It must be verified that the direct sum has a natural topology that indeed makes it a vector bundle.

Proof. Take vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ and recall that the direct sum on bundles as a set is given by the disjoint union of direct sums on fibers

$$E_1 \oplus E_2 = \bigcup_{x \in X} p_1^{-1}(x) \oplus p_2^{-1}(x).$$

This set is paired with with the projection $p : E_1 \oplus E_2 \to X$ given by $p : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto x$.

The topology on $E_1 \oplus E_2$ is defined in this paragraph. For each $x \in X$, the definition of vector bundle promises an open set U containing x over which both E_1 and E_2 are trivial. This provides trivializations $t_1 : p_1^{-1}(U) \to U \times V_1$ and $t_2 : p_2^{-1}(U) \to U \times V_2$ for vector spaces V_1 and V_2. Next, define the map $t_1 \oplus t_2 : p_1^{-1}(U) \oplus p_2^{-1}(U) \to U \times (V_1 \oplus V_2)$ as follows.

$$t_1 \oplus t_2 : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto t_1(p_1^{-1}(x)) \oplus t_2(p_2^{-1}(x))$$

Then, the topology on $p_1^{-1}(U) \oplus p_2^{-1}(U)$ is defined by requiring the map $t_1 \oplus t_2$ to be a homeomorphism. By letting x vary, this defines a topology over all of $E_1 \oplus E_2$. It must be verified, however, that this topology is well-defined.

Before the proof of well-defined, observe how this choice of topology gives that $E_1 \oplus E_2$ is a vector bundle. Firstly, this choice equips each fiber $p_1^{-1}(x) \oplus p_2^{-1}(x)$ with the typical topology of the direct sum of vector spaces. This ensures that the projection map $p : E_1 \oplus E_2 \to X$ is continuous. Next, the local triviality condition must be verified. Luckily the topology is built exactly so that $t_1 \oplus t_2$ is a trivialization. For any $x \in X$, the mapping $t_1 \oplus t_2$ defined on the appropriate U as described above satisfies all the conditions of a vector bundle homomorphism. Further, the defining condition that $t_1 \oplus t_2$ is a homeomorphism promises a continuous inverse and so $t_1 \oplus t_2$ is an isomorphism of vector bundles.

It only remains to show that the topology on $E_1 \oplus E_2$ is well-defined. In particular, it must be shown that the topology is independent of the choice of trivializations over a single open set U and that the open sets induce the same topology over their intersection. So, for $x \in X$ and corresponding $U \subset X$, consider two trivializations for each bundle: $t_1, t_1' : E_1 \mapsto U$ and $t_2, t_2' : E_1 \mapsto U$. Because each trivialization gives an isomorphism to the trivial bundle, the composition $t_1^{-1} \circ t_1' : p^{-1}(U) \to p^{-1}(U)$ is an isomorphism and similarly $t_2^{-1} \circ t_2' : p^{-1}(U) \to p^{-1}(U)$ is an isomorphism. Then composition $t_1' \circ t_1^{-1}$ is an isomorphism on $U \times V_1$ and similarly $t_2' \circ t_2^{-1}$ is an isomorphism on $U \times V_2$. It follows that the composition $(t_1' \oplus t_2') \circ (t_1 \oplus t_2)^{-1}$ is an isomorphism on $U \times (V_1 \otimes V_2)$, which implies that the choices $(t_1 \oplus t_2)$ and $(t_1' \oplus t_2')$ supply the same topology.

Finally, consider a separate set of open set $U' \subset X$. Then, taking the restrictions of the bundles $p^{-1}(U)$ and $p^{-1}(U')$ over the intersection $U \cap U'$ would only differ in the trivializations, which induce the same topology as shown in the previous paragraph. $\square$

In the above argument, the only part that appeals to the direct sum operation itself is the implicit assumption that the mapping $(v, w) \mapsto v \oplus w$ is continuous. This is also true for the tensor product, so a simple substitution of "$\otimes$" in place of "$\oplus$" in the above proof provides the needed verification for tensor product.

