Chapter 4

Fourier transforms

4.1 The complex inner product space $l^2(\mathbb{Z})$

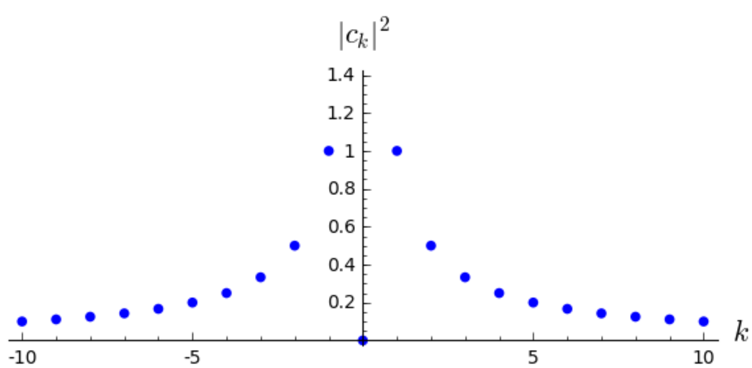
The (periodic) Fourier transform takes a function u in $\mathcal{L}^2([-L, L])$ and gives out an infinite list c_k of complex numbers. In order to study this process, we view a list of numbers c_k as a function where the inputs are integers and the outputs are complex numbers. We use the phrase "sequence" to describe these lists, but we think of them as functions with domain $\mathbb{Z}$.

For instance, in Example 3.19 we computed the Fourier coefficients for the function $u(x) = x$ to be the sequence

$$c_k = \begin{cases} (-1)^k \frac{L}{k\pi} i & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases} \tag{4.1}$$

LFT-line

We visualize this sequence by plotting $|c_k|$ versus k as follows. (For the purposes of plotting, we set $L = \pi$.)



The `Sage` code used to generate this plot is the following.

```
# counter k ranges from -n to n
n=10

# create a collection of points called "data"
data = [(k, k) for k in [-n..n]]

# fill in the values for negative k
for k in [-n..-1]:
    data[k+n] = (k,-1/k)

# fill in the values for positive k
for k in [1..n]:
    data[k+n] = (k,1/k)

# fill in value for k=0
data[n] = (0,0)

# display plot
list_plot(data,figsize=[6,3], size=30,axes_labels=("$k$","$|
   c_k|^2$"),ymax=1.4)
```

We define a complex inner product of two sequences $a = \{a_k\}$ and $b = \{b_k\}$ by

$$\langle a, b\rangle = \sum_{k=-\infty}^{\infty} \overline{a_k} b_k.$$

Using this inner product we define the norm of a sequence by

$$\|a\|^2 = \langle a, a\rangle = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

We define the vector space $l^2(\mathbb{Z})$ to be the collection of all sequences with finite norm. This vector space is called ***little el two***, though it is typically written symbolically and not with the name spelled out like this.

The sequence $c = \{c_k\}$ given by is in $l^2(\mathbb{Z})$ with norm given by

$$\|c\|^2 = \sum_{k\neq 0} \frac{L^2}{\pi^2 k^2} = \frac{2L^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{L^2}{3}.$$

This was computed using the Sage code

```
var('k')
show(sum(1/k^2, k,1,infinity))
```

Exercise 4.1.1. *Let c be the sequence of Fourier coefficients arising from the function $u(x) = x^2$, as computed in Example 3.20.*

1. *Make a plot to visualize the function c.*

2. *Compute $\|c\|^2$. How does this norm compare to the norm of u in $\mathcal{L}^2([-L, L])$? (They should differ by a factor of $2L$. Do they?)*

4.2 Fourier series as a linear transformation

The process of taking a function u in $\mathcal{L}^2([-L, L])$ and producing a sequence in $l^2(\mathbb{Z})$ can be viewed as a function, which we call f (for "Fourier"). The domain of f is $\mathcal{L}^2([-L, L])$, meaning that the inputs to the function f are themselves functions. The codomain is $l^2(\mathbb{Z})$, meaning that the outputs are also functions; in fact, the outputs are sequences.

Notice that both the domain and codomain of f are vector spaces. A function where both the domain and codomain are vector spaces is called a ***transformation***. Our particular transformation f we call the ***little Fourier transform***. We call it "little" because the outputs are in "little el two." (In §**??** below we define the "big" Fourier transform, which has outputs in "big el two".) • **section reference needed**

Let's introduce some notation. Given a function u in $\mathcal{L}^2([-L, L])$, we denote by $f(u)$ the sequence of Fourier coefficients. The k^{th} number in the sequence $f(u)$ we denote by $f(u)_k$. For example, if the function u is given by the formula $u(x) = x$, then $f(u)$ is the sequence given by the formula

$$f(u)_k = \begin{cases} (-1)^k \frac{L}{k\pi} i & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

More generally, we have a formula for $f(u)$ that comes from (3.12):

$$f(u)_k = \frac{1}{2L} \int_{-L}^{L} e^{-i\frac{k\pi}{L}y} u(y)\, dy. \tag{4.2}$$ `LFT-forward`

