CHAPTER 1

Vector Bundles

/*big idea to talk about throughout: doing things to the fibers and everything else works out.*/

1. Definition and Examples

To motivate vector bundles, consider any vector field any vector field over $\mathbb{R}^2$ and ask: what larger object is a home for this vector field? A vector field is certainly not a point in $\mathbb{R}^2$, so what larger object does the vector field lie inside of? To identify each vector of a vector field requires 4 numbers: two numbers (x, y) to identify the location of the vector within the topological space and two additional numbers $\binom{v_x}{v_y}$ to communicate the direction of the vector at this point. This suggests that this vector field rests inside of $\mathbb{R}^2 \times \mathbb{R}^2$ or something similar; denote this $T\mathbb{R}^2$ for now. Interpret $T\mathbb{R}^2$ as the *topological space* $\mathbb{R}^2$ with a copy of the *vector space* $\mathbb{R}^2$ at every point. There is an important distinction between the structure on the two sets $\mathbb{R}^2$. The topological space $\mathbb{R}^2$ is where each point (x, y) resides and the vector space $\mathbb{R}^2$ is where each vector $\binom{v_x}{v_y}$ resides. This distinction opens the door for changing the topological space $\mathbb{R}^2$ to any arbitrary topological space.

Now consider changing the topological space, which is perhaps better called the *base space*, to the sphere S^2. What would a field look like over S^2? Not much changes: a vector field would associate to each point in S^2 some vector in the plane $\mathbb{R}^2$ tangent to the sphere. Then, the whole space that the vector field lives inside is the topological space S^2 with a vector space $\mathbb{R}^2$ associated at every point.

/*need to include some figures for the motivation*/

Definition 1.1 (Vector Bundle). Take X as a topological space. Then, a topological space E paired with a continuous map $p : E \to X$ is a *vector bundle* over X if:

(i) For each $x \in X$, the preimage $p^{-1}(x)$ is a finite vector space with the appropriate subspace topology induced from E.
(ii) E is locally trivial; that is, for each $x \in X$, there exists an open neighborhood $U \subset X$ containing x such that the preimage is trivial. That is, $p^{-1}(U) \approx U \times V$ for a vector space V.

The topological space denoted X in the definition is called the *base space* and represents the topological spaces $\mathbb{R}^2$ and S^2 discussed earlier. Then, at each point in the base space X, there is the vector space $p^{-1}(x)$ which is called the *fiber* at X and is equivalent to a copy of the vector space $\mathbb{R}^2$ at a point of the sphere discussed previously. However, this construction of vector bundle is more general than the spaces TS^2 and $T\mathbb{R}^2$ as discussed earlier because each fiber does not have to be tangent to the topological space as in the following example

Example 1.2 (Cylinder). Take S^1 with the standard topology to be the base space. As a vector space, take $\mathbb{R}$ and consider the vector bundle given by the product $S^1 \times \mathbb{R}$. Giving $\mathbb{R}$ the standard topology induces the product topology on $S^1 \times \mathbb{R}$ and take the projection map $p : S^1 \times \mathbb{R} \to S^1$ given by $p : (x, v) \mapsto x$ to be the continuous projection map. Then, each preimage $p^{-1}(x)$ is a copy of the vector space $\mathbb{R}$ with the appropriate topology and thus this gives a vector bundle. In fact, this vector bundle should be visualized as a cylinder.

/*figure of a cylinder*/

The above construction of the cylinder demonstrates that fibers need not be tangent to the base space, but in fact fibers are not required to correspond to the dimension of the base space. The example of the cylinder is a specific case of the idea of a *trivial bundle* which is defined as follows.

Definition 1.3 (Trivial Bundle). Let X be a topological base space and let V be a vector space with a topology. Then, taking the product topology, $X \times V$ forms a topological space. This together with the projection map $p : X \times V \to X$ given by $p : (x, v) \mapsto x$ forms a vector bundle. This vector bundle is called a *trivial bundle*. If E is of dimension n, the trivial bundle is often denotes ε^n.

With this construction of the trivial bundle, note in the "locally trivial" condition in definition 1.1, the $X \times V$ is understood as the trivial bundle. However, there are many bundles that are not trivial bundles and the best example of such a bundle is the Mobius strip.

Example 1.4 (Mobius Strip). /*include example of Mobius Strip*/

/*Also give as examples, the formal construction of TS^2 and the normal bundle over S^2*/

To complete the category of vector bundles, a notion of homomorphisms between vector bundles is necessary. Vector bundles contain the structure of both a topological space and of many vector spaces, so a homomorphism of vector bundles aims to preserve both of these structures. These homomorphisms will be over the same base space and are defined as follows.

