2) 1. Rotation matrices:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_\psi = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}$$

$$R_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Small-angle approx.:

$$R_{\theta, \varepsilon} = \begin{bmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{\psi, \varepsilon} = \begin{bmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1 \end{bmatrix}$$

$$R_{\phi \varepsilon} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}$$

Only rotation about the z axis, R_θ, keeps the Lagrangian fixed/invariant. This invariance is global as rotation by any amount θ about the z axis will not change the Lagrangian.

Format of problem 2.1: (for use in understanding my written work)

1. Show R_θ transformation is locally invariant in L_ε

2. Show R_θ transformation is globally invariant in L_ε

3. Show R_ϕ and R_ψ transformations are not locally invariant in L_ε, and therefore cannot be globally invariant in L_ε

2.1) Check for local invariance of R_θ transformation using small-angle approximation:

$$q = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \quad q_\varepsilon = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \varepsilon \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

$$\dot q_\varepsilon = \frac{d}{dt}\left(q + \varepsilon G(q) \right)$$
$$= \dot q + \varepsilon \dot G q$$
$$= \begin{bmatrix} \dot x \\ \dot y \\ \dot z \end{bmatrix} + \varepsilon \begin{bmatrix} -\dot y \\ \dot x \\ 0 \end{bmatrix}$$

Substitute in the perturbed versions of q, q_ε, into the Lagrangian:

$$L = \frac{1}{2}m\left(\dot x^2 + \dot y^2 + \dot z^2\right) - \frac{1}{2}k\left(x^2 + y^2 + z^2\right) - mgz$$
$$L_\varepsilon = \frac{1}{2}m\left(\left(\dot x - \varepsilon \dot y\right)^2 + \left(\dot y + \varepsilon \dot x\right)^2 + \dot z^2\right)$$
$$- \frac{1}{2}k\left(\left(x - \varepsilon y\right)^2 + \left(y + \varepsilon x\right)^2 + z\right)^2 - mgz$$
$$= \frac{1}{2}m\left(\dot x^2 - \cancel{2\varepsilon \dot x \dot y} + \varepsilon^2 \dot y^2 + \dot y^2 + \cancel{2\varepsilon \dot x \dot y} + 2\varepsilon \dot x^2\right.$$
$$\left. + \dot z^2\right) - \frac{1}{2}k\left(x^2 - \cancel{2\varepsilon xy} + \varepsilon^2 y^2 + y^2 + \cancel{2\varepsilon xy} + \varepsilon^2 x^2\right)$$
$$- mgz$$
$$L_\varepsilon = \frac{1}{2}m\left(\dot x^2 + \dot y^2 + \varepsilon^2\left(\dot x^2 + \dot y^2\right) + \dot z^2\right) - \frac{1}{2}k\left(x^2 + y^2 + \varepsilon^2\left(x^2 + y^2\right)\right.$$
$$\left. + z^2\right) - mgz$$

2.1)

Subtract $L_\varepsilon - L$:

$$L_\varepsilon - L =$$

$$\left[\tfrac{1}{2}m\left(\cancel{\dot{x}^2} + \cancel{\dot{y}^2} + \varepsilon^2(\dot{x}^2+\dot{y}^2) + \cancel{\dot{z}^2}\right) - \tfrac{1}{2}k\left(\cancel{x^2}+\cancel{y^2}+\varepsilon^2(x^2+y^2) + \cancel{z^2}\right) - \cancel{mgz}\right]$$

$$- \left[\tfrac{1}{2}m\left(\cancel{\dot{x}^2}+\cancel{\dot{y}^2}+\cancel{\dot{z}^2}\right) - \tfrac{1}{2}k\left(\cancel{x^2}+\cancel{y^2}+\cancel{z^2}\right) - \cancel{mgz}\right]$$

$$L_\varepsilon = L + \tfrac{1}{2}m\varepsilon^2(\dot{x}^2+\dot{y}^2) - \tfrac{1}{2}k\varepsilon^2(x^2+y^2)$$

To check for local invariance: let $\frac{\partial}{\partial\varepsilon}(L_\varepsilon)\big|_{\varepsilon=0} \to 0$

$$\frac{\partial}{\partial\varepsilon}(L_\varepsilon)\Big|_{\varepsilon=0} = \varepsilon\left(m(\dot{x}^2+\dot{y}^2) - k(x^2+y^2)\right)\Big|_{\varepsilon=0}$$

$$= 0$$

The transformation corresponding to R_θ, or rotation about z axis, leaves the Lagrangian locally invariant.

