Hearing the Local Orientability of Orbifolds

Sean Richardson '20, Liz Stanhope

Department of Mathematical Sciences, Lewis & Clark College

Listening to Drums

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- However, if you hear a drum being played in another room, what can you say about it?
- Can you hear the shape of a drum?
- Can you hear the shape of mathematical objects called "orbifolds"?

What is an Orbifold?

Reflectional Symmetry

- We classify symmetries with the group of isometries that leave a pattern unchanged
- Figure 1 has group $\Gamma = \{e, r\}$
- One line in space is left unmoved by all $g \in \Gamma$.
- We can fold the space into $\mathbb{R}^2/\Gamma$

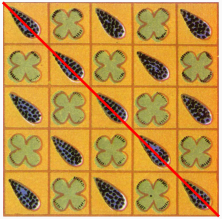

Figure 1: Reflectional Symmetry

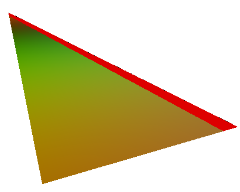

Figure 2: Folding the Pattern

Rotational Symmetry

- We classify symmetries through the group of isometries that leave a pattern unchanged
- Figure 3 has group $\Gamma = \{0^\circ, 90^\circ, 180^\circ, 270^\circ\}$
- One point is left unmoved by all $g \in \Gamma$
- We can twist the space into $\mathbb{R}^n/\Gamma$ creating a party hat

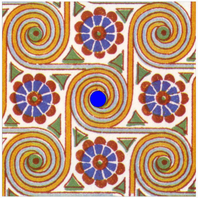

Figure 3: Rotational Symmetry

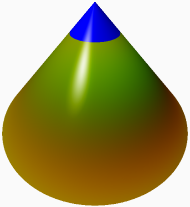

Figure 4: Folding the Pattern

Orbit-folding

- We fold a pattern such that no section occurs twice.
- Result has mirror edge and cone point strata.

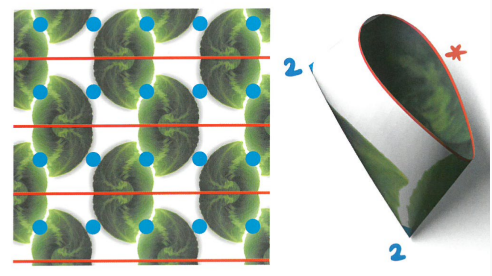

Figure 5: Folded Pattern from *The Symmetries of Things*

Orbifolds

- An *orbifold* is a generalization of a manifold.
- While a manifold has local structure $\mathbb{R}^n$, an orbifold is allowed local structure $\mathbb{R}^n/\Gamma$

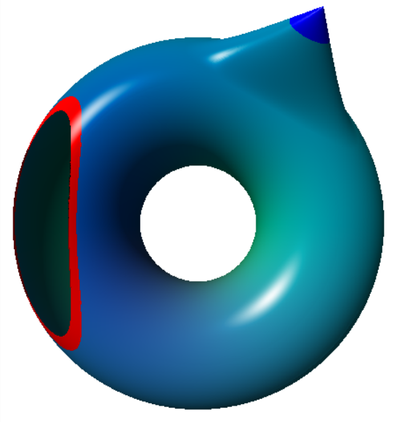

Figure 6: Orbifold Representation

What is the “Sound” of an Orbifold?

Sound Comes from Vibration

- Like physical objects, the “sound” of an orbifold is defined by how quickly it vibrates.

