Lutzer's Rotating Hammer

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1 Introduction

The paper "Hammer Juggling, Rotational Inertia, and Eigenvalues" by Carl Lutzer begins with a simple system: throwing a rotating hammer. If thrown vertically from the hilt, with the head down, (if you're coordinated), you'll catch in with the head down again. However, thrown with the head to the left side, it'll flip horizontal mid-air and you'll catch it with the head on the right side. The reason for this phenomena can be traced to angular velocity, and with a bit of math magic, we can derive information about the Moment of inertia of the system and explain this phenomena in mathematical terms.

2 The Rotating Hammer System (in physics terms)

Center of Mass: Imagine a stationary floating object spinning freely without the influence of gravity. Mass does not spontaneously move, so the average location of all the mass should be remain in the same point in space. This point in space is called the "Center of mass". Because the center of mass remains fixed in space, the object will always rotate about this point regardless of the direction of rotation.
Angular Velocity: In basic mechanics, there are two types of motion, rotational and translational. Translational is our normal concept of motion: a car moves down a road. Rotational is similar, if we imagine our car driving on a circle. The car moves radially, so it's more useful to think of movement as that with respect to the circle. We define angular velocity as velocity times $2\pi \cdot r$, where r is the radius. This measures how quickly an object completes a cycle. Now, rotating is just like transversing a circle, but the center is at the center of mass. Rotation about the center of mass is fixed, so it will be useful for us when analyzing our system.
- what else should we define?

3 The Math

3.1 The Approach

There is a natural way to coordinatize a spinning object into an orthornormal basis along three "principle axes". All calculations we perform are in reference to these three natural axes. Using Euler's equation allows a relationship between the angular acceleration and the angular velocity about these axes. This paired with the assumption of 0 torque being applied to the system gives rise to a first order system of differential equations. We find the equilibrium solutions, which correspond to rotations around each of the principle axes. We linearize the system near the equlibrium solutions to analyze the local behavior. In doing so, we find that equilibrium solutions about the shortest and longest axis are semi-stable centers. So, when a hammer is thrown about one of these axes, it's rotation pattern will stay fairly constant. On the other hand, the equilibrium solutions about the axis of intermidiate length are unstable saddles. So, when an object is rotated about this intermidiate axis will have have a rotation with a more chaotic appearance.

3.2 Principle axes and Euler's equation

First off, we must consider the angular velocity of our object. Since we live in three dimensions, our object will have three principle axis to rotate around, which we denote p_1, p_2, p_3. Each p refers to an arbitrary dimension, say p_1 is length, p_2 is width, and p_3 is height. Since this labeling is arbitrary, we're going to make an assumption that length$>$ width$>$ height. With this assumption we'll see the unstable rotation about p_2 as we will show. Now with now our orthonormal base, we can express rotational velocity as a linear combination, so that,

$$\omega = \alpha_1(t)p_1 + \alpha_2(t)p_2 + \alpha_3(t)p_3 \tag{1}$$

Note: ω is not a constant value, thus the α's are functions of time.

Next, we have to consider the Moment of Inertia Tensor (which is essentially the rotational mass), which is a 3x3 matrix, which holds our eigenvalues in each p direction

$$M = \begin{pmatrix} \lambda_1 \; 0 \; 0 \\ 0 \; \lambda_2 \; 0 \\ 0 \; 0 \; \lambda_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{2}$$

This matrix may seem arbitrary, but is necessary for Euler's equation of motion. Note: $\lambda_1 > \lambda_2 > \lambda_3$ due to our assumption of length$>$ width$>$ height. Now we can apply this to our system. Now, since we only throw the book and let it spin, we determine there must be no outside torque on our system. Thus Euler's equation in each direction becomes

$$0 = M\dot{\omega} + \omega \times M\omega \tag{3}$$

3.3 Translating Euler's equation into Differential Equation System

Let's do some computin'

