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Heavy-traffic analysis of mean response time under Shortest Remaining Processing Time

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ABSTRACT

Shortest Remaining Processing time (SRPT) has long been known to optimize the queue length distribution and the mean response time (a.k.a. flow time, sojourn time). As such, it has been the focus of a wide body of analysis. However, results about the heavy-traffic behavior of SRPT have only recently started to emerge. In this work, we characterize the growth rate of the mean response time under SRPT in the M/GI/1 system under general job size distributions. Our results illustrate the relationship between the job size tail and the heavy traffic growth rate of mean response time. Further, we show that the heavy traffic growth rate can be used to provide an accurate approximation for mean response time outside of heavy traffic regime.

1. Introduction

Shortest Remaining Processing Time (SRPT) has long been known to optimize the mean response time (a.k.a. flow time, sojourn time) in a single server queue [1]. As a result, there has been extensive research studying SRPT in a wide variety of models over the last 50 years. For example, SRPT has been studied in the M/GI/1 model [2], with MAP arrivals [3], with setup times [4], and in a variety of other settings [5–7]. Further, there has been renewed interest in SRPT recently as a result of a number of computer system designs based on SRPT-like policies, e.g., web servers [8,9], routers [10,11], wireless networks [12], and beyond. This renewed interest has led to new results studying the tail behavior [13–15], the heavy-traffic behavior [16–18], and the fairness of SRPT [19,20].

Despite this large literature, there are some simple properties of SRPT that are still not well understood. One such property is the focus of this paper: *How does the mean response time under SRPT scale in the heavy-traffic regime?*

It is perhaps surprising that this question is still not understood given its fundamental nature, especially since the mean response time, $E[T]$, was derived for the first time by Schrage and Miller [2] in 1966. However, the formula for the mean response time is complicated enough that the dependence of it on load, ρ, is not well understood. Specifically, there are a few papers that have derived the heavy-traffic growth rate of SRPT under specific job size distributions, i.e., under Exponential job sizes [17] and under Pareto job sizes [21,16]. Additionally, [21] studies the queue length distribution in the heavy-traffic regime ($\rho \to 1$) in the cases of Pareto and rapidly varying job size distributions. Further, for job size distributions with finite support, Down et al. derive a process-level fluid limit for the conditional response time of a tagged job [18]. However, there is no complete characterization of the heavy-traffic growth rate of $E[T]$ under general job size distributions.

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The contribution of the current paper is to provide a characterization of the heavy-traffic growth rate of $E[T]$ in an M/GI/1 preempt–resume queue (see Theorems 1–3). This characterization highlights the relationship between the growth rate and the tail of the job size distribution. Specifically, the heavy-traffic growth rate is shown to depend on the tail of a measure $G(x)$, which characterizes the truncated load, and the Matuszewska index of the job size distribution, which relates to the moment conditions of the job sizes. The results illustrate that SRPT provides an order of magnitude of improvement over other common scheduling policies such as Processor Sharing (PS) and First Come First Served (FCFS) if and only if the job size distribution is unbounded. Further, the results illustrate that a heavier-tail implies a slower growth rate. Additionally, once the tail is "heavy-enough" (i.e. job size have an infinite variance) the growth rate becomes (up to a constant factor) independent of the job size distribution.

In addition to the insight provided by the heavy-traffic growth rate of SRPT, we illustrate that the heavy-traffic analysis can be used to provide a simple approximation of $E[T]$ under SRPT, which is accurate even outside of the heavy-traffic regime. This simple approximation is useful when analyzing more complex models which have pieces that use SRPT. For example, this approximation has already been applied to attain results for a multi-queue load balancing model [22] and a speed scaling model [23]. In each case, analyzing the system using the exact form of $E[T]$ under SRPT would have resulted in only numeric results, but using the heavy-traffic approximation led to analytic results providing new insights.

The remainder of the paper is organized as follows. In Section 2, we introduce our notation, provide some background on the Matuszewska index, and discuss the prior work. In Section 3, we summarize our main results characterizing the heavy-traffic growth rate of $E[T]$ under SRPT. Additionally, we illustrate the application of our main results to two specific distributions: the Pareto and the Weibull, in order to highlight the usefulness of the heavy-traffic results as an approximation outside of the heavy-traffic regime. Next, in Section 4 we show details of the proofs. Finally, Section 5 provides some concluding remarks.

2. Preliminaries

We study the performance of SRPT in an M/GI/1 preempt–resume queue. Under SRPT at every instant, the job with the smallest remaining service time is scheduled. We assume the c.d.f. of job sizes, $F(x)$, is continuous. Denote by $E[T]$ the mean response time (a.k.a. sojourn time) under SRPT, which is the time from when a job enters the system until it completes service. Let $\bar{F}(x) = 1 - F(x)$, λ denote the arrival rate, and $\rho = \lambda E[X]$ be the load. Define $\rho(x) = \lambda \int_0^x t \mathrm{d}F(t)$ and $m_2(x) = \int_0^x t^2 \mathrm{d}F(t)$. Here $\rho(x)$ can be interpreted as the load made up by jobs with size $<x$ (ignoring all jobs with size x), $m_2(x)$ is the second moment of the jobs with size $<x$. Then the conditional mean response time for a job of size x, $E[T(x)]$, under $M/GI/1/SRPT$ was first derived by Schrage and Miller [2] and is equal to

$$E[T(x)] = \int_0^x \frac{\mathrm{d}t}{1-\rho(t)} + \frac{\lambda x^2 \bar{F}(x)}{2(1-\rho(x))^2} + \frac{\lambda m_2(x)}{2(1-\rho(x))^2}, \tag{1}$$

with $E[T] = E[E[T(x)]]$. Despite the existence of this result, due to its complex form, understanding the behavior of $E[T]$ under SRPT is difficult. For example, it is hard to determine the impact of job size variability and load on this formula. Further, calculating $E[T]$ numerically is non-trivial. The goal of this paper is to provide insight into the behavior of $E[T]$ by studying SRPT in heavy traffic.

Our main results are described in asymptotic notation. We say $f(x) = O(g(x))$ as $x \to a$, if and only if $\limsup_{x\to a} |f(x)/g(x)| < \infty$. Similarly, $f(x) = \Theta(g(x))$ denotes $0 < \liminf_{x\to a} |f(x)/g(x)| \le \limsup_{x\to a} |f(x)/g(x)| < \infty$, and $f(x) = o(g(x))$ denotes $\lim_{x\to a} |f(x)/g(x)| = 0$. To denote more accurate asymptotics, we write $f(x) \sim g(x)$ if and only if $\lim_{x\to a} |f(x)/g(x)| = 1$, and $f(x) \preceq g(x)$ if and only if $\limsup_{x\to a} |f(x)/g(x)| \le 1$. Other notations $\succeq$, $\prec$ and $\succ$ are defined in a similar way.