Definition 1.5 (Homomorphisms of Vector Bundles). Take two vector bundles E and F both with base space over X. Then, let $p : E \to X$ and $q : F \to X$ be the continuous maps. A mapping $\varphi : E \to F$ is a *homomorphism of vector bundles* if:

(i) $q\varphi = p$
(ii) $\varphi : E \to F$ is a homomorphism of topological spaces; that is, φ is continuous.
(iii) For each $x \in X$, the mapping $\varphi : p^{-1}(x) \to q^{-1}(x)$ is a homomorphism of vector spaces; that is, φ is a linear map between these vector spaces.

/*Include an example of homomorphism?*/

The definition of isomorphism for vector bundles carries over from the definition of isomorphism in category theory: a homomorphism with a homomorphism as an inverse. However, some of the homomorphism properties of the inverse follow automatically. For instance, take vector bundles $p : E \to X$ and $q : F \to X$ and a bijective homomorphism $\varphi : E \to F$. Then, it follows immediately from $q\varphi = p$ that $p\varphi^{-1} = q$, so this does not need to be checked. Additionally, a bijective linear map will have a linear inverse. Then, it does not need to be verified that $varphi^{-1}$ maps the fibers in a linear way because it is known that φ does. However, the continuous property of φ^{-1} does not follow automatically and is typically the most difficult part of isomorphism proofs. With these observations, an isomorphism can be defined in the following more practical way.

Definition 1.6 (Isomorphism). For two vector bundles $p : E \to X$ and $q : F \to X$, a map $\varphi : E \to F$ is defined to be an isomorphism if it is a bijective homomorphism with continuous inverse.

/*include an example of an isomorphism?*/

2. Direct Sum and Tensor Product on Vector Bundles

It is worth emphasizing that every point of a vector bundle E belongs to some fiber of the bundle. In fact, E as a set can be thought of as the disjoint union of only fibers. By taking the perspective of a vector bundle as the union of vector spaces, much of the structure of vector spaces extends to vector bundles. For instance, the fibers can be used to construct the direct sum operation in the following way.

Definition 1.7 (Direct Sum of Vector Bundles). Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles over X. Then, consider the disjoint unions of the direct sums of fibers

$$E_1 \oplus E_2 = \bigcup_{x \in X} p_1^{-1}(x) \oplus p_2^{-1}(x)$$

together with the projection mapping $p : E_1 \oplus E_2 \to X$ given by $p : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto x$. Then, $p : E_1 \oplus E_2 \to X$ when given a natural topology forms a vector bundle over X called the *direct sum* of E_1 and E_2.

/*mention intuition of pairs for direct sum bundle*/

Of course, a vector bundle has more structure than simply a union of vector spaces; in particular, vector bundles must be given a topology and must satisfy the local triviality condition. /*ref*/ gives the specifics of the "natural topology" referred to in the above definition along with this necessary proof of local triviality, but these verifications all work out. Because a vector bundle is built out of fibers, vector space properties such as the direct sum carry over naturally to vector bundles and the extra properties typically "all work out".

In this construction, consider some $x \in X$ and let $v_1 \in p_1^{-1}(x)$ and $v_2 \in p_2^{-1}(x)$ be elements of both fibers. Then, taking the direct sum of these vector spaces, these two vectors can be identified with $v_1 \oplus v_2$, which can also be thought of as simply (v_1, v_2).

A second similar construction by using the fibers is in the extension of the tensor product to vector bundles.

Definition 1.8 (Tensor Product of Vector Bundles). Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles over X. Then, consider the disjoint unions of all tensor products of the fibers

$$E_1 \otimes E_2 = \bigcup_{x \in X} p_1^{-1}(x) \otimes p_2^{-1}(x)$$

together with the projection mapping $p : E_1 \otimes E_2 \to X$ given by $p : p_1^{-1}(x) \otimes p_2^{-1}(x) \mapsto x$. Then, $p : E_1 \otimes E_2 \to X$ when given a natural topology forms a vector bundle over X called the *tensor product* of E_1 and E_2.

Again, the specifics of the "natural topology" and the verification of natural triviality all work out as explained in /*ref*/. The proof is identical to the proof for direct sum. It is worth mentioning that this construction can be generalized to other operations on vector spaces such as the dual and the exterior power, but these notes only require the direct sum and the tensor product.

Because the tensor product and direct sum are defined on each fibers, the properties of direct sum and tensor product on vector spaces carry over to analogous properties on vector bundles.

Claim 1.9. Listed below are properties of direct sum and tensor product over vector bundles.

(i) The direct sum between bundles is associative and commutative.
(ii) The trivial bundle of dimension 0 is an identity element for the direct sum. That is, $E \oplus \varepsilon^0 = E$.
(iii) The tensor product between bundles is associative and commutative.
(iv) The trivial dimension of dimension 1 is an identity element for the tensor product. That is, $E \otimes \varepsilon^1 = E$.
(v) The tensor product distributes over direct sum.

