

# FERMAT'S LAST THEOREM FOR REGULAR PRIMES

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## 1. INTRODUCTION

We prove Fermat's Last Theorem for regular primes and give some of the necessary background. It uses [Sam70, Mar18, Was82].

## 2. DISCRIMINANTS OF NUMBER FIELDS

**lemma 2.1.** *Let  $K$  be a number field,  $\alpha \in K$  and let  $\sigma_i$  be the embeddings of  $K$  into  $\mathbb{C}$ . Then*

$$\mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = \sum_i \sigma_i(\alpha) \quad N_{K/\mathbb{Q}}(\alpha) = \prod_i \sigma_i(\alpha)$$

**lemma 2.2.** *Let  $K$  be a number field with basis  $B = \{b_1, \dots, b_n\}$  and let  $\sigma_1, \dots, \sigma_n$  be the embeddings of  $K$  into  $\mathbb{C}$ . Now let  $M$  be the matrix*

$$\begin{pmatrix} \sigma_1(b_1) & \cdots & \sigma_1(b_n) \\ \vdots & & \vdots \\ \sigma_n(b_1) & \cdots & \sigma_n(b_n) \end{pmatrix}.$$

*Then*

$$\Delta(B) = \det(M)^2.$$

*Proof.* By Proposition 2.1 we know that  $\mathrm{Tr}_{K/\mathbb{Q}}(b_i b_j) = \sum_k \sigma_k(b_i) \sigma_k(b_j)$  which is the same as the  $(i, j)$  entry of  $M^t M$ . Therefore

$$\det(T_B) = \det(M^t M) = \det(M)^2.$$

□

**lemma 2.3.** *Let  $K$  be a number field and  $B = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$  for some  $\alpha \in K$ . Then*

$$\Delta(B) = \prod_{i < j} (\sigma_i(\alpha) - \sigma_j(\alpha))^2$$

*where  $\sigma_i$  are the embeddings of  $K$  into  $\mathbb{C}$ . Here  $\Delta(B)$  denotes the discriminant.*

*Proof.* First we recall a classical linear algebra result relating to the Vandermonde matrix, which states that

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{i < j} (x_i - x_j).$$

Combining this with Proposition 2.2 gives the result.  $\square$

**lemma 2.4.** *Let  $f$  be a monic irreducible polynomial over a number field  $K$  and let  $\alpha$  be one of its roots in  $\mathbb{C}$ . Then*

$$f'(\alpha) = \prod_{\beta \neq \alpha} (\alpha - \beta),$$

where the product is over the roots of  $f$  different from  $\alpha$ .

*Proof.* We first write  $f(x) = (x - \alpha)g(x)$  which we can do (over  $\mathbb{C}$ ) as  $\alpha$  is a root of  $f$ , where now  $g(x) = \prod_{\beta \neq \alpha} (x - \beta)$ . Differentiating we get

$$f'(x) = g(x) + (x - \alpha)g'(x).$$

If we now evaluate at  $\alpha$  we get the result.  $\square$

**lemma 2.5.** *Let  $K = \mathbb{Q}(\alpha)$  be a number field with  $n = [K : \mathbb{Q}]$  and let  $B = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ . Then*

$$\Delta(B) = (-1)^{\frac{n(n-1)}{2}} N_{K/\mathbb{Q}}(m'_\alpha(\alpha))$$

where  $m'_\alpha$  is the derivative of  $m_\alpha(x)$  (which we recall denotes the minimal polynomial of  $\alpha$ ).

