

# FERMAT'S LAST THEOREM FOR REGULAR PRIMES

## CONTENTS

1. Introduction	1
2. Discriminants of number fields	1
3. Cyclotomic fields	3
4. Fermat's Last Theorem for regular primes	5
References	7

## 1. INTRODUCTION

We prove Fermat's Last Theorem for regular primes and give some of the necessary background. It uses [Sam70, Mar18, Was82].

## 2. DISCRIMINANTS OF NUMBER FIELDS

We recall basic facts about the discriminant.

**Lemma 2.1.** *Let  $K$  be a number field,  $\alpha \in K$  and let  $\sigma_i$  be the embeddings of  $K$  into  $\mathbb{C}$ . Then*

$$N_{K/\mathbb{Q}}(\alpha) = \prod_i \sigma_i(\alpha)$$

.

*Proof.* The proof is standard.  $\square$

**Lemma 2.2.** *Let  $K$  be a number field,  $\alpha \in K$  and let  $\sigma_i$  be the embeddings of  $K$  into  $\mathbb{C}$ . Then*

$$\text{Tr}_{K/\mathbb{Q}}(\alpha) = \sum_i \sigma_i(\alpha).$$

*Proof.* The proof is standard.  $\square$

**Definition 2.3.** Let  $A, K$  be commutative rings with  $K$  and  $A$ -algebra. let  $B = \{b_1, \dots, b_n\}$  be a set of elements in  $K$ . The discriminant of  $B$  is defined as

$$\Delta(B) = \det \begin{pmatrix} \text{Tr}_{K/A}(b_1 b_1) & \cdots & \text{Tr}_{K/A}(b_1 b_n) \\ \vdots & & \vdots \\ \text{Tr}_{K/A}(b_n b_1) & \cdots & \text{Tr}_{K/A}(b_n b_n) \end{pmatrix}.$$

**Lemma 2.4.** *Let  $L/K$  be an extension of fields and let  $B = \{b_1, \dots, b_n\}$  be a  $K$ -basis of  $L$ . Then  $\Delta(B) \neq 0$ .*

*Proof.* The proof is standard.  $\square$

**Lemma 2.5.** *Let  $K$  be a number field and  $B, B'$  bases for  $K/\mathbb{Q}$ . If  $P$  denotes the change of basis matrix, then*

$$\Delta(B) = \det(P)^2 \Delta(B').$$

*Proof.* The proof is standard.  $\square$

**Lemma 2.6.** *Let  $K$  be a number field with basis  $B = \{b_1, \dots, b_n\}$  and let  $\sigma_1, \dots, \sigma_n$  be the embeddings of  $K$  into  $\mathbb{C}$ . Now let  $M$  be the matrix*

$$\begin{pmatrix} \sigma_1(b_1) & \cdots & \sigma_1(b_n) \\ \vdots & & \vdots \\ \sigma_n(b_1) & \cdots & \sigma_n(b_n) \end{pmatrix}.$$

*Then*

$$\Delta(B) = \det(M)^2.$$

*Proof.* By Lemma 2.2 we know that  $\text{Tr}_{K/\mathbb{Q}}(b_i b_j) = \sum_k \sigma_k(b_i) \sigma_k(b_j)$  which is the same as the  $(i, j)$  entry of  $M^t M$ . Therefore

$$\det(T_B) = \det(M^t M) = \det(M)^2.$$

$\square$

**Lemma 2.7.** *Let  $K$  be a number field and  $B = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$  for some  $\alpha \in K$ . Then*

$$\Delta(B) = \prod_{i < j} (\sigma_i(\alpha) - \sigma_j(\alpha))^2$$

*where  $\sigma_i$  are the embeddings of  $K$  into  $\mathbb{C}$ . Here  $\Delta(B)$  denotes the discriminant.*

*Proof.* First we recall a classical linear algebra result relating to the Vandermonde matrix, which states that