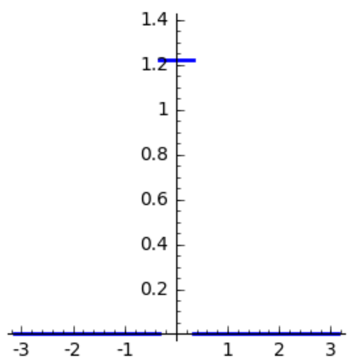
`ormalized-pulse` **Example 4.1.** *For any positive number $a < L$ let u be the function in $\mathcal{L}^2([-L, L])$ given by*

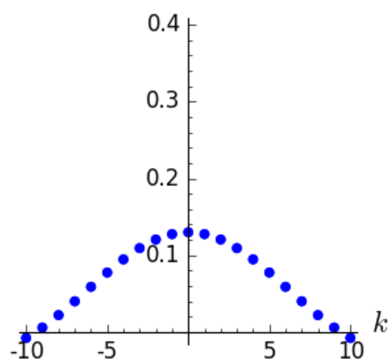
$$u(x) = \begin{cases} 0 & \text{if } |x| > a \\ \frac{1}{\sqrt{2a}} & \text{if } |x| \leq a. \end{cases}$$

*We call the function u the **normalized pulse** because $\|u\| = 1$. We compute the little Fourier transform of u to be given by*

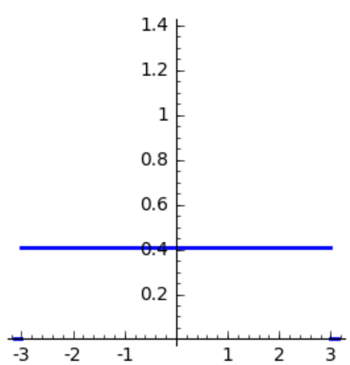
$$f(u)_k = \begin{cases} \dfrac{1}{\sqrt{2a}\, k\pi} \sin\left(\dfrac{k\pi a}{L}\right) & \text{if } k \neq 0 \\[2ex] \dfrac{\sqrt{2a}}{2L} & \text{if } k = 0. \end{cases}$$

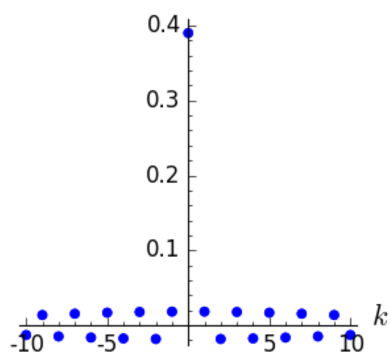
Since $f(u)$ is real-valued, we can plot $f(u)_k$ versus k. For the purposes of plotting, set $L = \pi$. When $a = 1/3$, the plots of u and $f(u)$ are the following.





When $a = 3$ the plots of u and $f(u)$ are the following.





The Sage code used to generate this second plot is the following.

```
# counter k ranges from -n to n
n=10
a=1/2
```

```
# hack define the normalized function
u1(x) = 1/sqrt(2*a)
u0(x) = 0

# create a collection of points called "data"
data = [(k, k) for k in [-n..n]]

# fill in the values for negative k
for k in [-n..-1]:
    data[k+n] = (k,sin(k*a)/(sqrt(2*a)*k*pi) )

# fill in the values for positive k
for k in [1..n]:
    data[k+n] = (k,sin(k*a)/(sqrt(2*a)*k*pi))

# fill in value for k=0
data[n] = (0,sqrt(2*a)/(2*pi))

# construct plots
xplot = plot(u1, (x,-a, a), thickness=2) + plot(u0, (x,-pi,
    -a), thickness=2)+ plot(u0, (x,a, pi), thickness=2) +
    plot(1.4,(x,0,1), thickness=0)
kplot = list_plot(data, size=30,axes_labels=("$k$",""),ymax
    = .4)

# display both plots side-by-side
mainplot = graphics_array((xplot,kplot))
show(mainplot, figsize = [6,3])
```

Notice that when a is small, then the pulse $u(x)$ is concentrated near $x = 0$, while values of k for which the transform $f(u)_k$ is (relatively) larger is spread out. Conversely, when a is large, then the pulse $u(x)$ is spread out, while the values of k for which $f(u)_k$ is larger is more concentrated. Note also the different vertical scales on the two plots.

Remark 4.2. *The previous example illustrates a more general feature of Fourier transforms.*

- *When k is small, the coefficients $f(u)_k$ tell us about the large scale structure of the function u. Thus if $f(u)_k$ is large when k is small, then there are large-scale structures present in the function u.*

  Recall that the frequency of the standing waves is dependent on the size of k, and that small values of k correspond to slower frequency oscillations. Thus that part of u determined by $f(u)_k$ with k small is often called the "low frequency part" of the function u.

- *Conversely, the coefficients $f(u)_k$ with k large correspond to fine-scale* *structure in the function u. If $f(u)_k$ is large when k is relatively large, then u has fine-scale structure present.*

Since larger values of k correspond to standing waves with faster oscillations, the part of u determined by $f(u)_k$ with k large is called the "high frequency part" of the function u.

