

FERMAT'S LAST THEOREM FOR REGULAR PRIMES

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1. INTRODUCTION

We prove Fermat's Last Theorem for regular primes and give some of the necessary background. It uses [Sam70, Mar18, Was82].

2. DISCRIMINANTS OF NUMBER FIELDS

lemma 2.1. *Let K be a number field, $\alpha \in K$ and let σ_i be the embeddings of K into \mathbb{C} . Then*

$$\mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = \sum_i \sigma_i(\alpha) \quad N_{K/\mathbb{Q}}(\alpha) = \prod_i \sigma_i(\alpha)$$

Definition 2.2. Let A, K be commutative rings with K and A -algebra. let $B = \{b_1, \dots, b_n\}$ be a set of elements in K . The discriminant of B is defined as

$$\Delta(B) = \det \begin{pmatrix} \mathrm{Tr}_{K/A}(b_1 b_1) & \cdots & \mathrm{Tr}_{K/A}(b_1 b_n) \\ \vdots & & \vdots \\ \mathrm{Tr}_{K/A}(b_n b_1) & \cdots & \mathrm{Tr}_{K/A}(b_n b_n) \end{pmatrix}.$$

lemma 2.3. *Let K be a number field and let $B = \{b_1, \dots, b_n\}$ be a set of elements in K . Then $\Delta(B) \neq 0$ if and only if the elements in B are linearly independent.*

lemma 2.4. *Let K be a number field and B, B' bases for K/\mathbb{Q} . If P denotes the change of basis matrix, then*

$$\Delta(B) = \det(P)^2 \Delta(B').$$

lemma 2.5. *Let K be a number field with basis $B = \{b_1, \dots, b_n\}$ and let $\sigma_1, \dots, \sigma_n$ be the embeddings of K into \mathbb{C} . Now let M be the matrix*

$$\begin{pmatrix} \sigma_1(b_1) & \cdots & \sigma_1(b_n) \\ \vdots & & \vdots \\ \sigma_n(b_1) & \cdots & \sigma_n(b_n) \end{pmatrix}.$$

Then

$$\Delta(B) = \det(M)^2.$$

Proof. By Proposition 2.1 we know that $\text{Tr}_{K/\mathbb{Q}}(b_i b_j) = \sum_k \sigma_k(b_i) \sigma_k(b_j)$ which is the same as the (i, j) entry of $M^t M$. Therefore

$$\det(T_B) = \det(M^t M) = \det(M)^2.$$

□

lemma 2.6. *Let K be a number field and $B = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ for some $\alpha \in K$. Then*

$$\Delta(B) = \prod_{i < j} (\sigma_i(\alpha) - \sigma_j(\alpha))^2$$

where σ_i are the embeddings of K into \mathbb{C} . Here $\Delta(B)$ denotes the discriminant.

Proof. First we recall a classical linear algebra result relating to the Vandermonde matrix, which states that

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{i < j} (x_i - x_j).$$

Combining this with Proposition 2.5 gives the result. □

lemma 2.7. *Let f be a monic irreducible polynomial over a number field K and let α be one of its roots in \mathbb{C} . Then*

$$f'(\alpha) = \prod_{\beta \neq \alpha} (\alpha - \beta),$$

where the product is over the roots of f different from α .

Proof. We first write $f(x) = (x - \alpha)g(x)$ which we can do (over \mathbb{C}) as α is a root of f , where now $g(x) = \prod_{\beta \neq \alpha} (x - \beta)$. Differentiating we get

$$f'(x) = g(x) + (x - \alpha)g'(x).$$

If we now evaluate at α we get the result. □

lemma 2.8. *Let $K = \mathbb{Q}(\alpha)$ be a number field with $n = [K : \mathbb{Q}]$ and let $B = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$. Then*

$$\Delta(B) = (-1)^{\frac{n(n-1)}{2}} N_{K/\mathbb{Q}}(m'_\alpha(\alpha))$$

where m'_α is the derivative of $m_\alpha(x)$ (which we recall denotes the minimal polynomial of α).