2.1) Check global invariance of R_θ transformation by modeling transformation without small-angle approximation:

$$q = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad q_\varepsilon = \begin{bmatrix} x\cos(\varepsilon) - y\sin(\varepsilon) \\ x\sin(\varepsilon) + y\cos(\varepsilon) \\ z \end{bmatrix}$$

$$\dot{q}_\varepsilon = \frac{d}{dt}(q_\varepsilon) = \begin{bmatrix} \dot{x}\cos(\varepsilon) - \dot{y}\sin(\varepsilon) \\ \dot{x}\sin(\varepsilon) + \dot{y}\cos(\varepsilon) \\ \dot{z} \end{bmatrix}$$

→ assume ε term has no dependence on the t

Lagrangian: substitute in q_ε and $\dot{q}_\varepsilon$ to see if perturbations change L

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(x^2 + y^2 + z^2) - mgz$$

$$L_{0,\varepsilon} = \frac{1}{2}m\left((\dot{x}\cos\varepsilon - \dot{y}\sin\varepsilon)^2 + (\dot{x}\sin\varepsilon + \dot{y}\cos\varepsilon)^2 + \dot{z}^2\right) - \frac{1}{2}k\left((x\cos\varepsilon - y\sin\varepsilon)^2 + (x\sin\varepsilon + y\cos\varepsilon)^2 + z^2\right) - mgz$$

$$= \frac{1}{2}m\Big[\dot{x}^2\cos^2\varepsilon - \cancel{2\dot{x}\dot{y}\cos\varepsilon\sin\varepsilon} + \dot{y}^2\sin^2\varepsilon + \dot{x}^2\sin^2\varepsilon + \cancel{2\dot{x}\dot{y}\cos\varepsilon\sin\varepsilon} + \dot{y}^2\cos^2\varepsilon + z^2\Big]$$
$$- \frac{1}{2}k\Big[x^2\cos^2\varepsilon - \cancel{2xy\cos\varepsilon\sin\varepsilon} + y^2\sin^2\varepsilon + x^2\sin^2\varepsilon + \cancel{2xy\cos\varepsilon\sin\varepsilon} + y^2\cos^2\varepsilon + z^2\Big]$$
$$- mgz$$

2.1)

$$L_{\theta,\varepsilon} = \frac{1}{2}m\left((\dot{x}^2)(\cos^2\varepsilon + \sin^2\varepsilon) + (\dot{y}^2)(\cos^2\varepsilon + \sin^2\varepsilon) + \dot{z}^2\right)$$
$$-\frac{1}{2}k\left((x^2)(\cos^2\varepsilon + \sin^2\varepsilon) + (y^2)(\cos^2\varepsilon + \sin^2\varepsilon) + z^2\right)$$
$$-mgz$$

$$L_{\theta\varepsilon} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(x^2 + y^2 + z^2) - mgz$$

$L_{\theta\varepsilon} - L = 0 \longrightarrow$ The Lagrangian of the system does not depend on the magnitude of perturbation ε.

2.1) Test of local invariance with rotation about x axis R_ϕ:

$$\begin{bmatrix} x_{new} \\ y_{new} \\ z_{new} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix}$$

Sub into the Lagrangian to see if it changes:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(x^2 + y^2 + z^2) - mgz$$

$$L_{\phi\varepsilon} = \frac{1}{2}m\left(\dot{x}^2 + (\dot{y} - \varepsilon\dot{z})^2 + (\dot{z} + \varepsilon\dot{y})^2\right)$$
$$-\frac{1}{2}k\left(x^2 + (y - \varepsilon z)^2 + (z + \varepsilon y)^2\right)$$
$$-mg(z + \varepsilon y)$$

2.1)

$$L_{\varphi,\varepsilon} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 - \cancel{2\varepsilon\dot{y}\dot{z}} + \varepsilon^2\dot{z}^2 + \dot{z}^2 + \cancel{2\varepsilon\dot{y}\dot{z}} + \varepsilon^2\dot{y}^2\right)$$
$$-\frac{1}{2}k\left(x^2 + y^2 - \cancel{2\varepsilon yz} + \varepsilon^2 z^2 + z^2 + \cancel{2\varepsilon yz} + \varepsilon^2 y^2\right)$$
$$-mg(z + \varepsilon y)$$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \varepsilon^2(\dot{y}^2 + \dot{z}^2)\right)$$
$$-\frac{1}{2}k\left(x^2 + y^2 + z^2 + \varepsilon^2(y^2 + z^2)\right)$$
$$-mg(z + \varepsilon y)$$

$$L_{\varphi,\varepsilon} = L + \varepsilon^2\left(\frac{1}{2}m(\dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(y^2 + z^2)\right) - mg\varepsilon y$$

