Chapter 6

Series solutions to PDEs

:series-solutions

We have now derived the "big three" partial differential equations: the wave equation, the heat equation, and Laplace's equation. In this part of the course we study these equations in the case where the spatial domain is bounded. As in the case of the one-dimensional wave equation, we are able to construct solutions using infinite series of eigenfunctions.

We focus in particular on situations where the spatial domain is very symmetric because for such domains one can understand the eigenfunctions rather explicitly. In less symmetric settings, there is generally no "formula" for the eigenfunctions and one must either rely on numerical methods or more abstract theory in order to draw conclusions.

We begin by studying the wave equation, focusing on the situations where the spatial domain is a rectangle, a disk, or a ball. We then study Laplace's equation on these same domains. The study of the heat equation in these settings we leave to the excursions in Chapter 8.

6.1 Waves on 2 and 3 dimensional domains

We now turn to the studying the initial boundary value problem for the wave equation in two and three dimensions. In this chapter we focus on the situation where the spatial domain Ω is a bounded region, and where we require that the Dirichlet boundary condition hold at Γ, the boundary of Ω.

23D-BVP Thus we seek functions $u(t, \mathbf{x})$, defined for $t \in [0, T]$ and $\mathbf{x} \in \Omega$, such that

$$\text{PDE} \qquad \frac{\partial^2 u}{\partial t^2} = \Delta u \qquad t \in [0, T], \quad \mathbf{x} \in \Omega \tag{6.1a}$$

23D-wave

$$\text{BC} \qquad u(t, \mathbf{x}) = 0 \qquad t \in [0, T], \quad \mathbf{x} \in \Gamma \tag{6.1b}$$

There are two problems of interest. The first is to find standing wave solutions to (6.1). The second is to solve the initial value problem where we also require

$$\text{IC} \qquad u(0, \mathbf{x}) = s(\mathbf{x}) \quad \frac{\partial u}{\partial t}(0, \mathbf{x}) = v(\mathbf{x}) \qquad \mathbf{x} \in \Omega. \tag{6.2}$$

23D-IC

for some pre-specified functions s and v.

Just as in the one-dimensional setting, we begin by looking for standing wave solutions to (6.1a) of the form $u(t, \mathbf{x}) = A(t)\psi(x)$. Inserting this in to (6.1a), we see that the functions A and ψ must satisfy

$$\frac{dA}{dt} = \lambda A \quad \text{and} \quad \Delta\psi = \lambda\psi$$

for some constant λ. Applying the Dirichlet boundary condition we furthermore see that we must have $\psi = 0$ along the boundary Γ. Thus we see that the spatial function ψ that describes the shape of standing wave solutions to the wave equation (with Dirichlet boundary conditions) must satisfy the eigenvalue-problem following eigenvalue problem:

$$\Delta\psi = \lambda\psi \qquad \text{on } \Omega \tag{6.3a}$$

$$\psi = 0 \qquad \text{on } \Gamma. \tag{6.3b}$$

The following exercise shows that there are only non-trivial solutions to (6.3) in the case that $\lambda < 0$.

Exercise 6.1.1. *Suppose that ψ satisfies* (6.3)*. Using the fact that $\Delta\psi = \operatorname{div}(\operatorname{grad}\psi)$ and the divergence theorem, show that*

$$\lambda \int_\Omega \psi^2 \, dV = \int_\Omega \psi \, \Delta\psi \, dV = -\int_\Omega \|\operatorname{grad}\psi\|^2 \, dV.$$

Explain why this implies that $\lambda \le 0$.

Suppose, then, that $\lambda = 0$. Explain why it must be that $\operatorname{grad}\psi = \mathbf{0}$ and thus that $\psi = 0$.

Finally, conclude that we must have $\lambda < 0$ in order to have a non-trivial solution to (6.3) *and that as a result we can set $\lambda = -\omega^2$ for some $\omega > 0$.*

If we make use of the inner product

$$\langle u, v\rangle = \int_\Omega u \, v \, dV \tag{6.4}$$ 23D-inner-product

we can also show that eigenfunctions with different eigenvalues are orthogonal.

al-eigenfunctions **Exercise 6.1.2.** *Suppose $\Delta\psi_1 = \lambda_1\psi_1$ and $\Delta\psi_2 = \lambda_2\psi_2$, and that both ψ_1, ψ_2 satisfy the Dirichlet boundary condition. Show that*

$$\lambda_1\langle\psi_1, \psi_2\rangle = \lambda_2\langle\psi_1, \psi_2\rangle.$$

Use this to conclude that if $\lambda_1 \neq \lambda_2$ then ψ_1 and ψ_2 are orthogonal.

In fact, just as is the case for Sturm-Liouville problems, the Laplace operator

Δ, when subjected to Dirichlet boundary conditions, is a negative, self-adjoint operator with an infinite list of eigenvalues that tend to negative infinity and with a corresponding list of eigenfunctions that form a complete orthogonal collection.

theorem:Hilbert-Schmidt **Theorem 6.1.** *(Hilbert-Schmidt Theorem) Suppose that Ω is a bounded domain in $\mathbb{R}^n$. Then there exists an infinite list of eigenvalues λ_k and eigenfunctions ψ_k satisfying* (6.3).

The eigenvalues satisfy $0 > \lambda_1 > \lambda_2 \ldots$ and are such that $\lambda_k \to -\infty$ as $k \to \infty$.

