Rotational Kinetic Energy and Rotational Inertia

21 Integral Expressions for Rotational Kinetic Energy

Take home messages:

- Geometry shows us that rotational kinetic energy can be calculated with a volume integral using density, body angular velocity, and particle positions. This calculation happens in the body frame.
- Spatial distribution of mass about the axis of rotation influences rotational kinetic energy, distribution of mass along the axis does not
- Velocity is coordinate dependent, energy is not

A rigid body may possess kinetic energy due to both translational and rotational motions. For translational motions finding the associated kinetic energy doesn't present too much difficulty, as all the mass in the body is traveling in the same direction. As we will see, computing rotational kinetic energy requires us to pay attention to how mass is spatially distributed in the body. Fortunately, our understanding of geometry and rigid body transformations helps in this regard.

Rotational Kinetic Energy as a Volume Integral

Consider a rigid body in pure rotation about its center of mass. To track it as it rotates, let's affix a body coordinate frame to the center of mass of the rigid body. Since we're considering pure rotation, we can specify that the origin of the stationary frame is also at this center of mass such that the origins of the two frames will always be coincident. Suppose a rotation $R(t)$ describes the transformation from the body frame to our specified stationary frame. That is, for a particle located at $\underline{r}_b$ in the body frame, its position in the stationary frame is

$$\underline{r}_s = R\,\underline{r}_b$$

Then the velocity of that particle is

$$\frac{d}{dt}\underline{r}_s = \dot{R}\,\underline{r}_b$$

where there is no second term from the product rule since $\underline{r}_b$ is constant. But what is $\dot{R}$? If we express the rotation as

$$R(t) = e^{\widehat{\underline{\omega}}\theta(t)}$$

for some normalized $\underline{\omega}$ and a scalar $\theta(t)$, then

$$\dot{R} = R\,\widehat{\underline{\omega}}\dot{\theta}(t)$$

where $\underline{\omega}\dot{\theta}(t)$ is the body angular velocity. Moving forward, we drop the assumption that $\underline{\omega}$ is normalized, such that

$$\dot{R} = R\,\widehat{\underline{\omega}}$$

and $\underline{\omega}$ denotes the body angular velocity ($\dot{\theta}$ will no longer appear). In this notation

$$\frac{d}{dt}\underline{r}_s = R\,\widehat{\underline{\omega}}\,\underline{r}_b$$

Now that we know how a single particle in the body moves, we proceed to calculating the rotational kinetic energy of the entire rigid body. Calculating this requires integrating the kinetic energies of differential volume elements over the entire volume of the body. This integral views the rigid body as an infinite sum of infinitely small particles (with associated infinitely small mass). The integral is simply stated

$$\mathrm{KE}_{\mathrm{rot}} = \frac{1}{2}\int_{\mathcal{V}} \rho\, v^2\, d\mathcal{V} = \frac{1}{2}\int_{\mathcal{V}} \rho\, \underline{v}^T\, \underline{v}\, d\mathcal{V}$$

where $\mathcal{V}$ is the volume of the rigid body, ρ is the density of the body (for now assumed constant),v signifies the magnitude of the velocity of a differential element $d\mathcal{V}$, and $\underline{v}$ signifies the velocity vector of a differential element $d\mathcal{V}$. Using our analysis above, we can plug in for $\underline{v}$ as

$$\begin{aligned}
\mathrm{KE}_{\mathrm{rot}} &= \frac{1}{2}\int_{\mathcal{V}} \rho\, (R\,\widehat{\underline{\omega}}\,\underline{r}_b)^T (R\,\widehat{\underline{\omega}}\,\underline{r}_b)\, d\mathcal{V} \\
&= \frac{1}{2}\int_{\mathcal{V}} \rho\, (\widehat{\underline{\omega}}\,\underline{r}_b)^T R^T R(\widehat{\underline{\omega}}\,\underline{r}_b)\, d\mathcal{V} \\
&= \frac{1}{2}\int_{\mathcal{V}} \rho\, (\widehat{\underline{\omega}}\,\underline{r}_b)^T (\widehat{\underline{\omega}}\,\underline{r}_b)\, d\mathcal{V}
\end{aligned}$$

Given the geometry and density of a rigid body, as well as its body angular velocity, this integral provides its rotational kinetic energy. Notice that our expression for the velocity of a particle depends R, implying some dependence on the choice of stationary frame. However, the $R^T R$ cancellation above yields an expression for kinetic energy that is independent of R. This should fit our intuition, that velocity is coordinate dependent but energy is not.