3. Pullback Bundles

The following construction, addresses pullback bundles. In the next chapter of this story, all of the arrows will suddenly point backwards as a contravariant functor emerges. The reason why the arrows will point backwards is due to pullback bundles.

Consider two base spaces X and Y where X has a vector bundle structure $p : E \to X$ but Y, unfortunately, has no such structure. However, Y can be given a vector bundle $q : F \to Y$ by stealing the structure of E through the association given by f. Specifically, each fiber $q^{-1}(y)$ can just take a copy of the fiber $p^{-1}(f(y))$.

Definition 1.10 (Pullback Bundle). Let $f : X \to Y$ be a mapping and $p : E \to X$ a bundle as defined above. Then there exists a unique bundle $f^*(p) : f^*(E) \to Y$ and a mapping $h : f^*(E) \to E$ such that h maps each fiber $(f^*(p))^{-1}(y)$ to the fiber $p^{-1}(f(y))$ as a vector space isomorphism. This bundle is called the *pullback bundle* and denoted $f^*(p) : f^*(E) \to Y$.

/*talk about existence and uniqueness proofs*/

However, there is a detail of well-defined to address. When given a vector bundle $p : E \to X$ with continuous functions $f : X \to Y$ and $g : Y \to Z$, how is the bundle structure on X pulled back to a bundle on Z? There are two options: $(f \circ g)^*(E)$ and $f^*(g^*(E))$. Luckily, the following claim shows that the two options are isomorphic and gives more pleasant properties of the pullback.

Claim 1.11. Listed below are important properties of pullbacks.

(i) $(f \circ g)^*(E) \approx g^*(f^*(E))$ for any bundle E and continuous functions f and g.
(ii) $\mathrm{Id}^*(E) \approx E$ for any vector bundle E over X and the identity mapping $\mathrm{Id} : X \to X$.
(iii) $f^*(\varepsilon^n) \approx \varepsilon^n$ for all continuous functions f and trivial bundles ε^n over the corresponding base spaces.
(iv) $f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)$ for all bundles E_1 and E_2 and continuous function f.
(v) $f^*(E_1 \otimes E_2) \approx f^*(E_1) \otimes f^*(E_2)$ with E_1 and E_2 vector bundles and f a continuous function.

Note that given a vector bundle $p : E \to X$, a function f from Y *to* X is necessary to induce a pullback bundle $f^*(E)$ over Y; not the other way around. The function must point this direction in order for $f^*(E)$ to effectively steal the structure of E. A function $f' : X \to Y$ would be rather useless, for this function may associate each point of the base space Y to multiple points in the base space X or non at all. However, given the function $f : Y \to X$, each point in Y is mapped to a single point in X and so the structure of E can be effectively stolen. This fact — that a function induces a vector bundle in the opposite direction — is half way to defining a contravariant functor.

4. Necessary Results on Vector Bundles

this section is still in progress

/*Canonical Line bundle over $\mathbb{R}P^1$ gives mobius band*/

/*A contractible implies the bundle over A is trivial*/.

Claim 1.12. For every bundle E over a compact Hausdorff space X, there exists a bundle E' over X such that $E \oplus E'$ is trivial.

This claim is central to the story, so a full proof /*I still need to do this*/ is provided in /*ref*/. The proof is long with many lemma's, so more useful than reading the full proof is to read the following summary of the proof's idea. Given the bundle $p : E \to X$ over a compact Hausdorff space, a huge trivial bundle T is constructed by using a topology theorem[1] that follows from the compact Hausdorff condition. The trivial bundle T is built exactly such that there is a convenient isomorphism from E to a subbundle E_0 in the huge trivial bundle. Another topology tool[2] allows the extension of a metric to vector bundles, which then gives a Gran-Schmidt orthogonalization process on vector bundles. The orthogonal complement of each fiber in E_0 gives a vector bundle $E_0^\perp$ such that $E_0 \oplus E_0^\perp = T$ and the desired conclusion follows from $E \cong E_0$.

