Thesis in K-Theory

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Contents

Chapter 1. Category Theory 5
- 1. Categories and Functors 5
- 2. Further Examples 9

Chapter 2. Algebra 11
- 1. Ring Completion 11
- 2. Packing Together Modules 14
- 3. Verifications 15

Chapter 3. Topology 19
- 1. The Category of Topological Spaces 19
- 2. Some Mappings and Corresponding Topologies 21
- 3. Nice Properties of Topological Objects 23
- 4. Operations on Topological Spaces 24
- 5. More Topology 25
- 6. Homotopy 26
- 7. Appendix 26

Chapter 4. Vector Bundles 29
- 1. Definition and Examples 29
- 2. Direct Sum and Tensor Product on Vector Bundles 31
- 3. Pullback Bundles 32
- 4. Necessary Results on Vector Bundles 33
- 5. Verifications 34

Chapter 5. The Definition of K-Theory 41
- 1. The K-Theory Functor $\mathbf{K}$ 41
- 2. The Reduced K-Theory Functor $\widetilde{\mathbf{K}}$ 44
- 3. Verifications 46

Chapter 6. K-Theory as a Cohomology Theory 51
- 1. Exact Sequences 51
- 2. Exact Sequences in K-Theory 51
- 3. K-Theory as a Cohomology Theory 54
- 4. Further Structure on the Cohomology Theory 55

Chapter 7. Division Algebra Application 57
- 1. The odd case 57

2. The even case 57

Bibliography 59

CHAPTER 1

Category Theory

1. Categories and Functors

A good starting spot is with several mathematical areas that the reader is already familiar with and that do not at first appear to have any connection with one another. The study of linear algebra focuses on two things: vector spaces and linear transformations between vector spaces. Set theory examines sets as well as mappings between sets. Group theory considers groups and homomorphisms between groups whereas ring theory focuses on rings and homorphisms between rings. Real analysis studies metric spaces together with continuous functions between metric spaces as well as manifolds paired with smooth mappings between these manifolds. A pattern emerges: each one of these topics have two things: some *objects* of study (vector spaces, metric spaces, manifolds, sets, groups, rings) as well a specific type of function within the objects of study (linear transformations, continuous functions, smooth mappings, set mappings, homomorphisms); these functions are generally referred to as *morphisms.* Any such object-morphism pair is called a *category* so long as it obeys some rules.

Definition 1.1 (Category). Let $\mathcal{O}$ denote a collection of objects and let $\mathcal{M}$ denote a collection of morphisms. Then, the pair $(\mathcal{O}, \mathcal{M})$ is called a *category* if:

(i) There is an identity element Id in the morphisms $\mathcal{M}$ that satisfies: $\mathrm{Id}(obj) = obj$ for all objects obj in the objects $\mathcal{O}$ and the composition law $f \circ \mathrm{Id} = f = \mathrm{Id} \circ f$ holds for all f in $\mathcal{M}$.

(ii) For a specific object obj, the collection of morphisms from obj to itself must contain the identity

(iii) composition is associative: for all $f, g, h \in \mathcal{M}$, $(f \circ g) \circ h = f \circ (g \circ h)$.

The above idea of a category takes all the specific object-morphisms pairs mentioned earlier and identifies the commonality between them. To get a feel for this formal notion of category, examine the following two categories.

Example 1.2 (Vector Spaces as a Category). The category of vector spaces has as objects the collection of all vector spaces and as morphisms the collection of all linear maps between vector spaces.

(i) The collection of all linear maps indeed includes the identity mapping. Here, the 1×1 identity matrix, the 2×2 identity matrix, the 3×3 identity matrix, and all of the others are representations of the same identity morphism. Indeed for any other linear map L, the composition requirement $\mathrm{Id} \circ L = L = L \circ \mathrm{Id}$ holds.

(ii) The collection of all linear maps from a particular vector space V to itself indeed includes the identity. In this case, fixing a basis for an n dimensional vector space allows the $n \times n$ identity matrix to represent this identity map.

(iii) The composition of linear maps, by the nature of functions, is associative.

Example 1.3 (Rings as a Category). The category of rings has as objects the collection of all rings and as morphisms the collection of all rings homomorphisms between rings.

(i) The collection of all ring homomorphisms indeed includes the identity. Here the identity is the mapping that takes elements to themselves and is indeed a ring homomorphism:

$$\begin{aligned} \mathrm{Id}(r+s) &= r+s = \mathrm{Id}(r)+\mathrm{Id}(s) \\ \mathrm{Id}(rs) &= rs = \mathrm{Id}(r)\,\mathrm{Id}(s) \\ \mathrm{Id}(1) &= 1 \end{aligned}$$

Further, for any other ring homomorphism φ, the composition requirement $\varphi \circ \mathrm{Id} = \varphi$ holds.

(ii) Additionally, the set of all ring homomorphisms from a particular ring R to itself includes the identity mapping. Taking the elements of R to themselves is a valid ring homomorphism from R to R.

(iii) Finally, the composition of ring homomorphisms, by the nature of functions, is associative.

So the rules posed in the definition of category seem to work out for specific examples. However, this general idea of a "category" is currently quite useless; however, pinning down the similarities between different categories allows for creating relationships between categories. The following defines a way to map one category to another.

Definition 1.4 (Functor). Consider two categories $\mathcal{C}_A = (\mathcal{O}_A, \mathcal{M}_A)$ and $\mathcal{C}_B = (\mathcal{O}_B, \mathcal{M}_B)$. Then, consider a mapping $\mathcal{F} : \mathcal{C}_A \to \mathcal{C}_B$, which maps $\mathcal{O}_A$ to $\mathcal{O}_B$ and $\mathcal{M}_A$ to $\mathcal{M}_B$. Then, $\mathcal{F}$ is called a *functor* if $\mathcal{F}$ preserves identity identity: $\mathcal{F}(\mathrm{Id}_A) = \mathrm{Id}_B$ as well as satisfies either one of the following two composition requirements:

- For, $f, g \in \mathcal{M}_A$, then $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. Here, $\mathcal{F}$ is called a *covariant functor*.
- For, $f, g \in \mathcal{M}_A$, then $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$. Here, $\mathcal{F}$ is called a *contravariant functor*.

Is there a good simple example of a functor? I could do the functor from rings to abelian group by forgetting multiplication.

The difference between covariant and contravariant functors becomes clearer when examining *commutative diagrams* as depicted in the included figures. Figure 1 shows the arrows pointing in the same direction and corresponds to a covariant functor. However, Figure 2 reverses the direction of the arrow with the application of the functor and represents a contravariant functor. Both of these diagrams represents a functor between to categories, say from category A to category B. Then, X and Y represent two objects in category A and f is a morphism from object X to object Y. Then $\mathcal{F}$ denotes a functor from category A to category B and so $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are objects in categories B; more specifically, $\mathcal{F}(X)$ is where the functor maps object X to and $\mathcal{F}(Y)$ is where the functor maps object Y to. The functor takes f to $\mathcal{F}(f)$, which represents a morphism between $\mathcal{F}(X)$ and $\mathcal{F}(Y)$, but keep in mind the direction of this mapping depends on the type of functor.

So, the direction of the arrows is preserved for covariant functors and reversed for contravariant functors. But how does this but how does this relate to the composition requirements as given in definition 1.4? Applying the functor on an additional object Z together with an additional morphism f as in Figures 3 and 4 gives a visual of the composition requirements. For covariant functors as in figure 3, the natural composition requirement is not surprising, $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. However, in the case of contravariant functors as depicted in Figure 4, the statement $\mathcal{F}(g) \circ \mathcal{F}(f)$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathcal{F}"] & Y \arrow[d, "\mathcal{F}"] \\
\mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(Y)
\end{tikzcd}$$

Figure 1. Covariant Functor

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathcal{F}"] & Y \arrow[d, "\mathcal{F}"] \\
\mathcal{F}(X) & \mathcal{F}(Y) \arrow[l, "\mathcal{F}(f)"']
\end{tikzcd}$$

Figure 2. Contravariant Functor

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[dd, "\mathcal{F}"] & \\
X \arrow[ur, "f"] \arrow[rr, "g\circ f"] \arrow[dd, "\mathcal{F}"] & & Z \arrow[dd, "\mathcal{F}"] \\
 & \mathcal{F}(Y) \arrow[dr, "\mathcal{F}(g)"] & \\
\mathcal{F}(X) \arrow[ur, "\mathcal{F}(f)"] \arrow[rr, "\mathcal{F}(g\circ f)"] & & \mathcal{F}(Z)
\end{tikzcd}$$

Figure 3. Covariant Functor Composition

does not make sense! It is impossible to apply $\mathcal{F}(f)$ and then immediately $\mathcal{F}(g)$ because the input space of $\mathcal{F}(g)$ is different than the output space of $\mathcal{F}(f)$. The relevant composition requirement then must be $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

The word isomorphism is used when working in rings, groups, manifolds, vector spaces, and various other settings. So perhaps it does not come as a surprise that category theory also provides general definition of isomorphism that carries over to all of these different categories.

$$\begin{tikzcd}
 & Y \arrow[rd, "g"] \arrow[dd, "\mathcal{F}"] & \\
X \arrow[ru, "f"] \arrow[rr, "g\circ f"] \arrow[dd, "\mathcal{F}"] & & Z \arrow[dd, "\mathcal{F}"] \\
 & \mathcal{F}(Y) \arrow[ld, "\mathcal{F}(f)"'] & \\
\mathcal{F}(X) & & \mathcal{F}(Z) \arrow[lu, "\mathcal{F}(g)"'] \arrow[ll, "\mathcal{F}(g\circ f)"']
\end{tikzcd}$$

Figure 4. Contravariant Functor Composition

Definition 1.5 (Isomorphism). Given a category $(\mathcal{O}, \mathcal{M})$ and two objects X and Y, then a morphism $\varphi : X \to Y$ is an *isomorphism* if there exists a morphism $\psi : Y \to X$ such that $\varphi \circ \psi = \mathrm{Id}$ and $\psi \circ \varphi = \mathrm{Id}$.

The above definition essentially says an isomorphism is a morphism that has a morphism as an inverse. For example, a linear map is an isomorphism if it has a linear inverse and a ring homomorphism is an isomorphism if its inverse is a ring homomorphism. However, in linear algebra, the definition of an isomorphism is often given as a bijective linear map. Similarly, in ring theory, an isomorphism is often defined as a bijective ring homomorphism. These definitions do not address the morphism properties of the inverse. However, in these specific cases, one can verify that the inverse of a bijective linear map L is always linear. For instance, the scalar verification would go as follows.

$$L^{-1}(\alpha x) = L^{-1}(\alpha L(x')) = L^{-1}(L(\alpha x')) = \alpha x' = \alpha L^{-1}(x)$$

Where the first step $x = L(x')$ for some x' uses surjectivity of L and the last step $x' = L^{-1}(x)$ uses the injectivity of L. A similar argument gives the additive property of linear transformations and

However, it is not always true that a bijective morphism will have a morphism as an inverse. In particular, a continuous bijection need not have a continuous inverse.

Category theory is, by design, abstract. Comfort with speaking in the language of category theory comes with practice and the following chapters will aid in practicing this language.

2. Further Examples

Definition 1.6 (Division Algebra). A division algebra $(A, +, \cdot)$ is such that $(A, +)$ forms an abelian group, there is a multiplicative identity, and every element has a two-sided multiplicative inverse.

Note that division algebras do not require commutativity or associativity the multiplication operation, which is what distinguishes the definition of division algebras from that of fields. Every field is then a division algebra, including the fields $\mathbb{R}$ and $\mathbb{C}$. However, there are examples of division algebras that are not fields.

Such an example of a non-commutative division algebra the *quaternions*, which considers the set $\mathbb{R}^4$ and denotes each element $(a, b, c, d) \in \mathbb{R}^4$ by $a + bi + cj + dk$. Applying the relations $i^2 = j^2 = k^2 = ijk = -1$ makes the set $\{\pm 1, \pm i, \pm j, \pm k\}$ a (noncommutative) group under multiplication. With this, multiplication of two elements of $\mathbb{R}^4$ is given by expanding the expression $(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k)$ with the distribution law, and simplifying using the operations of the group $\{\pm 1, \pm i, \pm j, \pm k\}$. With this construction, every element indeed has a multiplicative inverse. Specifically,

$$(a + bi + cj + dk)^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

Thus $\mathbb{R}^4$ can be equipped with this division algebra structure. The *Cayley octonions* are an example of a non-associative division algebra structure on the set $\mathbb{R}^8$. Considering the division algebras over the objects

Example 1.7 (Division Algebras as a Category). /*verify it forms a category*/

Definition 1.8 (Sphere as an H-Space). Let the sphere S^{n-1} denote the subset of $\mathbb{R}^n$ that is distance 1 from the origin. Then, S^{n-1} is an *H-space* if it is equipped with a continuous multiplication map $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$.

For example of an H-space, consider S^1 as the set of complex numbers with norm 1. This equips S^1 with a continuous multiplication map given by multiplication of complex numbers. This map is closed in S^1 because the product of two complex numbers with norm 1 will have norm 1 and thus this gives an H-space.

Example 1.9 (Sphere H-Spaces as a Category). /*verify H-Spaces form a category?*/

Example 1.10. /*A functor between division algebras and H-Spaces?*/

CHAPTER 2

Algebra

1. Ring Completion

An example of a category that the reader is likely unfamiliar with is the category of semirings. The objects in these categories are called semirings, which are simply rings without necessarily having an additive inverse. For completion, a formal definition of semiring follows.

Definition 2.1 (Semiring). A *semiring* is a set S paired with the binary operations $(+, \cdot)$ such that the following properties hold:

(i) The operation $+$ is associative and commutative
(ii) The operation $\cdot$ is associative
(iii) The operation $\cdot$ distributes over $+$
(iv) S has both an additive and multiplicative identity.

A simple example of a semiring is the set of nonnegative integers under the usual addition and multiplication operations. The element 0 is the additive identity and 1 is the multiplicative identity. In fact, this example of $\mathbb{N} \cup \{0\}$ has two additional nice properties: commutativity of multiplication and the cancellation property under addition. To be precise, the cancellation property promises that given elements a, b, and s in a semiring, the statement $a+s=b+s$ implies $a=b$. This section will focus on commutative semirings with the additive cancellation property.

To complete the category of semirings, the morphisms of a category must be discussed. In this case, the morphisms are referred to as homomorphisms of semirings and the definition follows.

Definition 2.2 (Homomorphism of Semirings). Take monoids S and R and consider a mapping $\varphi : S \to R$. Then, φ is a *homomorphism of semirings* if:

(i) $\varphi(a+b) = \varphi(a) + \varphi(b)$ for all $a, b \in S$.
(ii) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$
(iii) $\varphi(1) = 1$

Note homomorphisms between semirings follows the same structure as homomorphism between rings; in fact, a homomorphism of rings *is* a homomorphism of semiring, for rings are themselves semirings. In fact, even a mapping from a semiring S to a ring R could be considered a homomorphism of semirings if the mapping satisfies the necessary properties. Overall, the category of semirings is frustratingly close to the category of rings. Luckily, there is a functor from the category of commutative semirings with cancellation to the category of rings called *ring extension* – a way to expand the structure of a monoid into a fully fledged ring. K-Theory heavily relies on this functor, so pay particular attention to it.

The formal definition of ring extension is addressed shortly, but first consider the following example. Take the semiring of nonnegative integers; predictably, the ring extension of this example is the set of all integers. In this example, ring extension hinges on the fact we can map a pair of nonnegative integers (a, b) to an element of $\mathbb{Z}$ via the mapping $a - b$. In a semiring, there is no promise of subtraction, but the pair (a, b) can secretly represent the difference $a-b \in \mathbb{Z}$ through an equivalence relation.

Definition 2.3 (Ring Completion). Take commutative semiring S with additive cancellation. Then, a *ring completion* of of S is a commutative ring R together with an injective homomorphism $i : S \to R$ that satisfies the following property: for any commutative ring R' and corresponding homomorphism of semirings $\varphi : S \to R'$, there exists a unique homomorphism of rings such $\psi : R \to R'$ such that the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[d, "i"] & R' \\
R \arrow[ur, dashed, "\psi", "\exists!"'] &
\end{tikzcd}$$

Figure 1. The Universal Property

That is, $\psi \circ i = \varphi$.

There is still work to be done with this definition; it must still be verified that the above construction exists and is unique. The requirement that the above triangle commutes is the *universal property*, and throughout this chapter there will be many constructions using the universal property structure.

To get a better feel for this definition, recall the example of semiring of nonnegative integers extending into ring of all integers. In this case, the extension function $i : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ is given by the injective identity function $i(n) = n$. First, observe how the choice of $\mathbb{Z}$ as the extension fulfills the requirement of the definition. For instance, taking $R' = \mathbb{Z}/(2)$ and homomorphism $\varphi : n \mapsto n \mod 2$, then the homomorphism over the integers $\psi : z \mapsto z \mod 2$ satisfies the triangle, and it follows from the restrictions provided by the definition of a ring homomorphism that this is the unique choice of ψ. However, this is only one specific case. The homomorphism ψ will be unique regardless of the choice of R' and φ. This makes $\mathbb{Z}$ a valid group completion for the nonnegative integers. In fact, $\mathbb{Z}$ is the *unique* group completion and the proof of this is given now.

