NOT-SO-RANDOM STUFF FOR THE CURVATURE / RELATIVITY CLASS

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1. About the class

So, here is the deal. I am trained as a mathematician, differential geometer to be precise. The phrase *differential geometry* refers to this business of studying space / geometry / shapes and such using calculus. (Remember parametrizing surfaces, dealing with line, area and volume elements and such in Multivarible Calc? Yeah, that stuff.) (If you don't know what I am talking about then we've got a problem.)

Anyhow, I like to tell people that the basic premise of general relativity is that the geometry of our universe is dynamic and that it changes in whatever-you-call-time. And what better way to study change than using calculus, and general relativity than using differential geometry!

I know a fair amount (others can judge if I am an "expert" or not) about what is called mathematical general relativity. The astrophysical side of the story is cool and all, but that is not the story I fell in love with and that is not the story I intend to share with you.

For ten-ish weeks of the class I will teach you basics of differential geometry: the mathematical underpinnings of general relativity. This will be like a continuation of Calc 3 - pure geometry and calculus joy! This is, after all, a math class! I understand that there might be people in the audience who might not be too excited about this, and would much rather talk physics-ie stuff. Here is what I have to say to those people: this class is an opportunity to get someone who knows what they are doing to guide through all the math you need to know to read a more physics-ie book on GR. And so it is especially you who should be excited about all the math in the class! That math-ie stuff is the stuff you may never be able to get yourself to do on your own! Also, please no fancy physics talk until the end of the tenth week of class.

The last four-to-five-ish weeks we will look at general relativity per se. I promise to get you to the explanation of the Einstein equation (think of that photo where he is writing $R_{ik} = 0$ on the board!), and I promise to get you to the analysis of Schwarzschild space-time. (The latter is a theoretical prototype for the gravitational field / geometry near a relativistic massive object, potentially a black hole.) We should also be able to get to Friedman-Robertson-Walker cosmology, so that you can see the story about the big bang / big crunch. However – I don't promise I'll get you there; we'll see how people are doing and how tired people are getting. And I definitely don't promise much in terms of factual astrophysical information. Nope. That stuff belongs in Olin.

I also understand that there may be pure math people in this audience who may be concerned about the amount of physics needed to understand what is going on. I have one thing to say to those people: if I can understand and absolutely love this stuff, so can (and will!) you. I started out as a snotty pure math person who was poo-poo-ing anything applicable and who regarded anything physics-ie as "not rigorous" – mainly as a lame excuse for poor physical intuition. And here I am! And here you are!

2. Textbook stuff

I did teach this class before. Sort of. In 2013 Park City Math Institute (PCMI) ran a summer program on Geometric Analysis, which is a part of math that mathematical GR is more-or-less a part of. PCMI was a huge shindig with tons of mini-courses and lectures by fancy people, and somehow Paul and I got to teach courses for undergrads. (The students where selected internationally by organizers.) Your main textbook is the set of lecture notes I wrote for my PCMI course. It is available on my website, and on the first day of class I will give you the password to access it.

Warning: these are literally the notes I gave to the PCMI students, and there are things in there which relied on course work I assumed they had but which I cannot assume LC students had. So, there will be a certain amount of fill-in-the-blanks by me. For instance, we will start with "all the things Iva wish she could say in her Calc 3 course but she couldn't because Linear was not a prereq." Right before we go into relativity per se I'll go over basics of special relativity for those who have not had this in class yet, etc. This is all to say that: I expect you to come to and acquire notes from every single class as well as read every single line in the lecture notes. *I expect commitment from you, and you will get commitment from me!*

Here are some actual books you may want to know about.

- Woodhouse, N. M. J., *General Relativity*, published by Springer in 2007. This is a fantastic little math book, written especially for undergraduates. I highly recommend it – it will probably be of help in this course too!! The book was a great source of inspiration for me when I wrote my PCMI notes in 2013.

- Schutz, Bernard, *A First Course in General Relativity*, 2nd edition published by Cambridge University Press in 2009. I wish I utilized this book more, especially when I was first learning GR. In retrospect, the book is somewhat unique in that it combines mathematical rigor with astrophysical stuff. Check it out! It is meant to be accessible to undergraduates.

- Carroll, Sean, *Spacetime and Geometry: An Introduction to General Relativity*, published by Addison-Wesley in 2003. You will definitely find this book to be more advanced than our class. The book does have a more physics-ie point of view which I found very helpful on several occasions. If you want to keep learning GR, and you are interested in actual physics, this would be a good book to look into.

- Hartle, James B., *Gravity: An Introduction to Einstein's General Relativity*, published by Addison-Wesley in 2003. This is another advanced physics-ie book which I found very helpful. I would recommend it as a follow up to our course for those who are interested in physics.

- O'Neill, Barrett, *Semi-Riemannian Geometry with Applications to Relativity*, published by Academic Press in 1983. This is primarily a differential geometry textbook, and as such it should be easily readable by an advanced undergrad or beginning math-grad-school student. I am speaking from personal experience and experience of my research students many years ago!!

- Wald, Robert M., *General Relativity*, published by University of Chicago Press in 1984. This is the book I learned GR from!! It is an absolute classic at least amongst math GR people – we've all learned from it! Be warned though: the book is very much at a graduate level and (being published so long ago) it is not up to date on astrophysics-ie things.

- Hawking, S.W., Ellis, G.F.R, *The Large Scale Structure of Space-Time*, published by Cambridge University Press in 1975. This book complements Wald very nicely. If you find yourself going though Wald do yourself a favor and also go through Hawking and Ellis. I wish I discovered this book earlier – it would have been of great help.

- Misner, C.W., Thorne, K.S., Wheeler, J.A, *Gravitation*, published by W. H. Freeman in 1973. This is THE BOOK on so many levels. Its scope is mind-blowing. If you are a physics-ie person and want to learn "from the source" then learn from this book. Be warned: it will take a while to go through the book. It is huge!

3. Goals for the class

(1) You will deepen your understanding of how multivariable calculus and linear algebra are used to describe and study stuff around us or in a highly dimensional space.