Of course, there isn't really a sharp cutoff between "small" and "large" values of k, so the distinctions between high and low frequency are a bit arbitrary. But they are still a useful way to talk about a function.

The interpretation of $f(u)_k$ as the large and small frequency parts has an important application in physics and engineering. If you only care about some length scales (or frequency scales) below some threshold, then you can simply set $f(u)_k$ to zero for k larger than the corresponding threshold. This gives an approximation of u that is "good enough" for the situation at hand.

Finally, note that in this interpretation of the little Fourier transform the trigonometric functions $\cos\left(\frac{k\pi}{L}x\right)$ and $\sin\left(\frac{k\pi}{L}x\right)$ are considered to be "purely" at a single length scale because they are themselves the shapes of a single standing wave with periodic boundary conditions.

We now list three "mapping properties" of the little Fourier transformation. The first mapping property is that it is ***linear*** in the sense that

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \tag{4.3}$$

for all functions u, v and scalars α, β. This is a simple consequence of the formula (4.2).

The second mapping property is that the little Fourier transform is ***invertible***, meaning that there is a transform f^{-1} that takes in sequences in $l^2(\mathbb{Z})$ and gives out functions in $\mathcal{L}^2([-L, L])$ in such a way that

$$f^{-1}(f(u)) = u \text{ for all functions } u \text{ in } \mathcal{L}^2([-L, L]) \tag{4.4}$$

LFT-first-inver

and

$$f(f^{-1}(c)) = c \text{ for all sequences } c \text{ in } l^2(\mathbb{Z}). \tag{4.5}$$

LFT-second-inve

The formula for f^{-1} is given by

$$f^{-1}(c) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x}. \tag{4.6}$$

LFT-inverse

The first inversion property (4.4) follows from the fact that the Fourier series of a function converges to the function (except at jump points, which we can ignore). If we apply (4.6) to $f(u)$ we get

$$f^{-1}(f(u)) = \sum_{k=-\infty}^{\infty} f(u)_k e^{i\frac{k\pi}{L}x},$$

which we know converges to $u(x)$. The second inversion property (4.5) is discussed in more detail in §**??**.

• section reference needed

The third mapping property is that the little Fourier transform is an ***isomorphism***. This means that there is a direct relationship between the norm of a function u and the norm of its transform $f(u)$. In particular, we have

$$\|f(u)\|^2 = \frac{1}{2L}\|u\|^2. \tag{4.7}$$

`LFT-isomorphism`

This means that (up to an overall multiplicative factor) the little Fourier transform of a function is the "same size" as the function itself. The proof of this fact is discussed in §**??**.

• section reference needed
• needed

Exercise 4.2.1. *take previous FS examples and look at different parts of the series. What do low k terms do? What do high k terms do?*

Use a modification of the following code.

```
var('k')
start=20
stop =30

f(x) = (4/pi)* sum(sin((2*k+1)*x)/(2*k+1),k,start,stop)

plot( f(x), (x,0,4*pi), figsize = [4,3])
```

Exercise 4.2.2. *Set $L = 1$ and consider function*

$$u(x) = \begin{cases} (\#)(a - |x|) & \text{if } |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

1. *Find the number $(\#)$ so that $\|u\| = 1$.*
2. *Compute $f(u)$*
3. *Make a plot of $f(u)_k$ for large/small values of a.*
4. *Now set $a = 1/2$. Make a plot of the partial sum*

   $$\sum_{|k|\le 5} f(u)_k e^{ik\pi x}.$$

   How closely does this describe u?

5. *Still with $a = 1/2$. Make a plot of the partial sum*

$$\sum_{5 \le |k| \le 100} f(u)_k e^{ik\pi x}.$$

What parts of the function u does this describe?

4.3 Properties of little Fourier transform

We now establish two more properties of the little Fourier transform. We won't use these properties directly in this course. Rather, we are demonstrating the properties for the little Fourier transform in preparation for our upcoming discussion of the "big" Fourier transform.

The first property is the "multiplication property" of the little Fourier transform. Suppose we have two functions u, v in $\mathcal{L}^2([-L, L])$; consider the two sequences $f(u)$ and $f(v)$. We can multiply these two lists of numbers in a very simply way: just multiply $f(u)_k f(v)_k$ for each k. This raises an interesting question? What operation on the original functions u and v does this multiplication of $f(u)_k f(v)_k$ correspond to?

To answer this question, we use the formula (4.2) which implies, provided we periodically extend u, v, that

$$\begin{aligned} f(u)_k f(v)_k &= \frac{1}{(2L)^2} \int_{-L}^{L} \left[\int_{-L}^{L} u(x) v(y) e^{-i\frac{k\pi}{L}(x+y)}\, dx \right] dy \\ &= \frac{1}{2L} \int_{-L}^{L} \left[\frac{1}{2L} \int_{-L}^{L} u(z) v(x-z)\, dz \right] e^{i\frac{k\pi}{L}x}\, dx \end{aligned}$$

Notice that this is simply the little Fourier transform of the stuff in square brackets, which is a function of x.