Proof. By Proposition 2.6 we have $\Delta(B) = \prod_{i < j} (\alpha_i - \alpha_j)^2$ where $\alpha_k := \sigma_k(\alpha)$. Next, we note that the number of terms in this product is $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$. So if we write each term as $(\alpha_i - \alpha_j)^2 = -(\alpha_i - \alpha_j)(\alpha_j - \alpha_i)$ we get

$$\Delta(B) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^n \prod_{i \neq j} (\alpha_i - \alpha_j).$$

Now, by lemma 2.7 and Proposition 2.1 we see that

$$N_{K/\mathbb{Q}}(m'_\alpha(\alpha)) = \prod_{i=1}^n m'_\alpha(\alpha_i) = \prod_{i=1}^n \prod_{i \neq j} (\alpha_i - \alpha_j),$$

which gives the result. \square

lemma 2.9. *Let K be a number field and B, B' bases for K/\mathbb{Q} . If P denotes the change of basis matrix, then*

$$\Delta(B) = \det(P)^2 \Delta(B').$$

lemma 2.10. *If K is a number field and $\alpha \in \mathcal{O}_K$ then $\text{Tr}_{K/\mathbb{Q}}(\alpha)$ and $N_{K/\mathbb{Q}}(\alpha)$ are both in \mathbb{Z} .*

lemma 2.11. *Let K be a number field and $B = \{b_1, \dots, b_n\}$ be elements in \mathcal{O}_K , then $\Delta(B) \in \mathbb{Z}$.*

lemma 2.12. *Let K be a number field and $B = \{b_1, \dots, b_n\}$ be a basis for K/\mathbb{Q} consisting of algebraic integers. If B is not an integral basis then there exists an algebraic integer of the form*

$$\alpha = \frac{x_1 b_1 + \dots + x_n b_n}{p}$$

where p is a prime and $x_i \in \{0, \dots, p-1\}$ with not all x_i zero. Moreover, if $x_i \neq 0$ and we let B' be the basis obtained by replacing b_i with α , then

$$\Delta(B') = \frac{x_i^2}{p^2} \Delta(B).$$

In particular $p^2 \mid \Delta(B)$.

Proof. If B is not an integral basis then we can find some element $\phi \in \mathcal{O}_K$ such that

$$\phi = y_1 b_1 + \dots + y_n b_n$$

with not all the y_i in \mathbb{Z} . So, let N be the least common multiple of the denominators of the y_i (meaning $Ny_i \in \mathbb{Z}$ for all i). Now, let p be a prime factor of N . If we now consider $(N/p)\phi$ then all of the coefficients of b_i are in $\frac{1}{p}\mathbb{Z}$ (so they have denominator 1 or p .) and at least one of them has denominator p (since not all the y_i were in \mathbb{Z}). So by relabelling, wlog we can assume

$$\phi = y_1 b_1 + \dots + y_n b_n$$

with $y_i \in \frac{1}{p}\mathbb{Z}$

Now look at

$$\psi := [y_1]b_1 + \dots + [y_n]b_n$$

(here $[x]$ denotes the integer part of x). The both ψ and ϕ are algebraic integers (as the b_i are algebraic integers). Therefore, so is $\theta = \phi - \psi$. By construction, θ has coefficients of the form $\frac{x_i}{p} := y_i - [y_i]$ where $x_i \in \{0, \dots, p-1\}$ and not all the x_i are zero (since, again, not all the y_i were in \mathbb{Z}). This gives the first part of the lemma.

Now, assume $x_i \neq 0$, then let us replace $b_i \in B$ with θ to get a new basis B' which again consists of algebraic integers. Next, we note that the change of basis matrix from B to B' is

$$\begin{pmatrix} 1 & 0 & \dots & \frac{x_1}{p} & \dots & 0 \\ 0 & 1 & \dots & \frac{x_2}{p} & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{x_n}{p} & \dots & 1 \end{pmatrix}$$

(here the column of x_j/p 's is in the i -th column).

This matrix has determinant $\frac{x_i}{p}$. Therefore, by Proposition 2.9 we see that $\Delta(B') = \frac{x_i^2}{p^2} \Delta(B)$. But both $\Delta(B), \Delta(B')$ are in \mathbb{Z} by Proposition 2.11, therefore $p^2 \mid \Delta(B)$ giving the result. \square

lemma 2.13. *Let $K = \mathbb{Q}(\alpha)$ and α be an algebraic integer such that m_α satisfies Eisensteins Criterion for a prime p . Then none of the elements*

$$\phi = \frac{1}{p}(x_0 + x_1\alpha + \cdots + x_{n-1}\alpha^{n-1})$$

is an algebraic integer, where $n = \deg(m_\alpha)$ and $x_i \in \{0, \dots, p-1\}$.