(2) You will gain an understanding of how the concept of curvature of space(time) can be quantified.

(3) You will gain an understanding of Einstein equations, and their connection to Newtonian gravity.

(4) You will develop a more mathematically precise understanding of what a black hole might be.

(5) You will develop an ability to discuss the subject with some mathematical sophistication, and clearly beyond the level of popular science.

4. Schedule – **highly tentative**

Overview of multivariable calculus and linear algebra. August 31st - September 8th. It sounds like review, but most of it won't be. Trust me. Homework problems 1 – 8, due by the time I leave office on Thursday, September 9th.

Lecture 1 from PCMI notes. September 10 – 11. I will omit the whole projective plane mumbo-jumbo. Homework problems 9 – 11 due by the time I leave office on Tuesday, September 15.

Lecture 2 from PCMI notes. September 14 – 17. Homework problems 12 – 17 due by the time I leave office on Monday, September 21.

Lecture 3 from PCMI notes. September 18 – 24. Note: I will postpone the material on hyperbolic geometry, and talk about the exponential map from Lecture 4. Homework problems 18 – 22 due by the time I leave office on Tuesday, September 29.

Lecture 4 from PCMI notes. September 25 – October 1 or 2nd. Homework problems 23 – 29 due by the time I leave office on Friday October 2nd. Note: all these essay-ish homework problems will be expected to be typed up in LaTex. Save the files! You will need them for the final essay. No, seriously.

Iva's free-stylin' on the topic of hyperbolic geometry. October 2nd – October 5th or 6th. This is stuff and things which will hopefully give you context for your take-home exam.

Take-home exam. Will be given in class on October 2nd and will be due by 9 am on Wednesday October 7th. (Day before the Fall Break.)

Lecture 5 from PCMI notes. October 12 – October 16. I will probably absorb some of the tensor stuff in here, which is written up in Lecture 6. On the other hand, I intend to skip the Maximum Principle stuff which is in Lecture 5. Homework problems 30 – 38 due by the time I leave office on Tuesday October 20th.

Lecture 6 from PCMI notes. October 19 – October 23. Homework problems 39 – 44 due by the time I leave office on Tuesday October 27th. Mind the last two problems – they can take time!

Lecture 7 from PCMI notes. October 26 – October 30. Homework problems 45 – 50 due by the time I leave office on Tuesday November 3rd. Mind the last problem!

Lecture 8 from PCMI notes. November 2nd – November 6th. Homework problems 51 – 55 due by the time I leave office on Tuesday November 10th. Mind the first and the last problem – please don't wait until the last minute to do them.

Lecture 9 from PCMI notes. November 9th – 13th. Note that I will expand on special relativity, for the benefit of those who haven't seen this in class. Homework problems 56 – 62 due by the time I leave office on Tuesday November 17th. Yes, there is an "essay" again.

Lecture 10 from PCMI notes. November 16th – 20th. I'll add photon orbits and bending of light to what is already in the notes. Homework problems 63 – 65 due by the time I leave office on Tuesday November 24th. Note: you'll have to use Mathematica or something of the sort to compute stuff and things in this assignment.

Kruskal extension and black holes. November 23rd – November 24th. Note: these are the days before the Thanksgiving Break. Don't miss them! Homework problems 66 – 67 due by the time I leave office on Tuesday December 1st. "Essay" again!

FRW cosmology. November 30th – December 4th. The only homework (see "problem" 68) is to deal with the final essay; *it is due in class on December 8th.*

The final exam + essay +evaluations stuff and things. December 7th – December 8th.

5. Homework problems

(1) Let V denote the space of vectors which are tangent to the elliptic paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$. One can verify that V is 2-dimensional (tangent plane!) and that the vectors $\partial_1 = \langle 1, 0, 2\rangle$ and $\partial_2 = \langle 0, 1, 2\rangle$ with base-point at $(1, 1, 2)$ form a basis for V. (I don't expect you to verify this.) Are the following vectors in V:

$$\vec{v}_1 = \langle 1, -1, 0\rangle, \quad \vec{v}_2 = \langle 2, -1, 2\rangle, \quad \vec{v}_3 = \langle 1, 1, 2\rangle?$$

If so, find their coordinates with respect to the given basis.

(2) Let $\vec{v} = \langle 1, 0, 1, 0\rangle$ be a vector in $\mathbb{R}^4$ and let V denote the space of vectors orthogonal[1] to $\vec{v}$ in $\mathbb{R}^4$. (Assume the standard dot (inner) product.)

  (a) Are the following vectors in V:

  $$\vec{v}_1 = \langle 1, 1, -1, -1\rangle, \quad \vec{v}_2 = \langle 1, -1, 0, 0\rangle, \quad \vec{v}_3 = \langle 0, 1, 0, 1\rangle?$$

  (b) Find a basis for V; also verify that what you claim is a basis really is a basis.

  (c) For each of the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ which happen to be in V find the coordinates with respect to the basis you just found.

(3) Analyze the beejeebers out of linear transformations corresponding to the following matrices.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- Compute the kernel and the image, the rank and the nullity.

[1] This is usually called the orthogonal complement; you may have seen the notation $\vec{v}^{\perp}$ for it.

- Provide sketches in lower-dimensional settings; otherwise describe the effect of the transformations in words.
- When appropriate compute the eigenvalues and the eigenvectors; interpret them visually.
- Discuss if the transformations involved are 1-1 / onto or bijections. Please find the inverse in situations when the inverse exists.

(4) Consider the curvilinear transformation given by
$$(y^1, y^2) = \Phi_x^y(x^1, x^2) = \Big((x^1)^2 - (x^2)^2, 2(x^1)(x^2)\Big).$$
(a) Find the images of the points $(1, 0)$, $(1, -1)$, $(1, 1)$, $(1, -2)$ and $(1, 2)$.
(b) Compute the Jacobi matrix $D(\Phi_x^y)$ of the transformation, and evaluate it at points from part (4a) of this problem.
(c) Analyze the action of Φ_x^y in the neighborhood of the points from part (4a) of this problem. You may want to use the guidelines from Problem (3).
(d) Combine the above to get a visual sense of what the transformation Φ_x^y does to the line $x^1 = 1$. Sketch your conclusion.

