23 Body Velocities

Without any real motivation except for the fact that $R^{-1}\dot{R}$ is important in calculating the kinetic energy for pure rotation, look at $g^{-1}\dot{g}$, called the *body velocity* for $SE(n)$.

$$g^{-1}\dot{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & R^T\dot{p} \\ 0 & 0 \end{bmatrix}$$

Now look at kinetic energy.

$$KE = \frac{1}{2}m\|\dot{p}\|^2 + \frac{1}{2}\omega^T \mathcal{I}\omega$$

and note that since R preserves distances (and therefore R^T preserves distances), we can also write the kinetic energy as:

$$KE = \frac{1}{2}m\|R^T\dot{p}\|^2 + \frac{1}{2}\omega^T \mathcal{I}\omega.$$

So how do we write the kinetic energy in terms of the body velocity? Define the "hat" operator of a vector of the form $[v, \omega] \in \mathbb{R}^6$ to be

$$\begin{bmatrix} v \\ \omega \end{bmatrix}^{\hat{}} = \widehat{\begin{bmatrix} v \\ \omega \end{bmatrix}} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}.$$

Then we can take the "unhat" of a 4×4 matrix to get an element of $\mathbb{R}^6$. In particular define the body velocity to be the vector $V^b = \mathbb{R}^6$ that, when hatted, is $g^{-1}\dot{g}$,

$$V^b = (g^{-1}\dot{g})^{\check{}} = \begin{bmatrix} R^T\dot{p} \\ \omega \end{bmatrix}$$

which gives us

$$KE = \frac{1}{2}(V^b)^T \begin{bmatrix} mI_{n\times n} & 0 \\ 0 & \mathcal{I} \end{bmatrix} V^b$$

The terrific thing about this formulation of the kinetic energy is that you can compute the dynamics of a set of general interconnected rigid bodies by just looking at their inertial properties in the body frame and looking at the g tranformations that interconnect them.

Transforming $\mathcal{I}$

Look at the body velocity V^b of a particular body defined in terms of two different frames, the first to one part of a body and the second to another part of the body.

$$\hat{V}_B^b = g_{WB}^{-1}\dot{g}_{WB} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

$$\hat{V}_C^b = g_{WC}^{-1}\dot{g}_{WC} = (g_{WB}g_{BC})^{-1}\frac{d}{dt}(g_{WB}g_{BC}) = g_{BC}^{-1}\hat{V}_B^b g_{BC}.$$

(Note that $\dot{g}_{BC} = 0$ because g_{BC} is a transfromation within the body that does not vary with time.) Looking at the right hand side, $\hat{V}_C^b$ is a coordinate transformation of $\hat{V}_B^b$. Let's try to rewrite this without the "hat" operator.

$$\begin{aligned}
\hat{V}_C^b &= \begin{bmatrix} R^T & -R^Tp \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R^T & -R^Tp \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega}R & \hat{\omega}p + v \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R^T\hat{\omega}R & R^T(\hat{\omega}p + v) \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Now, using the fact that $\widehat{R^T\omega} = R^T\hat{\omega}R$ and $\hat{\omega}p = -\hat{p}\omega$, we get

$$\begin{aligned}
\hat{V}_C^b &= \begin{bmatrix} \widehat{R^T\omega} & R^T(-\hat{p}\omega + v) \\ 0 & 0 \end{bmatrix} \\
\Rightarrow V_C^b &= \begin{bmatrix} R^T(-\hat{p}\omega + v) \\ R^T\omega \end{bmatrix} \\
&= \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix} = \underbrace{\begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}}_{Adjoint} V_B^b.
\end{aligned}$$

Example

Assume that g_{WB} is of the form

$$g_{WB} = \begin{bmatrix} R_Z(\theta) & \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

and g_{BC} is of the form

$$g_{BC} = \begin{bmatrix} R_Z(\psi) & \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

where ψ, p_x and p_y are constants. Then you can verify that

$$V_C^b = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} V_B^b$$

Back to transforming $\mathcal{I}$

Why is this useful for transforming $\mathcal{I}$? Now look at kinetic energy (which we know is coordinate invariant).

$$\begin{aligned}
KE &= \frac{1}{2}(V_B^b)^T \mathcal{I}_{\mathcal{B}} V_B^b \\
&= \frac{1}{2}(V_C^b)^T \mathcal{I}_{\mathcal{C}} V_C^b \\
&= \frac{1}{2}\left(\begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} V_B^b \right)^T \mathcal{I}_{\mathcal{C}} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} V_B^b \\
&= \frac{1}{2}(V_B^b)^T \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}^T \mathcal{I}_{\mathcal{C}} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} V_B^b.
\end{aligned}$$

Hence, since this must be true for all possible V_B^b, we get that

$$\mathcal{I}_B = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}^T \mathcal{I}_{\mathcal{C}} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$$

Example

Consider the simple situation where g_{WB} is of the form

$$g_{WB} = \begin{bmatrix} R_Z(\theta) & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

and g_{BC} is of the form

$$g_{BC} = \begin{bmatrix} I_{3\times3} & \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

where L is constant. If the body is a point mass and frame C is located at the mass, then in the C frame $\mathcal{I}_C = diag(m, m, m, 0, 0, 0)$. However, in the B frame (the point around which the mass is moving), we get

$$\begin{aligned}
\mathcal{I}_B &= \begin{bmatrix} I_{3\times3} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & L \\ 0 & -L & 0 \end{bmatrix} \\ 0_{3\times3} & I_{3\times3} \end{bmatrix}^T \mathcal{I}_{\mathcal{C}} \begin{bmatrix} I_{3\times3} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & L \\ 0 & -L & 0 \end{bmatrix} \\ 0_{3\times3} & I_{3\times3} \end{bmatrix} \\
&= \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & Lm \\ 0 & 0 & m & 0 & -Lm & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Lm & 0 & L^2m & 0 \\ 0 & Lm & 0 & 0 & 0 & L^2m \end{bmatrix}
\end{aligned}$$

The kinetic energy calculation in both instances yields the same kinetic energy of $KE = \frac{1}{2}mL^2\dot{\theta}^2$.