*Proof.* By Proposition 2.3 we have  $\Delta(B) = \prod_{i < j} (\alpha_i - \alpha_j)^2$  where  $\alpha_k := \sigma_k(\alpha)$ . Next, we note that the number of terms in this product is  $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$ . So if we write each term as  $(\alpha_i - \alpha_j)^2 = -(\alpha_i - \alpha_j)(\alpha_j - \alpha_i)$  we get

$$\Delta(B) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^n \prod_{i \neq j} (\alpha_i - \alpha_j).$$

Now, by lemma 2.4 and Proposition 2.1 we see that

$$N_{K/\mathbb{Q}}(m'_\alpha(\alpha)) = \prod_{i=1}^n m'_\alpha(\alpha_i) = \prod_{i=1}^n \prod_{i \neq j} (\alpha_i - \alpha_j),$$

which gives the result.  $\square$

**lemma 2.6.** *Let  $K$  be a number field and  $B, B'$  bases for  $K/\mathbb{Q}$ . If  $P$  denotes the change of basis matrix, then*

$$\Delta(B) = \det(P)^2 \Delta(B').$$

**lemma 2.7.** *If  $K$  is a number field and  $\alpha \in \mathcal{O}_K$  then  $\text{Tr}_{K/\mathbb{Q}}(\alpha)$  and  $N_{K/\mathbb{Q}}(\alpha)$  are both in  $\mathbb{Z}$ .*

**lemma 2.8.** *Let  $K$  be a number field and  $B = \{b_1, \dots, b_n\}$  be elements in  $\mathcal{O}_K$ , then  $\Delta(B) \in \mathbb{Z}$ .*

**lemma 2.9.** *Let  $K$  be a number field and  $B = \{b_1, \dots, b_n\}$  be a basis for  $K/\mathbb{Q}$  consisting of algebraic integers. If  $B$  is not an integral basis then there*

exists an algebraic integer of the form

$$\alpha = \frac{x_1 b_1 + \cdots + x_n b_n}{p}$$

where  $p$  is a prime and  $x_i \in \{0, \dots, p-1\}$  with not all  $x_i$  zero. Moreover, if  $x_i \neq 0$  and we let  $B'$  be the basis obtained by replacing  $b_i$  with  $\alpha$ , then

$$\Delta(B') = \frac{x_i^2}{p^2} \Delta(B).$$

In particular  $p^2 \mid \Delta(B)$ .

*Proof.* If  $B$  is not an integral basis then we can find some element  $\phi \in \mathcal{O}_K$  such that

$$\phi = y_1 b_1 + \dots + y_n b_n$$

with not all the  $y_i$  in  $\mathbb{Z}$ . So, let  $N$  be the least common multiple of the denominators of the  $y_i$  (meaning  $N y_i \in \mathbb{Z}$  for all  $i$ ). Now, let  $p$  be a prime factor of  $N$ . If we now consider  $(N/p)\phi$  then all of the coefficients of  $b_i$  are in  $\frac{1}{p}\mathbb{Z}$  (so they have denominator 1 or  $p$ .) and at least one of them has denominator  $p$  (since not all the  $y_i$  were in  $\mathbb{Z}$ ). So by relabelling, wlog we can assume

$$\phi = y_1 b_1 + \dots + y_n b_n$$

with  $y_i \in \frac{1}{p}\mathbb{Z}$

Now look at

$$\psi := \lfloor y_1 \rfloor b_1 + \cdots + \lfloor y_n \rfloor b_n$$

(here  $\lfloor x \rfloor$  denotes the integer part of  $x$ ). The both  $\psi$  and  $\phi$  are algebraic integers (as the  $b_i$  are algebraic integers). Therefore, so is  $\theta = \phi - \psi$ . By construction,  $\theta$  has coefficients of the form  $\frac{x_i}{p} := y_i - \lfloor y_i \rfloor$  where  $x_i \in \{0, \dots, p-1\}$  and not all the  $x_i$  are zero (since, again, not all the  $y_i$  were in  $\mathbb{Z}$ ). This gives the first part of the lemma.

Now, assume  $x_i \neq 0$ , then let us replace  $b_i \in B$  with  $\theta$  to get a new basis  $B'$  which again consists of algebraic integers. Next, we note that the change of basis matrix from  $B$  to  $B'$  is

$$\begin{pmatrix} 1 & 0 & \cdots & \frac{x_1}{p} & \cdots & 0 \\ 0 & 1 & \cdots & \frac{x_2}{p} & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{x_n}{p} & \cdots & 1 \end{pmatrix}$$

(here the column of  $x_j/p$ 's is in the  $i$ -th column).

