Problems - 2013.12.05

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1. Show that the edges of a complete graph on $n \geq 4$ vertices can be colored red and blue such that there is no monochromatic Hamiltonian path.

2. The edges of a complete graph on $2n$ vertices have been colored red and blue. Prove that there is a monochromatic path of length n.

3. The points of $\mathbb{R}$ have been colored red and blue. Prove that for any $n \in \mathbb{N}$, there exists a monochromatic n-dimensional Hilbert cube, that is, there are numbers $\delta \in \mathbb{R}, \delta_1, \ldots, \delta_n \in \mathbb{R}^+$ such that the set

$$\left\{ \delta + \sum_{i=1}^{n} \varepsilon_i \delta_i \mid \forall i : \varepsilon_i \in \{0, 1\} \right\}$$

is monochromatic.

4. Prove Schur's theorem: for every $n \in \mathbb{N}$, if the prime p is large enough ($p > p_0(n)$), there exists a nontrivial solution of the Fermat Equation modulo p, that is, there exist integers x, y, z such that $p \nmid xyz$ and

$$x^n + y^n \equiv z^n \pmod{p}.$$

5. Prove that for any $n \in \mathbb{N}$, there exists $C(n) \in \mathbb{N}$ with the following property. Given $C(n)$ points on the plane such that no three of them are collinear, there exist n among them such that their convex hull has n vertices.

6. Let G be a complete directed graph (i.e. for any two vertices u, v, there is an edge either from u to v or from v to u). Prove that there is a directed Hamiltonian path in G.

7. Let G be a complete directed graph (i.e. for any two vertices u, v, there is an edge either from u to v or from v to u). Prove that the following statements are equivalent

   (i) there is a directed Hamiltonian cycle in G;

   (ii) for any partition of the vertices $V = U_1 \cup U_2$, there exist edges both from U_1 to U_2 and from U_2 to U_1.