Differential Equations

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Contents

Part I

Syllabus

Educational Goals

Goal 1: Develop an ability to express (or read about) a continuous phenomenon in the language of differential equations, and become aware of the three main strategies people employ when studying solutions of differential equations.

Goal 2: Perform equilibrium point analysis, including the analysis of the linearized version of the system near the equilibrium points.

Goal 3: Develop fluency in dealing with second order equations having to do with oscillations and / or conservation of energy.

Goal 4: Have the experience of learning, within the structured environment of the course, something beyond the content explicitly covered in class and then report on it.

Grading scheme

For each rubric under Educational Goals above you will receive a letter grade determined by your performance on the corresponding exams. The final course grade will be the weighted average of the above, with each goal worth 25% of the course grade.

For the description of letter grades and their numerical equivalents please refer to our College Catalog. Please note that a professor has a right to withdraw a student for the reasons of non-attendance. *I reserve the right to fail any student who exhibits an extreme lack of understanding of multivariable and vector calculus on the final exam. I also reserve the right to boost the course grade of any student who presents an impressive amount of progress throughout the semester.*

Textbook, calculators etc.

The main reference is **Differential Equations,** by **Blanchard, Devaney and Hall**. Any edition is fine. Note that the textbook is not required. **Do note that much of the course content will not be directly based on the textbook material.** I expect you to rely on your own lecture notes.

I expect you to have access to a calculator for basic homework and exam problems. In addition, I will expect you to develop an ability to use a more substantial computing software. Although I do not have a strong preference as to what you use, I will give you some basic introduction to a computing software called Sage. There will be two sessions in the **Dubach Lab** in which we would be going over basics of Sage: one on Tuesday **September 12th** and on Wednesday **September 13th**. We have the lab from **6 pm to 8pm**. You are expected to attend one of these sessions. More code information is included in the chapter on Sage in this packet.

Homework

Homework problems for the semester can be found later on in this packet. The point of the homework problems is to provide a structure and a pace for you to study the subject of differential equations. I have absolutely no desire to use homework assignments for grading purposes.

Homework solutions will be available for viewing in the SQRC – ask the tutors. It is your responsibility to make sure you are doing your problems correctly.

Homework will be due in the box by my office door labeled “Differential Equations”. My graders and I will go through the box about once a week and record attempts to complete the homework at the suggested pace. Students who are clearly behind (be that mathematically or schedule-wise) will be called in, or in extreme cases: asked to drop the class. **Admittance to midterm exams is subject to demonstrated effort to do the homework at the suggested pace.** Do me and yourself a favor – take this seriously.

Exams and such

MIDTERM EXAMS: There will be three midterm exams. **Only students who have demonstrated that they have attempted the homework will be allowed to take the exams.** To provide everyone with enough time to complete the problems **all exams will be evening exams**. The first midterm exam will take place on **Monday, October 2nd**, the second midterm exam will take place on **Thursday, November 2nd** and the third midterm exam will take place on **Wednesday, November 29th**. Students with class conflicts will be given an opportunity to take the exam at a different time. I recommend setting aside three hours for the exams, just in case.

FINAL PROJECT / EXAM: This course will have no final exam per se. Instead, the students will be turning in their final projects. The time to work on the final project is between December 1st and the official exam day, December 19th. The descriptions of the projects will be given in class on December 1st.

Day-to-day schedule

Wednesday, September 6th: Overview of analytic, qualitative and numeric methods.

Friday, September 8th: Learn how to read and write!

Monday, September 11th: Introducing ... stream plots!

Wednesday, September 13th: NO CLASS – lab day.

Friday, September 15th: Fundamental Theorem of Ordinary Differential Equations via slope fields.

Monday, September 18th: Fundamental Theorem of Ordinary Differential Equations – applications.

Wednesday, September 20th: Numerical Methods: The Euler Method.

Friday, September 22nd: Analytic Methods: Separable Equations.

Monday, September 25th: Analytic Methods: Propagators, day 1.

Wednesday, September 27th: Analytic Methods: Propagators, day 2.

Friday, September 29th: Exam review.

Monday, October 2nd: NO CLASS – EVENING EXAM!

Wednesday, October 4th: Introduction to First Order Systems.

Friday, October 6th: Introduction to First Order Systems, day 2.

Monday, October 9th: Linear Theory – Superposition Principle.

Wednesday, October 11: (Real) eigenvalues.

FALL BREAK.

Monday, October 16th: Real eigenvalues – practice.

Wednesday, October 18th: Complex eigenvalues.

Friday, October 20th: Complex eigenvalues – practice.

Monday, October 23rd: Linearization in general; application to ODE's.

Wednesday, October 25th: Linearization as it pertains to systems of ODE's.

Friday, October 27th: Practice.

Monday, October 30th: The method of nullclines.

Wednesday, November 1st: Exam Review.

Friday, November 3rd: NO CLASS – EXAM THE NIGHT BEFORE.

Monday, November 6th: Oscillations and second order ODEs.

Wednesday, November 8th: Linear oscillations without driving terms.

Friday, November 10th: Linear oscillations with driving terms, day 1.

Monday, November 13th: Linear oscillations with driving terms, day 2.

Wednesday, November 15th: Resonance.

Friday, November 17th: Non-linear oscillations and conservation of energy, day 1.

Monday, November 20th: Non-linear oscillations and conservation of energy, day 2.

Wednesday, November 22nd: Damped non-linear oscillations.

THANKSGIVING!!

Monday, November 27th: Exam review.

Wednesday, November 29th: NO CLASS – EVENING EXAM.

Friday, December 1st: Discussion of the final exam project, course evaluations.

Monday, December 4th: Time to work on the project.

Wednesday, December 6th: Time to work on the project.

Friday, December 8th: Time to work on the project.

Monday, December 11th: Time to work on the project.

Part II

Homework Problems

Chapter 1

Learning how to read and write differential equations

1.1 Overview of analytic, qualitative and numeric methods

Homework problems, due Thursday, September 7th

1. Consider the equation $e^{-x} = \alpha x$. The only thing you know about α is that it is some real number. Could be positive. Could be zero. Could be negative. You don't know.

   (a) **Qualitative Analysis:**

   i. Does the said equation have solutions? How many? What's your argument in the case when $\alpha > 0$? What if $\alpha = 0$? What if $\alpha < 0$?

   ii. What can you say about the solutions – how do they change with α? Do they increase? Decrease? Can you draw the graph where the independent variable is α and the dependent variable is the supposed solution of the equation $e^{-x} = \alpha x$? (If the answer is yes, then do draw it!) (Not funny, I know.)

   (b) **Numerical Approach:** Fix $\alpha = 2$, and try to get the numerical value of the supposed solutions of $e^{-x} = 2x$. Try to get it within 0.25 of whatever the solution actually is. Try it. See how it goes. Tell me about it.

   (c) **Analytical Approach:** Can you actually solve the equation $e^{-x} = \alpha x$ for x? Like ... get the actual value of x? What if $\alpha = 2$?

1.2 Learn how to read and write!

Homework problems, due Thursday, September 14th

1. Construct a differential equation which models the following situations.

   (a) An investment $y(t)$ grows with relative growth rate of 5% per year;

   (b) \$1000 is invested with annual interest rate of 5%;

(c) Continuous deposits are made into an account at the rate of \$1000 a year. In addition to these deposits, the account earns 7% interest per year.

(d) Illia takes out a loan with an annual interest rate of 6%. Continuous repayments are made totaling \$1000 per year.

(e) A fish population under ideal conditions grows at a relative growth rate k per year. The carrying capacity of their habitat is N and H fish are harvested each month.

(f) A fish population under ideal conditions grows at growth rate k per year. The carrying capacity of their habitat is N and one quarter of the fish population is harvested annually.

(g) Due to the pollution problems the relative growth rate of a fish population is a decreasing exponential function of time. Adjust your equation(s) from the previous problems accordingly.

