Chapter 2 summary

as $\mathbb{R}^3 \times S^2$. The workspace could be the reachable points in $\mathbb{R}^3 \times S^2$, or, to simplify visualization, the user could define the workspace to be the subset of $\mathbb{R}^3$ corresponding to the reachable Cartesian positions of the nozzle.

2.6 Summary

- A robot is mechanically constructed from links that are connected by various types of joint. The links are usually modeled as rigid bodies. An end-effector such as a gripper may be attached to some link of the robot. Actuators deliver forces and torques to the joints, thereby causing motion of the robot.

- The most widely used one-dof joints are the revolute joint, which allows rotation about the joint axis, and the prismatic joint, which allows translation in the direction of the joint axis. Some common two-dof joints include the cylindrical joint, which is constructed by serially connecting a revolute and prismatic joint, and the universal joint, which is constructed by orthogonally connecting two revolute joints. The spherical joint, also known as the ball-and-socket joint, is a three-dof joint whose function is similar to the human shoulder joint.

- The configuration of a rigid body is a specification of the location of all its points. For a rigid body moving in the plane, three independent parameters are needed to specify the configuration. For a rigid body moving in three-dimensional space, six independent parameters are needed to specify the configuration.

- The configuration of a robot is a specification of the configuration of all its links. The robot's configuration space is the set of all possible robot configurations. The dimension of the C-space is the number of degrees of freedom of a robot.

- The number of degrees of freedom of a robot can be calculated using Grübler's formula,

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i,$$

where $m = 3$ for planar mechanisms and $m = 6$ for spatial mechanisms, N is the number of links (including the ground link), J is the number of joints, and f_i is the number of degrees of freedom of joint i.

Chapter 2 summary

C-space

- A robot's C-space can be parametrized explicitly or represented implicitly. For a robot with n degrees of freedom, an explicit parametrization uses n coordinates, the minimum necessary. An implicit representation involves m coordinates with $m \geq n$, with the m coordinates subject to $m - n$ constraint equations. With an implicit parametrization, a robot's C-space can be viewed as a surface of dimension n embedded in a space of higher dimension m.

- The C-space of an n-dof robot whose structure contains one or more closed loops can be implicitly represented using k loop-closure equations of the form $g(\theta) = 0$, where $\theta \in \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$. Such constraint equations are called holonomic constraints. Assuming that θ varies with time t, the holonomic constraints $g(\theta(t)) = 0$ can be differentiated with respect to t to yield
$$\frac{\partial g}{\partial \theta}(\theta)\dot{\theta} = 0,$$
where $\partial g(\theta)/\partial\theta$ is a $k \times m$ matrix.

Holonomic/non-holonomic constraints

- A robot's motion can also be subject to velocity constraints of the form
$$A(\theta)\dot{\theta} = 0,$$
where $A(\theta)$ is a $k \times m$ matrix that cannot be expressed as the differential of some function $g(\theta)$. In other words, there does not exist any $g(\theta), g : \mathbb{R}^m \to \mathbb{R}^k$, such that
$$A(\theta) = \frac{\partial g}{\partial \theta}(\theta).$$
Such constraints are said to be nonholonomic constraints, or nonintegrable constraints. These constraints reduce the dimension of feasible velocities of the system but do not reduce the dimension of the reachable C-space. Nonholonomic constraints arise in robot systems subject to conservation of momentum or rolling without slipping.

- A robot's task space is a space in which the robot's task can be naturally expressed. A robot's workspace is a specification of the configurations that the end-effector of the robot can reach.

2.7 Notes and References

In the kinematics literature, structures that consist of links connected by joints are also called mechanisms or linkages. The number of degrees of freedom of a

Chapter 3 summary

properties

actions/changes

Rotations	Rigid-Body Motions
$R \in SO(3)$: 3×3 matrices $R^{\mathrm{T}}R = I, \det R = 1$	$T \in SE(3)$: 4×4 matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^{\mathrm{T}}$	$T^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$
change of coordinate frame: $R_{ab}R_{bc} = R_{ac}, \quad R_{ab}p_b = p_a$	change of coordinate frame: $T_{ab}T_{bc} = T_{ac}, \quad T_{ab}p_b = p_a$
rotating a frame {b}: $R = \mathrm{Rot}(\hat{\omega}, \theta)$ $R_{sb'} = RR_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ $R_{sb''} = R_{sb}R$: rotate θ about $\hat{\omega}_b = \hat{\omega}$	displacing a frame {b}: $T = \begin{bmatrix} \mathrm{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$ $T_{sb'} = TT_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ (moves {b} origin), translate p in {s} $T_{sb''} = T_{sb}T$: translate p in {b}, rotate θ about $\hat{\omega}$ in new body frame
unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$, where $\|\hat{\omega}\| = 1$	"unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$
	for a screw axis $\{q, \hat{s}, h\}$ with finite h, $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$
angular velocity is $\omega = \hat{\omega}\dot{\theta}$	twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$

continued...

