Spectral and Covering Properties of a Class of Directed Graphs

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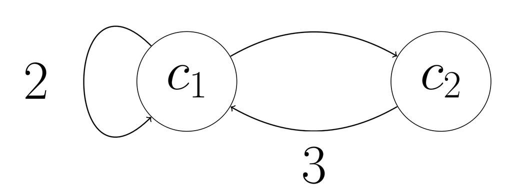
Example of an orbigraph

Figure 1 below shows a small orbigraph. It consists of two *vertices*, c_1 and c_2, connected by *arrows*. The loop-shaped arrow connecting vertex c_1 to itself is labeled by the number 2. This number is called the *weight* of the arrow. Note that the arrow from vertex c_2 to c_1 has weight 3.



$$M = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

Figure 1: A small 3-orbigraph & adjacency matrix M

Adjacency matrix of an orbigraph

An adjacency matrix is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in each position according to whether the graph vertices are adjacent or not.*Adjacency* in this case means whether or not two graph vertices are joined by an arrow.Although we are dealing with a directed graph, the adjacency matrix needs to be symmetric around the diagonal.
Matrix M (Figure 1) is the adjacency matrix of the orbigraph defined in Figure 1

Singular vertices in an orbigraph

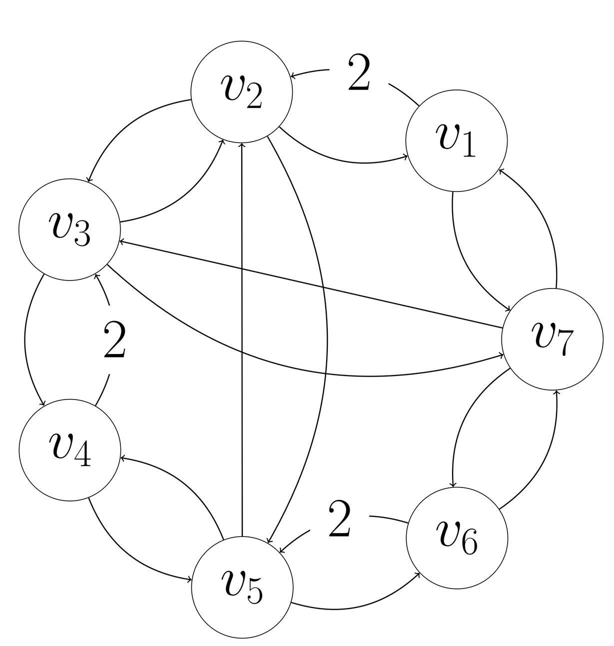


Figure 2: 3-orbigraph with 7 vertices

This orbigraph has 3 singular vertices. *A singular vertex* is a vertex with at least one *outgoing arrow*

Objectives

This project has had two main foci:

- In a Summer 2013 Rogers Research project Colin Gavin ('15) obtained the following bounds on the number of singular vertices, s, in a k-orbigraph in terms of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the orbigraph and the number of vertices, n, of the orbigraph.
$$\frac{\sum_i \lambda_i^2 - nk}{k^2 - k} \le s \le \sum_i \lambda_i^2 - nk$$
We showed that the upper and lower bounds provided in this result are *sharp*. That is, these bounds cannot be improved to give tighter control on the number of singular vertices.
- The second question that we considered is whether or not a connected k-orbigraph which admits a *countable* cover by a k-regular graph must in fact also have a *finite* cover by a k-regular graph. **We have a 'brute-force' argument that this is true for 2-orbigraphs with two and three vertices. Current work seeks to find a more elegant approach that might generalize to all orbigraphs.**

Sharpness result

Ex. 1: The orbigraph with the adjacency matrix below proves that the upper inequality is sharp.