There is only a limited amount of prior work studying the heavy-traffic behavior of SRPT. Recently, Bansal [17] characterized the heavy-traffic behavior of SRPT in the M/M/1. Specifically, he proves that for $\rho \in [2/3, 1)$ in $M/M/1$, $\frac{1/(18e)}{\mu(1-\rho)\log(1/(1-\rho))} \le E[T] \le \frac{7}{\mu(1-\rho)\log(1/(1-\rho))}$. Soon after, Bansal and Gamarnik [16] studied the mean response time of SRPT in an M/GI/1 queue with a Pareto job size distribution and showed that the heavy-traffic growth rate is $O((1-\rho)^{-(\alpha-2)/(\alpha-1)})$ if $\alpha > 2$ and $O(\log(1/(1-\rho)))$ if $1 < \alpha < 2$. Pechinkin [21] studies the heavy-traffic queue length distribution under SRPT given Pareto job sizes and rapidly varying job size distributions and characterizes the distributional limit in each case. Pechinkin does not explicitly consider the growth rate of $E[T]$, but does provide results for the case of Pareto job sizes in the text. A different sort of analysis was performed by Down et al., who derive a fluid limit for the conditional response time for a job of size x under job size distributions with finite support [18].

Our work extends [17,21,16] to the general M/GI/1 setting and provides an explicit characterization of the impact of the job size distribution on the heavy-traffic growth rate. Our results show that the heavy-traffic behavior of $E[T]$ under SRPT depends on a measure $G(x) = \rho(x)/\rho = \int_0^x t \mathrm{d}F(t)/E[X]$ and the Matuszewska index [24] of the tail of the job size distribution, $\bar{F}(x)$.

Definition 1. Let $f(\cdot)$ be positive,

- The upper Matuszewska index $\alpha(f)$ is the infimum of those α for which there exists a constant $C = C(\alpha)$ such that for each $\Lambda > 1$,

$$f(\lambda x)/f(x) \le C\{1+o(1)\}\lambda^{\alpha} \quad (x \to \infty) \text{ uniformly in } \lambda \in [1, \Lambda];$$

- The lower Matuszewska index $\beta(f)$ is the supremum of those β for which, for some $D = D(\beta) > 0$ and all $\Lambda > 1$,

$$f(\lambda x)/f(x) \ge D\{1+o(1)\}\lambda^{\beta} \quad (x \to \infty) \text{ uniformly in } \lambda \in [1, \Lambda];$$

Intuitively, the Matuszewska index bounds the function $f(x)$ by functions of the form $g(x) = Cx^{\mu}$, for some constant C. A function $f(x)$ with upper Matuszewska index α and lower Matuszewska index β means that $f(x)$ lies roughly between C_1x^{α} and C_2x^{β} as $x \to \infty$. For more details about the Matuszewska index, please refer to [25].

3. Main results and discussion

In this section we present our main results characterizing the heavy-traffic growth rate of $E[T]$ under SRPT. We defer the proofs of the results to Section 4, and instead focus in this section on the interpretation of the results. We divide the discussion into two sections: first we present theorems characterizing the heavy-traffic limits and second we present numeric results illustrating the usefulness of the heavy-traffic limits outside of the heavy-traffic regime.

3.1. Heavy-traffic results

The first theorem we present is a comparison of the heavy-traffic behavior of SRPT to the heavy-traffic behavior of other common policies—Processor Sharing (PS) and First Come First Served (FCFS). Recall that, in an $M/G/1$ system, the mean response time under PS (FCFS) scale with $\frac{1}{1-\rho}$, i.e., $E[T] = \Theta\left(\frac{1}{1-\rho}\right)$, regardless of the job size distribution as long as it has finite first (second) moment [26]. The following theorem shows that the growth rate of $E[T]$ for SRPT can be slower than that of PS and FCFS as $\rho \to 1$. Further, it shows that the growth rate depends on the job size distribution.

Theorem 1. *In an $M/GI/1$ SRPT queue, as $\rho \to 1$,*

$$E[T] = \begin{cases} \Theta\left(\dfrac{1}{1-\rho}\right) & F(x) \text{ has bounded support} \\ o\left(\dfrac{1}{1-\rho}\right) & \text{Otherwise}. \end{cases}$$

Theorem 1 shows that, not only does SRPT minimize $E[T]$, it also provides an improvement which is larger than a constant factor if and only if the job size distribution is unbounded. The following theorem shows that not only does $E[T]$ scale more slowly than PS and FCFS when the job size distribution is unbounded, but the improvement can be exponential. The growth rate of $E[T]$ can be characterized explicitly as follows:

Theorem 2. *In an $M/GI/1$ SRPT queue, if $F(x)$ has unbounded support, then as $\rho \to 1$,*

$$E[T] = \begin{cases} \Theta\left(\dfrac{1}{(1-\rho)G^{-1}(\rho)}\right) & \alpha(\bar{F}) < -2 \\ \Theta\left(\log\left(\dfrac{1}{1-\rho}\right)\right) & \beta(\bar{F}) > -2. \end{cases}$$

Theorem 2 illustrates the precise impact of the job size distribution on the growth rate of $E[T]$. It turns out that the growth rate is determined by $G(\cdot)$ and the Matuszewska index. Both of which are related to the tail of the job size distribution. It shows that job size distributions with heavier tails have $E[T]$ that increases more slowly as $\rho \to 1$. And, once the tail is heavy enough, the growth rate becomes $\Theta(\log(1/(1-\rho)))$, which is "independent" of the distribution (though the constant factor may be different). Note that the exact constant factor for $E[T]$ depends on the distribution. Thus, one cannot hope to provide a constant factor without using explicit information about the job size distribution. However, given special classes of distributions, it is possible to get the exact constant factor for $E[T]$. To illustrate this fact, the following theorem characterizes the asymptotic mean response time for regularly varying job size distributions (RV_{α}), which have $\bar{F}(x) = L(x)x^{\alpha}$ where $L(\cdot)$ is a slowly varying function (i.e. $L(ax)/L(x) \to 1$ as $x \to \infty$ for every $a > 0$).

Theorem 3. *In an $M/GI/1$ SRPT queue, if $\bar{F}(x) \sim RV_{\alpha}$, then as $\rho \to 1$,*

$$E[T] \sim \begin{cases} \dfrac{(\pi/(\alpha+1))}{2\sin(\pi/(\alpha+1))} \cdot \dfrac{E[X^2]}{(1-\rho)G^{-1}(\rho)} & \alpha < -2 \\ \dfrac{1}{(-\alpha)(\alpha+2)} \cdot E[X] \log\left(\dfrac{1}{1-\rho}\right) & \alpha > -2. \end{cases}$$

Note that for a regularly varying distribution $\bar{F}(x) \sim RV_\alpha$, we have $\alpha(\bar{F}) = \beta(\bar{F}) = \alpha$ and so Theorem 2 is still applicable. However, Theorem 3 provides a refined result. This theorem generalizes recent results from Bansal and Gamarnik [16] and Pechinkin [21], who each analyze the Pareto distribution, which is a special case of regularly varying distributions. Additionally, note that this result is valid for tails that are of rapid variation (corresponding to $\alpha = -\infty$), which satisfy $\bar{F}(xy)/\bar{F}(x) \to 0$ as $x \to \infty$ for $y > 1$. Most light-tailed distributions fit into this category.