2. Some quantity f changes with time, is measured in gallons, and is modeled by the differential equation
$$\frac{df}{dt} = k\,f\left(1 - \frac{f}{M}\right).$$
The time variable t is measured in months and the parameter M is also measured in gallons.

   (a) In what units is $\frac{df}{dt}$ measured?

   (b) In what units is $1 - \frac{f}{M}$ measured?

   (c) In what units is $f\left(1 - \frac{f}{M}\right)$ measured?

   (d) In what units is the parameter k measured?

3. A variable quantity $r = r(t)$ is measured in grams per liter and is modeled by the differential equation
$$\frac{dr}{dt} = -k \cdot \frac{r}{c+r}.$$
The time variable t is measured in seconds. Figure out the units for the parameters k and c.

4. We consider a large urn of coffee in cafeteria. A student is draining coffee in to her thermos while at the same time the brewing machine is adding fresh coffee. We want to keep track of the amount of caffeine (in milligrams) in the urn during this process.

   Suppose the following:

   - There is initially 4 liters of coffee in the urn, with a caffeine concentration of 100 mg per liter.
   - Starting at time $t = 0$ the brewing machine is adding extra strong coffee, which has a concentration of 250 mg per liter. This new coffee is being added at a rate of 25 mL per minute.

- Starting at time $t = 0$ the student begins draining coffee out of the urn at a rate of 30 mL per minute.

Write down a differential equation which models the amount of caffeine in the urn as a function of time.

5. In this problem we model the number of junk emails in Iva's inbox with a continuous function $J(t)$. Suppose the following:

   - When the semester started ($t = 0$), Iva had 6000 emails in her inbox; 4000 of them were junk.
   - Email is continuously flowing in to Iva's email inbox at a rate of 50 per day; 30% of the emails are junk.
   - Each day, Iva randomly picks 20 emails to deal with – once they have been dealt with, she moves them out of her inbox.

   Write down a differential equation describing the number of junk emails in Iva's inbox.

6. A big mixing vat contains 50 liters of a mixture in which the concentration of a certain chemical is 1.25 grams per liter. This mixture is being diluted by another mixture in which the concentration of the same chemical is 0.25 grams per liter. Each minute 6 liters of the less concentrated mixture are poured into the vat and 4 liters of the resulting new mixture are drained out. Construct a differential equation modeling this process.

1.3 Slope Fields and Such...

Homework problems, due Monday, September 18th

1. For each differential equation below you need to

   - Make plots (using Sage) of these solutions,
   - Generate a slope field plot (using Sage), and
   - Sketch some solution plots by hand on top of the slope field plot.

   Turn in the printouts of what you got. You should also write a sentence or two describing the long-term behavior of solutions.

   (a) $\dfrac{dy}{dt} = -y + t + 1;$

   (b) $\dfrac{dy}{dt} = y(9 - y^2);$

   (c) $\dfrac{dy}{dt} = y + 2t.$

2. Without using 'technology', match the plots of solutions in Figure 1.1 to the corresponding differential equation:

(a) $\frac{dy}{dt} = (y-1)(y-2e^t)$
(b) $\frac{dy}{dt} = (y+1)(y+e^t)$
(c) $\frac{dy}{dt} = y\left(1 - \frac{y}{2-\sin(t)}\right)$
(d) $\frac{dy}{dt} = y + \sin(t)$.

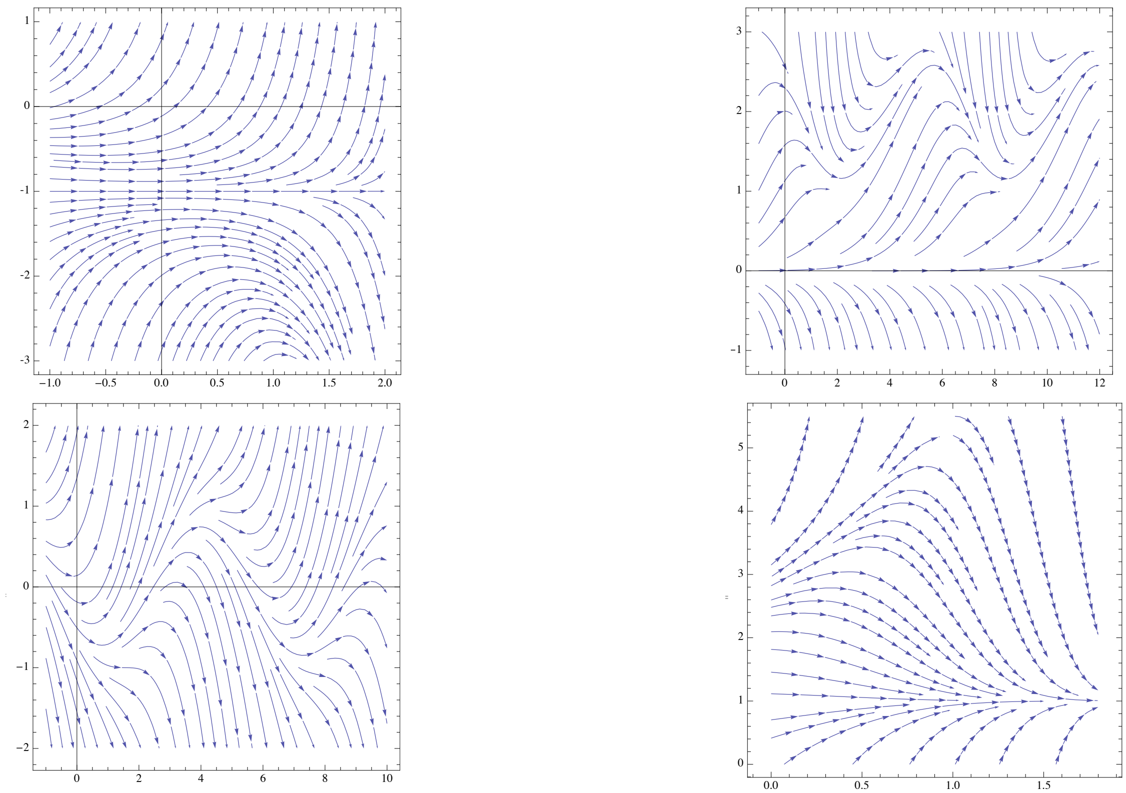


Figure 1.1: Plots showing various solutions to differential equations.

1.4 Initial Value Problem; The Fundamental Theorem of ODE's

Homework problems, due Thursday, September 21st

1. Verify that for any constant C, the function

$$y(t) = Ce^{6t} - \frac{1}{12} - \frac{1}{2}t$$

satisfies the differential equation

$$\frac{dy}{dt} = 6y + 3t.$$

Use this to solve the IVP

$$\frac{dy}{dt} = 6y + 3t \qquad y(0) = 10.$$

2. Consider the IVP
$$\frac{dy}{dt} = y^3, \qquad y(0) = 1.$$
   (a) Verify that the function
   $$y(t) = \sqrt{\frac{1}{1-2t}}$$
   solves this IVP.
   (b) Graph this solution.
   (c) State the domain on which this solution is defined.
   (d) Describe what happens to the solution as the independent variable approaches the endpoints of the domain. Why can't the solution be extended for more time?

3. Consider the initial value problem
$$\frac{dy}{dt} = y^{1/2} \qquad y(0) = 0.$$
   (a) Show that $y(t) = t^2/4$ is a solution to the IVP.
   (b) Show that $y(t) = 0$ is a solution to the IVP.
   (c) This IVP has two solutions. Have we violated the part of FTODE that states there is only one solution? Explain.

4. Find all equilibrium solutions of the following equations:
   (a) $\frac{dP}{dt} = 2P\left(1 - \frac{P}{10}\right) - 4.$
   (b) $\frac{dy}{dt} = y(1+y)(y - e^t).$
   (c) $\frac{dy}{dt} = (y + \sin t)(y+1).$

5. Consider the IVP
$$\frac{dy}{dt} = -y^3 + y^2 + 12y, \qquad y(0) = 1.$$
Perform the qualitative analysis of the differential equation, and based on your conclusions make an educated guess about the long-term fate of the solution of our IVP. You should check your answer with Sage and turn in the print out of the slope field.

