LEMMA 0.1. /*yikes, this lemma (lemma 1.2 in Hatcer) uses partition of unity... this is becoming a rabbit hole*/

Given a vector space V and a vector subspace, $V_0 \subset V$, the Gram-Schmidt orthogonalization process provides the orthogonal complement $V_0^{\perp}$ to the subspace V_0 in V. Further, it holds that $V_0 \oplus V_0^{\perp} = V$. An analogous result holds for vector bundles by applying the same process to each fiber.

LEMMA 0.2. Take a vector bundle $p : E \to X$ that has

/*assumes all V_i's are equal. Need to fix? Say it suffices to consider connected components*/

PROOF OF /*REF*/. The strategy of this proof is to construct a trivial vector space, later called $X \times \mathcal{V}$, that an isomorphic copy of the given vector bundle resides in. Then the result will follow by the above lemma.

Consider a vector bundle $p : E \to X$ where X is a compact Hausdorff topological space.. Each point $x \in X$ has a neighborhood U_x over which the bundle is trivial. By X compact Hausdorff, apply Urysohn's Lemma /*ref*/ on the closed sets $\{x\}$ and the complement $\overline{U_x}$. Urysohn's Lemma then promises a continuous function $\varphi_x : X \to [0,1]$ satisfying $\varphi_x^{-1}(\{0\}) \subset \overline{U_x}$ and $\varphi_x^{-1}(\{1\}) \subset \{1\}$. In other words, f evaluates to 0 outside of U_x and to 1 at x. Note that $\varphi_x^{-1}(0,1]$ contains X and is open by by φ_x continuous and the interval equipped with standard topology. Then $\varphi_x^{-1}(0,1]$ provides an open cover when allowing x to vary. By compactness there is a finite subcover; denote this subcover $\varphi_i^{-1}(0,1]$ and let the corresponding functions and neighborhoods be indexed φ_i and U_i.

Next, for each index define a function $g_i : E \to V$ as follows. Let $h_i : p^{-1}(U_i) \to U_i \times V$ be the trivialization as promised by the choice of U_i. Additionally, let $\pi_i : X \times V \to V$ be the projection from the trivial bundle to the corresponding vector component: $\pi_i : (x, v) \mapsto v$. Then, the function g_i is defined as follows.

$$g_i(e) = \begin{cases} \varphi_i(p(e)) \cdot (\pi_i \circ h_i(e)) \text{ if } p(e) \in U_i \\ 0 \text{ otherwise.} \end{cases}$$

Note g_i is continuous by g_i a composition of continuous functions and by φ_i is 0 outside of U_i. Importantly note that each g_i is a linear injection over the fibers of $\varphi_i^{-1}(0,1]$. Indeed, fix an $x_0 \in \varphi_i^{-1}(0,1] \subset U_i$ and take v_1, v_2 in the fiber $p^{-1}(x_0)$ such that $g_i(v_1) = g_i(v_2)$. That is,

$$\varphi_i(p(v_1)) \cdot (\pi_i \circ h_i(v_1)) = \varphi_i(p(v_2)) \cdot (\pi_i \circ h_i(v_2))$$

The fixed x gives $\varphi_i(p(v_1)) = \varphi_i(p(v_1)) = \varphi_i(x_0)$. This together with h_i an isomorphism and π_i an isomorphism over the fixed x_0 promises $v_1 = v_2$, confirming injectivity over the fibers. Linearity follows by π_i and h_i linear over the fibers.

Next, consider the vector space $\mathcal{V} = V \times V \times \cdots \times V$ with one copy of V for each of the indices i. Then, define the function $g : E \to \mathcal{V}$ given by $g : e \mapsto (g_1(e), g_2(e), \ldots, g_k(e))^T$. Note that g is a linear injection. Indeed, fix an $x_0 \in \varphi_i^{-1}(0,1] \subset U_i$ and take v_1, v_2 in the fiber $p^{-1}(x_0)$ such that $g(v_1) = g(v_2)$. By the collection $\varphi_i^{-1}(0,1]$ a cover, $x_0 \in \varphi_i^{-1}(0,1]$ for some i. But then,

$g_i(v_1) = g_i(v_2)$, which then provides the desired $v_1 = v_2$ confirming injectivity. Linearity follows by each individual g_i linear.

Finally consider the map $f : E \to X \times \mathcal{V}$ given by $f : e \mapsto (p(e), g(v))$. Now observe that the image of f is a subbundle of of $X \times \mathcal{V}$. The bundles takes the natural projection map of the larger trivial bundle and by linearity of g each fiber of the image has a vector space structure. It only remains to verify the local triviality condition. Indeed, for each $x_0 \in X$, the open cover promises $x_0 \in \varphi_i^{-1}(0,1]$ for some i. Then, consider the projection $X \times \mathcal{V}$ by mapping the vector component of (x, v) to the ith copy of V used to construct $\mathcal{V}$ and call the projection q. Then, a local trialization over the region is provided by $(x, v) \mapsto (x, q(v))$. With the verification that $\operatorname{Im} f$ indeed forms a vector bundle, and by injective implies bijective onto the image, lemma /*ref*/ applies and gives that the image is isomorphic to a subbundle of $X \times \mathcal{V}$. So, lemma /*ref*/ applies and promises a bundle E' such that $E \oplus E' = X \times \mathcal{V}$.

$\square$