Theorem 3 can be made even more explicit for the case of $\alpha < -2$ by specifying bounds on the behavior of $G^{-1}(\rho)$. In particular, based on properties of regular variation, we have $\bar{G}^{-1}(1/x) \sim RV_{-1/(\alpha+1)}$, thus there exists a slow varying function $L(x)$ such that $\bar{G}^{-1}(1/x) \sim L(x)x^{-1/(\alpha+1)}$, which implies $G^{-1}(\rho) = \bar{G}^{-1}(1-\rho) \sim L\left(\frac{1}{1-\rho}\right)(1-\rho)^{1/(\alpha+1)}$. Therefore, for $\alpha < -2$, we have the following simple bounds

$$\frac{1}{(1-\rho)^{\frac{\alpha+2}{\alpha+1}-\epsilon}} \prec E[T] \prec \frac{1}{(1-\rho)^{\frac{\alpha+2}{\alpha+1}+\epsilon}} \quad \forall \epsilon > 0.$$

A similar argument holds for the case of $\alpha(\bar{F}) < -2$ in Theorem 2 as well. Note that we can show $\alpha(\bar{G}^{-1}(1/x)) \le -1/(\alpha(\bar{F})+1)$ (see the proof of Theorem 2), which implies $G^{-1}(\rho) < (1-\rho)^{1/(\alpha(\bar{F})+1)-\epsilon}$ as $\rho \to 1$ for arbitrary small constant $\epsilon > 0$. Thus we have $E[T] = \omega((1-\rho)^{-(\alpha(\bar{F})+2)/(\alpha(\bar{F})+1)+\epsilon})$.

3.2. Beyond the heavy-traffic regime

The theorems that have just been presented characterize the heavy-traffic performance of SRPT, however, the heavy-traffic limits often provide simple descriptions of queuing models that can be used to develop approximations outside of the heavy-traffic regime, e.g., [27–29]. In this section, we illustrate that Theorems 2 and 3 can often be used in this manner. In particular, we use numeric experiments to illustrate that the form of $E[T]$ provides an accurate approximation of $E[T]$ outside of the heavy-traffic regime.

We focus our numeric experiments on the cases of the Pareto and Weibull job size distributions. The Pareto distribution is probably the most popular heavy-tailed distribution, and is widely used as a model for the tails of real world workloads, e.g., [30,31]. The Weibull distribution is another common job size distribution since it can mimic the behavior of many other statistical distributions such as the normal distribution, the exponential distribution, and other distributions with both increasing or decreasing failure rate. Note that the heavy traffic behavior of $E[T]$ under SRPT for Pareto distribution has been studied in [21,16], while the heavy traffic behavior for Weibull distribution is novel.

Example 1: *Pareto job size distribution*

The first example we consider is the case of Pareto job sizes. Take $X \sim \text{Pareto}(\alpha) \in RV_{-\alpha}$. Thus $G^{-1}(\rho) = x_m(1-\rho)^{\frac{1}{1-\alpha}}$, and we get the asymptotic mean response time for Pareto job size immediately from Theorem 3.

Corollary 1. *In an $M/GI/1$ SRPT queue with Pareto job sizes such that* $\bar{F}(x) = \left(\frac{x}{x_m}\right)^{-\alpha}$,

$$E[T] \sim \begin{cases} \dfrac{(\pi/(-\alpha+1))}{2\sin(\pi/(-\alpha+1))} \cdot \dfrac{E[X^2]}{x_m(1-\rho)^{\frac{\alpha-2}{\alpha-1}}} & \alpha > 2 \\ \dfrac{1}{\alpha(-\alpha+2)} \cdot E[X] \log\left(\dfrac{1}{1-\rho}\right) & \alpha < 2 \end{cases} \quad \textit{as } \rho \to 1.$$

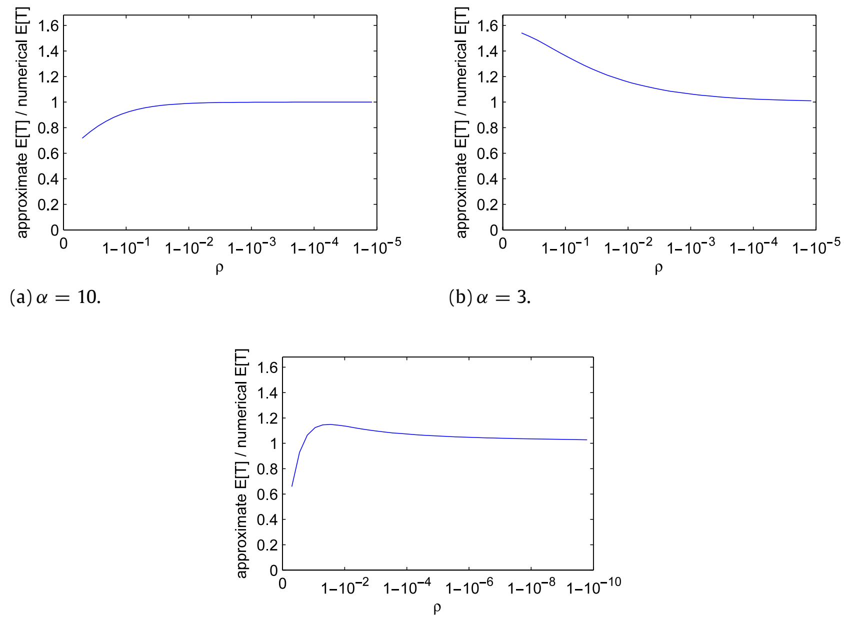
Now we use the above asymptotic formula in Corollary 1 as an approximation for $E[T]$ and compare it to numerical calculation of the exact mean response time in (1). Figs. 1 and 2 illustrate the results. These figures illustrate the results for Pareto job sizes with a wide variety of parameters: $\alpha = 10, 3, 1.5$. Fig. 1 shows the ratio of the approximate $E[T]$ to the numerical $E[T]$ as a function of $\rho \in [0.5, 1)$. This figure illustrates the accuracy in the heavy-traffic regime. Next, Fig. 2 shows the approximate $E[T]$ and the numerical $E[T]$ as a function of $\rho \in [0.5, 1)$ with ρ in linear scale, which illustrates that the asymptotic formula is a good approximation even outside of the heavy-traffic regime, especially when α is large.

Example 2: *Weibull job size distribution*

The second example we consider is the case of Weibull job sizes. Note that the $M/M/1$ is a special case of Weibull obtained by setting $\alpha = 1$. Further, notice that $X \sim \text{Weibull}(\alpha) \in RV_{-\infty}$. So, with a little calculation, we can obtain $G^{-1}(\rho)$ and then apply Theorem 3 to attain the following corollary.

