

ASYMPTOTIC EXPANSION OF THE HEAT KERNEL FOR ORBIFOLDS

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ABSTRACT. We study the relationship between the geometry and the Laplace spectrum of a Riemannian orbifold \mathcal{O} via its heat kernel; as in the manifold case, the time-zero asymptotic expansion of the heat kernel furnishes geometric information about \mathcal{O} . In the case of a *good* Riemannian orbifold (i.e., an orbifold arising as the orbit space of a manifold under the action of a discrete group of isometries), H. Donnelly [10] proved the existence of the heat kernel and constructed the asymptotic expansion for the heat trace. We extend Donnelly's work to the case of general compact orbifolds. Moreover, in both the good case and the general case, we express the heat invariants in a form that clarifies the asymptotic contribution of each part of the singular set of the orbifold. We calculate several terms in the asymptotic expansion explicitly in the case of two-dimensional orbifolds; we use these terms to prove that the spectrum distinguishes elements within various classes of two-dimensional orbifolds.

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1. INTRODUCTION

Roughly speaking, a topological orbifold is a space locally homeomorphic to an orbit space of a finite group action on \mathbb{R}^n . A smooth orbifold consists of a Hausdorff second countable topological space together with an atlas of coordinate charts realizing such local homeomorphisms and satisfying compatibility conditions (see Section 2). Orbifolds were introduced by Satake, then studied by Thurston because of their utility in the investigation of three-manifolds (e.g., a Seifert fibred three-manifold is naturally a generalized circle bundle over a two-orbifold); today,

Gordon and Webb were supported in part by NSF grants DMS-0072534, DMS-0306752, and DMS-0605247; Greenwald was supported in part by NSF ROA grants 0072533 and 9972304.

orbifolds arise naturally in diverse branches of mathematics and physics, including symplectic geometry, string theory, and vertex operator algebras.

We will be interested in orbifolds from a spectral-theoretic point of view. An orbifold endowed with a metric structure is a *Riemannian orbifold*. As in the manifold case, associated with every Riemannian metric is a Laplace operator acting on smooth functions on the orbifold. In the case of closed orbifolds, the Laplacian has a discrete spectrum. We study the relationship between the geometry and the Laplace spectrum of a closed orbifold via its heat kernel; as in the manifold case, the time-zero asymptotic expansion of the heat kernel furnishes geometric information about the orbifold.

Orbifolds began appearing sporadically in the spectral theory literature in the early 1990s, and have received more concentrated attention in the last five years. C. Farsi [16] showed that the spectrum of an orbifold determines its volume by proving that Weyl's asymptotic formula holds for orbifolds. Dryden and A. Strohmaier [13] showed that for a compact, negatively curved two-dimensional orbifold, the Laplace spectrum determines both the length spectrum and the orders of the singular points and vice versa; on the other hand, P. Doyle and J.P. Rossetti [12] gave (disconnected) examples of isospectral flat two-dimensional orbifolds with different length spectra and orders of singular points. Further investigations of the relationship between the lengths of closed geodesics and the spectrum were carried out by E. Stanhope and A. Uribe in [30]. It is natural to ask about the singularities that can appear in an isospectral family of orbifolds. Stanhope [29] showed that, in general, there can be at most finitely many isotropy types (up to isomorphism) in a set of isospectral Riemannian orbifolds that share a uniform lower bound on Ricci curvature. On the other hand, N. Shams, Stanhope, and Webb [28] constructed arbitrarily large (finite) isospectral sets of orbifolds satisfying this curvature condition whose isotropy types differ. Rossetti, D. Schueth, and M. Weilandt [26] recently constructed a pair of isospectral Riemannian orbifolds whose isotropy types have different orders.

For Riemannian manifolds, the asymptotic expansion of the heat kernel can be used to relate the geometry of the manifold to its spectrum. From the so-called *heat invariants* appearing in the asymptotic expansion, one can tell the dimension, the volume, and various quantities involving the curvature of the manifold. The heat kernel has been studied in various analogous or more general settings (e.g. [4, 5, 6, 11, 18, 25]).

In the case of a *good* Riemannian orbifold (i.e., an orbifold arising as the orbit space of a manifold under the action of a discrete group of isometries), H. Donnelly [10] proved the existence of the heat kernel and constructed the asymptotic expansion for the heat trace. We extend Donnelly's work to the case of general compact orbifolds. Moreover, in both the good case and the general case, we express the heat invariants in a form that clarifies the asymptotic contribution of each part of the singular set of the orbifold.

We calculate several terms in the asymptotic expansion explicitly in the case of two-dimensional orbifolds; we use these terms to prove that the spectrum distinguishes elements within various classes of two-dimensional orbifolds. In particular, within the class of all two-dimensional orbifolds with nonnegative Euler characteristic, the spectrum is a complete topological invariant. Additional results are obtained for triangular pillow orbifolds endowed with a hyperbolic structure, and for nonorientable two-dimensional orbifolds.

The paper is organized as follows. In Section 2, we give the background necessary for the rest of the paper, recalling several results that clarify the structure of the singular locus of an orbifold. Section 3 is devoted to the construction of the heat kernel on an arbitrary closed Riemannian orbifold by means of the construction of a parametrix. The existence of the heat kernel for closed orbifolds was shown previously by Y.-J. Chiang [8]; existence also follows from more general results for the heat kernel on Riemannian foliations (see [25]). However, we give a different construction in order to express the heat kernel in a convenient form that will allow us, in Section 4, to generalize Donnelly's asymptotic expansion. Section 5 is devoted to various applications of the heat expansion, including those mentioned in the previous paragraph.

