GRE PREP

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1. Precalculus

Bits and Bobs:

- *Parabola*: Let F be a point and D a fixed line that doesn't contain F. A parabola is the set of points in the plane equidistant from F and D. Further, for $y = \frac{1}{4p}x^2$, $F = (0, p)$ and D is given by $y = -p$.
- *Hyperbola*: The set of all points in plane such that the difference between the distances of two fixed points (the *foci*) is constant. Further, for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the foci are given by $(\pm c, 0)$ where $c = \sqrt{a^2 + b^2}$. (Can find asyptotes and vertices).
- *Elipse*: Thse set of all points in the plane
- Fundamental theorem of algebra.
- Rational roots Thm [18] To remember: $0 = a_1 x + a_0$ /*pf*/
- Conjugate radical roots theorem [18] /*remember with quadratic formula*/. /*pf*/
- Complex conjugate roots thm
- Sum of roots of polynomial is $-\frac{a_{n-1}}{a_n}$. /*pf*/
- Product of roots of polynomial is $(-1)^n \frac{a_0}{a_n}$. /*pf*/

Trig:

- $1 + \tan^2(\theta) = \sec^2(\theta)$.
- $1 + \cot^2(\theta) = \csc^2(\theta)$.
- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$.
- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.
- $\tan(\alpha + \beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$.
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta) = 2\cos^2(\theta) - 1$.
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2\theta}$
- $\sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1-\cos\theta}{2}}$.
- $\cos(\frac{\theta}{2}) = \pm\sqrt{\frac{1+\cos\theta}{2}}$.
- $\tan(\frac{\theta}{2}) = \frac{\sin\theta}{1+\cos\theta}$.

2. Calculus I and II

- sequence convergence rules
- limit convergence rules
- Squeeze theorem
- L'Hopital's rule
- Extreme & Intermediate value theorems
- Definition of derivative
- Derivative of inverse function: $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$ if $y_0 = f(x_0)$ /*how to remember*/
- Implicit differentiation
- Mean value theorem
- Integration by parts
- Fundamental Theorem of Calculus (both forms).
- Solids of revolution
- Arc length of curve $y(x)$ given by

$$s = \int_{x_i}^{x_f} \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\ dx \tag{1}$$

- derivative of $f(x)^{g(x)}$ things.
- $1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$. And can take $n \to \infty$ for $|x|$
- p-series
- Comparison test
- Ratio test
- Integral test
- Root test $\ldots \lim_{n\to\infty}(a_n)^{\frac{1}{n}}$.
- Interval of convergence of power series: use ratio test and solve for x. (endpoints must be checked case by case).
- Taylor Series given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{2}$$

- Taylor's theorem / Taylor Series error thing.

Derivatives to know:

- $\frac{d}{dx}(a^x) = \log(a)a^x$.
- $\frac{d}{dx}(\log_a(x)) = \frac{1}{x\log(a)}$.
- $\frac{d}{dx}(\tan x) = \sec^2(x)$.
- $\frac{d}{dx}(\cot x) = -\csc^2(x)$.
- $\frac{d}{dx}(\sec x) = \sec x \tan x$.
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$.
- $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1+x^2}}$.
- $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1+x^2}}$.

- $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

Integrals to know:

- $\int a^x dx = \frac{1}{\log a} a^x + c$.
- $\int \sec^2 x dx = \tan x + c$.
- $\int \csc^2 x dx = -\cot x + c$.
- $\int \sec x \tan x dx = \sec x + c$.
- $\int \csc x \cot x dx = -\csc x + c$.
- $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$.
- $\int \frac{1}{1+x^2} dx = \arctan x + c$.

Taylor Series to know:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $-1 < x < 1$
- $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$, $-1 < x < 1$
- $\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^n}{n}$, $-1 \le x < 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, all x.
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, all x
- $\cos x = \sum_{n=0}^{\infty} \frac{-1^n}{(2n)!} x^{2n}$, all x

Trig Substitution Method:
For integrals that contain $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{u^2-x^2}$. In particular:

If integrand contains	*Make this substitution*
$\sqrt{a^2-u^2}$	$x = a\sin\theta$
$\sqrt{a^2+u^2}$	$x = a\tan\theta$
$\sqrt{u^2-a^2}$	$x = a\sec\theta$

Partial Fractions Method:
For integrals of the form $\int \frac{P(x)}{Q(x)} dx$, with $P(x), Q(x)$ polynomials, $\deg(P) < \deg(Q)$. The method:

(1) First, factor $Q(x)$.
(2) Express $\frac{P(x)}{Q(x)} = \frac{A_1}{q_1(x)} + \frac{A_2}{q_2(x)} \ldots$. The q_i's are factors of Q. If Q factors with a term of th form $(ax+b)^n$, there will be n corresponding partial fractions $(ax+b), (ax+b)^2, \ldots$. If you get irreducible quadratic, have a degree 1 numerator.
(3) Solve for A_i's by multiplying both sides by corresponding q_i and plugging in root.

3. Multivariable Calculus

- Projection of $\mathbf{b}$ onto $\mathbf{a}$:
$$\mathbf{proj_a\, b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}.$$
- Triple product is volume of paraleltope
- Various vector product identities?
- Line equations in space
- Plane equations in space
- Various coordinate systems
- Tangent plane to surface and linear approximations
- Higher order approximations /*TODO*/
- Chain rule
- Gradient and properties
- Min / max problems /*2nd deriv test TODO*/
- Min / max problems with constraint: if possible combine constraint and function into simple function and apply calc 1.
- Otherwise …Lagrange multiplier method. Function f and constraint $g = c$. Then, impose $\nabla f = \lambda \nabla g$.
- Line integrals and arclength, integrating on V.F etc.
- Fundamental theorem of calc for line integrals.
- Green's theorem and applications.
- Weird cases /*[158–159]*/

4. Differential Equations

5. Number Theory

- Relating divisibility to digits rules
- Division algorithm
- gcd and lcm definitions and relation to prime factorizations (min and max).
- Also, $\gcd(a,b)\cdot \operatorname{lcm}(a,b) = ab$.
- Euclidean algorithm.
- Fermat's Little Theorem. If p prime, $p \not| a$, then
$$a^{p-1} \equiv 1 (\bmod\ p)$$

Diophantine Equation $ax + by = c$. Has solution iff $\gcd(a,b)|c$. Given solution (x_1, y_1) all solutions are given by:
$$x = x_1 + t\frac{b}{\gcd(a,b)} \text{ and } y = y_1 - t\frac{a}{\gcd(a,b)}$$
for $t \in \mathbb{Z}$.