Proving a given pt is a local minimizer of f:

objective is to show $\frac{\partial}{\partial x}(f) = \frac{\partial}{\partial y}(f) = 0$

and $\left(\frac{\partial^2}{\partial x^2} f, \frac{\partial^2}{\partial y^2} f\right) > 0$ (increasing)

$f(x,y) = \sin(x+y)\sin(x-y)$

$$\frac{\partial f}{\partial x} = (\sin(x+y))\frac{\partial}{\partial x}(\sin(x-y)) + \left(\frac{\partial}{\partial x}\sin(x+y)\right)(\sin(x-y))$$

$$= \sin(x+y)\cos(x-y)\cdot 1 + \cos(x+y)\sin(x-y)\cdot 1$$

identity: $\sin(a)\cos(b) + \cos(a)\sin(b) = \sin(A+B)$

$$\sin(x+y)\cos(x-y)\cdot 1 + \cos(x+y)\sin(x-y)\cdot 1 = \sin((x+y)+(x-y))$$

$$\frac{\partial f}{\partial x} = \sin(2x)$$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos(2x)$$

$$\frac{\partial f}{\partial y} = (\sin(x+y)) \frac{\partial}{\partial y}(\sin(x-y)) + \left(\frac{\partial}{\partial y}\sin(x+y)\right)(\sin(x-y))$$

$$= \sin(x+y)\cos(x-y)\cdot -1$$
$$+ \cos(x+y)\sin(x-y)$$

$$= \sin(x-y)\cos(x+y) - \cos(x-y)\sin(x+y)$$

$$\left(\sin(a)\cos(b) - \cos(a)\sin(b) = \sin(a-b)\right)$$

$$= \sin((x-y)-(x+y))$$
$$= \sin(-2y)$$
$$\frac{\partial f}{\partial y} = -\sin(2y)$$

$$\frac{\partial^2 f}{\partial y^2} = -2\cos(2y)$$

1st + 2nd derivatives at given pt:

$$\frac{\partial f}{\partial x} = \sin(2x)$$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos(2x)$$

$$\frac{\partial f}{\partial y} = -\sin(2y)$$

$$\frac{\partial^2 f}{\partial y^2} = -2\cos(2y)$$

$$f_x\left(0, \tfrac{\pi}{2}\right) = \sin(0) = 0$$

$$f_{xx}\left(0, \tfrac{\pi}{2}\right) = 2\cos(0) = 2$$

$$f_y\left(0, \tfrac{\pi}{2}\right) = -\sin(\pi) = 0$$

$$f_{yy}\left(0, \tfrac{\pi}{2}\right) = -2\cos(\pi) = 2$$

Function is at a local minimum in both x and y.

2) Equations of motion in (y_1, y_2) coords;

$$KE = \frac{1}{2}m_1 \dot{y}_1^2 + \frac{1}{2}m_2 \dot{y}_2^2$$

$$U = \frac{1}{2}k_1 y_1^2 + \frac{1}{2}k_2 (y_2 - y_1)^2$$

$$L = KE - U$$

$$= \frac{1}{2}m_1 \dot{y}_1^2 + \frac{1}{2}m_2 \dot{y}_2^2 - \frac{1}{2}k_1 y_1^2 - \frac{1}{2}k_2 (y_2 - y_1)^2$$

State variables:

$$q = [y_1 \; y_2]^T \longrightarrow \dot{q} = [\dot{y}_1 \; \dot{y}_2]^T$$

$$q_{ext} = [y_1 \; y_2 \; \dot{y}_1 \; \dot{y}_2]^T$$

$$\frac{\partial L}{\partial q} = \left[-k_1 y_1 + k_2(y_2 - y_1) \quad -k_2(y_2 - y_1)\right]$$

$$\frac{\partial L}{\partial \dot{q}} = \left[m_1 \dot{y}_1 \quad m_2 \dot{y}_2\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \left[m_1 \ddot{y}_1 \quad m_2 \ddot{y}_2\right]$$

$$LHS = \begin{bmatrix} -k_1 y_1 + k_2 (y_2 - y_1) - m_1 \ddot{y}_1 \\ -k_2 (y_2 - y_1) - m_2 \ddot{y}_2 \end{bmatrix}$$

$$\begin{bmatrix} -k_1 y_1 + k_2(y_2 - y_1) - m_1 \ddot{y}_1 \\ -k_2(y_2 - y_1) - m_2 \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Sol: $\ddot{y}_1 = \frac{1}{m_1}\left(-k_1 y_1 + k_2(y_2 - y_1)\right)$