/*example: $NS^2 \oplus TS^2$ is trivial*/

Example 1.13. For an example of the above theorem, consider the tangent bundle to S^2, denoted TS^2. As promises by theorem /*ref*/, the normal bundle to S^2, denoted NS^2, satisfies $TS^2 \oplus NS^2$ trivial. To see this, consider the space S^2 as embedded inside $\mathbb{R}^3$. Then elements of TS^n can be expressed $(x, v) \in S^2 \times \mathbb{R}^3$ and similarly, elements of NS^2 are given by $(x, n) \in S^2 \times \mathbb{R}^3$. Further, at a fixed point x, all vectors v in the tangent fiber will be orthogonal to the vectors n in the normal

[1] Urysohn's Lemma
[2] Partition of Unity

fiber by the definition of the bundles. Then elements of the direct sum $TS^2 \oplus NS^2$ can be expressed by $(x, v \oplus n)$ or simply (x, v, n). Then consider the isomorphism $\varphi : TS^2 \oplus NS^2 \to S^2 \times \mathbb{R}^3$ given by the isomorphism.

$$\varphi : (x, v, n) \mapsto (x, v + n)$$

The above mapping an isomorphism follows from the above continuous and a linear bijection. The inverse map to the above can be constructed by taking the projection of the vector component onto the normal and tangent subspaces, which is again continuous giving isomorphism. /*a picture would be nice here*/

/*example: M mobius band ... $M \oplus M$ trivial*/

/*Example: Mobius band with itself is stably trivial AND/OR tangent bundle over S^2 with normal bundle over S^2*/

5. Verifications

5.1. Direct Sum and Tensor Product Verifications. It must be verified that the direct sum has a natural topology that indeed makes it a vector bundle.

Proof. Take vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ and recall that the direct sum on bundles as a set is given by the disjoint union of direct sums on fibers

$$E_1 \oplus E_2 = \bigcup_{x \in X} p_1^{-1}(x) \oplus p_2^{-1}(x).$$

This set is paired with with the projection $p : E_1 \oplus E_2 \to X$ given by $p : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto x$.

The topology on $E_1 \oplus E_2$ is defined in this paragraph. For each $x \in X$, the definition of vector bundle promises an open set U containing x over which both E_1 and E_2 are trivial. This provides trivializations $t_1 : p_1^{-1}(U) \to U \times V_1$ and $t_2 : p_2^{-1}(U) \to U \times V_2$ for vector spaces V_1 and V_2. Next, define the map $t_1 \oplus t_2 : p_1^{-1}(U) \oplus p_2^{-1}(U) \to U \times (V_1 \oplus V_2)$ as follows.

$$t_1 \oplus t_2 : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto t_1(p_1^{-1}(x)) \oplus t_2(p_2^{-1}(x))$$

Then, the topology on $p_1^{-1}(U) \oplus p_2^{-1}(U)$ is defined by requiring the map $t_1 \oplus t_2$ to be a homeomorphism. By letting x vary, this defines a topology over all of $E_1 \oplus E_2$. It must be verified, however, that this topology is well-defined.

Before the proof of well-defined, observe how this choice of topology gives that $E_1 \oplus E_2$ is a vector bundle. Firstly, this choice equips each fiber $p_1^{-1}(x) \oplus p_2^{-1}(x)$ with the typical topology of the direct sum of vector spaces. This ensures that the projection map $p : E_1 \oplus E_2 \to X$ is continuous. Next, the local triviality condition must be verified. Luckily the topology is built exactly so that $t_1 \oplus t_2$ is a trivialization. For any $x \in X$, the mapping $t_1 \oplus t_2$ defined on the appropriate U as described above satisfies all the conditions of a vector bundle homomorphism. Further, the defining condition that $t_1 \oplus t_2$ is a homeomorphism promises a continuous inverse and so $t_1 \oplus t_2$ is an isomorphism of vector bundles.

It only remains to show that the topology on $E_1 \oplus E_2$ is well-defined. In particular, it must be shown that the topology is independent of the choice of trivializations over a single open set U and that the open sets induce the same topology over their intersection. So, for $x \in X$ and corresponding $U \subset X$, consider two trivializations for each bundle: $t_1, t_1' : E_1 \mapsto U$ and $t_2, t_2' : E_1 \mapsto U$. Because each trivialization gives an isomorphism to the trivial bundle, the composition $t_1^{-1} \circ t_1' : p^{-1}(U) \to p^{-1}(U)$ is an isomorphism and similarly $t_2^{-1} \circ t_2' : p^{-1}(U) \to p^{-1}(U)$ is an isomorphism. Then composition $t_1' \circ t_1^{-1}$ is an isomorphism on $U \times V_1$ and similarly $t_2' \circ t_2^{-1}$ is an isomorphism on $U \times V_2$. It follows that the composition $(t_1' \oplus t_2') \circ (t_1 \oplus t_2)^{-1}$ is an isomorphism on $U \times (V_1 \otimes V_2)$, which implies that the choices $(t_1 \oplus t_2)$ and $(t_1' \oplus t_2')$ supply the same topology.