6. Show that $y_1(t) = 1$ and $y_2(t) = 1 + 2e^{-t}$ are solutions of the differential equation
$$\frac{dy}{dt} = (1-y)(y - 2e^{-t}).$$
Based on the fact that $y_1(0) = 1$ and $y_2(0) = 3$ say something useful about solutions of the differential equation $\frac{dy}{dt} = (1-y)(y-2e^{-t})$ which obey the initial condition $1 < y(0) < 3$.

7. Consider the differential equation

$$\frac{dy}{dt} = y\left(\frac{y}{2e^t} - 1\right)$$

One can easily see that $y_1(t) = 0$ and $y_2(t) = 4e^t$ are both solutions of this differential equation.

(a) Produce a rough sketch of the slope field corresponding to this equation **without** using "technology". Pay particular attention to the slope field along the curve $y = 2e^t$, between the curves $y = 2e^t$ and $y = 4e^t$, between $y = 2e^t$ and $y = 0$, above the curve $y = 4e^t$ and below the line $y = 0$. Check your answer on Mathematica.

(b) A certain variable quantity $y(t)$ satisfies the differential equation in this problem. It is known that $y(0) \approx 4$, but the exact value of $y(0)$ is not known. What can you say about the long term behavior of this quantity i.e. about $\lim_{t\to+\infty} y(t)$?

1.5 The Euler Method

Homework problems, due Monday, September 25th

1. Create an approximate table of values for the solution of the IVP

$$\frac{dy}{dt} = \cos(y), \quad y(0) = 0$$

over the interval $0 \le t \le 1$. Please focus on the step-size of $\Delta t = 0.25$.

2. As you know, the function $y(t) = e^t$ is *the unique* solution of the IVP

$$\frac{dy}{dt} = y, \qquad y(0) = 1.$$

In this problem you get to investigate the power and the limitation of the Euler method.

(a) Make a table which shows *the actual values* of $y(0)$, $y(0.25)$, $y(0.5)$, $y(0.75)$ and $y(1)$.

(b) What does the Euler method for step-size of $\Delta t = 0.25$ predict for the values of $y(0)$, $y(0.25)$, $y(0.5)$, $y(0.75)$ and $y(1)$?

(c) What does the Euler method for step-size of $\Delta t = 0.05$ predict for the values of $y(0)$, $y(0.25)$, $y(0.5)$, $y(0.75)$ and $y(1)$?

(d) Make a table that lists the errors that the two approximation methods make. Then write a sentence or so interpreting what you see.

(e) Represent your findings graphically in the t–y plane. Then write a sentence or so interpreting what you see.

3. Consider the differential equation
$$\frac{dy}{dt} = y(y-3)(y-6).$$
   (a) Perform qualitative analysis of this equation.
   (b) In addition to the differential equation above consider the initial condition $y(0) = 1$. Perform Euler method with the step size $\Delta t = 0.5$ in order to understand $y(t)$ for $0 \leq t \leq 2$. Make sure your answer includes the table of values and the corresponding piece-wise linear graph.
   (c) In light of part (a) please discuss the reliability of your answer in part (b).

4. Consider the initial value problem
$$\frac{dy}{dt} = y^3, \qquad y(0) = 1.$$
   (a) Using the Euler's Method with $\Delta t = 0.5$, graph an approximate solution over the interval $0 \leq t \leq 1$.
   (b) What happens if you make Δt considerably smaller than above (e.g. $\Delta t = 0.05$)?
   (c) Verify that the function
$$y(t) = \frac{1}{\sqrt{1-2t}}$$
   is a solution to the IVP in this problem. Use this knowledge to interpret the findings in parts a) and b) of this problem.

1.6 Analytic Methods: Separable Equations

Homework due Monday, September 25th

1. Find the general solution of the following equations:
   (a) $\frac{dy}{dt} = y^3$;
   (b) $\frac{dy}{dt} = e^{-t}(2-y)$;
   (c) $\frac{dy}{dt} = y(3-y)$;
   (d) $\frac{dy}{dt} = y(3-y) - 2$.

2. A person makes an initial payment of \$3 000 to a retirement fund, and plans to contribute \$6 400 each year continuously for 40 years until the person is 65. The fund will earn 8% a year.
   (a) Find an expression for the value of the fund t years after the initial payment was made;
   (b) What is the value of the fund at age 65;
   (c) Assume that no payments are going to be made after the age 65 and that the same amount of money is going to be taken from the account each month. How big can this sum of money be in order for the retirement fund to last another 30 years?

3. Assuming unlimited resources some particular species of fish grows at a steady per capita rate of 200% per year. However, it is believed that the lake in which this species lives can only support up to 10 000 fish. The lake is private and the owners would like to (continually throughout the year) harvest about 4 800 fish. The current fish population in the lake is about 4 200. Find an expression for the number of fish as a function of time. Graph your solution. Would you agree that the owner is "overharvesting"?

4. A 100-gallon mixing vat is initially full of brine in which the concentration of salt is $0.5\ \frac{\text{lb}}{\text{gall}}$. Pure water is pumped into the tank at the rate of $3\frac{\text{gall}}{\text{min}}$. Simultaneously 3 gallons of brine per minute are being pumped out. The vat is kept thoroughly mixed at all times.

   (a) Find the expression for the amount (in lb's) of salt in the vat after t minutes;

   (b) When will the concentration of salt drop below $0.1\ \frac{\text{lb}}{\text{gall}}$?

1.7 Analytic Methods: Propagators

Homework due Thursday, September 28th

1. Verify (by direct computation) the properties of the propagator function $P(t, s)$:

   (a) $P(s, s) = 1$,

   (b) $\frac{d}{dt}P(t, s) = r(t)P(t, s)$,

   (c) $P(t, s) = P(s, t)^{-1}$,

   (d) $P(t_2, t_1)P(t_1, t_0) = P(t_2, t_0)$.

2. Solve the following initial value problems:

   (a) $\frac{dy}{dt} = 2y - t, \quad y(0) = 1$

   (b) $\frac{dy}{dt} = \frac{y}{1+2t} - 7, \quad y(0) = 1.$

   (c) $\frac{dy}{dt} = y + \sin(t), \quad y(0) = 1$

   (d) $\frac{dy}{dt} = \frac{y}{t} + t, \quad y(1) = 0.$

3. Suppose money is invested in a volatile market that has an annual growth rate of $r(t) = 0.01 + 0.05 \cos 10t$, where t is measured in years.

   (a) Make a plot of $r(t)$ over a 10 year time period. How should one interpret this growth rate?

   (b) Suppose there is an initial investment of \$100. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?

4. Suppose instead that no money is initially invested, but that one continuously adds to an investment at a rate of \$10 per year. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?

5. A 100-gallon mixing tank is full of pure water at time $t = 0$. Salty water of salt concentration $0.4\frac{\text{lb}}{\text{gal}}$ is being pumped into the tank at a decreasing rate of $e^{-0.05t}\frac{\text{gal}}{\text{min}}$. The resulting salty water is also being drained from the tank so that its volume is kept constant at 100 gallons. Assuming the tank is always thoroughly mixed, find the amount of salt in the tank (in pounds) at time t. What will the concentration of salt roughly become in the long run?

6. A 1 000-gallon tank is full of pure water. Salty water is being pumped into the tank at a decreasing rate of $\frac{1000}{10+t}$ gallons per hour; here the variable t denotes the number of hours since the beginning of the mixing process. The concentration of salt in the solution which is pumped into the tank is 0.01 pounds per gallon. The tank is constantly being mixed and drained so that the volume of the tank is maintained at 1 000 gallons. How much salt will there be in the tank in the long run?