Furthermore, the corresponding eigenfunctions ψ_k form a complete, orthogonal collection. In particular, if u is some function in $L^2(\Omega)$ then the sum

$$\sum_k \alpha_k \psi_k, \qquad \text{where} \qquad \alpha_k = \frac{\langle u, \psi_k \rangle}{\|\psi_k\|^2}, \tag{6.5}$$ 23D-Fourier-series

converges in norm to u.

Unfortunately, the proof of completeness of eigenfunctions is beyond the scope of this course. Most books on functional analysis and/or Hilbert spaces include this theorem; you might search the name of the theorem, or search the term "spectral decomposition."

We now show that Theorem 6.1 implies that we may solve the initial value problem (6.1) – (6.2). For each eigenvalue λ_k we can find functions $A_k(t)$ and $B_k(t)$ such that

$$A_k'' = \lambda_k A_k, \quad A_k(0) = 1, \quad A_k'(0) = 0,$$
$$B_k'' = \lambda_k B_k, \quad B_k(0) = 0, \quad B_k'(0) = 1.$$

Exercise 6.1.3. *Since each λ_k is negative, we know that we may write $\lambda_k = -\omega_k^2$ for some positive ω_k. Use this to find formulas for A_k and B_k.*

Using the linearity of the wave equation, we see that the most general solution to (6.1) is

$$u(t,\mathbf{x}) = \sum_k \left(\alpha_k A_k(t)\psi_k(\mathbf{x}) + \beta_k B_k(t)\psi_k(\mathbf{x})\right).$$

Thus the initial condition (6.2) is satisfied if

$$s = \sum_k \alpha_k \psi_k \quad \text{and} \quad v = \sum_k \beta_k \psi_k. \tag{6.6}$$

23D-reduced-ic

Since Theorem 6.1 implies that we may choose α_k and β_k to satisfy (6.6), we see that the initial value problem can be solved using combinations of standing wave solutions.

We conclude this discussion by emphasizing that, just as in the one-dimensional setting, understanding the standing wave solutions to the wave equation allows us to understand all solutions, as any solution can be expressed as the sum of standing wave solutions.

In the following sections, we analyze the eigenvalue problem (6.3), and the corresponding initial value problem for the wave equation, in a number of special circumstances. Our primary goal in each section is to understand the shape of the standing wave solutions. Thus we focus on the Dirichlet eigenvalue problem (6.3).

6.2 Wave equation on rectangular domains

rectangular-waves

As a first example, let us consider the case where the domain is a rectangle. In particular, we take $\Omega = [0, L] \times [0, M]$ for some positive numbers L and M. Our goal is to find solutions to (6.3) with $\lambda = -\omega^2$. Using Cartesian

2D-Cartesian-eigenproblem coordinates, we obtain the following problem:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega^2\psi, \tag{6.7a}$$ Cartesian-eigeneq

$$\psi(0, y) = 0 \quad \psi(L, y) = 0 \tag{6.7b}$$ Cartesian-BVx

$$\psi(x, 0) = 0 \quad \psi(x, M) = 0. \tag{6.7c}$$ Cartesian-BVy

Our plan is to look for "product solutions," meaning functions ψ of the form $\psi(x, y) = X(x)\,Y(y)$ for some functions X and Y. This is motivated by our success in finding such solutions for the one-dimensional wave equation. (Hey! It worked once... why not try it again?)

Plugging $\psi = XY$ in to (6.7a) yields

$$X''(x)\,Y(y) + X(x)\,Y''(y) = \lambda X(x)\,Y(y).$$

We re-arrange this in to the form

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)}.$$

Since the left side is only a function of X, and the right side is only a function of Y, we conclude that each side is constant. (It might be helpful to review the discussion at the beginning of §**??**.)

This means that there is some constant μ such that

$$X'' = \mu X \qquad \text{and } Y'' = (\lambda - \mu)Y.$$

Notice also that the boundary conditions (6.7b) and (6.7c) imply that

$$X(0) = 0, \quad X(L) = 0, \quad Y(0) = 0, \quad Y(M) = 0.$$

In particular, we see that the functions X and Y each satisfy a Dirichlet eigenvalue problem. This is great – we know exactly how to find the solutions!

Let's begin with the equation for X, which is

$$X''(x) = \mu X(x) \qquad X(0) = 0 \quad X(L) = 0. \tag{6.8}$$

Cartesian-X-problem

We know from §1.7 that we only obtain solutions when $\mu < 0$, in which case we set $\mu = -\nu^2$. The general solution to the ODE in (6.8) is constructed from

$$\cos(\nu x) \qquad \text{and} \qquad \sin(\nu x).$$

The boundary condition $X(0) = 0$ rules out the cosine solution, while the boundary condition $X(L) = 0$ implies that ν must be an integer multiple of π/L. Thus we obtain a list of solutions

$$X_l(x) = \sin\left(\frac{\pi l}{L}x\right),$$

together with corresponding values

$$\mu_l = -\left(\frac{\pi l}{L}\right)^2.$$

We now turn to the equation for Y. Since we have a long list of possible values for μ, we in fact have a list of eigenvalue problems for Y – a separate problem for each value of l! We need functions Y satisfying

