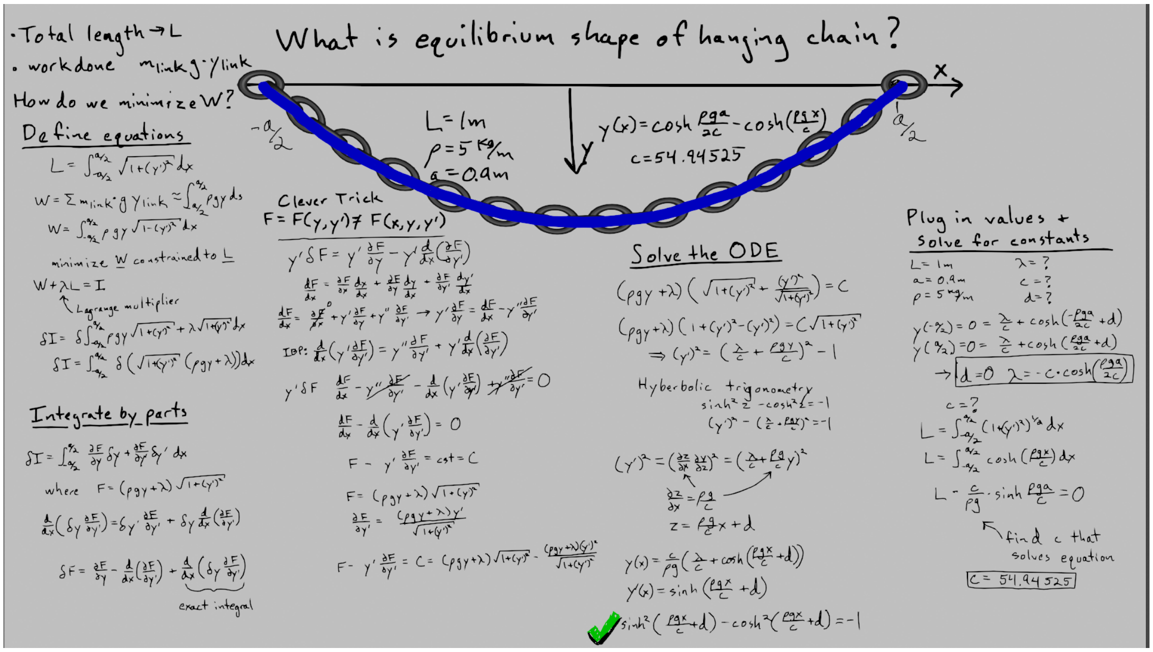
HW_03 - Nonlinear solutions



Creating a solution for the hanging chain, we reached a point where the constants required a nonlinear solution to an algebraic equation,