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{i < j} (x_i - x_j).$$

Combining this with Lemma 2.6 gives the result.  $\square$

**Lemma 2.8.** *Let  $f$  be a monic irreducible polynomial over a number field  $K$  and let  $\alpha$  be one of its roots in  $\mathbb{C}$ . Then*

$$f'(\alpha) = \prod_{\beta \neq \alpha} (\alpha - \beta),$$

*where the product is over the roots of  $f$  different from  $\alpha$ .*

*Proof.* We first write  $f(x) = (x - \alpha)g(x)$  which we can do (over  $\mathbb{C}$ ) as  $\alpha$  is a root of  $f$ , where now  $g(x) = \prod_{\beta \neq \alpha} (x - \beta)$ . Differentiating we get

$$f'(x) = g(x) + (x - \alpha)g'(x).$$

If we now evaluate at  $\alpha$  we get the result.  $\square$

**Lemma 2.9.** *Let  $K = \mathbb{Q}(\alpha)$  be a number field with  $n = [K : \mathbb{Q}]$  and let  $B = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ . Then*

$$\Delta(B) = (-1)^{\frac{n(n-1)}{2}} N_{K/\mathbb{Q}}(m'_\alpha(\alpha))$$

*where  $m'_\alpha$  is the derivative of  $m_\alpha(x)$  (which we recall denotes the minimal polynomial of  $\alpha$ ).*

*Proof.* By Lemma 2.7 we have  $\Delta(B) = \prod_{i < j} (\alpha_i - \alpha_j)^2$  where  $\alpha_k := \sigma_k(\alpha)$ . Next, we note that the number of terms in this product is  $1+2+\dots+(n-1) = \frac{n(n-1)}{2}$ . So if we write each term as  $(\alpha_i - \alpha_j)^2 = -(\alpha_i - \alpha_j)(\alpha_j - \alpha_i)$  we get

$$\Delta(B) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^n \prod_{i \neq j} (\alpha_i - \alpha_j).$$

Now, by lemmas 2.8 and 2.1 we see that

$$N_{K/\mathbb{Q}}(m'_\alpha(\alpha)) = \prod_{i=1}^n m'_\alpha(\alpha_i) = \prod_{i=1}^n \prod_{i \neq j} (\alpha_i - \alpha_j),$$

which gives the result.  $\square$

**Lemma 2.10.** *If  $K$  is a number field and  $\alpha \in \mathcal{O}_K$  then  $N_{K/\mathbb{Q}}(\alpha)$  is in  $\mathbb{Z}$ .*

*Proof.* The proof is standard.  $\square$

**Lemma 2.11.** *If  $K$  is a number field and  $\alpha \in \mathcal{O}_K$  then  $\text{Tr}_{K/\mathbb{Q}}(\alpha)$  is in  $\mathbb{Z}$ .*

*Proof.* The proof is standard.  $\square$

**Lemma 2.12.** *Let  $K$  be a number field and  $B = \{b_1, \dots, b_n\}$  be elements in  $\mathcal{O}_K$ , then  $\Delta(B) \in \mathbb{Z}$ .*

*Proof.* Immediate by 2.11.  $\square$

**Lemma 2.13.** *Let  $K = \mathbb{Q}(\alpha)$  be a number field, where  $\alpha$  is an algebraic integer. Let  $B = \{1, \alpha, \dots, \alpha^{[K:\mathbb{Q}]-1}\}$  be the basis given by  $\alpha$  and let  $x \in \mathcal{O}_K$ . Then  $\Delta(B)x \in \mathbb{Z}[\alpha]$ .*

*Proof.* See the Lean proof.  $\square$

**Lemma 2.14.** *Let  $K = \mathbb{Q}(\alpha)$  be a number field, where  $\alpha$  is an algebraic integer with minimal polynomial that is Eisenstein at  $p$ . Let  $x \in \mathcal{O}_K$  such that  $p^n x \in \mathbb{Z}[\alpha]$  for some  $n$ . Then  $x \in \mathbb{Z}[\alpha]$ .*