matrix operations:
(AB)^T = B^T * A^T

Chapter 3 summary

SO(3) <-> so(3)
SE(3) <-> se(3)
matrix exp. and log.
to go from velocities
to transformations

Rotations (cont.)	Rigid-Body Motions (cont.)
for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ identities, $\omega, x \in \mathbb{R}^3, R \in SO(3)$: $[\omega] = -[\omega]^{\mathrm{T}}, [\omega]x = -[x]\omega$, $[\omega][x] = ([x][\omega])^{\mathrm{T}}, R[\omega]R^{\mathrm{T}} = [R\omega]$	for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$ (the pair (ω, v) can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context)
$\dot{R}R^{-1} = [\omega_s], \;\; R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s], \;\; T^{-1}\dot{T} = [\mathcal{V}_b]$
	$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$ identities: $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$, $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1T_2}]$
change of coordinate frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b, \;\; \omega_a = R_{ab}\omega_b$	change of coordinate frame: $\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \;\; \mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$
exp coords for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$	exp coords for $T \in SE(3)$: $\mathcal{S}\theta \in \mathbb{R}^6$
$\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$ $R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$ $I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2$	$\exp : [\mathcal{S}]\theta \in se(3) \to T \in SE(3)$ $T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$ where $* =$ $(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$
$\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$ algorithm in Section 3.2.3.3	$\log : T \in SE(3) \to [\mathcal{S}]\theta \in se(3)$ algorithm in Section 3.3.3.2
moment change of coord frame: $m_a = R_{ab}m_b$	wrench change of coord frame: $\mathcal{F}_a = (m_a, f_a) = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{F}_b$

forces and moments,
wrenches

if an external wrench
(forces, torques) Fb
is being applied at
end effector, robot
needs to apply
wrench -Fb to resist it

3.6 Software

The following functions are included in the software distribution accompanying the book. The code below is in MATLAB format, but it is available in other languages. For more details on the software, consult the code and its documentation.

```
invR = RotInv(R)
```

Chapter 4 summary


```
      <inertia ixx="0.0171364731454" ixy="0.0" ixz="0.0"
               iyy="0.0171364731454" iyz="0.0" izz="0.033822"/>
    </inertial>
  </link>
  <link name="ee_link"/>
</robot>
```

Beyond the properties described above, a URDF can describe other properties of a robot, such as its visual appearance (including geometric models of the links) as well as simplified representations of link geometries that can be used for collision detection in motion planning algorithms.

4.3 Summary

- Given an open chain with a fixed reference frame $\{s\}$ and a reference frame $\{b\}$ attached to some point on its last link – this frame is denoted the end-effector frame – the forward kinematics is the mapping $T(\theta)$ from the joint values θ to the position and orientation of $\{b\}$ in $\{s\}$.

- In the Denavit–Hartenberg representation the forward kinematics of an open chain is described in terms of the relative displacements between reference frames attached to each link. If the link frames are sequentially labeled $\{0\}, \ldots, \{n+1\}$, where $\{0\}$ is the fixed frame $\{s\}$, $\{i\}$ is a frame attached to link i at joint i (with $i = 1, \ldots, n$), and $\{n+1\}$ is the end-effector frame $\{b\}$ then the forward kinematics is expressed as

  $$T_{0,n+1}(\theta) = T_{01}(\theta_1) \cdots T_{n-1,n}(\theta_n) T_{n,n+1}$$

  where θ_i denotes the joint i variable and $T_{n,n+1}$ indicates the (fixed) configuration of the end-effector frame in $\{n\}$. If the end-effector frame $\{b\}$ is chosen to be coincident with $\{n\}$ then we can dispense with the frame $\{n+1\}$.

- The Denavit–Hartenberg convention requires that reference frames assigned to each link obey a strict convention (see Appendix C). Following this convention, the link frame transformation $T_{i-1,i}$ between link frames $\{i-1\}$ and $\{i\}$ can be parametrized using only four parameters, the Denavit–Hartenberg parameters. Three of these parameters describe the kinematic structure, while the fourth is the joint value. Four numbers is the minimum needed to represent the displacement between two link frames.

Chapter 4 summary

final transformation matrix of system is given by initial config. mult. by transformations due to rotation about each joint

- The forward kinematics can also be expressed as the following product of exponentials (the space form),

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M,$$

where $\mathcal{S}_i = (\omega_i, v_i)$ denotes the screw axis associated with positive motion along joint i expressed in fixed-frame $\{s\}$ coordinates, θ_i is the joint-i variable, and $M \in SE(3)$ denotes the position and orientation of the end-effector frame $\{b\}$ when the robot is in its zero position. It is not necessary to define individual link frames; it is only necessary to define M and the screw axes $\mathcal{S}_1, \ldots, \mathcal{S}_n$.

