1) $R_1(\theta_1)$ and $R_2(\theta_2) \in S0(n)$

Prove $R_1(\theta_1) R_2(\theta_2) \in S0(n)$:

$\det(R_1) = \det(R_2) = 1$

$R_1^T R_1 = R_2^T R_2 = I$

$$(R_1 R_2)^T R_1 R_2 = R_2^T \overbrace{R_1^T R_1}^{I} R_2$$

$$= R_2^T R_2$$

$$\underline{(R_1 R_2)^T R_1 R_2 = I}$$

$$\det(R_1 R_2) = \det(R_1) \det(R_2)$$

$$= \underline{1 \cdot 1 = 1}$$

2) $g_1(x_1, y_1, \theta_1) \in SE(2)$

$$\longrightarrow g_1 = \begin{bmatrix} R_1(\theta_1) & \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ 0\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} R_1(\theta) \in S0(2), \\ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in IR^{2\times 1} \end{matrix}$$

$g_2(x_2, y_2, \theta_2) \in SE(2)$

$$\longrightarrow g_2 = \begin{bmatrix} R_2(\theta_2) & \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ 0\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} R_2(\theta) \in S0(2), \\ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in IR^{2\times 1} \end{matrix}$$

2) Show $g_1(x_1, y_1, \theta_1)\, g_2(x_2, y_2, \theta_2) \in SE(2)$.

$$g_1 g_2 = \begin{bmatrix} R_1(\theta_1) & \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2(\theta) & \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ 0\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_1(\theta_1) & P_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2(\theta_1) & P_2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 R_2 & R_1 P_2 + P_1 \\ 0 & 1 \end{bmatrix}$$

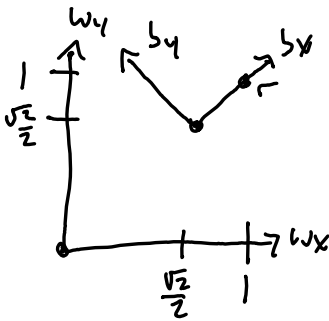$R_1 \& R_2 \in SO(2) \longrightarrow R_1 R_2 \in SO(2)$

$R_1 P_2 \ \& \ P_1 \in \mathbb{R}^{2 \times 1} \longrightarrow R_1 P_2 + P_1 \in \mathbb{R}^{2 \times 1}$

Therefore the product $g_1(x_1, y_1, \theta_1)\, g_2(x_2, y_2, \theta_2)$
has form

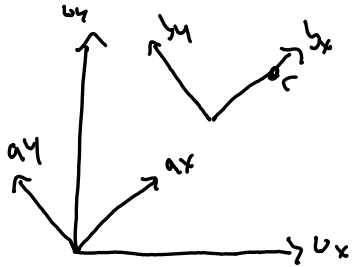$$\begin{bmatrix} R \in SO(2) & P \in \mathbb{R}^{2 \times 1} \\ 0 & 1 \end{bmatrix},$$

implying $g_1 g_2$ belongs to $SE(2)$.

3. Homogeneous transformation in $SE(2)$:

Take a change of coordinate from w frame to s frame of point r:

use an intermediate frame:

$$g_{wa} = \begin{bmatrix} R_{wa} & 0 \\ 0 & 1 \end{bmatrix} \qquad R_{wa} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$g_{AB} = \begin{bmatrix} I & P_{AB} \\ 0 & 1 \end{bmatrix} \qquad P_{AB} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\overline{r_w} = \begin{bmatrix} r_w \\ 1 \end{bmatrix} = g_A \, g_B \, \overline{r_B}$$

"r_i" = vector position of r in the "i" frame

"$\overline{r_i}$" = vector with a trailing 1

$$\overline{r_w} = \begin{bmatrix} R_{wa} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & P_{AB} \\ 0 & 1 \end{bmatrix} \overline{r_B}$$

3.

$$\bar{T}_W = \begin{bmatrix} R_{wa} & R_{wa}P_{ab} \\ 0 & 1 \end{bmatrix} \bar{T}_b :$$

a generalizing transformation has been decomposed into a
rotation and a translation.

The order of operations we should usually use
is rotation ⟶ translation, because if
we translated first, we'd be revolving our
coordinate representation about the origin on
a moment arm as well.

If the order were reversed:

$$G_{wc} = \begin{bmatrix} I & P_{wc} \\ 0 & 1 \end{bmatrix}, \quad P_{wc} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$G_{cB} = \begin{bmatrix} R_{cB} & P_{cB} \\ 0 & 1 \end{bmatrix}, \quad P_{cB} = \begin{bmatrix} \frac{\sqrt{2}}{2} - 1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$
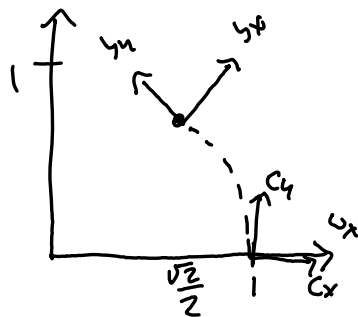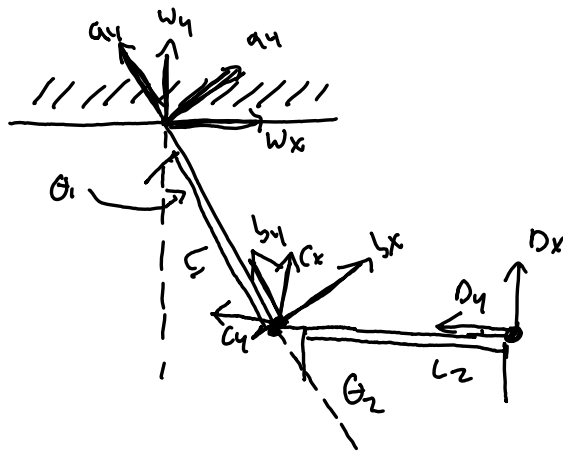

$$G_{wc} \, G_{cB} = \begin{bmatrix} I & P_{wc} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{cB} & P_{cB} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_{cB} & P_{cB} + P_{wc} \\ 0 & 1 \end{bmatrix}$$

difference:

the rotation in this
version has its own
associated translation
(R_{cB}, P_{cB}).

4. Diagram of how frames were defined:

Transformation matrices:

$$G_{WA} = \begin{bmatrix} R_{WA}(\theta_1) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad R_{WA} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$

$$G_{AB} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ -L_1 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$G_{BC} = \begin{bmatrix} R_{BC}(\theta) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad R_{BC} = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$$

$$G_{CD} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ -L_2 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$