This matrix has determinant  $\frac{x_i}{p}$ . Therefore, by Proposition 2.6 we see that  $\Delta(B') = \frac{x_i^2}{p^2} \Delta(B)$ . But both  $\Delta(B), \Delta(B')$  are in  $\mathbb{Z}$  by Proposition 2.8, therefore  $p^2 \mid \Delta(B)$  giving the result.  $\square$

**lemma 2.10.** *Let  $K = \mathbb{Q}(\alpha)$  and  $\alpha$  be an algebraic integer such that  $m_\alpha$  satisfies Eisenstein's Criterion for a prime  $p$ . Then none of the elements*

$$\phi = \frac{1}{p}(x_0 + x_1 \alpha + \cdots + x_{n-1} \alpha^{n-1})$$

*is an algebraic integer, where  $n = \deg(m_\alpha)$  and  $x_i \in \{0, \dots, p-1\}$ .*

*Proof.* We will only prove the case when  $m_\alpha$  is Eisenstein, since the proof of the more general case is identical.

Suppose for contradiction that  $\phi \in \mathcal{O}_K$  and let  $x_d$  be the first non-zero coefficient, so

$$\phi = \frac{1}{p}(x_d\alpha^d + x_{d+1}\alpha^{d+1} + \cdots + x_{n-1}\alpha^{n-1}) \in \mathcal{O}_K.$$

Now, rewrite this as  $\phi = \frac{1}{p}(x_d\alpha^d + \alpha^{d+1}\beta)$  for some  $\beta \in \mathcal{O}_K$ . Next, multiply through by  $\alpha^{n-1-d}$ , then we have

$$\frac{x_d\alpha^{n-1}}{p} + \frac{\alpha^n\beta}{p} \in \mathcal{O}_K.$$

Now, since  $m_\alpha$  satisfies Eisenstein at  $p$ , we see that  $\alpha^n = pf(\alpha)$  for some  $f \in \mathbb{Z}[x]$  and therefore the above gives us that

$$\frac{x_d\alpha^{n-1}}{p} + \beta f(\alpha) \in \mathcal{O}_K.$$

and thus

$$\frac{x_d\alpha^{n-1}}{p} \in \mathcal{O}_K.$$

Lets now calculate the norm of this:

$$N_{K/\mathbb{Q}}\left(\frac{x_d\alpha^{n-1}}{p}\right) = \frac{x_d^n N_{K/\mathbb{Q}}(\alpha)^{n-1}}{p^n}.$$

By Eisenstein the constant coefficient of  $m_\alpha$  is divisible by  $p$  but not  $p^2$ , so since the constant coefficient of  $m_\alpha$  is  $N_{K/\mathbb{Q}}(\alpha)$  we see that  $N_{K/\mathbb{Q}}(\alpha) = pa$  where  $p \nmid a$ . Therefore we have

$$N_{K/\mathbb{Q}}\left(\frac{x_d\alpha^{n-1}}{p}\right) = \frac{x_d^n p^{n-1} a^{n-1}}{p^n} = \frac{x_d^n a^{n-1}}{p}.$$

But this cant be in  $\mathbb{Z}$  since  $p$  doesn't divide  $x_d$  or  $a$ , and this gives us a contradiction since Proposition 2.7 says that the norm of an algebraic integer must be an integer. So  $\phi$  couldn't have been an algebraic integer.  $\square$

### 3. CYCLOTOMIC FIELDS

**lemma 3.1.** *For  $n$  any integer,  $\Phi_n$  (the  $n$ -th cyclotomic polynomial) is an irreducible polynomial of degree  $\varphi(n)$  (where  $\varphi$  is Euler's Totient function).*

