PUTNAM SUMMARY

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1. General

1.1. Working Flowchart.

(1) Understand the problem. (Play around with the problem in whatever way I find intuitive / test something concrete)
(2) Select good notation to symbolically represent the problem.
(3) Write down known information symbolically and draw a diagram if applicable.
(4) Write down what to show symbolically. Brainstorm what would be sufficient to prove.
(5) Brainstorm all relevant approaches to the problem (see approaches listed below).
(6) Identify relevant areas of math and brainstorm all relevant tools for the problem (see tools listed below).
(7) Choose what seems to be the best approach.
(8) Play around with tools on the problem with this approach *algebraically* (start with a specific/easier instance of the problem if possible). Make sure to incorporate all necessary information.
(9) If stuck, brainstorm possible instances of "if stuck" ideas.
(10) If appears to be a dead end, write down possible new tools/approaches, consider a new approach and repeat.

1.2. If Stuck.

- Consider special case / simplified version of problem. What would be the easiest thing to prove?
- Brainstorm intermediate steps to solving the problem. What would be sufficient to prove?
- Derive consequence of problem and try to show that first.
- Reformulate problem. (contrapositive, contradiction, substitution, clever manipulation).

1.3. General Approaches.

- Contradiction.
- Induction
- Construction
- Invariant
- Clever Reformulation
- Clever Choice
- Weaker Claim
- Stronger Claim

1.4. General Tools.

- Pigeonhole Principle

2. Number Theory

2.1. Approaches.

- Manipulate problem so we are dealing with integers on both sides.
- Modular arithmetic / even vs odd argument.

2.2. Tools.

- Fundamental theorem of arithmetic.
- gcd theorem.
- Chinese Remainder Theorem.
- Fermat-Euler Theorem / Fermat's Little Theorem.
- Division algorithm / euclidean algorithm for gcd.
- Euclid's Lemma
- Wilson's Theorem.
- Linear Diophantine Equation theory.
- Pell's Equation theory.

3. Algebra

3.1. Approaches.

- Algebraic manipulation of problem into simpler form.
- Imposing symmetry on the problem to make things (ex: factorization) more obvious.

3.2. Tools.

- Geometric sum formula (finite and infinite). Useful for factoring things, evaluating telescoping products and sums.
- $x^2 > 0$
- Division algorithm for polynomials
- Fundamental Theorem of Algebra.

- Roots and Divisibility relationship
- A polynomial can only have a finite number of roots (if can establish infinite pattern, use this!)
- Rational Roots Theorem
- Vieta's Relations: Given $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$, then .../**/
- Unique Factorization of Polynomials (even for multivariabled polynomials).
- $x^n - y^n =$
- $x^n + y^n =$
- $x^3 + y^3 + z^3 - 3xyz =$
- $(x+y)^p$ for primes p.
- Lagrange Identity
- Cauchy-Schwartz Inequality
- Bernoulli's Identity
- Cauchy's Inequality
- Triangle Inequality
- Reverse Triangle Inequality
- Euler's identities on products of sums of squares.
- Sophie Germaine Identity
- (other algebraic manipulations)

4. Geometry

4.1. Approaches.

4.2. Tools.

- Cross, dots products
- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a})\vec{b} - (\vec{b} \times \vec{a})\vec{c}$
- $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{u} \times \vec{v})$
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5. Linear Algebra

5.1. Approaches.

5.2. Tools.

- Cauchy-Schwartz Inequality
- m homogeneous linear equations with n variables. $m < n$ implies there exists a nontrivial solution.
- Trace
- Eigenstuff
- Sum of eigenvalues is the trace

6. Combinatorics and Graph Theory

6.1. Approaches.

- Count something two different ways
- Vector space dimension bound
- Homogeneous System of Linear Equations Method
- Probability Method

6.2. Tools.

- Pigeonhole Principle
- Inclusion-Exclusion Principle
- Binomial Theorem / Multinomial Theorem

7. Analysis, Calculus, and Diff eq

7.1. Approaches.

- Prove for $\mathbb{Z}$, then $\mathbb{Q}$, then $\mathbb{R}$.
- Apply definite integrals to differential equations
- Associate given differential equation with (Euler method inspired sequence) and vice versa

7.2. Tools.

- Fundamental Theorem of Calculus in various forms
- Mean Value Theorem
- Intermediate Value Theorem
- Definition of derivative
- Taylor's Theorem / Taylor Series
- $f(x) \leq g(x)$ on $a \leq x \leq b$, then we have $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.
- Integration by substitution: Basic techniques, trig substitution techniques, technique on simple fractions.
- Integration by parts techniques.
- Integration by partial fractions.
- Can take advantage of complex numbers to simplify integral.
- Continuity (multiple definitions)
- Every bounded monotonic sequence is convergent
- For every real number x, there is a rational sequence converging to x.
- Integral test. Specifically, if g is a positive decreasing function, $\int_1^{n+1} g(x)dx \leq g(1) + \cdots + g(n) \leq \int_0^n g(x)dx$. (In particular, ln).
- Separable differential equations solution
- Functions that grow at a more than proportional rate to function itself blow up in finite time.
- Additive functions
- Hamel Basis

8. Set Theory and Abstract Algebra

8.1. **Approaches.**

- Impose an order on your set.

8.2. **Tools.**

- Order, total order.