COMPLEX VARIABLES: EXAM 1

1 Geometry of the complex plane

Find **all** the values of the following, and present them in the $u+iv$-form. I specifically want you to approach this problem via the exponential (polar) form of complex numbers.

1. $(\sqrt{3}+i)^{2016}$.

2. $(e^4)^{1/4}$. (Why am I asking you this question?)

2 Elementary functions of complex variable

1. Express $\exp\left(\frac{\pi}{1+i}\right)$ in the $u+iv$-form. Simplify as far as possible.

2. Find all values of $\log(e^2)$ and $2\log(e)$. How do the two sets of values compare?

3. Express $\cos(\frac{\pi}{4}+i\ln(2))$ in the $u+iv$-form. Simplify as far as possible.

4. Find $P.V\,(1+i)^i$.

3 Visualization of functions of complex variables

Describe in geometric terms the effect of the mapping $f(z)=\sinh(z)$ on the horizontal strip

$$-\frac{\pi}{2}i \leq \operatorname{Im}(z) \leq \frac{\pi}{2}i.$$

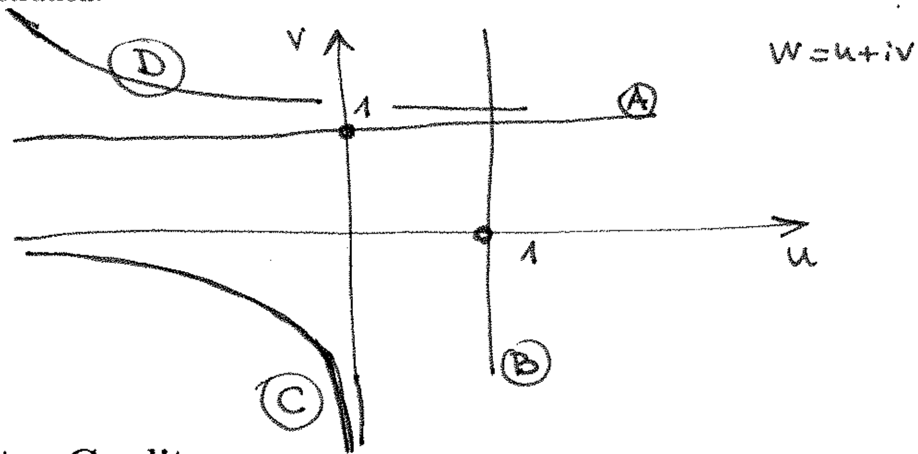
Specifically, sketch the *images* of the coordinate grid-lines of the "input" z-plane in the "output" w-plane.

4 Riemann surfaces and branches

1. Find a branch $f(z)$ of the multivalued $\sqrt[3]{\ }$ with a branch cut along the positive real axis such that
$$f(-1)=\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$
Give an explicit (piece-wise) formula relating your branch to P.V. $\sqrt[3]{z}$. Alternatively, provide an interpretation in terms of the Riemann surface. In any case, sketch the range for your branch.

(over)

2. Consider complex logarithm as a function defined on its Riemann surface. Find the pre-image, on the Riemann surface, of the following. Please provide an accompanying illustration!



5 Extra Credit:

There was this thing in the series part of the basic calculus sequence called absolute convergence. Together with what is known as the Comparison Test it states something like this: *If* $\sum a_n$ *is a convergent series with positive terms and if* b_n *is a sequence such that* $|b_n| \leq a_n$*, then* $\sum b_n$ *is also convergent.* The same argument used back in the day justifies this result in the context of complex variables. Now that you are reminded of this.....

Let s be a complex number, and consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\mathrm{P.V}\, n^s} = \frac{1}{\mathrm{P.V}\, 1^s} + \frac{1}{\mathrm{P.V}\, 2^s} + \frac{1}{\mathrm{P.V}\, 3^s} + \ldots.$$

For what values of s is this series convergent?