Initial conditions: X = 1, Y = 0, Z = 0
Time: from t = 0 to t = 30, Δt = 0.01

First, plot X, Y, and Z as a function of time. Also, display the trajectory of the system in 3-dimensional space using the "plot3" command. Describe what you see: the result should look like a mys

Question 3: Show that this system is chaotic, that is, that it displays sensitive dependence on initial conditions. Repeat Step 3 making a tiny change in X, Y, or Z (about 0.01 should do it) and plot X vs time for both runs, on the same graph. Show that eventually the two simulations diverge and become completely different.

Question 4: The Lorenz system is not chaotic for all choices of σ, b, and r. Try keeping σ and b constant, but reduce the value of r. At what r-value does the Lorenz system change from non-chaotic to chaotic behavior? Try to explain the changes in the real-world systemthat lead to this transition: refer to the first page for a reminder of what *r* represents.

This transition from non-chaotic to chaotic behavior is another kind of "bifurcation point". We first discussed bifurcations in the climate model from Assignment 2, where we found a transition from 1 steady climate state to 3 as we increased solar brightness.