*Proof.* See the Lean proof.  $\square$

### 3. CYCLOTOMIC FIELDS

**Lemma 3.1.** *For  $n$  any integer,  $\Phi_n$  (the  $n$ -th cyclotomic polynomial) is a polynomial of degree  $\varphi(n)$  (where  $\varphi$  is Euler's Totient function).*

*Proof.* The proof is classical.  $\square$

**Lemma 3.2.** *For  $n$  any integer,  $\Phi_n$  (the  $n$ -th cyclotomic polynomial) is an irreducible polynomial.*

*Proof.* The proof is classical.  $\square$

**Lemma 3.3.** *Let  $\zeta_p$  be a  $p$ -th root of unity for  $p$  an odd prime, let  $\lambda_p = 1 - \zeta_p$  and  $K = \mathbb{Q}(\zeta_p)$ . Then*

$$\Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) = \Delta(\{1, \lambda_p, \dots, \lambda_p^{p-2}\}) = (-1)^{\frac{(p-1)}{2}} p^{p-2}.$$

*Proof.* First note  $[K : \mathbb{Q}] = p - 1$ .

Since  $\zeta_p = 1 - \lambda_p$  we at once get  $\mathbb{Z}[\zeta_p] = \mathbb{Z}[\lambda_p]$  (just do double inclusion). Next, let  $\alpha_i = \sigma_i(\zeta_p)$  denote the conjugates of  $\zeta_p$ , which is the same as

the image of  $\zeta_p$  under one of the embeddings  $\sigma_i : \mathbb{Q}(\zeta_p) \rightarrow \mathbb{C}$ . Now by Proposition 2.7 we have

$$\begin{aligned} \Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) &= \prod_{i < j} (\alpha_i - \alpha_j)^2 = \prod_{i < j} ((1 - \alpha_i) - (1 - \alpha_j))^2 \\ &= \Delta(\{1, \lambda_p, \dots, \lambda_p^{p-2}\}) \end{aligned}$$

Now, by Proposition 2.9, we have

$$\Delta(\{1, \zeta_p, \dots, \zeta_p^{p-2}\}) = (-1)^{\frac{(p-1)(p-2)}{2}} N_{K/\mathbb{Q}}(\Phi'_p(\zeta_p))$$

Since  $p$  is odd  $(-1)^{\frac{(p-1)(p-2)}{2}} = (-1)^{\frac{(p-1)}{2}}$ . Next, we see that

$$\Phi'_p(x) = \frac{px^{p-1}(x-1) - (x^p-1)}{(x-1)^2}$$

therefore

$$\Phi'_p(\zeta_p) = -\frac{p\zeta_p^{p-1}}{\lambda_p}.$$

Lastly, note that  $N_{K/\mathbb{Q}}(\zeta_p) = 1$ , since this is the constant term in its minimal polynomial. Similarly, we see  $N_{K/\mathbb{Q}}(\lambda_p) = p$ . Putting this all together, we get

$$N_{K/\mathbb{Q}}(\Phi'_p(\zeta_p)) = \frac{N_{K/\mathbb{Q}}(p)N_{K/\mathbb{Q}}(\zeta_p)^{p-1}}{N_{K/\mathbb{Q}}(-\lambda_p)} = (-1)^{p-1}p^{p-2} = p^{p-2}$$

□

**Theorem 3.4.** *Let  $\zeta_p$  be a  $p$ -th root of unity for  $p$  an odd prime, let  $\lambda_p = 1 - \zeta_p$  and  $K = \mathbb{Q}(\zeta_p)$ . Then  $\mathcal{O}_K = \mathbb{Z}[\zeta_p] = \mathbb{Z}[\lambda_p]$ .*

*Proof.* We need to prove is that  $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$ . The inclusion  $\mathbb{Z}[\zeta_p] \subseteq \mathcal{O}_K$  is obvious. Let now  $x \in \mathcal{O}_K$ . By Lemma 2.13 and Proposition 3.3, there is  $k \in \mathbb{N}$  such that  $p^k x \in \mathbb{Z}[\zeta_p]$ . We conclude by Lemma 2.14. □