Proof of Uniqueness of Definition 2.3. Consider two ring completions (R, i) and (R', i'). It must be shown that R and R' are isomorphic. By (R, i) a ring completion and taking (R', i') to be a ring-homomorphism pair, the universal property in the definition of ring completion promises the existence of a unique homomorphism $\psi_1 : R \to R'$ such that $\psi_1 \circ i = i'$. Similarly, by swapping the roles of (R, i) and (R', i'), there exists a a unique homomorphism $\psi_2 \circ i' = i$. But then, the composition $\psi_2 \circ \psi_1 : R \to R$ satisfies the commutativity restriction $(\psi_2 \circ \psi_1) \circ i = i$. Thus $\psi_2 \circ \psi_1$

must be the unique map promised by the universal property by applying the universal property of ring completion (R, i) on (R, i) itself. However, the identity mapping also satisfies the condition $\mathrm{Id} \circ i = i$ and so the uniqueness conditions gives that $\psi_2 \circ \psi_1 = \mathrm{Id}$. See Figure 2 for a visual of this argument. The same argument gives that $\psi_1 \circ \psi_2 = \mathrm{Id}$ and thus ψ_1 and ψ_2 are inverses of one another. This gives that ψ_1 and ψ_2 are isomorphisms and so $R \cong R'$.

$$\begin{tikzcd}
R \arrow[rr, dashed, "\mathrm{Id}"] \arrow[ddr, dashed, "\psi_1"'] & & R \\
& S \arrow[ul, "i"'] \arrow[ur, "i"] \arrow[d, "i'"] & \\
& R' \arrow[uur, dashed, "\psi_2"'] &
\end{tikzcd}$$

Figure 2. Uniqueness of Ring Completion Argument

□

The above argument never appeals to the specific properties of rings and semirings; in fact, this argument applies to *all* definitions defines through the universal property. For every additional definition using the universal property in this chapter, uniqueness will follow automatically.

All that needs to be shown to justify a definition using the universal property is existence. For the case of semiring completion, this existence proof is given in section **??** of the Appendix. Here are the important takeaways from the proof. For a semiring S, the proof uses the equivalence relation $\sim$ on $S \times S$ given by $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. The motivation for this equivalence relation is that the integers can be created by all differences of the nonnegative numbers. In fact, one can think of this equivalence relation as "sneaky subtraction" stemming from the wish to express $a_1 - b_1 = a_2 - b_2$ without the explicit use of subtraction. From this equivalence relation, we get a natural addition on the equivalence classes that gives a commutative group structure. However, in order to get a well-defined multiplication, the semiring must have the additive cancellation property.

Rings are nicer than semirings; they have additive inverses and extensive theory. As shown above, every commutative semiring with cancellation extends to a unique ring; therefore, given a semiring with these properties, it is best to ditch the semiring and instead talk about the ring extension. Keep this motivation to extend an "incomplete" object into a nicer object in mind for the following constructions.

2. Packing Together Modules

I am still working on this section

Definition 2.4 (Module). Let M be a set, and let R be a commutative ring with identity. Further, take an additive operation $+ : M \times M \to M$ and a scalar multiplication from $R \times M$ to M. Then, M is a *module over* R if:

(i) $(r+s)m = rm + sm$ for all $r, s \in R$ and $m \in M$
(ii) $r(m+n) = rm + rn$ for all $r \in R$ and $m, n \in M$
(iii) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$
(iv) $1 \cdot m = m$ for all $m \in M$

Modules can be made into a category. Keeping in mind that modules are generalizations of vector spaces, the natural homomorphism to associate with with modules is a linear map as in the following definition.

Definition 2.5 (Module Homomorphism). Let R be a commutative ring and let M and N be R-modules. Then, a *homomorphism of modules* is a mapping $\varphi : M \to N$ such that

(i) $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ for all $m_1, m_2 \in M$.
(ii) $\varphi(rm) = r\varphi(m)$ for all $r \in R$ and $m \in M$.

Definition 2.6 (Direct Sum). Take commutative ring R with identity and consider a collection M_λ of R-modules, $\lambda \in I$ where I is an index set. Then, the *direct sum* of the collection M_λ, denoted $\oplus_{\lambda \in I} M_\lambda$, is the unique R-module such that:

(i) For all $\lambda \in I$, there is an inclusion map $i_\lambda : M_\lambda \to \oplus_{\lambda \in I} M_\lambda$.
(ii) The universal property is satisfied. That is, for any R-module N and homomorphisms of R-modules $\varphi_\lambda : M_\lambda \to N$ there exists a unique homomorphism of R-modules $\psi : \oplus_{\lambda \in I} M_\lambda$ such that the following diagram commutes. That is, $\psi \circ i_\lambda = \psi_\lambda$ for all $\lambda \in I$.

$$\begin{tikzcd}
M_\lambda \arrow[r, "\varphi_\lambda"] \arrow[d, "i_\lambda"] & N \\
\bigoplus_{\lambda \in I} M_\lambda \arrow[ur, dashed, "\psi", "\exists!"'] &
\end{tikzcd}$$

Figure 3. Universal Property of Direct Sum

Uniqueness of the direct sum follows directly from the universal property as mentioned in the ring completion section. /*comment on existence*/.

Recall the categories of vector spaces, abelian groups, and commutative rings all are special cases of modules. Then, the functor from each of these categories to the category of modules defines allows each category to borrow the direct sum operation on modules, which in turn defines a direct sum operation on each category. However, borrowing a module operation only promises that the resulting direct sum will be a module — not a vector space, abelian group, or commutative ring. For each individual category, it must be verified that the direct sum construction is an object in the same category. For example, the direct sum of vector spaces with field F promises an F-module which is luckily exactly equivalent to a vector space over F. Similarly, the direct sum of abelian groups promises a $\mathbb{Z}$-module which again is luckily exactly equivalent to an abelian group. However, showing that the direct sum of commutative rings results in a commutative ring takes more work to verify, for there is no predefined multiplication mapping on the direct sum /*prove this works and elaborate*/.

Definition 2.7 (Tensor Product). Take commutative ring R with identity and take M_1 and M_2 to be R-modules. Then, the *tensor product* of M_1 and M_2, denoted $M_1 \otimes M_2$ is the unique R-module such that:

(i) There is a bilinear map $b : M_1 \times M_2 \to M_1 \otimes M_2$
(ii) The universal property is satisfied. That is, for for any R-module N with corresponding bilinear map $\omega : M_1 \times M_2 \to N$, there exists a unique homomorphism of modules $\psi : M_1 \otimes M_2 \to N$ such that the following diagram commutes. That is, $\psi \circ b = \omega$.

$$\begin{tikzcd}
M_1 \times M_2 \arrow[r, "\omega"] \arrow[d, "b"] & N \\
M_1 \otimes M_2 \arrow[ur, dashed, "\psi", "\exists!"'] &
\end{tikzcd}$$

Figure 4. Universal Property of Tensor Products

Again, uniqueness of the direct sum follows automatically from the universal property. /*comment on existence*/.

/*extend to vector spaces, abelian groups, and commutative rings*/

3. Verifications

Proof of Existence of Definition 2.3. The existence of a ring completion is shown through an explicit construction. Take any commutative semiring with additive cancellation $(S, +, \cdot)$ and consider the equivalence relation $\sim$ on $S \times S$ defined as follows: for (a_1, b_1), (a_2, b_2) in $S \times S$, then let $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. The aim is to make the set of equivalence classes under $\sim$ into a ring.

First, define the additive operation $+$ by

$$[(a, b)] + [(c, d)] = [(a + c, b + d)]$$

Next, define the multiplicative operation $\cdot$ by

$$[(a, b)] \cdot [(c, d)] = [(ac + bd, ad + bc)]$$

This proof aims to verify that the set of equivalence classes $S \times S/\sim$ paired with the operations $(+, \cdot)$ forms a commutative ring that is a ring completion of S.

It must be verified that the additive operation is well defined, so consider elements (a_1, b_1), (a_2, b_2), (c_1, d_1), (c_2, d_2) in $S \times S$ such that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$. Then, I claim that $(a_1 + c_1, b_1 + d_1) \sim (a_2 + c_2, b_2 + d_2)$. Indeed, this satisfies the definition of the equivalence relation, for

$$\begin{aligned}&(a_1 + c_1) + (b_2 + d_2) = (a_1 + b_2) + (c_1 + d_2)\\&= (a_2 + b_1) + (c_2 + d_1) = (a_2 + c_2) + (b_1 + d_1)\end{aligned}$$

where the above computation used the substitutions $a_1 + b_2 = a_2 + b_1$ and $c_1 + d_2 = c_2 + d_1$ promised by the relations $(m_1, m_2) \sim (m'_1, m'_2)$ and $(l_1, l_2) \sim (l'_1, l'_2)$. This confirms that $+$ is well-defined on $(S \times S)/\sim$.

The transitivity and commutativity of $+$ on the equivalence classes follows immediately from the commutativity and transitivity of the operation $+$ on S.

Next, note that the additive identity in $(S \times S)/\sim$ is given by $[(0, 0)]$ where 0 denotes the identity element in S. Indeed, we have $[(a, b)] + [(0, 0)] = [(a, b)]$ for any element $[(a, b)]$.

The proposed ring has an inverse mapping for the addition operation. Consider an element $[(a, b)]$. Then, I claim the element $[(b, a)]$ forms the desired inverse. To see this, consider the sum $[(a+b, b+a)]$ and note that $(a + b) + 0 = 0 + (b + a)$, which shows $[(a + b, b + a)] = [(0, 0)]$.

It must be verified that the multiplicative operation is well-defined before verifying any further properties. Consider the elements $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$ in $S \times S$. It then must be verified that $(a_1c_1 + b_1d_1, a_1d_1 + b_1c_1) \sim (a_2c_2 + b_2d_2, a_2d_2 + b_2c_2)$. To accomplish this, consider the following $M_1, M_2 \in S$:

$$\begin{aligned}M_1 &= c_2(a_1 + b_1) + b_2(c_1 + d_1) + b_2c_2\\M_2 &= c_1(a_2 + b_2) + b_1(c_2 + d_2) + b_1c_1\end{aligned}$$

Next, observe that using the relations $a_1 + b_2 = a_2 + b_1$ and $c_1 + d_2 = c_2 + d_1$, it follows that $a_1c_1 + b_1d_1 + M_1 = a_2c_2 + b_2d_2 + M_2$.

$$\begin{aligned}&a_1c_1 + b_1d_1 + M_1 = a_1c_1 + b_1d_1 + c_2a_1 + c_2b_1 + b_2c_1 + b_2d_1 + b_2c_2\\&= (a_1 + b_2)(c_1 + c_2) + (d_1 + c_2)(b_1 + b_2)\\&= (a_2 + b_1)(c_1 + c_2) + (d_2 + c_1)(b_1 + b_2)\\&= a_2c_2 + b_2d_2 + c_1a_2 + c_1b_2 + b_1c_2 + b_1d_2 + b_1c_1 = a_2c_2 + b_2d_2 + M_2\end{aligned}$$

A similar process shows that $a_1d_1 + b_1c_1 + M_1 = a_2d_2 + b_2c_2 + M_2$. Then, summing the two results gives

$$(a_1c_1 + b_1d_1) + (a_2d_2 + b_2c_2) + (M_1 + M_2) = (a_2c_2 + b_2d_2) + (a_1d_1 + b_1c_1) + (M_1 + M_2)$$

Applying the additive cancellation property of S to the term $(M_1 + M_2)$ gives the desired relation and provides the conclusion $(a_1c_1 + b_1d_1, a_1d_1 + b_1c_1) \sim (a_2c_2 + b_2d_2, a_2d_2 + b_2c_2)$ and so the multiplicative operation is well defined.

The transitivity of the multiplicative operation follows directly from $+$ and $\cdot$ transitive in S. Similarly, the commutativity of the multiplicative operation follows directly from the commutativity of $+$ and $\cdot$ in S.

Next, note that the element $[(1,0)]$ acts as an identity element for the multiplicative operation. Indeed, $[(1,0)] \cdot [(a,b)] = [(a,b)]$ for any element $[(a,b)]$.

It only remains to show that $+$ distributes over $\cdot$ to verify that $S \times S/\sim$ forms a ring. Indeed, for elements $[(a,b)]$, $[(c,d)]$, $[(e,f)]$:

$$\begin{aligned}&[(e,f)] \cdot ([(a,b)] + [(c,d)]) = [(e,f)] \cdot [(a+c, b+d)]\\ &= [(ea + fb + ec + fd, eb + ed + fa + fe)]\\ &= [(ea + fb, eb + fa)] + [(ec + fd, ed + fc)] = [(e,f)] \cdot [(a,b)] + [(e,f)] \cdot [(e,d)]\end{aligned}$$

Thus we have that $(S \times S)/\sim$ forms a commutative ring under the proposed operations. However, it remains to show that $(S \times S)/\sim$ is a valid ring completion. The necessary inclusion map $i : S \to (S \times S)/\sim$ is given by $i(s) = [(s,0)]$. Then, take any ring R' and homomorphism $\varphi : S \to R'$; the existence and uniqueness of a commuting ring homomorphism $\psi : (S \times S)/\sim \to R'$ must be shown.

Uniqueness follows quickly from its homomorphism properties and the commutativity of the universal property. Indeed, take two commuting ring homomorphisms ψ and ψ' from $S \times S/\sim$ to R'. Then, the restrictions $\psi \circ i = \varphi$ and $\psi' \circ i = \varphi$ paired with i injective gives that $\psi = \psi'$ over the image $i(S)$. Then observe that any element $[(a,b)]$ is the composition of elements in $i(S)$ by $[(a,b)] = [(a,0)] - [(b,0)]$. Then, the homomorphism properties of rings extends ψ and ψ' to be equivalent over all of $(S \times S)/\sim$ giving uniqueness.

It only remains to show existence of the homomorphism. The map $\psi : [(a,b)] \mapsto \varphi(a) - \varphi(b)$ works. Commutativity follows easily, for $(\psi \circ i)(s) = \psi([(s,0)]) = \varphi(s)$ for all $s \in S$. Now, it must be verified that ψ is a homomorphism. So, consider elements $[(a,b)]$ and $[(c,d)]$ of the ring completion.

The following equality chain shows that the the additive property of φ gives the additive property of ψ.

$$\begin{aligned}&\psi([(a,b) + (c,d)]) = \psi([(a+c, b+d)]) = \varphi(a+c) - \varphi(b+d)\\ &= (\varphi(a) - \varphi(b)) + (\varphi(c) - \varphi(d)) = \psi([(a,b)]) + \psi([(c,d)])\end{aligned}$$

Similarly, the additive and multiplicative property of φ gives the multiplicative property of ψ.

$$\begin{aligned}&\psi([(a,b)] \cdot [(c,d)]) = \psi([(ac + bd, ad + bc)])\\ &= \varphi(ac + bd) - \varphi(ad + bc) = \varphi(a)\varphi(c) + \varphi(b)\varphi(d) - \varphi(b)\varphi(c) - \varphi(a)\varphi(d)\\ &= (\varphi(a) - \varphi(b))(\varphi(c) - \varphi(d)) = \psi([(a,b)]) \cdot \psi([(c,d)])\end{aligned}$$

Finally $\psi(1) = \psi([(1,0)]) = \varphi(1) = 1$, completing the proof.

□

CHAPTER 3

Topology

1. The Category of Topological Spaces

To begin this section, consider the following instance of a metric space. Note that a metric space is simply a set paired with a way of telling distance between two points. Note that a metric space gives rise to a notion of open sets.

EXAMPLE 3.1 (The Interval as a Metric Space). Consider the metric space that takes the interval $[0,1]$ as a set and define the distance between two points $x, y \in [0,1]$ to be $|x-y|$. This gives a notion of *open sets* on the interval. For example, the open interval $(1/3, 2/3)$ is an open set. In this case, the collection of all open sets is any finite intersection and infinite union of open intervals, where the whole set $[0,1]$ is also defined as open.

This section addresses a new category — the category of topological spaces. Topology looks at the open sets that come as a consequence of a metric and asks and asks: what if the *only* thing we knew were what subsets of a space are open and and nothing else. What then can we say about the space? And importantly, what can we learn from comparing these spaces along with their open sets?

Common terminology hints at this central role of open sets in topology. Something *has a topology* if it is knowm what sets are open, and a set is *topologized* by declaring what subsets are open. Below is a formal definition of a topological space. Keep in mind a topological space is simply a set together with a collection of open sets, often denoted $\mathcal{T}$, that follows some rules.