Chapter 2

First Order Systems

2.1 Introduction to First Order Systems

Homework due Monday, October 9th

1. Match the pictures in Figure 2.1 with the parametric equations below, and indicate the direction in which the given curves are traversed. You can use Sage to prompt you in the right direction. Make sure that in the end you learn something that can help you do such exercises without any help of "technology".

   (a) $x(t) = \cos(2t)$, $y(t) = \sin(2t)$;
   (b) $x(t) = 2\cos(t)$, $y(t) = \sin(t)$;
   (c) $x(t) = e^{-t/10}\sin(t)$, $y(t) = e^{-t/10}\cos(t)$;
   (d) $x(t) = e^{-t/10}\cos(t)$, $y(t) = e^{-t/10}\sin(t)$;
   (e) $x(t) = (1+\cos^2(5t))\cos(t)$, $y(t) = (1+\cos^2(5t))\sin(t)$;
   (f) $x(t) = e^t$, $y(t) = -e^t$;
   (g) $x(t) = \cos(t)$, $y(t) = \cos(t)$;
   (h) $x(t) = 20\cos(t)$, $y(t) = t$;
   (i) $x(t) = t$, $y(t) = \cos(2t)$.

2. Draw the curves with the following parametric equations; indicate the direction in which t increases. You can use Sage to prompt you in the right direction. Make sure that in the end you find a way of making the drawing without any use of "technology".

   (a) $x(t) = \cos\left(\frac{t}{2}\right)$, $y(t) = \sin\left(\frac{t}{2}\right)$;
   (b) $x(t) = e^t\cos(t)$, $y(t) = e^t\sin(t)$;
   (c) $x(t) = t\cos(t)$, $y(t) = t\sin(t)$;
   (d) $x(t) = 2e^t$, $y(t) = -3e^t$;
   (e) $x(t) = e^{-t}$, $y(t) = 2e^{-t}$.

3. Match the graphs in Figure 2.2 with the corresponding phase diagrams in Figure 2.3.

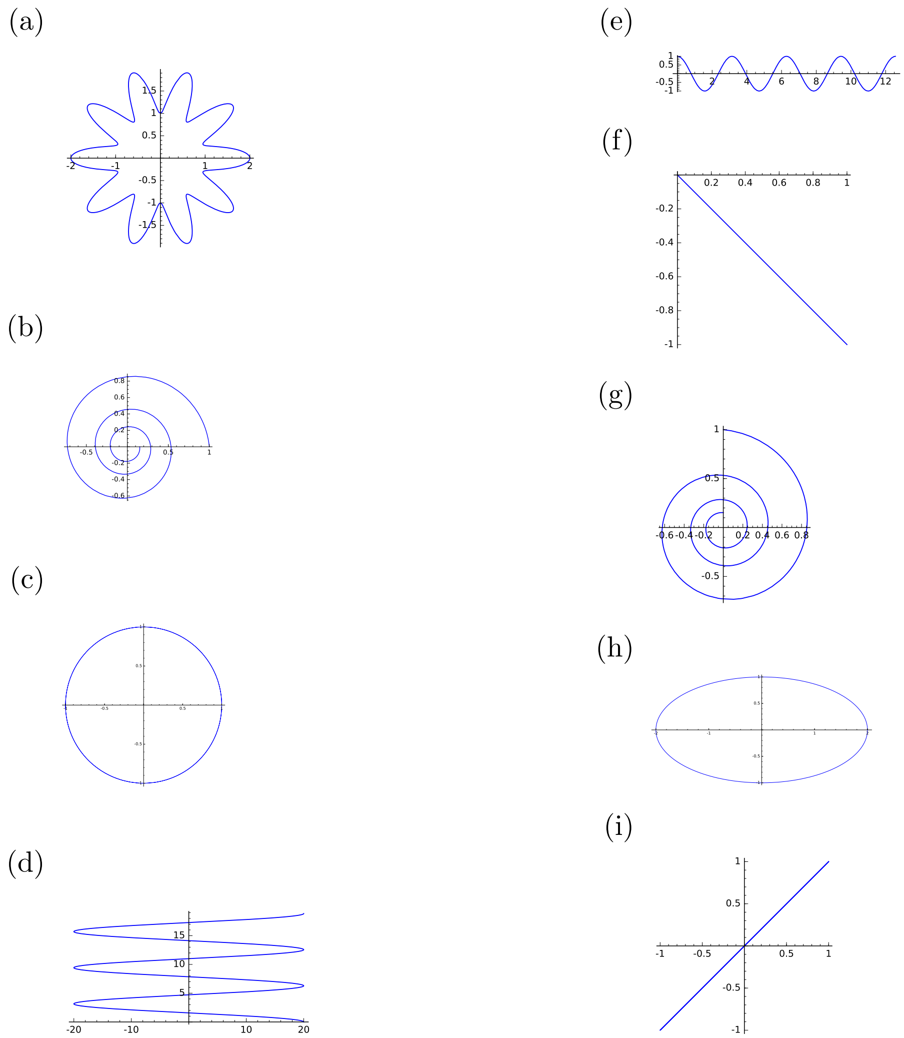


Figure 2.1: Parametric curves corresponding to the functions in Exercise 1.

4. Use Sage to draw the vector fields of the following system of differential equations. Then eyeball several solution curves in the x–y plane. Sketch the corresponding graphs of $x(t)$ and $y(t)$ as functions of t.

   (a) $$\begin{cases} \dfrac{dx}{dt} = x + 2y, \\[2ex] \dfrac{dy}{dt} = -x - y. \end{cases}$$

   (b) $$\begin{cases} \dfrac{dx}{dt} = x + 2y, \\[2ex] \dfrac{dy}{dt} = 4x + 3y. \end{cases}$$

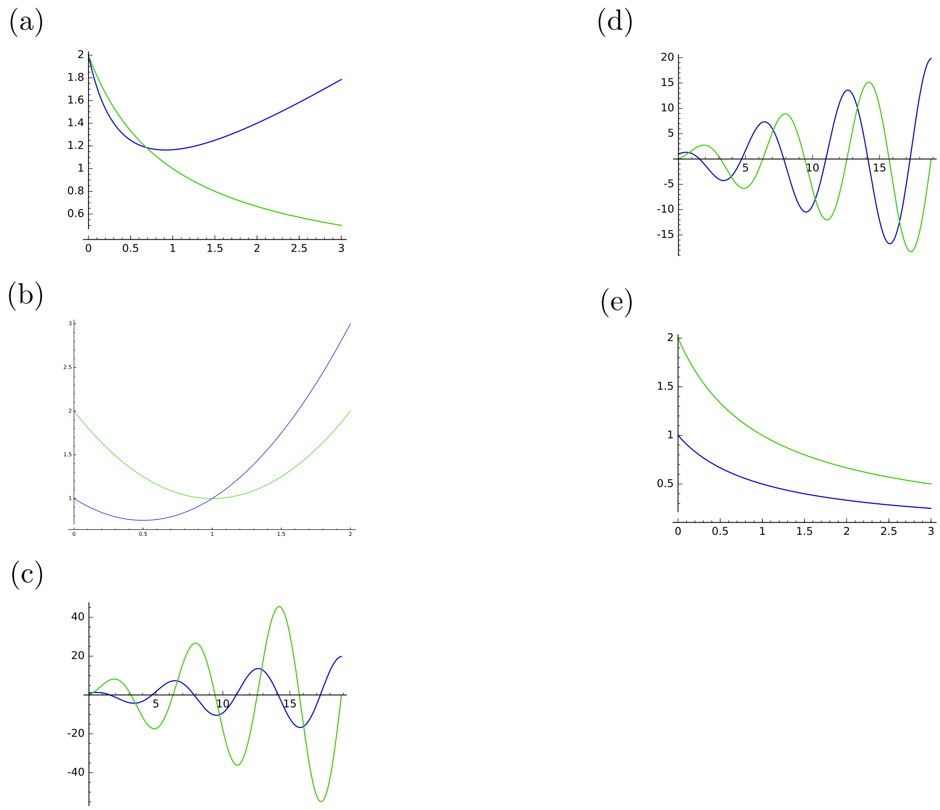


Figure 2.2: Graphs of the functions $x(t)$ and $y(t)$

5. Find all equilibrium solutions of the simple predator-prey model:

$$\frac{dP}{dt} = 2P - PR$$
$$\frac{dR}{dt} = -R + 4PR.$$

Have Sage plot the vector field for the system in a neighborhood of each equilibrium. Determine which equilibria look stable and which appear to be unstable. Finally, have Sage construct a numerical approximation of the solution having initial conditions $(P_0, R_0) = (1, 1)$.

