18 Euler Angles

How do we typically parameterize $SO(3)$—the space of rigid body rotations? Typically, particularly with mechanisms,[5] we use Euler angles, defined by:

$$R_Z = e^{\theta\hat{\omega}_Z} \quad R_X = e^{\psi\hat{\omega}_X} \quad R_Y = e^{\phi\hat{\omega}_Y}$$

where

$$\omega_Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \omega_X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \omega_Y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Note that Euler angles do not commute:

$$R_X R_Y R_Z \neq R_Y R_X R_Z$$

but both of these are common choices for parameterizing $SO(3)$.

Example: 3D mechanism

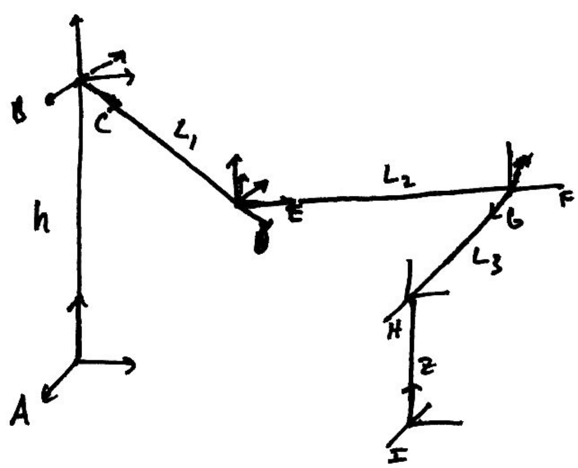

Figure 25: 3D mechanism

$$g_{AI} = g_{AB}g_{BC}g_{CD}g_{DE}g_{EF}g_{FG}g_{GH}g_{HI}$$

$$= \begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_Z(\theta_1) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_Z(\theta_2) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_Z(\theta_3) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Also, $q = (\theta_1, \theta_2, \theta_3, z)$ and $\frac{d}{dt}\begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} 0 \\ 0 \\ \dot{z} \end{bmatrix} \\ 0 & 0 \end{bmatrix}$ and $\frac{d}{dz}\begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ 0 & 0 \end{bmatrix}$

[5] In other instances, like modeling satellites in orbit, Euler angles can be disasterous because of singularities in their parameterization of $SO(3)$, but for our purposes they will be fine for almost everything we might come up with.