Proof. Suppose for contradiction that $\phi \in \mathcal{O}_K$ and let x_d be the first non-zero coefficient, so

$$\phi = \frac{1}{p}(x_d\alpha^d + x_{d+1}\alpha^{d+1} + \cdots + x_{n-1}\alpha^{n-1}) \in \mathcal{O}_K.$$

Now, rewrite this as $\phi = \frac{1}{p}(x_d\alpha^d + \alpha^{d+1}\beta)$ for some $\beta \in \mathcal{O}_K$. Next, multiply through by α^{n-1-d} , then we have

$$\frac{x_d\alpha^{n-1}}{p} + \frac{\alpha^n\beta}{p} \in \mathcal{O}_K.$$

Now, since m_α satisfies Eisenstein at p , we see that $\alpha^n = pf(\alpha)$ for some $f \in \mathbb{Z}[x]$ and therefore the above gives us that

$$\frac{x_d\alpha^{n-1}}{p} + \beta f(\alpha) \in \mathcal{O}_K.$$

and thus

$$\frac{x_d\alpha^{n-1}}{p} \in \mathcal{O}_K.$$

Lets now calculate the norm of this:

$$N_{K/\mathbb{Q}}\left(\frac{x_d\alpha^{n-1}}{p}\right) = \frac{x_d^n N_{K/\mathbb{Q}}(\alpha)^{n-1}}{p^n}.$$

By Eisenstein the constant coefficient of m_α is divisible by p but not p^2 , so since the constant coefficient of m_α is $N_{K/\mathbb{Q}}(\alpha)$ we see that $N_{K/\mathbb{Q}}(\alpha) = pa$ where $p \nmid a$. Therefore we have

$$N_{K/\mathbb{Q}}\left(\frac{x_d\alpha^{n-1}}{p}\right) = \frac{x_d^n p^{n-1} a^{n-1}}{p^n} = \frac{x_d^n a^{n-1}}{p}.$$

But this cant be in \mathbb{Z} since p doesn't divide x_d or a , and this gives us a contradiction since Proposition 2.10 says that the norm of an algebraic integer must be an integer. So ϕ couldn't have been an algebraic integer. \square

3. CYCLOTOMIC FIELDS

lemma 3.1. *For n any integer, Φ_n (the n -th cyclotomic polynomial) is an irreducible polynomial of degree $\varphi(n)$ (where φ is Euler's Totient function).*

Theorem 3.2. *Let ζ_p be a p -th root of unity for p an odd prime, let $\lambda_p = 1 - \zeta_p$ and $K = \mathbb{Q}(\zeta_p)$. Then $\mathcal{O}_K = \mathbb{Z}[\zeta_p] = \mathbb{Z}[\lambda_p]$ moreover*

$$\Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) = \Delta(\{1, \lambda_p, \dots, \lambda_p^{p-2}\}) = (-1)^{\frac{(p-1)}{2}} p^{p-2}.$$

Proof. First note $[K : \mathbb{Q}] = p - 1$.

Since $\zeta_p = 1 - \lambda_p$ we at once get $\mathbb{Z}[\zeta_p] = \mathbb{Z}[\lambda_p]$ (just do double inclusion). Next, let $\alpha_i = \sigma_i(\zeta_p)$ denote the conjugates of ζ_p , which is the same as the image of ζ_p under one of the embeddings $\sigma_i : \mathbb{Q}(\zeta_p) \rightarrow \mathbb{C}$. Now by Proposition 2.6 we have

$$\begin{aligned} \Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) &= \prod_{i < j} (\alpha_i - \alpha_j)^2 = \prod_{i < j} ((1 - \alpha_i) - (1 - \alpha_j))^2 \\ &= \Delta(\{1, \lambda_p, \dots, \lambda_p^{p-2}\}) \end{aligned}$$

Now, by Proposition 2.8, we have

$$\Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) = (-1)^{\frac{(p-1)(p-2)}{2}} N_{K/\mathbb{Q}}(\Phi'_p(\zeta_p))$$

Since p is odd $(-1)^{\frac{(p-1)(p-2)}{2}} = (-1)^{\frac{(p-1)}{2}}$. Next, we see that

$$\Phi'_p(x) = \frac{px^{p-1}(x-1) - (x^p-1)}{(x-1)^2}$$

therefore

$$\Phi'_p(\zeta_p) = -\frac{p\zeta_p^{p-1}}{\lambda_p}.$$

Lastly, note that $N_{K/\mathbb{Q}}(\zeta_p) = 1$, since this is the constant term in its minimal polynomial. Similarly, we see $N_{K/\mathbb{Q}}(\lambda_p) = p$. Putting this all together, we get

$$N_{K/\mathbb{Q}}(\Phi'_p(\zeta_p)) = \frac{N_{K/\mathbb{Q}}(p)N_{K/\mathbb{Q}}(\zeta_p)^{p-1}}{N_{K/\mathbb{Q}}(-\lambda_p)} = (-1)^{p-1}p^{p-2} = p^{p-2}$$