Example:

Rotation of a rigid block (a.k.a. rectangular prism). Consider a block with constant density ρ and dimensions 2ℓ, $2w$, and $2h$ aligned respectively with the x, y, and z axes of a coordinate frame placed at the center of the block. We will compute the kinetic energy of the block as it rotates about the z axis with an angular velocity of ω_1. More specifically, the body angular velocity of the block is $\underline{\omega}_1 = \begin{bmatrix} 0 & 0 & \omega_1 \end{bmatrix}^T$. Noting that in rectangular coordinates $\underline{r}_b = \begin{bmatrix} x & y & z \end{bmatrix}^T$, the body's rotational kinetic energy is

$$\begin{aligned}
\mathrm{KE}_{\mathrm{rot}} &= \frac{1}{2}\int_{\mathcal{V}} \rho\, (\widehat{\underline{\omega}}_1\,\underline{r}_b)^T (\widehat{\underline{\omega}}_1\,\underline{r}_b)\, d\mathcal{V} \\
&= \frac{1}{2}\int_{-h}^{h}\int_{-w}^{w}\int_{-\ell}^{\ell} \rho \begin{bmatrix} -y\omega_1 & x\omega_1 & 0 \end{bmatrix} \begin{bmatrix} -y\omega_1 \\ x\omega_1 \\ 0 \end{bmatrix} dxdydz \\
&= \frac{1}{2}\int_{-h}^{h}\int_{-w}^{w}\int_{-\ell}^{\ell} \rho\, \omega_1^2 (x^2+y^2) dxdydz
\end{aligned}$$

Crunching this integral by hand or with Mathematica produces

$$\begin{aligned}
\mathrm{KE}_{\mathrm{rot}} &= \frac{1}{2}\rho\,\omega_1^2\,\frac{8}{3}\,\ell w h(\ell^2+w^2) \\
&= \frac{1}{2} m_{\mathrm{tot}}\,\omega_1^2\,\frac{1}{3}\,(\ell^2+w^2) \\
&= \frac{1}{2} m_{\mathrm{tot}}\,\omega_1^2\,\frac{1}{12}\,((2\ell)^2+(2w)^2)
\end{aligned}$$

where m_{tot} is the total mass of the block, $m_{\text{tot}} = \rho(2\ell)(2w)(2h) = 8\rho\ell wh$. The last expression is included to express the answer in terms of the side lengths of the block, the way it may appear in other references that solve this problem (say, a physics book).

Notice that, if we fix m_{tot}, this answer is independent of h. That is, for this geometry the kinetic energy of a very tall block rotating will be the same as that for a very short block as long as they have the same total mass and angular velocity. Physically this is true because the dimension h does not influence how mass is distributed about the axis of rotation. Is this always true?

Example:

Rotation of a rigid block again. Using the same block and notation from the previous example, let's see how the computations are affected by changing the axis of rotation. This time, let's rotate the block about an axis in the yz plane that passes through an edge of the block. That is, use a body angular velocity of $\underline{\omega}_2 = \begin{bmatrix} 0 & \omega_2 \cos\theta & \omega_2 \sin\theta \end{bmatrix}^T$, where $\theta = \arctan \frac{h}{w}$. Moving forward we make use of the shorthand $\mathrm{s}_\theta = \sin\theta$ and $\mathrm{c}_\theta = \cos\theta$. Keep in mind s_θ and c_θ are constants in this problem. Now