$$Y''(y) = (\lambda - \mu_l)Y(y), \qquad Y(0) = 0, \quad Y(M) = 0.$$

Using the same reasoning as we applied to the equation for X, we obtain a list of solutions

$$Y_m(y) = \sin\left(\frac{\pi m}{M}y\right)$$

for which we have

$$\lambda - \mu_l = -\left(\frac{\pi m}{M}\right)^2.$$

Combining the functions X_l and Y_m, we obtain a collection of functions

$$\psi_{lm}(x,y) = \sin\left(\frac{\pi l}{L}x\right)\sin\left(\frac{\pi m}{M}y\right).$$

that satisfy the eigenvalue problem (6.7) with eigenvalue

$$\lambda_{lm} = -\pi^2\left(\left(\frac{l}{L}\right)^2 + \left(\frac{m}{M}\right)^2\right).$$

Exercise 6.2.1.

1. *For each function $\psi_{lm}(x,y)$ find the functions $A_{lm}(t)$ and $B_{lm}(t)$ so that*

$$u_{lm}(t,x,y) = A_{lm}(t)\psi_{lm}(x,y) \quad \text{and} \quad v_{lm}(t,x,y) = B_{lm}(t)\psi_{lm}(x,y)$$

are standing wave solutions to the two-dimensional wave equation on the rectangle with Dirichlet boundary conditions, and are such that

$$u_{lm}(0,x,y) = \psi_{lm}(x,y) \quad \text{and} \quad \frac{\partial v_{lm}}{\partial t}(0,x,y) = \psi_{lm}(x,y).$$

2. *What is the frequency at which the standing wave solutions u_{lm} and v_{lm} oscillate? Is it possible that two different u_{lm} solutions oscillate at the same frequency?*

3. *What do the functions $\psi_{lm}(x,y)$ look like? Draw a contour plot for "generic" values of l and m. Then describe the corresponding standing wave solutions $u_{lm}(t,x,y)$.*

Using the linearity of the wave equation, we see that

$$u(t,x,y) = \sum_{l,m}\left(\alpha_{lm}u_{lm}(t,x,y) + \beta_{lm}v_{lm}(t,x,y)\right)$$

is also a solution for any constants α_{lm} and β_{lm}. We would like to choose the constants in order to ensure that the initial conditions

$$u(0,x,y) = s(x,y) \quad \text{and} \quad \frac{\partial u}{\partial t}(0,x,y) = v(x,y). \tag{6.9}$$

Cartesian-IC

These conditions would be satisfied if

$$s(x,y) = \sum_{l,m} \alpha_{lm}\psi_{lm}(x,y) \quad \text{and} \quad v(x,y) = \sum_{l,m} \beta_{lm}\psi_{lm}(x,y).$$

As the eigenfunctions ψ_{lm} form a complete orthogonal collection, we can compute the coefficients using (6.5).

Exercise 6.2.2.

1. *Verify by direct computation that the functions ψ_{lm} form an orthogonal collection.*
2. *Compute $\|\psi_{lm}\|^2$.*
3. *Suppose $s(x,y) = x(x-L)y(y-M)$ and $v(x,y) = 0$. Find the function $u(t,x,y)$ that satisfies the corresponding IBVP for the wave equation.*

6.3 Wave equation on the disk

We now consider the case where the spatial domain Ω is the unit disk. We make use of polar coordinates (r,θ) and describe Ω by

$$0 \le r \le 1 \quad \text{and} \quad \pi \le \theta \le \pi. \tag{6.10}$$

polar-region

Using (5.5), we see that that in polar coordinates the eigenvalue problem (6.3) becomes

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = \lambda\psi \tag{6.11}$$

polar-eigenvalue-problem

together with the boundary condition

$$\psi(1,\theta) = 0. \tag{6.12}$$

polar-Dirichlet

While (6.12) does indeed describe the Dirichlet boundary condition for the unit disk, the parametrization of the disk using (6.10) has artificially introduced "boundaries" at $r = 0$, $\theta = -\pi$, and $\theta = \pi$. Since $\theta = -\pi$ and $\theta = \pi$ represent the same location on the disk, we require periodic boundary conditions

$$\psi(r, -\pi) = \psi(r, \pi) \quad \text{and} \quad \frac{\partial \psi}{\partial \theta}(r, -\pi) = \frac{\partial \psi}{\partial \theta}(r, \pi). \tag{6.13}$$

polar-periodicBC

We furthermore require that $\psi(r, \theta)$ be "reasonable" at $r = 0$.

Just as in the previous section, we look for functions solutions that are in product form.

exercise:polar-reduced-ode

Exercise 6.3.1. *Suppose that $\psi(r, \theta) = R(r)\Theta(\theta)$. Show that* (6.11) *becomes*

$$\frac{r^2 R''(r) + rR'(r) - \lambda r^2 R(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}.$$

Conclude that each side must be equal to the same constant, which we call $-\mu$, and thus that

$$r^2 R''(r) + rR'(r) + (\mu - \lambda r^2)R(r) = 0, \tag{6.14}$$

first-Bessel

$$\Theta''(\theta) = -\mu\Theta(\theta). \tag{6.15}$$

Theta-ODE

Finally, explain why Θ must satisfy periodic boundary conditions, while R must satisfy $R(1) = 0$ and $R(0)$ is "reasonable."