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
plt.style.use('fivethirtyeight')
```

1. $y(x) = \cosh\frac{\rho g a}{2c} - \cosh\frac{\rho g x}{c}$
2. $L = \int_{-a/2}^{a/2} \cosh\frac{\rho g x}{c} dx \rightarrow L = \frac{c}{\rho g}\sinh\frac{\rho g a}{c}$

The second equation does not have an "analytical" solution. Where "analytical" refers to an equation with seperable input/output. What you need is a "numerical" solution to equation 2:

what c will satisfy this equation?

$$f(c) = L - \frac{c}{\rho g}\sinh\frac{\rho g a}{c} = 0$$

These problems often come up when engineering systems have large displacements or large rotations that cannot be ignored. One way to approach this problem is to *guess* the solution. You could try:

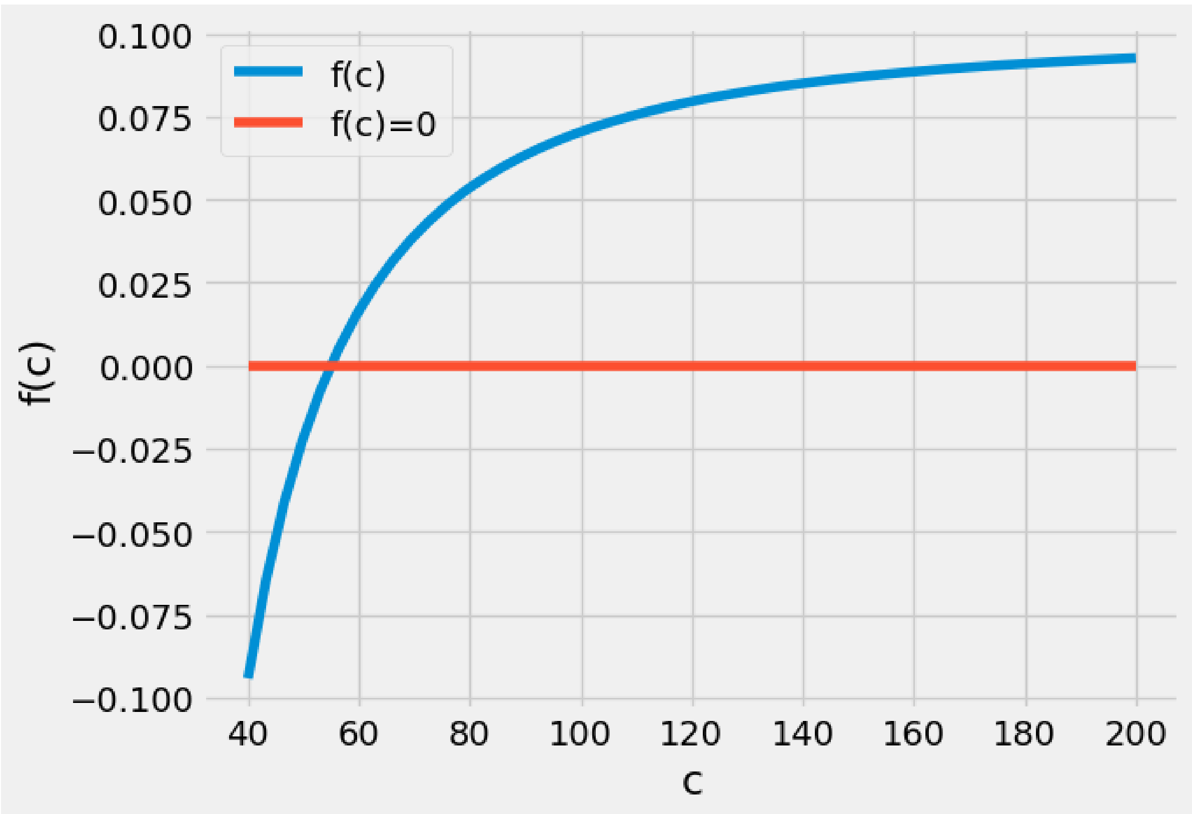
c=	f(c)
c=0	$-\infty$

c=	f(c)
c=50	-0.022
c=100	0.07
c=200	0.09

If you happen to guess numbers that change the sign of $f(c)$, then you know one interval where $f(c_{solution}) = 0$ must have been true. I find it helps to plot the function to see where the solution may exist

In [ ]:
```
g = 9.81
L = 1
a = 0.9
rho = 5
F = lambda c: L-c/rho/g*np.sinh(rho*g*a/c)
c = np.linspace(40,200)
plt.plot(c,F(c), label='f(c)')
plt.plot(c,np.zeros(c.shape), label='f(c)=0')
plt.legend()
plt.xlabel('c')
plt.ylabel('f(c)')
```

Out[ ]:
```
Text(0, 0.5, 'f(c)')
```



Numerical solution

We can use `fsolve` to automate the guess-and-check method. You need 2 things:

1. a function `f(c)` that returns the result $f(c) = L - \frac{c}{\rho g} \sinh \frac{\rho g a}{c} = 0$
2. an initial guess, `c_0`

Numerical solutions always require an initial guess for the solution and they will iterate until your function `f(c_sol)` ≈ 0.

> **Note**: `fsolve` has more advanced features than 'guess-and-check', but at its core it uses algorithms to reduce the number of guesses and checks.

Define `f(c)` with `lambda`

In Python, you can use the `lambda` function to create functions in one line. The other way to create a function is using `def`.

> **Note**: `def` is a much richer way to create functions in Python. We will use it later when we want more involved functions.

Here, you define the function `f(c)` with `lambda`:

In [ ]:
```
g = 9.81
L = 1
a = 0.7
rho = 5