6. Find all equilibrium solutions of the predator-prey model:

$$\frac{dP}{dt} = P\big(1 - P\big) - 10PR$$
$$\frac{dR}{dt} = -\frac{1}{27}R + \frac{1}{9}PR.$$

Have Sage plot the vector field for the system in a neighborhood of each equilibrium. Determine which equilibria look stable and which appear to be unstable. Finally, have Sage construct a numerical approximation of the solution having initial conditions $(P_0, R_0) = (.5, .5)$.

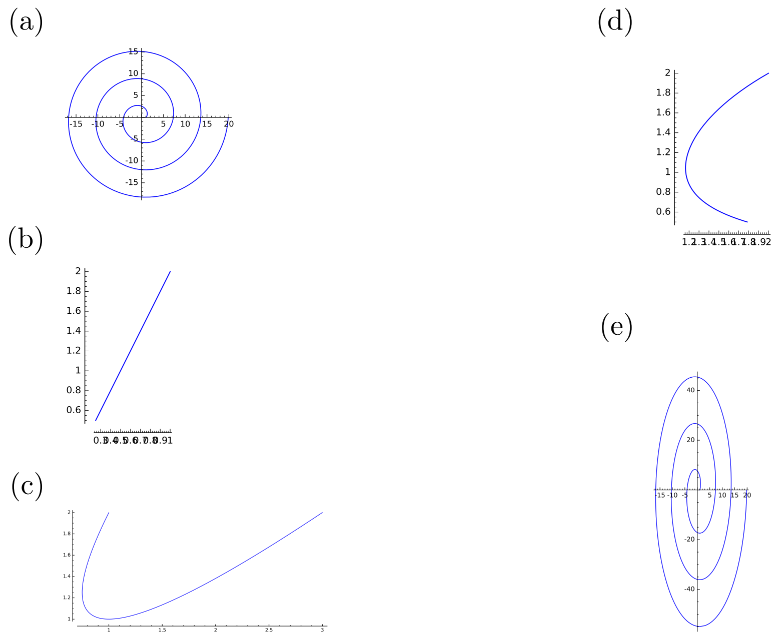


Figure 2.3: Phase diagrams showing the trajectories $(x(t), y(t))$.

2.2 Linear Theory – Superposition Principles

Homework due Thursday, October 12th

1. Re-write the following functions in a matrix form.

   (a) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x \end{pmatrix}$

   (b) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$

2. Verify that $Y_1(t) = e^{6t}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $Y_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are both solutions of the system equations
$$\frac{d}{dt}Y = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} Y.$$

3. Consider the system of equations
$$\frac{dY}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y.$$

   (a) Verify that $Y_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $Y_2(t) = e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are both solutions of this system equations.

   (b) By the Superposition Principle you now know infinitely many solutions of the system. What are they?

(c) Solve the IVP:
$$\frac{dY}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2.3 (Real) eigenstuff

Homework due Thursday, October 19th

1. Study each of the following "systems" by addressing the following questions:

   - Find the general solution of the system;
   - Draw the phase diagram for the system without any use of "technology". Then check your answer with Sage.
   - Discuss the long-term fate of the solutions of the system. Your answer potentially depends on the initial condition $Y(0)$.
   - Discuss the stability of the equilibrium solution $Y(t) = 0$.

   (a) $\dfrac{dY}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} Y$

   (b) $\dfrac{dY}{dt} = \begin{pmatrix} 6 & 9 \\ 3 & 0 \end{pmatrix} Y$;

   (c) $\dfrac{dY}{dt} = \begin{pmatrix} 0.41 & 0.12 \\ 0.12 & 0.34 \end{pmatrix} Y$;

   (d) $\dfrac{dY}{dt} = \begin{pmatrix} 4 & 15 \\ -2 & -7 \end{pmatrix} Y$;

2. A market researcher established that the daily profits of two competing stores, *Ethel's Knick-knack Heaven* and *Irma's Antiques*, relate to each other as in the system:
$$\frac{dx}{dt} = 6x - 2y$$
$$\frac{dy}{dt} = -2x + 3y$$
here $x(t)$ denotes the daily profit of Ethel's store and $y(t)$ denotes the daily profit of Irma's store. The time is measured in months.

   (a) Draw the phase portrait of this system. Use the least amount of computations possible.
   (b) Use the phase portrait only to discuss the long-term fate of these stores if their current profits are:
      i. $x_0 = \$90$ and $y_0 = \$240$
      ii. $x_0 = \$100$ and $y_0 = \$120$
      iii. $x_0 = \$95$ and $y_0 = \$180$
   (c) Use the phase portrait to find the relationship between the current profits which would allow both stores to stay in business.
   (d) Find (analytically) the particular solution of the model corresponding to the current profits of $x_0 = \$90$ and $y_0 = \$240$. Use the knowledge of this particular solution to determine when Ethel's profits are going to be biggest. How big is this profit?

(e) Do the same for Irma's store and the current profits of $x_0 = \$95$ and $y_0 = \$180$.

3. Two 100 gallon mixing tanks TANK_1 and TANK_2 are connected to each other with two pipes, PIPE_1 and PIPE_2. The tanks are both completely filled with salty water. The salty water from TANK_1 flows through PIPE_1 to TANK_2 at the (continuous) rate of $8\frac{\text{gal}}{\text{hr}}$. The salty water from TANK_2 flows through PIPE_2 to TANK_1 at the (continuous) rate of $2\frac{\text{gal}}{\text{hr}}$. The volume of TANK_1 is kept constant by continuous adding of pure water at the rate of 6 gallons per hour. Likewise, the volume of TANK_2 is kept constant by continuous draining at the rate of 6 gallons per hour. Everything is always kept 'perfectly well mixed.'

   (a) Let $s_1(t)$ and $s_2(t)$ be the amount of salt (in pounds, say) in TANK_1 and TANK_2 (respectively). Write a linear system of differential equations modeling $s_1(t)$ and $s_2(t)$.

   (b) Draw the phase portrait of this system.

   (c) What can you say about "the fate of" s_1 and s_2? Is it true that

$$\lim_{t\to+\infty} s_1(t) = \lim_{t\to+\infty} s_2(t) = 0 \quad ?$$

   Which of the two tanks will have more salt in the long run? Does your answer depend on how much salt the tanks initially had?

2.4 Complex Eigenvalues

Homework due Monday, October 23rd

1. Put the following expressions into the form $a + b\,i$, where a,b are real numbers.

   (a) $(1 + 2i)(3 - 4i)$

   (b) $\dfrac{1}{2+i} + \dfrac{1}{1-2i}$

   (c) $\dfrac{2+3i}{1+i}$

   (d) $e^{\frac{i\pi}{4}}$

   (e) $e^{-1+\pi\,i}$

   (f) $2e^{1+i} + 2e^{1-i}$

   (g) $e^{-(2+\pi i)t}$

2. Solve the quadratic equation $x^2 + x + 1 = 0$ within the set of complex numbers.

3. Solve the system of equations:

$$x + iy = 2$$
$$2ix - y = 3i.$$

4. Find the eigenvalues and the eigenvectors of the matrix
$$\begin{pmatrix} -2 & -9 \\ 1 & -2 \end{pmatrix}.$$

5. Assume that a, b and t are some real numbers with $b \neq 0$.
   (a) Identify the real and the imaginary part of the expression $e^{(a+ib)t}$.
   (b) Treat the real and the imaginary part as individual functions of the independent t variable: $f(t)$ and $g(t)$. Graph these two functions. Note that the graphs will look radically different depending on whether a is positive, zero, or negative. Explore all three situations.