(5) Consider the curvilinear transformation given by
$$(y^1, y^2, y^3) = \Phi_x^y(x^1, x^2) = \Big(x^1 \cos(x^2), x^1 \sin(x^2), x^2\Big).$$
You should be able to recognize that the image of this transformation traces out a helicoidal surface in a three dimensional space.
(a) Find the image of the point $(1, 0)$ under Φ_x^y.
(b) Compute the Jacobi matrix $D(\Phi_x^y)$, at evaluate it at the point $(1, 2\pi)$.
(c) Analyze the kernel and the image, the rank and the nullity, of the transformation determined by $D(\Phi_x^y)$.
(d) What is the tangent space to the helicoidal surface at the point $(1, 0, 2\pi)$?
(e) Mimic the process outlined above to determine the tangent space to the helicoidal surface at $(1, 1, \pi/4)$.

(6) Consider the mapping $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by
$$\begin{aligned} g(\langle x^1, x^2, x^3\rangle, \langle y^1, y^2, y^3\rangle) &= 2x^1y^1 + x^1y^2 + x^2y^1 + x^2y^2 + 4x^3y^3 \\ &= \begin{pmatrix} x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}. \end{aligned}$$
(a) Verify that g defines an inner product on $\mathbb{R}^3$, and find an orthonormal basis for it.

(b) Let A and B be points $(-1, 1, 0)$ and $(0, -1, 1)$, respectively. Find the distance between A and B using the inner product g. Bonus points if you can also do it with respect to the orthonormal basis you discovered in problem (6a).

(7) Consider the mapping $Q : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} Q(\langle x^1, x^2\rangle, \langle y^1, y^2\rangle) =& x^1y^1 + 2x^1y^2 + 2x^2y^1 + x^2y^2 \\ =& \begin{pmatrix} x^1 & x^2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}; \end{aligned}$$

it can easily be shown that this map is symmetric and bilinear.

(a) Find a basis which diagonalizes Q.

(b) Verify that Q is non-degenerate. What is its signature?

(c) Find an orthonormal basis for Q, and express Q with respect to this basis.

(d) Describe, in as much detail as you can, the shape traced out by points (x^1, x^2) in $\mathbb{R}^2$ for which

$$Q(\langle x^1, x^2\rangle, \langle x^1, x^2\rangle) = 1.$$

(e) Describe, in as much detail as you can, the shape traced out by points (x, y, z) in $\mathbb{R}^3$ for which

$$z = x^2 + 4xy + y^2 = Q(\langle x, y\rangle, \langle x, y\rangle).$$

(8) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = 3x + 3y - x^2 - xy - y^2.$$

(a) Find the gradient and the Hessian of the function f at $(x, y) = (1, 1)$.

(b) Find the quadratic Taylor expansion of the function at $(x, y) = (1, 1)$.

(c) Does the function f have a minimum or a maximum at $(x, y) = (1, 1)$?

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(9) Consider the polar coordinates (r, θ) in the Euclidean plane, and consider the line element $ds^2 = dr^2 + f(r)^2 d\theta^2$ for some positive function $f(r)$.

(a) Find a formula for the length of $r(t) = t$, $\theta(t) = \theta_*$, $0 < t < \varrho$.

(b) Find a formula for the length of $r(t) = \varrho$, $\theta(t) = t$, $-\pi < t < \pi$ in terms of f.

(c) Which geometry corresponds to the following choices of f?

- $f(r) = r$;
- $f(r) = \sin(r)$, under the restriction $r \in (0, \pi)$;
- $f(r) = 1$;

(d) Based on the above provide a visual description of the geometries corresponding to the following choices of f.

- $f(r) = \frac{1}{2}r$;
- $f(r) = 2r$;
- $f(r) = 1 + 2r$;
- $f(r) = 2 + \sin(r)$
- $f(r) = e^r$;
- $f(r) = \sinh(r) = \frac{1}{2}(e^r - e^{-r})$.

(10) (a) Provide a computation which shows that the Euclidean line element for $\mathbb{R}^3 \setminus \{(0,0,0)\}$ in spherical coordinates (r, θ, ϕ) takes the form of $ds^2 = dr^2 + r^2 d\Theta^2$, where $d\Theta^2$ denotes the spherical line element $d\Theta^2 = d\phi^2 + \sin^2\phi\, d\theta^2$.

(b) In analogy with exercise 9, provide a visual description of the geometries on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ corresponding to the following line elements.

- $ds^2 = dr^2 + \sin^2 r\, d\Theta^2$;
- $ds^2 = dr^2 + \sinh^2 r\, d\Theta^2$.

(11) (a) Develop formulas for a stereographic coordinatization of S^n.

(b) Provide a computation which shows that the standard line element on S^n takes the form of

$$ds^2 = \frac{4}{(1 + (x^1)^2 + \cdots + (x^n)^2)^2}\left(d(x^1)^2 + \cdots + d(x^n)^2\right)$$

in the coordinates arising from the stereographic projection.

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(12) Consider the the line element $ds^2 = \frac{dx^2+dy^2}{y^2}$ on the upper half-plane $y > 0$. Use this line element to compute the lengths of the following paths joining $(-1, 1)$ and $(1, 1)$.

(a) Let c_1 be the "straight" line segment $x(t) = t,\ \ y(t) = 1,\ \ -1 \le t \le 1$.

(b) Let c_2 be the "circular arc" $x(t) = \sqrt{2}\cos(t),\ \ y(t) = \sqrt{2}\sin(t)$ with $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$.

Which of the two paths is shorter?

(13) This problem investigates how a change of coordinates

$$\Phi_x^y : (x^1, x^2, ..., x^n) \mapsto (y^1, y^2, ..., y^n)$$

effects the coordinate expressions for vector fields and the components of a Riemannian metric.