Corollary 2. *In an $M/GI/1$ SRPT queue with Weibull job sizes such that* $\bar{F}(x) = e^{-\mu x^\alpha}$,

$$E[T] \sim \frac{E[X^2]}{2} \frac{1}{(1-\rho)\cdot\mu^{-1/\alpha}\log\left(\frac{1}{1-\rho}\right)^{1/\alpha}} \quad \textit{as } \rho \to 1.$$

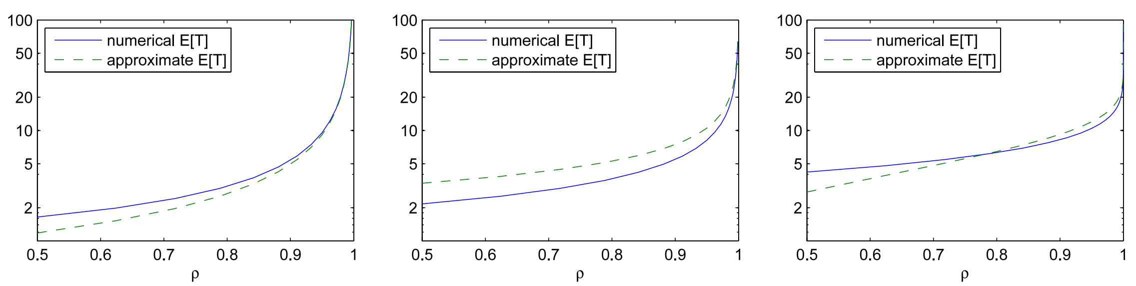


(a) $\alpha = 10$.

(b) $\alpha = 3$.

(c) $\alpha = 1.5$.

Fig. 1. Ratio of the approximate $E[T]$ to the numerical $E[T]$ as a function of ρ in the case of Pareto job sizes in the heavy-traffic regime.



(a) $\alpha = 10$.

(b) $\alpha = 3$.

(c) $\alpha = 1.5$.

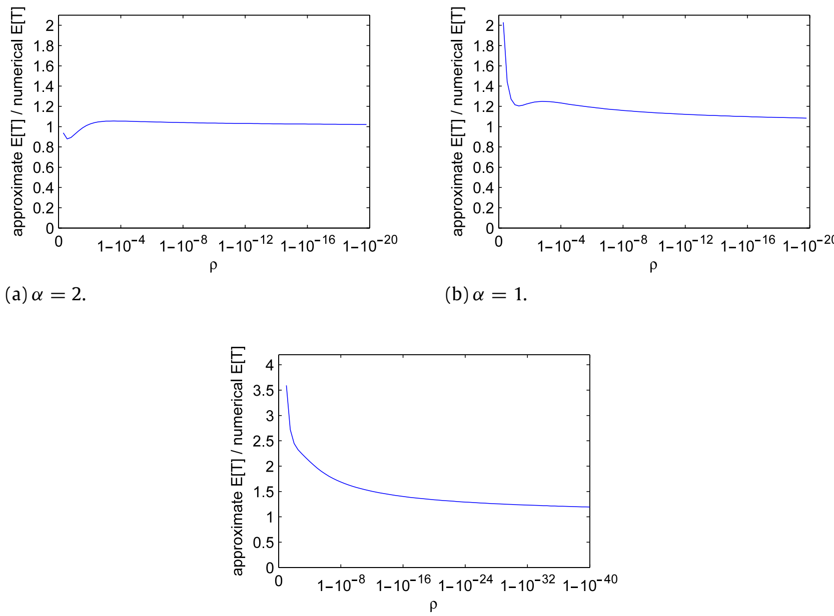
Fig. 2. Comparison of approximate $E[T]$ with numeric $E[T]$ as a function of ρ in the case of Pareto job sizes outside of the heavy-traffic regime.

Proof. Note that $\bar{F}(x) = \mathrm{e}^{-\mu x^{\alpha}} \sim R_{-\infty}$. Substituting $\alpha = -\infty$ into Theorem 3, we get $\lim_{\gamma\to-\infty} \frac{(\pi/(\gamma+1))}{2\sin(\pi/(\gamma+1))} = \frac{1}{2}$. Now let us show that $G^{-1}(\rho) \sim \mu^{-1/\alpha} \log\left(\frac{1}{1-\rho}\right)^{1/\alpha}$. Based on L'Hospital Rule, we get

$$\lim_{y\to\infty} \frac{\log\left(\frac{1}{1-G(y)}\right)}{y^{\alpha}} = \lim_{y\to\infty} \frac{G'(y)}{(1-G(y))\alpha y^{\alpha-1}} = \lim_{y\to\infty} \frac{y^{2-\alpha}F'(y)}{\left(E[x] - \int_0^y t\,\mathrm{d}F(t)\right)\alpha} = \mu.$$

Let $\rho = G(y)$, then we get $\lim_{\rho\to 1} \log\left(\frac{1}{1-\rho}\right)^{1/\alpha} / G^{-1}(\rho) = \mu^{1/\alpha}$. $\square$

Again, we now use the above asymptotic formula in Corollary 2 as an approximation for $E[T]$ and compare it to numerical calculation of the exact mean response time in (1). Figs. 3 and 4 show the numerical $E[T]$ and the approximate $E[T]$ for Weibull distributions with various parameters ($\alpha = 0.5, 1, 2$) as a function of $\rho \in [0.5, 1)$ for the first two subplots and $\rho \in [0.9, 1)$ for the last subplot. Again, the approximation seems accurate even when ρ is not very large in the cases of $\alpha = 1, 2$; however, in the case of $\alpha = 0.5$, the approximation is very poor outside of the heavy-traffic regime. One intuition why the case with smaller α has worse approximation is as follows: As α decreases, the tail of the distribution becomes heavier. Thus the measure $G(x)$ grows slower, which implies that $G^{-1}(\rho)$ grows faster. Therefore the mean response time $E[T]$ grows slower. Since the heavy-traffic formula we derived makes sense only when $E[T]$ is large, we need ρ much closer to 1 to have the approximation accurate.

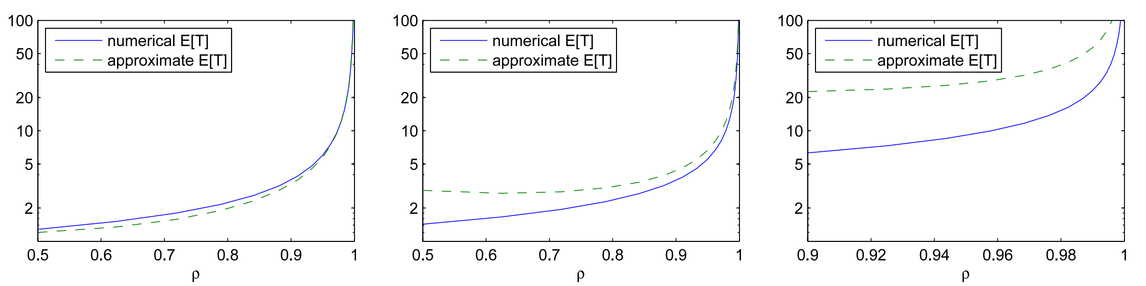


(a) $\alpha = 2$.

(b) $\alpha = 1$.

(c) $\alpha = 0.5$.

Fig. 3. Ratio of the approximate $E[T]$ to the numerical $E[T]$ as a function of ρ in the case of Weibull job sizes in the heavy-traffic regime.



(a) $\alpha = 2$.

(b) $\alpha = 1$.

(c) $\alpha = 0.5$.

Fig. 4. Comparison of approximate $E[T]$ with numeric $E[T]$ as a function of ρ in the case of Weibull job sizes outside of the heavy-traffic regime.

