Complex Topological K-Theory

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Introduction to Vector Bundles

Vector Fields

/*use vector fields in $\mathbb{R}^2$ and question of what do these vector fields live as motivation*/

Vector Fields on Sphere

/*extend to tangent vector fields on a sphere. Break away from the vector fields and emphasize the object itself*/

The Cylinder

/*Show construction of cylinder*/

The Mobius Band

/*Show construction of Mobius Band*/

Trivial Bundle

/*Emphasize cylinder as an example of a trivial bundle.
Necessary for giving definition of vector bundle*/

Topology Interlude

Continuous Functions and Open Sets

/*Give picture of a continuous and noncontinuous function $\mathbb{R} \to \mathbb{R}$ and emphasize the connection between continuity and open sets*/

Continuous Functions between Objects

/*Generalize to a visual example of looking at the open sets and continuous functions between two topological spaces*/

Some Formal Topology

/*Briefly give the formal definitions of a topological space and continuous functions in topology*/

Back to Vector Bundles

The Definition of a Vector Bundle

/*Now ready to give a formal definition of V.B. Do this with some visual example as an aid*/

How to Think About Vector Bundles

/*Emphasize what vector bundles are: topological objects but also a bunch of vector spaces shoved together. Introduce the vocabulary of fiber here*/

Homomorphisms on Vector Bundles

/*Give definition of a homomorphism between vector bundles with an example as a visual aid*/

The Objective

/*Formulate an objective for the talk: Take a topological object and use the vector bundles over that object to create a ring*/

Algebra Interlude

Definition of a Ring

/*Give very brief definition of a ring for those unfamiliar*/

The Direct Sum Operation on Vector Spaces

/*Give some kind of definition of direct sum... will probably end up using the cartesian product*/

The Tensor Product Operation on Vector Spaces

/*Give some kind of definition of tensor product. Honestly not sure how to do this so that it is accessible. I could say "tensor product is a thing that gives distribution over direct sum" and leave it there...?*/

Properties of Direct Sum and Tensor Product

/*Give list of nice properties of tensor product an direct sum (associativity, commutativity, distributativity...)*/

Extending Vector Space Operations to Vector Bundles

Extending Direct Sum to Vector Bundles

/*Emphasize that a V.B. is a simply a bunch of fibers, so can simply apply the direct sum operation to each fiber*/
/*Mention that there is more to do (give a topology and check local triviality, but it all works out)*/

Example of Direct Sum of Vector Bundles

/*Some sort of example of direct sum... not sure what a good choice here would be*/

Extending Tensor Product to Vector Bundles

/*Say that tensor product is basically the same thing as direct sum ... apply the T.P. operation to each fiber*/

The Ring Properties of Vector Bundles

/*side by side comparison of properties of vector bundles and properties of a ring. Emphasize lack of additive identity*/

Another Algebra Interlude

Semirings

/*def of semiring, example of $\mathbb{N} \cup \{0\}$*/

Ring Extension

/*$\mathbb{N} \cup \{0\}$ extends to $\mathbb{Z}$. This generalizes given ... commutativity of multiplication and additive cancellation law*/

The Semiring Properties of Vector Bundles

/*side by side comparison of properties of vector bundles and properties of ring. Emphasize lack of cancellation law*/

Quest for the Additive Cancellation Property

Useful Tool

/*Mention that given certain constraints, for any bundle E there exists a bundle E' such that $E \oplus E'$ is trivial*/ /*Include visual example of this... the best visual example might be the normal and tangent bundles on a sphere*/

An Attempt at Using the Tool

/* Show some work trying to use this as a cancellation property:

$$E \oplus F \cong E' \oplus F$$
$$E \oplus (F \oplus F') \cong E' \oplus (F \oplus F')$$
$$E \oplus \varepsilon^n \cong E' \oplus \varepsilon^n$$

*/

A Convenient Equivalence Relation

/* the work on the previous slide motivates the equivalence relation $E \approx_s E'$ if $E \oplus \varepsilon^n \cong E' \oplus \varepsilon^n$ for some n. */
/*mention the complications of introducing an equivalence relation (well-defined) but it all works out*/

The Definition of K-Theory

K-Theory

/*A summary slide that gives the definition of K-Theory*/

???

/*ideas on how to expand:

- Incorporate category theory and talk about K-Theory as a functor. Would require introducing pullback bundles and putting more emphasis on homomorphisms.
- Introduce cohomology theory and explain how K-Theory extends to a cohomology theory (would probably require the above)
- Work towards the division algebra application. A semi-manageable goal would be the proof showing odd dimensions are impossible (would probably require the above)

*/