1 Abstract

Given some object, such as a drum, there exists a spectrum of fundamental frequencies determined by physics. However, if the drum was in a neighboring room and you could only listen to these frequencies, is it possible to deduce the drum's shape? In other words, "Can you hear the shape of a drum?". In this research project, we ask a similar question but for abstract mathematical objects. The vibrational frequencies for abstract objects correspond to the eigenvalue spectrum of the Laplace operator associated to the object.
The abstract shapes we study are called *orbifolds*. An orbifold is a multidimensional object that is allowed to have some "trouble spots," which are tied to the symmetries allowed in n-dimensional space. We ask: Given the Laplace spectrum of an unknown orbifold $\mathcal{O}$, what properties of $\mathcal{O}$ are determined? We show that one can hear the *local orientability* of an orbifold. That is, we can use the Laplace spectrum to detect trouble spots associated to orientation reversing symmetries of n-dimensional space.

2 Symmetries in Space

2 examples.. symmetrical b/c can perform some action on the piece of paper that leaves the paper unchanged.
- Γ.
- isometries
- orientation preserving
- orientation reversing.

Introduce strata?

3 Orbifold

3.1 Intuitive

We consider abstract objects called *orbifolds*. Visually, we represent 2 dimensional orbifolds as a surface in space with a few "trouble spots".
- 2 dimensions: torus with "trouble spots"
- In this case, the trouble spots correspond to symmetries of 2 dimensional space
- rotational symmetry
- reflectional symmetry

Gets "folded" in half or "twisted" like a party hat, resulting in

3.2 Formal

Formally, an *orbifold* is a generalization of the Riemannian Manifold (some n dimensional surface). While a manifold requires local structure of $\mathbb{R}^n$, an orbifold allows local structure of $\mathbb{R}^n/\Gamma$ where Γ is a group of isometries.

4 Laplace

4.1 Intuitive/motivation

When you pluck a string, the string produces a sound determined by a specific resonance frequency it can vibrate at. Actually, the string can vibrate at a *spectrum* of resonance frequencies. Furthermore, all objects, not just strings, vibrate at a certain spectrum as observed by the sounds that drums produce. Mathematically, we express these frequencies through the *Laplace Spectra*.

4.2 Formal

The Laplace Spectra follows from the solutions to the following PDE called the "wave equation".

$$\Delta u(t, \mathbf{x}) = \frac{\partial u(t, \mathbf{x})}{\partial t}$$

(formal definition of the Laplace Spectra)
I will now give a more mathematically grounded definition of the Laplace Spectra. Consider some manifold $\mathcal{M}$. Let $u(t, \mathbf{x})$ be the displacement of some point x on $\mathcal{M}$ at time t from equilibrium. (picture). We then define the energy of the manifold $E_{\mathcal{M}}$ to behigher the faster the surface moves and the more stretched it is. /*get exact formula?*/ In applying conservation of energy ($\frac{\partial E_{\mathcal{M}}}{\partial t} = 0$), we derive the following PDE known as the wave equation

$$\Delta u(t, \mathbf{x}) = \frac{d^2 u}{dt^2} \tag{1}$$

We are looking for fundamental frequencies, which we define to be waves of the form $u(t, \mathbf{x}) = A(t)\psi(\mathbf{x})$. With this, we break down the wave equation into the following.

$$\Delta\psi(\mathbf{x}) = -\lambda\psi(\mathbf{x}) \text{ and } \Delta A(t) = -\lambda A(t) \tag{2}$$

For some λ. Only discrete values of λ solve this equation. These values are represented $\lambda_1, \lambda_2, \ldots$ and called the Laplace Spectra.

5 The Question:

Given a specific drum, it is possible to deduce what sound the drum will make when hit. However, consider the reverse: if you hear a drum in the neighboring room, is it possible to deduce the drum. Put nicely by /**/, can you hear the shape of a drum? This is what the field of *Inverse Spectral Geometry* attempts to answer. In this research, we apply Inverse Spectral Geometry techniques to Orbifolds.
Formally, consider some laplace spectra $\lambda_1, \lambda_2, \ldots$ (as defined in /**/) belonging to some unknown orbifold $\mathcal{O}$ (as defined in /**/). From the laplace spectra, what properties can we deduce about $\mathcal{O}$?

6 Asymptotic Heat Expansion Technique

6.1 Intuitive

Consider heating a point $\mathbf{x}$ on some orbifold $\mathcal{O}$ with a match. Then, allow the heat to disperse around $\mathcal{O}$. The temperature of the initial value will decrease.

6.2 Formal

The solution to the following PDE (the heat equation)

$$-\Delta u(t, \mathbf{x}) = \frac{\partial u(t, \mathbf{x})}{\partial t}$$

is of the form $u(t, \mathbf{x}) = \int_{\mathcal{M}} K(t, \mathbf{x}, \mathbf{y}) \mu_0 dvol \mathcal{M}_y$.

$$\mathrm{Tr}(K) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sim a_0 t^1$$

7 Results

7.1 Locally Orientable

First, we define *locally orientability*.

7.1.1 Intuitive

Intuitively, ...

7.1.2 Formal

If for $g \in \Gamma$, $\det(g) = -1$. Γ is orientation reversing. A chart $(., ., .)$ on $\mathcal{O}$ is defined to be *orientable* if there exists some orientation reversing operation in Γ.