f = lambda c: L-c/rho/g*np.sinh(rho*g*a/c)
```

In [ ]:
```
f(40)
```

Out[ ]:
```
0.21081614830202033
```

Solve `f(c_sol)=0` with `fsolve`

The numerical solver, `fsolve`, is part of the `scipy.optimize` library. Import the function with the `from` ... `import` -command.

In [ ]:
```
from scipy.optimize import fsolve
```

Now, you can solve for the value of `c_sol` that creates a solution to `f(c_sol)=0`. Use the function, `f` and an initial guess, `c0=40`.

In [ ]:
```
c0 = 40
c_sol = fsolve(f, c0)

print('c_sol = {} and f(c_sol) = {}'.format(c_sol[0], f(c_sol)))
```

```
c_sol = 22.67152264101508 and f(c_sol) = [1.56541446e-14]
```

Plug into catenary equation

Now, you have a solution for c that describes the hanging chain. Plug it into the original equation

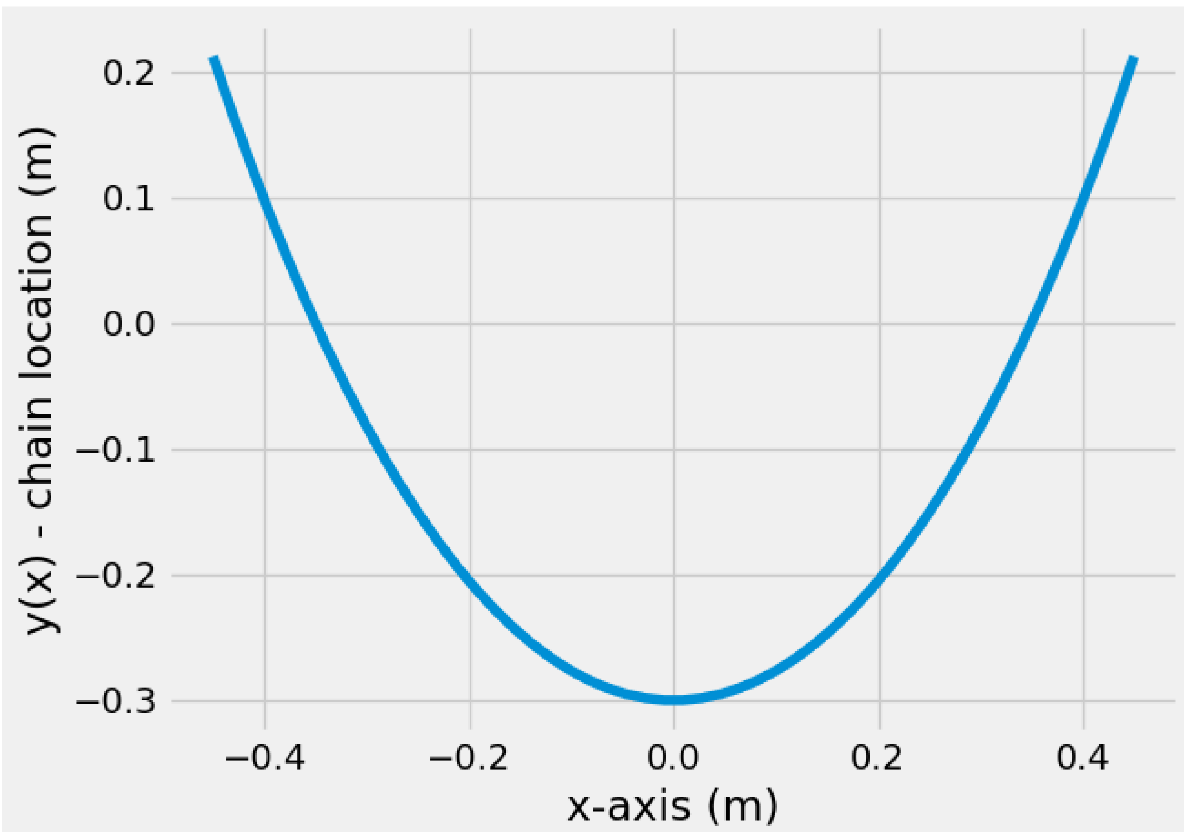
1. $y(x) = \cosh\frac{\rho g a}{2c} - \cosh\frac{\rho g x}{c}$

and plot the final shape.

In [ ]:
```
x = np.linspace(-0.9/2,0.9/2)
y = np.cosh(9.81*5*0.7/2/c_sol[0])-np.cosh(9.81*5*x/c_sol[0])
```

In [ ]:
```
plt.plot(x,-y)
plt.xlabel('x-axis (m)')
plt.ylabel('y(x) - chain location (m)')
```

Out[ ]:
```
Text(0, 0.5, 'y(x) - chain location (m)')
```



Problem 1

Plot the solution for two hanging chains, the same as we did above:

$g = 9.81\ m/s/s\ L = 1\ m\ rho = 5\ kg/m$

1. $a = 0.9\ m$

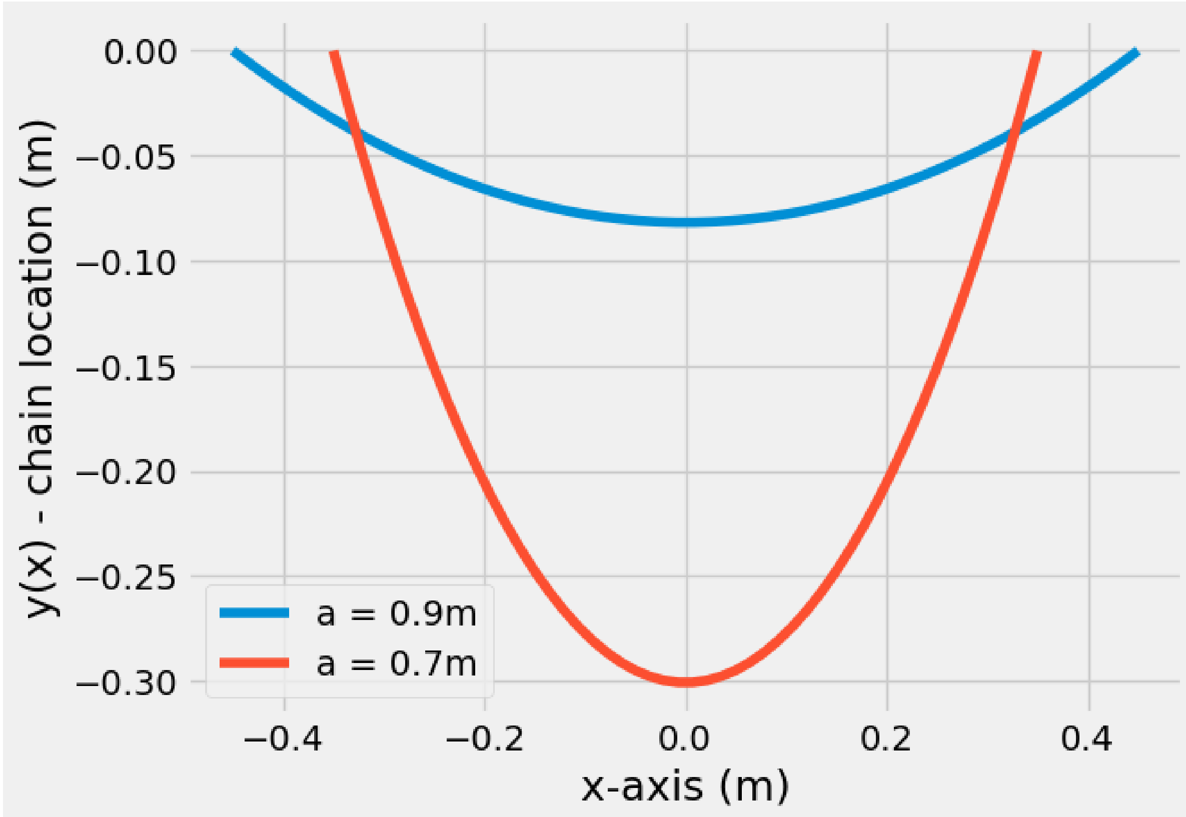
2. $a = 0.7\ m$

In [ ]:
```
g = 9.81
L = 1
a_1 = 0.9
rho = 5