Finally, consider a separate set of open set $U' \subset X$. Then, taking the restrictions of the bundles $p^{-1}(U)$ and $p^{-1}(U')$ over the intersection $U \cap U'$ would only differ in the trivializations, which induce the same topology as shown in the previous paragraph. □

In the above argument, the only part that appeals to the direct sum operation itself is the implicit assumption that the mapping $(v, w) \mapsto v \oplus w$ is continuous. This is also true for the tensor product, so a simple substitution of "$\otimes$" in place of "$\oplus$" in the above proof provides the needed verification for tensor product.

Proof of claim /*ref*/. Verifying each claim requires establishing an isomorphism φ over two bundles, say $p : E \to X$ and $q : F \to X$. The approach will be to establish a vector space isomorphism between the fibers, which gives necessary properties of vector bundle isomorphism except for continuity and continuity of inverse. To deal with the continuity conditions, the strategy is to show local continuity at every point as descried in /*ref*/. It then suffices to show that for every $x \in X$, there is an open neighborhood U such that the restricted function $\varphi : p^{-1}(U) \to q^{-1}(U)$ is continuous in both directions.

(i) For associativity of the direct product, consider vector bundles E_1, E_2, E_3 over a base space X with corresponding projection maps p_1, p_2, and p_3. An isomorphism $\varphi : (E_1 \oplus E_2) \oplus E_3 \to E_1 \oplus (E_2 \oplus E_3)$ must be constructed. Let φ be the linear bijective function defined on the fibers by

$$\varphi : (p_1^{-1}(x) \oplus p_2^{-1}(x)) \oplus p_3^{-1}(x) \mapsto p_1^{-1}(x) \oplus (p_2^{-1}(x) \oplus p_3^{-1}(x))$$

For the continuity conditions, fix a point $x \in X$. Then, choose an open set $U \subset X$ small enough such that the local triviality conditions are satisfied by both direct sum bundles. Then, noting the vector space isomorphism $(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3)$, continuity in both directions is given by the following composition of isomorphisms

$$\begin{aligned} &(p_1^{-1}(U) \oplus p_2^{-1}(U)) \oplus p_3^{-1}(U) \to U \times (V_1 \oplus V_2) \oplus V_3 \\ &\to U \times V_1 \oplus (V_2 \oplus V_3) \to p_1^{-1}(U) \oplus (p_2^{-1}(U) \oplus p_3^{-1}(U)) \end{aligned}$$

The proof for commutativity follows in a near identical way. The difference being that an isomorphism $\varphi : E_1 \oplus E_2 \to E_2 \oplus E_1$ is considered with the mapping between fibers $\varphi : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto p_2^{-1}(x) \oplus p_1^{-1}(x)$ and the vector space isomorphism $V_1 \oplus V_2 \cong V_2 \oplus V_1$ is considered instead.

(ii) Verifying that ε^0 is the identity element under direct sum requires establishing an isomorphism $\varphi : E \oplus \varepsilon^0 \to E$. This follows in the same way as the previous claims, but uses the mapping of fibers $\varphi : p^{-1}(x) \oplus \{0\} \mapsto p^{-1}(x)$ and uses the vector space isomorphism $V \oplus \{0\} \cong V$.
(iii) The proofs for associativity and commutativity of the tensor product is given by a substitution of "$\otimes$" for "$\oplus$" in the corresponding direct sum proofs.
(iv) The proof that ε^1 acts as an identity element over the tensor product follows similarly to the identity proof over direct sum. The difference being that here an isomorphism $\varphi : E \otimes \varepsilon^1 \to E$ is established by the mapping of fibers $\varphi : p^{-1}(x) \otimes V^1 \mapsto p^{-1}(x)$ where V^1 represents a one dimensional vector space. This proof additionally uses the vector space isomorphism $V \oplus V^1 \cong V$.
(v) Finally, the proof for distributivity establishes a vector space isomorphism $\varphi : E_1 \otimes (E_2 \oplus E_3) \to (E_1 \otimes E_2) \oplus (E_1 \otimes E_3)$ given by the linear bijection on the fibers

$$\varphi : p_1^{-1}(x) \otimes (p_2^{-1}(x)) \oplus p_3^{-1}(x)) \mapsto p_1^{-1}(x) \otimes p_2^{-1}(x) \oplus p_1^{-1}(x) \otimes p_3^{-1}(x)$$

and later uses the isomorphism on vector spaces $V_1 \otimes (V_2 \oplus V_3) \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$.

□

5.2. Pullback Bundle Verifications. /*I should really break up this proof into Lemma's, but I do not want to dive back into this proof*/

Proof of Existence for Definition 1.10. The proof of existence is by an explicit construction. Specifically, for a continuous function of topological spaces $f : X \to Y$ in addition to the vector bundle $p : E \to X$ consider the vector bundle $q : F \to Y$ where F is the following set.

$$F = \{(y, e) \in Y \times E : f(y) = p(e)\}$$

Further, let q be the mapping $q : (y, e) \mapsto y$.