**Theorem 3.2.** *Let  $\zeta_p$  be a  $p$ -th root of unity for  $p$  an odd prime, let  $\lambda_p = 1 - \zeta_p$  and  $K = \mathbb{Q}(\zeta_p)$ . Then  $\mathcal{O}_K = \mathbb{Z}[\zeta_p] = \mathbb{Z}[\lambda_p]$  moreover*

$$\Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) = \Delta(\{1, \lambda_p, \dots, \lambda_p^{p-2}\}) = (-1)^{\frac{(p-1)}{2}} p^{p-2}.$$

*Proof.* First note  $[K : \mathbb{Q}] = p - 1$ .

Since  $\zeta_p = 1 - \lambda_p$  we at once get  $\mathbb{Z}[\zeta_p] = \mathbb{Z}[\lambda_p]$  (just do double inclusion). Next, let  $\alpha_i = \sigma_i(\zeta_p)$  denote the conjugates of  $\zeta_p$ , which is the same as the image of  $\zeta_p$  under one of the embeddings  $\sigma_i : \mathbb{Q}(\zeta_p) \rightarrow \mathbb{C}$ . Now by

Proposition 2.3 we have

$$\begin{aligned}\Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) &= \prod_{i < j} (\alpha_i - \alpha_j)^2 = \prod_{i < j} ((1 - \alpha_i) - (1 - \alpha_j))^2 \\ &= \Delta(\{1, \lambda_p, \dots, \lambda_p^{p-2}\})\end{aligned}$$

Now, by Proposition 2.5, we have

$$\Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) = (-1)^{\frac{(p-1)(p-2)}{2}} N_{K/\mathbb{Q}}(\Phi'_p(\zeta_p))$$

Since  $p$  is odd  $(-1)^{\frac{(p-1)(p-2)}{2}} = (-1)^{\frac{(p-1)}{2}}$ . Next, we see that

$$\Phi'_p(x) = \frac{px^{p-1}(x-1) - (x^p-1)}{(x-1)^2}$$

therefore

$$\Phi'_p(\zeta_p) = -\frac{p\zeta_p^{p-1}}{\lambda_p}.$$

Lastly, note that  $N_{K/\mathbb{Q}}(\zeta_p) = 1$ , since this is the constant term in its minimal polynomial. Similarly, we see  $N_{K/\mathbb{Q}}(\lambda_p) = p$ . Putting this all together, we get

$$N_{K/\mathbb{Q}}(\Phi'_p(\zeta_p)) = \frac{N_{K/\mathbb{Q}}(p)N_{K/\mathbb{Q}}(\zeta_p)^{p-1}}{N_{K/\mathbb{Q}}(-\lambda_p)} = (-1)^{p-1}p^{p-2} = p^{p-2}$$

So the last thing we need to prove is that  $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$ . From the calculation we just did, the only prime dividing the discriminant is  $p$ , therefore Lemma 2.9 tells us the only prime we need to check is  $p$ . But from Lemma 2.10 we know that dividing by  $p$  won't give us any new integral elements, so this must be an integral basis which give the result.  $\square$

**lemma 3.3.** *Let  $\alpha$  be an algebraic integer all of whose conjugates have absolute value one. Then  $\alpha$  is a root of unity.*

**lemma 3.4.** *Any unit  $u$  in  $\mathbb{Z}[\zeta_p]$  can be written in the form  $\beta\zeta_p^k$  with  $k$  an integer and  $\beta \in \mathbb{R}$ .*

**lemma 3.5.** *Let  $p$  be a prime and  $n = p^k$ . Then*

$$p = u(1 - \zeta_n)^{\varphi(n)}$$

where  $u \in \mathbb{Z}[\zeta_n]^\times$ .