Exercise 6.3.1 reduces the problem of finding standing wave solutions to finding solutions to the ordinary differential equations (6.14) and (6.15), subject to the appropriate boundary conditions.

The problem for Θ is a version of the periodic boundary value problem that we encountered in Chapter **??**.

Exercise 6.3.2. *Show that* (6.15) *has solutions satisfying periodic boundary conditions only when μ is one of the numbers*

$$\mu_n = -n^2$$

and that the corresponding solutions are

$$a_n \cos(n\theta) + b_n \sin(n\theta).$$

Suggestion: Argue by cases on μ. First consider what happens when $\mu = 0$. Then consider what happens when $\mu > 0$. Finally consider what happens when $\mu < 0$.

We now address the differential equation (6.14). In fact, we must find a solution for each different value of μ. Our strategy is to try to express (6.14) as a Sturm-Liouville problem. Using the fact that $\mu = -n^2$, we re-write the equation as

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2}{r^2}R = \lambda R. \tag{6.16}$$ n-Bessel

We see from Exercise 3.6.1 that in order for the left side to be a self-adjoint operator, we require a weight function w that satisfies

$$\frac{1}{r} = \frac{1}{w}\frac{d}{dr}\left[1\, w\right].$$

It is easy to see that $w = r$ is a function that satisfies this condition, and is also positive on the domain $0 < r < 1$. Thus we may write (6.14) in the form

$$\frac{d}{dr}\left[r\frac{dR}{dr}\right] - \frac{n^2}{r}R = \lambda r R, \tag{6.17}$$ SL-Bessel

which corresponds to the self-adjoint form (3.13) with $p = r$. The equation (6.17) also satisfies theconditions (3.14) on the domain $0 < r < 1$, in particular the positivity condition. Finally, we see that by imposing the boundary conditions

$$R(0) \text{ is finite} \quad \text{and} \quad R(1) = 0, \tag{6.18}$$ SL-Bessel-BC

then the condition (3.16) holds. Thus (6.18) is an admissible boundary condition for (6.17).

We may now conclude from the Theorem 3.13 that for each value of n there exists an infinite sequence of eigenvalues

$$0 > \lambda_{n1} > \lambda_{n2} > \dots$$

such that $\lambda_{nk} \to -\infty$ as $k \to \infty$ and and corresponding eigenfunctions $R_{nk}(r)$ comprising a complete orthogonal collection of functions. Here orthogonality is with respect to the inner product with weight $w = r$;

$$\langle R_{nk}, R_{nl} \rangle_w = \int_0^1 R_{nk}(r)\, R_{ml}(r)\, r\, dr = \begin{cases} 0 & \text{if } k \neq l, \\ \|R_{nk}\|_w^2 & \text{if } k = l. \end{cases} \tag{6.19}$$

Bessel-IP

Using the functions R_{nk}, we are able to construct solutions to the eigenvalue problem (6.11) satisfying the Dirichlet boundary condition (6.12), namely

$$\psi_{nk}^c(r,\theta) = R_{nk}(r)\cos(n\theta) \quad \text{and} \quad \psi_{nk}^s(r,\theta) = R_{nk}(r)\sin(n\theta) \tag{6.20}$$

polar-eigenfunctio

having eigenvalue negative λ_{nk}.

We know from the general discussion at the beginning of the chapter that the eigenfunctions (6.20) are orthogonal. In the following exercise, we see that this orthogonality comes from the orthogonality of the functions Θ and R.

Exercise 6.3.3.

1. *Consider two of the cosine-type solutions, ψ_{nk}^c and ψ_{ml}^c. Show that*

$$\langle \psi_{nk}^c, \psi_{ml}^c \rangle = \left(\int_{-\pi}^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta \right) \left(\int_0^1 R_{nk}(r)\, R_{ml}(r)\, r\, dr \right). \tag{6.21}$$

polar-ortho-coord

2. *Explain why the θ integral in* (6.21) *is zero unless $n = m$.*

3. *Suppose that $n = m$. Explain why in this case the second integral in* (6.21) *is zero unless $k = l$.*

4. *Why is it that we never need to compute*

$$\int_0^1 R_{nk}(r)\, R_{ml}(r)\, r\, dr$$

in the case that $n \neq m$*?*

5. *Explain why similar reasoning applies to* $\langle \psi^c_{nk}, \psi^s_{ml} \rangle$ *and* $\langle \psi^s_{nk}, \psi^s_{ml} \rangle$*.*

While the Sturm-Liouville theorem tells us that the functions $R_{nk}(r)$ exist, and are orthogonal, it does not tell us much else about these functions. In the remainder of this section, we learn more about these functions by constructing a power series expression for them.

Since we know that each eigenvalue λ is negative, we set $\lambda = -\omega^2$ for some $\omega > 0$. It is convenient to make the change of variables $x = \omega r$. By the chain rule we compute

$$\frac{dR}{dr} = \frac{dR}{dx}\,\frac{dx}{dr} = \omega \frac{dR}{dx} \qquad \text{and thus } \frac{d^2R}{dr^2} = \omega^2 \frac{d^2R}{dx^2}.$$

Making use of these, we see that (6.16) becomes

$$x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0, \tag{6.22}$$ Bessel

which we want to solve for $x > 0$, and for which we want $x = \omega$ to be a root. We furthermore require that R be finite at $x = 0$.