Check if $\frac{d}{d\varepsilon} L_{\varphi,\varepsilon}\Big|_{\varepsilon=0} = 0$:

$$\frac{d}{d\varepsilon} L_{\varphi,\varepsilon}\Big|_{\varepsilon=0} = \varepsilon\left(m(\dot{y}^2 + \dot{z}^2) - k(y^2 + z^2)\right) - mgy\Big|_{\varepsilon=0}$$
$$= \underline{\underline{mgy}} \neq 0.$$

The transformation R_φ, rotation about the x axis, is not locally invariant. R_ψ is also not locally invariant as its Lagrangian $L_{\psi,\varepsilon}$ has term $L_{\varphi\varepsilon} = \ldots - mg(z - \varepsilon x)$. $\frac{d}{d\varepsilon} L_{\varphi,\varepsilon}\Big|_{\varepsilon=0}$ has term $\underline{\underline{mgx}} \neq 0$ that leaves $L_{\varphi,\varepsilon}$ not locally invariant.

2.2)

Linearized transformation for $R_\theta \cdot q$:

$$q_\varepsilon = q + \varepsilon G(q) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \varepsilon \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix};$$

$L(q_\varepsilon, \dot{q}_\varepsilon) - L(q, \dot{q}) = 0$; see calculations in 2.1 for derivation of that result.

2.3) Conserved Quantity via Noether:

$$P_{\text{conserved}} = \frac{\partial L}{\partial \dot{Q}} G(q)$$

$$\frac{\partial L}{\partial \dot{Q}} = \begin{bmatrix} m\dot{x} \\ m\dot{y} \\ m\dot{z} \end{bmatrix}^T, \qquad G(q) = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \qquad (q_\varepsilon = q + \varepsilon G(q))$$

$$P_{\text{cons}} = \begin{bmatrix} m\dot{x} & m\dot{y} & m\dot{z} \end{bmatrix} \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

$$\boxed{P_{\text{cons}} = m(x\dot{y} - y\dot{x})} = \text{angular momentum about the } z \text{ axis.}$$

The rationale behind conservation of this value is that there are no forces or torques, conservative or non-conservative, that act on the particle about the z axis.

3) Find conserved momentum in cart-pendulum system using Noether:

1) Find a transformation $G(q)$ that leaves $L(q,\dot{q})$ locally invariant

2) Test local invariance with $L(q+\varepsilon Q(q))$ or $L(R(q,\varepsilon))$; global invariance with general, non-small approximations

3) Find $p_{cons} = \frac{\partial L}{\partial \dot{q}} G(q)$

1) State vector: $[x_m, \theta]^T$

Transformation $G(q)$ that leaves $L(q,\dot{q})$ locally invariant: $[1 \quad 0]^T$

2) Test local invariance:

$$L = \frac{1}{2} M \dot{x}_m^2 - mgR\cos\theta + \frac{1}{2} m R^2 \dot{\theta}^2 + mR\cos(\theta)\dot{\theta}\dot{x}_m + \frac{1}{2} m \dot{x}_m^2$$

$$\frac{dL}{d\dot{x}} = (M+m)\dot{x}_m + mR\cos(\theta)\dot{\theta}$$

$$\frac{dL}{d\dot{\theta}} = mR^2\dot{\theta} + mR\cos\theta(\dot{x}_m)$$

3) 2. $q_\varepsilon = q + \varepsilon G(q)$

$$= \begin{bmatrix} x_m \\ \theta \end{bmatrix} + \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x + \varepsilon \\ \theta \end{bmatrix}$$

$L_\varepsilon = L(q_\varepsilon, \dot{q}_\varepsilon)$

$$L_\varepsilon = \frac{1}{2} M \dot{x}_m^2 - mgR\cos\theta + \frac{1}{2} mR^2\dot{\theta}^2 + mR\cos(\theta)\dot{\theta}\dot{x}_m + \frac{1}{2} m \dot{x}_m^2 \rightarrow L_\varepsilon - L = 0$$

The system is both locally and globally invariant due to $G(q)$ as the Lagrangian does not depend on ε.

3. Find conserved quantity P_{cons}:

$$P_{cons} = \frac{\partial L}{\partial \dot{q}} G(q)$$

$$= \begin{bmatrix} (M+m)\dot{x}_m + mR\cos(\theta)\dot{\theta} \\ mR^2\dot{\theta} + mR\cos\theta(\dot{x}_m) \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{P_{cons} = (M+m)\dot{x}_m + mR\cos(\theta)\dot{\theta}}$$