f_1 = lambda c: L-c/rho/g*np.sinh(rho*g*a_1/c)

a_2 = 0.7

f_2 = lambda c: L-c/rho/g*np.sinh(rho*g*a_2/c)

c0 = 40
c_sol_f_1 = fsolve(f_1, c0)
c_sol_f_2 = fsolve(f_2, c0)

print('c_sol_f_1 = {} and f(c_sol_f_1) = {}'.format(c_sol_f_1[0], f_1(c_sol_f_1)))
print('c_sol_f_2 = {} and f(c_sol_f_2) = {}'.format(c_sol_f_2[0], f_2(c_sol_f_2)))
```

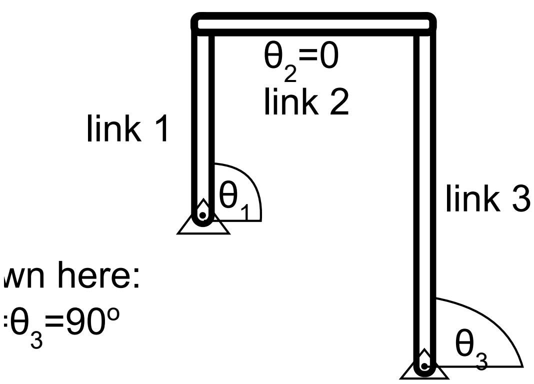
```
c_sol_f_1 = 54.94525819265913 and f(c_sol_f_1) = [-4.88498131e-15]
c_sol_f_2 = 22.67152264101508 and f(c_sol_f_2) = [1.56541446e-14]
```

In [ ]:
```
x_1 = np.linspace(-a_1/2,a_1/2)
y_1 = np.cosh(9.81*5*a_1/2/c_sol_f_1[0])-np.cosh(9.81*5*x_1/c_sol_f_1[0])
plt.plot(x_1,-y_1, label='a = 0.9m')
plt.xlabel('x-axis (m)')
plt.ylabel('y(x) - chain location (m)')

x_2 = np.linspace(-a_2/2,a_2/2)
y_2 = np.cosh(9.81*5*a_2/2/c_sol_f_2[0])-np.cosh(9.81*5*x_2/c_sol_f_2[0])
plt.plot(x_2,-y_2, label='a = 0.7m')
plt.xlabel('x-axis (m)')
plt.ylabel('y(x) - chain location (m)')