The quantity in square brackets above is called the ***convolution product*** of the functions u and v, and is given the symbol $u * v$. Thus the convolution product is defined by the formula

$$(u * v)(x) = \frac{1}{2L} \int_{-L}^{L} u(z) v(x-z)\, dz.$$

The convolution product is a way to multiply two functions together and obtain another function, but it is very different from the "usual" way of multiplying functions. (The "usual" way of multiplying functions is also called "pointwise multiplication.") The ***multiplication property*** of the little Fourier transform is that the convolution product of two functions is taken to the pointwise product of the transforms, and is expressed in the following formula:

$$f(u * v)_k = f(u)_k f(v)_k.$$

The convolution product of two functions is perhaps a little bit difficult to interpret. Before we attempt to do so, let's consider two examples.

Example 4.3. *Fix two positive integers $p \neq q$ and let u, v be the functions given by*

$$u(x) = \cos\left(\frac{p\pi}{L}x\right) \qquad v(x) = \cos\left(\frac{q\pi}{L}x\right)$$

We compute the convolution

$$\begin{aligned}(u * v)(x) &= \frac{1}{2L}\int_{-L}^{L} u(z)v(x-z)\,dz \\ &= \frac{1}{2L}\int_{-L}^{L} \cos\left(\frac{p\pi}{L}z\right)\cos\left(\frac{q\pi}{L}(x-z)\right)\,dz \\ &= 0\end{aligned}$$

Notice that $f(u)_k = 0$ unless $k = \pm p$ and $f(v)_k = 0$ unless $k = \pm q$. Thus it is easy to see from the multiplication property of little Fourier transform that we must have $f(u)_k f(v)_k = 0$ for each k. This is consistent with our calculation that $u * v = 0$.

We interpret this example to mean that the convolution of two functions with totally different length (or frequency) scales is zero.

Example 4.4. *For each local x_0 in the interval $[-L, L]$ and each small number a let u_{x_0} be the pulse function of size A concentrated at x_0, given by*

$$u_{x_0}(x) = \begin{cases} 0 & \text{if } |x - x_0| > a \\ A\frac{L}{a} & \text{if } |x - x_0| \leq a. \end{cases}$$

Let v be any function in $\mathcal{L}^2([-L, L])$. We compute

$$(u_{x_0} * v)(x) = \frac{A}{2a}\int_{(x-x_0)-a}^{(x-x_0)+a} v(y)\,dy.$$

If we take a to be very small, then we have

$$(u_{x_0} * v)(x) \approx Av(x - x_0).$$

Thus if we take a function concentrated near x_0 and convolve it with v, the result is approximately the function v, but shifted by x_0.

If we think of a generic function u as being approximately built by the sum of pulses (of various strengths) at each location, then we can interpret the convolution product $u * v$ as giving us the sum of the various shifts of v, with each shift having the strength of the corresponding pulse.

The image that emerges from the two examples above is the following:

- On one hand, the convolution product can be seen as a measure of the extent to which the various frequencies (or length scales) of the two functions overlap. This is made clear by the multiplication property formula.
- On the other hand, the convolution product can be understood as a "shift to each point and scale by the size of the pulse at that point" which I think of as a "maximally dispersed multiplication."

The second property of the little Fourier transform that we present in this section is the ***derivative property***, which states that

$$f(u')_k = i\left(\frac{k\pi}{L}\right) f(u)_k. \tag{4.8}$$ LFT-derivative

We can interpret this formula to mean that the process of taking a derivative of a function corresponds to multiplying each of the frequency amplitudes by a number that is small for large scale (low frequency) structures and large for small scale (high frequency) structures. This makes sense because small scale structures of functions have, by definition, fluctuations on shorter length scales and thus contribute more to the derivative of a function.

Exercise 4.3.1. *Let u and v be functions in $\mathcal{L}^2([-L, L])$ that are periodically extended to functions on the entire real line. Show that $u*v = v*u$, meaning that the convolution product is commutative.*

Exercise 4.3.2 (Relies on Exercise 3.7.1)**.** *Compute the convolution of square wave and triangle wave.*

Exercise 4.3.3. *Verify that* (4.8) *holds.* [*Hint: use* (4.2).]

4.4 The (big) Fourier transform

The little Fourier transform is used to study functions with domain $[-L, L]$. We now develop tools for studying functions with domain all of $\mathbb{R}$. Our approach is to take the theory for $\mathcal{L}^2([-L, L])$ and take the limit as $L \to$

∞. Before we get in to technical details, let's do a rough (or "formal") calculation.