$$
\begin{aligned}
\mathrm{KE}_{\text{rot}} &= \frac{1}{2}\int_{\mathcal{V}} \rho\,(\widehat{\underline{\omega}}_2\,\underline{r}_b)^T(\widehat{\underline{\omega}}_2\,\underline{r}_b)\,d\mathcal{V} \\
&= \frac{1}{2}\int_{-h}^{h}\int_{-w}^{w}\int_{-\ell}^{\ell} \rho \begin{bmatrix} \omega_2(z\,\mathrm{c}_\theta - y\,\mathrm{s}_\theta) & \omega_2\,x\,\mathrm{s}_\theta & -\omega_2\,x\,\mathrm{c}_\theta \end{bmatrix} \begin{bmatrix} \omega_2(z\,\mathrm{c}_\theta - y\,\mathrm{s}_\theta) \\ \omega_2\,x\,\mathrm{s}_\theta \\ -\omega_2\,x\,\mathrm{s}_\theta \end{bmatrix} dxdydz \\
&= \frac{1}{2}\int_{-h}^{h}\int_{-w}^{w}\int_{-\ell}^{\ell} \rho\,\omega_2^2(z^2\mathrm{c}_\theta^2 - 2zy\mathrm{c}_\theta\mathrm{s}_\theta + y^2\mathrm{s}_\theta^2 + x^2)dxdydz \\
&= \frac{1}{2}\rho\,\omega_2^2\,\frac{8}{3}\ell wh(h^2\mathrm{c}_\theta^2 + w^2\mathrm{s}_\theta^2 + \ell^2) \\
&= \frac{1}{2}m_{\text{tot}}\,\omega_2^2\,\frac{1}{3}\,(h^2\mathrm{c}_\theta^2 + w^2\mathrm{s}_\theta^2 + \ell^2) \\
&= \frac{1}{2}m_{\text{tot}}\,\omega_2^2\,\frac{1}{12}\,((2h)^2\mathrm{c}_\theta^2 + (2w)^2\mathrm{s}_\theta^2 + (2\ell)^2)
\end{aligned}
$$

In this case our result depends on all three of the block dimensions, ℓ, w, and h. This might have been expected, as changing any one of these dimensions does change the distribution of mass about the axis of rotation.

Also, this example gives some insight into the complexity of the general case. If the body angular velocity $\underline{\omega}$ is an arbitrary vector (not aligned with any particular axis or plane), then every entry in the vector $\widehat{\underline{\omega}}\,\underline{r}_b$ will be a binomial, and the number of terms in the volume integral will grow to 9. Even with Mathematica to help us, we'd rather not repeatedly perform these integrals every time the body angular velocity changes. Can we use some geometric properties to simplify the dependence of rotational kinetic energy on $\underline{\omega}$?

22 The Rotational Inertia Tensor

Take home messages:

- Body angular velocity can be isolated from the last lecture's volume integral. The remaining integral gives the body's inertia tensor.
- Placing the body frame at the center of mass simplifies the expression of total kinetic energy when both translations and rotations are present
- With m and $\mathcal{I}$ in hand, total kinetic energy is just the sum of a quadratic in $\underline{\dot{p}}$ and a quadratic in $\underline{\omega}$.

Defining Rotational Inertia

Let's return to our integral expression of rotational kinetic energy.

$$\mathrm{KE}_{\mathrm{rot}} = \frac{1}{2}\int_{\mathcal{V}} \rho\,(\widehat{\underline{\omega}}\,\underline{r}_b)^T(\widehat{\underline{\omega}}\,\underline{r}_b)\,d\mathcal{V}$$

We can manipulate this integral by utilizing the following property of the hat operator, $\widehat{\underline{\omega}}\,\underline{r}_b = -\widehat{\underline{r}}_b\,\underline{\omega}$. If this is not immediately apparent, it can easily be verified by hand. We can make use of this fact to produce

$$\begin{aligned}
\mathrm{KE}_{\mathrm{rot}} &= \frac{1}{2}\int_{\mathcal{V}} \rho\,(\widehat{\underline{\omega}}\,\underline{r}_b)^T(\widehat{\underline{\omega}}\,\underline{r}_b)\,d\mathcal{V} \\
&= \frac{1}{2}\int_{\mathcal{V}} \rho\,(-\widehat{\underline{r}}_b\,\underline{\omega})^T(-\widehat{\underline{r}}_b\,\underline{\omega})\,d\mathcal{V} \\
&= \frac{1}{2}\int_{\mathcal{V}} \rho\,(\widehat{\underline{r}}_b\,\underline{\omega})^T(\widehat{\underline{r}}_b\,\underline{\omega})\,d\mathcal{V} \\
&= \frac{1}{2}\int_{\mathcal{V}} \rho\,\underline{\omega}^T\,\widehat{\underline{r}}_b^T\,\widehat{\underline{r}}_b\,\underline{\omega}\,d\mathcal{V}
\end{aligned}$$