plt.legend()
```

Out[ ]:
```
<matplotlib.legend.Legend at 0x2d4bcf938b0>
```



Problem 2



In the four-bar linkage show above there are 3 bodies moving in 2D (9 DOF) and 4 pins (8 constraints). The linkage configuration is constrained by the two nonlinear equations

1. $l_1 \sin\theta_1 + l_2 \sin\theta_2 - l_3 \sin\theta_3 - d_y = 0$
2. $l_1 \cos\theta_1 + l_2 \cos\theta_2 - l_3 \cos\theta_3 - d_x = 0$

If you have one of the angles,θ_1, you can use equations 1 and 2 to solve for the other two angles, $\theta_2\ and \theta_3$ using `fsolve` only now the input is a vector with two values and the output is a vector with two values.

$$\bar{f}\,(\bar{x}) = \left[\begin{array}{c} f_1(\theta_2,\ \theta_3) \\ f_2(\theta_2,\ \theta_3) \end{array}\right] = \left[\begin{array}{c} l_1 \sin\theta_1 + l_2 \sin\theta_2 - l_3 \sin\theta_3 - d_y \\ l_1 \cos\theta_1 + l_2 \cos\theta_2 - l_3 \cos\theta_3 - d_x \end{array}\right]$$

The linkage system has the following properties:

- link 1: $l_1 = 0.5\ m$
- link 2: $l_2 = 1\ m$
- link 3: $l_3 = 1\ m$

when $\theta_1 = 90^o$, $\theta_2 = 0^o$, and $\theta_3 = 90^o$. So the two grounded pins have a fixed relative position, $r_{3/1} = d_x\hat{i} + d_y\hat{j} = 1\hat{i} - 0.5\hat{j}$.

Below, the definition of `Fbar` is defined for $\bar{f}\left(\bar{x}\right)$ and the function is satisfied for $\theta_1 = \theta_3 = 90^o$ and $\theta_2 = 0^o$. Then, the links are plotted with `rx` and `ry`, where

- $rx = \begin{bmatrix} 0 \\ l_1\cos(\theta_1) \\ l_1\cos(\theta_1) + l_2\cos(\theta_2) \\ l_1\cos(\theta_1) + l_2\cos(\theta_2) - l_3\cos(\theta_3) \end{bmatrix}$
- $ry = \begin{bmatrix} 0 \\ l_1\sin(\theta_1) \\ l_1\sin(\theta_1) + l_2\sin(\theta_2) \\ l_1\sin(\theta_1) + l_2\sin(\theta_2) - l_3\sin(\theta_3) \end{bmatrix}$

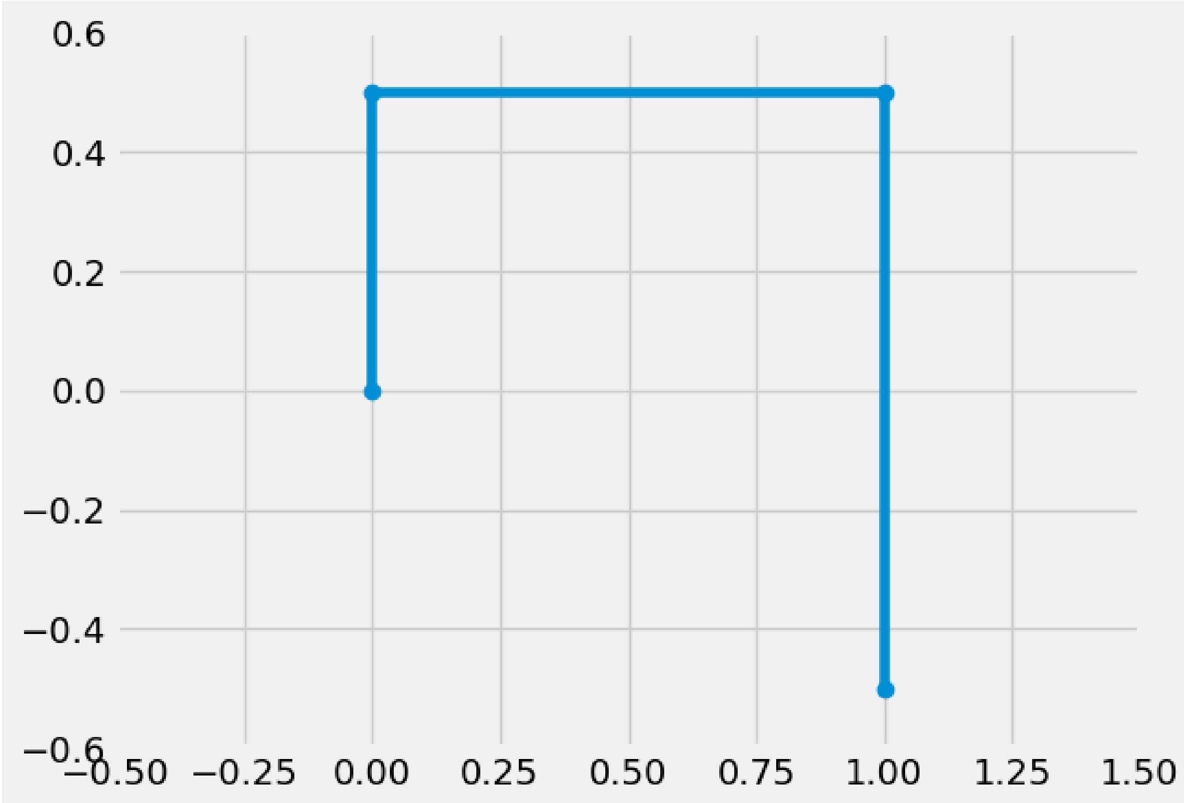
In [ ]:
```
l1 = 0.5
l2 = 1
l3 = 1
a1 = np.pi/2
dy = -0.5
dx = 1
Fbar = lambda x: np.array([l1*np.sin(a1)+l2*np.sin(x[0])-l3*np.sin(x[1])-dy,
                           l1*np.cos(a1)+l2*np.cos(x[0])-l3*np.cos(x[1])-dx])
```

In [ ]:
```
x90 = np.array([0,np.pi/2])
Fbar(x90)
```

Out[ ]:
```
array([ 0.00000000e+00, -1.11022302e-16])
```

In [ ]:
```
rx = np.array([0,
               l1*np.cos(a1),
               l1*np.cos(a1)+l2*np.cos(x90[0]),
               l1*np.cos(a1)+l2*np.cos(x90[0])-l3*np.cos(x90[1])])
ry = np.array([0,
               l1*np.sin(a1),
               l1*np.sin(a1)+l2*np.sin(x90[0]),
               l1*np.sin(a1)+l2*np.sin(x90[0])-l3*np.sin(x90[1])])