Suppose we have a function u that is in $\mathcal{L}^2([-L, L])$ for any $L > 0$. The formula (4.4) means that for most values of x we have

$$u(x) = \sum_{k=-\infty}^{\infty} f(u)_k e^{i\frac{k\pi}{L}x} = \sum_{k=-\infty}^{\infty} \left(\frac{f(u)_k}{\pi/L} \right) e^{i\frac{k\pi}{L}x} \frac{\pi}{L}. \tag{4.9}$$ motivate-FT

We interpret the term in round brackets as follows. The numerator $f(u)_k$ represents how much of frequency $k\frac{\pi}{L}$ is present in the function u, while the denominator π/L represents a single "unit" of frequency. Thus the term in round brackets can be viewed as the "amount of frequency per unit frequency".

As L gets very large, the gap π/L between the possible frequencies shrinks towards zero. So it makes sense to think of frequency as a continuous variable rather than a discrete variable. Let ξ be the continuous frequency variable. In the limit $L \to \infty$ we have $\pi/L \to d\xi$ and thus $k\pi/L \to \xi$. Using this, together with the formula for the little Fourier transform, we see that in the limit $L \to \infty$ we have

$$\frac{f(u)_k}{\pi/L} = \frac{1}{2\pi} \int_{-L}^{L} u(y) e^{-i\frac{k\pi}{L}y}\, dy \to \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) e^{-i\xi y}\, dy,$$

This last expression is a function of ξ, which we again interpret as the "normalized" amount of frequency ξ present in the function u.

The discussion in the previous paragraph motivates us to define the "big" ***Fourier transform*** of a function u to be the function $\widehat{u}(\xi)$ defined by

$$\widehat{u}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) e^{-i\xi y}\, dy. \tag{4.10}$$ FT-formula

The Fourier transform $\widehat{u}(\xi)$ tells us how much of frequency ξ is present in a u having domain $\mathbb{R}$. You should think of this as analogous to the Fourier coefficient c_k telling us how much of frequency $k\pi/L$ is in a function u on the domain $[-L, L]$. The key idea is that since u is defined on all of $\mathbb{R}$ and not on an interval of finite width, all frequencies are possible!

We continue our formal calculation as follows. If we discretize the ξ axis by tiny intervals of width $\Delta\xi = \frac{\pi}{L}$, then from (4.9) we have

$$u(x) = \sum_{k=-\infty}^{\infty} \left(\frac{f(u)_k}{\pi/L} \right) e^{i\frac{k\pi}{L}x} \frac{\pi}{L} = \sum_{k=-\infty}^{\infty} \widehat{u}(k\Delta\xi) e^{i(k\Delta\xi)x} \Delta\xi.$$

In the limit as $L \to \infty$ (and thus $\Delta\xi \to 0$), the right hand side is precisely the Riemann sum for the integral

$$\int_{-\infty}^{\infty} \widehat{u}(\xi) e^{i\xi x}\, d\xi.$$

Thus we expect that, for a reasonably nice functions u, to have the formula

$$u(x) = \int_{-\infty}^{\infty} \widehat{u}(\xi) e^{i\xi x}\, d\xi. \tag{4.11}$$ FT-recovery

We should interpret this last equation to mean that we can construct the function u by summing over all frequencies ξ the amount $\widehat{u}(\xi)$ of that frequency present times the periodic function $e^{i\xi x}$ having that frequency. This is analogous to the formula coming from the Fourier series.

We conclude this section with the remark that all of the formal calculations done above can be massaged into proper mathematical statements. However, doing so requires some technicalities that are beyond the scope of this course. You are encouraged to take the complex variables course, and also the real analysis course, before returning to these issues.

HW:FT-unit-pulse ★**Exercise 4.4.1.**

1. *Find a constant C so that the function*

$$u(x) = \begin{cases} C & \text{if } |x| \le a \\ 0 & \text{otherwise} \end{cases}$$

   has norm $\|u\| = 1$.

2. *Compute the Fourier transform of u.*

3. *What happens to $\widehat{u}$ as $a \to \infty$? What happens as $a \to 0$?*

Exercise 4.4.2. *Compute the Fourier transform of the following functions.*

1. $u(x) = e^{-|x|}$

2. $u(x) = e^{-ax^2}$, *where a is some positive constant. [Hint: Complete the square.]*

4.5 Properties of the Fourier transform

In this section we study the Fourier transform from the perspective of a transformation from one inner product space to another. The main point is that the Fourier transform has properties analogous to the properties of the little Fourier transform.

First we define $\mathcal{L}^2(\mathbb{R})$ to be the vector space of all piecewise-smooth functions u with domain $\mathbb{R}$ such that

$$\|u\|^2 = \int_{-\infty}^{\infty} |u(x)|^2 \, dx < \infty.$$

The function u may be real-valued or complex valued. Notice that such functions must decay to zero as $x \to \pm\infty$. Thus whenever we compute using integration by parts we may disregard the boundary term[1].

For such functions we define an inner product by

$$\langle u, v \rangle = \int_{-\infty}^{\infty} \overline{u(x)}\, v(x) \, dx.$$

It is easy to see that this makes $\mathcal{L}^2(\mathbb{R})$ in to a complex inner product space.