$$N = \begin{bmatrix} 0&2&0&0&0&0&1 \\ 1&0&1&0&1&0&0 \\ 0&1&0&1&0&0&1 \\ 0&0&2&0&1&0&0 \\ 0&1&0&1&0&1&0 \\ 0&0&0&0&2&0&1 \\ 1&0&1&0&0&1&0 \end{bmatrix} \quad \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = -1-\sqrt{3} \\ \lambda_3 = -2 \\ \lambda_4 = -1 \\ \lambda_5 = 1 \\ \lambda_6 = 1 \\ \lambda_7 = \sqrt{3}-1 \end{matrix}$$

$$\frac{\sum_{i=1}^7 \lambda_i^2 - (7)(3)}{(3)^2 - (3)} \le (3) \le \sum_{i=1}^7 \lambda_i^2 - (7)(3)$$

$$\frac{1}{2} \le 3 \le 3$$

Ex. 2: The orbigraph with the adjacency matrix below proves that the lower inequality is sharp.

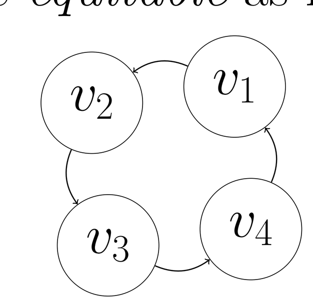
$$N = \begin{bmatrix} 0&2&0 \\ 2&0&0 \\ 0&0&2 \end{bmatrix} \quad \begin{matrix} \lambda_1 = -2 \\ \lambda_2 = 2 \\ \lambda_3 = 2 \end{matrix}$$

$$\frac{\sum_{i=1}^3 \lambda_i^2 - (3)(2)}{(2)^2 - (2)} \le (3) \le \sum_{i=1}^3 \lambda_i^2 - (3)(2)$$

$$3 \le 3 \le 6$$

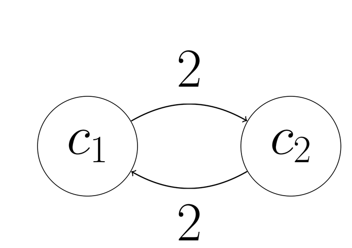
Coverings by equitable partitions

Divide the vertices of a graph into a partition. The graph formed by collapsing all vertices in a partition element to a single vertex, with adjacent partition elements connected by the corresponding number of edges, is a *quotient graph*. Here we need the partition to be *equitable* as in the example below.



$$R = \begin{bmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{bmatrix}$$

Figure 3: Equitable partition of a graph & adjacency matrix



$$Q = \begin{bmatrix} 0&2 \\ 2&0 \end{bmatrix}$$

Figure 4: Orbigraph quotient of partitioned graph above

Graph R with quotient graph Q is said to *cover* Q.

Work in progress

We seek to prove that all 2-orbigraphs with 2 vertices have a finite cover using the equation $PQ = RP$, where Q is the 2×2 adjacency matrix of a given 2-orbigraph, P is the $n \times 2$ partition matrix of the cover, and R is the $n \times n$ adjacency matrix of the orbigraph's cover.

Example: Show that for any Q we can find the corresponding matrix P.

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

$$P = \begin{bmatrix} a_1 & (1-a_1) \\ a_2 & (1-a_2) \\ \ldots & \ldots \\ a_n & (1-a_n) \end{bmatrix} \quad R = \begin{bmatrix} 0\ 1\ 0 \ldots 0\ 0\ 1 \\ 1\ 0\ 1\ 0 \ldots 0\ 0 \\ \ldots \ \ldots. \ \ldots. \ \ldots \\ 0\ 0 \ldots 0\ 1\ 0\ 1 \\ 1\ 0\ 0 \ldots 0\ 1\ 0 \end{bmatrix}$$

Acknowledgements

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Formal definition of an orbigraph

A ***k*-orbigraph** is a weighted, directed graph Γ where the adjacency matrix A satisfies the following:

1. All entries in A are non-negative integers.
2. All row sums of A equal k.
3. Letting A_{ij} denote the entry in row i and column j of A, we require the symmetry-like condition:
$$A_{ij} > 0 \text{ if and only if } A_{ji} > 0.$$

Larger context of this project