It must be shown that F is a vector bundle satisfying the defining property of the pullback bundle. However, in order for F to be a vector bundle, it must be given extra structure.

Let F have the natural choice of topology induced by Y and E; specifically F takes the subspace topology of the product $Y \times E$.

Next, define the vector space structure over F as follows. Consider a fixed $y \in Y$ and fiber $q^{-1}(y)$. Note for each element (y, e) of the fiber, the condition $f(y) = p(e)$ restricts the elements of E to be in the vector space $p^{-1}(f(y))$. Then, borrowing the vector space structure from $p^{-1}(f(y))$ gives the natural definition of addition and scalar multiplication by a scalar α.

$$\alpha(y, v) = (y, \alpha v)$$
$$(y, v) + (y, w) = (y, v + w)$$

It follows from the vector space structure on $p^{-1}(f(y))$ that $q^{-1}(y)$ will satisfy all the necessary axioms to be a vector space.

Finally, the construction is complete and it must now be verified that F is indeed a vector bundle. Firstly, the definition of product topology promises that the projection q will be continuous.

Additionally, the above construction of the vector space structure over F promises that each fiber $q^{-1}(y)$ will be continuous.

It remains to show that F is locally trivial so fix a point $y \in Y$. By definition, E is locally trivial and so has a neighborhood U containing $g(y)$ such that $p^{-1}(U)$ is locally trivial. This promises a trivializing isomorphism $t : p^{-1}(U) \to U \times V$ for some vector field V. Note that this trivial bundle comes with the projection map $p' : U \times V \to U$ given by $p' : (u, v) \to u$. Define the mapping $t_1 : E \to U$ to be the composition of t with the projection onto the first factor and take $t_2 : E \to V$ to be the same composition but onto the second factor. This allows for the representation of the trivialization by $t : e \mapsto (t_1(e), t_2(e))$. Applying the condition $p' \circ t = p$ (given by t a homomorphism) to the representation gives the conclusion $t_1(e) = p(e)$ and thus allows for the simplification

$$t : e \mapsto (p(e), t_2(e))$$

After unpacking the promised trivialization on $E|_U$, a trivialization on $F|_{f^{-1}(U)}$ can now be constructed. Specifically, let the trivialization $\tau : F|_{f^{-1}(U)} \to f^{-1}(U) \times V$ be given by the following.

$$\tau : (y, e) \mapsto (y, t_2(e))$$

Additionally note that the bundle $f^{-1}(U) \times V$ comes equipped with a projection map q'.

It must now be shown that τ is an isomorphism of vector spaces. Observe that τ satisfies all the properties of a vector bundle homomorphism. First, τ continuous follows from t_2 continuous. The property $q' \circ \tau = q$ follows by

$$(q' \circ \tau)((y, e)) = q'((y, t_2(e))) = y = q((y, e)).$$

The last property of a homomorphism is that is linearity over the fibers. To see this, fix a $y \in U$ and notice that t linear over $p^{-1}(f(y))$ gives that t_2 is linear.

$$\begin{aligned}(f(y), t_2(\alpha v + \beta w)) &= t(\alpha v + \beta w) = \alpha t(v) + \beta t(w) \\ &= \alpha(f(y), t_2(v)) + \beta(f(y), t_2(w)) = (f(y), \alpha t_2(v) + \beta t_2(w))\end{aligned}$$

where the above computation used the $p(e) = f(y)$ as well as the predefined vector space structure of the trivial bundle. By a similar computation, t_2 linear gives that τ is linear over the fiber and thus a homomorphism.

To get that τ is an isomorphism, it suffices to show that that the inverse function is continuous. An explicit expression for $\tau^{-1} : f^{-1}(U) \times V$ follows.

$$\tau^{-1} : (y, v) \mapsto (y, t^{-1}(f(y), v))$$

Using $t \circ t^{-1} = \mathrm{Id}$ and $t^{-1} \circ t = \mathrm{Id}$, it follows that the above is indeed the inverse expression. Further, t^{-1} continuous gives that τ^{-1} continuous and so τ is an isomorphism, completing the verification of F a vector bundle.

It still remains to show that F has the defining property of the pullback. For this, take the function $h : F \to E$ to be the projection onto E.

$$h : (y, e) \mapsto e$$

Next, fix an element $y \in Y$ and consider the fiber $q^{-1}(y)$. The restriction $f(y) = p(e)$ ensures that $h((y, v)) = v$ is an element of $p^{-1}(f(y))$. Finally, the conclusion that h is a linear map from the fiber $q^{-1}(y)$ to the fiber $p^{-1}(f(y))$ follows quickly from the vector space structure of E.

$$h((y, \alpha v + \beta w)) = \alpha v + \beta w = \alpha h((y, v)) + \beta h((y, w))$$

Concluding the proof.