The Fourier transform can be viewed as a linear transformation from $\mathcal{L}^2(\mathbb{R})$ to $\mathcal{L}^2(\mathbb{R})$, where the input of the transformation is a function u and the output is $\widehat{u}$. We use the symbol $\mathcal{F}$ for this transformation; thus $\widehat{u} = \mathcal{F}[u]$. Thus (4.10) can also be written

$$\mathcal{F}[u](\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) e^{-i\xi y} \, dy. \tag{4.12}$$ FT-define

Remark 4.5. *There are several conventions about where to put the factor of 2π in the definition of the Fourier transform. When reading a paper or book it is a good idea to check the convention being used.*

We now take a look at several properties of the Fourier transform. The first two properties are linearity and invertibility. It is easy to see from the formula (4.12) that the Fourier transform is ***linear***, by which we mean that

$$\mathcal{F}[\alpha u + \beta v] = \alpha \mathcal{F}[u] + \beta \mathcal{F}[v]$$

for any functions u, v in $\mathcal{L}^2(\mathbb{R})$ and scalars α, β.

[1]Technically, we are using the fact that the Schwartz class is dense in $L^2(\mathbb{R})$.

The formula (4.11) says that if we define

$$\mathcal{F}^{-1}[\widehat{u}](x) = \int_{-\infty}^{\infty} \widehat{u}(\xi) e^{i\xi x} d\xi \tag{4.13}$$

FT-inverse

then

$$\mathcal{F}^{-1}[\mathcal{F}[u]] = u.$$

Thus we see that the Fourier transform is ***invertible***.

The next property is that the Fourier transform is an ***isomorphism***, meaning that the norm of $\mathcal{F}[u]$ is a multiple of the norm of u. To see this we compute, using (4.12) and (4.13), that

$$\begin{aligned}
\|\widehat{u}\|^2 &= \langle \widehat{u}, \widehat{u} \rangle \\
&= \int_{-\infty}^{\infty} \overline{\widehat{u}(\xi)} \widehat{u}(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} \overline{\widehat{u}(\xi)} \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) e^{-i\xi y}\, dy\, d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) \int_{-\infty}^{\infty} \overline{\widehat{u}(\xi) e^{i\xi y}} d\xi\, dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) \overline{\mathcal{F}^{-1}[\widehat{u}](y)}\, dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) \overline{u(y)}\, dy \\
&= \frac{1}{2\pi} \|u\|^2.
\end{aligned}$$

ex:FT-triangle

Example 4.6. *Consider the triangle function*

$$u(x) = \begin{cases} 1 + x & \text{if } -1 \le x \le 0 \\ 1 - x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

We compute the Fourier transform of u to be

$$\begin{aligned}
\widehat{u}(\xi) &= \frac{1}{2\pi} \int_{-1}^{0} (1 + x) e^{-ix\xi}\, dx + \frac{1}{2\pi} \int_{0}^{1} (1 - x) e^{-ix\xi}\, dx \\
&= \frac{2(1 - \cos(\xi))}{\xi^2}.
\end{aligned}$$

The Sage code used to compute this is:

```
var('x,z')
show(integral( (1+x)*exp(-I*x*z), (x,-1,0))
     + integral( (1-x)*exp(-I*x*z), (x,0,1)))
```

(Notice that we had to do some simplification by hand after using Sage!) Even though it appears that $\widehat{u}(\xi)$ is not defined at zero, we can see (using Taylor series) that the function $\widehat{u}$ is actually defined there, with $\widehat{u}(0) = 1$.

It is easy to compute $\|u\|^2 = 2/3$. We can have Sage compute $\|\widehat{u}\|^2$, verifying the isomorphism property that $\|\widehat{u}\|^2 = 2\pi\|u\|^2 = 4\pi/3$. The Sage code is

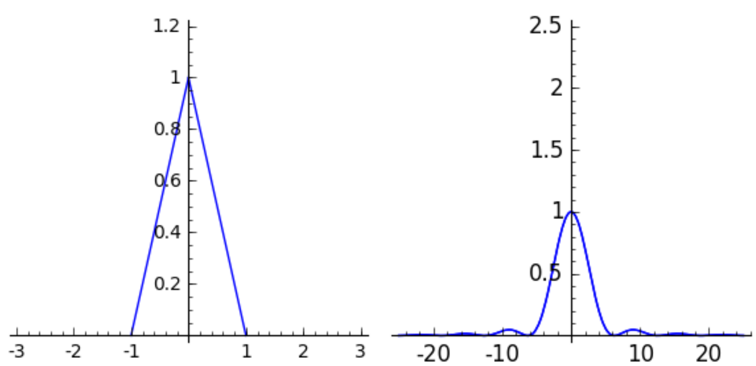
```
var('x,z')
show(integral((2*(1-cos(z))/z^2)^2,(z,-infinity, infinity)))
```

We now explore how the Fourier transform $\widehat{u}$ changes when we change the function u. We consider two types of changes: scaling and translation. First we consider scaling. Fix a function u and define a function v by $v(x) = u(ax)$, where $a > 0$ is some constant. Graphically, the function v takes u and stretches it horizontally by a factor of $1/a$. (You can see this by observing that $v(1) = u(a)$. Computing using change of variables we see that