**lemma 3.6.** *Let  $R$  be a Dedekind domain,  $p$  a prime and  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$  ideals such that*

$$\mathfrak{a}\mathfrak{b} = \mathfrak{c}^p$$

and suppose  $\mathfrak{a}, \mathfrak{b}$  are coprime. Then there exist ideals  $\mathfrak{e}, \mathfrak{d}$  such that

$$\mathfrak{a} = \mathfrak{e}^p \quad \mathfrak{b} = \mathfrak{d}^p \quad \mathfrak{e}\mathfrak{d} = \mathfrak{c}$$

#### 4. FERMAT'S LAST THEOREM FOR REGULAR PRIMES

**Theorem 4.1.** *Let  $p \geq 5$  be a prime number,  $\zeta_p$  a  $p$ -th root of unity and  $K = \mathbb{Q}(\zeta_p)$ . Assume that we have  $x, y, z \in \mathbb{Z}$  with  $\gcd(xyz, p) = 1$  and such that*

$$x^p + y^p = z^p.$$

(1) We cannot have

$$x \equiv y \equiv -z \pmod{p}$$

and, therefore, without loss of generality, we may assume

$$x \not\equiv y \pmod{p}.$$

(2) For  $i \neq j$  we can write

$$(\zeta_p^i - \zeta_p^j) = u(1 - \zeta_p)$$

with  $u$  a unit in  $\mathbb{Z}[\zeta_p]$ . From this it follows that the ideals

$$(x + y), (x + \zeta_p y), (x + \zeta_p^2 y), \dots, (x + \zeta_p^{p-1} y)$$

are pairwise coprime.

(3) Let  $\alpha \in \mathbb{Z}[\zeta_p]$ . Then  $\alpha^p$  is congruent to an integer modulo  $p$ .

(4) Let  $\alpha \in \mathbb{Z}[\zeta_p]$ . Suppose that  $\alpha = a_0 + a_1 \zeta_p + \dots + a_{p-1} \zeta_p^{p-1}$  with at least one  $a_i \neq 0$ . Then there is an integer  $n$  such that  $\alpha/n \in \mathbb{Z}[\zeta_p]$ , then  $n$  divides each  $a_i$ .

(5) Let  $i \in \{1, \dots, p-1\}$ . Suppose that  $x + y\zeta_p^i = u\alpha^p$  with  $u \in \mathbb{Z}[\zeta_p]^\times$  and  $\alpha \in \mathbb{Z}[\zeta_p]$ . Then there is an integer  $k$  such that

$$x + y\zeta_p^i - \zeta_p^{2k} x - \zeta_p^{2k-i} y \equiv 0 \pmod{p}.$$

*Proof.* (1) Reducing modulo  $p$ , using Fermat's little theorem, you get that if  $x \equiv y \equiv -z \pmod{p}$  then  $3z \equiv 0 \pmod{p}$ . But since  $p > 3$  this means  $p|z$  but this contradicts  $\gcd(xyz, p) = 1$ . Now, if  $x \equiv y \pmod{p}$  then  $x \not\equiv -z \pmod{p}$  we can relabel  $y, z$  so that wlog  $x \not\equiv y \pmod{p}$  (this uses that  $p$  is odd).

(2) Lemma 3.5 gives that  $u$  is a unit. So all that needs to be proved is that the ideals are coprime. Assume not, then for some  $i \neq j$  we have some prime ideal  $\mathfrak{p}$  dividing  $(x + y\zeta_p^i)$  and  $(x + y\zeta_p^j)$ . It must then also divide their sum and their difference, so we must have  $\mathfrak{p} | (1 - \zeta_p)$  or  $\mathfrak{p} | y$ . Similarly,  $\mathfrak{p}$  divides  $\zeta_p^j(x + y\zeta_p^i) - \zeta_p^i(x + y\zeta_p^j)$  so  $\mathfrak{p}$  divides  $x$  or  $(1 - \zeta_p)$ . We can't have  $\mathfrak{p}$  dividing  $x, y$  since they are coprime, therefore  $\mathfrak{p} | (1 - \zeta_p)$ . We know that since  $(1 - \zeta_p)$  has norm  $p$  it must be a prime ideal, so  $\mathfrak{p} = (1 - \zeta_p)$ . Now, note that  $x + y \equiv x + y\zeta_p^i \equiv 0 \pmod{\mathfrak{p}}$ . But since  $x, y \in \mathbb{Z}$  this means we would have  $x + y \equiv 0 \pmod{p}$ , which implies  $z^p \equiv 0 \pmod{p}$  which contradicts our assumptions.