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \; 0 \; 0 \\ 0 \; \lambda_2 \; 0 \\ 0 \; 0 \; \lambda_3 \end{pmatrix} \begin{pmatrix} \dot{\alpha_1} \\ \dot{\alpha_2} \\ \dot{\alpha_3} \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \times \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1\dot{\alpha_1} \\ \lambda_2\dot{\alpha_2} \\ \lambda_3\dot{\alpha_3} \end{pmatrix} + \begin{pmatrix} (\lambda_3 - \lambda_2)\alpha_2\alpha_3 \\ (\lambda_1 - \lambda_3)\alpha_1\alpha_3 \\ (\lambda_2 - \lambda_1)\alpha_1\alpha_2 \end{pmatrix} \tag{5}$$

Now we have our basis equations that we will derive our eigenvalues and equations of motion from.

$$\begin{aligned} \lambda_1\dot{\alpha_1} &= (\lambda_2 - \lambda_3)\alpha_2\alpha_3 \\ \lambda_2\dot{\alpha_2} &= (\lambda_3 - \lambda_1)\alpha_1\alpha_3 \\ \lambda_3\dot{\alpha_3} &= (\lambda_1 - \lambda_2)\alpha_1\alpha_2 \end{aligned}$$

3.4 Analysis of System of Differential Equations

We begin to analyze the system of differential equations by finding the equilibrium values. The system is at equilibrium when the angular vecolcity in each of the three directions is 0. So, $\dot{\alpha_1} = \dot{\alpha_2} = \dot{\alpha_3} = 0$ or $\vec{w} = \vec{0}$. Equilibrium occurs in three different situations:

- $\alpha_2 = \alpha_3 = 0$ for any α_1
- $\alpha_1 = \alpha_3 = 0$ for any α_2
- $\alpha_1 = \alpha_2 = 0$ for any α_3

So, we find that we do not simply have equilibrium points. Rather, the freedom of one alpha to vary in each case implies that the system has three equilibrium *lines*. Furthermore, because two of the α_i values are 0 in each case, it is the $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ axes that hold all equlibrium values. To determine the local behavior of the solution about each equlibrium point, we linearize the equation into something we can deal with at each point.

The Jacobi matrix takes the following form,

$$DT = \begin{pmatrix} \partial_{\alpha_1}\omega_1 & \partial_{\alpha_2}\omega_1 & \partial_{\alpha_3}\omega_1 \\ \partial_{\alpha_1}\omega_2 & \partial_{\alpha_2}\omega_2 & \partial_{\alpha_3}\omega_2 \\ \partial_{\alpha_1}\omega_3 & \partial_{\alpha_2}\omega_3 & \partial_{\alpha_3}\omega_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\lambda_1}(\lambda_2 - \lambda_3)\alpha_3 & \frac{1}{\lambda_1}(\lambda_2 - \lambda_3)\alpha_2 \\ \frac{1}{\lambda_2}(\lambda_3 - \lambda_1)\alpha_3 & 0 & \frac{1}{\lambda_2}(\lambda_3 - \lambda_1)\alpha_1 \\ \frac{1}{\lambda_3}(\lambda_3 - \lambda_1)\alpha_2 & \frac{1}{\lambda_3}(\lambda_3 - \lambda_1)\alpha_1 & 0 \end{pmatrix}$$

So,

$$\begin{pmatrix}\dot{\omega_1}\\ \dot{\omega_2}\\ \dot{\omega_3}\end{pmatrix} \approx \begin{pmatrix} 0 & \frac{1}{\lambda_1}(\lambda_2-\lambda_3)\alpha_3 & \frac{1}{\lambda_1}(\lambda_2-\lambda_3)\alpha_2 \\ \frac{1}{\lambda_2}(\lambda_3-\lambda_1)\alpha_3 & 0 & \frac{1}{\lambda_2}(\lambda_3-\lambda_1)\alpha_1 \\ \frac{1}{\lambda_3}(\lambda_3-\lambda_1)\alpha_2 & \frac{1}{\lambda_3}(\lambda_3-\lambda_1)\alpha_1 & 0 \end{pmatrix} \cdot \begin{pmatrix}\alpha_1-\mathrm{eq}_1\\ \alpha_2-\mathrm{eq}_2\\ \alpha_3-\mathrm{eq}_3\end{pmatrix}$$

About some equlibrium point $\vec{\mathrm{eq}} = \langle \mathrm{eq}_1,\ \mathrm{eq}_2,\ \mathrm{eq}_3\rangle$ with respect to the basis p_1, p_2, p_3.

