COMPLEX VARIABLES: IVA'S SUMMARY OF LIMITS AND CONTINUITY

The most basic concept in calculus is that of convergence of sequences.

Let $\{z_n\}$ denote a sequence of complex numbers; this basically means we are dealing with a list $z_1, z_2, z_3, \ldots$ although it need not always be the case that z_1 is the first term in this list.

We say that $\lim_{n\to\infty} z_n = z_*$ whenever the sequence of *real* numbers $\{|z_n - z_*|\}$ converges to 0:

$$\lim_{n\to\infty} z_n = z_* \longleftrightarrow \lim_{n\to\infty} |z_n - z_*| = 0.$$

In situations like this we also write $z_n \to z_*$.

Assuming the standard notation $z_n = x_n + iy_n$, it is clear that each sequence of complex numbers $\{z_n\}$ corresponds to two sequences of real numbers: $\{x_n\}$ and $\{y_n\}$. If $z_n \to z_*$ with $z_* = x_* + iy_*$, then one must also have

$$x_n \to x_* \ \text{ and } \ y_n \to y_*.$$

To understand where this fact is coming from one should observe that the following triangle-type inequalities hold:

$$|x_n - x_*| \leq |z_n - z_*|, \quad |y_n - y_*| \leq |z_n - z_*|;$$

convergence of $|z_n - z_*|$ to zero necessitates the convergences $|x_n - x_*| \to 0$ and $|y_n - y_*| \to 0$ by virtue of the so-called Squeeze Theorem. In fact, the Squeeze Theorem and yet another triangle inequality type fact, namely

$$|z_n - z_*| \leq |x_n - x_*| + |y_n - y_*|,$$

lead us to the realization that if $x_n \to x_*$ and $y_n \to y_*$ then also $|z_n - z_*| \to 0$ i.e $z_n \to z_*$. In other words, we have $z_n \to z_*$ when and only when $x_n \to x_*$ and $y_n \to y_*$.

From this standpoint it is not hard to convince oneself that algebraic properties of limits of sequences which apply in the context of real variables apply in the context of complex variables as well. For example, if $\{z_n\}$ and $\{w_n\}$ are convergent then so are $\{z_n \pm w_n\}$, $\{z_n \cdot w_n\}$ etc; the value of the respective limits is found through the application of the corresponding algebraic operations. This phenomenon is responsible for the fact that $z_n \to z_*$ guarantees $z_n^2 \to z_*^2$ etc.

From now on assume that $f(z)$ is a function defined on a domain D and that $\{z_n\}$ is some sequence of elements of D. Furthermore, let z_* be a point which is either in D or is on the boundary of D in the sense that one can approach z_* from within D:

$$z_n \to z_* \ \text{ for some sequence } \{z_n\} \text{ from } D \text{ with } z_n \neq z_*.$$

If for all sequences $\{z_n\}$ from D with $z_n \to z_*$ (and $z_n \neq z_*$) the sequences $\{f(z_n)\}$ converge to one and the same complex number L then we write $\lim\limits_{z \to z_*} f(z) = L$. When the point z_* is clear from context we shorten the notation to $f(z) \to L$.

Main example: Let $f(z) = \text{P.V}\, z^{\frac{1}{2}}$ and $z_n = e^{i(\pi - \frac{1}{n})}$; note that $z_n \to -1$ from the "upper side". We have

$$f(z_n) = e^{i(\frac{\pi}{2} - \frac{1}{2n})} = \cos\left(\frac{\pi}{2} - \frac{1}{2n}\right) + i \sin\left(\frac{\pi}{2} - \frac{1}{2n}\right).$$

Relying on the fact that $\cos(\theta)$ and $\sin(\theta)$ are continuous functions of the real variable θ, we conclude that

$$f(z_n) \to \cos(\tfrac{\pi}{2}) + i \sin(\tfrac{\pi}{2}) = i.$$

On the other hand if we consider the sequence $z_n = e^{-i(\pi - \frac{1}{n})} \to -1$, where the convergence is happening from the "under-side", the same type of reasoning leads us

$$f(z_n) \to \cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2}) = -i.$$

Thus, for $f(z) = \text{P.V}\, z^{\frac{1}{2}}$ we have that $\lim\limits_{z \to -1} f(z)$ does not exist. This is happening despite the fact that $f(z)$ is defined at -1, namely $f(-1) = i$.

Remark: If one is utilizing the point of view where $w = f(z)$ is seen as $u(x, y) + iv(x, y)$, the convergence $f(z) \to L$ is seen to be equivalent to convergences

$$\lim_{(x,y) \to (x_*, y_*)} u(x, y) = \text{Re}(L), \quad \lim_{(x,y) \to (x_*, y_*)} v(x, y) = \text{Im}(L).$$

Once again, the algebraic rules of limits carry through. However, as the function $\text{P.V} z^{\frac{1}{2}}$ shows one is not necessarily able to come with the limit inside of a function. This brings us to our big definition of the day:

Definition: A function $f(z)$ defined at a point z_* is said to be *continuous at* z_* whenever $\lim\limits_{z \to z_*} f(z)$ exists and satisfies

$$\lim_{z \to z_*} f(z) = f(z_*).$$

The phrase *continuous* is reserved for functions f which are continuous at each and every point of its domain.

An investigation of the properties of $\text{P.V}\, z^{\frac{1}{2}}$ in the vicinity of the negative portion of the real axis shows that this function is <u>not</u> continuous at any of the points along the negative half of the real axis. This also means that one is not to take the continuity of elementary functions of complex variable for granted!

In view of the algebraic properties of limits it is easy to see that algebraic combinations of continuous functions are continuous, wherever defined. By the definition of continuity we may now say the same about their compositions.