COMPLEX VARIABLES: STUDY GUIDE FOR EXAM 1

Complex Plane

1. How do we represent complex numbers geometrically?

2. What is meant under the following terms?

   (a) $\mathrm{Re}(z)$;
   (b) $\mathrm{Im}(z)$;
   (c) $|z|$;
   (d) $\bar{z}$;
   (e) $\arg(z)$;
   (f) $\mathrm{Arg}(z)$.

3. Each of the following describes a region in the complex plane. Sketch them!

   (a) $|z - 2i| \leq 3$;

   (b) $|z + i| \geq 1$;

   (c) $1 \leq \mathrm{Im}(z) \leq 3$;

   (d) $\mathrm{Re}(z) \leq 0$;

   (e) $1 \leq |2z - 3| \leq 3$.

4. There are some properties that modulus and conjugate obey in regards to addition, multiplication, etc. For instance, we have a property that states $|z_1 z_2| = |z_1||z_2|$. Review such properties and get to a point where you can easily prove them.

5. What does the Triangle Inequality say, and what – roughly speaking – are the reasons why it holds?

6. Express the following complex numbers in the exponential form $z = |z| \exp(i\theta)$. Use the principle value θ of the argument of z. For each number provide a little drawing which illustrates both $|z|$ and θ.

   (a) $z = 2 - 2i$;

   (b) $z = -3i$;

   (c) $z = -1 + i$;

   (d) $z = -4$.

7. Exponential form of complex numbers yields a nice geometric interpretation of the product, the quotient, the n-th power and the n-th root of complex numbers. What is this interpretation? In particular, how can one most easily compute and visualize the following?

   (a) $(1+i)^5$;

   (b) The sequence $\xi_n = (1+i)^n$;

   (c) $8^{\frac{1}{3}}$;

   (d) $i^{\frac{1}{5}}$;

   (e) $(-1)^{\frac{1}{4}}$;

   (f) $(-1+\sqrt{3}i)^{\frac{1}{2}}$.

8. Let $n \geq 2$ be an integer, and let $\omega = \exp(\frac{2\pi i}{n})$. Argue that $1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}$ lists all n-th roots of 1.

Basic elementary functions

1. How did we, in this class, define the following?

   (a) $\exp(z)$;

   (b) $\log(z)$;

   (c) $\mathrm{Log}(z)$;

   (d) z^s;

   (e) $\sin(z)$;

   (f) $\cos(z)$;

   (g) $\sinh(z)$;

   (h) $\cosh(z)$.

2. Both exponential and trigonometric functions are periodic. Make sure you understand this periodicity. Also make sure you understand what all we can say about z_1 and z_2 if $\exp(z_1) = \exp(z_2)$.

3. Compute the following:

   (a) $\exp(-\frac{\pi i}{4})$;

   (b) $\exp(1 + i\frac{\pi}{2})$;

   (c) $\exp(-1 - \pi i)$;

(d) $\sin(i)$;

(e) $\sinh(i\pi)$;

(f) $\sin(\frac{\pi}{2} + i)$;

(g) $\cos(i\pi)$;

(h) $\cosh(2\pi i)$;

(i) $\tan(-\frac{\pi}{2} + 2i)$;

(j) $\mathrm{Log}(-1)$;

(k) $\mathrm{Log}(-i)$;

(l) $\mathrm{Log}(-2 - 2i)$;

(m) $\log(-1 + \sqrt{3}i)$;

(n) $\log(3 - 4i)$;

(o) $\mathrm{Log}(2 + i)$;

(p) $P.V(-1)^i$;

(q) $(1 - i)^{-i}$;

(r) $P.V(-i)^{\frac{1}{\pi}}$;

(s) $(-1 + i)^{3i}$;

(t) $P.V(\sqrt{3} + i)^{-3i}$.

4. What algebraic (i.e. pertaining to addition, multiplication etc) properties of $\exp(z)$, $\mathrm{Log}(z)$ and $\log(z)$ do you know? Get to a point where you can utilize these properties fluently.

5. Compute the following, assuming all complex powers refer to their principle values.

$$i^{\frac{3}{2}}, \quad (i^3)^{\frac{1}{2}}, \quad (i^{\frac{1}{2}})^3.$$

Comment on what you observe. What all can you say about the relationship between $(z^{s_1})^{s_2}$ and $z^{s_1 s_2}$?

6. There are identities addressing $\sin(z \pm w)$, $\cos(z \pm w)$, $\sin(2z)$, etc. Get to a point where you are able to deduce these identities in a short amount of time.

7. The function $\mathrm{Log}(z)$ exhibits a discontinuity along the negative portion of the real axis. Get to a point where you understand the issue and are able to explain it to other people.

8. The principle value of the logarithm, $\mathrm{Log}(z)$, is discontinuous along the negative portion of the real axis. Comment on the continuity of $f(z) = P.V z^s$. For what s is the power function continuous and for what s is it discontinuous (along the negative portion of the real axis)?