Acknowledgments. The initial inspiration for this project came from Shunhui Zhu, and the authors thank him for sharing his curiosity. We also thank Iosif Polterovich for helpful discussions and encouragement, and Alejandro Uribe and Liz Stanhope for alerting us to the frame bundle approach. The first named author acknowledges the Centre Interfacultaire Bernoulli in Lausanne, Switzerland, and the Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Instituto Superior Técnico, in Lisbon, Portugal, for their support during her work on this project.

We thank Naveed Bari for pointing out a gap in the statement of Theorem 5.1 that appeared in the original version [14] of this article. (The gap has been corrected in this new version.)

2. ORBIFOLDS AND THEIR SINGULAR SETS

2.1. DEFINITION.

- (i) An *orbifold chart* on a topological space X consists of a connected open subset \tilde{U} of \mathbb{R}^n , a finite group G_U acting on \tilde{U} by diffeomorphisms, and a mapping π_U from \tilde{U} onto an open subset U of X inducing a homeomorphism from the orbit space $G_U \backslash \tilde{U}$ onto U . We will always assume that the group G_U acts effectively on \tilde{U} .

- (ii) An *embedding* $\lambda : (\tilde{U}, G_U, \pi_U) \rightarrow (\tilde{U}', G_{U'}, \pi_{U'})$ between orbifold charts with $U \subseteq U'$ is a smooth embedding $\lambda : \tilde{U} \rightarrow \tilde{U}'$ such that the diagram

$$\begin{array}{ccc}
 \tilde{U} & \xrightarrow{\lambda} & \tilde{U}' \\
 \pi_U \downarrow & & \downarrow \pi_{U'} \\
 G_U \backslash \tilde{U} & & G_{U'} \backslash \tilde{U}' \\
 \sim \downarrow & & \downarrow \sim \\
 U & \hookrightarrow & U'
 \end{array}$$

commutes. Two charts (\tilde{U}, G_U, π_U) and $(\tilde{U}', G_{U'}, \pi_{U'})$ on X are said to be *compatible* if, for each point $x \in U \cap U'$, there exists an orbifold chart $(\tilde{U}'', G_{U''}, \pi_{U''})$ with $U'' \subset U \cap U'$ and smooth embeddings $(\tilde{U}'', G_{U''}, \pi_{U''}) \rightarrow (\tilde{U}, G_U, \pi_U)$ and $(\tilde{U}'', G_{U''}, \pi_{U''}) \rightarrow (\tilde{U}', G_{U'}, \pi_{U'})$.

- (iii) An n -dimensional *orbifold atlas* \mathcal{A} on X is a compatible family of n -dimensional orbifold charts whose images form a covering of X . A *refinement* \mathcal{A}' of an orbifold atlas \mathcal{A} is an orbifold atlas each of whose charts embeds into a chart of \mathcal{A} . Two orbifold atlases are said to be *equivalent* if they have a common refinement. Every orbifold atlas is equivalent to a unique maximal one. An *orbifold* is a Hausdorff, second countable topological space together with a maximal orbifold atlas.
- (iv) Let \mathcal{O} be an orbifold. A point x of \mathcal{O} is said to be *singular* if for some (hence every) orbifold chart (\tilde{U}, G_U, π_U) about x , the points in the inverse image of x in \tilde{U} have nontrivial isotropy in G_U . The isomorphism class of the isotropy group, called the *abstract isotropy type* of x , is independent both of the choice of point in the inverse image of x in \tilde{U} and of the choice of chart (\tilde{U}, G_U, π_U) about x . Points that are not singular are called *regular*.

2.2. REMARKS.

- (i) The notion of orbifold generalizes slightly the notion of V -manifold introduced by Satake [27]; V -manifolds are orbifolds for which the singular set has codimension at least two.
- (ii) Given an embedding $\lambda : (\tilde{U}, G_U, \pi_U) \rightarrow (\tilde{U}', G_{U'}, \pi_{U'})$ as in Definition 2.1, there exists a homomorphism $\tau : G_U \rightarrow G_{U'}$ such that $\lambda \circ \gamma = \tau(\gamma) \circ \lambda$ for all $\gamma \in G_U$. This was proven by Satake for V -manifolds and was generalized to orbifolds by Moerdijk and Pronk [21]. From our convention that the group actions on each chart are effective, it follows that the homomorphism τ is injective.
- (iii) There are some subtle differences among the definitions of *orbifold* in the literature; in particular, some authors do not require that the finite group actions in the orbifold charts be effective. The definition we use is that in [21]. While these distinctions become significant when formulating the

correct notion of the category of smooth orbifolds, they play no role in our computations.

An orbifold is said to be *good* if it is the orbit space of a manifold under the smooth action of a discrete group; otherwise, it is said to be *bad*. In particular, every point in an orbifold has a neighborhood that is a good orbifold. Even a bad orbifold can be expressed globally as the quotient of a manifold by a group action, although not a discrete group action. This is done by introducing a Riemannian structure and constructing the “bundle” of orthonormal frames. The orthonormal frame bundle is actually an *orbibundle*, the appropriate notion of vector bundle over an orbifold as described in [20] or [31]. However, its total space is a smooth manifold with an action of the orthogonal group, and one recovers the original orbifold as the orbit space of this orthogonal action. We will review this construction below after first establishing some notation in the setting of arbitrary group actions.

2.3. DEFINITION. (See [15, Chapter 2].)

- (i) Consider a smooth proper action of a Lie group H on a smooth manifold M . For $\tilde{x} \in M$, let $\text{Iso}_H(\tilde{x})$ denote the subgroup of H that fixes \tilde{x} . Define an equivalence relation on M by $\tilde{x} \equiv \tilde{y}$ if $\text{Iso}_H(\tilde{x})$ and $\text{Iso}_H(\tilde{y})$ are conjugate. Each equivalence class is called an *H-orbit type*. Note that the equivalence classes are invariant under the action of H .