DEFINITION 3.2 (Topological Space). Given a set X and $\mathcal{T} \subseteq \mathcal{P}(X)$, the pair $(X, \mathcal{T})$ is called a *topological space* if:

(i) The empty set $\emptyset$ and the full set X are in $\mathcal{T}$.
(ii) For every set $\mathcal{U}_\alpha$ in $\mathcal{T}$ where $\alpha \in I$, the infinite union $\cup_{\alpha \in I} \mathcal{U}_\alpha$ is in $\mathcal{T}$.
(iii) For any two subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{T}$, the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is in $\mathcal{T}$.

For $\mathcal{U} \in \mathcal{T}$, the element $\mathcal{U}$ is called an *open* set, and its complement $\overline{\mathcal{U}}$ is called a *closed set.*

Compare the above definition to the open sets in a metric space. Recall that in a metric space in which the open sets are given by a metric, it follows that the infinite union of open sets are open and the finite intersection of open sets are open. Topology focuses more directly on the open sets themselves and so makes these union and intersection properties defining characteristics of open sets. Further, note that in a metric space, a closed set has its own definition and it follows that open sets and closed sets are complements of one another. Here however, a closed set is by definition the complement of an open set. Topology describes the same structure as metric spaces, but from a different angle. It should be emphasized, however, that topology is more general than the notion of a metric space. Some topological spaces can be given a metric, but some can not. Below is the same interval

Example 3.3 (The Interval as a Topological Space). Consider the topological space by taking the set $[0,1]$ and let the open sets $\mathcal{T}$ be defined by any combination of a finite number of intersections and an arbitrary number of unions on open intervals, and additionally let $[0,1] \in \mathcal{T}$. This indeed forms a topological space because:

(i) $\emptyset$ and $[0,1]$ are open.
(ii) For any collection of open sets $\mathcal{U}_\alpha$ with $\alpha \in I$, each $\mathcal{U}_\alpha$ is the union of open intervals, so the union of the collection $\cup_{\alpha \in I}\mathcal{U}_\alpha$ reduces to the union of open sets and thus gives an open set by definition.
(iii) For any two open sets $\mathcal{U}_1$ and $\mathcal{U}_2$, both $\mathcal{U}_1$ and $\mathcal{U}_2$ is the union of open intervals, but then consider the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$. This is indeed open, for it is a combination of open intervals with only a single intersection operation.

Thus the pair $[0,1]$ and $\mathcal{T}$ indeed forms a topological space. This topology is induced by the metric that describes physical distance as described in example 3.1, so this topology is called the *standard topology.*

The category of topological spaces is still incomplete — topological objects require some notion of morphisms between them. The inspiration for a good choice of morphism comes from the notion of continuous maps between metric spaces. In metric spaces, one equivalent way to define continuity is through open sets. This definition fits nicely in topology, so the category of topology borrows this definition to define continuity between topological spaces, which will be the morphisms of the category.

Definition 3.4 (Continuous Function). Take two topological spaces and corresponding open sets $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. A function between the spaces $f : X \to Y$ is called *continuous* if for every open set $V \in \mathcal{T}_Y$, its inverse image is open: $f^{-1}(V) \in \mathcal{T}_X$.

As an example of a continuous function, consider the following mapping between intervals.

Example 3.5. As an example of a continuous function, consider the interval I with the standard topology and let the function $f : I \to I$ be given by $f(x) = 0$. To see that this map is continuous, take any set V in the codomain. Consider two cases: $0 \in V$ and $0 \notin V$. In the $0 \in V$ case, then the inverse image is the entire domain, that is $f^{-1}(V) = I$, and the domain as a whole is open. In the case that $0 \notin V$, then the inverse image is the empty set, in other words $f^{-1}(V) = \emptyset$, and the empty set is open. Thus, f is continuous.

Note that the above example does not actually depend on the topology I. This is typically not the case; in fact, the topology typically has a large impact on whether a function is continuous.

Example 3.6. Take two topological objects $(I_1, \mathcal{T}_1)$ and $(I_2, \mathcal{T}_2)$ where I_1 and I_2 denote the interval and $\mathcal{T}_1$ and $\mathcal{T}_2$ will be discussed later. Then, let $f : I_1 \to I_2$ be a function given by $f(x) = x$. Now consider two possibilities:

- Firstly, take $\mathcal{T}_1$ to be the power set $\mathcal{P}(I_1)$ and take $\mathcal{T}_2 = \{\emptyset, I_2\}$. Then, the preimage of every set V in I_2 is indeed open, for every set in I_1 is open and thus f is continuous.

- Next, swap the topologies. Take $\mathcal{T}_1 = \{\emptyset, I_1\}$ and $\mathcal{T}_2 = \mathcal{P}(I_1)$. Then, many open sets have a non-open preimage. For instance, the set $\{0\} \subset \mathcal{T}_2$ is open in this topology, but the preimage is given by $\{0\} \subset I_1$, which is not one of the two open sets I_1. Thus this map is not continuous.

As hinted at before, the category of topological spaces is similar to the category of metric spaces. Given a metric space, the objects can be translated to topological objects and the continuous functions between metric spaces can interpreted as continuous functions between topological objects. In fact, this is a functor from the category of metric spaces to the category of topological spaces. This functor can be used to define topological objects as in the following example.

Example 3.7 (S^2). Consider the unit sphere as understood as a metric space. That is, take the set $X \subset \mathbb{R}^3$ defined by $X = \{(x, y, z) \in \mathbb{R} | x^2 + y^2 + z^2 = 1\}$. Then, consider the standard metric d in $\mathbb{R}^3$ and the pair (X, d) defines the unit sphere as a metric space. By considering all open sets $\mathcal{T}$ as defined by the metric d, the pair (X, d) then defines the topology of the two dimensional sphere S^2. Take particular note that this process is due to the functor from metric spaces to topological spaces.

This process of defining topological spaces through metric spaces provides many fundamental topological objects that we can build upon. In particular, by using the standard metric of $\mathbb{R}^n$, we can get the topology of an interval I, the topology of any sphere S^n, and the topology of $\mathbb{R}^n$ itself.

Recall that an isomorphism as generally defined in category theory is a morphism that has an inverse morphism, suggesting the following definition in topology.

Definition 3.8 (Isomorphism). A function f between topological spaces is an *isomorphism* if f is continuous with a continuous inverse.

/*example of isomorphism?*/

2. Some Mappings and Corresponding Topologies

This section addresses three useful mappings in topology: the quotient map, the inclusion map, and the product map. Ideally, these three maps will be continuous functions so that they fit into category theory. However, as demonstrated in example 3.6, changing the topology can *make* a function continuous. So, each of these three mappings has a corresponding topology that provides continuity. With the completion of this section, there will be no need to stress about whether a quotient map, inclusion map, or product map is continuous — the assumed topology will make it continuous.

A quotient map is a map from a topological X to a quotient $X/\sim$ by some equivalence relation $\sim$ on X. Let Y denote the set of equivalence classes $X/\sim$. Because elements of Y are subsets of X, there is a natural choice of topology $\mathcal{T}_Y$. For a single equivalence class $[x] \in Y$, the natural choice is to say $[x] \in \mathcal{T}_Y$ exactly when $[x] \in \mathcal{T}_X$. In general, any group of equivalence classes $\cup_\alpha [x_\alpha]$ in Y should be open in Y exactly when $\cup_\alpha [x_\alpha]$ is open in X. This topology makes the mapping $x \mapsto [x]$

continuous. However, this quotient map is actually a better starting point to define the quotient topology, but note that the following definition is inspired by the above.

Definition 3.9 (Quotient Map and Quotient Topology). Consider topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ along with a surjective function $q : X \to Y$. Further restrict q such that $V \in \mathcal{T}_Y$ if and only if $q^{-1}(V) \in \mathcal{T}_X$. Then, q is a *quotient map*. Note that in the case that $\mathcal{T}_Y$ is not defined, a specified quotient map defines a *quotient topology* on Y.

Note that q requires exactly the sets such that it is always continuous. The condition $V \in \mathcal{T}_Y$ if and only if $q^{-1}(V) \in \mathcal{T}_X$ is analogous to the condition $\cup_\alpha [x_\alpha] \in T_Y$ if and only if $\cup_\alpha [x_\alpha] \in \mathcal{T}_X$ discussed previously. However, the quotient map is a more natural starting point and gives the equivalence relation immediately by $x_0 \sim x_1$ if and only if $q(x_0) = q(x_1)$.

/*example of quotient map. collapsing $A \subset X$ to a point example?*/

Along with the quotient topology and corresponding quotient map, there is a subspace topology and corresponding inclusion map. Consider a topological space $(X, \mathcal{T}_X)$ and an open subset $A \subset X$ with no pre-defined topology. However, the subset A can borrow topology from the larger space X in a natural way: define $\mathcal{T}_A$ by $\mathcal{U} \in \mathcal{T}_A$ exactly when $U \in T_X$. However, note that this definition only works when A is open in X because for A to be a topological space, the whole space must be open. A construction that works when A is not open is given by $\mathcal{T}_A = \{\mathcal{U} \cap A | \mathcal{U} \in \mathcal{T}_X\}$. This gives the same open sets as previously discussed for A open, but also defines a valid topology for A closed. This topology is constructed exactly so that the inclusion map $i : A \to X$ given by $i(a) = a$ is continuous. However, defining the inclusion map itself serves as a better starting point, but the following definition is motivated by the above.

Definition 3.10 (Inclusion Map and Subspace Topology). Take topological spaces $(A, \mathcal{T}_A)$ and $(X, \mathcal{T}_X)$. Then, an injective map $i : A \hookrightarrow X$ is called an *inclusion map* when $\mathcal{U} \in \mathcal{T}_A$ if and only if there exists a $V \in \mathcal{T}_X$ such that $V \cap i(A) = i(\mathcal{U})$. In the case that $\mathcal{T}_A$ is not defined, a specified inclusion map defines a *subspace topology* on A.

The defined subspace topology is equivalent to the topology discussed previously, and note this construction is perfectly engineered so that the inclusion map is continuous. Take an inclusion map $i : A \to X$ and an open set $V \subset X$. The parts of X that do not contain the image of A is irrelevant to the inverse image, so $i^{-1}(V)$ is equivalent to $i^{-1}(i(A) \cap V)$. Then, by definition, this corresponds to an open subset $\mathcal{U}$ in A, giving continuity.

/*example for interval and $\mathbb{R}$*/

So, the quotient map is paired with the quotient topology and the inclusion map is paired with the subspace topology. Now consider the projection map, which is paired with the product topology.

Definition 3.11 (Projection). Let $X_1, \ldots, X_n$ be spaces with topologies $\mathcal{T}_1, \ldots, \mathcal{T}_n$. Then the *projection onto the k^{th} factor* is given by the following.

$$p : X_1 \times \cdots \times X_k \times \cdots \times X_N \to X_K$$
$$p : (x_1, \ldots, x_k, \ldots, x_n) \mapsto x_k$$

However, whether such a projection is continuous depends on the choice of topology for the product $X_1 \times \cdots \times X_n$. The natural choice for this topology is the product topology, which is best defined through a universal property.

Definition 3.12 (Product Topology). Let $X_1, \ldots, X_n$ be spaces with topologies $\mathcal{T}_1, \ldots, \mathcal{T}_n$. Then, consider the family of projections onto the k^{th} factor $p_k : X_1, \ldots, X_n \to X_k$. Then, the product topology $\mathcal{T}_\times$ is the unique topology that satisfies the following universal property of product topology. That is, the product topology is such that each p_k is continuous, and for any topological space Y together with a family of continuous map $f_k : Y \to X_k$ there exists a unique continuous map $f : Y \to X_1 \times \cdots \times X_n$ such that the following diagram commutes for each k. That is, $f_k = p_k \circ f$ for each k.

$$\begin{tikzcd}
X_k & & Y \arrow[ll, "f_k"'] \arrow[dll, dashed, "f"', "\exists!"] \\
X_1 \times \cdots \times X_n \arrow[u, "p_k"'] & &
\end{tikzcd}$$

Figure 1. Universal Property of Product Topology

This definition still requires verifications of existence and uniqueness. Uniqueness follows automatically from the universal property; the proof follows similarly here even though the arrows are pointing in the opposite directions-. /*still need to discuss existence*/.

The bottom line with the product topology is that the universal defines the topology of $X_1 \times \cdots \times X_n$ exactly so that each of the projections p_k is continuous. So, assuming the product topology, there is no need to stress about whether p_k is continuous — it will be just as the inclusion and quotient maps will be.

3. Nice Properties of Topological Objects

Topology does not assume a lot — only any set and a notion of open sets. This results in there being a lot of possible topological objects. Some of these objects are intuitive and have nice properties, but many of these objects are unintuitive and often difficult to work with. This section will aim to identify two nice properties — Hausdorff and compact. Restricting spaces to having these two nice properties will throw out the troublesome topological spaces that would obstruct the remainder of the story.

First, note a nice property of the topological space $\mathbb{R}^2$ with the standard topology (which is defined by the standard metric). For any two distinct points $p, q \in \mathbb{R}^2$, it is possible to draw two tiny open sets — one surrounding around p and one surrounding q — such that the two open sets do not intersect. This corresponds to p and q having some distance between them — some "space" to themselves.

However, not all topological objects have this property. For instance, consider the same set $\mathbb{R}^2$ with a ridiculous metric: the distance between any two points is 0. The visual here is that all of $\mathbb{R}^2$ has been squashed into a single point. This then results in a ridiculous topology: $\mathcal{T} = \{\emptyset, \mathbb{R}^2\}$, containing only the empty set and the full set with nothing else. In this case, given any two points p and q it is impossible to draw two disjoint open sets where one contains p and the other contains q. The reason for this is that there is literally no distance between p and q. The best way to avoid working with this topological object is by imposing the Hausdorff condition. The definition follows, but intuitively think of the Hausdorff condition as requiring that any two distinct points have a nonzero distance between them.

Definition 3.13 (Hausdorff). Let X be a topological space. We call X *Hausdorff* if for any pair of distinct points $p, q \in X (p \neq q)$, there exists open sets U_p with $p \in U_p$ and U_q with $q \in U_q$ such that $U_p \cap U_q = \emptyset$.

The Hausdorff property is required for the story to move forward. Another necessary property is *compactness*, which prevents the topological spaces from getting too "big". For example, consider the interval I with the standard metric in comparison to $\mathbb{R}$ with the standard metric. The space $\mathbb{R}$ is unbounded and in some sense bigger than I. The following definition characterizes this difference using only open sets.

Definition 3.14 (Compact). A topological space X is *compact* if every open cover of X has a finite subcover. That is, for any cover $X = \cup_{\alpha \in I} U_\alpha$ with open $U_\alpha \subset X$ for all $\alpha \in I$, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots U_{\alpha_n}$.

With this criterion, spaces such as D^n and S^n are compact and spaces such as $\mathbb{R}^n$ are not compact. To practice, consider the following verification that $\mathbb{R}$ is not compact.

Example 3.15 ($\mathbb{R}^2$ is not compact). In order to show $\mathbb{R}$ is not compact, it suffices to bring up a specific infinite open cover and prove it does not have a finite subcover. So consider the following infinite cover:

$$C = \{(k-1, k+1) \times \mathbb{R} : k \in \mathbb{Z}\}$$

However, consider removing any element $(k-1, k+1)$ from the set. Then, the resulting set $C \setminus \{(k-1, k+1)\}$ does not contain the point k in any of its sets and so it does not cover all of $\mathbb{R}$. Then, there is no way to reduce the cover and thus there is no finite subcover.

The rest of this chapter is not worth reading currently

4. Operations on Topological Spaces

This section is still in progress

The cone operation C takes some topological space and expands it by making it into a cone. Before giving the formal definition, here is an illustration. Begin with S^0 — the set of two points. Then, as is illustrates in figure 2, $C(S^0)$ would be create a low dimensional cone shape which is isomorphic to the interval I. Then, $C(I)$ would create a cone, which is isomorphic to the two dimensional disk D^2.

/*Figure with cone operations*/

Figure 2. Sequence of Cone Operations

/*figure with suspension operations*/

Figure 3. Sequence of Suspension Operations

The formal definition of the cone operation combines the product and quotient topologies.

Definition 3.16 (Cone Operation). Take a topological space X. Then, taking the interval $I = [0, 1]$, the cone operation C is given by

$$C(X) = (X \times I)/(X \times \{0\})$$

Note the quotient of a topological space is given in example **??**. Intuitively, this cone operation takes extends the topological space into a new dimension by crossing with the interval and then "pinches" the top.

A relative to the cone operation is the suspension operation S. If the cone "cone-ifies" a topological space, the suspension operation "sphere-ifies" the topological space. Again, before giving the formal definition, consider an example sequence of suspension operations. Following Figure 3 and beginning with S^0, then $S(S^0)$ gives the circle S^1 and applying the suspension again gives $S(S^1)$ gives the sphere S^2.

The formal definition of the suspension follows uses a similar construction as the cone operation.