So the last thing we need to prove is that $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$. From the calculation we just did, the only prime dividing the discriminant is p , therefore Lemma 2.12 tells us the only prime we need to check is p . But from Lemma 2.13 we know that dividing by p won't give us any new integral elements, so this must be an integral basis which give the result. \square

lemma 3.3. *Let α be an algebraic integer all of whose conjugates have absolute value one. Then α is a root of unity.*

lemma 3.4. *Let p be a prime, $K = \mathbb{Q}(\zeta_p)$ $\alpha \in K$ such that there exists $n \in \mathbb{N}$ such that $\alpha^n = 1$, then $\alpha = \pm \zeta_p^k$ for some k .*

Proof. If n is different to p then K contains a $2pn$ -th root of unity. Therefore $\mathbb{Q}(\zeta_{2pn}) \subset K$, but this cannot happen as $[K : \mathbb{Q}] = p - 1$ and $[\mathbb{Q}(\zeta_{2pn}) : \mathbb{Q}] = \varphi(2np)$. \square

lemma 3.5. *Any unit u in $\mathbb{Z}[\zeta_p]$ can be written in the form $\beta \zeta_p^k$ with k an integer and $\beta \in \mathbb{R}$.*

lemma 3.6. *Let p be a prime and $n = p^k$. Then*

$$p = u(1 - \zeta_n)^{\varphi(n)}$$

where $u \in \mathbb{Z}[\zeta_n]^\times$.

lemma 3.7. *Let R be a Dedekind domain, p a prime and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ ideals such that*

$$\mathfrak{a}\mathfrak{b} = \mathfrak{c}^p$$

and suppose $\mathfrak{a}, \mathfrak{b}$ are coprime. Then there exist ideals $\mathfrak{e}, \mathfrak{d}$ such that

$$\mathfrak{a} = \mathfrak{e}^p \quad \mathfrak{b} = \mathfrak{d}^p \quad \mathfrak{e}\mathfrak{d} = \mathfrak{c}$$

4. FERMAT'S LAST THEOREM FOR REGULAR PRIMES

lemma 4.1. *Let $p \geq 5$ be a prime number, ζ_p a p -th root of unity and $x, y \in \mathbb{Z}$ coprime.*

For $i \neq j$ we can write

$$(\zeta_p^i - \zeta_p^j) = u(1 - \zeta_p)$$

with u a unit in $\mathbb{Z}[\zeta_p]$. From this it follows that the ideals

$$(x + y), (x + \zeta_p y), (x + \zeta_p^2 y), \dots, (x + \zeta_p^{p-1} y)$$

are pairwise coprime.

Proof. Lemma 3.6 gives that u is a unit. So all that needs to be proved is that the ideals are coprime. Assume not, then for some $i \neq j$ we have some prime ideal \mathfrak{p} dividing by $(x + y\zeta_p^i)$ and $(x + y\zeta_p^j)$. It must then also divide their sum and their difference, so we must have $\mathfrak{p} | (1 - \zeta_p)$ or $\mathfrak{p} | y$. Similarly, \mathfrak{p} divides $\zeta_p^j(x + y\zeta_p^i) - \zeta_p^i(x + y\zeta_p^j)$ so \mathfrak{p} divides x or $(1 - \zeta_p)$. We can't have \mathfrak{p} dividing x, y since they are coprime, therefore $\mathfrak{p} | (1 - \zeta_p)$. We know that since $(1 - \zeta_p)$ has norm p it must be a prime ideal, so $\mathfrak{p} = (1 - \zeta_p)$. Now, note that $x + y \equiv x + y\zeta_p^i \equiv 0 \pmod{\mathfrak{p}}$. But since $x, y \in \mathbb{Z}$ this means we would have $x + y \equiv 0 \pmod{p}$, which implies $z^p \equiv 0 \pmod{p}$ which contradicts our assumptions. \square

lemma 4.2. *Let p be a prime number, ζ_p a p -th root of unity and $\alpha \in \mathbb{Z}[\zeta_p]$. Then α^p is congruent to an integer modulo p .*

Proof. Just use $(x + y)^p \equiv x^p + y^p \pmod{p}$ and that ζ_p is a p -th root of unity. \square

lemma 4.3. *Let $p \geq 5$ be a prime number, ζ_p a p -th root of unity and $\alpha \in \mathbb{Z}[\zeta_p]$. If there is an integer n such that $\alpha/n \in \mathbb{Z}[\zeta_p]$, then n divides each a_i .*