(a) Let $\{\partial_i\}$ and $\{\partial_{i'}\}$ be the coordinate vector fields corresponding to the x and the y coordinates, respectively. Consider the decompositions $V = \sum_i V^i \partial_i = \sum_{i'} V^{i'} \partial_{i'}$ of a vector field V. Find a formula relating the functions V^i to the functions $V^{i'}$.

(b) Let g_{ij} and $g_{i'j'}$ be the metric components with respect to the x and y coordinates, respectively. Find a formula relating $g_{i'j'}$ to g_{ij}.

(14) *You are expected to use / apply the general framework developed in Problem 13 while solving the following.* Consider the coordinatization of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ using spherical coordinates (r, θ, ϕ).

(a) Express the vector fields ∂_r, ∂_θ and ∂_ϕ in terms of Cartesian ∂_x, ∂_y and ∂_z, and vice-versa.

(b) A vector field V on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ can be decomposed in terms of spherical as well as Cartesian coordinates:

$$V = V^r\partial_r + V^\theta\partial_\theta + V^\phi\partial_\phi \;\text{ while }\; V = V^x\partial_x + V^y\partial_y + V^z\partial_z.$$

Find formulas which relate the coordinates V^r, V^θ and V^ϕ to the Cartesian coordinates V^x, V^y, V^z, and vice-versa.

(15) Let $g(.,.)$ be some (not necessarily Euclidean) inner-product operation on $\mathbb{R}^n$, let the matrix G have entries $g_{ij} = g(\partial_i, \partial_j)$, and let L denote a $n \times n$ matrix and its corresponding linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$. We say that L is an *isometry* of $g(.,.)$ if

$$g(L\vec{v}, L\vec{w}) = g(\vec{v}, \vec{w}).$$

(a) Show that an isometry L preserves distances with respect to g.

(b) Show that $L^T GL = G$.

(c) The above formula should be highly related to that in exercise 13b. Explain why this is not a coincidence.

(16) (a) The *diameter* of a compact Riemannian manifold (M, g) is the maximum value of $d(P, Q)$ for $P, Q \in M$. What is the diameter of the standard unit sphere S^2? What about S^n?

(b) We define the circle of radius r centered at the point P of a 2-dimensional Riemannian manifold (M, g) as the set of all points Q with $d(P, Q) = r$. Describe and compute the circumference of the circle of radius r centered at the North Pole of the standard unit sphere S^2.

(17) Compute the volume of S^4.

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(18) (a) Compute the Christoffel symbols for the standard unit 2-dimensional sphere with respect to the standard spherical coordinates (θ, ϕ).

(b) Express the geodesic equations on the standard unit sphere, and verify that a particular great circle is a geodesic.

(19) Let (r, θ) denote the standard polar coordinates, and consider the metric $ds^2 = dr^2 + f(r)^2 d\theta^2$, where $f(r)$ is some positive, unknown but nice function.

(a) Compute the Christoffel symbols of this metric. *Note:* I expect you to check your answers with problem 18 above. Basically: ϕ corresponds to r and $\sin(\phi)$ corresponds to $f(r)$, or something like that.

(b) Express the geodesic equations for this metric.

(c) Let r_0 be a value where f reaches a local extremum. Show that

$$r(t) = r_0, \quad \theta(t) = t, \quad 0 \le t \le 2\pi$$

is a geodesic. Interpret visually. (I expect to see some generic pictures here.)

(20) Consider the standard unit sphere centered at the origin, and let P denote the "North Pole" $(0, 0, 1)$. In addition, consider the following tangent vectors

$$\vec{v}_1 = \langle 1, 1, 0 \rangle, \quad \vec{v}_2 = \langle 2, 2, 0 \rangle$$

to the sphere at P. Find $\exp_P(\vec{v}_1)$ and $\exp_P(\vec{v}_2)$.

(21) Show that any solution $\gamma(t) = (x^1(t), \ldots, x^n(t))$ of the geodesic equation satisfies

$$\tfrac{d}{dt}(|\dot{\gamma}|_g^2) = 0.$$

In particular, show that geodesics are necessarily parametrized in such a way that their tangent vectors are of constant length.

(22) This problem addresses surfaces which for a given boundary minimize the enclosed surface area; such surfaces are higher dimensional analogues of length minimizers. Critical surfaces for area functionals $S \mapsto \text{Area}(S)$ are called *minimal surfaces*.

(a) Consider $x^{n+1} = f(x^1, \ldots, x^n)$ for $(x^1, \ldots, x^n) \in \Omega$; this defines an n-dimensional surface S in Euclidean $\mathbb{R}^{n+1}$. Show that the area functional is given by

$$\int_\Omega \sqrt{1 + |\text{grad} f|^2}\, dx^1 \ldots x^n.$$

(b) Show that S is a minimal surface if and only if

$$\text{div}\left(\frac{1}{\sqrt{1 + |\text{grad} f|^2}} \,\text{grad} f \right) = 0.$$

Note: You are expected to mimic the calculus of variations procedure which brought us to the geodesic equation.

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(23) (a) Show that for a given function f and a given vector field V the value of $\sum_i V^i \partial_i f$ does not depend of the choice of coordinates.

(b) Let $\{\partial_i\}$ and $\{\partial_{i'}\}$ be the coordinate vector fields corresponding to the x and the y coordinates, respectively, and let V and W be two vector fields. Prove the following equality.

$$\sum_{ij}(V^i\partial_i W^j - W^i\partial_i V^j)\partial_j = \sum_{i'j'}(V^{i'}\partial_{i'}W^{j'} - W^{i'}\partial_{i'}V^{j'})\partial_{j'}.$$

(24) Compute the Lie brackets of the following vector fields on $\mathbb{R}^2$. Note: the letters x, y refer to the standard Cartesian coordinates while r, θ refer to the standard polar coordinates.

(a) $[x\partial_y, y\partial_x]$

(b) $[x\partial_x + y\partial_y, -y\partial_x + x\partial_y]$

(c) $[r\partial_r, \partial_\theta]$

(d) $[r\partial_r, \frac{1}{r}\partial_r]$.

(25) Prove the following properties of the Lie bracket. You should assume α is some differentiable function.