Notice the body angular velocity $\underline{\omega}$ is independent of the volume integral over $d\mathcal{V}$. Thus the $\underline{\omega}$'s can be pulled outside the integral producing

$$\mathrm{KE}_{\mathrm{rot}} = \frac{1}{2}\underline{\omega}^T\left(\int_{\mathcal{V}} \rho\,\widehat{\underline{r}}_b^T\,\widehat{\underline{r}}_b\,d\mathcal{V}\right)\underline{\omega}$$

The quantity inside the parentheses is known as the inertia tensor $\mathcal{I}$ for a rigid body. If computed once, it can be used for all time to relate body angular velocity and rotational kinetic energy. In the Cartesian coordinates of our two examples, $\underline{r}_b = \begin{bmatrix} x & y & z \end{bmatrix}^T$ and thus

$$\mathcal{I} = \int_{\mathcal{V}} \rho\,\widehat{\underline{r}}_b^T\,\widehat{\underline{r}}_b\,d\mathcal{V} = \int_{\mathcal{V}} \rho \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dxdydz$$

Combining Rotations and Translations

Suppose we have a rigid body in both translational and rotational motion relative to the stationary frame. Keeping the body frame affixed to the center of mass, it is now the pair $(\underline{p}, R)$ that gives the position and orientation of the body relative to the stationary frame. For a particle at $\underline{r}_b$, its velocity is given by $\dot{\underline{p}} + \dot{R}\,\underline{r}_b = \dot{\underline{p}} + R\,\widehat{\underline{\omega}}\,\underline{r}_b$. Noting this, the body's total kinetic energy is

$$
\begin{aligned}
\mathrm{KE}_{\mathrm{tot}} &= \frac{1}{2}\int_{\mathcal{V}} \rho\,(\dot{\underline{p}} + R\,\widehat{\underline{\omega}}\,\underline{r}_b)^T(\dot{\underline{p}} + R\,\widehat{\underline{\omega}}\,\underline{r}_b)\,d\mathcal{V} \\
&= \frac{1}{2}\int_{\mathcal{V}} \rho\,\left(\dot{\underline{p}}^T\dot{\underline{p}} + 2\dot{\underline{p}}^T R\,\widehat{\underline{\omega}}\,\underline{r}_b + (R\,\widehat{\underline{\omega}}\,\underline{r}_b)^T(R\,\widehat{\underline{\omega}}\,\underline{r}_b)\right)\,d\mathcal{V} \\
&= \frac{1}{2}\int_{\mathcal{V}} \rho\,\left(\dot{\underline{p}}^T\dot{\underline{p}} + 2\dot{\underline{p}}^T R\,\widehat{\underline{\omega}}\,\underline{r}_b + \underline{\omega}^T\,\widehat{\underline{r}}_b^T\,\widehat{\underline{r}}_b\,\underline{\omega}\right)\,d\mathcal{V}
\end{aligned}
$$

Examining specifically the middle term

$$
\frac{1}{2}\int_{\mathcal{V}} \rho\,2\dot{\underline{p}}^T R\,\widehat{\underline{\omega}}\,\underline{r}_b\,d\mathcal{V} = \left(\dot{\underline{p}}^T R\,\widehat{\underline{\omega}}\right)\int_{\mathcal{V}} \rho\,\underline{r}_b\,d\mathcal{V} = \left(\dot{\underline{p}}^T R\,\widehat{\underline{\omega}}\right) 0 = 0
$$

where the integral vanishes because we have placed the body frame at the center of mass. Simplifying the remaining nonvanishing terms

$$
\begin{aligned}
\mathrm{KE}_{\mathrm{tot}} &= \frac{1}{2}\int_{\mathcal{V}} \rho\,\left(\dot{\underline{p}}^T\dot{\underline{p}} + \underline{\omega}^T\,\widehat{\underline{r}}_b^T\,\widehat{\underline{r}}_b\,\underline{\omega}\right)\,d\mathcal{V} \\
&= \frac{1}{2}\left(\dot{\underline{p}}^T\dot{\underline{p}}\right)\int_{\mathcal{V}} \rho\,d\mathcal{V} + \frac{1}{2}\underline{\omega}^T\left(\int_{\mathcal{V}} \rho\,\widehat{\underline{r}}_b^T\,\widehat{\underline{r}}_b\,d\mathcal{V}\right)\underline{\omega} \\
&= \frac{1}{2}m\,\dot{\underline{p}}^T\dot{\underline{p}} + \frac{1}{2}\underline{\omega}^T\,\mathcal{I}\,\underline{\omega}
\end{aligned}
$$