$\ddot{y}_2 = \frac{1}{m_2}\left(-k_2(y_2 - y_1)\right)$

Equations of motion in terms of θ_1, θ_2:

$$KE = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

$$U = \frac{1}{2} k_1 y_1^2 + \frac{1}{2} k_2 (y_2 - y_1)^2$$

$$y_1 = L\tan(\theta_1)$$

$$y_2 = L\tan(\theta_1 + \theta_2)$$

$$\dot{y}_1 = L\sec^2(\theta_1)\,\dot{\theta}_1$$

$$\dot{y}_2 = L\sec^2(\theta_1 + \theta_2)\,(\dot{\theta}_1 + \dot{\theta}_2)$$

this will quickly become a mess so I will let Sympy do it

3. Newton's Equations in Non-Inertial Reference Frames:

Reference frame using x will accelerate

$$q = [x_1, x_2]$$
$$\dot{q} = [\dot{x}_1, \dot{x}_2]$$

$$KE_{sys} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2 + \dot{x}_1)^2$$
$$U_{sys} = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$$

$$L = KE - U$$
$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2 + \dot{x}_1)^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2$$

Find E-L equations using this frame + prove predictions are different from y_1, y_2 frame

3. Apply coordinate change to $\ddot{x}$ equations to compare properly to y equations:

$$x_1 = y_1$$
$$x_2 = y_2 - y_1$$

$$\ddot{x}_1 = \ddot{y}_1$$
$$\ddot{x}_2 = \ddot{y}_2 - \ddot{y}_1$$

$$\ddot{x}_1 = -\frac{1}{m_1}\left(k_1 x_1 + k_2 x_2\right)$$
$$\ddot{y}_1 = -\frac{1}{m_1}\left(k_1 y_1 + k_2 (y_2 - y_1)\right)$$

$$\ddot{x}_2 = \frac{1}{m_1}\left(k_1 x_1 - k_2 x_2\right) - \frac{1}{m_2} k_2 x_2$$
$$= \frac{1}{m_1}\left(k_1 y_1 - k_2 (y_2 - y_1)\right) - \frac{1}{m_2} k_2 (y_2 - y_1)$$
$$= \frac{k_1}{m_1} y_1 - \frac{k_2}{m_1} y_2 + \frac{k_2}{m_1} y_1 - \frac{k_2}{m_2} y_2 + \frac{k_2}{m_2} y_1$$
$$\ddot{y}_2 - \ddot{y}_1 = y_1\left(\frac{k_1}{m_1} + \frac{k_2}{m_1} + \frac{k_2}{m_2}\right) + y_2\left(-\frac{k_2}{m_1} - \frac{k_2}{m_2}\right)$$

$$\ddot{y}_2 = \ddot{y}_1 + \text{"}$$
$$= \cancel{-\frac{k_1}{m_1} y_1} - \frac{k_2}{m_1} y_2 + \cancel{\frac{k_2}{m_1}} y_1$$
$$+ \cancel{\frac{k_1}{m_1}} y_1 + \cancel{\frac{k_2}{m_1}} y_1 + \frac{k_2}{m_2} y_1$$
$$- \frac{k_2}{m_1} y_2 - \frac{k_2}{m_2} y_2$$
$$= -\frac{2k_2}{m_1} y_2 + \frac{k_2}{m_2} y_1 - \frac{k_2}{m_2} y_2 \; ?$$

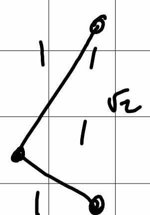
doesn't make sense, I must have made an error

4.

Constrained E-L Equations:

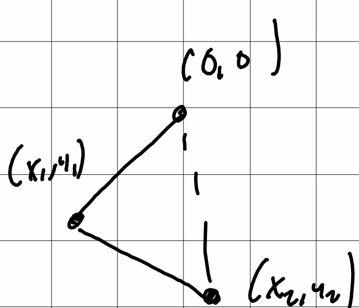
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \lambda \nabla \phi$$

System in question:



Dist. from origin must stay at $\sqrt{2}$

Constraint equation expression:



$$x_2^2 + y_2^2 = 2$$

$$x_2^2 + y_2^2 - 2 = 0$$

Constraint side of equals:

$$\phi = R_1^2 + 2R_1R_2\cos(\theta_2(t)) + R_2^2 - 2$$

↳ assume Python's calculations are right

$$\frac{1}{2R_1R_2}\frac{d\phi}{dt} = -\sin(\theta_2)\dot{\theta}_2$$

$$\frac{1}{2R_1R_2}\frac{d^2\phi}{dt^2} = -(\cos\theta_2)(\dot{\theta}_2)^2 - (\sin\theta_2)(\ddot{\theta}_2)$$

$$\nabla\phi = \begin{bmatrix} \frac{d\phi}{d\theta_1} \\ \frac{d\phi}{d\theta_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2R_1R_2\sin(\theta_2(t)) \end{bmatrix}$$

$$\lambda\nabla\phi = \begin{bmatrix} 0 \\ -2\lambda R_1R_2\sin(\theta_2(t)) \end{bmatrix}$$

$$LHS - \lambda\nabla\phi$$

$$= LHS + \begin{bmatrix} 0 \\ 2\lambda R_1R_2\sin(\theta_2(t)) \end{bmatrix}$$

solved equations: idk, but I would think θ_1 would oscillate back and forth + θ_2 would not vary