(a) $[V, W] = -[W, V]$;

(b) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$;

(c) $[V, \alpha W] = (\nabla_V \alpha)W + \alpha[V, W]$.

(26) Let (r, θ, ϕ) denote the standard spherical coordinates in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Assume the standard Levi-Civita connection for Euclidean metric on $\mathbb{R}^3$. Compute

$$\nabla_{r\partial_r}(\sin(\phi)\partial_\theta), \quad \nabla_{\sin(\phi)\partial_\theta}\partial_\phi, \quad \nabla_{\partial_\phi}(r\partial_r)$$

in two different ways:

- Using Christoffel symbols;
- Using the formula

$$\begin{aligned} g(\nabla_U V, W) = (1/2)\{&\nabla_U(g(V, W)) + \nabla_V(g(W, U)) - \nabla_W(g(U, V)) \\ &+ g([U, V], W) - g([V, W], U) + g([W, U], V)\}. \end{aligned}$$

(27) Prove that Levi-Civita connection is torsion-free: $\nabla_V W - \nabla_W V = [V, W]$.

(28) Find parallel transport of $\vec{v}_0$ along the curve γ, for the following choices of $\vec{v}_0$ and γ. In each case, illustrate what parallel transport does along the curve.

(a) γ is a North-South meridian on the surface of S^2 whose tangent vector at the North Pole $(0, 0, 1)$ is $\langle 1, 0, 0\rangle$; $\vec{v}_0$ is the vector $\langle 0, 1, 0\rangle$ based at the North Pole.

(b) γ is a North-to-South meridian on the surface of S^2 whose tangent vector at the North Pole is $\langle 1, 0, 0\rangle$, followed by a South-to-North meridian whose tangent vector at the South Pole is $\langle 0, 1, 0\rangle$; $\vec{v}_0$ is as above. Be clear about what happens to $\vec{v}_0$ as it completes the loop all the way back to the North Pole.

(c) γ is the counterclockwise contour of the geodesic triangle on the surface of S^2 with vertices at the North Pole $(0, 0, 1)$, the Front Pole $(1, 0, 0)$ and the East Pole $(0, 1, 0)$; $\vec{v}_0 = \langle 1, 0, 0\rangle$ is tangential to the side of the triangle leaving the North Pole. Be clear about what happens to $\vec{v}_0$ as it completes the loop all the way back to the North Pole.

(d) γ is the counterclockwise contour of some other geodesic triangle on the surface of S^2 with one vertex at the North Pole $(0, 0, 1)$; $\vec{v}_0 = \langle 1, 0, 0\rangle$ is tangential to the side of the triangle leaving the North Pole. Be clear about what happens to $\vec{v}_0$ as it completes the loop all the way back to the North Pole.

(29) Material covered in this homework assignment has several very deep concepts. *Take the time to reflect on what they actually mean.* Take the time to think about the fact that you may not be able to keep your "reference frame" constant (parallel) and that you may have to compensate for the fact that the "reference frame" is changing. (In plain words: think about trying to measure something in a lab and your measuring device keeps changing on you!) Take the time to think about the idea of parallel transport; do you see the qualitative idea of curvature manifesting itself in final answers to problems like (28c)? "Bonus points" if you can connect all of this to geodesics. Bottom line? Articulate a couple of paragraphs on the subject; they need to be in plain words, full sentences, and all of that. Type them up (LaTeX anyone?) and save the file. You will find it handy for the essay at the end of the semester. Trust me.

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(30) (a) Show that the map $\flat$ introduced in Lecture 4 is $1 - 1$.

(b) Let ω be a covector field. Show that the vector field $\sum_{ij}(g^{ij}\omega(\partial_j))\partial_i$ is invariant under the change of coordinates.

(31) Let $\sharp$ be the map introduced in Lecture 4 and addressed above in Problem (30b). Show that

$$(\omega_1, \omega_2) \mapsto g(\omega_1^\sharp, \omega_2^\sharp)$$

defines an inner-product on covector fields.

(32) Show the following features of the maps $\sharp$ and $\flat$ with respect to a metric g. Whenever you see L take it to be a $(1, 1)$-tensor.

(a) $(V, \omega) \mapsto g(L(\omega^\sharp), V)$ is a bilinear mapping taking a vector field and a covector field to a scalar field.

(b) $\omega \mapsto L(\omega^\sharp)_\flat$ is a linear map taking covector fields to covector fields.

(33) (a) Let T be a $(2,1)$-tensor. Show that $\mathbf{T}(U,V,W) = g(T(U,V),W)$ is a $(3,0)$-tensor.

(b) In analogy to the above, formulate a correspondence between $(k,1)$-tensors and $(k+1,0)$ tensors. Also, find an explicit formula for the inverse correspondence.

(c) Let T be a $(2,1)$-tensor. Show that $\mathbf{T}(U,V,\omega) = \omega(T(U,V))$ is a multilinear map taking 2 vector fields and one covector field to a scalar field.

(d) Explain how one can think of $(k,1)$-tensors and multilinear maps taking k vector fields and one covector field to a scalar field.

(e) Propose a way of think about $(k,2)$-tensors or, more generally, (k,i)-tensors.

(34) (a) Let T be a $(3,0)$-tensor field. Let T_{ijk} and $T_{i'j'k'}$ be the components with respect to two different sets of coordinates (let's say x's and y's). Find a formula relating $g_{i'j'}$ to g_{ij}. (Compare your answer to what you got in the homework problem 13b.)

(b) Repeat the above for a $(2,1)$-tensor field T and its components T^k_{ij} and $T^{k'}_{i'j'}$.

(c) Propose (but do not bother to prove!) a relationship between the components of more general (k,i)-tensors.

(35) Let L be a $(1,1)$-tensor.

(a) Show that the quantity $\mathrm{Tr}(L) = \sum_i L^i_i$ is independent of the choice of coordinates. (You may want consult problem (34).)

(b) Let $\{\partial_j\}$ is orthonormal with respect to the Euclidean inner-product $\langle .,. \rangle$. Show that $\mathrm{Tr}(L) = \sum_i \langle L\partial_i, \partial_i \rangle$.