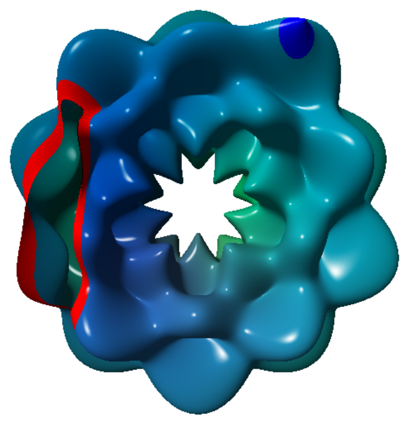

Figure 7: Orbifold Vibration

The Wave Equation

- Let $u(t, \mathbf{x})$ be the amplitude of the wave at time t and position $\mathbf{x}$
- The orbifold will vibrate according to the wave equation,

$$\Delta\psi = \frac{\partial^2 u}{\partial t^2}$$

- Assuming $u(t, \mathbf{x}) = A(t)\psi(\mathbf{x})$ for standing wave solutions results in

$$\Delta\psi(\mathbf{x}) = -\lambda\psi(\mathbf{x})$$

Laplace Spectra

$$\Delta\psi(\mathbf{x}) = -\lambda\psi(\mathbf{x})$$

- Eigenvalue solutions form a discrete spectrum $\lambda_1, \lambda_2, \lambda_3, \ldots$ such that $\sqrt{\lambda_i}$ is a fundamental frequency.
- This spectrum $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called the Laplace Spectra.
- The Laplace Spectra defines "sound" on an orbifold.

Research Question

- "Can you hear the shape of an orbifold?"
- **What properties of an unknown orbifold $\mathcal{O}$ are determined by it's known Laplace Spectra?**
  - Can you hear the types/amount of strata on an orbifold?
  - Can an orbifold and a manifold have the same Laplace Spectra?

Heat Expansion

Intuition of Heat Expansion

Consider:

- Heat up a point $\mathbf{x}$ on an orbifold $\mathcal{O}$
- Then, allow the heat to disperse
- The point cools down with time

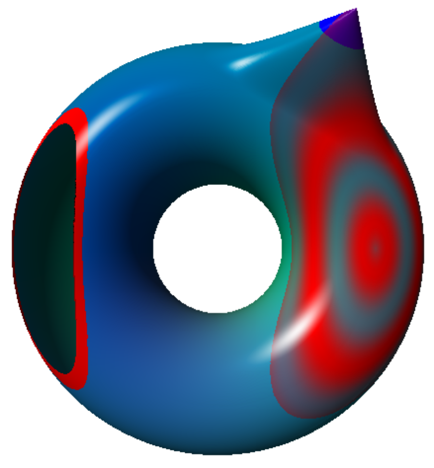

Figure 8: Heat Dispersing from $\mathbf{x}$

Intuition of Heat Expansion

We study a specific function:

- How does a point cool with time on average for every point in the orbifold $\mathcal{O}$?
- We approximate this function for small values of time.
- This function is related to the Laplace Spectrum.

The Heat Equation

- Let $u(t, \mathbf{x})$ be the heat at time t and position $\mathbf{x}$ on an orbifold $\mathcal{O}$
- Heat will spread through $\mathcal{O}$ according to the heat equation
$$\Delta u = \frac{\partial u}{\partial t}$$
- The solution to the heat equation has the form,
$$u(t, \mathbf{x}) = \int_M K(t, \mathbf{x}, \mathbf{y}) \mu_0(y) dvolM_y$$
- Consider the function $K(t, \mathbf{x}, \mathbf{y})$, the "heat kernel"

Heat Kernel

- If heat starts at $\mathbf{p}$, there is $K(t, \mathbf{p}, \mathbf{q})$ heat at $\mathbf{q}$ at time t.
- Taking the trace of K relates to the elements of the Laplace Spectra.

$$\mathrm{Tr}(K) = \int_M K(t, \mathbf{x}, \mathbf{x}) dvolM = \sum_{j=0}^{\infty} e^{-\lambda_j t}$$

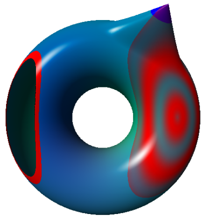

Figure 9: Heat Dispersing from $\mathbf{p}$

Asymptotic Expansion of Heat Kernel

- The Heat trace has a useful approximation for small values of time.