As a side note, observe that $p \circ h = f \circ q$ follows by the condition $f(y) = p(e)$ in the construction.

$$(p \circ h)((y, e)) = p(e) = f(y) = (f \circ q)((y, e))$$

This justifies drawing the commutative diagram /*ref*/ which hopefully helps in keeping track of variables for this proof.

□

PROOF OF /*REF*/. The strategy for proving each of the following isomorphisms is to take advantage of the uniqueness property. If it can be shown that one side of the isomorphism satisfies the defining property of pullback for the other side, then they must be isomorphic by uniqueness.

(i) For topological spaces X, Y, Z let $g : Z \to Y$ and $f : Y \to X$ be continuous functions and let $p : E \to X$ be a vector bundle. By definition, the bundles $f^*(E)$ and $g^*(f^*(E))$ come equipped with maps $h_g : g^*(f^*(E)) \to f^*(E)$ and $h_f : f^*(E) \to E$ that isomorphically map fibers to corresponding fibers. Then, the composition $h_f \circ h_g : g^*(f^*(E)) \to E$ isomorphically maps fibers to corresponding fibers. Further, the bundle $g^*(f^*(E))$ comes equipped with a projection mapping r into the base space Z. Thus, the triple $g^*(f^*(E))$, $h_f \circ h_g$, and r satisfy the defining characteristics of the pullback bundle $(f \circ g)^*(E)$, giving isomorphism by uniqueness.

(ii) Take the mapping $\mathrm{Id} : X \to X$ for a topological space X with a bundle $p : E \to X$. Then, the bundle E itself with the identity mapping $\mathrm{Id} : E \to E$ isomorphically maps fibers to fibers and comes equipped with the projection mapping p to X. Then, the triple E, $\mathrm{Id} : E \to E$, and p satisfy the defining characteristics of the pullback $\mathrm{Id}^*(E)$ which promises the isomorphism $E \cong \mathrm{Id}^*(E)$ by uniqueness.

(iii) Let $f : Y \to X$ be a continuous function between topological spaces and consider the trivial bundle $p : X \times V \to X$ over X with the regular projection p. Then, consider the trivial bundle over $q : Y \times V \to Y$ over Y with the regular projection q. Then, the mapping $h : Y \times V \to X \times V$ given by $h : (y, v) \mapsto (f(y), v)$ gives the identity mapping over each fiber and is thus a linear isomorphism of the fibers. Thus, the triple $Y \times V$, h, and q satisfies the defining properties of $f^*(X \times V)$ and thus uniqueness promises an isomorphisms between the trivial bundles $Y \times V \cong f^*(X \times V)$. Note that the trivial pullback is over the same vector space.

(iv) Next, take $f : Y \to X$ to be a continuous function between topological spaces. Further, let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles. The pullbacks $f^*(E_1)$ and $f^*(E_2)$ then come with mappings $h_1 : f^*(E_1) \to E_1$ and $h_2 : f^*(E_2) \to E_2$ that are isomorphisms on the fibers. Then, the direct sum of the pullbacks has a mapping $h : f^*(E_1) \oplus f^*(E_2) \to E_1 \oplus E_2$ defined on the fibers by $h : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto h_1(p_1^{-1}(x)) \oplus h_2(p_2^{-1}(x))$ which is also an isomorphism on the fibers. Additionally note that the direct sum comes equipped with a projection mapping p onto Y. Thus the triple $f^*(E_1) \oplus f^*(E_2)$, h, and p satisfy the defining properties of the pullback $f^*(E_1 \oplus E_2)$ giving the isomorphism $f^*(E_1) \oplus f^*(E_2) \cong f^*(E_1 \oplus E_2)$ by uniqueness.

(v) The proof for the distributivity of pullback over tensor product is identical to preceding such proof for direct sum, differing only by replacing each "$\oplus$" with "$\otimes$".

□

5.3. Other Verifications.

LEMMA 1.14. /*yikes, this lemma (lemma 1.2 in Hatcer) uses partition of unity... this is becoming a rabbit hole*/

Given a vector space V and a vector subspace, $V_0 \subset V$, the Gram-Schmidt orthogonalization process provides the orthogonal complement $V_0^{\perp}$ to the subspace V_0 in V. Further, it holds that $V_0 \oplus V_0^{\perp} = V$. An analogous result holds for vector bundles by applying the same process to each fiber.