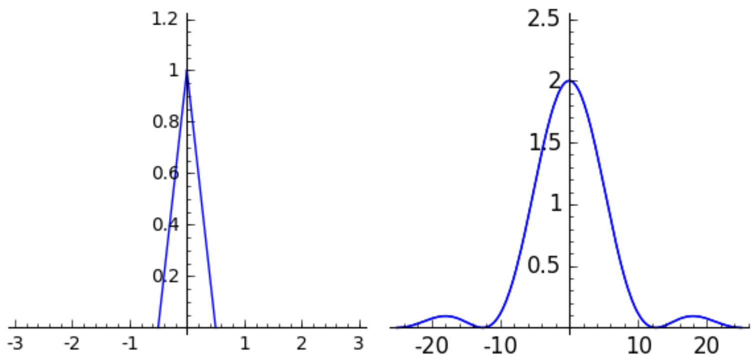
$$\widehat{v}(\xi) = \frac{1}{a}\widehat{u}(\xi/a).$$

We interpret this as follows: If a function u is stretched horizontally by a factor of $1/a$, then the Fourier transform is stretched horizontally by a factor of a and vertically by a factor of $1/a$.

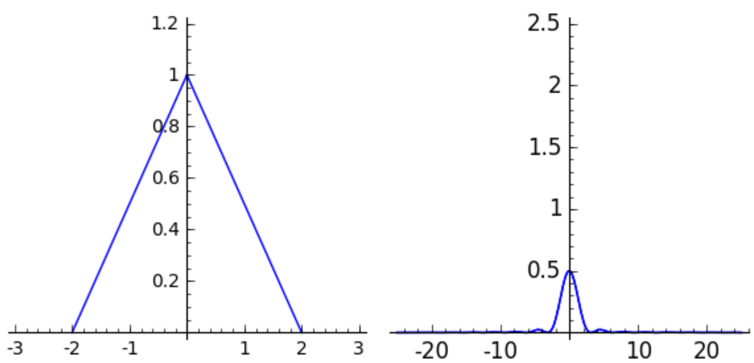
Example 4.7. *Consider the triangle function u defined in Example 4.6. The following image shows the plot of u on the right and the plot of $\widehat{u}$ on the left.*



We now stretch u *by a factor of* $1/2$*. The following image shows a plot of* $u(2x)$ *on the left and the plot of* $\frac{1}{2}\widehat{u}(\xi/2)$ *on the right. Notice that the height of the Fourier transform has increased, and the spread has also increased.*



Finally, we stretch u *by a factor of* 2*. The following image shows a plot of* $u(x/2)$ *on the left and the plot of* $2\widehat{u}(2\xi)$ *on the right. Notice that the height of the Fourier transform has decreased, and the spread has also decreased.*



From this example we see how concentrating a function leads to its Fourier transform becoming more spread out, and vice versa.

The Sage code used to generate the above images is the following.

```
var('x,z')
a = 1/2

uplot = plot( (1+a*x), (x,-1/a,0)) + plot( 1-a*x, (x,0,1/a))
     + plot(1.2, (x,-3,3),thickness=0)

hatu(z) = 2*(1-cos(z))/z^2
```

```
hatuplot = plot( a*hatu(z/a) , (z,-25,25))+ plot(2.5, (x
    ,-3,3),thickness=0)

mainplot = graphics_array((uplot,hatuplot))
show(mainplot, figsize = [6,3])
```

We now consider the effects of translating a function. Fix a function u and define a function v by $v(x) = u(x-b)$ where b is some real number. Graphically, the function v takes u and shifts it to the right by amount b. (If b is negative then the shift is to the left.) We can compute (see Exercise 4.5.1)

$$\widehat{v}(\xi) = e^{-ib\xi}\widehat{u}(\xi). \tag{4.14}$$

FT-phase-shift

The last property of the Fourier transform we discuss is the multiplication property. Recall that in the case of the little Fourier transform, pointwise multiplication of the transformed functions corresponded to the convolution product of the original functions. The same is true for the big Fourier transform. For two functions u and v in $\mathcal{L}^2(\mathbb{R})$ we define the ***confolution product*** to be the function $u * v$ defined by

$$(u * v)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(y)v(x-y)\,dy.$$

We claim that

$$\mathcal{F}[u * v] = \mathcal{F}[u] \cdot \mathcal{F}[v]. \tag{4.15}$$

FT-convolution

The verification of this fact is the task of Exercise 4.5.2.

ify-phase-shift

Exercise 4.5.1. *Verify the formula* (4.14)*. Hint: compute directly using the definition* (4.12)*.*