**Lemma 3.5.** *Let  $\alpha$  be an algebraic integer all of whose conjugates have absolute value one. Then  $\alpha$  is a root of unity.*

*Proof.* Lemma 1.6 of [Was82]. □

**Lemma 3.6.** *Let  $p$  be a prime,  $K = \mathbb{Q}(\zeta_p)$   $\alpha \in K$  such that there exists  $n \in \mathbb{N}$  such that  $\alpha^n = 1$ , then  $\alpha = \pm \zeta_p^k$  for some  $k$ .*

*Proof.* If  $n$  is different to  $p$  then  $K$  contains a  $2pn$ -th root of unity. Therefore  $\mathbb{Q}(\zeta_{2pn}) \subset K$ , but this cannot happen as  $[K : \mathbb{Q}] = p-1$  and  $[\mathbb{Q}(\zeta_{2pn}) : \mathbb{Q}] = \varphi(2np)$ . □

**Lemma 3.7.** *Any unit  $u$  in  $\mathbb{Z}[\zeta_p]$  can be written in the form  $\beta \zeta_p^k$  with  $k$  an integer and  $\beta \in \mathbb{R}$ .*

*Proof.* See the Lean proof. □

**Lemma 3.8.** *Let  $p$  be an odd prime,  $\zeta_p$  a primitive  $p$ -th root of unity and let  $K = \mathbb{Q}(\zeta_p)$ . Then for any  $i, j \in 0, \dots, p-1$  with  $i \neq j$ , there exists a unit  $u \in \mathcal{O}_K^\times$  such that  $\zeta_p^i - \zeta_p^j = u * (\zeta_p - 1)$ .*

*Proof.* This is Ex 34 in chapter 2 of [Mar18]. □

**Lemma 3.9.** *Let  $R$  be a Dedekind domain,  $p$  a prime and  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$  ideals such that*

$$\mathfrak{a}\mathfrak{b} = \mathfrak{c}^p$$

*and suppose  $\mathfrak{a}, \mathfrak{b}$  are coprime. Then there exist ideals  $\mathfrak{e}, \mathfrak{d}$  such that*

$$\mathfrak{a} = \mathfrak{e}^p \quad \mathfrak{b} = \mathfrak{d}^p \quad \mathfrak{e}\mathfrak{d} = \mathfrak{c}$$

*Proof.* It follows from the unique decomposition of ideals in a Dedekind domain.  $\square$

#### 4. FERMAT'S LAST THEOREM FOR REGULAR PRIMES

**Lemma 4.1.** *Let  $p \geq 5$  be an prime number,  $\zeta_p$  a  $p$ -th root of unity and  $x, y \in \mathbb{Z}$  coprime.*

*For  $i \neq j$  with  $i, j \in 0, \dots, p-1$  we can write*

$$(\zeta_p^i - \zeta_p^j) = u(1 - \zeta_p)$$

*with  $u$  a unit in  $\mathbb{Z}[\zeta_p]$ . From this it follows that the ideals*

$$(x + y), (x + \zeta_p y), (x + \zeta_p^2 y), \dots, (x + \zeta_p^{p-1} y)$$

*are pairwise coprime.*

*Proof.* Lemma 3.8 gives that  $u$  is a unit. So all that needs to be proved is that the ideals are coprime. Assume not, then for some  $i \neq j$  we have some prime ideal  $\mathfrak{p}$  dividing by  $(x + y\zeta_p^i)$  and  $(x + y\zeta_p^j)$ . It must then also divide their sum and their difference, so we must have  $\mathfrak{p} | (1 - \zeta_p)$  or  $\mathfrak{p} | y$ . Similarly,  $\mathfrak{p}$  divides  $\zeta_p^j(x + y\zeta_p^i) - \zeta_p^i(x + y\zeta_p^j)$  so  $\mathfrak{p}$  divides  $x$  or  $(1 - \zeta_p)$ . We can't have  $\mathfrak{p}$  dividing  $x, y$  since they are coprime, therefore  $\mathfrak{p} | (1 - \zeta_p)$ . We know that since  $(1 - \zeta_p)$  has norm  $p$  it must be a prime ideal, so  $\mathfrak{p} = (1 - \zeta_p)$ . Now, note that  $x + y \equiv x + y\zeta_p^i \equiv 0 \pmod{\mathfrak{p}}$ . But since  $x, y \in \mathbb{Z}$  this means we would have  $x + y \equiv 0 \pmod{p}$ , which implies  $z^p \equiv 0 \pmod{p}$  which contradicts our assumptions.  $\square$

