15 Rigid Body Motions

What is the Euler-Lagrange *Algorithm*?

1. choose coordinates q
2. calculate kinetic energy KE_i for each point i
3. calculate potential energy V_i for each point i
4. calculate ϕ_j for each constraint and impact j
5. calculate the external forces F_k
6. plug all of this into the Euler-Lagrange Equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \lambda \nabla \phi + F$$
$$\frac{d^2}{dt^2}\phi(q(t)) = 0$$

So what do we not know how to do at this point? We don't know how to represent all this data in a reasonably efficient manner!

Rigid Body Motions

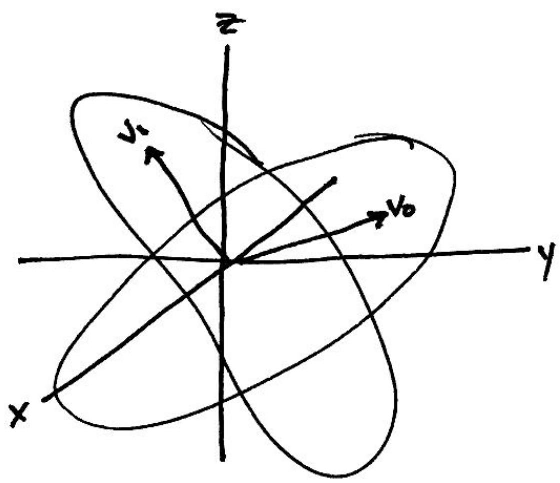

Figure 19: pic of a body at two different orientations with points v_0 and v_1

Assume that we have a linear map relating v_0 and v_1 so that $Av_0 = v_1$. A rigid body motion should satisfy the requirement that the magnitude of v_0 should equal v_1. That is, $\|v_0\| = \|v_1\| \Rightarrow \|v_0\|^2 = \|v_1\|^2$

$$\begin{aligned}
&\Rightarrow \langle v_0, v_0\rangle = \langle v_1, v_1\rangle \\
&\Rightarrow v_0^T v_0 = v_1^T v_1 \\
&\Rightarrow v_0^T v_0 = (Av_0)^T (Av_0) \\
&\Rightarrow v_0^T v_0 = v_0^T A^T A v_0 \quad \forall \quad v_0 \\
&\Rightarrow A^T A = I.
\end{aligned}$$

Therefore, A is an element of $O(n)$ (called the "orthogonal group"). They are called orthogonal because $A^T = A^{-1}$.

Moreover, since $\det(I) = 1$ and $\det(A^T A) = \det(A)^2$, we know that $\det(A) = \pm 1$. If $\det(A) = -1$, A is a reflection. We are interested in the case where $\det(A) = +1$. This set is called the "special orthogonal group" and will turn out to be all the rigid body rotations when $n = 3$.

Note that the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is an element of $SO(2)$ because

$$\det(R(\theta)) = \cos^2\theta + \sin^2\theta = 1$$

and $R^T R =$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \left[\begin{array}{c|c} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \hline -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

HW problem: show that rigid body motions also preserve angles between v_0 and v_1 by using the fact that the angle θ between v_0 and v_1 is defined by

$$\cos\theta = \frac{\langle v_0, v_1 \rangle}{\|v_0\| \|v_1\|}$$

16 Matrix Representations of Rigid Body Motions

Overview of Matrix Representations of rigid body motion in $\mathbb{R}^2$ and $\mathbb{R}^3$

There are four fundamental spaces of interest

1. $SO(2)$ planar rotations (parameterized by θ)
2. $SE(2)$ planar rotations and translations (parameterized by x, y, θ)
3. $SO(3)$ 3D rotations (locally parameterized by θ, ψ, ϕ—roll, pitch, yaw)
4. $SE(3)$ 3D rotations and translations (locally parameterized by $x, y, z, \theta, \psi, \phi$)

Let's take a look at $SO(2)$

$$SO(2) = \left\{R | R^T R = I \& \det(R) = 1\right\} = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} | \theta \in \mathbb{R} \right\}.$$

Things $SO(2)$ does:

1. takes a point $r \in \mathbb{R}^2$ and rotates it
2. transforms the coordinates form one frame to another
3. represents a rigid body motion or trajectory