(3) This is easy. Just use  $(x + y)^p \equiv x^p + y^p \pmod{p}$  and that  $\zeta_p$  is a  $p$ -th root of unity.

(4) Looking at  $\alpha = a_0 + a_1 \zeta_p + \dots + a_{p-1} \zeta_p^{p-1}$ , if one of the  $a_i$ 's is zero and  $\alpha/n \in \mathbb{Z}[\zeta_p]$ , then  $\alpha/n = \sum_i a_i/n \zeta_p^i$ . Now, as  $\alpha/n \in \mathbb{Z}[\zeta_p]$ , pick the basis of  $\mathbb{Z}[\zeta_p]$  which does not contain  $\zeta_p$  (which is possible as any subset of  $\{1, \zeta_p, \dots, \zeta_p^{p-1}\}$  with  $p-1$  elements forms a basis of  $\mathbb{Z}[\zeta_p]$ ). Then  $\alpha = \sum_i b_i \zeta_p^i$  where  $b_i \in \mathbb{Z}$ . Therefore comparing coefficients, we get the result.

(5) Using lemma 3.4 we have  $(x + y\zeta_p^i) = \beta \zeta_p^k \alpha^p$  which is equivalent to  $\beta \zeta_p^k a \pmod{p}$  with  $a$  and integer (using part 3). Now, if we consider the complex conjugate we have  $\overline{(x + y\zeta_p^i)} \equiv \beta \zeta_p^{-k} a \pmod{p}$ . Looking at  $(x + y\zeta_p^i) - \zeta_p^{2k} \overline{(x + y\zeta_p^i)}$  then gives the result.  $\square$

**Definition 4.2.** A prime number  $p$  is called regular if it does not divide the class number of  $\mathbb{Q}(\zeta_p)$ .

**Theorem 4.3.** *Let  $p$  be a regular prime. Then*

$$x^p + y^p = z^p$$

*has no solutions with  $x, y, z \in \mathbb{Z}$  and  $\gcd(xyz, p) = 1$ .*

*Proof.* First thing is to note that if  $x^p + y^p = z^p$  then

$$z^p = (x + y)(x + \zeta_p y) \cdots (x + y\zeta_p^{p-1})$$

as ideals. Then since we know the ideals are coprime, then by lemma 3.6 we have that each  $(x + y\zeta_p^i) = \mathfrak{a}^p$ , for  $\mathfrak{a}$  some ideal. Note that,  $[\mathfrak{a}^p] = 1$  in the class group. Now, since  $p$  does not divide the size of the class group we have that  $[\mathfrak{a}] = 1$  in the class group, so its principal. So we have  $x + y\zeta_p^i = u_i \alpha_i^p$  with  $u_i$  a unit. So by 4.1 part 5 we have some  $k$  such that  $x + y\zeta_p - \zeta_p^{2k} x - \zeta_p^{2k-1} y \equiv 0 \pmod{p}$ . If  $1, \zeta_p, \zeta_p^{2k}, \zeta_p^{2k-1}$  are distinct, then 4.1 part 4 (which uses that  $p > 3$ ) says that  $p$  divides  $x, y$ , contrary to our assumption. So they cannot be distinct, but checking each case leads to a contradiction, therefore there cannot be any such solutions.  $\square$

**Theorem 4.4.** *Let  $p$  be a regular prime. Then*

$$x^p + y^p = z^p$$

*has no solutions with  $x, y, z \in \mathbb{Z}$  and  $xyz \neq 0$ .*

## REFERENCES

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