Definition 3.17 (Suspension Operation). Take a topological space X. Then, taking the interval $I = [0, 1]$, the suspension operation S is given by

$$C(X) = (X \times I)/(X \times \{0, 1\})$$

And so the suspension operation takes the Cartesian product with the interval and "pinches" two sides together.

/*Wedge Sum*/ /*Smash Product*/ /*Think of good examples for the wedge sum and smash product*/

5. More Topology

This section is still in progress

/*Do I need to define it more general as subsets?*/

Definition 3.18 (Compact). A topological space X is *compact* if every open cover of X has a finite subcover. That is, for any cover $X = \cup_{\alpha \in I} U_\alpha$ with open $U_\alpha \subset X$ for all $\alpha \in I$, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots U_{\alpha_n}$.

/*motivation*/

Definition 3.19 (Hausdorff). Let X be a topological space. We call X *Hausdorff* if for any pair of distinct points $p, q \in X (p \neq q)$, there exists open sets U_p with $p \in U_p$ and U_q with $q \in U_q$ such that $U_p \cap U_q = \emptyset$.

/*examples*/ /*Motivation for normal. Normal is stronger than Hausdorff.*/

Definition 3.20 (Normal). Let X be a topological space. We call X *Normal* if for any two disjoint closed subsets $A \subset X$ and $B \subset X$, there exists open sets U_A with $A \subset U_A$ and $U_B \subset B$ and $U_A \cap U_B = \emptyset$.

/*motivation: will be dealing primarily with compact Hausdorff spaces*/

Theorem 3.21. Every compact Hausdorff space is normal.

/*How difficult is this proof? I need this, should I prove it?*/

/*Topological quotient of Hausdorff space with closed subspace is compact Hausdorff*/

6. Homotopy

This section is still in progress

/*Work towards definition of contractible*/

Definition 3.22 (Homotopic Functions). Take Topological spaces X and Y with functions $f, g : X \to Y$. Then, we say *f is homotopic to g* if there exists a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. We denote the homotopy with $f \simeq g$.

Intuitively, two functions are homotopic if we can continuously deform the functions into one another.

/*give a *visual* example. $f, g : I \to \mathbb{R}^2$ with $f(x) = (x,0)$, $g(y) = (0,y)$ could work well? But perhaps too trivial. A better one could be $f, g : I \to \mathbb{R}^2$ with $f(x) = (\cos(x), \sin(x))$ and $g(x) = (2\cos(x), 2\sin(x))$ The second example would build intuition for the first homology group, which I perhaps should include in the document*/.

Definition 3.23 (Homotopic Spaces). Take topological spaces X and Y. We say X and Y are homotopy equivalent if there exist continuous functions $F : X \to Y$ and $G : X \to Y$ such that $F \circ G \simeq \mathrm{Id}_Y$ and $G \circ F \simeq \mathrm{Id}_X$. We denote the homotopy with $X \simeq Y$.

Example 3.24. I claim that $\mathbb{R}^2 \setminus \{(0,0)\} \simeq S^1$.

7. Appendix

This section is still in progress

7.1. Continuity as a Local Property.

/*description of continuity as a local property and importance*/

/*Reformulate below to be more along the lines of a restriction; more useful later and more intuitive now.*/

Definition 3.25 (Local Continuity). Let $f : X \to Y$ be a function between two topological spaces and take $x \in X$. Then. f is *locally continuous at x* if there exists an open neighborhood V_x containing $f(x)$ such that all open $V' \subset V_x$ satisfies $f^{-1}(V')$ open.

Claim 3.26. A function $f : X \to Y$ is continuous if and only if it is locally continuous at every point in X.

Proof. For the forward direction, assume a continuous function between topological spaces $f : X \to Y$ and consider an arbitrary point $x \in X$. Then, by continuity the entire set Y provides the necessary open set about $f(x)$.

For the reverse direction, assume a function $f : X \to Y$ is locally continuous at every point $x \in X$ and let V_x denote the promised open neighborhood for each x. Then, fix an open set $V \subset Y$. Consider the union of preimages

$$\bigcup_{x\in X} f^{-1}(V \cap V_x).$$

Note that by V and V_x open, each intersection $V \cap V_x$ is open, and by $V \cap V_x \subset V_x$, the definition of local continuity gives $f^{-1}(V \cap V_x)$ open and thus the entire expression P is open. Finally, note that P is equivalent to $f^{-1}(V)$. Indeed, for each $x \in f^{-1}(V)$, $x \in f^{-1}(V \cup V_x)$ and so x is contained in the union ensuring that $f^{-1}(V)$ is contained in the union. Similarly, for each x in the union, consider the corresponding V_x and observe that $x \in f^{-1}(V \cap V_x) = f^{-1}(V) \cap f^{-1}(V_x)$, thus $x \in f^{-1}(V)$ and so the union is contained in the preimage. Thus, the union and and $f^{-1}(V)$ and by the union open it follows that $f^{-1}(V)$ is open. □

7.2. Important Topological Facts.

/*Note this is weaker than the actual Lemma, but all that is needed currently*/

Theorem 3.27 (One Direction of Urysohn's Lemma). Let $(X, \mathcal{T})$ be a compact Hausdorff topological space. Then for any disjoint closed subsets A and B, there exists a continuous function $f : X \to [0, 1]$ such that

$$A \subset f^{-1}(\{0\}) \text{ and } B \subset f^{-1}(\{1\}).$$

CHAPTER 4

Vector Bundles

/*big idea to talk about throughout: doing things to the fibers and everything else works out.*/

1. Definition and Examples

To motivate vector bundles, consider any vector field any vector field over $\mathbb{R}^2$ and ask: what larger object is a home for this vector field? A vector field is certainly not a point in $\mathbb{R}^2$, so what larger object does the vector field lie inside of? To identify each vector of a vector field requires 4 numbers: two numbers (x, y) to identify the location of the vector within the topological space and two additional numbers $\binom{v_x}{v_y}$ to communicate the direction of the vector at this point. This suggests that this vector field rests inside of $\mathbb{R}^2 \times \mathbb{R}^2$ or something similar; denote this $T\mathbb{R}^2$ for now. Interpret $T\mathbb{R}^2$ as the *topological space* $\mathbb{R}^2$ with a copy of the *vector space* $\mathbb{R}^2$ at every point. There is an important distinction between the structure on the two sets $\mathbb{R}^2$. The topological space $\mathbb{R}^2$ is where each point (x, y) resides and the vector space $\mathbb{R}^2$ is where each vector $\binom{v_x}{v_y}$ resides. This distinction opens the door for changing the topological space $\mathbb{R}^2$ to any arbitrary topological space.

Now consider changing the topological space, which is perhaps better called the *base space*, to the sphere S^2. What would a field look like over S^2? Not much changes: a vector field would associate to each point in S^2 some vector in the plane $\mathbb{R}^2$ tangent to the sphere. Then, the whole space that the vector field lives inside is the topological space S^2 with a vector space $\mathbb{R}^2$ associated at every point.

/*need to include some figures for the motivation*/

Definition 4.1 (Vector Bundle). Take X as a topological space. Then, a topological space E paired with a continuous map $p : E \to X$ is a *vector bundle* over X if:

(i) For each $x \in X$, the preimage $p^{-1}(x)$ is a finite vector space with the appropriate subspace topology induced from E.

(ii) E is locally trivial; that is, for each $x \in X$, there exists an open neighborhood $U \subset X$ containing x such that the preimage is trivial. That is, $p^{-1}(U) \approx U \times V$ for a vector space V.

The topological space denoted X in the definition is called the *base space* and represents the topological spaces $\mathbb{R}^2$ and S^2 discussed earlier. Then, at each point in the base space X, there is the vector space $p^{-1}(x)$ which is called the *fiber* at X and is equivalent to a copy of the vector space $\mathbb{R}^2$ at a point of the sphere discussed previously. However, this construction of vector bundle is more general than the spaces TS^2 and $T\mathbb{R}^2$ as discussed earlier because each fiber does not have to be tangent to the topological space as in the following example

Example 4.2 (Cylinder). Take S^1 with the standard topology to be the base space. As a vector space, take $\mathbb{R}$ and consider the vector bundle given by the product $S^1 \times \mathbb{R}$. Giving $\mathbb{R}$ the standard topology induces the product topology on $S^1 \times \mathbb{R}$ and take the projection map $p : S^1 \times \mathbb{R} \to S^1$ given by $p : (x, v) \mapsto x$ to be the continuous projection map. Then, each preimage $p^{-1}(x)$ is a copy of the vector space $\mathbb{R}$ with the appropriate topology and thus this gives a vector bundle. In fact, this vector bundle should be visualized as a cylinder.

/*figure of a cylinder*/

The above construction of the cylinder demonstrates that fibers need not be tangent to the base space, but in fact fibers are not required to correspond to the dimension of the base space. The example of the cylinder is a specific case of the idea of a *trivial bundle* which is defined as follows.

Definition 4.3 (Trivial Bundle). Let X be a topological base space and let V be a vector space with a topology. Then, taking the product topology, $X \times V$ forms a topological space. This together with the projection map $p : X \times V \to X$ given by $p : (x, v) \mapsto x$ forms a vector bundle. This vector bundle is called a *trivial bundle*. If E is of dimension n, the trivial bundle is often denotes ε^n.

With this construction of the trivial bundle, note in the "locally trivial" condition in definition 4.1, the $X \times V$ is understood as the trivial bundle. However, there are many bundles that are not trivial bundles and the best example of such a bundle is the Mobius strip.

Example 4.4 (Mobius Strip). /*include example of Mobius Strip*/

/*Also give as examples, the formal construction of TS^2 and the normal bundle over S^2*/

To complete the category of vector bundles, a notion of homomorphisms between vector bundles is necessary. Vector bundles contain the structure of both a topological space and of many vector spaces, so a homomorphism of vector bundles aims to preserve both of these structures. These homomorphisms will be over the same base space and are defined as follows.

Definition 4.5 (Homomorphisms of Vector Bundles). Take two vector bundles E and F both with base space over X. Then, let $p : E \to X$ and $q : F \to X$ be the continuous maps. A mapping $\varphi : E \to F$ is a *homomorphism of vector bundles* if:

(i) $q\varphi = p$
(ii) $\varphi : E \to F$ is a homomorphism of topological spaces; that is, φ is continuous.
(iii) For each $x \in X$, the mapping $\varphi : p^{-1}(x) \to q^{-1}(x)$ is a homomorphism of vector spaces; that is, φ is a linear map between these vector spaces.

/*Include an example of homomorphism?*/

The definition of isomorphism for vector bundles carries over from the definition of isomorphism in category theory: a homomorphism with a homomorphism as an inverse. However, some of the homomorphism properties of the inverse follow automatically. For instance, take vector bundles $p : E \to X$ and $q : F \to X$ and a bijective homomorphism $\varphi : E \to F$. Then, it follows immediately from $q\varphi = p$ that $p\varphi^{-1} = q$, so this does not need to be checked. Additionally, a bijective linear map will have a linear inverse. Then, it does not need to be verified that $varphi^{-1}$ maps the fibers in a linear way because it is known that φ does. However, the continuous property of φ^{-1} does not follow automatically and is typically the most difficult part of isomorphism proofs. With these observations, an isomorphism can be defined in the following more practical way.

Definition 4.6 (Isomorphism). For two vector bundles $p : E \to X$ and $q : F \to X$, a map $\varphi : E \to F$ is defined to be an isomorphism if it is a bijective homomorphism with continuous inverse.

/*include an example of an isomorphism?*/

2. Direct Sum and Tensor Product on Vector Bundles

It is worth emphasizing that every point of a vector bundle E belongs to some fiber of the bundle. In fact, E as a set can be thought of as the disjoint union of only fibers. By taking the perspective of a vector bundle as the union of vector spaces, much of the structure of vector spaces extends to vector bundles. For instance, the fibers can be used to construct the direct sum operation in the following way.

Definition 4.7 (Direct Sum of Vector Bundles). Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles over X. Then, consider the disjoint unions of the direct sums of fibers

$$E_1 \oplus E_2 = \bigcup_{x \in X} p_1^{-1}(x) \oplus p_2^{-1}(x)$$

together with the projection mapping $p : E_1 \oplus E_2 \to X$ given by $p : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto x$. Then, $p : E_1 \oplus E_2 \to X$ when given a natural topology forms a vector bundle over X called the *direct sum* of E_1 and E_2.

/*mention intuition of pairs for direct sum bundle*/

Of course, a vector bundle has more structure than simply a union of vector spaces; in particular, vector bundles must be given a topology and must satisfy the local triviality condition. /*ref*/ gives the specifics of the "natural topology" referred to in the above definition along with this necessary proof of local triviality, but these verifications all work out. Because a vector bundle is built out of fibers, vector space properties such as the direct sum carry over naturally to vector bundles and the extra properties typically "all work out".

In this construction, consider some $x \in X$ and let $v_1 \in p_1^{-1}(x)$ and $v_2 \in p_2^{-1}(x)$ be elements of both fibers. Then, taking the direct sum of these vector spaces, these two vectors can be identified with $v_1 \oplus v_2$, which can also be thought of as simply (v_1, v_2).

A second similar construction by using the fibers is in the extension of the tensor product to vector bundles.

Definition 4.8 (Tensor Product of Vector Bundles). Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles over X. Then, consider the disjoint unions of all tensor products of the fibers

$$E_1 \otimes E_2 = \bigcup_{x \in X} p_1^{-1}(x) \otimes p_2^{-1}(x)$$

together with the projection mapping $p : E_1 \otimes E_2 \to X$ given by $p : p_1^{-1}(x) \otimes p_2^{-1}(x) \mapsto x$. Then, $p : E_1 \otimes E_2 \to X$ when given a natural topology forms a vector bundle over X called the *tensor product* of E_1 and E_2.

Again, the specifics of the "natural topology" and the verification of natural triviality all work out as explained in /*ref*/. The proof is identical to the proof for direct sum. It is worth mentioning that this construction can be generalized to other operations on vector spaces such as the dual and the exterior power, but these notes only require the direct sum and the tensor product.

Because the tensor product and direct sum are defined on each fibers, the properties of direct sum and tensor product on vector spaces carry over to analogous properties on vector bundles.

Claim 4.9. Listed below are properties of direct sum and tensor product over vector bundles.

(i) The direct sum between bundles is associative and commutative.
(ii) The trivial bundle of dimension 0 is an identity element for the direct sum. That is, $E \oplus \varepsilon^0 = E$.
(iii) The tensor product between bundles is associative and commutative.
(iv) The trivial dimension of dimension 1 is an identity element for the tensor product. That is, $E \otimes \varepsilon^1 = E$.
(v) The tensor product distributes over direct sum.

3. Pullback Bundles

The following construction, addresses pullback bundles. In the next chapter of this story, all of the arrows will suddenly point backwards as a contravariant functor emerges. The reason why the arrows will point backwards is due to pullback bundles.

Consider two base spaces X and Y where X has a vector bundle structure $p : E \to X$ but Y, unfortunately, has no such structure. However, Y can be given a vector bundle $q : F \to Y$ by stealing the structure of E through the association given by f. Specifically, each fiber $q^{-1}(y)$ can just take a copy of the fiber $p^{-1}(f(y))$.

Definition 4.10 (Pullback Bundle). Let $f : X \to Y$ be a mapping and $p : E \to X$ a bundle as defined above. Then there exists a unique bundle $f^*(p) : f^*(E) \to Y$ and a mapping $h : f^*(E) \to E$ such that h maps each fiber $(f^*(p))^{-1}(y)$ to the fiber $p^{-1}(f(y))$ as a vector space isomorphism. This bundle is called the *pullback bundle* and denoted $f^*(p) : f^*(E) \to Y$.

/*talk about existence and uniqueness proofs*/

However, there is a detail of well-defined to address. When given a vector bundle $p : E \to X$ with continuous functions $f : X \to Y$ and $g : Y \to Z$, how is the bundle structure on X pulled back to a bundle on Z? There are two options: $(f \circ g)^*(E)$ and $f^*(g^*(E))$. Luckily, the following claim shows that the two options are isomorphic and gives more pleasant properties of the pullback.

Claim 4.11. Listed below are important properties of pullbacks.

(i) $(f \circ g)^*(E) \approx g^*(f^*(E))$ for any bundle E and continuous functions f and g.
(ii) $\mathrm{Id}^*(E) \approx E$ for any vector bundle E over X and the identity mapping $\mathrm{Id} : X \to X$.
(iii) $f^*(\varepsilon^n) \approx \varepsilon^n$ for all continuous functions f and trivial bundles ε^n over the corresponding base spaces.
(iv) $f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)$ for all bundles E_1 and E_2 and continuous function f.
(v) $f^*(E_1 \otimes E_2) \approx f^*(E_1) \otimes f^*(E_2)$ with E_1 and E_2 vector bundles and f a continuous function.