plt.plot(rx,ry,'o-')
plt.axis([-0.5, 1.5, -0.6, 0.6])
```

Out[ ]:
```
(-0.5, 1.5, -0.6, 0.6)
```



Your goal:

Change the angle to $\theta_1 = 45^o,\ 135^o,\ and\ 180^o$. Plot the three configurations like above.
Use `fsolve` to find $\theta_2\ and\ \theta_3$.

In [ ]:
```
# your work here

l1 = 0.5
l2 = 1
l3 = 1
dy = -0.5
dx = 1

a_45 = np.pi/4
a_135 = 3*np.pi/4
a_180 = np.pi

Fbar_a_45 = lambda x: np.array([l1*np.sin(a_45)+l2*np.sin(x[0])-l3*np.sin(x[1])-dy,
                                l1*np.cos(a_45)+l2*np.cos(x[0])-l3*np.cos(x[1])-dx])
Fbar_a_135 = lambda x: np.array([l1*np.sin(a_135)+l2*np.sin(x[0])-l3*np.sin(x[1])-d
                                l1*np.cos(a_135)+l2*np.cos(x[0])-l3*np.cos(x[1])-dx])
Fbar_a_180 = lambda x: np.array([l1*np.sin(a_180)+l2*np.sin(x[0])-l3*np.sin(x[1])-d
                                l1*np.cos(a_180)+l2*np.cos(x[0])-l3*np.cos(x[1])-dx])


x_a_45 = np.array([0, a_45])
x_a_135 = np.array([0, a_135])
x_a_180 = np.array([0, a_180])

thetas_45 = fsolve(Fbar_a_45, x_a_45)
```

```
thetas_135 = fsolve(Fbar_a_135, x_a_135)
thetas_180 = fsolve(Fbar_a_180, x_a_180)
```

In [ ]:
```
thetas23 = [thetas_45, thetas_135, thetas_180]
thetas1 = [a_45, a_135, a_180]

rx_array = []
ry_array = []

for index in range(len(thetas1)):
    rx_array.append(np.array([0,
                l1*np.cos(thetas1[index]),
                l1*np.cos(thetas1[index])+l2*np.cos(thetas23[index][0]),
                l1*np.cos(thetas1[index])+l2*np.cos(thetas23[index][0])-l3*np.cos(t
    ry_array.append(np.array([0,
                l1*np.sin(thetas1[index]),
                l1*np.sin(thetas1[index])+l2*np.sin(thetas23[index][0]),
                l1*np.sin(thetas1[index])+l2*np.sin(thetas23[index][0])-l3*np.sin(t

    plt.plot(rx_array[index],ry_array[index],'o-', label=r'$\theta_1 = ${}'.format(

plt.legend(bbox_to_anchor = [1.3, .3])
```

Out[ ]: <matplotlib.legend.Legend at 0x2d4c0173b50>