Proof of claim /*ref*/. Verifying each claim requires establishing an isomorphism φ over two bundles, say $p : E \to X$ and $q : F \to X$. The approach will be to establish a vector space isomorphism between the fibers, which gives necessary properties of vector bundle isomorphism except for continuity and continuity of inverse. To deal with the continuity conditions, the strategy is to show local continuity at every point as descried in /*ref*/. It then suffices to show that for every $x \in X$, there is an open neighborhood U such that the restricted function $\varphi : p^{-1}(U) \to q^{-1}(U)$ is continuous in both directions.

(i) For associativity of the direct product, consider vector bundles E_1, E_2, E_3 over a base space X with corresponding projection maps p_1, p_2, and p_3. An isomorphism $\varphi : (E_1 \oplus E_2) \oplus E_3 \to E_1 \oplus (E_2 \oplus E_3)$ must be constructed. Let φ be the linear bijective function defined on the fibers by

$$\varphi : (p_1^{-1}(x) \oplus p_2^{-1}(x)) \oplus p_3^{-1}(x) \mapsto p_1^{-1}(x) \oplus (p_2^{-1}(x) \oplus p_3^{-1}(x))$$

For the continuity conditions, fix a point $x \in X$. Then, choose an open set $U \subset X$ small enough such that the local triviality conditions are satisfied by both direct sum bundles. Then, noting the vector space isomorphism $(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3)$, continuity in both directions is given by the following composition of isomorphisms

$$\begin{aligned}(p_1^{-1}(U) \oplus p_2^{-1}(U)) \oplus p_3^{-1}(U) &\to U \times (V_1 \oplus V_2) \oplus V_3 \\ &\to U \times V_1 \oplus (V_2 \oplus V_3) \to p_1^{-1}(U) \oplus (p_2^{-1}(U) \oplus p_3^{-1}(U))\end{aligned}$$

The proof for commutativity follows in a near identical way. The difference being that an isomorphism $\varphi : E_1 \oplus E_2 \to E_2 \oplus E_1$ is considered with the mapping between fibers $\varphi : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto p_2^{-1}(x) \oplus p_1^{-1}(x)$ and the vector space isomorphism $V_1 \oplus V_2 \cong V_2 \oplus V_1$ is considered instead.

(ii) Verifying that ε^0 is the identity element under direct sum requires establishing an isomorphism $\varphi : E \oplus \varepsilon^0 \to E$. This follows in the same way as the previous claims, but uses the mapping of fibers $\varphi : p^{-1}(x) \oplus \{0\} \mapsto p^{-1}(x)$ and uses the vector space isomorphism $V \oplus \{0\} \cong V$.

(iii) The proofs for associativity and commutativity of the tensor product is given by a substitution of "$\otimes$" for "$\oplus$" in the corresponding direct sum proofs.

(iv) The proof that ε^1 acts as an identity element over the tensor product follows similarly to the identity proof over direct sum. The difference being that here an isomorphism $\varphi : E \otimes \varepsilon^1 \to E$ is established by the mapping of fibers $\varphi : p^{-1}(x) \otimes V^1 \mapsto p^{-1}(x)$ where V^1 represents a one dimensional vector space. This proof additionally uses the vector space isomorphism $V \oplus V^1 \cong V$.

(v) Finally, the proof for distributivity establishes a vector space isomorphism $\varphi : E_1 \otimes (E_2 \oplus E_3) \to (E_1 \otimes E_2) \oplus (E_1 \otimes E_3)$ given by the linear bijection on the fibers

$$\varphi : p_1^{-1}(x) \otimes (p_2^{-1}(x)) \oplus p_3^{-1}(x)) \mapsto p_1^{-1}(x) \otimes p_2^{-1}(x) \oplus p_1^{-1}(x) \otimes p_3^{-1}(x)$$

and later uses the isomorphism on vector spaces $V_1 \otimes (V_2 \oplus V_3) \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$.

$\square$