Proof. Looking at $\alpha = a_0 + a_1\zeta_p + \dots + a_{p-1}\zeta_p^{p-1}$, if one of the a_i 's is zero and $\alpha/n \in \mathbb{Z}[\zeta_p]$, then $\alpha/n = \sum_i a_i/n\zeta_p^i$. Now, as $\alpha/n \in \mathbb{Z}[\zeta_p]$, pick the basis of $\mathbb{Z}[\zeta_p]$ which does not contain ζ_p (which is possible as any subset of $\{1, \zeta_p, \dots, \zeta_p^{p-1}\}$ with $p - 1$ elements forms a basis of $\mathbb{Z}[\zeta_p]$). Then $\alpha = \sum_i b_i \zeta_p^i$ where $b_i \in \mathbb{Z}$. Therefore comparing coefficients, we get the result. \square

lemma 4.4. *Let $p \geq 5$ be a prime number, ζ_p a p -th root of unity and $\alpha \in \mathbb{Z}[\zeta_p]$. Suppose that $x + y\zeta_p^i = u\alpha^p$ with $u \in \mathbb{Z}[\zeta_p]^\times$ and $\alpha \in \mathbb{Z}[\zeta_p]$. Then there is an integer k such that*

$$x + y\zeta_p^i - \zeta_p^{2k}x - \zeta_p^{2k-i}y \equiv 0 \pmod{p}.$$

Proof. Using lemma 3.5 we have $(x + y\zeta_p^i) = \beta\zeta_p^k\alpha^p$ which is equivalent to $\beta\zeta_p^k a \pmod{p}$ with a and integer 4.2). Now, if we consider the complex conjugate we have $\overline{(x + y\zeta_p^i)} \equiv \beta\zeta_p^{-k}a \pmod{p}$. Looking at $(x + y\zeta_p^i) - \zeta_p^{2k}\overline{(x + y\zeta_p^i)}$ then gives the result. \square

lemma 4.5. *Let $p \geq 5$ be an prime number, ζ_p a p -th root of unity and $K = \mathbb{Q}(\zeta_p)$. Assume that we have $x, y, z \in \mathbb{Z}$ with $\gcd(xyz, p) = 1$ and such that*

$$x^p + y^p = z^p.$$

Then without loss of generality, we may assume x, y, z are pairwise coprime and

$$x \not\equiv y \pmod{p}.$$

Proof. The first part is easy.

Reducing modulo p , using Fermat's little theorem, you get that if $x \equiv y \equiv -z \pmod{p}$ then $3z \equiv 0 \pmod{p}$. But since $p > 3$ this means $p|z$ but this contradicts $\gcd(xyz, p) = 1$. Now, if $x \equiv y \pmod{p}$ then $x \not\equiv -z \pmod{p}$ we can relabel y, z so that $wlog\ x \not\equiv y \pmod{p}$ (this uses that p is odd). \square

Definition 4.6. A prime number p is called regular if it does not divide the class number of $\mathbb{Q}(\zeta_p)$.

Theorem 4.7. *Let p be an odd regular prime. Then*

$$x^p + y^p = z^p$$

has no solutions with $x, y, z \in \mathbb{Z}$ and $\gcd(xyz, p) = 1$.

Proof. First thing is to note that if $x^p + y^p = z^p$ then

$$z^p = (x + y)(x + \zeta_p y) \cdots (x + y\zeta_p^{p-1})$$

as ideals. Then since by 4.1 we know the ideals are coprime, then by lemma 3.7 we have that each $(x + y\zeta_p^i) = \mathfrak{a}^p$, for \mathfrak{a} some ideal. Note that, $[\mathfrak{a}^p] = 1$ in the class group. Now, since p does not divide the size of the class group we have that $[\mathfrak{a}] = 1$ in the class group, so its principal. So we have $x + y\zeta_p^i = u_i \alpha_i^p$ with u_i a unit. So by 4.4 we have some k such that $x + y\zeta_p - \zeta_p^{2k}x - \zeta_p^{2k-1}y \equiv 0 \pmod{p}$. If $1, \zeta_p, \zeta_p^{2k}, \zeta_p^{2k-1}$ are distinct, then 4.3 (which uses that $p > 3$) says that p divides x, y , contrary to our assumption. So they cannot be distinct, but checking each case leads to a contradiction, therefore there cannot be any such solutions. \square

Theorem 4.8. *Let p be an odd regular prime. Then*

$$x^p + y^p = z^p$$

has no solutions with $x, y, z \in \mathbb{Z}$ and $p|xyz$.

Theorem 4.9. *Let p be an odd regular prime. Then*

$$x^p + y^p = z^p$$

has no solutions with $x, y, z \in \mathbb{Z}$ and $xyz \neq 0$.

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