Note that given a vector bundle $p : E \to X$, a function f from Y *to* X is necessary to induce a pullback bundle $f^*(E)$ over Y; not the other way around. The function must point this direction in order for $f^*(E)$ to effectively steal the structure of E. A function $f' : X \to Y$ would be rather useless, for this function may associate each point of the base space Y to multiple points in the base space X or non at all. However, given the function $f : Y \to X$, each point in Y is mapped to a single point in X and so the structure of E can be effectively stolen. This fact — that a function induces a vector bundle in the opposite direction — is half way to defining a contravariant functor.

4. Necessary Results on Vector Bundles

this section is still in progress

/*Canonical Line bundle over $\mathbb{R}P^1$ gives mobius band*/

/*A contractible implies the bundle over A is trivial*/.

Claim 4.12. For every bundle E over a compact Hausdorff space X, there exists a bundle E' over X such that $E \oplus E'$ is trivial.

This claim is central to the story, so a full proof /*I still need to do this*/ is provided in /*ref*/. The proof is long with many lemma's, so more useful than reading the full proof is to read the following summary of the proof's idea. Given the bundle $p : E \to X$ over a compact Hausdorff space, a huge trivial bundle T is constructed by using a topology theorem[1] that follows from the compact Hausdorff condition. The trivial bundle T is built exactly such that there is a convenient isomorphism from E to a subbundle E_0 in the huge trivial bundle. Another topology tool[2] allows the extension of a metric to vector bundles, which then gives a Gran-Schmidt orthogonalization process on vector bundles. The orthogonal complement of each fiber in E_0 gives a vector bundle $E_0^{\perp}$ such that $E_0 \oplus E_0^{\perp} = T$ and the desired conclusion follows from $E \cong E_0$.

/*example: $NS^2 \oplus TS^2$ is trivial*/

Example 4.13. For an example of the above theorem, consider the tangent bundle to S^2, denoted TS^2. As promises by theorem /*ref*/, the normal bundle to S^2, denoted NS^2, satisfies $TS^2 \oplus NS^2$ trivial. To see this, consider the space S^2 as embedded inside $\mathbb{R}^3$. Then elements of TS^n can be expressed $(x, v) \in S^2 \times \mathbb{R}^3$ and similarly, elements of NS^2 are given by $(x, n) \in S^2 \times \mathbb{R}^3$. Further, at a fixed point x, all vectors v in the tangent fiber will be orthogonal to the vectors n in the normal fiber by the definition of the bundles. Then elements of the direct sum $TS^2 \oplus NS^2$ can be expressed by $(x, v \oplus n)$ or simply (x, v, n). Then consider the isomorphism $\varphi : TS^2 \oplus NS^2 \to S^2 \times \mathbb{R}^3$ given by the isomorphism.

---

[1] Urysohn's Lemma

[2] Partition of Unity

$$\varphi : (x, v, n) \mapsto (x, v + n)$$

The above mapping an isomorphism follows from the above continuous and a linear bijection. The inverse map to the above can be constructed by taking the projection of the vector component onto the normal and tangent subspaces, which is again continuous giving isomorphism. /*a picture would be nice here*/

/*example: M mobius band ... $M \oplus M$ trivial*/

/*Example: Mobius band with itself is stably trivial AND/OR tangent bundle over S^2 with normal bundle over S^2*/

5. Verifications

5.1. Direct Sum and Tensor Product Verifications. It must be verified that the direct sum has a natural topology that indeed makes it a vector bundle.

Proof. Take vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ and recall that the direct sum on bundles as a set is given by the disjoint union of direct sums on fibers

$$E_1 \oplus E_2 = \bigcup_{x \in X} p_1^{-1}(x) \oplus p_2^{-1}(x).$$

This set is paired with with the projection $p : E_1 \oplus E_2 \to X$ given by $p : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto x$.

The topology on $E_1 \oplus E_2$ is defined in this paragraph. For each $x \in X$, the definition of vector bundle promises an open set U containing x over which both E_1 and E_2 are trivial. This provides trivializations $t_1 : p_1^{-1}(U) \to U \times V_1$ and $t_2 : p_2^{-1}(U) \to U \times V_2$ for vector spaces V_1 and V_2. Next, define the map $t_1 \oplus t_2 : p_1^{-1}(U) \oplus p_2^{-1}(U) \to U \times (V_1 \oplus V_2)$ as follows.

$$t_1 \oplus t_2 : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto t_1(p_1^{-1}(x)) \oplus t_2(p_2^{-1}(x))$$

Then, the topology on $p_1^{-1}(U) \oplus p_2^{-1}(U)$ is defined by requiring the map $t_1 \oplus t_2$ to be a homeomorphism. By letting x vary, this defines a topology over all of $E_1 \oplus E_2$. It must be verified, however, that this topology is well-defined.

Before the proof of well-defined, observe how this choice of topology gives that $E_1 \oplus E_2$ is a vector bundle. Firstly, this choice equips each fiber $p_1^{-1}(x) \oplus p_2^{-1}(x)$ with the typical topology of the direct sum of vector spaces. This ensures that the projection map $p : E_1 \oplus E_2 \to X$ is continuous. Next, the local triviality condition must be verified. Luckily the topology is built exactly so that $t_1 \oplus t_2$ is a trivialization. For any $x \in X$, the mapping $t_1 \oplus t_2$ defined on the appropriate U as described above satisfies all the conditions of a vector bundle homomorphism. Further, the defining condition that $t_1 \oplus t_2$ is a homeomorphism promises a continuous inverse and so $t_1 \oplus t_2$ is an isomorphism of vector bundles.

It only remains to show that the topology on $E_1 \oplus E_2$ is well-defined. In particular, it must be shown that the topology is independent of the choice of trivializations over a single open set U and that the open sets induce the same topology over their intersection. So, for $x \in X$ and corresponding $U \subset X$, consider two trivializations for each bundle: $t_1, t_1' : E_1 \mapsto U$ and $t_2, t_2' : E_1 \mapsto U$. Because each trivialization gives an isomorphism to the trivial bundle, the composition $t_1^{-1} \circ t_1' : p^{-1}(U) \to p^{-1}(U)$ is an isomorphism and similarly $t_2^{-1} \circ t_2' : p^{-1}(U) \to p^{-1}(U)$ is an isomorphism. Then composition $t_1' \circ t_1^{-1}$ is an isomorphism on $U \times V_1$ and similarly $t_2' \circ t_2^{-1}$ is an isomorphism on $U \times V_2$. It follows that the composition $(t_1' \oplus t_2') \circ (t_1 \oplus t_2)^{-1}$ is an isomorphism on $U \times (V_1 \otimes V_2)$, which implies that the choices $(t_1 \oplus t_2)$ and $(t_1' \oplus t_2')$ supply the same topology.

Finally, consider a separate set of open set $U' \subset X$. Then, taking the restrictions of the bundles $p^{-1}(U)$ and $p^{-1}(U')$ over the intersection $U \cap U'$ would only differ in the trivializations, which induce the same topology as shown in the previous paragraph. □

In the above argument, the only part that appeals to the direct sum operation itself is the implicit assumption that the mapping $(v, w) \mapsto v \oplus w$ is continuous. This is also true for the tensor product, so a simple substitution of "$\otimes$" in place of "$\oplus$" in the above proof provides the needed verification for tensor product.

Proof of claim /*ref*/. Verifying each claim requires establishing an isomorphism φ over two bundles, say $p : E \to X$ and $q : F \to X$. The approach will be to establish a vector space isomorphism between the fibers, which gives necessary properties of vector bundle isomorphism except for continuity and continuity of inverse. To deal with the continuity conditions, the strategy is to show local continuity at every point as descried in /*ref*/. It then suffices to show that for every $x \in X$, there is an open neighborhood U such that the restricted function $\varphi : p^{-1}(U) \to q^{-1}(U)$ is continuous in both directions.

(i) For associativity of the direct product, consider vector bundles E_1, E_2, E_3 over a base space X with corresponding projection maps p_1, p_2, and p_3. An isomorphism $\varphi : (E_1 \oplus E_2) \oplus E_3 \to E_1 \oplus (E_2 \oplus E_3)$ must be constructed. Let φ be the linear bijective function defined on the fibers by

$$\varphi : (p_1^{-1}(x) \oplus p_2^{-1}(x)) \oplus p_3^{-1}(x) \mapsto p_1^{-1}(x) \oplus (p_2^{-1}(x) \oplus p_3^{-1}(x))$$

For the continuity conditions, fix a point $x \in X$. Then, choose an open set $U \subset X$ small enough such that the local triviality conditions are satisfied by both direct sum bundles. Then, noting the vector space isomorphism $(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3)$, continuity in both directions is given by the following composition of isomorphisms

$$\begin{aligned}&(p_1^{-1}(U) \oplus p_2^{-1}(U)) \oplus p_3^{-1}(U) \to U \times (V_1 \oplus V_2) \oplus V_3 \\ &\to U \times V_1 \oplus (V_2 \oplus V_3) \to p_1^{-1}(U) \oplus (p_2^{-1}(U) \oplus p_3^{-1}(U))\end{aligned}$$

The proof for commutativity follows in a near identical way. The difference being that an isomorphism $\varphi : E_1 \oplus E_2 \to E_2 \oplus E_1$ is considered with the mapping between fibers $\varphi : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto p_2^{-1}(x) \oplus p_1^{-1}(x)$ and the vector space isomorphism $V_1 \oplus V_2 \cong V_2 \oplus V_1$ is considered instead.

(ii) Verifying that ε^0 is the identity element under direct sum requires establishing an isomorphism $\varphi : E \oplus \varepsilon^0 \to E$. This follows in the same way as the previous claims, but uses the mapping of fibers $\varphi : p^{-1}(x) \oplus \{0\} \mapsto p^{-1}(x)$ and uses the vector space isomorphism $V \oplus \{0\} \cong V$.

(iii) The proofs for associativity and commutativity of the tensor product is given by a substitution of "$\otimes$" for "$\oplus$" in the corresponding direct sum proofs.

(iv) The proof that ε^1 acts as an identity element over the tensor product follows similarly to the identity proof over direct sum. The difference being that here an isomorphism $\varphi : E \otimes \varepsilon^1 \to E$ is established by the mapping of fibers $\varphi : p^{-1}(x) \otimes V^1 \mapsto p^{-1}(x)$ where V^1 represents a one dimensional vector space. This proof additionally uses the vector space isomorphism $V \oplus V^1 \cong V$.

(v) Finally, the proof for distributivity establishes a vector space isomorphism $\varphi : E_1 \otimes (E_2 \oplus E_3) \to (E_1 \otimes E_2) \oplus (E_1 \otimes E_3)$ given by the linear bijection on the fibers

$$\varphi : p_1^{-1}(x) \otimes (p_2^{-1}(x)) \oplus p_3^{-1}(x)) \mapsto p_1^{-1}(x) \otimes p_2^{-1}(x) \oplus p_1^{-1}(x) \otimes p_3^{-1}(x)$$

and later uses the isomorphism on vector spaces $V_1 \otimes (V_2 \oplus V_3) \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$.

□

5.2. Pullback Bundle Verifications. /*I should really break up this proof into Lemma's, but I do not want to dive back into this proof*/

Proof of Existence for Definition 4.10. The proof of existence is by an explicit construction. Specifically, for a continuous function of topological spaces $f : X \to Y$ in addition to the vector bundle $p : E \to X$ consider the vector bundle $q : F \to Y$ where F is the following set.

$$F = \{(y, e) \in Y \times E : f(y) = p(e)\}$$

Further, let q be the mapping $q : (y, e) \mapsto y$.

It must be shown that F is a vector bundle satisfying the defining property of the pullback bundle. However, in order for F to be a vector bundle, it must be given extra structure.

Let F have the natural choice of topology induced by Y and E; specifically F takes the subspace topology of the product $Y \times E$.

Next, define the vector space structure over F as follows. Consider a fixed $y \in Y$ and fiber $q^{-1}(y)$. Note for each element (y, e) of the fiber, the condition $f(y) = p(e)$ restricts the elements of E to be in the vector space $p^{-1}(f(y))$. Then, borrowing the vector space structure from $p^{-1}(f(y))$ gives the natural definition of addition and scalar multiplication by a scalar α.

$$\alpha(y, v) = (y, \alpha v)$$
$$(y, v) + (y, w) = (y, v + w)$$

It follows from the vector space structure on $p^{-1}(f(y))$ that $q^{-1}(y)$ will satisfy all the necessary axioms to be a vector space.

Finally, the construction is complete and it must now be verified that F is indeed a vector bundle. Firstly, the definition of product topology promises that the projection q will be continuous.

Additionally, the above construction of the vector space structure over F promises that each fiber $q^{-1}(y)$ will be continuous.

It remains to show that F is locally trivial so fix a point $y \in Y$. By definition, E is locally trivial and so has a neighborhood U containing $g(y)$ such that $p^{-1}(U)$ is locally trivial. This promises a trivializing isomorphism $t : p^{-1}(U) \to U \times V$ for some vector field V. Note that this trivial bundle comes with the projection map $p' : U \times V \to U$ given by $p' : (u, v) \to u$. Define the mapping $t_1 : E \to U$ to be the composition of t with the projection onto the first factor and take $t_2 : E \to V$ to be the same composition but onto the second factor. This allows for the representation of the trivialization by $t : e \mapsto (t_1(e), t_2(e))$. Applying the condition $p' \circ t = p$ (given by t a homomorphism) to the representation gives the conclusion $t_1(e) = p(e)$ and thus allows for the simplification

$$t : e \mapsto (p(e), t_2(e))$$

After unpacking the promised trivialization on $E|_U$, a trivialization on $F|_{f^{-1}(U)}$ can now be constructed. Specifically, let the trivialization $\tau : F|_{f^{-1}(U)} \to f^{-1}(U) \times V$ be given by the following.

$$\tau : (y, e) \mapsto (y, t_2(e))$$

Additionally note that the bundle $f^{-1}(U) \times V$ comes equipped with a projection map q'.

It must now be shown that τ is an isomorphism of vector spaces. Observe that τ satisfies all the properties of a vector bundle homomorphism. First, τ continuous follows from t_2 continuous. The property $q' \circ \tau = q$ follows by

$$(q' \circ \tau)((y, e)) = q'((y, t_2(e))) = y = q((y, e)).$$

The last property of a homomorphism is that is linearity over the fibers. To see this, fix a $y \in U$ and notice that t linear over $p^{-1}(f(y))$ gives that t_2 is linear.

$$\begin{aligned}(f(y), t_2(\alpha v + \beta w)) &= t(\alpha v + \beta w) = \alpha t(v) + \beta t(w)\\ &= \alpha(f(y), t_2(v)) + \beta(f(y), t_2(w)) = (f(y), \alpha t_2(v) + \beta t_2(w))\end{aligned}$$

where the above computation used the $p(e) = f(y)$ as well as the predefined vector space structure of the trivial bundle. By a similar computation, t_2 linear gives that τ is linear over the fiber and thus a homomorphism.

To get that τ is an isomorphism, it suffices to show that that the inverse function is continuous. An explicit expression for $\tau^{-1} : f^{-1}(U) \times V$ follows.

$$\tau^{-1} : (y, v) \mapsto (y, t^{-1}(f(y), v))$$

Using $t \circ t^{-1} = \mathrm{Id}$ and $t^{-1} \circ t = \mathrm{Id}$, it follows that the above is indeed the inverse expression. Further, t^{-1} continuous gives that τ^{-1} continuous and so τ is an isomorphism, completing the verification of F a vector bundle.

It still remains to show that F has the defining property of the pullback. For this, take the function $h : F \to E$ to be the projection onto E.

$$h : (y, e) \mapsto e$$

Next, fix an element $y \in Y$ and consider the fiber $q^{-1}(y)$. The restriction $f(y) = p(e)$ ensures that $h((y,v)) = v$ is an element of $p^{-1}(f(y))$. Finally, the conclusion that h is a linear map from the fiber $q^{-1}(y)$ to the fiber $p^{-1}(f(y))$ follows quickly from the vector space structure of E.

$$h((y, \alpha v + \beta w)) = \alpha v + \beta w = \alpha h((y,v)) + \beta h((y,w))$$

Concluding the proof.

As a side note, observe that $p \circ h = f \circ q$ follows by the condition $f(y) = p(e)$ in the construction.

$$(p \circ h)((y,e)) = p(e) = f(y) = (f \circ q)((y,e))$$

This justifies drawing the commutative diagram /*ref*/ which hopefully helps in keeping track of variables for this proof.

□

PROOF OF /*REF*/. The strategy for proving each of the following isomorphisms is to take advantage of the uniqueness property. If it can be shown that one side of the isomorphism satisfies the defining property of pullback for the other side, then they must be isomorphic by uniqueness.