Visualization of functions

1. Describe in geometric terms the effect of the following mappings.

   (a) $f(z) = iz$;

   (b) $f(z) = -2 + (1+i)z$;

   (c) $f(z) = z^2$ for z in the right half-plane;

   (d) $f(z) = iz^2 + 1$ for z in the right half-plane;

   (e) $f(z) = P.V z^{\frac{1}{2}}$

   (f) $f(z) = z^3$ near $z = 1 + i$;

   (g) $f(z) = P.V z^{\frac{1}{3}}$ near $z = i$;

   (h) $f(z) = \frac{1}{z^2}$ near $z = 1 + i$;

   (i) $f(z) = \text{Log}(z)$ near $z = i$;

   (j) $f(z) = \exp(z)$ on the left half-plane ($\text{Re}(z) \leq 0$);

   (k) $f(z) = \exp(z)$ on the rectangle given by $-1 \leq \text{Re}(z) \leq 1$, $-\frac{\pi}{2} \leq \text{Im}(z) \leq \frac{\pi}{2}$;

   (l) $f(z) = -\frac{i}{2}\exp(z)$ on the rectangle $1 \leq \text{Re}(z) \leq 4$, $0 \leq \text{Im}(z) \leq \pi$;

   (m) $f(z) = \text{Log}(2z)$ on the unit disk centered at the origin;

   (n) $f(z) = 2i\text{Log}(z) + i$ on the first quadrant;

   (o) $f(z) = (i\text{Log}(z) + \pi)^2$ on the first quadrant.

2. Consider the mapping $f(z) = \text{P.V.}(z^2 - 1)^{1/2}$ on the right half plane $\text{Re}(z) \geq 0$. Describe in geometric terms the effect of this map.

3. Consider the mapping $f(z) = \cos(z)$ on the vertical strip $0 \leq \text{Re}(z) \leq \pi$. Describe the geometric effect of this mapping.

Riemann surfaces and branches of multivalued functions

1. Get to a point where you can visualize and work with the Riemann surfaces for the multivalued roots and logarithm.

2. Consider complex logarithm as a function defined on its Riemann surface. Find the pre-images, on the Riemann surface, of the following subsets of the complex plane. Please provide accompanying illustrations!

   (a) The pre-image of the imaginary axis.

(b) The pre-image of the vertical strip $-1 \leq \mathrm{Re}(w) \leq 1$.

(c) The pre-image of the real axis.

(d) The pre-image of the vertical strip $0 \leq \mathrm{Im}(w) \leq 4\pi$.

3. Find branches $f(z)$ of the multivalued logarithmic function satisfying the following conditions; give an explicit (piece-wise) formula relating the branches to $\mathrm{Log}(z)$, and interpret in terms of the Riemann surface.

   (a) $f(z)$ is continuous at $z = -1$ and attains the value of πi there.

   (b) $f(z)$ is continuous at $z = 1$ and attains the value of $4\pi i$ there.

   (c) f is continuous at $z = -i$ and attains the value of $-5\frac{\pi}{2}i$ there.

   (d) f is continuous along the negative real axis and and around $z = 1$ where it attains the value of $f(1) = -2\pi i$;

   (e) f is continuous along the negative real axis and and around $z = -i$ where it attains the value of $f(-i) = -\frac{\pi}{2}i$;

4. Find a branch f of the multivalued square root satisfying the following conditions. Give an explicit (piece-wise) formula relating the branches to $P.V z^{\frac{1}{2}}$, sketch the range for your branch and emphasize the location of the branch cut.

   (a) f is continuous at $z = -1$ and attains the value of $-i$ there;

   (b) f is continuous along both negative and positive parts of the real axis, and attains the value of $f(1) = -1$.

5. Find a branch f of the multivalued cube root satisfying the following conditions. Give an explicit (piece-wise) formula relating the branches to $P.V z^{\frac{1}{3}}$, sketch the range for your branch and emphasize the location of the branch cut.

   (a) f is continuous at $z = -1$ and attains the value of -1 there;

   (b) f is continuous along both negative and positive parts of the real axis, and attains the value of $f(i) = -i$.

Functions of complex variable defined through series

1. For a sequence of complex numbers a_n the formula $f(z) = \sum_n a_n(z - z_0)^n$ defines a function of complex variable. What can you say about the domain of such a function?

2. Find the domain of the following functions. (You are not expect to worry about the boundaries of the domains.)

   (a) $f(z) = \sum_{n=0}^{\infty} \left(\frac{2z+1}{3}\right)^n$;

(b) $f(z) = \sum_{n=1}^{\infty} \frac{n}{2^n}(z+i)^n$;

(c) $f(z) = \sum_{n=1}^{\infty} n^2(z-2i)^{-n}$;

(d) $f(z) = \sum_{n=1}^{\infty} \frac{2^n+1}{n}(z-2)^{-n}$;

(e) $f(z) = \sum_{n=1}^{\infty} \frac{n!}{(2n)!} z^n$;

(f) $f(z) = \sum_{n=1}^{\infty} n!\, z^n$;

(g) $f(z) = ... - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + ...$

3. Functions of the form $f(z) = \sum_k a_k(z-z_0)^k$ are holomorphic inside of their disk of convergence. How will you compute their derivatives?

Holomorphic functions and Cauchy-Riemann equations

1. Get to a point where you can summarize in two sentences the reasons why Cauchy-Riemann equations hold. Also, make sure you can very quickly recall what these equations state.

2. Investigate the existence of $f'(z)$ for the following functions of the complex variable $z = x + iy$. When the derivative does exist, please find an expression for it.

(a) $f(z) = \bar{z}$;

(b) $f(z) = |z|^2$;

(c) $f(x+iy) = (x^2+y^2) + 2ixy$;

(d) $f(x+iy) = (x^2-y^2) - 2ixy$;

(e) $f(x+iy) = (e^x + e^{-x})\cos(y) + i(e^x - e^{-x})\sin(y)$.

3. Find all holomorphic functions $f(z)$ for which $\mathrm{Re}(f(z)) = \ln(|z|)$.