$$\mathrm{Tr}(K) \overset{t\to 0}{\sim} \frac{a}{t} + \frac{b}{\sqrt{t}} + c + d\sqrt{t} + et + \dots$$

- Because we now know $\mathrm{Tr}(K) = \sum_j e^{-\lambda_j t}$,

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} \overset{t\to 0}{\sim} \frac{a}{t} + \frac{b}{\sqrt{t}} + \dots$$

- A different value of any coefficient $a, b, c, \dots$ implies a different Laplace Spectra $\lambda_1, \lambda_2, \lambda_3, \dots$

Asymptotic Expansion of Heat Kernel

Meaning of $\sum_{j=0}^{\infty} e^{-\lambda_j t} \overset{t\to 0}{\sim} \frac{a}{t} + \frac{b}{\sqrt{t}} + \ldots$ graphically:

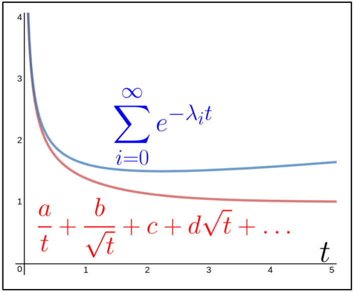

Figure 10: Asymptotic Expansion Approximation

Heat Expansion Coefficients

We find the coefficients of the heat expansion through the following expression:

$$(4\pi t)^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k(\mathcal{O}) t^k$$

$$+ \sum_{N \in S(\mathcal{O})} \frac{(4\pi t)^{-\dim(N)/2}}{|\operatorname{Iso}(N)|} \sum_{k=0}^{\infty} t^k \int_N \sum_{\gamma \in \operatorname{Iso}^{\max}(\tilde{N})} b_k(\gamma, x) dvol_N$$

Notes:

- The strata in the orbifold affect the expansion
- Even strata and odd strata behave differently

Result

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- We say $\mathcal{O}$ is *locally non-orientable* if $\mathcal{O}$ contains some non-orientable local structure, otherwise $\mathcal{O}$ is *orientable*.
- Local orientability can be interpreted through the existence of a short orientation reversing path.

We Can Hear Local Orientability

- We found that you can hear the local orientability of an orbifold $\mathcal{O}$.
- There exist no locally orientable orbifold and locally non-orientable orbifold with the same Laplace Spectra.

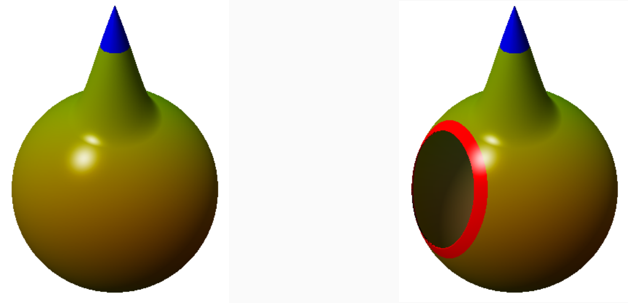

Figure 11: Two Orbifolds with Different Laplace Spectra

Proof Framework

- Let $\mathcal{O}_o$ be some locally orientable odd orbifold and let $\mathcal{O}_n$ be some locally non-orientable odd orbifold.
- $\mathcal{O}_o$ will have no even strata while $\mathcal{O}_n$ will have at least one even strata.

	$\dots$	t^{-1}	$t^{-1/2}$	t^0	$t^{1/2}$	t^1	$\dots$
$\mathcal{O}_o$	$\dots$	0	$\#$	0	$\#$	0	$\dots$
$\mathcal{O}_n$	$\dots$	d_{-1}	$\#$	d_0	$\#$	d_1	$\dots$

Table 1: Heat expansion coefficients

We are able to prove that at least one d_i is non-zero. So, $\mathcal{O}_o$ and $\mathcal{O}_n$ have distinct heat expansions, which implies different Laplace Spectra.

Questions?