We will say that the H -orbit type of \tilde{x} *dominates* that of \tilde{y} if $\text{Iso}_H(\tilde{x})$ is conjugate to a subgroup of $\text{Iso}_H(\tilde{y})$.

- (ii) Let $\pi : M \rightarrow H \backslash M$ be the projection onto the orbit space. Let $p \in H \backslash M$. As \tilde{p} ranges over the H -orbit $\pi^{-1}(p)$ in M , the stabilizer $\text{Iso}_H(\tilde{p})$ ranges over a conjugacy class of subgroups of H . We will denote this conjugacy class of subgroups by $\text{Iso}_H(p)$ and refer to it as the *H-isotropy type* of p . Define an equivalence relation on $H \backslash M$ by $p \equiv q$ if $\text{Iso}_H(p) = \text{Iso}_H(q)$. The equivalence classes will be called *H-isotropy equivalence classes*. Note that π carries points of the same H -orbit type in M to points of the same H -isotropy equivalence class in $H \backslash M$.

We will say that the H -isotropy equivalence class of p *dominates* that of q if the groups making up the conjugacy class $\text{Iso}_H(p)$ are conjugate to subgroups of those in $\text{Iso}_H(q)$.

By an abuse of notation, we will write $|\text{Iso}_H(p)|$ to mean the order of each of the groups making up the isotropy type $\text{Iso}_H(p)$. We will refer to this quantity as the order of the H -isotropy at p .

2.4. DEFINITION.

- (i) A *Riemannian structure* on an orbifold \mathcal{O} is an assignment to each orbifold chart (\tilde{U}, G_U, π_U) of a G_U -invariant Riemannian metric $g_{\tilde{U}}$ on \tilde{U} satisfying the compatibility condition that each embedding λ appearing in Definition 2.1 is isometric. Every orbifold admits Riemannian structures.
- (ii) We will say that an orbifold chart (\tilde{U}, G_U, π_U) on a Riemannian orbifold \mathcal{O} is a *distinguished* chart of radius r if \tilde{U} is a convex geodesic ball of radius r . In this case, U is a convex geodesic ball in \mathcal{O} . The entire group

G_U fixes the center \tilde{p} of \tilde{U} , so the abstract isotropy type of $p := \pi_U(\tilde{p})$ is represented by G_U .

2.5. REMARK. Recall that for a Riemannian manifold M and a point $p \in M$, the *convexity radius at p* is the largest positive real number $r(p)$ for which the geodesic ball of radius ϵ about p is geodesically convex for all $\epsilon < r(p)$. If M is compact, the infimum r of $\{r(p) : p \in M\}$ is positive and is called the *convexity radius* of M . For a point p in an orbifold \mathcal{O} , we may define the convexity radius at p to be the largest real number $r(p)$ such that \mathcal{O} admits a distinguished chart of radius ϵ centered at p for all $\epsilon < r(p)$. It is immediate that $r(p)$ is positive. Moreover, if \mathcal{O} is compact, then an elementary argument shows that the infimum r of $\{r(p) : p \in \mathcal{O}\}$ is positive; r is called the *convexity radius* of \mathcal{O} .

2.6. ORTHONORMAL FRAME BUNDLE. We give a brief description of the orthonormal frame bundle of a Riemannian orbifold. See [1] for more details. First consider a good Riemannian orbifold $\mathcal{O} = G \backslash M$, where M is a Riemannian manifold and G is a discrete group acting by isometries on M . Let $F(M) \rightarrow M$ be the orthonormal frame bundle of M . Each element $\gamma \in G$, being an isometry of M , induces a diffeomorphism γ_* of $F(M)$ carrying fibers to fibers; thus we obtain an action of G on $F(M)$ covering the action of G on M . The orthonormal frame bundle $F(\mathcal{O})$ of \mathcal{O} is defined to be $G \backslash F(M) \rightarrow \mathcal{O}$. The *fiber* of $F(\mathcal{O}) \rightarrow \mathcal{O}$ over $x \in \mathcal{O}$ is the preimage of x in $G \backslash F(M)$. The right action of $O(n)$ on $F(M)$ commutes with the left action of G , and hence descends to a right $O(n)$ -action on $F(\mathcal{O})$. For a bad orbifold, the orthonormal frame bundle is defined in such a way that its restriction to any good neighborhood $U \cong G_U \backslash \tilde{U}$ is the orthonormal frame bundle of the good orbifold U .

The orthonormal frame bundle of \mathcal{O} is a smooth manifold as well as an orbifold on which the orthogonal group $O(n)$ acts smoothly on the right, preserving fibers. In particular, the orbifold \mathcal{O} is the orbit space $F(\mathcal{O})/O(n)$ of the right action of $O(n)$ on the manifold $F(\mathcal{O})$.

2.7. NOTATION AND REMARKS. Let \mathcal{O} be an orbifold. Endow \mathcal{O} with a Riemannian metric and let $F(\mathcal{O})$ be the associated orthonormal frame bundle as in 2.6. The $O(n)$ -action on the fiber of $F(\mathcal{O})$ over a point $x \in \mathcal{O}$ is free if and only if x is a regular point of \mathcal{O} . In particular, for each singular point x of \mathcal{O} , viewed as an element of $F(\mathcal{O})/O(n)$, the $O(n)$ -isotropy type of x is a non-trivial conjugacy class $\text{Iso}_{O(n)}(x)$ of subgroups of $O(n)$. (See Definition 2.3.)