4. Proofs

In this section we prove the results summarized in Section 3. To get started, it is useful to decompose $E[T(x)]$ under SRPT in (1) as follows. Note that a similar decomposition technique has been used in [32,21] as well. Denote

$$\tilde{R}(x) = \int_0^x \frac{\mathrm{d}t}{1-\rho(t)} + \frac{\lambda x^2 \bar{F}(x)}{2(1-\rho(x))^2}, \qquad \tilde{W}(x) = \frac{\lambda m_2(x)}{2(1-\rho(x))^2}.$$

Then $E[T] = E[E[T(x)]] = E[\tilde{R}] + E[\tilde{W}]$ and we can derive the bounds for $E[\tilde{R}]$ and $E[\tilde{W}]$ separately. First, for $E[\tilde{R}]$ we have

$$E[\tilde{R}] = \int_0^\infty \tilde{R}(x)\mathrm{d}F(x) = \int_0^\infty \frac{\bar{F}(x)}{1-\rho(x)}\mathrm{d}x + \frac{\lambda}{2}\int_0^\infty \frac{x^2\bar{F}(x)}{(1-\rho(x))^2}\mathrm{d}F(x). \tag{2}$$

Now consider that

$$\log\frac{1}{1-\rho} = \lambda\int_0^\infty \frac{x\mathrm{d}F(x)}{1-\rho(x)} = \lambda\int_0^\infty \frac{\bar{F}(x)}{1-\rho(x)}\mathrm{d}x + \lambda^2\int_0^\infty \frac{x^2\bar{F}(x)}{(1-\rho(x))^2}\mathrm{d}F(x). \tag{3}$$

Comparing Eq. (2) to Eq. (3), we can see that

$$\frac{1}{2\lambda}\log\frac{1}{1-\rho} < E[\tilde{R}] < \frac{1}{\lambda}\log\frac{1}{1-\rho}.$$

In the heavy-traffic regime, i.e., $\lambda E[X] = \rho \to 1$, it becomes,

$$\frac{E[X]}{2}\log\frac{1}{1-\rho} < E[\tilde{R}] \preceq E[X]\log\frac{1}{1-\rho}. \tag{4}$$

A similar result on $E[\tilde{R}]$ can also be found in [32,21]. However, it is more difficult to obtain the bound of $E[\tilde{W}]$, which depends on the job size distribution. We rewrite the formulation of $E[\tilde{W}]$ as follows and focus on this formulation for the rest of the analysis.

$$E[\tilde{W}] = \int_0^\infty \tilde{W}(x)\mathrm{d}F(x) = \int_0^\infty \frac{\lambda m_2(x)}{2(1-\rho(x))^2}\mathrm{d}F(x) = \int_0^\infty \frac{m_2(x)}{2(1-\rho(x))^2 x}\mathrm{d}\rho(x).$$

We now begin by proving Theorem 1.

Proof of Theorem 1. If $F(x)$ has bounded support, i.e., $\forall x, x \le x_m$, note that $m_2(x)$ is non-decreasing in x, thus for a given $x_p \in (0, x_m)$, we have

$$E[\tilde{W}] \ge \int_{x_p}^{x_m} \frac{m_2(x_p)}{2(1-\rho(x))^2 x_m}\mathrm{d}\rho(x) = \frac{m_2(x_p)}{2x_m}\left.\frac{1}{1-\rho(x)}\right|_{x_p}^{x_m}. \tag{5}$$

On the other hand,

$$E[\tilde{W}] \le C_1 + \int_{x_p}^{x_m} \frac{m_2(x_m)}{2(1-\rho(x))^2 x_p}\mathrm{d}\rho(x) = C_1 + \frac{E[X^2]}{2x_p}\left.\frac{1}{1-\rho(x)}\right|_{x_p}^{x_m}. \tag{6}$$

As $\rho(x_m) = \rho \to 1$ for some constant C_1, Eqs. (5) and (6) imply that $E[\tilde{W}] = \Theta\left(\frac{1}{1-\rho}\right)$. Therefore $E[T] = E[\tilde{R}] + E[\tilde{W}] = \Theta\left(\frac{1}{1-\rho}\right)$ for $F(x)$ with bounded support.

If $F(x)$ has unbounded support, then for any given $\epsilon > 0$ we can find an x_0 so that $\bar{G}(x_0) < \epsilon$. Further

$$\lambda m_2(x) = x\rho(x) - \int_0^x \rho(t)\mathrm{d}t = \int_0^x (\rho - \rho(t))\mathrm{d}t - x(\rho - \rho(x)) \le \rho\int_0^x \bar{G}(t)\mathrm{d}t.$$

Based on this inequality, we have

$$\limsup_{x\to\infty}\frac{m_2(x)}{x} \le \limsup_{x\to\infty}\frac{E[X]\int_0^x \bar{G}(t)\mathrm{d}t}{x} = \limsup_{x\to\infty}\frac{C_2 + E[X]\int_{x_0}^x \bar{G}(t)\mathrm{d}t}{x} \le E[X]\epsilon.$$

Therefore, there exists an x_p so that for all $x > x_p$, $m_2(x)/x \le E[X]\epsilon + \epsilon$. As $\rho \to 1$, we get

$$E[\tilde{W}] = \int_0^\infty \frac{m_2(x)}{x}\frac{\mathrm{d}\rho(x)}{2(1-\rho(x))^2} = C_3 + \int_{x_p}^\infty \frac{m_2(x)}{x}\frac{\mathrm{d}\rho(x)}{2(1-\rho(x))^2} \preceq \frac{E[X]\epsilon + \epsilon}{2}\frac{1}{1-\rho},$$

in which ϵ can be arbitrarily small. Thus $E[\tilde{W}] = o\left(\frac{1}{1-\rho}\right)$ for $F(x)$ with unbounded support. □

To prove Theorem 2, we need a few basic results about Matuszewska indices, which we summarize here. First note that, by definition of $\alpha(f)$ and $\beta(f)$, we have $\beta(1/f) = -\alpha(f)$. Additionally we will make use of the following lemmas.

Lemma 1 ([25], *Karamata's Theorem' for One-Sided Indices, Part* 1)**.** *Let f be positive and locally integrable on $[X, \infty)$, and set $\tilde{f}(x) = \int_X^x f(t)/t\mathrm{d}t$. If $\beta(f) > 0$ then $\liminf_{x\to\infty} f(x)/\tilde{f}(x) > 0$. And $\limsup_{x\to\infty} f(x)/\tilde{f}(x) \ge \alpha(f)$.*

Lemma 2 ([25], *Karamata's Theorem' for One-Sided Indices, Part* 2)**.** *Let f be positive and measurable, and set $\tilde{f}(x) = \int_x^\infty f(t)/t\mathrm{d}t \le \infty$. If $\alpha(f) < 0$ then $\tilde{f}(x) < \infty$ for all large x.*

Lemma 3 ([33])**.** *Let $f : (0, \infty) \to R$ be non-decreasing and unbounded above, $f^{\leftarrow}(x) = \inf\{y \in [X, \infty) : f(y) > x\}$. $\alpha(f) < \infty$ if and only if $\beta(f^{\leftarrow}) > 0$ and $\beta(f^{\leftarrow}) = 1/\alpha(f)$.*

Lemma 4 ([25])**.** *Let f be positive. If $\alpha(f) < \infty$ then for every $\alpha > \alpha(f)$ there exist positive constants C, X such that $f(y)/f(x) \le C(y/x)^\alpha$ $(y \ge x \ge X)$.*

Now we prove two technical lemmas before we prove Theorem 2.