If we do rotate around an axis, perfectly, say α_1, we have zeros in the other two directions, which means no other rotational motion. However, in the real world it is impossible to perfectly flip an object. We will say that we will rotate the object very close to a principle axis, such that the other axes will be approximately zero. So, we use the approximation of the system through the Jacobi Matrix to analyze this approximation.

3.5 Eigenvalues of Stable Rotation

For the three axes of the system, stable rotation occurs when rotating around either the smallest or largest axis(p_1 *or* p_3). We will do the math for p_1 since p_3 is a similar derivation. This means we are considering equilibrium case where $\alpha_2 = \alpha_3 = 0$ for any α_1

Let's begin the math then. We consider our approximately perfect throw, such that $\alpha_1(0) \neq 0$ and that $\alpha_2(0) \approx \alpha_2(0) \approx 0$. Thus our equation for p_1 motion becomes,

$$\begin{aligned} 0 &= \lambda_1\dot{\alpha_1} + (\lambda_3-\lambda_2)\alpha_2\alpha_3 \\ 0 &= \lambda_1\dot{\alpha_1} + \approx 0 \\ 0 &\approx \lambda_1\dot{\alpha_1} \end{aligned}$$

This implies that α_1 is constant with time(or nearly so), so when applying to the Jacobian, we can eliminate one dimension, yielding,

$$\begin{pmatrix}\dot{\alpha_2}\\ \dot{\alpha_3}\end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\lambda_2}(\lambda_3-\lambda_1)\alpha_1 \\ \frac{1}{\lambda_3}(\lambda_1-\lambda_2)\alpha_1 & 0\end{pmatrix}\begin{pmatrix}\alpha_2\\ \alpha_3\end{pmatrix} \tag{6}$$

Thus we have a new matrix with new eigenvalues to consider. We'll name this new equation $\dot{x} = Ax$ for simplicity. We now compute the eigenvalues of A to be,

$$\lambda_A = \pm i\sqrt{\frac{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)\alpha_1^2}{\lambda_2\lambda_3}} \tag{7}$$

Note: We pop out an i because of our assumption of $\lambda_1 > \lambda_2 > \lambda_3$. Thus $(\lambda_1 - \lambda_2)$ is a negative quantity, so we rewrite as $-(\lambda_2 - \lambda_1)$ to keep the inside positive, which in turn, pops out an i. For simplicity, we will denote $\lambda_A = \pm i\Theta$. Now, from differential equations, we know that a solution to x(t) is

$$x(t) = c_1 e^{i\Theta t} \tag{8}$$

3.6 Eigenvalues of Unstable Rotation

Now we look at the special case when we rotate about p_2. Proceeding from before, we now have,

$$0 \approx \lambda_2\dot{\alpha_2}$$

Once again, this implies

To achieve our unstable, rotating phenomena, we now look at rotating about $\lambda 2$. Here, we look at rotating about the middle weighted axis.

Note: Incomplete section: this section proceeds with the same calculations from before, but then yields real eigenvalues for the motion equation: thus exponential growth. This explains the wobbly nature of the thrown hammer

3.7 Confirming Positive Eigenvalues

This section will dive into the math to explain why the eigenvalues are forced positive. Now, the author puts this in different terms than our previous courses, so we will need to recontextualize this to understand. We also might put this with our jacobi matrix computations instead of at this point in the paper.It's honestly just a bunch of weird matrix manipulation

3.8 Deriving Euler's equation of Motion

this setion just takes the idea of torque in relation to Euler's equation to rederive Euler's equation of motion. The main implication this section derives is that we see the system from the basis point of view, then it is spun about us

4 Ideas to take this further

- Behavior at different rotational velocities
- Consider what happens when there are equal eigenvalues
- Generalize to n dimensions