(i) For topological spaces X, Y, Z let $g : Z \to Y$ and $f : Y \to X$ be continuous functions and let $p : E \to X$ be a vector bundle. By definition, the bundles $f^*(E)$ and $g^*(f^*(E))$ come equipped with maps $h_g : g^*(f^*(E)) \to f^*(E)$ and $h_f : f^*(E) \to E$ that isomorphically map fibers to corresponding fibers. Then, the composition $h_f \circ h_g : g^*(f^*(E)) \to E$ isomorphically maps fibers to corresponding fibers. Further, the bundle $g^*(f^*(E))$ comes equipped with a projection mapping r into the base space Z. Thus, the triple $g^*(f^*(E))$, $h_f \circ h_g$, and r satisfy the defining characteristics of the pullback bundle $(f \circ g)^*(E)$, giving isomorphism by uniqueness.

(ii) Take the mapping $\mathrm{Id} : X \to X$ for a topological space X with a bundle $p : E \to X$. Then, the bundle E itself with the identity mapping $\mathrm{Id} : E \to E$ isomorphically maps fibers to fibers and comes equipped with the projection mapping p to X. Then, the triple E, $\mathrm{Id} : E \to E$, and p satisfy the defining characteristics of the pullback $\mathrm{Id}^*(E)$ which promises the isomorphism $E \cong \mathrm{Id}^*(E)$ by uniqueness.

(iii) Let $f : Y \to X$ be a continuous function between topological spaces and consider the trivial bundle $p : X \times V \to X$ over X with the regular projection p. Then, consider the trivial bundle over $q : Y \times V \to Y$ over Y with the regular projection q. Then, the mapping $h : Y \times V \to X \times V$ given by $h : (y,v) \mapsto (f(y), v)$ gives the identity mapping over each fiber and is thus a linear isomorphism of the fibers. Thus, the triple $Y \times V$, h, and q satisfies the defining properties of $f^*(X \times V)$ and thus uniqueness promises an isomorphisms between the trivial bundles $Y \times V \cong f^*(X \times V)$. Note that the trivial pullback is over the same vector space.

(iv) Next, take $f : Y \to X$ to be a continuous function between topological spaces. Further, let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles. The pullbacks $f^*(E_1)$ and $f^*(E_2)$ then come with mappings $h_1 : f^*(E_1) \to E_1$ and $h_2 : f^*(E_2) \to E_2$ that are isomorphisms on the fibers. Then, the direct sum of the pullbacks has a mapping $h : f^*(E_1) \oplus f^*(E_2) \to E_1 \oplus E_2$ defined on the fibers by $h : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto h_1(p_1^{-1}(x)) \oplus h_2(p_2^{-1}(x))$ which is also an isomorphism on the fibers. Additionally note that the direct sum comes equipped with a projection mapping p onto Y. Thus the triple $f^*(E_1) \oplus f^*(E_2)$, h, and p satisfy the defining properties of the pullback $f^*(E_1 \oplus E_2)$ giving the isomorphism $f^*(E_1) \oplus f^*(E_2) \cong f^*(E_1 \oplus E_2)$ by uniqueness.

(v) The proof for the distributivity of pullback over tensor product is identical to preceding such proof for direct sum, differing only by replacing each "$\oplus$" with "$\otimes$".

□

5.3. Other Verifications.

LEMMA 4.14. /*yikes, this lemma (lemma 1.2 in Hatcer) uses partition of unity... this is becoming a rabbit hole*/

Given a vector space V and a vector subspace, $V_0 \subset V$, the Gram-Schmidt orthogonalization process provides the orthogonal complement $V_0^\perp$ to the subspace V_0 in V. Further, it holds that $V_0 \oplus V_0^\perp = V$. An analogous result holds for vector bundles by applying the same process to each fiber.

LEMMA 4.15. Take a vector bundle $p : E \to X$ that has

/*assumes all V_i's are equal. Need to fix? Say it suffices to consider connected components*/

PROOF OF /*REF*/. The strategy of this proof is to construct a trivial vector space, later called $X \times \mathcal{V}$, that an isomorphic copy of the given vector bundle resides in. Then the result will follow by the above lemma.

Consider a vector bundle $p : E \to X$ where X is a compact Hausdorff topological space.. Each point $x \in X$ has a neighborhood U_x over which the bundle is trivial. By X compact Hausdorff, apply Urysohn's Lemma /*ref*/ on the closed sets $\{x\}$ and the complement $\overline{U_x}$. Urysohn's Lemma then promises a continuous function $\varphi_x : X \to [0,1]$ satisfying $\varphi_x^{-1}(\{0\}) \subset \overline{U_x}$ and $\varphi_x^{-1}(\{1\}) \subset \{1\}$. In other words, f evaluates to 0 outside of U_x and to 1 at x. Note that $\varphi_x^{-1}(0,1]$ contains X and is open by by φ_x continuous and the interval equipped with standard topology. Then $\varphi_x^{-1}(0,1]$ provides an open cover when allowing x to vary. By compactness there is a finite subcover; denote this subcover $\varphi_i^{-1}(0,1]$ and let the corresponding functions and neighborhoods be indexed φ_i and U_i.

Next, for each index define a function $g_i : E \to V$ as follows. Let $h_i : p^{-1}(U_i) \to U_i \times V$ be the trivialization as promised by the choice of U_i. Additionally, let $\pi_i : X \times V \to V$ be the projection from the trivial bundle to the corresponding vector component: $\pi_i : (x, v) \mapsto v$. Then, the function g_i is defined as follows.

$$g_i(e) = \begin{cases} \varphi_i(p(e)) \cdot (\pi_i \circ h_i(e)) \text{ if } p(e) \in U_i \\ 0 \text{ otherwise.} \end{cases}$$

Note g_i is continuous by g_i a composition of continuous functions and by φ_i is 0 outside of U_i. Importantly note that each g_i is a linear injection over the fibers of $\varphi_i^{-1}(0,1]$. Indeed, fix an $x_0 \in \varphi_i^{-1}(0,1] \subset U_i$ and take v_1, v_2 in the fiber $p^{-1}(x_0)$ such that $g_i(v_1) = g_i(v_2)$. That is,

$$\varphi_i(p(v_1)) \cdot (\pi_i \circ h_i(v_1)) = \varphi_i(p(v_2)) \cdot (\pi_i \circ h_i(v_2))$$

The fixed x gives $\varphi_i(p(v_1)) = \varphi_i(p(v_1)) = \varphi_i(x_0)$. This together with h_i an isomorphism and π_i an isomorphism over the fixed x_0 promises $v_1 = v_2$, confirming injectivity over the fibers. Linearity follows by π_i and h_i linear over the fibers.

Next, consider the vector space $\mathcal{V} = V \times V \times \cdots \times V$ with one copy of V for each of the indices i. Then, define the function $g : E \to \mathcal{V}$ given by $g : e \mapsto (g_1(e), g_2(e), \ldots, g_k(e))^T$. Note that g is a linear injection. Indeed, fix an $x_0 \in \varphi_i^{-1}(0, 1] \subset U_i$ and take v_1, v_2 in the fiber $p^{-1}(x_0)$ such that $g(v_1) = g(v_2)$. By the collection $\varphi_i^{-1}(0, 1]$ a cover, $x_0 \in \varphi_i^{-1}(0, 1]$ for some i. But then, $g_i(v_1) = g_i(v_2)$, which then provides the desired $v_1 = v_2$ confirming injectivity. Linearity follows by each individual g_i linear.

Finally consider the map $f : E \to X \times \mathcal{V}$ given by $f : e \mapsto (p(e), g(v))$. Now observe that the image of f is a subbundle of of $X \times \mathcal{V}$. The bundles takes the natural projection map of the larger trivial bundle and by linearity of g each fiber of the image has a vector space structure. It only remains to verify the local triviality condition. Indeed, for each $x_0 \in X$, the open cover promises $x_0 \in \varphi_i^{-1}(0, 1]$ for some i. Then, consider the projection $X \times \mathcal{V}$ by mapping the vector component of (x, v) to the ith copy of V used to construct $\mathcal{V}$ and call the projection q. Then, a local trialization over the region is provided by $(x, v) \mapsto (x, q(v))$. With the verification that $\operatorname{Im} f$ indeed forms a vector bundle, and by injective implies bijective onto the image, lemma /*ref*/ applies and gives that the image is isomorphic to a subbundle of $X \times \mathcal{V}$. So, lemma /*ref*/ applies and promises a bundle E' such that $E \oplus E' = X \times \mathcal{V}$.

□

CHAPTER 5

The Definition of K-Theory

K-Theory is a functor from the category of topological spaces to the category of rings. Topological spaces are messy, making it difficult to understand properties about topological spaces and homomorphisms between topological spaces. However, groups and rings are simple algebraic objects with much structure — an easier object to analyze.

The K-Theory functor first considers all possible vector bundles over a topological space. Looking at every possible bundle is too much information, but it turns out that after simplifying the set to equivalence classes of vector bundles, a ring structure emerges.

There are two veins of K-Theory; the difference is in the equivalence classes used to reduce the set of vector bundles. First, there is the *K-Theory* of a topological space X, denoted $K(X)$. In this case, the equivalence classes have a natural semiring structure and the ring is defined through the ring extension. Secondly, there is the *reduced K-Theory* of a topological space X, denoted $\widetilde{K}(X)$, which has bigger equivalence classes. In reduced K-Theory, the equivalence classes themselves can be made directly into a ring. In both K-Theory and reduced K-Theory, the functor is contravariant.

1. The K-Theory Functor K

/*give motivation of this equivalence relation — it is what gives us the cancellation property*/

The equivalence relation used in K-Theory to simplify the set of vector bundles is as follows.

DEFINITION 5.1 (Stably Isomorphic). Define the equivalence relation $\approx_s$ on vector bundles over the same base space such that for bundles E_1 and E_2, $E_1 \approx_s E_2$ if $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$ for some n where ε^n denotes the n dimensional trivial bundle. Here, E_1 and E_2 are said to be *stably isomorphic.*

This equivalence relation gives a natural semiring structure on the equivalence classes.

CLAIM 5.2. Take compact Hausdorff base space X. The set of all stably isomorphic equivalence classes over the vector bundles on X forms a commutative semiring with cancellation when taking the direct sum $\oplus$ as the additive operation and the tensor product $\otimes$ as the multiplicative operation. This semiring is denoted $J(X)$.

Proving the above claim takes some work, but the full proof is given in section 3.1 of this chapter. Most of the proof is routine verification, but getting the cancellation property and verifying that multiplication is well-defined appeals to /*ref*/, which is where the compact Hausdorff condition is used. Because this semiring is commutative with the cancellation property, it is most convenient to consider the unique commutative ring promised through ring completion.

Definition 5.3 (K-Theory of a Topological Object). Take compact Hausdorff base space X and let $J(X)$ denote the commutative semiring with cancellation as described in claim. Then, the ring completion of $J(X)$ is the *K-Theory of* X and is denoted $K(X)$.

Next, consider the following computations of K-Theory on simple topological spaces.

Example 5.4 (K-Theory of a point). Consider as a topological space a single point $\{x_0\}$. The only choice of vector bundles on $\{x_0\}$ are the trivial bundles of each dimension. That is, the set $\left\{\varepsilon^0, \varepsilon^1, \varepsilon^2, \dots\right\}$. No two trivial bundles will be in the same stable isomorphism class, giving the set of equivalence classes $\left\{[\varepsilon^0], [\varepsilon^1], [\varepsilon^2], \dots\right\}$, which is isomorphic to the semiring $\mathbb{N}$. So, the ring $K(\{x_0\})$ is the ring completion of $\mathbb{N}$. That is, $K(\{x_0\}) \cong \mathbb{Z}$.

Example 5.5 (K-Theory of n points). Consider the topological space of n disconnected points $\{x_0, x_2, \dots, x_{n-1}\}$. Then, each point can have a fiber of any dimension, and the choice of fibers is independent of one another. This gives that the set of all vector bundles is isomorphic to the set $\mathbb{N}^n$. Then, any arbitrary vector bundle over the space can be denoted $(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}})$ where the first element in the tuple represents the bundle over the first disconnected point, the second element represents the bundle over the second, and so on. The equivalence classes can then be represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}})]$, and in this case every vector bundle is its own equivalence class, and so this is isomorphic to the semiring $\mathbb{N}^n$, which has ring completion $\mathbb{Z}^n$. Thus, $K(\{x_0, x_1, \dots, x_{n-1}\}) \cong \mathbb{Z}^n$.

However, defining the K-theory on topological objects only brings the operation half way to being a functor. Functors map objects to objects but also morphisms to morphism. Just as K-Theory brings topological spaces to rings, K-Theory must bring continuous functions between topological spaces to homomorphisms of rings. In this case, K-Theory is a contravariant functor and so reverses the direction of the mapping.

Claim 5.6. Take topological spaces X and Y with a continuous function $f : X \to Y$. Let $J(X)$ denote the semiring as described in claim 5.2 and let $K(Y)$ be the K-Theory of Y. Further, define the function $J(f) : J(X) \to K(Y)$ defined on an equivalence class $[E] \in J(X)$ by

$$J(f) : [E] \mapsto [f^*(E)]$$

where f^* denotes the pullback as defined in /*ref*/. Then, $J(f)$ is a well-defined homomorphisms of semirings.

Verifying the above follows easily from the properties of pullback given in /*ref*/. The full proof is given in section 3.2. Note that this is the point where the contravariant property emerges. Because the elements of the semiring is equivalence classes of vector bundles, a homomorphism consists of mappings from one vector bundle to another vector bundle. This is best done through the induced vector bundle by the pullback which, as discussed previously, must be done in the reverse direction.

Definition 5.7 (K-Theory of a Topological Morphism). For compact Hausdorff spaces X and Y with a continuous function $f : X \to Y$, let $J(f) : J(X) \to J(Y)$ denote the homomorphism of semirings as described in claim 5.6. Further, let $i_X : J(X) \to K(X)$ and $i_Y : J(Y) \to K(Y)$ denote ring completions. Then, the *K-Theory of* f is the unique homomorphism of rings $K(f) : K(X) \to$

$K(Y)$ such that the following diagram commutes as promised by the universal property. That is, that $K(f) \circ i_X = i_Y \circ J(f)$.

$$\begin{tikzcd}
J(Y) \arrow[d, "i_Y"] & J(X) \arrow[l, "J(f)"'] \arrow[d, "i_X"] \arrow[dl, "i_Y \circ J(f)"] \\
K(Y) & K(X) \arrow[l, dashed, "K(f)"]
\end{tikzcd}$$

Figure 1. Definition of $K(f)$ through Universal Property

And that is the definition of K-Theory! Figure 2 denotes a diagram of the K-Theory functor. Next, observe how the K-Theory functor on the following examples of concrete topological spaces.

Example 5.8. This example examines the K-Theory of the inclusion from the space with one point to the space with n points. So, Take the topological space of n points $\{x_0, x_1, \ldots, x_{n-1}\}$ and consider the subspace $\{x_0\} \subset \{x_0, \ldots, x_{n-1}\}$. Then, let $i : \{x_0\} \to \{x_0, \ldots, x_{n-1}\}$ be the inclusion map.

Recall that the equivalence classes of vector bundles over x_0 can be represented $[\varepsilon^n]$ and the equivalence classes over $\{x_0, x_1, \ldots, x_{n-1}\}$ can be represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})]$. Then, by definition the function $J(i) : J(\{x_0, \ldots, x_{n-1}\}) \to J(\{x_0\})$ is given by

$$J(i) : [(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})] \mapsto [i^*(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})]$$

Recall that the pullback of the inclusion is simply the restriction to the space in the domain. In this case, that is the restriction to the point x_0 and so function $J(i)$ is

$$J(i) : [(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})] \mapsto [\varepsilon^{k_0}]$$

Now step away from the vector bundles themselves and let $J(i)$ denote the semiring homomorphism $J(i) : \mathbb{N}^n \to \mathbb{N}$ given by $J(i) : (k_0, k_1, \ldots, k_{n-1}) \mapsto k_0$. Finally, consider the ring completions $\mathbb{Z}^n$ and $\mathbb{Z}$. The universal property then promises a unique ring homomorphism $K(i) : \mathbb{Z}^n \to \mathbb{Z}$. In this case, the ring homomorphism is given by $K(i) : (k_0, k_1, \ldots, k_{n-1}) \mapsto k_0$, but this time, each k_i can take on the value of any integer. See figure 3 for a visual of this construction.

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "K"] & X \arrow[d, "K"] \\
K(Y) & K(X) \arrow[l, "K(f)"']
\end{tikzcd}$$

Figure 2. The K-Theory Functor

$$\begin{tikzcd}
\{x_0\} \arrow[r, "i"] \arrow[d] & \{x_0, \dots, x_{n-1}\} \arrow[d] \\
\mathbb{N} \arrow[d] & \mathbb{N}^n \arrow[l, "J(i)"'] \arrow[d] \\
\mathbb{Z} & \mathbb{Z}^n \arrow[l, dashed, "k_0 \leftarrowtail (k_0, \dots, k_{n-1})"', "K(i)"]
\end{tikzcd}$$

Figure 3. K-Theory of a Morphism Example

2. The Reduced K-Theory Functor $\widetilde{\mathbf{K}}$

There is another closely related vein of K-Theory called *reduced K-Theory.* Reduced K-Theory is a is a functor from the category of topological spaces to the category of abelian groups. However, with as assumption discussed later, this functor can be extended to the category of commutative rings (but not necessarily with identity). Reduced K-Theory uses a stronger equivalence relation, which gives fewer equivalence classes.