**Lemma 4.2.** *Let  $p$  be an prime number,  $\zeta_p$  a  $p$ -th root of unity and  $\alpha \in \mathbb{Z}[\zeta_p]$ . Then  $\alpha^p$  is congruent to an integer modulo  $p$ .*

*Proof.* Just use  $(x + y)^p \equiv x^p + y^p \pmod{p}$  and that  $\zeta_p$  is a  $p$ -th root of unity.  $\square$

**Lemma 4.3.** *Let  $p$  be an prime number,  $\zeta_p$  a  $p$ -th root of unity and  $\alpha \in \mathbb{Z}[\zeta_p]$  with  $\alpha = \sum_i a_i \zeta_p^i$ . Let us suppose that there is  $i$  such that  $a_i = 0$ . If  $n$  is an integer that divides  $\alpha$  in  $\mathbb{Z}[\zeta_p]$ , then  $n$  divides each  $a_i$ .*

*Proof.* Looking at  $\alpha = a_0 + a_1 \zeta_p + \dots + a_{p-1} \zeta_p^{p-1}$ , if one of the  $a_i$ 's is zero and  $\alpha/n \in \mathbb{Z}[\zeta_p]$ , then  $\alpha/n = \sum_i a_i/n \zeta_p^i$ . Now, as  $\alpha/n \in \mathbb{Z}[\zeta_p]$ , pick the basis of  $\mathbb{Z}[\zeta_p]$  which does not contain  $\zeta_p$  (which is possible as any subset of  $\{1, \zeta_p, \dots, \zeta_p^{p-1}\}$  with  $p-1$  elements forms a basis of  $\mathbb{Z}[\zeta_p]$ ). Then  $\alpha = \sum_i b_i \zeta_p^i$  where  $b_i \in \mathbb{Z}$ . Therefore comparing coefficients, we get the result.  $\square$

**Lemma 4.4.** *Let  $p \geq 3$  be an prime number,  $\zeta_p$  a  $p$ -th root of unity and  $\alpha \in \mathbb{Z}[\zeta_p]$ . Let  $x$  and  $y$  be integers such that  $x + y\zeta_p^i = u\alpha^p$  with  $u \in \mathbb{Z}[\zeta_p]^\times$  and  $\alpha \in \mathbb{Z}[\zeta_p]$ . Then there is an integer  $k$  such that*

$$x + y\zeta_p^i - \zeta_p^{2k}x - \zeta_p^{2k-i}y \equiv 0 \pmod{p}.$$

*Proof.* Using lemma 3.7 we have  $(x + y\zeta_p^i) = \beta\zeta_p^k\alpha^p$  which is congruent modulo  $p$  to  $\beta\zeta_p^k a \pmod{p}$  for some integer  $a$  by 4.2. Now, if we consider the complex conjugate we have  $\overline{(x + y\zeta_p^i)} \equiv \beta\zeta_p^{-k} a \pmod{p}$ . Looking at  $(x + y\zeta_p^i) - \zeta_p^{2k} \overline{(x + y\zeta_p^i)}$  then gives the result.  $\square$

**Lemma 4.5.** *Let  $p \geq 3$  be an prime number,  $\zeta_p$  a  $p$ -th root of unity and  $K = \mathbb{Q}(\zeta_p)$ . Assume that we have  $x, y, z \in \mathbb{Z}$  with  $\gcd(xyz, p) = 1$  and such that*

$$x^p + y^p = z^p.$$

*This is the so called "case I". To prove Fermat's last theorem, we may assume that:*

- $p \geq 5$ ;
- $x, y, z$  are pairwise coprime;
- $x \not\equiv y \pmod{p}$ .