LEMMA 1.15. Take a vector bundle $p : E \to X$ that has

/*assumes all V_i's are equal. Need to fix? Say it suffices to consider connected components*/

PROOF OF /*REF*/. The strategy of this proof is to construct a trivial vector space, later called $X \times \mathcal{V}$, that an isomorphic copy of the given vector bundle resides in. Then the result will follow by the above lemma.

Consider a vector bundle $p : E \to X$ where X is a compact Hausdorff topological space.. Each point $x \in X$ has a neighborhood U_x over which the bundle is trivial. By X compact Hausdorff, apply Urysohn's Lemma /*ref*/ on the closed sets $\{x\}$ and the complement $\overline{U_x}$. Urysohn's Lemma then promises a continuous function $\varphi_x : X \to [0,1]$ satisfying $\varphi_x^{-1}(\{0\}) \subset \overline{U_x}$ and $\varphi_x^{-1}(\{1\}) \subset \{1\}$. In other words, f evaluates to 0 outside of U_x and to 1 at x. Note that $\varphi_x^{-1}(0,1]$ contains X and is open by by φ_x continuous and the interval equipped with standard topology. Then $\varphi_x^{-1}(0,1]$ provides an open cover when allowing x to vary. By compactness there is a finite subcover; denote this subcover $\varphi_i^{-1}(0,1]$ and let the corresponding functions and neighborhoods be indexed φ_i and U_i.

Next, for each index define a function $g_i : E \to V$ as follows. Let $h_i : p^{-1}(U_i) \to U_i \times V$ be the trivialization as promised by the choice of U_i. Additionally, let $\pi_i : X \times V \to V$ be the projection from the trivial bundle to the corresponding vector component: $\pi_i : (x, v) \mapsto v$. Then, the function g_i is defined as follows.

$$g_i(e) = \begin{cases} \varphi_i(p(e)) \cdot (\pi_i \circ h_i(e)) \text{ if } p(e) \in U_i \\ 0 \text{ otherwise.} \end{cases}$$

Note g_i is continuous by g_i a composition of continuous functions and by φ_i is 0 outside of U_i. Importantly note that each g_i is a linear injection over the fibers of $\varphi_i^{-1}(0,1]$. Indeed, fix an $x_0 \in \varphi_i^{-1}(0,1] \subset U_i$ and take v_1, v_2 in the fiber $p^{-1}(x_0)$ such that $g_i(v_1) = g_i(v_2)$. That is,

$$\varphi_i(p(v_1)) \cdot (\pi_i \circ h_i(v_1)) = \varphi_i(p(v_2)) \cdot (\pi_i \circ h_i(v_2))$$

The fixed x gives $\varphi_i(p(v_1)) = \varphi_i(p(v_1)) = \varphi_i(x_0)$. This together with h_i an isomorphism and π_i an isomorphism over the fixed x_0 promises $v_1 = v_2$, confirming injectivity over the fibers. Linearity follows by π_i and h_i linear over the fibers.

Next, consider the vector space $\mathcal{V} = V \times V \times \cdots \times V$ with one copy of V for each of the indices i. Then, define the function $g : E \to \mathcal{V}$ given by $g : e \mapsto (g_1(e), g_2(e), \ldots, g_k(e))^T$. Note that g is a linear injection. Indeed, fix an $x_0 \in \varphi_i^{-1}(0, 1] \subset U_i$ and take v_1, v_2 in the fiber $p^{-1}(x_0)$ such that $g(v_1) = g(v_2)$. By the collection $\varphi_i^{-1}(0, 1]$ a cover, $x_0 \in \varphi_i^{-1}(0, 1]$ for some i. But then, $g_i(v_1) = g_i(v_2)$, which then provides the desired $v_1 = v_2$ confirming injectivity. Linearity follows by each individual g_i linear.

Finally consider the map $f : E \to X \times \mathcal{V}$ given by $f : e \mapsto (p(e), g(v))$. Now observe that the image of f is a subbundle of of $X \times \mathcal{V}$. The bundles takes the natural projection map of the larger trivial bundle and by linearity of g each fiber of the image has a vector space structure. It only remains to verify the local triviality condition. Indeed, for each $x_0 \in X$, the open cover promises $x_0 \in \varphi_i^{-1}(0, 1]$ for some i. Then, consider the projection $X \times \mathcal{V}$ by mapping the vector component of (x, v) to the ith copy of V used to construct $\mathcal{V}$ and call the projection q. Then, a local trialization over the region is provided by $(x, v) \mapsto (x, q(v))$. With the verification that $\operatorname{Im} f$ indeed forms a vector bundle, and by injective implies bijective onto the image, lemma /*ref*/ applies and gives that the image is isomorphic to a subbundle of $X \times \mathcal{V}$. So, lemma /*ref*/ applies and promises a bundle E' such that $E \oplus E' = X \times \mathcal{V}$.

$\square$