Lemma 5. *In an $M/GI/1$ SRPT queue, if the job size distribution has unbounded support and $E[X^2] < \infty$, then*

$$\frac{E[X^2]}{2}\int_0^\rho \frac{\mathrm{d}y}{(1-y)^2 G^{-1}(y)} > E[\tilde{W}] = \Omega\left(\frac{1}{(1-\rho)G^{-1}(\rho)}\right) \quad \textit{as } \rho \to 1.$$

Proof. Let us prove the lower bound of $E[\tilde{W}]$ first. Since $m_2(x)$ is non-decreasing in x, for a given $x_0 > 0$, we get

$$E[\tilde{W}] = \int_0^\infty \frac{m_2(x)}{2(1-\rho(x))^2 x} \mathrm{d}\rho(x) > \frac{m_2(x_0)}{2} \int_{x_0}^\infty \frac{\mathrm{d}\rho(x)}{(1-\rho(x))^2 x}.$$

Denote $y = \rho(x)$ and assume that function $\rho(x)$ is invertible, then we have

$$E[\tilde{W}] > \frac{m_2(x_0)}{2} \int_{\rho_0}^{\rho} \frac{\mathrm{d}y}{(1-y)^2 \rho^{-1}(y)} \geq \frac{m_2(x_0)}{2} \int_{\rho_0}^{\rho^2} \frac{\mathrm{d}y}{(1-y)^2 \rho^{-1}(y)}. \tag{7}$$

Remember that $G(x) = \rho(x)/\rho$, which implies $\rho^{-1}(z) = G^{-1}(z/\rho)$. Thus

$$E[\tilde{W}] > \frac{m_2(x_0)}{2} \int_{\rho_0}^{\rho^2} \frac{\mathrm{d}y}{(1-y)^2 G^{-1}\left(\frac{y}{\rho}\right)} > \frac{m_2(x_0)}{2G^{-1}(\rho)} \int_{\rho_0}^{\rho^2} \frac{\mathrm{d}y}{(1-y)^2} = \frac{m_2(x_0)}{2G^{-1}(\rho)} \frac{1}{1-y}\bigg|_{\rho_0}^{\rho^2}.$$

Thus we have $E[\tilde{W}] = \Omega\left(\frac{1}{(1-\rho)G^{-1}(\rho)}\right)$.

For the upper bound of $E[\tilde{W}]$:

$$E[\tilde{W}] = \int_0^\infty \frac{m_2(x)}{2(1-\rho(x))^2 x} \mathrm{d}\rho(x) < \frac{E[X^2]}{2} \int_0^\infty \frac{\mathrm{d}\rho(x)}{(1-\rho(x))^2 x}.$$

Denote $y = \rho(x)$ and remember that $\rho^{-1}(z) = G^{-1}(z/\rho) \geq G^{-1}(z)$, we have

$$E[\tilde{W}] < \frac{E[X^2]}{2} \int_0^\rho \frac{\mathrm{d}y}{(1-y)^2 \rho^{-1}(y)} \leq \frac{E[X^2]}{2} \int_0^\rho \frac{\mathrm{d}y}{(1-y)^2 G^{-1}(y)}. \quad \square \tag{8}$$

Another lemma we need in order to prove Theorem 2 is the following.

Lemma 6. $\alpha(\bar{G}) \leq \alpha(\bar{F}) + 1$; $\beta(\bar{G}) \geq \beta(\bar{F}) + 1$.

Proof. Since $\bar{G}(z) = 1 - G(z) = -\int_z^\infty t\mathrm{d}\bar{F}(t)/E[X]$, integration by parts gives

$$E[X]\bar{G}(\lambda x) = \int_{\lambda x}^\infty \bar{F}(t)\mathrm{d}t + \lambda x \bar{F}(\lambda x) = \lambda \int_x^\infty \bar{F}(\lambda u)\mathrm{d}u + \lambda x \bar{F}(\lambda x).$$

Let $\mu = \alpha(\bar{F})$. By definition of the Matuszewska index, we have $\bar{F}(\lambda x)/\bar{F}(x) \leq C\{1 + o(1)\}\lambda^\mu$ $(x \to \infty)$ uniformly in $\lambda \in [1, \Lambda]$. Thus

$$\begin{aligned} E[X]\bar{G}(\lambda x) &\leq C\{1+o(1)\}\lambda^{\mu+1}\left(\int_x^\infty \bar{F}(u)\mathrm{d}u + x\bar{F}(x)\right) \\ &= C\{1+o(1)\}\lambda^{\mu+1}E[X]\bar{G}(x) \quad \text{as } x \to \infty. \end{aligned}$$

Thus $\alpha(\bar{G}) \leq \mu + 1$. Similarly we can prove $\beta(\bar{G}) \geq \beta(\bar{F}) + 1$. $\square$

Now we are ready to prove Theorem 2. The proof consists of two parts. First, we prove the case of $\alpha(\bar{F}) < -2$ and then the case of $\beta(\bar{F}) > -2$.

Proof of Theorem 2 for $\alpha(\bar{F}) < -2$. Let us prove the lower bound first. Based on Lemma 5, we need only to prove that $E[X^2] < \infty$. Note that

$$\lambda m_2(x) = x\rho(x) - \int_0^x \rho(t)\mathrm{d}t = \int_0^x (\rho - \rho(t))\mathrm{d}t - x(\rho - \rho(x)) \leq \rho \int_0^x \bar{G}(t)\mathrm{d}t.$$

It is sufficient to prove that $\int_0^\infty \bar{G}(t)\mathrm{d}t < \infty$.

Based on Lemma 6 we have $\alpha(\bar{G}(x)) \leq \alpha(\bar{F}(x)) + 1 < -1$, which implies $\alpha(x\bar{G}(x)) < 0$ by the definition of the Matuszewska index. Therefore we get $\int_0^\infty \bar{G}(t)\mathrm{d}t < \infty$ by Lemma 2.

For the upper bound, we first prove the upper bound for $E[\tilde{W}]$, and then prove that $E[\tilde{R}]$ is dominated by $E[\tilde{W}]$ as $\rho \to 1$. To prove $E[\tilde{W}] = O\left(\frac{1}{(1-\rho)G^{-1}(\rho)}\right)$, based on Lemma 5, it is sufficient to prove $\lim\sup_{\rho\to 1} \int_0^\rho \frac{(1-\rho)G^{-1}(\rho)}{(1-y)^2 G^{-1}(y)}\mathrm{d}y < \infty$.

