Hearing the Local Orientability of Orbifolds

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Background

An orbifold is a generalization of a *manifold*, some n dimensional surface. Recall a manifold is defined through an atlas of compatible charts such that each chart maps $\widetilde{U} \subset \mathbb{R}^n$ onto some subset of the manifold. Note that each $\widetilde{U}$ must be open and connected. The definition of an orbifold follows similarly.

Definition (Orbifold)**.** For some orbifold $\mathcal{O}$ we take a group of isometries Γ and some subset $\widetilde{U} \subset \mathbb{R}^n$. Then, we have some mapping π_u the corresponding topological quotient $\widetilde{U}/\Gamma$ onto some subset $U \subset \mathcal{O}$. The tuple $(\widetilde{U}, \Gamma, \pi_U)$ is called a *chart*, and $\mathcal{O}$ is a valid orbifold if it has an atlas of compatible charts.

We can then provide an orbifold with a Riemannian metric by patching together Riemannian metrics on the local charts such that each metric is invariant under the group action of each chart. In this paper, all referenced orbifolds have an implied Riemannian metric.

Definition (Strata)**.** This construction gives rise to a natural partitioning of some orbifold $\mathcal{O}$ into a collection of submanifolds such that: each submanifold N is connected, and each point in N has local structure with exactly the same isometry group. We denote this group $\mathrm{Iso}(N)$ and we call each such submanifold a *strata* of $\mathcal{O}$.

There is a subclass of strata called *primary singular strata* or simple *primary strata*. For the purposes of the paper, a strata N is a *primary strata* or simply *primary strata* if there exists some isometry $\gamma \in \mathrm{Iso}(N)$ such that $\dim(\mathrm{Fix}(\gamma)) = \dim(N)$. Note that the Fix operator denotes the set of points where each point is mapped to itself under the isometry.

Example. Take the group of isometries $G = \{\mathbb{Z} \times \mathbb{Z}, e, r_x, r_y, R_{180}\}$. Here, $\mathbb{Z} \times \mathbb{Z}$ denotes the translation lattice by integers, e denotes the identity, r_x and r_y are reflections across the x and y axes respectively, and R_{180} denotes a rotation by 180°. Note that without the translation lattice, this is the dihedral group D_2. Now, we consider the topological quotient $\mathbb{R}^2/G$. This topological quotient forms a valid orbifold $\mathcal{O}$. Orbifold $\mathcal{O}$ has fundamental domain of a square with side lengths $\frac{1}{2}$; $\mathcal{O}$ has 4 strata of dimension 1 (mirror edges); and $\mathcal{O}$ has 4 strata of dimension 0 (corner reflector). Each mirror edge has chart with isotropy group $\Gamma = \{e, r_x\}$ or $\Gamma = \{e, r_y\}$, and each corner reflector has the full dihedral group $\Gamma = \{e, r_x, r_y, R_{180}\}$. In the case of the mirror edges, $\mathrm{Fix}(r_x)$ and $\mathrm{Fix}(r_y)$ is of dimension 1, the same dimension as the strata making each mirror edge a primary strata. Similarly, $\mathrm{Fix}(R_{180})$ is of dimension 0, the same dimension of the corner reflectors, so each corner reflector is a primary strata. See 'Asymptotic Expansion of the Heat Kernel for Orbifolds' 2.15 for more detail[3].

Definition (Laplace spectra)**.** We now define the notion of *Laplace spectra*. First, we define the *Laplacian Operator* Δ such that $\Delta f := -\,\mathrm{grad}(\mathrm{div}(f))$ for some function f. Then, for some orbifold $\mathcal{O}$ we consider the function ψ defined over all $\mathbf{x} \in \mathcal{O}$. Then, consider the eigenvalue equation $\Delta\psi(\mathbf{x}) = -\lambda\psi(\mathbf{x})$. Applying appropriate boundary conditions gives rise to a discrete unbounded list of real eigenvalue solutions $0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \dots$. We call this resulting sequence $\{\lambda_i\}$ the *Laplace spectrum*.

Research Question

We can now state our research question precisely as: given the Laplace Spectrum of some orbifold $\mathcal{O}$, what properties can we deduce about $\mathcal{O}$? Or equivalently: what orbifolds can we guarantee will have different Laplace spectra?

Laplace spectra are a mathematical formalization of resonance frequencies, so an intuitive interpretation of the question is: what properties can we "hear" in an orbifold? Mark Kac popularized this type of question in the article 'Can one hear the shape of a drum?'[6].

Result

We present a definition in preparation for our resulting theorem.

Definition (Local Orientability)**.** For some chart in an orbifold $\mathcal{O}$, we take the corresponding subset $\widetilde{U} \subset \mathcal{O}$ and topological quotient with respect to group Γ. Then, we say that $\widetilde{U}$ is *orientable* if all elements Γ are orientation-preserving isometries. An orbifold $\mathcal{O}$ is *locally orientable* if every chart on $\mathcal{O}$ is orientable. Conversely, an orbifold $\mathcal{O}$ is *locally non-orientable* if there exists a single chart on $\mathcal{O}$ that is not orientable.

Now, we present the result of our research.

Theorem. A locally orientable orbifold and a locally non-orientable orbifold will have a different Laplace spectra. Or, you can hear the local orientability of an orbifold.

