17 Euler's Theorem and Matrix Exponentials

Euler's theorem states that any rotation can be represented as a rotation of θ around a fixed axis ω.

Let $\omega \in \mathbb{R}^3$ and $\|\omega\| = 1$. For $\omega \in \mathbb{R}^3$ define the "hat" operator

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Note that for any vector ω, $\hat{\omega} = -\hat{\omega}^T$ (i.e., it is skew symmetric). Hence, we can talk about the "hat operator"

$$\hat{\ }\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$$

and the "unhat" operator:

$$\check{\ }\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^3$$

that takes a skew symmetric 3×3 matrix and returns a 3×1 vector.

Matrix Exponentials

Matrix exponentials are useful because they allow us to characterize rotations in terms of the hat operator. In particular, $e^x$is just a function if $x \in \mathbb{R}$, but how do we define the exponential if $x \in \mathbb{R}^{n\times n}$? If X is a square matrix, we can always multiply it be other square matrices, so we can define the matrix exponential through its Taylor expansion:

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

Example: let $\Omega \in \mathbb{R}$ and

$$\hat{\omega} = \begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix}$$

Then

$$e^{\hat{\omega}} = I + \hat{\omega} + \frac{1}{2!}\hat{\omega}^2 + \frac{1}{3!}\hat{\omega}^3 + \frac{1}{4!}\hat{\omega}^4 + \cdots$$

and

$$\hat{\omega}^2 = \begin{bmatrix} -\Omega^2 & 0 \\ 0 & -\Omega^2 \end{bmatrix} \quad \hat{\omega}^3 = \begin{bmatrix} 0 & \Omega^3 \\ -\Omega^3 & 0 \end{bmatrix} \quad \hat{\omega}^4 = \begin{bmatrix} \Omega^4 & 0 \\ 0 & \Omega^4 \end{bmatrix}$$

which means that

$$e^{\hat{\omega}} = \begin{bmatrix} 1 - \frac{1}{2!}\Omega^2 + \frac{1}{4!}\Omega^4 \cdots & -\Omega + \frac{1}{3!}\Omega^3 \cdots \\ \Omega - \frac{1}{3!}\Omega^3 \cdots & 1 - \frac{1}{2!}\Omega^2 + \frac{1}{4!}\Omega^4 \cdots \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega \\ \sin\Omega & \cos\Omega \end{bmatrix}.$$

Hence, $e^{\hat{\omega}}$ is the rotation of a body. Moreover, $e^{\hat{\omega}t}$ is the rotation of a body rotating at a constant angular velocity Ω.

More generally, if $\omega \in \mathbb{R}^3$ then $e^{\hat{\omega}\theta} \in SO(3)$. In this, ω is the axis of rotation and θ is the angle of rotation. Also, note that in this example we made $\hat{\omega}$ a 2×2 matrix, but we could have just as easily defined $\omega = [1, 0, 0]^T$ and found $\hat{\omega}$ to be

$$\hat{\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\Omega \\ 0 & \Omega & 0 \end{bmatrix}$$

and then we would get that

$$e^{\hat{\omega}t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega t & -\sin\Omega t \\ 0 & \sin\Omega t & \cos\Omega t \end{bmatrix}.$$

(This would be a rotation in the (y, z) plane around the x axis–therefore the x component of a transformed point stays constant.)

Now, how do we use this? First, let's say that we want to take the derivative of $R = e^{\hat{\omega}\theta}$ with respect to θ. Then we get that

$$\begin{aligned}
\frac{\partial R}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(I + \hat{\omega}\theta + \frac{1}{2!}(\hat{\omega}\theta)^2 + \frac{1}{3!}(\hat{\omega}\theta)^3 + \frac{1}{4!}(\hat{\omega}\theta)^4 + \cdots\right) \\
&= \hat{\omega} + \frac{1}{1!}\hat{\omega}^2\theta + \frac{1}{2!}\hat{\omega}^3\theta^2 + \frac{1}{3!}\hat{\omega}^4\theta^3 + \cdots \\
&= \hat{\omega}\left(II + \hat{\omega}\theta + \frac{1}{2!}(\hat{\omega}\theta)^2 + \frac{1}{3!}(\hat{\omega}\theta)^3 + \cdots\right) \\
&= \hat{\omega}R(= R\hat{\omega}).
\end{aligned}$$

Moreover, if we want to differentiate R with respect to time, we just apply chain rule:

$$\frac{d}{dt}R = \frac{d}{dt}e^{\hat{\omega}\theta} = \frac{\partial e^{\hat{\omega}\theta}}{\partial \theta}\frac{\partial \theta}{dt} = \hat{\omega}R\dot{\theta}.$$

OK, so how do we use this to calculate the dynamics in term of the configuration q?