Denote $\mu = \alpha(\bar{G}(x)) < -1$, which implies $\beta(1/\bar{G}(x)) = -\mu > 1$. Based on Lemma 3, we get $\alpha(\bar{G}^{-1}(1/x)) = -1/\mu$, thus $\beta(1/\bar{G}^{-1}(1/x)) = 1/\mu$, which gives $\beta(x/\bar{G}^{-1}(1/x)) = 1 + 1/\mu > 0$. By Lemma 1, we have $\lim\inf_{z\to\infty} \frac{z/\bar{G}^{-1}(1/z)}{\int_1^z 1/\bar{G}^{-1}\left(\frac{1}{u}\right)\mathrm{d}u} > 0$, i.e., $\limsup_{z\to\infty}\int_1^z \frac{\bar{G}^{-1}\left(\frac{1}{z}\right)}{z\bar{G}^{-1}(\frac{1}{u})}\mathrm{d}u < \infty$. (Notice that this result holds even for $\mu = -\infty$.) Denote $u = 1/(1-y)$, $z = 1/(1-\rho)$, we get

$$\limsup_{\rho\to 1}\int_0^{\rho}\frac{(1-\rho)G^{-1}(\rho)}{(1-y)^2G^{-1}(y)}\mathrm{d}y = \limsup_{z\to\infty}\int_1^z\frac{\bar{G}^{-1}\left(\frac{1}{z}\right)}{z\bar{G}^{-1}\left(\frac{1}{u}\right)}\mathrm{d}u < \infty.$$

Therefore, we have $E[\tilde{W}] = O\left(\frac{1}{(1-\rho)G^{-1}(\rho)}\right)$.

Now let us prove that $E[\tilde{R}]$ is dominated by $E[\tilde{W}]$ as $\rho \to 1$. Remember $E[\tilde{R}] = \Theta\left(\log\left(\frac{1}{1-\rho}\right)\right)$ and $E[\tilde{W}] = \Omega\left(\frac{1}{(1-\rho)G^{-1}(\rho)}\right)$, thus we need only to prove that $\lim_{\rho\to 1}\log\left(\frac{1}{1-\rho}\right)(1-\rho)G^{-1}(\rho) = \lim_{y\to\infty}\log y\bar{G}^{-1}(1/y)/y = 0$. Denote $g(y) = \log y\bar{G}^{-1}(1/y)/y$. We have shown that $\alpha(\bar{G}^{-1}(1/x)) = -1/\mu$, thus $\alpha(g(y)) = -1/\mu - 1 < 0$. Based on Lemma 4, $g(y) \le Cg(X)(y/X)^{\alpha(g(y))+\epsilon}$ for some constant C and X. Thus $\lim_{y\to\infty} g(y) = 0$, which means that $E[\tilde{R}]$ is dominated by $E[\tilde{W}]$. This completes our proof. □

Proof of Theorem 2 for $\beta(\bar{F}) > -2$. First, let us prove $E[X^2] = \infty$ by contradiction. Suppose $E[X^2] < \infty$, then $E[X^2] = \int_0^\infty t^2 f(t)\mathrm{d}t = 2\int_0^\infty t\bar{F}(t)\mathrm{d}t$, $m_2(x) = \int_0^x t^2 f(t)\mathrm{d}t = 2\int_0^x t\bar{F}(t)\mathrm{d}t - x^2\bar{F}(x)$. Thus

$$E[X^2] - m_2(x) \ge x^2\bar{F}(x) \Rightarrow \forall x\, x^2\bar{F}(x) < E[X^2].$$

However, we know that $\beta(x^2\bar{F}(x)) > 0$, which means $\liminf_{x\to\infty} x^2\bar{F}(x)$ is unbounded. Contradiction. Therefore, $E[X^2] = \infty$.

Now let us prove the bounds for $E[T]$. The lower bound is simply followed by seeing $E[T] > E[\tilde{R}]$. For the upper bounds, we have

$$\lambda m_2(x) = x\rho(x) - \int_0^x \rho(t)\mathrm{d}t = \int_0^x(\rho - \rho(t))\mathrm{d}t - x(\rho - \rho(x)).$$

Therefore,

$$\frac{\lambda m_2(x)}{x(\rho-\rho(x))} = \frac{\int_0^x(\rho-\rho(t))\mathrm{d}t}{x(\rho-\rho(x))} - 1 = \frac{\int_0^x\bar{G}(t)\mathrm{d}t}{x\bar{G}(x)} - 1. \tag{9}$$

Since $\beta(\bar{G}(x)) \ge \beta(\bar{F}(x)) + 1 > -1$, we have $\beta(x\bar{G}(x)) > 0$. By Lemma 1, $\liminf_{z\to\infty}\frac{z\bar{G}(z)}{\int_1^z\bar{G}(u)\mathrm{d}u} > 0$, which implies $\limsup_{x\to\infty}\frac{\lambda m_2(x)}{x(\rho-\rho(x))} < \infty$, Therefore

$$E[\tilde{W}] = \int_0^\infty\frac{m_2(x)}{2(1-\rho(x))^2x}\mathrm{d}\rho(x) < \int_0^\infty\frac{\lambda m_2(x)}{x(\rho-\rho(x))}\cdot\frac{\mathrm{d}\rho(x)}{2\lambda(1-\rho(x))} = O\left(\log\frac{1}{1-\rho}\right).$$

Finally, $E[T] = E[\tilde{R}] + E[\tilde{W}] = \Theta\left(\log\frac{1}{1-\rho}\right)$. □

Next we are going to prove Theorem 3. Again, the proof consist of two cases: first $\alpha < -2$ and then $\alpha > -2$.

Proof of Theorem 3 for $\alpha < -2$. Since Theorem 2 is still applicable for regular varying job size distribution, we know that $E[\tilde{R}]$ is dominated by $E[\tilde{W}]$ for $\alpha < -2$, thus $E[T] \sim E[\tilde{W}]$. Eqs. (7) and (8) imply

$$E[\tilde{W}] \sim \frac{E[X^2]}{2}\int_{\rho_0}^{\rho}\frac{\mathrm{d}y}{(1-y)^2\rho^{-1}(y)} = \frac{E[X^2]}{2}\int_{\frac{1}{1-\rho_0}}^{\frac{1}{1-\rho}}\frac{\mathrm{d}u}{\rho^{-1}\left(1-\frac{1}{u}\right)}.$$

The last equation is obtained by defining $u = 1/(1-y)$. Now remember that $G(x) = \rho(x)/\rho$, which implies $G^{-1}(y) = \rho^{-1}(\rho y)$, thus the above equation becomes

$$E[\tilde{W}] \sim \frac{E[X^2]}{2}\int_{\frac{1}{1-\rho_0}}^{\frac{1}{1-\rho}}\frac{\mathrm{d}u}{G^{-1}\left(\left(1-\frac{1}{u}\right)/\rho\right)} = \frac{E[X^2]}{2}\int_{\frac{1}{1-\rho_0}}^{\frac{1}{1-\rho}}\frac{\mathrm{d}u}{\bar{G}^{-1}\left(1-\frac{1}{\rho}+\frac{1}{\rho u}\right)}.$$

Define $v = (1-\rho)u$ and substitute it into the above formulation, then

$$E[\tilde{W}] \sim \frac{1}{1-\rho}\frac{E[X^2]}{2}\int_0^1\frac{\mathrm{d}v}{\bar{G}^{-1}\left(\frac{1-\rho}{\rho}\left(\frac{1}{v}-1\right)\right)} = \frac{1}{(1-\rho)G^{-1}(\rho)}\frac{E[X^2]}{2}\int_0^1\frac{\bar{G}^{-1}(1-\rho)}{\bar{G}^{-1}\left(\frac{1-\rho}{\rho}\left(\frac{1}{v}-1\right)\right)}\mathrm{d}v.$$