(36) Let T be a $(2,0)$-tensor, and let (as in Problem (33b)) L be the corresponding $(1,1)$-tensor:

$$T(V,W) = g(L(V),W)).$$

Show that $\mathrm{Tr}(L) = \sum_{ij} g^{ij} T_{ij}$.

(37) Find expressions for the gradient of a function f, the divergence of a vector field X and the Laplacian of a function f in standard

- polar;
- cylindrical;
- spherical

coordinates. A point of clarification: when dealing with the divergence of a vector field X in, say, spherical coordinates assume that X takes the form of $X = X^r\partial_r + X^\theta\partial_\theta + X^\phi\partial_\phi$ and have your expression depend on X^r, X^θ, X^ϕ and their derivatives.

(38) Let f be a spherically symmetric function on $\mathbb{R}^n$, i.e. let f be a function on $\mathbb{R}^n$ which depends only on $r = |x|$. Show that Euclidean Laplacian satisfies $\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r}\frac{\partial f}{\partial r}$.

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(39) Let T be a $(2,1)$-tensor. Prove the following formula.

$$(\nabla_{\partial_i} T)^l_{jk} = \partial_i (T^l_{jk}) + \textstyle\sum_m \Gamma^l_{im} T^m_{jk} - \sum_m \Gamma^m_{ij} T^l_{mk} - \sum_m \Gamma^m_{ik} T^l_{jm}.$$

(40) Use the property $\nabla_V W - \nabla_W V = [V, W]$ and the definition of the bracket to show that $\nabla^2 f$ is symmetric.

(41) Use Theorem 12 from lecture notes to show that

$$\langle \nabla_W \mathrm{grad}(f), V\rangle = \nabla^2 f(V, W)$$

for all V and W.

(42) How does $\nabla_V \nabla_W T$ relate to $\nabla_W \nabla_V T$ if T is

(a) a covector field?

(b) a covariant 2-tensor field?

(c) a $(1,1)$-tensor field?

(43) Compute the non-vanishing curvature components of

(a) the 2-dimensional sphere of radius ϱ in $\mathbb{R}^3$;

(b) the upper half-plane model of the hyperbolic plane.

Important: Take a photo of your answers you will need them for the following assignment!!!!

(44) *Take the time to think* about the concept of curvature tensor. Developing a thorough understanding of curvature is a long-term process. Nonetheless, you should attempt to articulate to yourself – in as plain words as possible – the relationship between the lack of symmetry of Hessian and features of parallel transport, and the relationship between features of parallel transport and the qualitative idea of curvature. (For the latter you may want to remind yourself of the last essay and what you wrote regarding connections to problem (28c).) Anyhow, write a paragraph or so about this: typed up, full sentences, file saved for future, the whole nine yards. You know that Iva's big on these essay things.

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(45) Compute the following.

(a) The curvature tensor, the sectional curvature and the Ricci curavture of the sphere of radius ϱ in $\mathbb{R}^{n+1}$.

(b) The curvature tensor, the sectional curvature and the Ricci curavture of the unit disk $|\mathbf{x}| < 1$ with the metric $ds^2 = \frac{4K^2}{(1-|\mathbf{x}|^2)^2}|d\mathbf{x}|^2$.

(46) Let π be a 2-dimensional subspace of the tangent space at a point of an n-dimensional manifold (M, g). Show that the value of

$$\frac{R(X, Y, Y, X)}{|X|_g^2 |Y|_g^2 - g(X, Y)^2}$$

is independent of the choice of a basis $\{X, Y\}$ of π.

(47) Use curvature symmetries to show the following identity. The derivatives are to be evaluated at $(t, s) = (0, 0)$.

$$\begin{aligned} R(W, V, X, Y) = &\frac{1}{6}\partial^2_{ts} R(V + tX, W + sY, W + sY, V + tX) \\ &- \frac{1}{6}\partial^2_{ts} R(V + tY, W + sX, W + sX, V + tY). \end{aligned}$$

(Note: what appears on the right-hand-side are sectional curvatures; the point of the problem is that sectional curvatures determine the full curvature tensor.)

(48) (a) Let V and W be two tangent vectors. Prove that the value of

$$\sum_i R(\partial_i, V, W, \partial_i)$$

is independent of the choice of orthonormal basis $\{\partial_i\}$ of tangent vectors.

(b) Prove the coordinate formula $\mathrm{Ricci}(V, W) = \sum_{ij} g^{ij} R(\partial_i, V, W, \partial_j)$.

(49) Show that on 3-dimensional manifolds Ricci curvature determines sectional curvature. Hint: For orthonormal bases $\{e_1, e_2, e_3\}$ we have that $\mathrm{Ricci}(e_1, e_1) = \mathscr{K}(\mathrm{Span}(e_1, e_2)) + \mathscr{K}(\mathrm{Span}(e_1, e_3))$. What can you say about $\mathrm{Ricci}(e_2, e_2)$ and $\mathrm{Ricci}(e_3, e_3)$? Can you get a solve $\mathscr{K}(\mathrm{Span}(e_1, e_2))$ as a linear combination of Ricci curvatures? Play with this!

(50) *Take the time to think* about the analogy between Hooke's Law / differential equation for the harmonic oscillator and the Jacobi equation for geodesic sprays. Write a paragraph or so about this: typed up, full sentences, file saved for future, the whole nine yards. Try to be non-technical, although I understand that at this point that can be difficult. Physics people: if you pouch this "essay" I am sending you back to Olin!

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(51) Verify the formulae (see Theorem 20) for the circumference of small circles and the surface area of small disks on 2-dimensional spheres through a direct computation.

(52) Do this next problem with a partner or two!!

- Acquire four or five sheets of paper tiled with regular triangles. You should get them from me, but I am sure they'll be more floating around in the math department or SQRC. Get a hold of some scissors and tape. Then cut some of your sheets into individual triangles. Note that taping the triangles together so that at each vertex exactly six of them meet reproduces our (Euclidean) sheet of paper.