6. (a) Find the explicit solution of the following IVP.
$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$
   (b) Suppose $x_0 = 1$ and $y_0 = 2$. What is the corresponding solution to the IVP?

7. Study each of the following systems by addressing the following questions:
   - Find the general solution of the system;
   - Draw the phase diagram for the system without any use of "technology". Then check your answer with Sage.
   - Draw the solution curves in the t–x and the t–y planes.
   - Discuss the long-term fate of the solutions of the system. Your answer potentially depends on the initial condition $Y(0)$.
   - Discuss the stability of the equilibrium solution $Y(t) = 0$.

   (a) $\dfrac{dY}{dt} = \begin{pmatrix} 6 & 9 \\ -5 & -6 \end{pmatrix} Y$;

   (b) $\dfrac{dY}{dt} = \begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix} Y$;

   (c) $\dfrac{dY}{dt} = \begin{pmatrix} 5 & -3 \\ 6 & -1 \end{pmatrix} Y$.

8. A market researcher established that the daily profits of two competing stores, *Ethel's Knick-knack Heaven* and *Irma's Antiques*, relate to each other as in the system:
$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = -5x + 3y$$
here $x(t)$ denotes the daily profit of Ethel's store and $y(t)$ denotes the daily profit of Irma's store. The time is measured in months.

(a) Draw the phase portrait of this system. Use the least amount of computations possible.

(b) Use the phase portrait only to discuss the long-term fate of these stores.

9. The following problem concerns profits of two nearby stores; the model we shall use is based on a linear system of equations. The first store, the profit of which at time t we label by $x(t)$, was successful on its own until recently when a new store opened nearby. This second store, the profit of which at time t we label by $y(t)$, offers a lot of low-quality cheap merchandise. If it wasn't for the customers of the old store dropping by periodically the second store would not be able to survive.

   The model applicable to these two stores is:

$$\begin{cases} \dfrac{dx}{dt} = x - 5y \\[2ex] \dfrac{dy}{dt} = 2x - y. \end{cases}$$

   Currently, the "profits" of both stores are negative. Determine if the stores will ever recover, and what their long term fate is. Your supporting evidence should at least include a phase diagram.

2.5 Linearization and systems of ODEs

Part 1 of the homework; due Thursday, October 26th

1. Linearize the following functions in the vicinity of the indicated points:

   (a) $f(x) = x\left(1 - \frac{x}{3}\right)$ near $x = 3$;

   (b) $f(x) = \sqrt{x}$ near $x = 4$;

   (c) $f(x) = \frac{1}{\sqrt{x}}$ near $x = 1$;

   (d) $f(x, y) = x + y + xy$ near $x = 1$, $y = 2$;

   (e) $f(x, y) = x(1 - x) - xy$ near $x = 1$, $y = 0$;

   (f) $f(x, y) = \frac{x}{y}$ near $x = 2$, $y = 1$;

   (g) $f(x, y) = \frac{xy}{x+y}$ near $x = y = 1$.

2. Use the ideas of linearization to numerically estimate the following **without** the use of 'technology.' **Then** 'check' your estimates on your calculator.

   (a) $(-0.9)^3$

   (b) $\frac{1}{\sqrt{4.01}}$;

   (c) $\frac{1}{1.95}$;

   (d) $\sqrt[3]{8.09}$

   (e) $\sin(0.1)$;

   (f) $\ln(0.98)$.

3. Equilibrium solutions of the differential equation
$$\frac{dy}{dt} = y\left(1 - \frac{y^2}{4}\right)$$
are $y_1(t) = -2$, $y_2(t) = 0$ and $y_3(t) = 2$.
   (a) Linearize the differential equation near y_1. What are the general solutions of the linearized equation? What conclusions can you draw about the original equation? In particular what can you say about the stability of the equilibrium solution $y_1(t)$?
   (b) Repeat the above procedure for $y_2(t)$.
   (c) Repeat for $y_3(t)$.

4. Find all equilibrium solutions of the differential equation
$$\frac{dy}{dt} = \sin y.$$
Organize your solutions in to a list $y_1(t)$, $y_2(t)$, $y_3(t)$, ...
   (a) Linearize the differential equation near the y_1. What are the general solutions of the linearized equation? What conclusions can you draw about the original equation? In particular what can you say about the stability of the equilibrium solution $y_1(t)$?
   (b) Repeat the above for $y_2(t)$ and $y_3(t)$.
   (c) Continue down the list of equilibria until you spot a pattern. What is the pattern? Illustrate your finding with a stream plot.

Part 2 of the homework; due Monday, October 30th

1. Perform the equilibrium point analysis of the following predator-prey model:
$$\frac{dx}{dt} = x - xy \qquad \frac{dy}{dt} = -2y + 2xy$$
That is, execute the following steps.
   - Find all equilibrium solutions of the system.
   - Linearize the system near each equilibrium.
   - Understand the linearized models using eigenstuff.
   - As much as possible, piece the phase portraits of the linearized systems together to get an approximate phase portrait of the full system.

2. Repeat the above for the predator-prey model
$$\frac{dP}{dt} = 0.3P\big(1 - \frac{P}{100}\big) - 0.06PR \qquad \frac{dR}{dt} = -0.4R + 0.01PR.$$

3. Repeat the above for the predator-prey model:

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{100}\right) - 0.005xy \qquad \frac{dy}{dt} = \frac{y}{2}\left(1 - \frac{y}{200}\right) + 0.01xy.$$

4. Consider the non-linear system

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -x + (1 - x^2 - y^2)y.$$

   (a) There is one equilibrium solution of this system – find it!

   (b) Linearize the system near this equilibrium, and draw the phase portrait of the linearized system.

   (c) Make an educated guess about the phase portrait of the non-linear system. *For your own benefit do this without any help of "technology".*

   (d) Show that $x(t) = \sin(t)$, $y(t) = \cos(t)$ is a solution of the non-linear system. What is the phase diagram of this solution?

   (e) Now make an educated guess about the phase portrait of the non-linear system. (Remember: phase curves for nice systems do not intersect!!!)

   (f) Construct a phrase plot using `Sage`.

   (g) Comment on what you learned about linearization from this problem.

2.6 The Method of Nullclines

Homework due Thursday, November 2nd

1. For each of the predator-prey models from the previous assignment due the following.

   - Find the nullclines of the system.
   - Use `Sage` to construct a plot of the vector field for the system. Draw the nullclines and equilibrium points on your plot using a pencil.
   - Write a couple sentences describing how "typical" solutions behave.

Chapter 3

Second Order Equations

3.1 Oscillations

Homework due Thursday, November 9th

1. Re-write the following second order equations as the first order system. Then use `Sage` to generate the phase portraits of the oscillators. Use the phase portrait to discuss the behavior of a 'typical' solution $y(t)$.

   (a) $\dfrac{d^2y}{dt^2} + 4y = 0$;

   (b) $\dfrac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 3$;

   (c) $\dfrac{d^2y}{dt^2} + 3y^2 = 3$.

2. Consider the second order differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

   (a) Find the general solution to this differential equation.

   (b) Solve the initial value problem

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

3. Show that the superposition principle also holds for **non-constant coefficient homogeneous linear equations**, which take the form

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = 0$$

   for functions a, b, c.

4. Consider the differential equation

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0.$$

(a) Find those values of α for which the function $y(t) = t^\alpha$ solves the differential equation.

(b) Use the superposition principle to solve the IVP:

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0, \qquad y(1) = 2, \quad y'(1) = 4.$$

5. Find the general solution of the following equations:

(a) $\dfrac{d^2y}{dt^2} + \omega^2 y = 0;$

(b) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 0;$

(c) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} = 0;$

(d) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = 0;$

(e) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0.$

3.2 Linear oscillations with driving terms

Homework due Thursday, November 16th

1. Consider the non-homogeneous equation:

$$\frac{d^2y}{dt^2} + 9y = 9.$$

This equation arises from studying a frictionless oscillator with constant forcing.