Based on the properties of regular variation, $\bar{F}(z) \sim RV_\alpha$ gives $\bar{G}(z) \sim RV_{\alpha+1}$, which implies $\bar{G}^{-1}(1/z) \sim RV_{-1/(\alpha+1)}$. (Notice that this result holds even for $\alpha = -\infty$.) Based on Theorem 2, we know the integral is bounded, thus we can interchange the limit and the integral. As $\rho \to 1$, we have $\frac{\bar{G}^{-1}(1-\rho)}{\bar{G}^{-1}\left(\frac{1-\rho}{\rho}\left(\frac{1}{v}-1\right)\right)} \sim \frac{1}{(1/v-1)^{-1/(\alpha+1)}}$. Therefore

$$E[\tilde{W}] \sim \frac{1}{(1-\rho)G^{-1}(\rho)} \frac{E[X^2]}{2} \int_0^1 (1/v-1)^{1/(\alpha+1)} \mathrm{d}v = \frac{1}{(1-\rho)G^{-1}(\rho)} \frac{E[X^2](\pi/(\alpha+1))}{2\sin(\pi/(\alpha+1))}. \quad \square$$

Proof of Theorem 3 for $\alpha > -2$**.** Similar to the case of $\alpha < -2$, we have $\bar{G}(z) \sim RV_{\alpha+1}$. By Karamata's Theorem, $x\bar{G}(x)/\int_0^x \bar{G}(t)\mathrm{d}t \to \alpha+2$ as $x \to \infty$. Substituting this result into Eq. (9) we get

$$\lim_{x\to\infty} \frac{\lambda m_2(x)}{x(\rho-\rho(x))} = \lim_{x\to\infty} \left(\frac{\int_0^x \bar{G}(t)\mathrm{d}t}{x\bar{G}(x)} - 1 \right) = \frac{-\alpha-1}{\alpha+2}. \tag{10}$$

Based on Eq. (10), we can calculate $E[\tilde{W}]$ as follows.

$$E[\tilde{W}] = \int_0^\infty \frac{(\rho-\rho(x))}{2\lambda(1-\rho(x))^2} \frac{\lambda m_2(x)}{x(\rho-\rho(x))} \mathrm{d}\rho(x) \sim \frac{-\alpha-1}{2\lambda(\alpha+2)} \int_{x_0}^\infty \frac{\rho-\rho(x)}{(1-\rho(x))^2} \mathrm{d}\rho(x).$$

Define $y = \rho(x)$. The above formulation becomes

$$E[\tilde{W}] \sim \frac{-\alpha-1}{2\lambda(\alpha+2)} \int_{x_0}^\rho \frac{\rho-y}{(1-y)^2} \mathrm{d}y \sim \frac{-\alpha-1}{2(\alpha+2)} E[X] \log\left(\frac{1}{1-\rho}\right).$$

For $E[\tilde{R}]$, by comparing Eqs. (2) and (3), we have

$$E[\tilde{R}] = \frac{1}{2} \int_0^\infty \frac{\bar{F}(x)}{1-\rho(x)} \mathrm{d}x + \frac{1}{2\lambda} \log \frac{1}{1-\rho}.$$

Let us now focus on the first term of the above formulation. By Karamata's Theorem, $x\bar{F}(x)/\int_x^\infty \bar{F}(t)\mathrm{d}t \to -(\alpha+1)$ as $x \to \infty$. Thus

$$\rho - \rho(x) = \lambda \left(\int_x^\infty \bar{F}(t)\mathrm{d}t + x\bar{F}(x) \right) \sim -\lambda\alpha \int_x^\infty \bar{F}(t)\mathrm{d}t \quad \text{as } x \to \infty.$$

Therefore

$$\int_0^\infty \frac{\bar{F}(x)}{1-\rho(x)} \mathrm{d}x \sim \int_{x_0}^\infty \frac{\bar{F}(x)}{(1-\rho) - \lambda\alpha \int_x^\infty \bar{F}(t)\mathrm{d}t} \mathrm{d}x.$$

By denoting $y = \int_x^\infty \bar{F}(t)\mathrm{d}t$, the above formulation becomes

$$\int_0^\infty \frac{\bar{F}(x)}{1-\rho(x)} \mathrm{d}x \sim \int_{y_0}^0 \frac{-1}{(1-\rho) - \lambda\alpha y} \mathrm{d}y \sim -\frac{1}{\lambda\alpha} \log\left(\frac{1}{1-\rho}\right).$$

Therefore, $E[\tilde{R}]$ has the following asymptotic form:

$$E[\tilde{R}] \sim \frac{\alpha-1}{2\alpha} E[X] \log\left(\frac{1}{1-\rho}\right).$$

Combining the asymptotic forms of $E[\tilde{R}]$ and $E[\tilde{W}]$, we finally get

$$E[T] = E[\tilde{R}] + E[\tilde{W}] \sim \frac{1}{(-\alpha)(\alpha+2)} E[X] \log\left(\frac{1}{1-\rho}\right). \quad \square$$

5. Concluding remarks

In this work, we tightly characterize the growth rate of $E[T]$ under SRPT in the M/GI/1 queue under general job size distributions (Theorems 1–3). This provides some interesting insight into the behavior of SRPT. For example, SRPT provides an order of magnitude improvement over PS and FCFS if and only if the job size distribution is unbounded. Further, if the distribution is unbounded the growth rate depends delicately on the tail of the job size distribution through the measure $G(x)$ when the job size distribution has a finite second moment. However, if the job size distribution has an infinite second moment, then the growth rate of $E[T]$ is independent (up to a constant) of the job size distribution. An important direction for future work is to try to extend these results and insights to the GI/GI/1 SRPT queue.

To illustrate the heavy traffic results, we used numeric experiments in the cases of the Pareto and Weibull job sizes. These experiments illustrate that the heavy-traffic growth rates can often be used to provide accurate approximations of $E[T]$ even outside of heavy-traffic. Thus, the heavy-traffic results provide simple, accurate approximations for use in more complicated models. These results have already led to new approximate analysis of SRPT in load balancing [22] and speed scaling [23] settings.

Finally, the characterization of the growth rate of SRPT is especially important because of the optimality of SRPT. The results in this paper provide a baseline with which to compare the performance of other policies. Without these results it has been difficult to understand the degree of suboptimality, i.e., the ''competitive ratio'', of other scheduling disciplines when compared to the optimal $E[T]$. Using the results in this paper it is now possible to ask, and hopefully answer, a number of new interesting questions. For example, we have shown that PS and FCFS are within a constant factor of the optimal $E[T]$ if and only if the job size distribution is bounded. A similar question can be asked for many other scheduling policies. One such question is: under which distributions is FB within a constant factor of optimal? And more generally, when is it possible for a scheduling policy that does not use job size information to be within a constant factor of optimal? We know that it is impossible under $M/M/1$, but that FCFS and PS achieve it under bounded job sizes distribution. Some partial results on this question can be found in [32,34].

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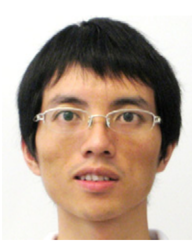

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