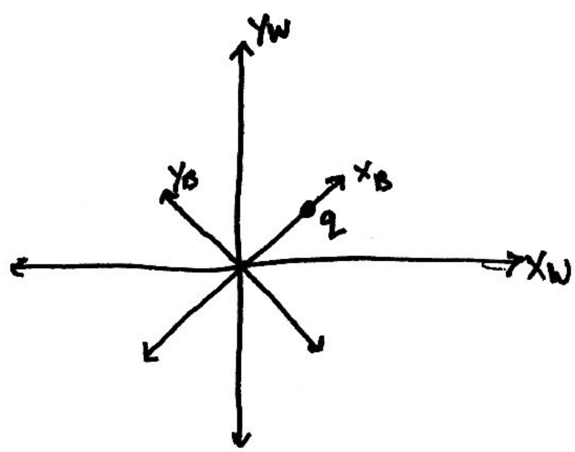

Figure 20: Two frames rotated by θ

$$r_W = \begin{bmatrix} r_x \\ r_y \end{bmatrix}_W = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix}_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} r_B$$

e.g., let $\theta = 45^\circ, r_B = (1, 0)^T$

$$= \begin{bmatrix} \frac{1}{\sqrt{(2)}} & -\frac{1}{\sqrt{(2)}} \\ \frac{1}{\sqrt{(2)}} & \frac{1}{\sqrt{(2)}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{(2)}} \\ \frac{1}{\sqrt{(2)}} \end{bmatrix}$$

So, given $\theta(t)$ and r_B, we can get r_W. Moreover, since we know that $r_W = R(\theta) r_B$, we can find the velocity of the point r in inertial coordinates by looking at $\frac{d}{dt} r_W = \frac{d}{dt}(R(\theta)) r_B$ (because r_B is constant because it is attached to the body). This means we could calculate the kinetic energy using the rigid body rotation $R(\theta)$ instead of doing trigonometry.

It is worth noting that $SO(2)$ is equivalent to the unit-magnitude complex numbers.

Now let's look at $SE(2)$

$SE(2)$ is parameterized by x, y, θ.

$$SE(2) = \left\{ (R, p) | R^T R = I \& \det(R) = 1 \& p \in \mathbb{R}^2 \right\} = \left\{ \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} p_x \\ p_y \end{bmatrix} \right) | (p_x, p_y, \theta) \in \mathbb{R}^3 \right\}.$$

We transform points r_B in the plane by multiplying them by R and adding $p = (p_x, p_y)$.

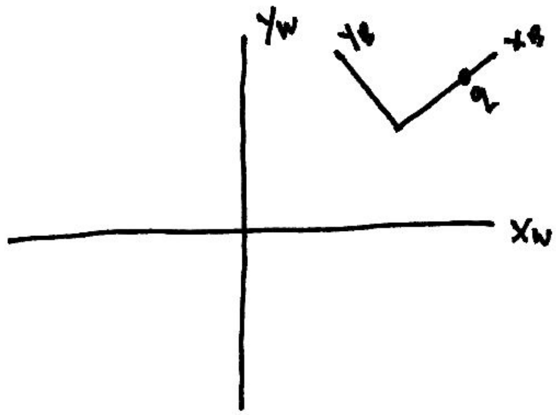

Figure 21: Two frames in the plane

So, for instance, one might want to transform $r_B = (1, 0)$ to r_W using the rigid body transformation $(R(\theta = 45°), (1, 1)^T)$.

$$r_W = \begin{bmatrix} r_x \\ r_y \end{bmatrix}_W = R(\theta = 45°) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{\sqrt{(2)}} \\ 1 + \frac{1}{\sqrt{(2)}} \end{bmatrix}$$

Moreover, we might want to be able to rewrite this operation so that it only uses matrix multiplication. Define the matrix $g(R, p)$ to be

$$g(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad R \in \mathbb{R}^{2\times 2} \quad p \in \mathbb{R}^{2\times 1} \quad 0 \in \mathbb{R}^{1\times 2} \quad 1 \in \mathbb{R}^{1\times 1}$$

and define

$$\bar{r} = \begin{bmatrix} r_x \\ r_y \\ 1 \end{bmatrix}.$$

What happens when we multiply $\bar{r}$ by $g(R, p)$?

$$g(R, p)\bar{r} = \begin{bmatrix} Rr + p \\ 1 \end{bmatrix}$$

which, if we "unbar" r gives us the rigid body transformation $Rr + p$. This g is called the *homogeneous representation* of the rigid body transformation and the $\bar{r}$ is called the *homogeneous representation* of the point r.