Exercise 6.3.4. *The Sturm-Liouville theorem tells us that there is, in fact, an increasing list of values for* ω *that lead to a solution to our eigenvalue problem. This means that for each value of* n*, there should be an infinite number of roots* $x = \omega_{nk}$ *of the solution* $R(x)$ *to* (6.22)*.*

To see this, suppose that $R(x) = \frac{1}{\sqrt{x}} S(x)$ *for some function* $S(x)$*. Plug this expression in to* (6.22) *and show that* S *must satisfy*

$$\frac{d^2S}{dx^2} = -\left(1 + \frac{1 - 4n^2}{4x^2}\right) S. \tag{6.23}$$ mod-Bessel

Conclude that for large values of x *any solutions to* (6.23) *should be oscillating roughly in the way that cosine and sine do, and thus that* $S(x)$ *must have an infinite number of roots.*

Further conclude that $R(x) \sim \frac{1}{\sqrt{x}}$*(cosine or sine) as* x *gets large.*

We now investigate the behavior of solutions to (6.22) when x is small. Suppose $R(x)$ is a solution to (6.22) and that $R(x) = x^p + R_{\text{rem}}(x)$ for some power p and function $R_{\text{rem}}(x)$ that vanishes to higher order as $x \to 0$ in the sense that

$$\lim_{x\to 0} \frac{R_{\text{rem}}(x)}{x^p} = 0, \quad \lim_{x\to 0} \frac{R'_{\text{rem}}(x)}{x^{p-1}} = 0, \quad \text{etc.}$$

Plugging $R(x) = x^p + R_{\text{rem}}(x)$ in to (6.22) yields

$$(p^2 + n^2)x^p + x^2 R''_{\text{rem}}(x) + xR'_{\text{rem}}(x) + (x^2 - n^2)R_{\text{rem}}(x) + x^{p+2} = 0.$$

Dividing by x^p and taking the limit as $x \to 0$ we conclude that $p^2 - n^2 = 0$ and thus $p = \pm n$. Since we require $R(x)$ to be bounded at $x = 0$, we conclude that $R(x) = x^n + \dots$ when x is small.

The differential equation (6.22) is actually a famous one – it is called ***Bessel's equation***, and the solutions to it are called ***Bessel functions of order*** n and are given the symbol $J_n(x)$. Most mathematical handbooks devote several pages to these functions, which can actually be defined for any value of n. Here, however, we focus on the case when n is a positive integer.

We begin by seeking a power series expression for the solution $J_n(x)$ to (6.22) of the form

$$J_n(x) = x^n \sum_{k=0}^{\infty} a_k x^k; \tag{6.24}$$

Bessel-anzatz

the reason for the factor of x^n in front of the sum is that we expect the behavior of J_n to be x^n as $x \to 0$.

Exercise 6.3.5.

1. *Plug* (6.24) *in to* (6.22) *and show that, after re-indexing, the result is*

$$a_1(2n+1)x + \sum_{k=2}^{\infty} \left[a_k(2n+k)k + a_{k-2}\right] x^{k+n} = 0.$$

2. *Use the previous step to conclude that* $a_1 = 0$ *and that the coefficients satisfy the recurrence relation*

$$a_k = \frac{a_{k-2}}{(k+2n)k}.$$

   Conclude from this that, in fact, all of the odd-indexed coefficients are zero.

3. *Since the odd-indexed coefficients are zero, we can write* (6.24) *as*

$$J_n(x) = x^n \sum_{l=0}^{\infty} a_{2l} x^{2l}.$$

   Show, using the recurrence relation from the previous step, that

$$a_{2l} = \frac{a_0(-1)^l n!}{(l+n)!\, l!} \frac{1}{2^{2l}}.$$

4. *Show how to choose* a_0 *in order to achieve*

$$J_n(x) = x^n \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+n)!\, l!} \left(\frac{x}{2}\right)^{2l}.$$

5. *For* $n = 1, 2, 3, \ldots 10$ *use technology to plot the approximations*

$$x^n \sum_{l=0}^{100} \frac{(-1)^l}{(l+n)!\, l!} \left(\frac{x}{2}\right)^{2l}$$

   on the domain $0 \leq x \leq 30$.

   (a) *Do the solutions oscillate?*
   (b) *What happens to the roots as* n *increases?*

(c) *Compare the plots with the function* $\sqrt{\frac{2}{\pi x}}$. *What do you observe?*

As you see in the previous exercise (and as predicted by the Sturm-Liouville theorem), the Bessel function $J_n(x)$ has an infinite list of roots; we denote these roots by $\omega_{n1}, \omega_{n2}, \omega_{n3}, \ldots$ so that ω_{nk} is the kth root of the nth order Bessel function $J_n(x)$. Many software packages can easily find (or have as built-in functions) the roots ω_{nk}.

We now return to the problem that motivated all this Bessel stuff – finding our standing wave eigenfunctions. Recall that $R(r) = R(\omega x)$ where ω was a root of the function that solved (6.22). Since we now know that the solutions to (6.22) are the Bessel functions, and have a list of all their roots, we know that

$$R_{nk}(r) = J_n(\omega_{nk} r).$$

Using this, we see that our standing wave eigenfunctions are

$$\psi^c_{nk}(r,\theta) = J_n(\omega_{nk} r)\cos(n\theta) \quad \text{and} \quad \psi^s_{nk}(r,\theta) = J_n(\omega_{nk} r)\sin(n\theta).$$

Exercise 6.3.6.