- The product of exponentials formula can also be written in the equivalent body form,

$$T(\theta) = M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n},$$

where $\mathcal{B}_i = [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_i$, $i = 1, \ldots, n$; $\mathcal{B}_i = (\omega_i, v_i)$ is the screw axis corresponding to joint axis i, expressed in $\{b\}$, with the robot in its zero position.

- The Universal Robot Description Format (URDF) is a file format used by the Robot Operating System and other software for representing the kinematics, inertial properties, visual properties, and other information for general tree-like robot mechanisms, including serial chains. A URDF file includes descriptions of joints, which connect a parent link and a child link and fully specify the kinematics of the robot, as well as descriptions of links, which specify its inertial properties.

4.4 Software

Software functions associated with this chapter are listed in MATLAB format below.

`T = FKinBody(M,Blist,thetalist)`
Computes the end-effector frame given the zero position of the end-effector `M`, the list of joint screws `Blist` expressed in the end-effector frame, and the list of joint values `thetalist`.

`T = FKinSpace(M,Slist,thetalist)`
Computes the end-effector frame given the zero position of the end-effector `M`, the list of joint screws `Slist` expressed in the fixed-space frame, and the list of joint values `thetalist`.

Chapter 5 summary

5.5 Summary

- Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M.$$

  The space Jacobian $J_s(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$, via $\mathcal{V}_s = J_s(\theta)\dot{\theta}$. The ith column of $J_s(\theta)$ is given by

$$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i),$$

  for $i = 2, \ldots, n$, with the first column $J_{s1} = \mathcal{S}_1$. The screw vector J_{si} for joint i is expressed in space-frame coordinates, with the joint values θ assumed to be arbitrary rather than zero.

- Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T(\theta) = M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n}.$$

  The body Jacobian $J_b(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the end-effector body twist $\mathcal{V}_b = (\omega_b, v_b)$ via $\mathcal{V}_b = J_b(\theta)\dot{\theta}$. The ith column of $J_b(\theta)$ is given by

$$J_{bi}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i),$$

  for $i = n-1, \ldots, 1$, with $J_{bn} = \mathcal{B}_n$. The screw vector J_{bi} for joint i is expressed in body-frame coordinates, with the joint values θ assumed to be arbitrary rather than zero.

- The body and space Jacobians are related via

$$\begin{aligned} J_s(\theta) &= [\mathrm{Ad}_{T_{sb}}] J_b(\theta), \\ J_b(\theta) &= [\mathrm{Ad}_{T_{bs}}] J_s(\theta), \end{aligned}$$

  where $T_{sb} = T(\theta)$.

- Consider a spatial open chain with n one-dof joints that is assumed to be in static equilibrium. Let $\tau \in \mathbb{R}^n$ denote the vector of the joint torques and forces and $\mathcal{F} \in \mathbb{R}^6$ be the wrench applied at the end-effector, in either space- or body-frame coordinates. Then τ and $\mathcal{F}$ are related by

$$\tau = J_b^{\mathrm{T}}(\theta)\mathcal{F}_b = J_s^{\mathrm{T}}(\theta)\mathcal{F}_s.$$

Chapter 5 summary

- A kinematically singular configuration for an open chain, or more simply a kinematic singularity, is any configuration $\theta \in \mathbb{R}^n$ at which the rank of the Jacobian is not maximal. For six-dof spatial open chains consisting of revolute and prismatic joints, some common singularities include (i) two collinear revolute joint axes; (ii) three coplanar and parallel revolute joint axes; (iii) four revolute joint axes intersecting at a common point; (iv) four coplanar revolute joints; and (v) six revolute joints intersecting a common line.
- The manipulability ellipsoid describes how easily the robot can move in different directions. For a Jacobian J, the principal axes of the manipulability ellipsoid are defined by the eigenvectors of JJ^{T} and the corresponding lengths of the principal semi-axes are the square roots of the eigenvalues.
- The force ellipsoid describes how easily the robot can generate forces in different directions. For a Jacobian J, the principal axes of the force ellipsoid are defined by the eigenvectors of $(JJ^{\mathrm{T}})^{-1}$ and the corresponding lengths of the principal semi-axes are the square roots of the eigenvalues.
- Measures of the manipulability and force ellipsoids include the ratio of the longest principal semi-axis to the shortest; the square of this measure; and the volume of the ellipsoid. The first two measures indicate that the robot is far from being singular if they are small (close to 1).

5.6 Software

Software functions associated with this chapter are listed below.

`Jb = JacobianBody(Blist,thetalist)`
Computes the body Jacobian $J_b(\theta) \in \mathbb{R}^{6\times n}$ given a list of joint screws $\mathcal{B}_i$ expressed in the body frame and a list of joint angles.

`Js = JacobianSpace(Slist,thetalist)`
Computes the space Jacobian $J_s(\theta) \in \mathbb{R}^{6\times n}$ given a list of joint screws $\mathcal{S}_i$ expressed in the fixed space frame and a list of joint angles.

5.7 Notes and References

One of the key advantages of the PoE formulation is in the derivation of the Jacobian; the columns of the Jacobian are simply the (configuration-dependent)