Our realization of the orbifold \mathcal{O} as a global quotient of a manifold (namely $F(\mathcal{O})$) by an action of $O(n)$ depends, of course, on the Riemannian metric. However, it is not difficult to show that the conjugacy class $\text{Iso}_{O(n)}(x)$ of subgroups of $O(n)$ is actually independent of the choice of Riemannian metric used in the construction. This conjugacy class of subgroups of $O(n)$ will henceforth be denoted $\text{Iso}(x)$ (without the subscript $O(n)$ except when needed for clarity) and will be referred to as the *isotropy type* of the singular point x of \mathcal{O} . Its cardinality $|\text{Iso}(x)|$ will be called the *order* of the isotropy at x . Similarly, the equivalence classes of elements of \mathcal{O} with the same isotropy type will be called *isotropy equivalence classes*, without mention of $O(n)$.

The subgroups of $O(n)$ in the conjugacy class $\text{Iso}(x)$ lie in the isomorphism class defined by the abstract isotropy type of x given in Definition 2.1. Indeed, let (\tilde{U}, G_U, π_U) be an orbifold chart with $x \in U$. Let $\tilde{x} \in \tilde{U}$ with $\pi_U(\tilde{x}) = x$. With respect to any choice of Riemannian metric on \mathcal{O} (and associated Riemannian metric on \tilde{U}), the group G_U acts isometrically on \tilde{U} and thus acts on the left on the orthonormal frame bundle $F(\tilde{U})$. The subgroup $\text{Iso}_{G_U}(\tilde{x})$ leaves invariant the fiber of $F(\tilde{U})$ over \tilde{x} . For each q in this fiber, define a homomorphism $\sigma_q : \text{Iso}_{G_U}(\tilde{x}) \rightarrow O(n)$ by the condition $\gamma(q) = (q)\sigma_q(\gamma)$ where $\gamma(\cdot)$ and $(\cdot)\sigma_q(\gamma)$ denote the left action of γ and right action of $\sigma_q(\gamma) \in O(n)$ on the fiber. By 2.6, the restriction of $F(\mathcal{O})$ to U is given by $G_U \backslash F(\tilde{U})$. Letting $\rho : F(\tilde{U}) \rightarrow G_U \backslash F(\tilde{U})$ be the projection, then σ_q maps $\text{Iso}_{G_U}(\tilde{x})$ isomorphically to the stabilizer of $\rho(q)$ in $O(n)$. This stabilizer is a representative of the conjugacy class $\text{Iso}(x)$, while $\text{Iso}_{G_U}(\tilde{x})$ represents the abstract isotropy type of x .

2.8. DEFINITION. A smooth **stratification** of a manifold or orbifold M is a locally finite partition of M into locally closed submanifolds, called the *strata*, satisfying the following condition: For each stratum N , the closure of N is the union of N with a collection of lower dimensional strata.

2.9. REMARKS.

- (i) For any stratification of an orbifold (or manifold) \mathcal{O} , the strata of maximal dimension are open in \mathcal{O} and their union has full measure in \mathcal{O} .
- (ii) The stratifications that we will discuss below are Whitney stratifications. As we will not explicitly use the additional properties of Whitney stratifications here, we omit the definition and refer the reader to [15]. The notion of Whitney stratification can be defined in the more general setting of spaces that can be at least locally embedded in a smooth manifold. As discussed in [15], the orbit space of a proper Lie group action on a smooth manifold has this property.

2.10. PROPOSITION. [15] *Given a smooth action of a Lie group H on a manifold M , then:*

- (i) *The connected components of the H -orbit types form a Whitney stratification of M . The closure of a stratum \tilde{N} is made up of the union of \tilde{N} with a collection of lower dimensional strata, each lying in an H -orbit type strictly dominated by that of \tilde{N} .*
- (ii) *The connected components of the H -isotropy equivalence classes in $H \backslash M$ form a Whitney stratification of $H \backslash M$. The closure of a stratum N is made up of the union of N with a collection of lower dimensional strata, each lying in an H -isotropy equivalence class strictly dominated by that of N . The map π carries each stratum in M onto a stratum in $H \backslash M$.*
- (iii) *For $x \in H \backslash M$ and N the stratum through x , there exists a neighborhood U of x in $H \backslash M$ such that the isotropy equivalence class of each element of the complement of N in U strictly dominates that of x .*
- (iv) *If M is compact, then the stratifications of M and of $H \backslash M$ are finite.*

2.11. COROLLARY. *Let \mathcal{O} be an orbifold. Then the action of $O(n)$ on the frame bundle $F(\mathcal{O})$ gives rise to a (Whitney) stratification of \mathcal{O} . The strata are connected components of the isotropy equivalence classes in \mathcal{O} . The set of regular points of \mathcal{O} intersects each connected component \mathcal{O}_0 of \mathcal{O} in a single stratum comprising an open dense submanifold of \mathcal{O}_0 .*

2.12. NOTATION.

- (i) We will refer to the strata of \mathcal{O} in Corollary 2.11 as \mathcal{O} -strata.
- (ii) If (\tilde{U}, G_U, π_U) is an orbifold chart on \mathcal{O} , then the action of G_U on \tilde{U} gives rise to stratifications both of \tilde{U} and of U as in Proposition 2.10. We will refer to these as \tilde{U} -strata and U -strata, respectively.

2.13. PROPOSITION. *Let \mathcal{O} be a Riemannian orbifold and (\tilde{U}, G_U, π_U) be an orbifold chart. Then:*

- (i) *The U -strata are precisely the connected components of the intersections of the \mathcal{O} -strata with U .*
- (ii) *Any two elements of the same \tilde{U} -stratum have the same stabilizers in G_U (not just conjugate stabilizers).*
- (iii) *If H is a subgroup of G_U , then each connected component W of the fixed point set $\text{Fix}(H)$ of H in \tilde{U} is a closed submanifold of \tilde{U} . Any \tilde{U} -stratum that intersects W nontrivially lies entirely in W . Thus the stratification of \tilde{U} restricts to a stratification of W .*

Proof. (i) A consequence of Proposition 2.10 is that each U -stratum, respectively \mathcal{O} -stratum, is a connected component of the set of all points in U , respectively \mathcal{O} , having G_U -isotropy, respectively $O(n)$ -isotropy, of a given order. Since by 2.7, the order of the G_U -isotropy of each $x \in U$ is equal to the order of the $O(n)$ -isotropy of x , statement (i) follows.