1. *What does a typical plot of* $J_n(\omega_{nk} r)$ *look like? What does a typical plot of* $\cos(nr)$ *look like?*

2. *Use the previous part to figure out what the contour plots of* $\psi^c_{nk}(r,\theta)$ *are. What does the number* n *indicate? What does the number* k *indicate?*

3. *Get a piece of technology to either plot the function* $\psi^c_{nk}(r,\theta)$, *or plot its contours, for various values of* n *and* k. *Describe the results.*

6.4 Wave equation on the ball★

This section is written so that it can be used as a take-home exam and/or a *longer homework assignment.*

In this exploration you consider the wave equation on the unit ball subject to Dirichlet boundary conditions. Be warned: There are a lot of functions and change of variables running around this exploration – it is imperative that you keep your work organized!

Working in spherical coordinates (r, θ, ϕ), the unit ball is described by

$$0 \leq r \leq 1, \quad -\pi \leq \theta \leq \pi, \quad \text{and} \quad 0 \leq \phi \leq \pi.$$

We require periodic boundary conditions at $\theta = \pm\pi$, and that functions are finite at $r = 0$ and at $\phi = 0, \pi$.

1. Express the eigenvalue problem $\Delta\psi = \lambda\psi$ in spherical coordinates.

2. Assume that the eigenfunction takes the product form $\psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$, and recall that we can write $\lambda = -\omega^2$ for some positive constant ω. Show that this implies that

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + (\omega^2 r^2 + \mu)R = 0 \qquad (6.25) \quad \boxed{\texttt{3D-R}}$$

and

$$\frac{1}{\sin^2\phi}\frac{\partial^2 Y}{\partial\theta^2} + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial Y}{\partial\phi}\right] = \mu Y \qquad (6.26) \quad \boxed{\texttt{3D-Y}}$$

for some constant μ.

3. It is possible to show that $\mu < 0$ using an integration-by-parts argument. To do this, multiply the Y equation by Y and then integrate over the sphere using the spherical area element $dA = \sin\phi \, d\theta \, d\phi$. Integrate by parts in order to show that we must have $\mu < 0$ in order to have a nontrivial solution Y.

4. We now suppose that the function Y takes product form. Assume that

$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$ and show that the functions Θ and Φ must satisfy

$$\frac{d^2\Theta}{d\theta^2} = \nu\Theta \tag{6.27} \qquad \boxed{\texttt{3D-Theta}}$$

$$\sin\phi \frac{d}{d\phi}\left[\sin\phi \frac{d\Phi}{d\phi}\right] = (\mu \sin^2\phi - \nu)\Phi \tag{6.28} \qquad \boxed{\texttt{3D-Phi}}$$

for some constant ν.

Take a minute to review the logic here: The plan is to first solve (6.27), which involves understanding what are the possible values of ν. With this information in hand, you will turn to (6.28). The functions Φ and Θ are to be used in constructing the functions Y. Finally, you will address (6.25).

5. Explain why the function Θ must satisfy periodic boundary conditions on the interval $-\pi \leq \theta \leq \pi$, and thus that (6.27) only has solutions when $\nu = -m^2$ for some positive integer m. What are the corresponding functions Θ?

6. Now you get to address (6.28). Make the change of variables $x = \cos\phi$ in order to obtain

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Phi}{dx}\right] - \frac{m^2}{1-x^2}\Phi = \mu\Phi, \tag{6.29} \qquad \boxed{\texttt{ass-Legendre-ode}}$$

which holds for $-1 < x < 1$. Explain why, for each fixed integer m, the equation (6.29) satisfies the hypotheses of the Sturm-Liouville theorem with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

Conclude that for each fixed m there exists a sequence of negative eigenvalues and corresponding orthogonal eigenfunctions satisfying (6.29).

The eigenfunctions are called ***associated Legendre functions*** and are given the symbol $P_{l,m}(x)$; the corresponding eigenvalues are $\mu_{l,m} =$

$-l(l+1)$. Verify that $P_{0,0}(x) = 1$ and that

$$\begin{array}{ll} P_{1,0}(x) = x & P_{2,0}(x) = \frac{3}{2}x^2 - \frac{1}{2} \\ P_{1,1}(x) = \sqrt{1-x^2} & P_{2,1}(x) = 3x\sqrt{1-x^2} \\ P_{2,2}(x) = 3(1-x^2) & P_{3,2}(x) = 15x(1-x^2) \end{array}$$

satisfy (6.29). (It turns out that one must have $l \geq m$ in order to have nonzero $P_{l,m}$.)

7. Conclude from the above that for each m and $l \geq m$ there exists $\Phi_{l,m}(\phi) = P_{l,m}(\cos\phi)$ satisfying (6.28) with $\nu = -m^2$ and $\mu = -l(l+1)$. Use this to construct functions $Y_{l,m}(\theta,\phi)$ satisfying (6.26). These functions are called the ***spherical harmonics***. List the first few spherical harmonics. Draw, as best you can, the contour plots of the first few harmonics on the sphere.