- try to understand the problem better to be able to revise
- check results by plotting energy of system over time

maybe constrain it so that theta2d = 0 b/c that's a true fact of the double-pend system

can check w/ those results

Hessian matrix new hand calcs:

$f = \sin(x+y)\,\sin(x-y)$

$\frac{\partial f}{\partial x} = (\sin(x+y))(\cos(x-y)) + (\cos(x+y))(\sin(x-y))$

$= \sin(a)\cos(b) + \cos(a)\sin(b)$

$\hookrightarrow$ via trig $= \sin(a+b)$

$= \sin(x+y+x-y)$

$= \sin(2x)$

$\frac{\partial f}{\partial y} = (\sin(x+y))(-\cos(x-y)) + (\cos(x+y))(\sin(x-y))$

$= \sin(a)\cos(b) - \cos(a)(\sin(b))$

$\hookrightarrow = \sin(a-b)$

$= \sin(x-y)\cos(x+y) - \cos(x-y)\sin(x+y)$

$= \sin(x-y-x-y)$

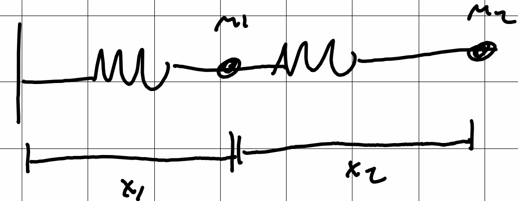
$= \sin(-2y)$

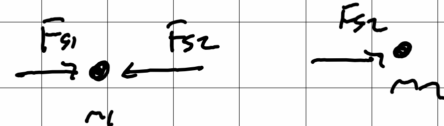
$\frac{\partial^2 f}{\partial x^2} = 2\cos(2x)$

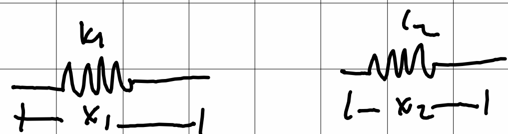
$\frac{\partial^2 f}{\partial y^2} = -2\cos(2y)$

$\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x} = \underline{0}$

3. Redo; let's derive Newton's 2nd Law for this system







$F_{s1} = k_1 x_1$ $\qquad$ $F_{s2} = k_2 x_2$

Sum of forces;

$$\Sigma F_1 = k_1 x_1 - k_2 x_2$$

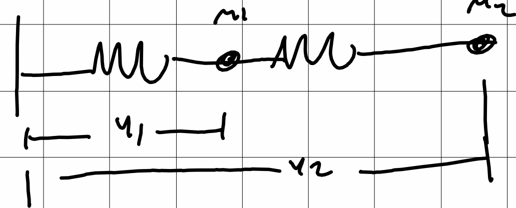
$$m\ddot{x}_1 = k_1 x_1 - k_2 x_2$$

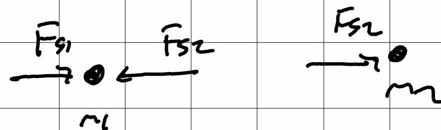
$$\Sigma F_2 = k_2 x_2$$

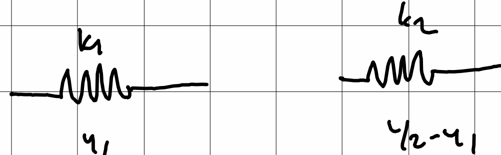
$$m(\ddot{x}_1 + \ddot{x}_2) = k_2 x_2$$

$$m\ddot{x}_2 = 2k_2 x_2 - k_1 x_1$$

In Cartesian inertial reference frame:







$\Sigma F_1 = F_{s1} - F_{s2}$

$m\ddot{y}_1 = k_1 y_1 - k_2(y_2 - y_1)$

$\Sigma F_2 = F_{s2}$

$m\ddot{y}_2 = k_2(y_2 - y_1)$

if I compare $m\ddot{x}_2$ and $m\ddot{y}_2$ after the coord changes I get and diff. results. unless if I don't factor in that $\ddot{x}_2 = \ddot{y}_2 - \ddot{y}_1$,

Change coordinates of x frame to y frame to compare results.

$m\ddot{x}_1 = k_1 x_1 - k_2 x_2 \longrightarrow m\ddot{y}_1 = k_1 y_1 - k_2(y_2 - y_1)$

$m(\ddot{x}_1 + \ddot{x}_2) = k_2 x_2 \longrightarrow m\ddot{y}_2 = k_2(y_2 - y_1)$?

gives same result. or should we compare $m\ddot{x}_2$ and $m\ddot{y}_2$?