THESIS OUTLINE

1. Category Theory

- ✓ Definition of Category
- ✓ Definition of Functor
- ✓ Category of Rings and/or Groups
- ○ Category of sets
- ✓ Category of Vector Spaces
- ✓ Definition of isomorphism
- ○ Category of abelian group sequence?
- ○ Category of pairs?

2. Algebra

2.1. **Ring Completion.**

- ✓ Ring completion definition
- ✓ Ring completion verification
- ✓ Ring completion Example
- ○ (for n-points): ring completion of $(\mathbb{N} \cup \{0\})^n$

2.2. **Packing together Modules.**

- ✓ Direct sum on modules definition
- • Verifications for direct sum
- • Extension to other categories
- • Example for vector spaces: $\mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m}$
- ✓ Tensor product on modules definition
- • Verifications for tensor product
- • Extension to other categories
- • Example for vector spaces: $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}$
- ○ Example/Result: $\mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta^2) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$
- ○ Idea of a *outer product* i.e. *external product* to make μ less jarring later.

3. Topology

3.1. **Basic Definitions.**

- ✓ Definition of topological space
- ✓ Definition of HM. i.e. cnts function

3.2. **Category of Topological Spaces.**

- ✓ Definition of the Category
- ✓ Formal comparison to metric space with functor
- ✓ Construction of S^2 as example

3.3. **Mappings in Topology.**

- ✓ Definition of quotient map and quotient topology
- ○ Example: I to S^1
- ✓ Definition of inclusion map and subspace topology
- ○ Example of $\mathbb{R}P^1$
- • Definition of $\mathbb{C}P^1$

- Theorem: $\mathbb{C}P^1 \cong S^2$

3.4. **Operations on Topological Spaces.**

- Wedge Sum
- Smash Product
- Cone Definition
- Example: D^n construction
- Suspension Definition
- Example: S^n construction
- Reduced Suspension
- Relevant sequence construction.
- Discuss the "union" sign.

3.5. **More Topology.**

✓ Definition of Compact
✓ Definition of Hausdorff
○ Definition of Normal
○ Theorem: Every Compact Hausdorff Space is Normal

3.6. **Homotopy Things.**

- Homotopic Functions
- Homotopic Spaces
- Contractible definition
- Examples: those necessary for ch. 6 costruction

3.7. **Appendix.**

○ Urysohn Lemma

4. Vector Bundles

4.1. **Basic Definition and Examples.**

∼ Motivation: vector fields
✓ Definition of V.B.
✓ Brief definition of fiber.
○ Brief definition of section.
- Example: Definition of trivial bundle of dimension n: ε^n.
- Example: Tangent Bundle over S^1 or S^2.
- Example: Mobius band

○ Canonical line bundle over $\mathbb{R}P^1$ gives mobius band

4.2. **Category Theory of Vector Bundles.**

✓ Definitions of homomorphism, isomorphism.
✓ Explanation of Category
- Definition of restriction
- Vereification that restrction is a vector bundle.

✓ Definition of $\oplus$ on vector bundles
✓ Properties of $\oplus$ on vector bundles (required for K-Theory def)
✓ Verifications for $\oplus$
✓ Definition of $\otimes$ on vector bundles.
✓ Propertues of $\oplus$ on vector bundles (required for K-Theory def)
✓ Verifications for $\otimes$
✓ Definition of pullback bundle
✓ Verifications for pullback bundles
✓ Important properties of pullback bundles
✓ Verification of properties of pullbacks

- Note restriction as example of pullback

4.3. **Necessary Results on Vector Bundles.**

- Theorem: If H is canonical line bundle on $\mathbb{C}P^1$, $(H \otimes H) \oplus 1 \cong H \oplus H$
- ○ Verification for above.
- ✓ Theorem: For every vector bundle E, there exists an E' such that $E \oplus E'$ is trivial.
- ○ Verification for above.
- ○ Example of above. Mobius band with itself.
- Pullback from homotopic map gives an isomorphism.
- Corrolarry: A contractible implies the bundle over A is trivial.

5. Definition of K-Theory

5.1. **The K-Theory Functor K.**

- ✓ Definition of K functor on topological spaces.
- ✓ Verification that above gives ring.
- ✓ Definition of K functor on homomorphisms.
- ✓ Verification that above gives homomorphism of rings.
- ✓ Verification that above is a functor.
- Simple Examples

5.2. **The Reduced K-Theory Functor $\widetilde{K}$.**

- Definition of $\widetilde{K}$ functor on topological spaces.
- Verification that above gives ring.
- Definition of $\widetilde{K}$ functor on homomorphisms.
- Verification that above gives homomorphism of rings.
- Verification that above is a functor.
- Simple Examples
- Theorem: $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$

5.3. **Notation.**

- Describing elements of K and $\widetilde{K}$
- pullback notation

6. K-Theory as a cohomomology theory

6.1. **Exact Sequences.**

- Short exact sequence definition
- State splitting Lemma
- ○ Verify splitting lemma
- Verify $\widetilde{K}$ and K relations claimed previously
- Discuss that (X, A) induces a short exact sequence
- Verify that the above is indeed a short exact sequence
- Extend above sequence to a long sequence
- Talk about homotopy equivalences and K-Theory at some point.

6.2. **Extending to a cohomology theory.**

- Discussion of what constitues a cohomology theory and why the reader should care?
- The construction

6.3. **Patterns of K-Theory.**

- Definition of the external product
- The external product $\widetilde{K}(S^{2k}) \otimes \widetilde{K}(X) \to \widetilde{K}(S^{2k} \wedge X)$ is an isomorphism.

- Bott Periodicity...
- $\widetilde{K}(S^n)$ is $\mathbb{Z}$ for n even and 0 for n odd.
- $K(S^{2k}) \otimes K(X) \to K(S^{2k} \times X)$ is an isomorphism.

7. Division Algebra Application

7.1. **Division Algebras and Paralizable Spheres.**

- Define division algebras
- reduce division algera problem to a K-Theory problem
- ○ Define paralizable spheres
- ○ reduce paralizable spheres to a K-Theory problem

7.2. **Even Case.**

7.3. **Odd Case.**

8. Overall Themes and Ideas

- Tell the reader right at the beginning the plan to get a ring out of a space and remind the reader of this goal all the way through.
- I may add a brief introductiony-ish thing that provides a brief summary (similar to what my slide presentation will be) of the direction that the book is going. It feels a bit harsh to push the reader into category theory and hope they survive until chapter 4 to finally see vector bundles... This way I can also talk of the motivation to what I address along the way. I can use the word vector bundle earlier to give motivation and do a better job of giving a story.