8. Finally, we are ready to address (6.25). Set $\mu = -l(l+1)$ and make the change of variable $x = \omega r$ in order to obtain

$$x^2 \frac{d^2 R}{dx^2} + 2x\frac{dR}{dx} + (x^2 - l(l+1))R = 0.$$

Then make the change of variables $R(x) = \frac{1}{\sqrt{x}} S(x)$. Show that $S(x)$ must satisfy

$$x^2 \frac{d^2 S}{dx^2} + x\frac{dS}{dx} + \left(x^2 - \left[l(l+1) + \frac{1}{4}\right]\right) S = 0, \qquad (6.30)$$ half-Bessel

which is ***Bessel's equation*** with order $l(l+1) + \frac{1}{2}$. The solutions are called ***Bessel functions*** and are given the symbol $J_{l(l+1)+\frac{1}{4}}(x)$.

9. It is straightforward to find power series expressions for $J_{l(l+1)+\frac{1}{2}}(x)$, following the procedure used in the previous section. At this stage, however, you should simply …

10. Each Bessel function $J_{l(l+1)+\frac{1}{4}}$ has an infinite number of roots $\omega_{l,k}$. Use these roots to construct solutions $R_{l,k}(r)$ satisfying (6.25) with

$\mu = -l(l+1)$ and $\omega = \omega_{l,k}$. Sketch $R_{l,k}(r)$ for the first few l, k.

11. Conclude this whole discussion by constructing standing wave forms $\Psi_{l,m,k}(r, \theta, \phi)$ using the functions $R_{l,k}$ and $Y_{l,m}$ obtained above. Give the explicit form for the first few wave shapes. How should one understand these shapes geometrically and physically?

12. Look up the the "orbital table" in the Wikipedia article "Atomic Orbital" https://en.wikipedia.org/wiki/Atomic_orbital#Orbitals_table. What do you notice?

6.5 Laplace's equation

ch:solve-laplace-equation

In Chapter **??** we saw that the gradient of the Dirichlet energy was

$$\operatorname{grad} E[u] = -\Delta u.$$

Thus critical points of the Dirichlet energy must satisfy

$$\Delta u = 0; \tag{6.31}$$

LaplaceEquation

this equation is called ***Laplace's equation***. In Chapter **??**, Laplace's equation arose as part of the boundary value problem (5.4):

$$\begin{aligned} \Delta u &= 0 \quad \text{on the domain } \Omega, \\ u &= b \quad \text{on the boundary } \Gamma. \end{aligned} \tag{6.32}$$

Laplace-BVP

In the following, we first show some general properties of solutions to the boundary value problem (6.32). Then we consider the situation where Ω is a rectangle in $\mathbb{R}^2$ and construct series solutions the boundary value problem. Finally, we conclude this chapter with a discussion of the relationship between Laplace's equation and the wave equation.

We begin our study of (6.32) by showing uniqueness of solutions. Suppose that both u_1 and u_2 satisfy (6.32). The function $w = u_1 - u_2$ satisfies

$$\Delta w = 0 \quad \text{and} \quad w|_\Gamma = 0.$$

Using (5.2) we see that

$$\operatorname{div}(w \operatorname{grad} w) = \operatorname{grad} w \cdot \operatorname{grad} w + w\Delta w = \|\operatorname{grad} w\|^2.$$

Integrating this over the region Ω yields

$$\int_\Omega \operatorname{div}(w \operatorname{grad} w)\, dV = \int_\Omega \|\operatorname{grad} w\|^2\, dV.$$

Using the divergence theorem, and the fact that $w = 0$ along the boundary Γ, we see that

$$\int_\Omega \|\operatorname{grad} w\|^2\, dV = \int_\Gamma w \operatorname{grad} w \cdot \hat{n}\, dA = 0.$$

Since $\|\operatorname{grad} w\|^2 \geq 0$ we conclude that in fact we must have $\operatorname{grad} w = \mathbf{0}$. This, in turn, means that w is a constant function. Since $w = 0$ at the boundary, it must mean that $w = 0$, which means that $u_1 = u_2$.

We now show that solutions to (6.32) do not attain internal maximum/minimum points (unless u is constant). In particular, any solution to (6.32) satisfies

$$\max_\Omega u = \max_\Gamma u; \tag{6.33}$$

MaximumPrinciple

this fact is called the ***weak maximum principle***.

To see why (6.33) is true, consider the function $u + \epsilon e^x$, where ϵ is any small positive number and x is the Cartesian coordinate. Since u satisfies (6.32) we have

$$\Delta(u + \epsilon e^x) = \epsilon e^x > 0. \tag{6.34}$$

Thus by the second derivative test, $u + \epsilon e^x$ cannot have a local maximum at any point in the interior of the region Ω. Thus we conclude that for every

$\epsilon > 0$ we have

$$\max_{\Omega}(u + \epsilon e^x) = \max_{\Gamma}(u + \epsilon e^x).$$

Taking the limit as $\epsilon \to 0$ gives us (6.33).

Exercise 6.5.1. *Adapt the line of reasoning that lead to* (6.33) *in order to show the following "minimum principle": If u satisfies $\Delta u = 0$ then*

$$\min_{\Omega} u = \min_{\Gamma} u.$$

We have shown that if solutions to (6.32) exist then are unique and that solutions are bounded above (and below) by the maximum (and minimum) of the boundary value function b. In the next section we show how to construct solutions to (6.32) in the case that Ω is a rectangle.