Simply put, we proved that orbifolds with a group containing orientation-reversing operations will always have a different Laplace spectra than orbifolds that have no such group.
This theorem is a generalization of Theorem 5.1 from 'Asymptotic Expansion'[3]. Theorem 5.1 concludes a manifold and an orbifold with specific strata have different Laplace spectra. This Theorem 5.1 follows from our slightly stronger local orientability theorem. Additionally, we use the concept of local orientability to present the theorem, which is a simpler formulation of the result.

Asymptotic Heat Expansion

To approach this problem, we use the asymptotic heat expansion technique as discussed in 'Asymptotic Expansion Orbifolds'[3]. In approximating how heat moves through the orbifold, we obtain a series of the form $H(t) = \sum_{k=-\dim(\mathcal{O})}^{\infty} h_{k/2} t^{k/2}$. Every orbifold has a such a series. Importantly, 'Asymptotic Expansion' demonstrates that two orbifolds have the same Laplace spectra exactly when the orbifolds have the same heat expansion.
In combining 4.5, 4.7, and 4.8 in 'Asymptotic Expansion' we have that an orbifold $\mathcal{O}$ with collection of strata $S(\mathcal{O})$ has the following heat expansion:

$$H(t) = (4\pi t)^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k(\mathcal{O}) t^k + \sum_{N \in S(\mathcal{O})} \frac{(4\pi t)^{-\dim(N)/2}}{|\operatorname{Iso}(N)|} \sum_{k=0}^{\infty} t^k \int_N \sum_{\gamma \in \operatorname{Iso}^{\max}(\widetilde{N})} b_k(\gamma, x) dvol_N \qquad (1)$$

There are a few important take-aways from this formula. Without loss of generality, we study odd dimensional orbifolds. Firstly, the strata in the orbifold affect the expansion. Importantly, for an odd dimension orbifold, the only way for an expansion to have a non-zero t^k term where k is some integer is by containing some *even dimensional strata.* This fact is essential in proving the result.

Proof

The proof of this theorem is a strong representation of what I specifically contributed to the project. My research mentor Liz Stanhope and I worked together on this theorem in the dimension 3 case. Then, I conjectured that the theorem holds in the n dimensional case and presented a rough proof. My research mentor pointed out various gaps in the proof that we filled in together, resulting in a finished proof. Below I present a simplified version of this proof which is the primary achievement of the research.

Proof. Take $\mathcal{O}_o$ to be a locally orientable orbifold and $\mathcal{O}_n$ to be a locally non-orientable orbifold. Then, we claim that $\mathcal{O}_o$ and $\mathcal{O}_n$ have different Laplace spectra.

It is known that orbifolds of different dimensions have different Laplace spectra, thus we conclude $\dim(\mathcal{O}_o) = \dim(\mathcal{O}_n)$. Without loss of generality, we take this dimension to be odd.
We first claim that every integer power coefficient in the heat expansion of $\mathcal{O}_o$ is 0. We proceed by the method of contradiction under the assumption there exists some non-zero coefficient. Then, as noted previously, it follows from Equation 1 that $\mathcal{O}_o$ has some even dimensional primary strata N. As discussed in the definition, the nature of primary strata implies there exists some isometry $\gamma \in \mathrm{Iso}(N)$ such that $\dim(\mathrm{Fix}(\mathrm{Iso}(N))) = \dim(N)$. But, because N is even and $\dim(\mathcal{O})$ is odd, this implies that γ is orientation reversing, which contradicts the locally orientable nature of $\mathcal{O}$. Thus, every integer power coefficient in the heat expansion is 0, verifying the claim.
We next claim that at least one integer power coefficient in $\mathcal{O}_n$ is non-zero. Firstly, $\mathcal{O}_n$ must have at least one non-orientable chart $(\widetilde{U}, \Gamma, \pi_U)$ by the construction of $\mathcal{O}_n$. So, there is some orientation-reversing operation $\gamma \in \Gamma$. Because $\mathcal{O}_n$ is odd, it follows that $\mathrm{Fix}(\gamma)$ is even. With a small argument, it follows from 2.14 in [3] that there exists at least one primary strata of the dimension of $\mathrm{Fix}(\gamma)$, which is even. Then, we consider the strata of the maximal dimension n. It follows from Equation 1 that only strata of maximal dimension affect the $-n/2$ term in the expansion. Furthermore, the $-n/2$ term contributed by each strata of maximal dimension strictly positive. Thus the final $-n/2$ is simply a sum of positive values and thus is strictly greater than 0. Note that n is even, so we have a non-zero integer power coefficient in $\mathcal{O}_n$, verifying the claim.
Thus, we know that there exists at least one term in the heat expansion of $\mathcal{O}_o$ and $\mathcal{O}_n$ that differ. Thus, as discussed above, the two orbifolds have different Laplace spectra, concluding the proof of this theorem. □

Future Work

We have looked into applying the asymptotic heat expansion machinery to the specific class of three dimensional flat orbifolds. Conway names this class of orbifolds *platycosms*[5] and provides a complete listing of the corresponding crystallographic space group in the book *The Symmetries of Things*[1]. In finding heat expansion coefficients of various platycosms, we could potentially conclude that some platycosms have different Laplace spectra. This would involve classifying all primary strata in each of the 230 different platycosms, which are necessary to compute heat expansion terms. It appears that it is possible to automate this process with computing.

References

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