(ii) This follows because G_U is discrete and the \tilde{U} -strata are connected.

(iii) The first statement is true for the fixed point set of any smooth proper action by a compact group on a manifold [15]. The second statement follows from (ii). \square

2.14. NOTATION AND REMARKS. Let \mathcal{O} be a Riemannian orbifold and (\tilde{U}, G_U, π_U) be an orbifold chart. Let \tilde{N} be a \tilde{U} -stratum in \tilde{U} . By Proposition 2.13, all the points in \tilde{N} have the same isotropy group in G_U ; we will refer to this group as the isotropy group of \tilde{N} , denoted $\text{Iso}(\tilde{N})$.

Given a \tilde{U} -stratum \tilde{N} , denote by $\text{Iso}^{\max}(\tilde{N})$ the set of all $\gamma \in \text{Iso}(\tilde{N})$ such that \tilde{N} is open in the fixed point set $\text{Fix}(\gamma)$ of γ . We will say that \tilde{N} is a *singular stratum* if $\text{Iso}(\tilde{N})$ is non-trivial and that it is a *primary singular stratum* if $\text{Iso}^{\max}(\tilde{N})$ is non-trivial.

For $\gamma \in G_U$, Proposition 2.13 tells us that each component W of the fixed point set $\text{Fix}(\gamma)$ of γ (equivalently, the fixed point set of the cyclic group generated by γ) is a manifold stratified by a collection of \tilde{U} -strata. By Remark 2.9(i), the strata in W of maximal dimension are open and their union has full measure in

W . In particular, the union of those \tilde{U} -strata \tilde{N} for which $\gamma \in \text{Iso}^{\max}(\tilde{N})$ has full measure in $\text{Fix}(\gamma)$.

We will similarly say that a stratum of \mathcal{O} is *singular* if it lies in the singular set of \mathcal{O} . Let N be a singular stratum, let (\tilde{U}, G_U, π_U) be an orbifold chart whose image intersects N non-trivially, and let \tilde{N} be a stratum in \tilde{U} with $\pi_U(\tilde{N}) \subset N$. We define $\text{Iso}^{\max}(N)$ to be the isomorphism class of $\text{Iso}^{\max}(\tilde{N})$ and say that N is *primary* if $\text{Iso}^{\max}(N)$ is non-trivial. One easily checks that this definition is independent of the choice of chart.

2.15. EXAMPLE. On \mathbb{R}^2 , let r_x and r_y denote the reflections across the x -axis and y -axis, respectively, and let r_0 denote the rotation through angle π about the origin 0. Then $G := \{r_x, r_y, r_0, Id\}$ is a Klein four group acting isometrically on \mathbb{R}^2 . The quotient of \mathbb{R}^2 by the semi-direct product of G with the lattice \mathbb{Z}^2 of translations is a closed orbifold \mathcal{O} , whose underlying space is a square of side length $\frac{1}{2}$. The points on the boundary of the square are singular points (not boundary points) of the orbifold, comprising eight **strata**: each corner point forms a single stratum with **isotropy of order four**, while each open edge forms a stratum with isotropy of order two. The strata of codimension one are called *reflectors* or *mirrors*, and the single point strata are called *dihedral points* or *corner reflectors*. The intersection U of the square with a disk of radius less than $\frac{1}{2}$ centered at one of the corners is the image of an orbifold chart (\tilde{U}, G, π_U) where \tilde{U} is a disk in \mathbb{R}^2 centered at the origin and G is the Klein four group above. The \tilde{U} -strata of this action consist of the single point 0 and the intersections of the disk \tilde{U} with the positive and negative x -axis and the positive and negative y -axis. If \tilde{N} is the intersection of \tilde{U} with one of the half axes, then $\text{Iso}(\tilde{N})$ consists of a reflection and the identity, while $\text{Iso}^{\max}(\tilde{N})$ contains only the reflection. For $\tilde{N} = \{0\}$, we have $\text{Iso}(\tilde{N}) = G$, but $\text{Iso}^{\max}(\tilde{N}) = \{r_0\}$.

2.16. EXAMPLE. Let r_x, r_y , and r_z denote the rotation of \mathbb{R}^3 through angle π about the x -, y -, or z -axis, respectively. Then $G := \{r_x, r_y, r_z, Id\}$ is a Klein four group acting isometrically on \mathbb{R}^3 . It is convenient to view the nontrivial elements of G as diagonal matrices, with

$$r_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad r_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \text{and } r_z = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The quotient of \mathbb{R}^3 by G is an orbifold. The axes project to primary 1-dimensional strata and the origin projects to a 0-dimensional stratum for which $\text{Iso}^{\max}(N)$ is trivial.

3. CONSTRUCTION OF THE HEAT EXPANSION

In this section, we address the heat kernel on closed Riemannian orbifolds.

3.1. PROPOSITION. *Let \mathcal{O} be a closed Riemannian orbifold. The Laplacian Δ of \mathcal{O} has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \dots$, with each eigenvalue having finite multiplicity. The normalized eigenfunctions φ_j are C^∞ and form an orthonormal basis of $L^2(\mathcal{O})$.*

This proposition was proved in the case of V -manifolds (as defined in Remark 2.2) by Y.-J. Chiang in [8]. For an orbifold \mathcal{O} which is not a V -manifold, those strata of the singular set of codimension one are called *reflectors*. By doubling along all reflectors one obtains a V -manifold X that doubly covers \mathcal{O} . Thus \mathcal{O} is the quotient of a V -manifold X by a \mathbb{Z}_2 action. Since the eigenfunctions on \mathcal{O} are then the \mathbb{Z}_2 -invariant eigenfunctions on X , Proposition 3.1 follows immediately.