(a) Find a particular solution of this equation.

(b) Find the homogeneous solution.

(c) Based on the above find the general solution of the equation.

(d) Solve the homogeneous IVP

$$\frac{d^2y}{dt^2} + 9y = 0, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 3.$$

(e) Solve the in-homogeneous IVP

$$\frac{d^2y}{dt^2} + 9y = 9, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 3.$$

(f) Graph the homogeneous solution and the solution of the IVP in the ty-plane, paying particular attention to long-term behavior of the graphs.

(g) How would you in words describe the effect the forcing has on the oscillator? (That is, compare the two graphs and comment in words.)

2. Repeat the previous problem for the following IVPs:

(a) $\frac{d^2y}{dt^2} + 9y = 10e^{-t}, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = -7.$

(b) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 1$, $y(0) = 1$, $\frac{dy}{dt}(0) = 0$. Note: this equation represents an oscillator with friction.

(c) $\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 6y = 4e^{-2t}$, $y(0) = 0$, $\frac{dy}{dt}(0) = 0$. Note: this equation represents an oscillator with "anti-friction".

(d) $\dfrac{d^2y}{dt^2} + 16y = 7\sin(3t)$, $y(0) = 0$, $\frac{dy}{dt}(0) = 0$.

(e) $\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t)$, $y(0) = 0$, $\frac{dy}{dt}(0) = 5$.

3. Find general solutions of the following differential equations.

(a) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 2t + 1$.

(b) $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + y = 1 + e^{-t}$.

(c) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = 5\sin(t)$.

4. Devise a recipe for finding a forced response of the oscillator with polynomial forcing $f(t) = a_nt^n + a_{n-1}t^{n-1} + ... + a_1t + a_0$.

3.3 Resonance

Homework due Monday, November 20th

1. Consider the equation

$$\frac{d^2y}{dt^2} + \omega^2 y = \cos(\omega t).$$

(a) Show that looking for a particular solution of the form $y_p(t) = a\cos(\omega t)$ yields an equation that cannot be satisfied.

(b) Show by direct computation that $y_p(t) = \frac{t}{2\omega}\sin(\omega t)$ actually is a particular solution.

(c) Find the general solution.

2. Find the general solution to $\dfrac{d^2y}{dt^2} + 4y = 3\cos(2t)$.

3. In this exercise, we study another type of resonance. Consider the equation

$$\frac{d^2y}{dt^2} - 2b\frac{dy}{dt} + y = 0$$

where b is some parameter with $0 < b \leq 1$.

(a) Assume that $0 < b < 1$. Find the solution satisfying the initial conditions $y(0) = 0$, $\frac{dy}{dt}(0) = 1$.

(b) Show that in the limit as $b \to 1$ we have $y(t) \to te^t$.

(c) Now set $b = 1$ in the equation and verify that te^t is a particular solution.

(d) Find the general solution in the case when $b = 1$.

(e) In the case that $b = 1$, we say our equation is *resonant*. Why is the term "resonant" appropriate in this case?

4. Find the general solution to $\dfrac{d^2y}{dt^2} - 8\dfrac{dy}{dt} + 16y = 0$.

3.4 Non-linear oscillations and the conservation of energy

Homework due before you leave for the Thanksgiving Break

1. Suppose we have the forced oscillator

$$\frac{d^2x}{dt^2} + 9x = 10$$

(a) Write this as a first order system.

(b) Show that $H = \frac{1}{2}v^2 + \frac{9}{2}x^2 - 10x$ is a conserved quantity.

(c) Draw the energy diagram for the equation.

(d) Draw the phase portrait for the equation.

(e) Discuss the long-term behavior of solutions to the system, based on your diagram & portrait.

(f) Find the general solution to the differential equation. Does the behavior match what the pictures predict?

2. Construct a Hamiltonian system, and an initial condition, for which the corresponding solution traverses the hyperbola

$$\frac{1}{2}v^2 - \frac{1}{2}x^2 = 50.$$

3. Use conservation of energy to analyze the behavior of solutions to the differential equation

(a) $\frac{d^2x}{dt^2} = \frac{1}{(x+1)^2}$

(b) $\frac{d^2x}{dt^2} = e^{-x} - 1$

4. Use conservation of energy to analyze the behavior of solutions to the differential equation arising from the following potential functions

(a) $V(x) = x^2 - 2x + 1$

(b) $V(x) = \frac{1}{x}$.

(c) $V(x) = \frac{1}{x} + x$.

(d) $V(x) = \frac{-1}{1+x^2}$.

5. In this exercise you explore a system of differential equations that have a conserved quantity that is not an energy.

   (a) Show that $Q = \frac{x}{y}$ is a conserved quantity for the system

   $$\frac{dx}{dt} = x^2 - xy \qquad \frac{dy}{dt} = xy - y^2$$

   (b) Use the conserved quantity Q to draw the phase portrait of the system.

3.5 Damped non-linear oscillations

Homework due Monday, November 27th

1. The equation

   $$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x^2 = 3$$

   can be perceived as a model for some non-linear oscillation with friction (damping).

   (a) Write down the corresponding frictionless (undamped) system. What is the conserved energy for that system?

   (b) Show that the energy for the damped system is monotone decreasing.

   (c) Draw the energy diagram for the damped system.

   (d) Then discuss the possibilities for the long-term fate of the solutions.

2. Repeat the previous problem for the equation

   $$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 3x^2 = -12.$$

Part III

Resources

Chapter 4

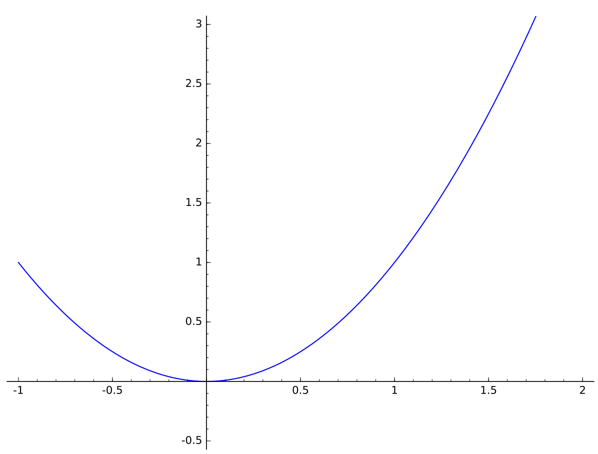
A little something about using Sage

4.1 Graphing in Sage

4.1.1 Plotting functions

An example of a Sage code which graphs the function $f(t) = t^2$ over the domain $-1 \leq t \leq 2$ is given below. Note the added option for the range of the output values.

