Proof. Take a compact Hausdorff space X. It must be verified that the set of stable isomorphism classes of vector bundles over X with operations defined by the direct sum $\oplus$ and the tensor product $\otimes$ indeed satisfies all the properties of a commutative semiring with additive cancellation.

Before proceeding further, it must be verified that addition is well defined. So, take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X. Then, take nonnegative integers n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Then it follows that $E_1 \oplus F_1 \approx_s E_2 \oplus F_2$ by the following chain of equalities.

$$(E_1 \oplus F_1) \oplus \varepsilon^{n+m} \approx (E_1 \oplus \varepsilon^n) \oplus (F_1 \oplus \varepsilon^m) \approx (E_2 \oplus \varepsilon^n) \oplus (F_2 \oplus \varepsilon^m) \approx (E_2 \oplus F_2) \oplus \varepsilon^{n+m}$$

Where the equivalence $\varepsilon^{n+m} \approx \varepsilon^n \oplus \varepsilon^m$ /*reference*/ was used.

With addition well defined, the associativity and commutativity of addition follows directly from the associativity and commutativity of the direct sum on vector bundles /*reference*/. Further, the result $E \oplus \varepsilon^0 \cong E$ for any vector bundle E /*ref*/ makes the equivalence class $[\varepsilon^0]$ the additive identity.

The additive cancellation follows from /*reference $E \oplus E'$ trivial result*/, which applies here by X compact Hausdorff. Indeed, take bundles E, F, and S over X such that $[E] + [S] = [F] + [S]$. First note that in the case of S trivial, $[E] = [F]$ by definition. Otherwise, by /*ref*/, there exists a bundle S such that $S \oplus S'$ is trivial. Adding $[S']$ to both sides reduces the expression to the first case with $[E] + [S \oplus S'] = [F] + [S \oplus S']$, giving $[E] = [F]$ as desired.

Before proceeding with any multiplicative verifications, it must be verified that the tensor product $\otimes$ gives a well defined multiplicative operation. So, again take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X and nonnegative integers n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Next, define the bundle M by

$$M \approx \varepsilon^n \otimes (F_1 \oplus \varepsilon^m) \oplus \varepsilon^m \otimes (E_1 \oplus \varepsilon^n) \approx \varepsilon^n \otimes (F_2 \oplus \varepsilon^m) \oplus \varepsilon^m \otimes (E_2 \oplus \varepsilon^n)$$

Next, observe that M is constructed exactly so that the relation $E_1 F_1 \oplus M \approx E_2 F_2 \oplus M$ holds:

$$E_1 \otimes F_1 \oplus M \approx (E_1 \oplus \varepsilon^n)(F_1 \oplus \varepsilon^m) \oplus \varepsilon^n \varepsilon^m \approx (E_2 \oplus \varepsilon^n)(F_2 \oplus \varepsilon^m) \oplus \varepsilon^n \varepsilon^m \approx E_2 \otimes F_2 \oplus M$$

So, take M' to be the bundle such that $M \oplus M'$ is trivial as promised by /*ref*/. Then, the desired conclusion follows easily, giving that multiplication is well-defined.

$$E_1 \otimes F_1 \oplus (M \oplus M') = E_2 \otimes F_2 \oplus (M \oplus M')$$

With multiplication well defined, the associativity and commutativity of multiplication follows directly from the associativity and commutativity of the tensor product on vector bundles /*reference*/. Similarly, the distributivity of $\otimes$ over $\oplus$ in vector bundles /*ref*/ gives that the defined multiplication distributes over the defined addition. Finally, the result $E \otimes \varepsilon^1 \cong E$ for any vector bundle E /*ref*/ makes the equivalence class $[\varepsilon^1]$ the multiplicative identity.

$\square$