Let's rewrite the transformation of $r_B = (1, 0)$ to r_W using the rigid body transformation $(R(\theta = 45°), (1, 1)^T)$.

$$\bar{r}_W = \begin{bmatrix} r_x \\ r_y \\ 1 \end{bmatrix}_W = \begin{bmatrix} \frac{1}{\sqrt{(2)}} & -\frac{1}{\sqrt{(2)}} & 1 \\ \frac{1}{\sqrt{(2)}} & \frac{1}{\sqrt{(2)}} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{\sqrt{(2)}} \\ 1 + \frac{1}{\sqrt{(2)}} \\ 1 \end{bmatrix}$$

We aren't yet seeing any major advantages of rigid body transformations, but we will. In particular, composition of rigid body transformations is achieved through matrix multiplication!

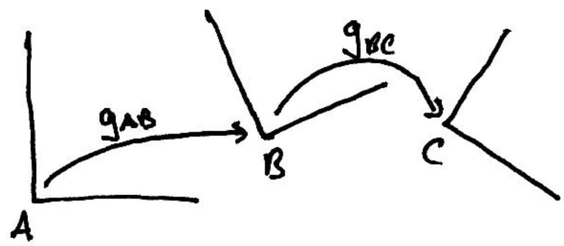

Figure 22: multiple transformations in the plane

So assume that we have a bunch of transformations (like the multiple link pendulum). Then the transformation

$$g_{AC} = g_{AB}g_{BC} = \begin{bmatrix} R_{AB} & p_{AB} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{BC} & p_{BC} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{AB}R_{BC} & R_{AB}p_{BC} + p_{AB} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{AC} & p_{AC} \\ 0 & 1 \end{bmatrix}.$$

This means that elements of $SE(2)$ compose (through matrix multiplication) to form other elements of $SE(2)$—assuming that multiplying two rotation matrices together yields another rotation matrix (something you will have to prove on the homework).

Lastly, note that we could have just as easily been looking at the case of $n = 3$. The only things that change are that that R is no longer parameterized by a single variable θ and $p \in \mathbb{R}^3$.

Example: multiple transformations

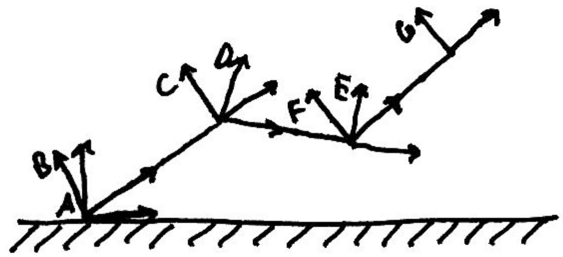

Figure 23: multiple transformations for a robot arm

What do these transformations look like?

$$g_{AB} = \begin{bmatrix} R(\theta_1) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \quad g_{BC} = \begin{bmatrix} I_{2\times 2} & \begin{bmatrix} L \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \quad g_{CD} = \begin{bmatrix} R(\theta_2) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$g_{DE} = g_{BC} \quad g_{EF} = \begin{bmatrix} R(\theta_3) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \quad g_{FG} = g_{BC}$$

and we have

$$r_A = g_{AB}\, \underbrace{g_{BC}\, \underbrace{g_{CD}\, \underbrace{g_{DE}\, \underbrace{g_{EF}\, \underbrace{g_{FG} r_G}_{r_F}}_{r_E}}_{r_D}}_{r_C}}_{r_B}$$

and if we wanted to calculate the velocity of the point q relative to A, we would differentiate both sides:

$$\frac{d}{dt} r_A = \frac{d}{dt}(g_{AB} g_{BC} g_{CD} g_{DE} g_{EF} g_{FG}) r_G$$

Something to think about–is it easy to differentiate this with respect to time? How would you use product rule and chain rule?