- Now tape the triangles together so that at each vertex exactly five of the triangles meet. What surface are you reproducing? Write a paragraph long explanation on how this exercise relates to Theorem 20. You may want to combine what you have to say into your essay – see Problem 55 below.

- Now tape the triangles together so that at each vertex exactly seven of the triangles meet. The extremely ruffly surface you are creating is a discretized version of a particular geometry. Which geometry is that?

(53) (a) Write up the proof of Theorem 22 in all the gory details.

(b) Compute the asymptotic formula for the "surface area" of the geodesic sphere $S^M_{\mathbf{r}}(P)$ of small radius $\mathbf{r}$ centered at the point $P \in M$.

(54) Prove that

$$\mathrm{vol}(S^{2n}) = \frac{(4\pi)^n (n-1)!}{(2n-1)!}, \quad \mathrm{vol}(S^{2n+1}) = 2\frac{\pi^{n+1}}{n!}$$

by using the volume element formula (4) from Lecture 8.

(55) Essay time! Prepare an essay in which you are explaining different concept(s) of curvature (sectional, Ricci, scalar) to somebody with basic understanding of mathematics (e.g somebody at the level of basic calculus). Illustrate the subject of Riemannian geometry on the example of comparison theorems. I expect at least a page, but it may take more. Type the words up, but draw the pictures in.

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(56) Let $\langle .,. \rangle$ be a Lorentzian inner-product on a vector space V of dimension $n+1$. Let $\vec{u}$ be a fixed non-zero (space-like, time-like or null) vector. Investigate the orthogonal complement of $\vec{u}$ following the guidelines given below.

(a) Show that the linear map $\vec{v} \mapsto \langle \vec{u}, \vec{v} \rangle$ is of rank 1 and nullity n; conclude that the orthogonal complement

$$\vec{u}^{\perp} = \{\vec{v} \in V \,\big|\, \langle \vec{v}, \vec{u} \rangle = 0\}$$

is a subspace of dimension n.

(b) The restriction of the inner-product $\langle .,. \rangle$ to the orthogonal complement $\vec{u}^{\perp}$ is clearly still symmetric and bilinear. Is it still non-degenerate? If so, determine its signature. (Note: your answer will vary depending on whether $\vec{u}$ is space-like, time-like or null.) (If you get confused you may want to compare with the specific numeric problem below.)

(57) Consider $\mathbb{R}^{1+1}$ with the Minkowski inner-product

$$\mathfrak{m}(\vec{x},\vec{y}) = \mathfrak{m}(\langle x^1, x^2\rangle, \langle y^1, y^2\rangle) = -x^1y^1 + x^2y^2.$$

For each of the vectors

$$\vec{x}_1 = \langle 2, 1\rangle, \quad \vec{x}_2 = \langle 1, 1\rangle, \quad \vec{x}_3 = \langle 1, 2\rangle$$

in $\mathbb{R}^{1+1}$:

(a) Determine if the vector if space-like, time-like or null.

(b) Compute the orthogonal complement of the vector.

(c) Sketch the vector and its orthogonal complement. Make sure your sketch is well labeled.

(58) You are running on a relativistic train with the constant relative speed of v_1; the train on the other hand travels straight at the constant relative speed of v_2 relative to the ground. Prove that you are moving at the constant relative speed of

$$v = \frac{v_1 + v_2}{1 + v_1 v_2}$$

relative to the ground.

(59) A train of restlength of 300 meters travels a straight stretch of track past a station of restlength 200 meters; the train passes the station at constant relative speed of $v = \sqrt{8}/3 \approx .94$.

(a) How long is the station from the perspective of the station master? How long is the train from the perspective of the stationmaster?

(b) According to the stationmaster, how long did it take for the train to pass the station?

(c) How long is the station from the perspective of the train conductor? How long is the train from the perspective of the train conductor?

(d) According to the train conductor, how long did it take for the train to pass the station?

If it helps with diagrams and such, feel free to assume that the conductor is in the front of the train and that the stationmaster is at the far end of the station.

(60) The Conservation Law of for Energy-Momentum implies that in a collision the total incoming energy-momentum vector equals the total outgoing energy-momentum:

$$\sum_{i=1}^{n} \vec{P}_{i,\text{in}} = \sum_{j=1}^{m} \vec{P}_{j,\text{out}}.$$

(Here $\vec{P}_{i,\text{in}}$ are energy-momentum vectors of n particles going into the collision, and $\vec{P}_{j,\text{out}}$ are energy-momentum vectors of m particles leaving the collision.) Assume you have two blobs of relativistic putty of mass m_1 and m_2. Suppose the two blobs are about to collide with relative speed v, and stick together.

(a) Use the Conservation of Energy-Momentum to show that the new blob will have the mass of

$$m = \sqrt{m_1^2 + \frac{2}{\sqrt{1-v^2}} m_1 m_2 + m_2^2}.$$

(b) Is the mass of the new blob bigger or smaller than the two masses m_1 and m_2 put together? Use the above to justify your answer.

(61) This problem investigates taking the trace in the presence of a general non-degenerate (and thus not necessarily positive-definite) inner-product. Throughout the problem $\{\partial_i\}$ denotes a (pseudo-)orthonormal basis, while $\epsilon_i = \langle \partial_i, \partial_i \rangle$.

(a) Let L be a linear map; recall that $\operatorname{Tr}(L) = \sum_i L_i^i$. Show that for pseudo-orthonormal bases $\{\partial_i\}$ one also has:

$$\operatorname{Tr}(L) = \sum_i \epsilon_i \langle L\partial_i, \partial_i \rangle.$$

(b) Show that the value of $\sum_i \epsilon_i R(\partial_i, V, W, \partial_i)$ is independent of the choice of a pseudo-orthonormal basis $\{\partial_i\}$. This common value defines the Ricci tensor, $\operatorname{Ricci}(V, W)$.

(c) How would you go about defining the scalar curvature Scal of a pseudo-Riemannian manifold? State and prove the corresponding result.

(62) *Take the time to reflect* on the derivation of the Einstein vacuum equation Ricci $= 0$. Articulate *your own account* of the story; type the thing up, save the file, turn a print-out in, business as usual.

