/*I should really break up this proof into Lemma's, but I do not want to dive back into that*/

Proof of Existence for Definition ??. The proof of existence is by an explicit construction. Specifically, for a continuous function of topological spaces $f : X \to Y$ in addition to the vector bundle $p : E \to X$ consider the vector bundle $q : F \to Y$ where F is the following set.

$$F = \{(y, e) \in Y \times E : f(y) = p(e)\}$$

Further, let q be the mapping $q : (y, e) \mapsto y$.

It must be shown that F is a vector bundle satisfying the defining property of the pullback bundle. However, in order for F to be a vector bundle, it must be given extra structure.

Let F have the natural choice of topology induced by Y and E; specifically F takes the subspace topology of the product $Y \times E$.

Next, define the vector space structure over F as follows. Consider a fixed $y \in Y$ and fiber $q^{-1}(y)$. Note for each element (y, e) of the fiber, the condition $f(y) = p(e)$ restricts the elements of E to be in the vector space $p^{-1}(f(y))$. Then, borrowing the vector space structure from $p^{-1}(f(y))$ gives the natural definition of addition and scalar multiplication by a scalar α.

$$\alpha(y, v) = (y, \alpha v)$$
$$(y, v) + (y, w) = (y, v + w)$$

It follows from the vector space structure on $p^{-1}(f(y))$ that $q^{-1}(y)$ will satisfy all the necessary axioms to be a vector space.

Finally, the construction is complete and it must now be verified that F is indeed a vector bundle. Firstly, the definition of product topology promises that the projection q will be continuous. Additionally, the above construction of the vector space structure over F promises that each fiber $q^{-1}(y)$ will be continuous.

It remains to show that F is locally trivial so fix a point $y \in Y$. By definition, E is locally trivial and so has a neighborhood U containing $g(y)$ such that $p^{-1}(U)$ is locally trivial. This promises a trivializing isomorphism $t : p^{-1}(U) \to U \times V$ for some vector field V. Note that this trivial bundle comes with the projection map $p' : U \times V \to U$ given by $p' : (u, v) \to u$. Define the mapping $t_1 : E \to U$ to be the composition of t with the projection onto the first factor and take $t_2 : E \to V$ to be the same composition but onto the second factor. This allows for the representation of the trivialization by $t : e \mapsto (t_1(e), t_2(e))$. Applying the condition $p' \circ t = p$ (given by t a homomorphism) to the representation gives the conclusion $t_1(e) = p(e)$ and thus allows for the simplification

$$t : e \mapsto (p(e), t_2(e))$$

After unpacking the promised trivialization on $E|_U$, a trivialization on $F|_{f^{-1}(U)}$ can now be constructed. Specifically, let the trivialization $\tau : F|_{f^{-1}(U)} \to f^{-1}(U) \times V$ be given by the following.

$$\tau : (y, e) \mapsto (y, t_2(e))$$

Additionally note that the bundle $f^{-1}(U) \times V$ comes equipped with a projection map q'.

It must now be shown that τ is an isomorphism of vector spaces. Observe that τ satisfies all the properties of a vector bundle homomorphism. First, τ continuous follows from t_2 continuous. The property $q' \circ \tau = q$ follows by

$$(q' \circ \tau)((y, e)) = q'((y, t_2(e))) = y = q((y, e)).$$

The last property of a homomorphism is that is linearity over the fibers. To see this, fix a $y \in U$ and notice that t linear over $p^{-1}(f(y))$ gives that t_2 is linear.

$$\begin{aligned}(f(y), t_2(\alpha v + \beta w)) &= t(\alpha v + \beta w) = \alpha t(v) + \beta t(w) \\ &= \alpha(f(y), t_2(v)) + \beta(f(y), t_2(w)) = (f(y), \alpha t_2(v) + \beta t_2(w))\end{aligned}$$

where the above computation used the $p(e) = f(y)$ as well as the predefined vector space structure of the trivial bundle. By a similar computation, t_2 linear gives that τ is linear over the fiber and thus a homomorphism.

To get that τ is an isomorphism, it suffices to show that that the inverse function is continuous. An explicit expression for $\tau^{-1} : f^{-1}(U) \times V$ follows.

$$\tau^{-1} : (y, v) \mapsto (y, t^{-1}(f(y), v))$$

Using $t \circ t^{-1} = \mathrm{Id}$ and $t^{-1} \circ t = \mathrm{Id}$, it follows that the above is indeed the inverse expression. Further, t^{-1} continuous gives that τ^{-1} continuous and so τ is an isomorphism, completing the verification of F a vector bundle.

It still remains to show that F has the defining property of the pullback. For this, take the function $h : F \to E$ to be the projection onto E.

$$h : (y, e) \mapsto e$$

Next, fix an element $y \in Y$ and consider the fiber $q^{-1}(y)$. The restriction $f(y) = p(e)$ ensures that $h((y, v)) = v$ is an element of $p^{-1}(f(y))$. Finally, the conclusion that h is a linear map from the fiber $q^{-1}(y)$ to the fiber $p^{-1}(f(y))$ follows quickly from the vector space structure of E.

$$h((y, \alpha v + \beta w)) = \alpha v + \beta w = \alpha h((y, v)) + \beta h((y, w))$$

Concluding the proof.

As a side note, observe that $p \circ h = f \circ q$ follows by the condition $f(y) = p(e)$ in the construction.

$$(p \circ h)((y, e)) = p(e) = f(y) = (f \circ q)((y, e))$$

This justifies drawing the commutative diagram /*ref*/ which hopefully helps in keeping track of variables for this proof.

$\square$

Proof. . $\square$