Definition 5.9. Define the equivalence relation $\sim$ on vector bundles E_1 and E_2 over the same base space such that $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^m$ for some n and m.

Then, this equivalence class immediately gives rise to the desired group.

Definition 5.10. Take a compact Hausdorff topological space X and let $\widetilde{K}(X)$ denote the set of all equivalence classes under the relation $\sim$ as described in definition 5.9. Then, define the group operation by the direct sum $\oplus$ operation on the elements. This forms a well-defined abelian group and is called the *reduded K-Theory* of X.

The verification that $\widetilde{K}(X)$ indeed forms a well-defined group is straight-forward, but it is worth noting that the existence of inverses uses /*ref*/, which requires the compact Hausdorff condition.

Consider some simple computation of reduced K-Theory.

Example 5.11 (Reduced K-Theory of a Point). Again as a topological space a single point $\{x_0\}$. The only choice of vector bundles on $\{x_0\}$ is the set trivial bundles $\left\{\varepsilon^0, \varepsilon^1, \varepsilon^2, \dots\right\}$. In this case, however, each trivial bundle is in the same isomorphism class, so the set of equivalence classes has only the identity element ε^0. So, the reduced K-Theory of a point $\widetilde{K}(\{x_0\})$ is the trivial group.

It must still be addressed how reduced K-Theory maps continuous topological maps to group morphisms, which again makes use of the pullback.

Definition 5.12 (Reduced K-Theory of a Topological Morphism). Let $f : Y \to X$ denote a continuous function between topological spaces. Then the induced mapping $\widetilde{K}(f) : \widetilde{K}(X) \to \widetilde{K}(Y)$

is defined by

$$\widetilde{K}(f) : [E] \mapsto [f^*(E)]$$

where the equivalence classes are with respect to the relation $\sim$ as in definition 5.9.

/*verify well-defined*/

It first must be verified that the above mapping of functions is well-defined, which is done in /*ref*/. But once well-defined is out of the way, the properties of pullback $\mathrm{Id}^*(E) \approx E$ and $(f \circ g)^*(E) \approx g^*(f^*(E))$ immediately give that $\widetilde{K}$ obeys the rules for functors.

Now consider the following example, which demonstrates an important relationship between the functors K and $\widetilde{K}$.

Example 5.13 (Reduced K-Theory of n points). Consider the topological space of n disconnected points $\{x_0, x_2, \ldots, x_{n-1}\}$. Again, each fiber is free to have a vector space of any dimension, so any arbitrary vector bundle over the space can be denoted $(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})$. The equivalence classes under $\sim$ are then represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})]_\sim$. In this case, every equivalence class has more than one element. In particular, $(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}}) \sim (\varepsilon^{l_0}, \varepsilon^{l_1}, \ldots, \varepsilon^{l_{n-1}})$ if there exists bundles ε^k and ε^l such that

$$(\varepsilon^{k_0} \oplus \varepsilon^k, \varepsilon^{k_1} \oplus \varepsilon^k, \ldots, \varepsilon^{k_{n-1}} \oplus \varepsilon^k) \approx (\varepsilon^{l_0} \oplus \varepsilon^l, \varepsilon^{l_1} \oplus \varepsilon^l, \ldots, \varepsilon^{l_{n-1}} \oplus \varepsilon^k)$$

By simply ditching the "ε" symbol, there is an isomorphism to equivalence classes of n tuples of integers $[(k_0, k_1, \ldots, k_{n-1})]$ where $(k_0, k_1, \ldots, k_{n-1}) \sim (l_0, l_1, \ldots, l_{n-1})$ if there exists integers k and l such that

$$(k_0 + k, k_1 + k, \ldots, k_{n-1} + k) = (l_0 + l, l_1 + l, \ldots, l_{n-1} + l)$$

Additionally, the group operation is element wise addition as is taken from the representation with "ε". Note that all of the k's and l's are allowed to be any integer but they originally represented the dimension of a trivial bundle, so it seems they should only be nonnegative integers. However, expanding the elements to integers does not change the group, for every new element containing a negative integer will land in a preexisting equivalence class.

And so, elements in reduced K-Theory are of the form of equivalence classes $[(k_0, k_1, \ldots, k_{n-1})]$ with the relation as defined earlier.

However, this does not clear up what this reduced K-Theory is isomorphic to. The bottom line is that the reduced K-Theory of n points is isomorphic to the group $\mathbb{Z}^{n-1}$. There are a few ways to see this, but the most educational is with the following.

Fix the point $x_0 \in \{x_0, \ldots, x_{n-1}\}$ and consider the K-Theory groups $K(\{x_0, \ldots, x_{n-1}\})$ and $K(\{x_0\})$. The goal will be to construct a homomorphism $\varphi : \widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \to K(\{x_0, \ldots, x_{n-1}\})$ and use φ with the mapping $K(i) : K(\{x_0, \ldots, x_{n-1}\})$ as defined previously. Overall, this will give the chain of mappings as shown in figure 4. Additionally takes note of the isomorphisms $K(\{x_0, \ldots, x_{n-1}\}) \cong \mathbb{Z}^n$ and $K(\{x_0\}) \cong \mathbb{Z}$.

Now, define φ on the discussed representations on the K-Theory and reduced K-Theory of n points as follows.

$$\varphi : [(k_0, k_1, \ldots, k_{n-1})] \to (0, k_1 - k_0, \ldots, k_{n-1} - k_0)$$

The verification that φ is indeed a group homomorphism follows easily. /*ref*/. Now recall that $K(i)$ is given by the following.

$$K(i) : (l_0, l_1, \ldots, l_{n-1}) \mapsto l_0$$

Now make a two observations. Firstly, $\mathrm{Im}(\varphi) = \ker(K(i))$. Secondly, φ is injective. The proofs for both of these are brief and given in /*ref*/. φ injective gives $\widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \cong \mathrm{Im}(\varphi)$ which then gives the relationship

$$\widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \cong \ker(K(i))$$

Lastly note that element of the kernel are of the form $(0, l_1, \ldots, l_{n-1})$ for any choice of l's, so the kernel is isomorphic to $\mathbb{Z}^{n-1}$. Overall, this gives $\widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \cong \mathbb{Z}^{n-1}$.

$$\begin{tikzcd}
\widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \arrow[r, "\varphi"] & K(\{x_0, \ldots, x_{n-1}\}) \arrow[r, "K(i)"] & K(\{x_0\})
\end{tikzcd}$$

Figure 4. Chain of Homomorphisms for n-points example

The above example found the reduced K-Theory by demonstrating that $\widetilde{K}(\{x_0, \ldots, x_{n-1}\})$ is isomorphic to $\ker(K(i))$. In fact, a relationship like this exists in general. For any topological space X with a fixed point $x_0 \in X$ and inclusion $i : x_0 \to X$, the relationship $\widetilde{K}(X) \cong \ker(K(i))$ holds. The proof of this uses tools developed in the next chapter, but the above example gives a taste of the proof. A consequence of this, however, is that $K(i)$ is a ring homomorphism, so $\ker(K(i))$ is an ideal. Then $\widetilde{K}(X)$ is isomorphic to this ideal and thus can be given a multiplication. Thus, we can consider $\widetilde{K}(X)$ to be a ring, but not necessarily with identity.

The computed examples hint at another relationship between K-Theory and reduced K-Theory. Note that for a point $\{x_0\}$, $K(\{x_0\}) \cong \mathbb{Z}$ and $\widetilde{K}(\{x_0\}) \cong \{0\}$. The computations for a collection of points $\{x_0, x_1, \ldots, x_{n-1}\}$ showed $K(\{x_0, x_1, \ldots, x_{n-1}\}) \cong \mathbb{Z}^n$ and $\widetilde{K}(\{x_0, x_1, \ldots, x_{n-1}\}) \cong \mathbb{Z}^{n-1}$. Note that $\mathbb{Z} \cong \{0\} \oplus Z$ and more generally, $\mathbb{Z}^n \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}$. More generally, it is true that $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ for any topological space X, but this proof again uses techniques of the following chapter.

3. Verifications

3.1. Semiring Verification.

Proof. Take a compact Hausdorff space X. It must be verified that the set of stable isomorphism classes of vector bundles over X with operations defined by the direct sum $\oplus$ and the tensor product $\otimes$ indeed satisfies all the properties of a commutative semiring with additive cancellation.

Before proceeding further, it must be verified that addition is well defined. So, take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X. Then, take nonnegative integers n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Then it follows that $E_1 \oplus F_1 \approx_s E_2 \oplus F_2$ by the following chain of equalities.

$$(E_1 \oplus F_1) \oplus \varepsilon^{n+m} \approx (E_1 \oplus \varepsilon^n) \oplus (F_1 \oplus \varepsilon^m) \approx (E_2 \oplus \varepsilon^n) \oplus (F_2 \oplus \varepsilon^m) \approx (E_2 \oplus F_2) \oplus \varepsilon^{n+m}$$

Where the equivalence $\varepsilon^{n+m} \approx \varepsilon^n \oplus \varepsilon^m$ /*reference*/ was used.

With addition well defined, the associativity and commutativity of addition follows directly from the associativity and commutativity of the direct sum on vector bundles /*reference*/. Further, the result $E \oplus \varepsilon^0 \cong E$ for any vector bundle E /*ref*/ makes the equivalence class $[\varepsilon^0]$ the additive identity.

The additive cancellation follows from /*reference $E \oplus E'$ trivial result*/, which applies here by X compact Hausdorff. Indeed, take bundles E, F, and S over X such that $[E] + [S] = [F] + [S]$. First note that in the case of S trivial, $[E] = [F]$ by definition. Otherwise, by /*ref*/, there exists a bundle S such that $S \oplus S'$ is trivial. Adding $[S']$ to both sides reduces the expression to the first case with $[E] + [S \oplus S'] = [F] + [S \oplus S']$, giving $[E] = [F]$ as desired.

Before proceeding with any multiplicative verifications, it must be verified that the tensor product $\otimes$ gives a well defined multiplicative operation. So, again take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X and nonnegative integers n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Next, define the bundle M by

$$M \approx \varepsilon^n \otimes (F_1 \oplus \varepsilon^m) \oplus \varepsilon^m \otimes (E_1 \oplus \varepsilon^n) \approx \varepsilon^n \otimes (F_2 \oplus \varepsilon^m) \oplus \varepsilon^m \otimes (E_2 \oplus \varepsilon^n)$$

Next, observe that M is constructed exactly so that the relation $E_1 F_1 \oplus M \approx E_2 F_2 \oplus M$ holds:

$$E_1 \otimes F_1 \oplus M \approx (E_1 \oplus \varepsilon^n)(F_1 \oplus \varepsilon^m) \oplus \varepsilon^n \varepsilon^m \approx (E_2 \oplus \varepsilon^n)(F_2 \oplus \varepsilon^m) \oplus \varepsilon^n \varepsilon^m \approx E_2 \otimes F_2 \oplus M$$

So, take M' to be the bundle such that $M \oplus M'$ is trivial as promised by /*ref*/. Then, the desired conclusion follows easily, giving that multiplication is well-defined.

$$E_1 \otimes F_1 \oplus (M \oplus M') = E_2 \otimes F_2 \oplus (M \oplus M')$$

With multiplication well defined, the associativity and commutativity of multiplication follows directly from the associativity and commutativity of the tensor product on vector bundles /*reference*/. Similarly, the distributivity of $\otimes$ over $\oplus$ in vector bundles /*ref*/ gives that the defined multiplication distributes over the defined addition. Finally, the result $E \otimes \varepsilon^1 \cong E$ for any vector bundle E /*ref*/ makes the equivalence class $[\varepsilon^1]$ the multiplicative identity.

□

3.2. Homomorphism of Semirings Verification.

PROOF. Let $f : X \to Y$ denote a continuous function between two compact Hausdorff spaces.

First it must be verified that $J(f)$ is well-defined. Specifically, it must be shown that if $[E_1] = [E_2]$, then $J(f)([E_1]) = J(f)([E_2])$. That is, it must be shown that $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$ for some n implies

$f^*(E_1) \approx_s f^*(E_2)$. First, note the following application of the distributivity of pullback over direct sum /*ref*/.

$$f^*(E_1) \oplus f^*(\varepsilon^n) \approx f^*(E_1 \oplus \varepsilon^n) \approx f^*(E_1 \oplus \varepsilon^n) \approx f^*(E_2) \oplus f^*(\varepsilon^n)$$

The result that the pullback of a trivial bundle is trivial combined with the above confirms $f^*(E_1) \approx_s f^*(E_2)$ and so $J(f)$ is well-defined.

With $J(f)$ well-defined, verifying that $J(f)$ is a semiring homomorphism follows easily from the properties of pullback. Specifically, the distributivity of pullback over direct sum directly gives the distributivity of $J(f)$ over the defined addition. Similarly, the distributivity of pullback over tensor product gives that $J(f)$ distributes over the defined multiplication. Lastly, the property that f^* maps the bundle ε^1 over X to the bundle ε^1 over Y implies that $J(f)$ maps the multiplicative identity to the multiplicative identity. □

3.3. K-Theory Functor Satisfies Contravariant Composition Law.

Proof. Let X, Y, and Z denote compact Hausdorff spaces and take $f : X \to Y$ and $g : Y \to Z$ be continuous functions between them.

Further denote the semirings as defined in claim /*ref*/ by $J(X)$, $J(Y)$, and $J(Z)$. Additionally, let $i_X : J(X) \to K(X)$, $i_Y : J(Y) \to K(Y)$, and $i_Z : J(Z) \to K(Z)$ denote the ring completions of each semiring as in definition /*ref*/. Further, let $J(g) : J(Z) \to J(Y)$ and $J(f) : J(Y) \to J(X)$ denote the homomorphism of semirings as described in claim /*ref*/. Finally, let the functions $K(g) : K(Z) \to K(Y)$, $K(f) : K(Y) \to K(X)$, and $K(f \circ g) : K(Z) \to K(X)$ be the unique functions such that the following composition identities hold.

$$K(f) \circ i_X = i_Y \circ J(f)$$
$$K(g) \circ i_Y = i_Z \circ J(g)$$
$$K(f \circ g) \circ i_X = i_Z \circ J(f \circ g)$$

Additionally note that $J(f \circ g) = J(g) \circ J(f)$ follows from the pullback property of /*ref*/ $(f \circ g)^*(E) = g^*(f^*(E))$ on a bundle E by the following computation on an element $[E] \in J(X)$.

$$J(f \circ g)([E]) = [(f \circ g)^*(E)] = [g^*(f^*(E))] = J(g)([f^*(E)]) = J(g)(J(f)([E]))$$

Substitutions of the preceding result together with the earlier composition identities allows for the following result.

$$(K(g) \circ K(f)) \circ i_X = K(g) \circ (i_Y \circ J(f)) = (i_Z \circ J(Y)) \circ J(f) = i_Z \circ J(f \circ g)$$

And so $K(f) \circ K(g)$ fulfills the defining property of $K(f \circ g)$. Because the function $K(f \circ g)$ is the unique function fulfilling this property, it must be that $K(f) \circ K(g) = K(f \circ g)$. Figure 5 provides a visual aid for this argument.

□

$$\begin{tikzcd}
 & J(Y) \arrow[dl, "J(g)"'] \arrow[dd, "i_Y"] & \\
J(Z) \arrow[dd, "i_Z"] & & J(X) \arrow[ul, "J(f)"'] \arrow[ll, "J(f\circ g)"'] \arrow[dd, "i_X"] \\
 & K(Y) \arrow[dl, dashed, "K(g)"'] & \\
K(Z) & & K(X) \arrow[ul, dashed, "K(f)"'] \arrow[ll, dashed, "K(f\circ g)"']
\end{tikzcd}$$

Figure 5. K-Theory Contravariant Composition Argument

3.4. Reduced K-Theory forms Group.

Proof. First it must be verified that the direct sum operation $\oplus$ is well-defined on the equivalence classes. So, consider vector bundles $E_1 \sim E_2$ and $F_1 \sim F_2$. Then let n_1, m_1, n_2, m_2 be the numbers such that $E_1 \oplus \varepsilon^{n_1} \approx E_2 \oplus \varepsilon^{n_2}$ and $F_1 \oplus \varepsilon^{m_1} \approx F_2 \oplus \varepsilon^{m_2}$. It then follows that
$E_1 \oplus F_1 \approx E_2 \oplus F_2$
$(E_1 \oplus F_1) \oplus (\varepsilon^{n_1+m_1}) \approx (E_1 \oplus \varepsilon^{n_1}) \oplus (F_1 \oplus \varepsilon^{m_1}) \approx (E_2 \oplus \varepsilon^{n_2}) \oplus (F_2 \oplus \varepsilon^{m_2}) \approx (E_1 \oplus F_1) \oplus (\varepsilon^{n_2+m_2})$
Where the above computation used $\varepsilon^{n+m} \approx \varepsilon^n \oplus \varepsilon^m$.