:FT-convolution

Exercise 4.5.2. *Verify the formula* (4.15)*. Hint: compute directly using the definition* (4.12)*, changing the order of integration, and changing variables.*

vative-property

★**Exercise 4.5.3.** *Verify the **derivative property** for Fourier transform:*

$$\mathcal{F}[u'](\xi) = -i\xi\mathcal{F}[u](\xi). \tag{4.16}$$

4.6 Fourier transform and the wave equation

We now use the Fourier transform to address the initial value problem for the wave equation on the real line. That is, we seek a function $u(t,x)$ that is defined for $t \geq 0$ and all real numbers x satisfying the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

satisfying the initial conditions

$$u(0, x) = u_0(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(0, x) = v_0(x)$$

for some given functions u_0 and v_0, and satisfying the infinite string boundary condition from (2.1):

$$u(t, x) \to 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t, x) \to 0 \qquad \text{as} \qquad x \to \pm\infty.$$

Our approach is the apply the Fourier transform to the entire problem. Using Exercise 4.5.3 we see that the transform of wave equation is

$$\frac{d^2 \widehat{u}}{dt^2}(t, \xi) = -\xi^2 \widehat{u}(t, \xi). \tag{4.17}$$

FT-wave

This is paired with the initial conditions

$$\widehat{u}(0, \xi) = \widehat{u}_0(\xi) \qquad \text{and} \qquad \frac{d\widehat{u}}{dt}(0, \xi) = \widehat{v}_0(\xi). \tag{4.18}$$

FT-wave-ic

This is fantastic – the Fourier transform has converted the wave equation into an ordinary differential equation (for each frequency ξ).

Using the methods from the differential equations course we see that the general solution to (4.17) is

$$\widehat{u}(t, \xi) = \alpha \cos(\xi t) + \beta \sin(\xi t).$$

Applying the initial conditions (4.18), we see that the solution to the transformed initial value problem is

$$\widehat{u}(t, \xi) = \widehat{u}_0(\xi) \cos(\xi t) + \widehat{v}_0(\xi) \frac{\sin(\xi t)}{\xi}.$$

It now remains to transform this back to the physical space variables.

We first focus on the term $\widehat{u}_0(\xi) \cos(\xi t)$, which we write as

$$\widehat{u}_0(\xi) \cos(\xi t) = \frac{1}{2} \left(e^{it\xi} \widehat{u}_0(\xi) + e^{-it\xi} \widehat{u}_0(\xi) \right).$$

Notice that each of the terms in the parentheses take the form (4.14). Thus we see that

$$\widehat{u}_0(\xi) \cos(\xi t) = \frac{1}{2} \left(\mathcal{F}[\widehat{u}_0(x + t)] + \mathcal{F}[\widehat{u}_0(x - t)] \right).$$

In other words

$$\mathcal{F}^{-1}[\widehat{u}_0(\xi)\cos(\xi t)] = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right). \tag{4.19}$$

FT-wave-shape-term

We interpret this last expression as follows. Suppose we have initial conditions where $v_0 = 0$. Then the solution to the wave equation on the line consists of two traveling waves, each having the shape as the half-sized initial condition, with one wave moving to the left and one moving to the right. At time $t = 0$ the two waves line up perfectly to form the initial shape of u_0.

Example 4.8. *Suppose our initial conditions are given by*

$$u_0(x) = e^{-x^2} \qquad \textit{and} \qquad v_0(x) = 0.$$

Then the corresponding solution to the initial value problem is

$$u(t,x) = \frac{1}{2}\left(e^{-(x+t)^2} + e^{-(x-t)^2}\right),$$

which we can view as the left-moving traveling wave

$$\frac{1}{2}e^{-(x+t)^2}$$

combined with the right-moving wave

$$\frac{1}{2}e^{-(x-t)^2}.$$

We now address the term

$$\widehat{v}_0(\xi)\frac{\sin(\xi t)}{\xi}.$$

Notice that

$$\widehat{v}_0(\xi)\frac{\sin(\xi t)}{\xi} = \frac{1}{2}\int_{-t}^{t} \widehat{v}_0(\xi)\cos(\xi\tau)\,d\tau.$$

Thus we can use the same technique as above in order to express this as the time-integral of a left-moving translation and a right-moving translation. In Exercise 4.6.1 you show that the result is that

$$\mathcal{F}^{-1}\left[\widehat{v}_0(\xi)\frac{\sin(\xi t)}{\xi}\right](t,x) = \frac{1}{2}\int_{x-t}^{x+t} v_0(z)\,dz. \tag{4.20}$$

FT-wave-velocity-term

Combining (4.20) with (4.19) we see that the solution to the entire initial value problem is

$$u(t,x) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} v_0(z)\,dz.$$

Notice that at location x and time t the value of the solution depends on the initial velocity over the interval $[x-t, x+t]$ and the value of the initial shape at the edge of that interval. This fact, known as *Huygen's principle*, captures the finite speed of propagation for the wave equation.

HW:FT-wave-velocity-term

Exercise 4.6.1. *Verify the formula* (4.20)*.*

Exercise 4.6.2. *Find the solution to the wave equation with initial conditions*

$$u_0(x) = 0 \quad \textit{and} \quad v_0(x) = \begin{cases} 1 & \textit{if } |x| \le 2 \\ 0 & \textit{otherwise.} \end{cases}$$

For which values of t, x is the solution $u(t,x) = 0$?