3.2. DEFINITION. Set

$$\mathbb{R}_+ = [0, \infty)$$

and

$$\mathbb{R}_+^* = (0, \infty).$$

We say that $K : \mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$ is a *fundamental solution* of the heat equation, or *heat kernel*, if it satisfies:

- (i) K is C^0 in the three variables, C^1 in the first, and C^2 in the second;
- (ii) $(\frac{\partial}{\partial t} + \Delta_x)K(t, x, y) = 0$ where Δ_x is the Laplacian with respect to the second variable;
- (iii) $\lim_{t \rightarrow 0^+} K(t, x, \cdot) = \delta_x$ for all $x \in \mathcal{O}$.

By the same argument as in the manifold case (see [2], III.E.2), Proposition 3.1 implies:

3.3. COROLLARY. *If a heat kernel exists, then it is unique and is given by*

$$K(t, x, y) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y).$$

Chiang proved the existence of the heat kernel on a compact V -manifold (from which existence on an arbitrary closed orbifold trivially follows) by proving the convergence of $\sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y)$. She also showed that the heat kernel can be approximated on good neighborhoods by the Dirichlet heat kernel on the local manifold covering. The existence also follows from more general results on existence of the heat kernel for the basic Laplacian on Riemannian foliations [23]. However, in order to apply Donnelly's results on the heat trace for good orbifolds to obtain the terms in the asymptotic expansion of arbitrary orbifolds in an applicable form, we will not assume the earlier existence results for the heat kernel or heat trace. We instead construct a parametrix and then follow the standard construction of the heat kernel from the parametrix as in [2]. Our construction of the parametrix and consequently the heat kernel will use directly the local structure of orbifolds as quotients of manifolds by finite group actions.

3.4. DEFINITION. A *parametrix* for the heat operator on \mathcal{O} is a function $H : \mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$ satisfying:

- (i) $H \in C^\infty(\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O})$;
- (ii) $(\frac{\partial}{\partial t} + \Delta_x)H(t, x, y)$ extends to a function in $C^0(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$;
- (iii) $\lim_{t \rightarrow 0^+} H(t, x, \cdot) = \delta_x$ for all $x \in \mathcal{O}$.

Recall that the heat kernel on a closed n -dimensional Riemannian manifold M has an asymptotic expansion along the diagonal in $M \times M$ as $t \rightarrow 0^+$ of the form

$$(3.5) \quad K(t, x, x) \sim (4\pi t)^{-\frac{n}{2}} (u_0(x, x) + tu_1(x, x) + t^2 u_2(x, x) + \dots)$$

where the u_i are local Riemannian invariants defined in a neighborhood of the diagonal in $M \times M$. Letting ζ be a cut-off function that is identically one near the diagonal, then for $m > \frac{n}{2}$, the function

$$(3.6) \quad K^{(m)}(t, x, y) = \zeta(x, y) (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(x, y)^2}{4t}} (u_0(x, y) + \dots + t^m u_m(x, y))$$

is a parametrix for the heat operator on M .

3.7. REMARK. In what follows, we shall take a local covering of our orbifold \mathcal{O} by distinguished charts and piece together a parametrix for the heat operator on \mathcal{O} from the expressions in (3.6). The key to piecing together the parametrix on \mathcal{O} is to note that while the parametrix $K^{(m)}$ in (3.6) is globally defined on M , the three defining conditions of a parametrix in Definition 3.4 are satisfied locally as well as globally by $K^{(m)}$. Indeed the first two conditions are trivially local.

The third condition is local in the following sense: The expression $e^{-\frac{d(x, y)^2}{4t}}$ goes to zero uniformly as $t \rightarrow 0^+$ when $d(x, y)$ is bounded away from zero. Thus for $f \in C^0(M)$, we have

$$f(x) = \lim_{t \rightarrow 0^+} \int_M K^{(m)}(t, x, y) f(y) dy = \lim_{t \rightarrow 0^+} \int_W K^{(m)}(t, x, y) f(y) dy,$$

where W is any neighborhood of x .

We will also use the fact (see [2]) that if $m > \frac{n}{2} + 2l$, then $(\frac{\partial}{\partial t} + \Delta_x)K^{(m)}(t, x, y)$ extends to a function in $C^l(\mathbb{R}_+ \times M \times M)$. Moreover, the extension is of class C^{2l} in the last two variables.

3.8. NOTATION. Let \mathcal{O} be an orbifold of dimension n . Fix $\epsilon > 0$ so that for each $x \in \mathcal{O}$, there exists a distinguished coordinate chart of radius ϵ centered at x . Cover \mathcal{O} with finitely many such charts $(\widetilde{W}_\alpha, G_\alpha, \pi_\alpha)$, $\alpha = 1, \dots, s$. (Here we write G_α for G_{W_α} and π_α for π_{W_α} .) Let p_α be the center of W_α and \tilde{p}_α the center of \widetilde{W}_α . Let U_α , respectively V_α , be the geodesic ball of radius $\frac{\epsilon}{4}$, respectively $\frac{\epsilon}{2}$, centered at p_α , and let \tilde{U}_α and \tilde{V}_α be the corresponding balls centered at \tilde{p}_α in \widetilde{W}_α . We may assume that the family of balls $\{U_\alpha\}_{1 \leq \alpha \leq s}$ still covers \mathcal{O} .

