# 1. Existence of Ring Completion

Proof of Existence of Definition ??.

The existence of a ring completion is shown through an explicit construction. Take any commutative semiring with additive cancellation $(S, +, \cdot)$ and consider the equivalence relation $\sim$ on $S \times S$ defined as follows: for $(a_1, b_1)$, $(a_2, b_2)$ in $S \times S$, then let $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. The aim is to make the set of equivalence classes under $\sim$ into a ring.

First, define the additive operation $+$ by

$$[(a, b)] + [(c, d)] = [(a + c, b + d)]$$

Next, define the multiplicative operation $\cdot$ by

$$[(a, b)] \cdot [(c, d)] = [(ac + bd, ad + bc)]$$

This proof aims to verify that the set of equivalence classes $S \times S/\sim$ paired with the operations $(+, \cdot)$ forms a commutative ring that is a ring completion of $S$.

It must be verified that the additive operation is well defined, so consider elements $(a_1, b_1)$, $(a_2, b_2)$, $(c_1, d_1)$, $(c_2, d_2)$ in $S \times S$ such that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$. Then, I claim that $(a_1 + c_1, b_1 + d_1) \sim (a_2 + c_2, b_2 + d_2)$. Indeed, this satisfies the definition of the equivalence relation, for

$$\begin{aligned} (a_1 + c_1) + (b_2 + d_2) = (a_1 + b_2) + (c_1 + d_2) \\ = (a_2 + b_1) + (c_2 + d_1) = (a_2 + c_2) + (b_1 + d_1) \end{aligned}$$

where the above computation used the substitutions $a_1 + b_2 = a_2 + b_1$ and $c_1 + d_2 = c_2 + d_1$ promised by the relations $(m_1, m_2) \sim (m_1', m_2')$ and $(l_1, l_2) \sim (l_1', l_2')$. This confirms that $+$ is well-defined on $(S \times S)/\sim$.

The transitivity and commutativity of $+$ on the equivalence classes follows immediately from the commutativity and transitivity of the operation $+$ on $S$.

Next, note that the additive identity in $(S \times S)/\sim$ is given by $[(0, 0)]$ where $0$ denotes the identity element in $S$. Indeed, we have $[(a, b)] + [(0, 0)] = [(a, b)]$ for any element $[(a, b)]$.

The proposed ring has an inverse mapping for the addition operation. Consider an element $[(a, b)]$. Then, I claim the element $[(b, a)]$ forms the desired inverse. To see this, consider the sum $[(a+b, b+a)]$ and note that $(a + b) + 0 = 0 + (b + a)$, which shows $[(a + b, b + a)] = [(0, 0)]$.

It must be verified that the multiplicative operation is well-defined before verifying any further properties. Consider the elements $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$ in $S \times S$. It then must be verified that $(a_1c_1 + b_1d_1, a_1d_1 + b_1c_1) \sim (a_2c_2 + b_2d_2, a_2d_2 + b_2c_2)$. To accomplish this, consider the following $M_1, M_2 \in S$:

$$\begin{aligned} M_1 = c_2(a_1 + b_1) + b_2(c_1 + d_1) + b_2c_2 \\ M_2 = c_1(a_2 + b_2) + b_1(c_2 + d_2) + b_1c_1 \end{aligned}$$

Next, observe that using the relations $a_1 + b_2 = a_2 + b_1$ and $c_1 + d_2 = c_2 + d_1$, it follows that $a_1c_1 + b_1d_1 + M_1 = a_2c_2 + b_2d_2 + M_2$.

$$\begin{aligned}
&a_1c_1 + b_1d_1 + M_1 = a_1c_1 + b_1d_1 + c_2a_1 + c_2b_1 + b_2c_1 + b_2d_1 + b_2c_2 \\
&= (a_1 + b_2)(c_1 + c_2) + (d_1 + c_2)(b_1 + b_2) \\
&= (a_2 + b_1)(c_1 + c_2) + (d_2 + c_1)(b_1 + b_2) \\
&= a_2c_2 + b_2d_2 + c_1a_2 + c_1b_2 + b_1c_2 + b_1d_2 + b_1c_1 = a_2c_2 + b_2d_2 + M_2
\end{aligned}$$

A similar process shows that $a_1d_1 + b_1c_1 + M_1 = a_2d_2 + b_2c_2 + M_2$. Then, summing the two results gives

$$(a_1c_1 + b_1d_1) + (a_2d_2 + b_2c_2) + (M_1 + M_2) = (a_2c_2 + b_2d_2) + (a_1d_1 + b_1c_1) + (M_1 + M_2)$$

Applying the additive cancellation property of $S$ to the term $(M_1 + M_2)$ gives the desired relation and provides the conclusion $(a_1c_1 + b_1d_1, a_1d_1 + b_1c_1) \sim (a_2c_2 + b_2d_2, a_2d_2 + b_2c_2)$ and so the multiplicative operation is well defined.

