5.5 Summary

- Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M.$$

The space Jacobian $J_s(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$, via $\mathcal{V}_s = J_s(\theta)\dot{\theta}$. The ith column of $J_s(\theta)$ is given by

$$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i),$$

for $i = 2, \ldots, n$, with the first column $J_{s1} = \mathcal{S}_1$. The screw vector J_{si} for joint i is expressed in space-frame coordinates, with the joint values θ assumed to be arbitrary rather than zero.

- Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n}.$$

The body Jacobian $J_b(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the end-effector body twist $\mathcal{V}_b = (\omega_b, v_b)$ via $\mathcal{V}_b = J_b(\theta)\dot{\theta}$. The ith column of $J_b(\theta)$ is given by

$$J_{bi}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i),$$

for $i = n-1, \ldots, 1$, with $J_{bn} = \mathcal{B}_n$. The screw vector J_{bi} for joint i is expressed in body-frame coordinates, with the joint values θ assumed to be arbitrary rather than zero.

- The body and space Jacobians are related via

$$\begin{aligned} J_s(\theta) &= [\mathrm{Ad}_{T_{sb}}]J_b(\theta), \\ J_b(\theta) &= [\mathrm{Ad}_{T_{bs}}]J_s(\theta), \end{aligned}$$

where $T_{sb} = T(\theta)$.

- Consider a spatial open chain with n one-dof joints that is assumed to be in static equilibrium. Let $\tau \in \mathbb{R}^n$ denote the vector of the joint torques and forces and $\mathcal{F} \in \mathbb{R}^6$ be the wrench applied at the end-effector, in either space- or body-frame coordinates. Then τ and $\mathcal{F}$ are related by

$$\tau = J_b^{\mathrm{T}}(\theta)\mathcal{F}_b = J_s^{\mathrm{T}}(\theta)\mathcal{F}_s.$$

- A kinematically singular configuration for an open chain, or more simply a kinematic singularity, is any configuration $\theta \in \mathbb{R}^n$ at which the rank of the Jacobian is not maximal. For six-dof spatial open chains consisting of revolute and prismatic joints, some common singularities include (i) two collinear revolute joint axes; (ii) three coplanar and parallel revolute joint axes; (iii) four revolute joint axes intersecting at a common point; (iv) four coplanar revolute joints; and (v) six revolute joints intersecting a common line.

- The manipulability ellipsoid describes how easily the robot can move in different directions. For a Jacobian J, the principal axes of the manipulability ellipsoid are defined by the eigenvectors of JJ^{T} and the corresponding lengths of the principal semi-axes are the square roots of the eigenvalues.

- The force ellipsoid describes how easily the robot can generate forces in different directions. For a Jacobian J, the principal axes of the force ellipsoid are defined by the eigenvectors of $(JJ^{\mathrm{T}})^{-1}$ and the corresponding lengths of the principal semi-axes are the square roots of the eigenvalues.

- Measures of the manipulability and force ellipsoids include the ratio of the longest principal semi-axis to the shortest; the square of this measure; and the volume of the ellipsoid. The first two measures indicate that the robot is far from being singular if they are small (close to 1).

5.6 Software

Software functions associated with this chapter are listed below.

`Jb = JacobianBody(Blist,thetalist)`
Computes the body Jacobian $J_b(\theta) \in \mathbb{R}^{6\times n}$ given a list of joint screws $\mathcal{B}_i$ expressed in the body frame and a list of joint angles.

`Js = JacobianSpace(Slist,thetalist)`
Computes the space Jacobian $J_s(\theta) \in \mathbb{R}^{6\times n}$ given a list of joint screws $\mathcal{S}_i$ expressed in the fixed space frame and a list of joint angles.

5.7 Notes and References

One of the key advantages of the PoE formulation is in the derivation of the Jacobian; the columns of the Jacobian are simply the (configuration-dependent)