1. First, to find velocities of point masses based on their geometry , we need to be able to evaluate the time derivatives of rigid body transformations

2. Second, we'll need to take derivatives with respect to $\dot{q}$ based on derivatives of the rigid body transformations (because those are the only things that depend on q).

3. Lastly, we'll need to take derivatives with respect to q based on derivatives of the rigid body transformations (again, because those are the only things that depend on q).

Example: single pendulum

$$\begin{aligned}
q &= \theta \\
(x, y)^T &= R(-\frac{\pi}{2})R(\theta)(L, 0) \\
(\dot{x}, \dot{y})^T &= R(-\frac{\pi}{2})\dot{R}(\theta)(L, 0) \\
&= R(-\frac{\pi}{2})R(\theta)\hat{\omega}\dot{\theta}(L, 0)
\end{aligned}$$

So, the kinetic energy and potential energy can be described by

$$
\begin{aligned}
KE &= \frac{1}{2}m(\dot{x},\dot{y})\cdot(\dot{x},\dot{y})^T \\
&= \frac{1}{2}m(R(-\frac{\pi}{2})R(\theta)\hat{\omega}\dot{\theta}(L,0))^T R(-\frac{\pi}{2})R(\theta)\hat{\omega}\dot{\theta}(L,0) \\
&= \frac{1}{2}m(L,0)^T\dot{\theta}\hat{\omega}^T R(\theta)^T R(-\frac{\pi}{2})^T R(-\frac{\pi}{2})R(\theta)\hat{\omega}\dot{\theta}(L,0) \\
&= \frac{1}{2}m(L,0)^T\dot{\theta}\hat{\omega}^T\hat{\omega}\dot{\theta}(L,0) \\
&= \frac{1}{2}m(L,0)^T\hat{\omega}^T\hat{\omega}(L,0)\dot{\theta}^2 \\
&= \frac{1}{2}mL^2\dot{\theta}^2
\end{aligned}
$$

and

$$V = mgy = mg\left[R(-\frac{\pi}{2})R(\theta)(L,0)\right]_2 = -mgL\cos\theta$$

where the subscript 2 indicates that we are looking at the second element of the vector.

Notice that we are never writing down sin and cos expressions, so we are entirely avoiding the trigonometry associated with this problem. So, at minimum, this is really helpful in terms of evaluating the kinetic energy and the potential energy in a way that doesn't involve trigonometric identities.

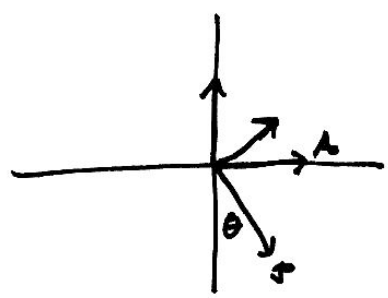

Figure 24: Simple rotation

Now, to evaluate the Euler-Lagrange Equations, we simply need to apply chain rule again to these expressions.

Let

$$
\begin{aligned}
a &= R(-\frac{\pi}{2})R(\theta)\begin{bmatrix} L \\ 0 \end{bmatrix} \\
a' &= \frac{\partial a}{\partial \theta} = R(-\frac{\pi}{2})R(\theta)\hat{\omega}\begin{bmatrix} L \\ 0 \end{bmatrix} \\
a'' &= \frac{\partial^2 a}{\partial \theta^2} = R(-\frac{\pi}{2})R(\theta)\hat{\omega}^2\begin{bmatrix} L \\ 0 \end{bmatrix}
\end{aligned}
$$

where $\hat{\hat{\omega}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. With these definitions, we have $\dot{a} = a'\dot{\theta}$. Hence, we get

$$L = KE - V = \frac{1}{2}m(a'\dot{\theta})^T a'\dot{\theta} - mga_y = \frac{1}{2}m\dot{\theta}^T a'^T a'\dot{\theta} - mga_y$$

where a_y is the "y" component of the vector a—that is, $a_y = [0\ 1] \cdot a$. Now, evaluate the Euler-Lagrange Equations:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} &= 0 \\
\frac{d}{dt}\left(m(a'\dot\theta)^T a'\right) - \left[\frac{1}{2}m\dot\theta\left(a''^T a' + a'^T a''\right)\dot\theta - mga'_y\right] &= 0 \\
m\left[\left[\dot\theta^T a''\dot\theta + a'\ddot\theta\right]^T a' + (a'\dot\theta)^T a''\dot\theta\right] - \left[\frac{1}{2}m\dot\theta\left(a''^T a' + a'^T a''\right)\dot\theta - mga'_y\right] &= 0 \\
\text{after simplifying by multiplying out matrices}& \\
mgL\sin\theta + mL^2\ddot\theta &= 0
\end{aligned}$$

Again, we did not take any explicit derivatives of sin and cos terms.