The transitivity of the multiplicative operation follows directly from $+$ and $\cdot$ transitive in $S$. Similarly, the commutativity of the multiplicative operation follows directly from the commutativity of $+$ and $\cdot$ in $S$.

Next, note that the element $[(1, 0)]$ acts as an identity element for the multiplicative operation. Indeed, $[(1, 0)] \cdot [(a, b)] = [(a, b)]$ for any element $[(a, b)]$.

It only remains to show that $+$ distributes over $\cdot$ to verify that $S \times S/\sim$ forms a ring. Indeed, for elements $[(a, b)]$, $[(c, d)]$, $[(e, f)]$:

$$\begin{aligned}
&[(e, f)] \cdot ([(a, b)] + [(c, d)]) = [(e, f)] \cdot [(a + c, b + d)] \\
&= [(ea + fb + ec + fd, eb + ed + fa + fe)] \\
&= [(ea + fb, eb + fa)] + [(ec + fd, ed + fc)] = [(e, f)] \cdot [(a, b)] + [(e, f)] \cdot [(e, d)]
\end{aligned}$$

Thus we have that $(S \times S)/\sim$ forms a commutative ring under the proposed operations. However, it remains to show that $(S \times S)/\sim$ is a valid ring completion. The necessary inclusion map $i : S \to (S \times S)/\sim$ is given by $i(s) = [(s, 0)]$. Then, take any ring $R'$ and homomorphism $\varphi : S \to R'$; the existence and uniqueness of a commuting ring homomorphism $\psi : (S \times S)/\sim \to R'$ must be shown.

Uniqueness follows quickly from its homomorphism properties and the commutativity of the universal property. Indeed, take two commuting ring homomorphisms $\psi$ and $\psi'$ from $S \times S/\sim$ to $R'$. Then, the restrictions $\psi \circ i = \varphi$ and $\psi' \circ i = \varphi$ paired with $i$ injective gives that $\psi = \psi'$ over the image $i(S)$. Then observe that any element $[(a, b)]$ is the composition of elements in $i(S)$ by $[(a, b)] = [(a, 0)] - [(b, 0)]$. Then, the homomorphism properties of rings extends $\psi$ and $\psi'$ to be equivalent over all of $(S \times S)/\sim$ giving uniqueness.

It only remains to show existence of the homomorphism. The map $\psi : [(a, b)] \mapsto \varphi(a) - \varphi(b)$ works. Commutativity follows easily, for $(\psi \circ i)(s) = \psi([(s, 0)]) = \varphi(s)$ for all $s \in S$. Now, it must be verified that $\psi$ is a homomorphism. So, consider elements $[(a, b)]$ and $[(c, d)]$ of the ring completion.

The following equality chain shows that the the additive property of $\varphi$ gives the additive property of $\psi$.

$$\begin{aligned}&\psi([(a,b)+(c,d)]) = \psi([(a+c,b+d)]) = \varphi(a+c)-\varphi(b+d)\\&= (\varphi(a)-\varphi(b))+(\varphi(c)-\varphi(d)) = \psi([(a,b)])+\psi([(c,d)])\end{aligned}$$

Similarly, the additive and multiplicative property of $\varphi$ gives the multiplicative property of $\psi$.

$$\begin{aligned}&\psi([(a,b)]\cdot[(c,d)]) = \psi([(ac+bd,ad+bc)])\\&= \varphi(ac+bd)-\varphi(ad+bc) = \varphi(a)\varphi(c)+\varphi(b)\varphi(d)-\varphi(b)\varphi(c)-\varphi(a)\varphi(d)\\&= (\varphi(a)-\varphi(b))(\varphi(c)-\varphi(d)) = \psi([(a,b)])\cdot\psi([(c,d)])\end{aligned}$$

Finally $\psi(1) = \psi([(1,0)]) = \varphi(1) = 1$, completing the proof.

$\square$