6.6 Laplace's equation on the rectangle

Suppose that Ω is the rectangle in $\mathbb{R}^2$ given by

$$0 \leq x \leq L \quad \text{and} \quad 0 \leq y \leq M.$$

Working in Cartesian coordinates, we want to find a solution $u(x, y)$ to the boundary value problem (6.32), which we write as

$$\begin{gathered} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u(x,0) = b_1(x) \qquad u(x,M) = b_3(x) \\ u(0,y) = b_4(y) \qquad u(L,y) = b_2(y), \end{gathered} \tag{6.35}$$

Laplace-Rectangle-

for some pre-specified functions b_1, b_2, b_3, b_4. The key to our approach is the following.

uce-rectangular-Laplace-BVP **Exercise 6.6.1.** *Suppose we have functions u_1, u_2, u_3, u_4 such that*

- *u_1 satisfies* (6.35) *with b_2, b_3, b_4 all set equal to zero,*

- *u_2 satisfies* (6.35) *with b_1, b_3, b_4 all set equal to zero,*

etc. Show that

$$u = u_1 + u_2 + u_3 + u_4 \tag{6.36}$$

solves (6.35).

In view of Exercise 6.6.1, it is enough to consider a boundary value problem on the rectangle where the boundary function b is zero on three of the four sides of the rectangle. For simplicity, we suppose that in (6.35) we have b_1 is some pre-specified function and $b_2 = 0$, $b_3 = 0$, $b_4 = 0$. Thus we want a function $u_1(x, y)$ satisfying

$$\begin{gathered} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u(x,0) = b_1(x) \qquad u(x,M) = 0 \\ u(0,y) = 0 \qquad u(L,y) = 0. \end{gathered} \tag{6.37}$$

Laplace-Rectangle-ReducedBV

We look for solutions to (6.37) by first looking for product solutions $u(x, y) = X(x)Y(y)$. Plugging this in to Laplace's equation yields

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

Thus we deduce that

$$X''(x) = \lambda X(x) \quad \text{and} \quad Y''(y) = -\lambda Y(y)$$

for some constant λ.

The boundary conditions at $x = 0$ and $x = L$ imply that

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

Thus we see that, just as in Section 6.2 we have a list of possible functions

$$X_l(x) = \sin\left(\frac{\pi l}{L}x\right)$$

with corresponding eigenvalues

$$\lambda_l = -\left(\frac{\pi l}{L}\right)^2. \tag{6.38}$$

Laplace-Rectangle-

The boundary condition at $y = M$ implies that

$$Y(M) = 0,$$

but we postpone for a moment a discussion of the boundary condition at $y = 0$. Using (6.38) we see that for each l we seek a function $Y_l(y)$ that satisfies

$$Y_l''(y) = \left(\frac{\pi l}{L}\right)^2 Y_l(y) \qquad \text{and } Y_l(M) = 0.$$

It is easy to check that the function

$$Y_l(y) = \sinh\left(\frac{\pi l}{L}(y - M)\right)$$

suffices. Thus we find that for each $l = 1, 2, 3, \ldots$ we have the function

$$\sin\left(\frac{\pi l}{L}x\right)\sinh\left(\frac{\pi l}{L}(y - M)\right)$$

that satisfies Laplace's equation as well as three of the four boundary conditions. Using linearity, we see that the function

$$u_1(x, y) = \sum_{l=1}^{\infty} a_l \sin\left(\frac{\pi l}{L}x\right)\sinh\left(\frac{\pi l}{L}(y - M)\right) \tag{6.39}$$

ReducedSolution

also solves Laplace's equation and three of the boundary conditions.

We now choose the constants a_l in order to satisfy the boundary condition at $y = 0$. This requires

$$b_1(x) = \sum_{l=1}^{\infty} a_l \sin\left(\frac{\pi l}{L} x\right) \sinh\left(\frac{\pi l}{L}(-M)\right).$$

Since the functions $X_l(x) = \sin\left(\frac{\pi l}{L} x\right)$ form a complete orthogonal collection, we know that

$$b_1(x) = \sum_{l=1}^{\infty} \alpha_l \sin\left(\frac{\pi l}{L} x\right),$$

where

$$\alpha_l = \frac{\langle b_1, X_l \rangle}{\|X_l\|^2}.$$

Thus we choose

$$a_l = -\sinh\left(\frac{\pi l}{L} M\right) \frac{\langle b_1, X_l \rangle}{\|X_l\|^2}. \tag{6.40}$$ ReducedCoefficients

The function $u_1(x, y)$ given by (6.39), where the coefficients are a_l are given by (6.40), solves the reduced boundary value problem (6.37). In order to solve the boundary value problem (6.35), we simply construct functions u_2, u_3, and u_4 satisfying analogous reduced boundary value problems; see Exercise 6.6.1.

Exercise 6.6.2. *Solve the boundary value problem* (6.35) *on the square* $[0, 1] \times [0, 1]$ *with* $b_1(x) = 1$, $b_2(y) = -1$, $b_3(x) = 1$ *and* $b_4(y) = -1$.

6.6.1 Laplace's equation and the wave equation

Paul didn't cover this last time... but it could be interesting if there is time.

6.6.2 Optional: Neumann boundary condition for Laplace's equation

Paul didn't cover this last time... but it could be interesting if there is time.