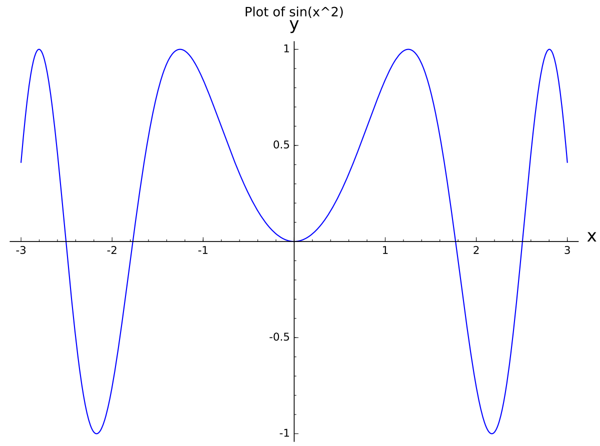
```
var('t')
f(t) = t^2
plot(f(t),(t,-1,2), ymin=-.5, ymax = 3)
```



Below is an alternative with some labels, and a way of graphing several curves simultaneously. You can play with the options, and see the effects.

```
plot(sin(x^2), (x, -3, 3), title='Plot␣of␣sin(x^2)', axes_labels=['x','y'
    ])
```

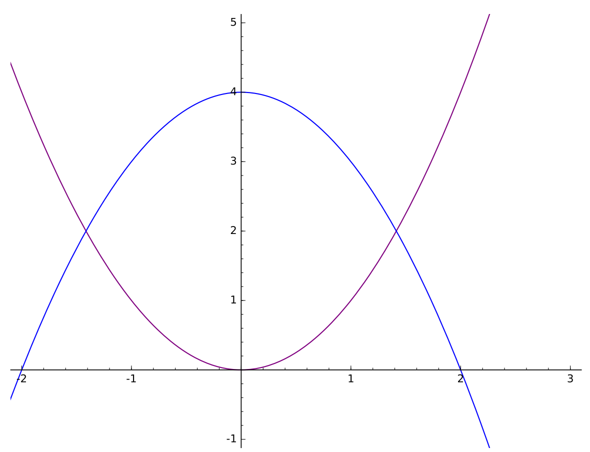
```
var('t')
f(t) = t^2
g(t) = 4-t^2
```




```
fplot = plot(f(t), (t,-10,10), color='purple')
gplot = plot(g(t), (t,-10,10), color='blue')

mainplot = fplot + gplot

mainplot.show(xmin = -2, xmax=3, ymin=-1, ymax=5, axes_labels=['$t$','␣'
    ])
```



4.1.2 Plotting vector fields

The Sage code used to generate the plot in Figure 4.1 is

```
var('t,y')
slopePlot = plot_slope_field(y*(1-y), (t,0,2), (y,-1,2),axes_labels=['$t$
    ','$y$'])
slopePlot.show()
```

The code used to generate the plot in Figure 4.2 is

```
var('t,y')
```

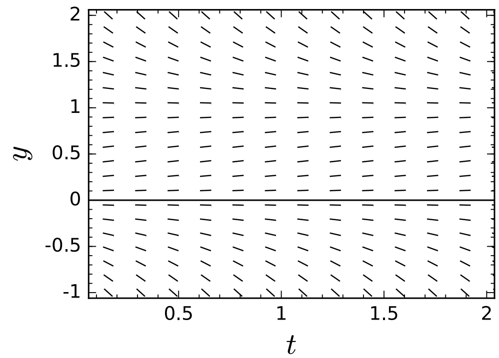


Figure 4.1: Plot of the slopes of solutions to $\frac{dy}{dt} = y(1-y)$

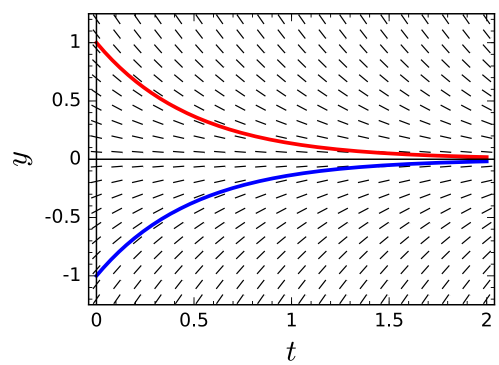


Figure 4.2: The slope field and the plots of two solutions superimposed on it. Notice that the solution in red and the solution in blue both follow the slope field lines.

```
slopePlot = plot_slope_field(-2*y, (t,0,2), (y,-1.2,1.2),axes_labels=['
    $t$','$y$'])
solnPlot1 = plot(exp(-2*t),(t,0,2),color='red',thickness=2)
solnPlot2 = plot(-exp(-2*t),(t,0,2),color='blue',thickness=2)
mainPlot = slopePlot + solnPlot1 + solnPlot2
mainPlot.show()
```

In order to have Sage generate a `pdf` file containing the graphic, simply replace `mainPlot.show()` by the command

```
mainPlot.save(filename="basic-model-slope-plot.pdf",figsize=[4,3])
```

The following piece of code "connects the dots" without you needing the know what the solutions of an ODE are. Sort of. Not really. Don't ask why it needs to be so complicated. I get angry when I think about it. The picture is in Figure connect-dots.

```
var('t,x,y')

Field = vector([y,(1-x^2)*y-x])
InitialCondition = [0,.1,.1]
EndTime=20

```



```
NumSoln = desolve_system_rk4(Field, [x,y], ics=InitialCondition, ivar=t,
    end_points=EndTime)
ParPlot = list_plot([[j,k] for i,j,k in NumSoln], plotjoined=true)

FieldPlot = plot_vector_field(Field/Field.norm(),(x,-3,3),(y,-4,4))

MainPlot = FieldPlot + ParPlot
MainPlot.show()
```

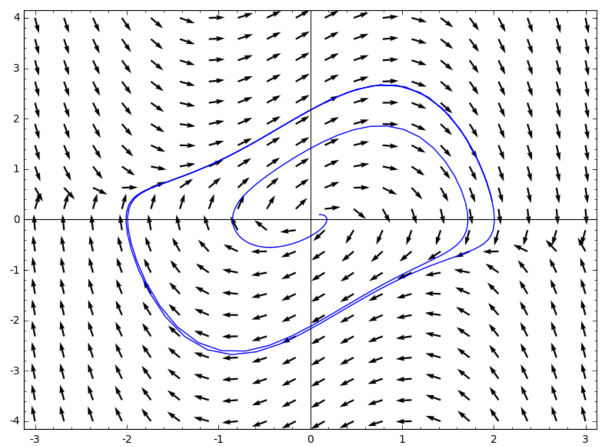


Figure 4.3: Trying to connect the dots....

4.1.3 Parametric curves

Suppose we want Sage to make a parametric plot of the functions

$$y_1(t) = \sin(2t) \qquad y_2(t) = \sin(3t)$$

for $0 \leq t \leq 2\pi$. We can accomplish this with the following code:

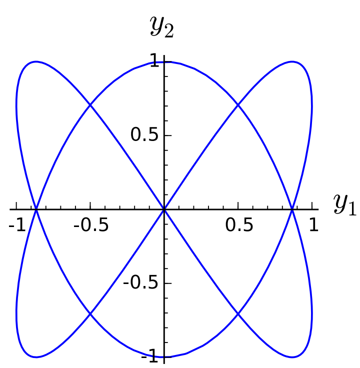


Figure 4.4: Example of a parametric plot.

```
var('t')
y1(t) = sin(2*t)
```

```
y2(t) = sin(3*t)
parPlot=parametric_plot( (y1(t),y2(t)),(t,0,2*pi),axes_labels=['$y_1$','
    $y_2$'])
parPlot.show()
```

The resulting plot appears in Figure 4.4

4.1.4 The Euler Method

Here is an example of an Euler Method code.

The thing you are creating (called `eulerData`) is a list of ordered pairs (t, y). It is organized as follows: the quantity `eulerData[2]` tells us the entries (t_2, y_2). If you just want t_2 you say `eulerData[2][0]`. If you want y_2 you say `eulerData[2][1]`. (Blame computer people for starting to count from 0.) Thus, the line

```
eulerData[k][1]= eulerData[k-1][1] + deltaT*f(eulerData[k-1][1])
```

below is a translation of $y_k = y_{k-1} + \Delta T \cdot f(y_{k-1})$. (The line

```
eulerData = [[k*deltaT,y0] for k in range(0,steps+1)]
```

is initializing the list. Think of it as setting computer space for the list and filling in your first educated guess into the spots.)

```
var('y')
f(y) = y^2

y0 = 1
deltaT = 0.1
steps=5

eulerData = [[k*deltaT,y0] for k in range(0,steps+1)]

for k in [1..steps]:
    eulerData[k][1]= eulerData[k-1][1] + deltaT*f(eulerData[k-1][1])

eulerPlot = list_plot(eulerData, color="red", plotjoined=true,marker='.',
    axes_labels=['$t$','$y$'])
eulerPlot.show()
```