For each α and each nonnegative integer m , we define $\tilde{H}_\alpha^{(m)} : \mathbb{R}_+^* \times \widetilde{W}_\alpha \times \widetilde{W}_\alpha$ by

$$\tilde{H}_\alpha^{(m)}(t, \tilde{x}, \tilde{y}) = (4\pi t)^{-n/2} e^{-d(\tilde{x}, \tilde{y})^2/4t} (u_0(\tilde{x}, \tilde{y}) + \dots + t^m u_m(\tilde{x}, \tilde{y}))$$

where the u_i are the invariants in (3.5). Since each $\gamma \in G_\alpha$ is an isometry of \widetilde{W}_α , we have $u_i(\gamma\tilde{x}, \gamma\tilde{y}) = u_i(\tilde{x}, \tilde{y})$ for all $\tilde{x}, \tilde{y} \in \widetilde{W}_\alpha$. It follows that the function

$$(t, \tilde{x}, \tilde{y}) \mapsto \sum_{\gamma \in G_\alpha} \widetilde{H}_\alpha^{(m)}(t, \tilde{x}, \gamma\tilde{y})$$

is G_α -invariant in both \tilde{x} and \tilde{y} and thus descends to a well-defined function, which we denote by $H_\alpha^{(m)}$, on $\mathbb{R}_+^* \times W_\alpha \times W_\alpha$.

Let $\psi_\alpha : \mathcal{O} \rightarrow \mathbb{R}$ be a C^∞ cut-off function, which is identically one on V_α and is supported in W_α . Let $\{\eta_\alpha : \alpha = 1, \dots, s\}$ be a partition of unity on \mathcal{O} with the support of η_α contained in \overline{U}_α . Define $H^{(m)} : \mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$ by

$$(3.9) \quad H^{(m)}(t, x, y) = \sum_{\alpha=1}^s \psi_\alpha(x) \eta_\alpha(y) H_\alpha^{(m)}(t, x, y).$$

We will show that $H^{(m)}$ is a parametrix for the heat kernel on \mathcal{O} when $m > \frac{n}{2}$.

3.10. LEMMA. $H^{(m)} \in C^\infty(\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O})$.

Lemma 3.10 is immediate.

3.11. LEMMA. *Let l be a nonnegative integer. Then*

- (i) $\left(\frac{\partial}{\partial t} + \Delta_x\right)H^{(m)}(t, x, y)$ extends to a function in $C^l(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$ if $m > \frac{n}{2} + 2l$. (It is moreover of class C^{2l} in the last two variables.)
- (ii) For any given $T > 0$ and for each $m > \frac{n}{2}$, there exists a constant A such that $\left|\left(\frac{\partial}{\partial t} + \Delta_x\right)H^{(m)}(t, x, y)\right| < At^{m-\frac{n}{2}}$ when $0 < t < T$.

Proof. (i) Let $l \geq 0$ and suppose $m > \frac{n}{2} + 2l$. By Remark 3.7, the function $\left(\frac{\partial}{\partial t} + \tilde{\Delta}_x\right)\widetilde{H}_\alpha^{(m)}(t, \tilde{x}, \tilde{y})$ on $\mathbb{R}_+^* \times \widetilde{W}_\alpha \times \widetilde{W}_\alpha$ extends to $C^l(\mathbb{R}_+ \times \widetilde{W}_\alpha \times \widetilde{W}_\alpha)$. (Here, we are using the notation $\tilde{\Delta}$ for the Laplacian on \widetilde{W}_α for all choices of α .) Thus the same is true for $\sum_{\gamma \in G_\alpha} \left(\frac{\partial}{\partial t} + \tilde{\Delta}_x\right)\widetilde{H}_\alpha^{(m)}(t, \tilde{x}, \gamma\tilde{y})$, and hence it follows that $\left(\frac{\partial}{\partial t} + \Delta_x\right)H_\alpha^{(m)}(t, x, y)$ extends to $C^l(\mathbb{R}_+ \times W_\alpha \times W_\alpha)$.

Now consider the function

$$f_\alpha(t, x, y) := \left(\frac{\partial}{\partial t} + \Delta_x\right)(\psi_\alpha(x) \eta_\alpha(y) H_\alpha^{(m)}(t, x, y)).$$

Noting that ψ_α and η_α are compactly supported inside W_α , we may view f_α as a function on $\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O}$ which is zero whenever x or y lies outside of W_α . We show f_α extends to $C^l(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$. Since $\psi_\alpha \equiv 1$ on V_α , it follows immediately from the previous paragraph that f_α extends to $C^l(\mathbb{R}_+ \times V_\alpha \times \mathcal{O})$. Moreover $f_\alpha \equiv 0$ on $\mathbb{R}_+^* \times \mathcal{O} \times (\mathcal{O} \setminus U_\alpha)$ and so f_α also extends smoothly (to zero) on $\mathbb{R}_+ \times \mathcal{O} \times (\mathcal{O} \setminus U_\alpha)$. Finally for $(x, y) \in (\mathcal{O} \setminus V_\alpha) \times \overline{U}_\alpha$, we have $d(x, y) \geq \frac{\epsilon}{4}$. Thus, as $t \rightarrow 0^+$, $f_\alpha(t, x, y)$ and all its derivatives converge to zero uniformly for $(x, y) \in (\mathcal{O} \setminus V_\alpha) \times \overline{U}_\alpha$. Thus $H^{(m)}$ extends to a function in $C^l(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$. The parenthetical statement in (1) similarly follows from Remark 3.7.