With the group operation well-defined, the associativity and commutativity of the operation follows from direct sum associative and commutative on bundles.

The identity element in the group is given by the equivalence class $[\varepsilon^0]$, which is the set of all trivial bundles. Indeed, $[E] + [\varepsilon^0] = [E \oplus \varepsilon^0] = [E]$.

It only remains to show the existence of inverses, which appeals to /*ref*/. Then take any element $[E]$ and consider the promised bundle E' such that $E \oplus E' \approx \varepsilon^n$ for some trivial bundle of dimension n. Then, the element $[E']$ is the inverse element.

$$[E] + [E'] = [E \oplus E'] = [\varepsilon^n] = [\varepsilon^0]$$

$\square$

CHAPTER 6

K-Theory as a Cohomology Theory

1. Exact Sequences

Definition 6.1 (Short Exact Sequence). Let A, B, and C be abelian groups. Then let $\psi : A \to B$ be an injective homomorphism and let $\varphi : B \to C$ be a surjective homomorphism. Further suppose that $\ker \varphi = \operatorname{Im} \psi$. Then, the pair of homomorphisms ψ and φ are called *exact* and the sequence $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$ is called a *short exact sequence.*

Note that an exact sequence does not need to be short. A sequence of arbitrary length can *exact* so long as each adjacent pair of homomorphisms is exact. Of particular note is the *long exact sequence*, which is a sequence of groups extending infinitely in both directions such that each adjacent pair of homomorphism is exact.

$$\ldots \to A^{-2} \xrightarrow{\alpha} A^{-1} \xrightarrow{\beta} A^0 \xrightarrow{\gamma} A^1 \xrightarrow{\delta} A^2 \to \ldots$$

The above definitions of exact sequences used groups, but replacing the word "groups" with "commutative rings" or even with "modules" gives a definition of exact sequence for other categories.

Exact sequences are useful. Recall that the computation of the reduced K-Theory for n points in example 5.13 makes use of an exact sequence. The following result is partially responsible for why exact sequences are so useful.

Claim 6.2 (Splitting Lemma). Take a short exact sequence $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$. Then, the following three statements are equivalent:

(i) There exists a homomorphism $\alpha : B \to A$ such that $\alpha \circ \psi$ is the identity on A.
(ii) There exists a homomorphism $\beta : C \to B$ such that $\varphi \circ \beta$ is the identity on C.
(iii) There is an isomorphism $A \oplus C \cong B$.

/*verify and describe how this implies K and $\widetilde{K}$ properties*/

2. Exact Sequences in K-Theory

However, exact sequences has a bigger role in K-Theory than highlighting the relationship between K-Theory and reduced K-Theory. Given a pair (X, A) of topological spaces with particular properties, the K-Theory functor induces a natural long exact sequence of groups with nice properties. This mapping from a pair (X, A) to a long exact sequence is the idea of *cohomology.* Constructing this long exact sequence, however, requires some work beginning with the short exact sequence in the following claim.

Claim 6.3. Take the pair of compact Hausdorff topological spaces (X, A) where $A \subset X$ to be a closed subset. Then consider the inclusion and quotient maps $i : A \to X$ with $q : X \to X/A$. This can also be written $A \xrightarrow{i} X \xrightarrow{q} X/A$. This sequence of morphisms between topological spaces induces a sequence of morphisms between commutative rings (possibly without identity) $\widetilde{K}(X/A) \xrightarrow{q^*} \widetilde{K}(A) \xrightarrow{i^*} \widetilde{K}(X)$ as depicted in 1. The sequence of ring homomorphisms is exact.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\widetilde{K}"] & X \arrow[r, "q"] \arrow[d, "\widetilde{K}"] & X/A \arrow[d, "\widetilde{K}"] \\
\widetilde{K}(A) & \widetilde{K}(X) \arrow[l, "i^*"'] & \widetilde{K}(X/A) \arrow[l, "q^*"']
\end{tikzcd}$$

Figure 1. Inducing Sequence of Rings

/*verify the above and talk about proof*/

This short exact sequence can be extended into a slightly longer exact sequence. First consider the sequence of spaces beginning with A and X such that each additional space is given by the disjoint union of the previous space together with the cone of the space two steps back. The disjoint unions allow for natural inclusions between spaces as depicted in the following diagram.

$$A \xrightarrow{i} X \xrightarrow{i} X \cup CA \xrightarrow{i} (X \cup CA) \cup CX \xrightarrow{i} ((X \cup CA) \cup CX) \cup C(X \cup CA)$$

The strategy will be to apply claim 6.3 to show that each adjacent pair of ring homomorphisms induced by $\widetilde{K}$ is exact. However, applying this claim requires taking the topological quotient of each space by collapsing the subspace two steps back to a point.

$$X \xrightarrow{q} X/A \qquad X \cup CA \xrightarrow{q} (X \cup CA)/CX \qquad (X \cup CA) \cup CX \xrightarrow{q} ((X \cup CA) \cup CX)/(CX)$$

So, the above diagrams define the inclusion maps and show the necessary quotient maps to incorporate, but claim 6.3 requires a relationship between the inclusion and quotient maps that is currently missing from this construction. For this, consider the relationship between spaces given by the following additional quotient maps. /*discuss these quotient maps or at least the form of these quotient maps earlier?*/

$$X \cup CA \xrightarrow{Q} X/A \qquad (X \cup CA) \cup CX \xrightarrow{Q} (X \cup CA)/CX$$

$$((X \cup CA) \cup CX) \cup C(X \cup CA) \xrightarrow{Q} ((X \cup CA) \cup CX)/(CX)$$

The following result comes to the rescue.

Claim 6.4. /*claim to justify that vertical maps induce isomorphisms... multiple ways to do this. Can actually avoid homotopy here if need be. Come back to this section of argument later*/

/*finish this part of argument*/

The top two rows of Figure 2 summarizes all of the maps talked about so far. /*some commutativity needs to be shown in order to write this*/. /*some typos in the following — fix later*/

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "Q"] & X \arrow[r, "i"] \arrow[d, "Q"] \arrow[dr, "q"] & X \cup CA \arrow[r, "i"] \arrow[d, "Q"] \arrow[dr, "q"] & (X \cup CA) \cup CX \arrow[r, "i"] \arrow[d, "Q"] \arrow[dr, "q"] & (X \cup CA) \cup CX \cup C(X \cup CA) \arrow[d, "Q"] \\
A \arrow[d, "\approx"] & X \arrow[d, "\approx"] & X/A \arrow[d, "\approx"] & (X \cup CA)/(CX) \arrow[d, "\approx"] & ((X \cup CA) \cup CX)/(CX) \arrow[d, "\approx"] \\
A & X & X/A & SA & SX
\end{tikzcd}$$

Figure 2. Relationship between Inclusion and Quotient Maps

Figure 2 also includes the following additional pleasant isomorphisms /*talk about these isomorphisms or the form of these isomorphisms previously*/

$$(C \cup CA)/CX \approx SA \qquad\qquad ((C \cup CA) \cup CX)/(CX) \approx SX$$

Next, applying the $\widetilde{K}$ functor on all of these maps will take the inclusion maps and quotient maps to ring homomorphisms in the opposite directions. Additionally, $\widetilde{K}$ will take the isomorphisms to ring isomorphisms and will additionally take each map denoted with a "Q" to a ring isomorphism by claim 6.4. This is summarized in Figure 3. Note in particular that claim 6.3 promises that each i^*q^* composition gives a short exact sequence. /*give the particular sequences?*/. Then, by viewing each inclusion map as the composition of an isomorphism with a quotient map, the entire top row is then exact. By further composing with the vertical isomorphisms provides a simple short exact sequence as shown in figure 4. However, while this sequence is useful, a cohomology theory requires more structure.

$$\begin{tikzcd}
\widetilde{K}(A) \arrow[d, "\cong"] & \widetilde{K}(X) \arrow[l, "i^*"'] \arrow[d, "\cong"] & \widetilde{K}(X \cup CA) \arrow[l, "i^*"'] \arrow[d, "\cong"] & \widetilde{K}((X \cup CA) \cup CX) \arrow[l, "i^*"'] \arrow[d, "\cong"] & \widetilde{K}(\dots) \arrow[l, "i^*"'] \arrow[d, "\cong"] \\
\widetilde{K}(A) \arrow[d, "\cong"] & \widetilde{K}(X) \arrow[d, "\cong"] & \widetilde{K}(X/A) \arrow[ul, "q^*"'] \arrow[d, "\cong"] & \widetilde{K}((X \cup CA)/(CX)) \arrow[ul, "q^*"'] \arrow[d, "\cong"] & \widetilde{K}(\dots) \arrow[ul, "q^*"'] \arrow[d, "\cong"] \\
\widetilde{K}(A) & \widetilde{K}(X) & \widetilde{K}(X/A) & \widetilde{K}(SA) & \widetilde{K}(SX)
\end{tikzcd}$$

Figure 3. Applying the Functor $\widetilde{K}$

$$\begin{tikzcd}
\widetilde{K}(SX) \arrow[r] & \widetilde{K}(SA) \arrow[r] & \widetilde{K}(X/A) \arrow[r] & \widetilde{K}(X) \arrow[r] & \widetilde{K}(A)
\end{tikzcd}$$

Figure 4. The Resulting Exact Sequence

3. K-Theory as a Cohomology Theory

A cohomology theory requires an infinite collection of functors where each functor takes a fixed subcategory topological space pairs to the category of abelian groups. K-Theory considers the subcategory of compact Hausdorff space-subspace pairs — all pairs of the form (A, X) discussed earlier. A cohomology theory must further satisfy the Eilenberg-Steenrod cohomology axioms to some degree. But at a minimum, the functors must take homotopic continuous maps to equivalent homomorphisms, and additionally induce an exact sequence extending infinitely in both directions. A cohomology theory that satisfies these minimum requirements is called a *reduced cohomology theory.* By building upon the work in the previous section, reduced K-Theory can be extended into a reduced cohomology theory.

The first step to develop this cohomology theory is to extend the exact sequence developed in the previous section into an exact sequence that extends infinitely in both directions. Figure 4 induces a slightly longer exact sequence given the initial compact Hausdorff space-subspace pair (X, A), but repeating the same process for the pair (SX, SA) will give a longer sequence. This can be extended infinitely to the left giving the infinite exact sequence. /*note commutativity of quotient and suspension*/

$$\cdots \to \widetilde{K}(S^2X) \to \widetilde{K}(S^2A) \to \widetilde{K}(SX/SA)) \to \widetilde{K}(SX) \to \widetilde{K}(SA) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A)$$

However, getting the sequence to extend infinitely to the right requires the following theorem.

Theorem 6.5. (Bott Periodicity) For any compact Hausdorff space X there is an isomorphism $\widetilde{K}(X) \cong \widetilde{K}(S^2X)$.

/*give the idea of this proof, but will not prove*/

Applying Bott periodicity to the infinite exact sequence induced by the pair (X, A) immediately gives $\widetilde{K}(X) \cong \widetilde{K}(S^2X)$, $KR(A) \cong KR(S^2A)$. In fact, there are only 6 distinct group isomorphism classes in the infinite exact sequence. This justifies drawing the following cyclic commutative diagram. A cyclic exact sequence can be interpreted as an exact sequence extending infinitely in both direction by continuing around the cycle infinitely in both directions. However, before unraveling the cyclic sequence, adopt the following change in notation to negative indices. From now on, denote $\widetilde{K}(S^kX)$ by $\widetilde{K}^{-k}(X)$, denote $\widetilde{K}(S^kA)$ by $\widetilde{K}^{-k}(A)$ and denote $\widetilde{K}(S^kX/S^kA)$ by $\widetilde{K}^{-k}(X, A)$. However, by Bott periodicity and as depicted in Figure 5, there are 6 distinct group isomorphism classes:

$$\begin{tikzcd}
\widetilde{K}(X/A) \arrow[r] & \widetilde{K}(X) \arrow[r] & \widetilde{K}(A) \arrow[d] \\
\widetilde{K}(SA) \arrow[u] & \widetilde{K}(SX) \arrow[l] & \widetilde{K}(SX/SA) \arrow[l]
\end{tikzcd}$$

Figure 5. Cyclic Exact Sequence

$$\widetilde{K}^{-1}(X, A) \qquad \widetilde{K}^{-1}(X) \qquad \widetilde{K}^{-1}(A) \qquad \widetilde{K}^{0}(X, A) \qquad \widetilde{K}^{0}(X) \qquad \widetilde{K}^{0}(A)$$

This allows for the following definition for positive indices. Specifically, let $\widetilde{K}^k(A)$ be $\widetilde{K}^0(A)$ for even k and $\widetilde{K}^1(A)$ for odd k. Doing the same for $\widetilde{K}^k(X)$ and $\widetilde{K}^k(X, A)$ gives the following infinite sequence of groups.

$$\cdots \to \widetilde{K}^{-1}(X) \to \widetilde{K}^{-1}(A) \to \widetilde{K}^0(X, A) \to \widetilde{K}^0(X) \to \widetilde{K}^0(A) \to \widetilde{K}^1(X, A) \to \widetilde{K}^1(X) \to \cdots$$

This infinite sequence is indeed exact, for it is only a reformulation of the cyclic exact sequence shown in figure 5 and each functor $\widetilde{K}^k$ preserves homotopy by $\widetilde{K}$ preserves homotopy. /*technically need to talk about suspension as a functor ...*/

/*include computations to demonstrate spaces are not the same? Add something here to make it seem immediately useful?*/

/*extend to K*/

4. Further Structure on the Cohomology Theory

/*motivation for external product*/

Definition 6.6 (External Product). Let X and Y be compact Hausdorff spaces and let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the natural projections. Then, the *external product* is a mapping $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ and is defined by $\mu(a \otimes b) = p_1^*(a)p_2^*(b)$.

/*need to do all of the work to get $K(S^{2k}) \otimes K(X) \to K(S^{2k} \times X)$ ISM*/

/*extend to rings / address external product / graded stuff if time?*/

CHAPTER 7

Division Algebra Application

Recall from /*ref*/ that a division algebra is a ring with multiplicative inverses. Further recall that a division algebra structure in $\mathbb{R}^n$ induces an H-space structure over the sphere S^{n-1} by considering the subset of $\mathbb{R}^n$ with norm 1. And so if a division algebra structure on $\mathbb{R}^n$ exists, then there is an H-space structure on the sphere S^{n-1}. As Bott periodicity hints at, K-Theory has a close relationship with spheres, making K-Theory a good tool to examine the existence of H-spaces on spheres. This chapter will use K-Theory to show that an H-space structure cannot exist on any sphere other than S^0, S^1, S^3, and S^7. This conclusion regarding H-spaces together with the explicit examples of the reals, the complex numbers, the quaternions, and the octonions gives the following theorem.

Theorem 7.1. $\mathbb{R}^n$ is a division algebra only when n is 1, 2, 4, or 8.

1. The odd case

The argument will first rule out the possibility of a division algebra structure on odd dimensions, so assume for the purpose of contradiction that there is a division algebra structure on $\mathbb{R}^{2k+1}$ for some nonnegative k. It then follows that S^{2k} is an H-Space, which promises a continuous mapping $\mu : S^{2k} \times S^{2k} \to S^{2k}$. Additionally let p_1 denote the projection from $S^{2k} \times S^{2k}$ to the first factor and let p_2 be the projection to the second factor. Applying the K-Theory functor to the H-space multiplication mapping gives a homomorphism between rings $\mu^* : K(S^{2k} \times S^{2k}) \to K(S^{2k})$ /*ref...do this step as an example earlier*/. This puts the homomorphism in the form $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ such that $\alpha = p_1^*(\gamma)$ and $\beta = p_2^*(\gamma)$. Note that γ is the generator of the ring and in particular, γ is $H - 1$ where H denotes the canonical line bundle over the space.

The contradiction will arise in analyzing the quantity $\mu^*(\gamma)$. To accomplish this, define $i_1 : S^{2k} \to S^{2k} \times S^{2k}$ by $i_1 : x \mapsto (x, e)$ where e is the identity element of S^{2k} as an H-space. Note that $\mu \circ i_1$ is the identity, giving $i_1^* \circ \mu^*$ is the identity, so studying i_1^* will give information about μ^*.

It follows from the definition of i_1 that $p_1 \circ i_1 = \mathrm{Id}$ and so $i_1^* \circ p_1^*$ is identity. Plugging in α to both sides and recalling the definition of α then gives:

$$i^*(\alpha) = \gamma$$

In a similar way, it follows that $p_2 \circ i_1$ is a constant function always mapping to the H-space identity e. Denote this const

2. The even case

Bibliography

[1] M. Atiyah. *K-theory*. CRC Press, 2018.

[2] A. Hatcher. *Vector Bundles and K-Theory*. 2003. http://www.math.cornell.edu/~hatcher.