1) $R_1(\theta_1)$ and $R_2(\theta_2) \in SO(n)$

Prove $R_1(\theta_1) R_2(\theta_2) \in SO(n)$:

$$\det(R_1) = \det(R_2) = 1$$
$$R_1^T R_1 = R_2^T R_2 = I$$

$$(R_1 R_2)^T R_1 R_2 = R_2^T \overbrace{R_1^T R_1}^{I} R_2$$
$$= R_2^T R_2$$
$$\underline{(R_1 R_2)^T R_1 R_2 = I}$$

$$\det(R_1 R_2) = \det(R_1)\det(R_2)$$
$$= \underline{\underline{1 \cdot 1 = 1}}$$

2) $g_1(x_1, y_1, \theta_1) \in SE(2)$

$$\longrightarrow g_1 = \begin{bmatrix} R_1(\theta_1) & \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ 0\ 0 & 1 \end{bmatrix}. \quad R_1(\theta) \in SO(2), \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

$g_2(x_2, y_2, \theta_2) \in SE(2)$

$$\longrightarrow g_2 = \begin{bmatrix} R_2(\theta_2) & \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ 0\ 0 & 1 \end{bmatrix}. \quad R_2(\theta) \in SO(2), \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

continued $\longrightarrow$

2) Show $g_1(x_1, y_1, \theta_1)\, g_2(x_2, y_2, \theta_2) \in SE(2)$.

$$g_1 g_2 = \begin{bmatrix} R_1(\theta_1) & \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ 0 \; 0 & 1 \end{bmatrix} \begin{bmatrix} R_2(\theta) & \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ 0 \; 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_1(\theta_1) & P_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2(\theta_1) & P_2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 R_2 & R_1 P_2 + P_1 \\ 0 & 1 \end{bmatrix}$$

R_1 & $R_2 \in SO(2) \longrightarrow R_1 R_2 \in SO(2)$

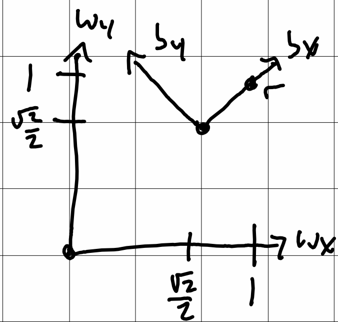
$R_1 P_2$ & $P_1 \in \mathbb{R}^{2\times 1} \longrightarrow R_1 P_2 + P_1 \in \mathbb{R}^{2\times 1}$

Therefore the product $g_1(x_1, y_1, \theta_1)\, g_2(x_2, y_2, \theta_2)$ has form

$$\begin{bmatrix} R \in SO(2) & P \in \mathbb{R}^{2\times 1} \\ 0 & 1 \end{bmatrix},$$

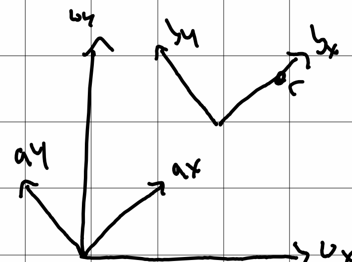
implying $g_1 g_2$ belongs to $SE(2)$.

3. Homogeneous transformation in $SE(2)$:



Take a change of coordinate from w frame to b frame of point r:

use an intermediate frame:



$$g_{wa} = \begin{bmatrix} R_{wa} & 0 \\ 0 & 1 \end{bmatrix}, \quad R_{wa} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$g_{AB} = \begin{bmatrix} I & P_{AB} \\ 0 & 1 \end{bmatrix} \qquad P_{AB} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{r}_w = \begin{bmatrix} r_w \\ 1 \end{bmatrix} = G_A G_B \bar{r}_B$$

"r_i" = vector position of r in the "i" frame

"$\bar{r}_i$" = vector with a trailing 1

$$\bar{r}_w = \begin{bmatrix} R_{wa} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & P_{AB} \\ 0 & 1 \end{bmatrix} \bar{r}_B$$

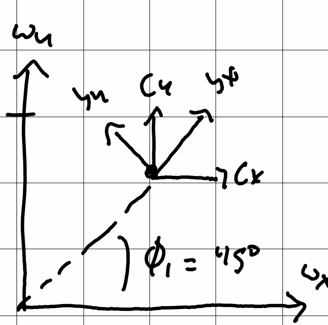
continued →

3. $\bar{r}_W = \begin{bmatrix} R_{wa} & R_{wa}P_{ab} \\ 0 & 1 \end{bmatrix} \bar{r}_B$:

a generalized transformation has been decomposed into a rotation and a translation.

The order of operations we should usually use is rotation $\longrightarrow$ translation, because if we translated first, both the rotation and translation would be in terms of our configuration variable, which takes more work.

If the order were reversed:



$$G_{WC} = \begin{bmatrix} I & P_{WC} \\ 0 & 1 \end{bmatrix}, \; P_{WC} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

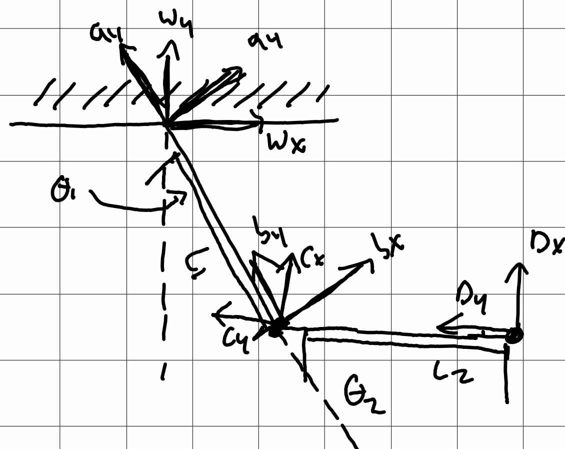
$$G_{CB} = \begin{bmatrix} R_{CB} & 0 \\ 0 & 1 \end{bmatrix}, \; R_{CB} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$G_{WC}G_{CB} = \begin{bmatrix} I & P_{WC} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_{CB} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\bar{r}_W = \begin{bmatrix} R_{CB} & P_{WC} \\ 0 & 1 \end{bmatrix} \bar{r}_B$$

difference: this version has a rotation relative to the "c" frame, and a translation expressed in the world frame.

4. Diagram of how frames were defined:



Transformation matrices:

$$G_{WA} = \begin{bmatrix} R_{WA}(\theta_1) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad R_{WA} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$

$$G_{AB} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ -L_1 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$G_{BC} = \begin{bmatrix} R_{BC}(\theta) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad R_{BC} = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$$

$$G_{CD} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ -L_2 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$