*Proof.* The first part is easy.

Reducing modulo  $p$ , using Fermat's little theorem, you get that if  $x \equiv y \equiv -z \pmod{p}$  then  $3z \equiv 0 \pmod{p}$ . But since  $p > 3$  this means  $p|z$  but this contradicts  $\gcd(xyz, p) = 1$ . Now, if  $x \equiv y \pmod{p}$  then  $x \not\equiv -z \pmod{p}$  we can relabel  $y, z$  so that wlog  $x \not\equiv y \pmod{p}$  (this uses that  $p$  is odd).  $\square$

**Definition 4.6.** A prime number  $p$  is called regular if it does not divide the class number of  $\mathbb{Q}(\zeta_p)$ .

**Theorem 4.7.** *Let  $p$  be an odd regular prime. Then*

$$x^p + y^p = z^p$$

*has no solutions with  $x, y, z \in \mathbb{Z}$  and  $\gcd(xyz, p) = 1$ .*

*Proof.* For  $p = 3$  use the standard elementary arguments, so assume  $p \geq 5$ .

First thing is to note that if  $x^p + y^p = z^p$  then

$$z^p = (x + y)(x + \zeta_p y) \cdots (x + y\zeta_p^{p-1})$$

as ideals. Then since by 4.1 we know the ideals are coprime, then by lemma 3.9 we have that each  $(x + y\zeta_p^i) = \mathfrak{a}^p$ , for  $\mathfrak{a}$  some ideal. Note that,  $[\mathfrak{a}^p] = 1$  in the class group. Now, since  $p$  does not divide the size of the class group we have that  $[\mathfrak{a}] = 1$  in the class group, so its principal. So we have  $x + y\zeta_p^i = u_i \alpha_i^p$  with  $u_i$  a unit. So by 4.4 we have some  $k$  such that  $x + y\zeta_p - \zeta_p^{2k} x - \zeta_p^{2k-1} y \equiv 0 \pmod{p}$ . If  $1, \zeta_p, \zeta_p^{2k}, \zeta_p^{2k-1}$  are distinct, then 4.3 says that (since  $p \geq 5$ )  $p$  divides  $x, y$ , contrary to our assumption. So they cannot be distinct, but checking each case leads to a contradiction, therefore there cannot be any such solutions.  $\square$

**Theorem 4.8.** *Let  $p$  be a regular prime and let  $u \in \mathbb{Z}[\zeta_p]^\times$ . If  $u \equiv a \pmod{p}$  for some  $a \in \mathbb{Z}$ , then there exists  $v \in \mathbb{Z}[\zeta_p]^\times$  such that  $u = v^p$ .*

**Theorem 4.9.** *Let  $h, h^+$  denote the class number of  $\mathbb{Q}(\zeta_p)$  and  $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$  respectively. If  $p \nmid h$  then  $p \nmid h^+$ .*

**Theorem 4.10.** *Let  $p$  be an odd regular prime. Then*

$$x^p + y^p = z^p$$

*has no solutions with  $x, y, z \in \mathbb{Z}$  and  $p|xyz$ .*

**Theorem 4.11.** *Let  $p$  be an odd regular prime. Then*

$$x^p + y^p = z^p$$

*has no solutions with  $x, y, z \in \mathbb{Z}$  and  $xyz \neq 0$ .*

#### REFERENCES

- [Mar18] Daniel A. Marcus. *Number fields*. Universitext. Springer, Cham, 2018. Second edition of [MR0457396], With a foreword by Barry Mazur.
- [Sam70] Pierre Samuel. *Algebraic theory of numbers*. Translated from the French by Allan J. Silberger. Houghton Mifflin Co., Boston, Mass., 1970.
- [Was82] Lawrence C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.