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(63) Find Ricci curvature components of $g = -E(r)dt^2 + F(r)dr^2 + r^2 g_{S^2}$.

(64) As discussed in Lecture 10, equatorial trajectories of massive particles / timelike geodesics have the effective potential of

$$\mathcal{V}_{\text{eff}}(r) = -\tfrac{M}{r} + \tfrac{J^2}{2r^2} - \tfrac{MJ^2}{r^3}$$

and the total "energy" of

$$\tilde{\mathscr{E}} = \tfrac{1}{2}\dot{r}^2 + \mathscr{V}_{\text{eff}}(r).$$

For the rest of this exercise you should use explicit values of $M = 1$ and $J = 5$. *You are expected to use Mathematica or some other software to get the values asked for below.*

(a) Sketch $\mathscr{V}_{\text{eff}}(r)$. *Note:* It is impossible to sketch this to scale.

(b) Does $\mathscr{V}_{\text{eff}}(r)$ have any intercepts, (local) maximum and minimum values? If so, compute them and add them to your sketch.

(c) What kind of a trajectory does a particle with energy $\tilde{\mathscr{E}} = -0.02$ have? Is there a minimum value of r (call it r_{min}) along this trajectory? What about the maximum value r_{max} of r? In other words, how close and how does this particle get relative to the "star" centered at $r = 0$?

(d) Repeat the above for the following energy levels:

$$\tilde{\mathscr{E}} = -0.01, \quad \tilde{\mathscr{E}} = 0, \quad \tilde{\mathscr{E}} = 0.01, \quad \tilde{\mathscr{E}} = 0.05,$$
$$\tilde{\mathscr{E}} = 0.1, \quad \tilde{\mathscr{E}} = 0.1516, \quad \tilde{\mathscr{E}} = 0.2, \quad \tilde{\mathscr{E}} = 1.$$

(e) What is the perihelion advance predicted by the theory presented in Lecture 10?

(65) This exercise guides you through a more thorough analysis of the perihelion advance and related phenomena. As usual, assume equatorial trajectories.

(a) Combine the two conversation laws:

$$\tilde{\mathscr{E}} = \tfrac{1}{2}(\dot{r})^2 - \tfrac{M}{r} + \tfrac{J^2}{2r^2} - \tfrac{MJ^2}{r^3}, \quad J = r^2\dot{\theta},$$

into a formula for $\frac{d\theta}{dr}$. Your formula should only involve $\tilde{\mathscr{E}}$, M, J and r.

(b) Use the above to justify the perihelion advance formula

$$\Delta\theta = -2\pi + J\sqrt{2}\int_{r_{\text{min}}}^{r_{\text{max}}} \left(\tilde{\mathscr{E}} + \frac{M}{r} - \frac{J^2}{2r^2} + \frac{MJ^2}{r^3}\right)^{-1/2} \frac{dr}{r^2}.$$

Here r_{min} (respectively, r_{max}) denotes the value where the orbit corresponding to energy level $\tilde{\mathscr{E}}$ is closest (respectively, furthest) from the "star".

(c) Determine the exact value of the perihelion advance for all applicable orbits from the homework problem (64). *Note:* Your answers should be somewhat approximate to the value for perihelion advance you got in problem (64).

(d) Reflect on the integral formula given above, and modify it to a formula which can tell us how many times a particle in a crash or fly-by orbit goes around "the star".

(e) Apply the formula you just developed to all applicable trajectories from the homework problem (64).

(f) What patterns do you see in all the numerical answers you got? In particular, what patterns do you see as you increase the energy level $\tilde{\mathscr{E}}$? Bonus points if you can make a comparison to the Newtonian theory.

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(66) Recall from class (!) that the equatorial photon orbits / null geodesics of Schwarzschild space-time have the effective potential of

$$\mathscr{V}_{\text{eff}}(r) = \frac{J^2}{2r^2}\left(1 - \frac{2M}{r}\right)$$

and the total "energy" of

$$\tilde{\mathscr{E}} = \tfrac{1}{2}\mathscr{E}^2 = \tfrac{1}{2}\dot{r}^2 + \mathscr{V}_{\text{eff}}(r).$$

Next, suppose that a photon is emitted from a (stationary) light source located at the point where $(r, \theta) = (r_0, 0)$ with $2M < r_0 < 3M$. Suppose furthermore that the photon is emitted in the direction which forms angle ψ with the line $\theta = 0$.

(a) What does your common sense tell you about the fate of this photon? What happens to it when $\psi \approx 0$? What about $\psi \approx \pi$? What about $\psi \approx \frac{\pi}{2}$? $\psi \approx \frac{\pi}{4}$?

(b) Explain why at the light source we must have

$$r_0\dot{\theta} = \tan\psi \cdot \frac{\dot{r}}{\sqrt{1 - \frac{2M}{r_0}}}.$$

(c) Combine the above identities to justify the following relationship between r_0, ψ and $\tilde{\mathscr{E}}$:

$$\tan^2(\psi) = \frac{\mathscr{V}_{\text{eff}}(r_0)}{\tilde{\mathscr{E}} - \mathscr{V}_{\text{eff}}(r_0)}.$$

(d) Based on the graph of the effective potential, what can you say about the minimum value of $\tilde{\mathscr{E}}$ for which the photon does *not* to crash into $r = 0$?

(e) What is the maximum value of ψ for which the photon is *not* on a crash-orbit? (*Note*: The value you are looking for is really a formula in r_0 and M.) (*Also note*: This is highly related to the first question in this problem.)

(f) How much light escapes to infinity? How does your answer to this question change as r_0 changes? Most importantly, what happens as $r_0 \to 2M$?

(67) *Take the time to reflect* on this whole black hole business. Articulate *your own account* of the story; type the thing up, save the file, turn a print-out in, business as usual.

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(68) Take the time to read through all of your itty-bitty essays and all of the feedback I have given you on them. Now is a perfect time to combine them all into one nice-to-read document. Note: I will use this essay heavily when determining final grades. By the way, bonus points if you can work FRW cosmology into your essay!

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