(ii) The u_i are constructed so that

$$\left(\frac{\partial}{\partial t} + \tilde{\Delta}_{\tilde{x}}\right) \tilde{H}_{\alpha}^{(m)}(t, \tilde{x}, \tilde{y}) = (4\pi t)^{-n/2} e^{-d(\tilde{x}, \tilde{y})^2/4t} t^m \tilde{\Delta}_{\tilde{x}} u_m(t, \tilde{x}, \tilde{y})$$

(see [2]). Since the u_i are C^∞ functions, it follows that there exists a constant B_α such that

$$\left(\frac{\partial}{\partial t} + \tilde{\Delta}_{\tilde{x}}\right) \tilde{H}_{\alpha}^{(m)}(t, \tilde{x}, \tilde{y}) < B_\alpha t^{m-\frac{n}{2}}$$

on $(0, T] \times \tilde{W}_\alpha \times \tilde{W}_\alpha$. Consequently,

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) H_{\alpha}^{(m)}(t, x, y) < |G_\alpha| B_\alpha t^{m-\frac{n}{2}}$$

on $(0, T] \times W_\alpha \times W_\alpha$ and

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) (\psi_\alpha(x) \eta_\alpha(y) H_{\alpha}^{(m)}(t, x, y)) < |G_\alpha| B_\alpha t^{m-\frac{n}{2}}$$

on $(0, T] \times V_\alpha \times \mathcal{O}$, since $\psi_\alpha \equiv 1$ on V_α . Once again, for x outside of V_α and y in the support of η_α , we have $d(x, y) \geq \frac{\epsilon}{4}$, and thus $(\frac{\partial}{\partial t} + \Delta_x)(\psi_\alpha(x) \eta_\alpha(y) H_{\alpha}^{(m)}(t, x, y))$ can be bounded in terms of any power of t on $(0, T] \times (\mathcal{O} \setminus V_\alpha) \times \mathcal{O}$.

Statement (ii) now follows from (3.9). \square

3.12. REMARK. When $m > \frac{n}{2} + 2l$, one obtains bounds on the partial derivatives of order at most l of $(\frac{\partial}{\partial t} + \Delta_x) H_{\alpha}^{(m)}(t, x, y)$ by an argument analogous to that used in the proof of Lemma 3.11(ii).

3.13. LEMMA. Let m be any nonnegative integer. For $f \in C^\infty(\mathcal{O})$ and $x \in \mathcal{O}$, we have

$$\lim_{t \rightarrow 0^+} \int_{\mathcal{O}} H^{(m)}(t, x, y) f(y) dy = f(x).$$

I.e., $\lim_{t \rightarrow 0^+} H^{(m)}(t, x, y) = \delta_x(y)$. Moreover, if N is a topological space and f is a continuous function on $N \times \mathcal{O}$, then the convergence of $\int_{\mathcal{O}} H^{(m)}(t, x, y) f(p, y) dy$ to $f(p, x)$ is locally uniform on $N \times \mathcal{O}$.

Proof. Let $\tilde{\psi}_\alpha$ and $\tilde{\eta}_\alpha$ be the lifts of ψ_α and η_α to \tilde{W}_α .

Since $\text{supp}(\eta_\alpha) \subset \bar{U}_\alpha \subset W_\alpha$, we have

$$\begin{aligned} & \int_{\mathcal{O}} \psi_\alpha(x) \eta_\alpha(y) H_{\alpha}^{(m)}(t, x, y) f(p, y) dy \\ (3.14) \quad &= \frac{\psi_\alpha(x)}{|G_\alpha|} \sum_{\gamma \in G_\alpha} \int_{\tilde{W}_\alpha} \tilde{H}_{\alpha}^{(m)}(t, \tilde{x}, \gamma \tilde{y}) \tilde{\eta}_\alpha(\tilde{y}) \tilde{f}_\alpha(p, \tilde{y}) d\tilde{y} \end{aligned}$$

where \tilde{f}_α is a lift of $f|_{N \times W_\alpha}$ to $N \times \tilde{W}_\alpha$ and \tilde{x} is an arbitrarily chosen point in the preimage of x under the map $\tilde{W}_\alpha \rightarrow W_\alpha$.

We change variables in each of the integrals in the right side of (3.14), letting $\tilde{u} = \gamma(\tilde{y})$. Since γ is an isometry and since $\tilde{\eta}_\alpha$ and $\tilde{f}_\alpha(p, \cdot)$ are γ -invariant, each integral in the summand is equal to

$$(3.15) \quad \int_{\tilde{W}_\alpha} \tilde{H}_\alpha^{(m)}(t, \tilde{x}, \tilde{u}) \tilde{\eta}_\alpha(\tilde{u}) \tilde{f}_\alpha(p, \tilde{u}) d\tilde{u}.$$

As $t \rightarrow 0^+$, the integral (3.15) above converges to $\tilde{\eta}_\alpha(\tilde{x}) \tilde{f}_\alpha(p, \tilde{x}) = \eta_\alpha(x) f(p, x)$ (see Remark 3.7). Moreover, this convergence is locally uniform on $N \times \tilde{W}_\alpha$ (see [2]). Noting that $\psi_\alpha \equiv 1$ on the support of η_α , it follows that both sides of (3.14) converge to $\eta_\alpha(x) f(p, x)$ as $t \rightarrow 0^+$, and the convergence is locally uniform on $N \times W_\alpha$. Since both sides of (3.14) are identically zero when x lies outside of $\text{supp}(\psi_\alpha) \subset W_\alpha$, we thus have locally uniform convergence to $\eta_\alpha(x) f(p, x)$ on all of $N \times \mathcal{O}$.

Finally it follows from (3.9) that

$$\lim_{t \rightarrow 0^+} \int_{\mathcal{O}} H^{(m)}(t, x, y) f(p, y) dy = \sum_{\alpha} \eta_\alpha(x) f(p, x) = f(p, x)$$

locally uniformly. \square

3.16. PROPOSITION. $H^{(m)}$ is a parametrix for the heat operator on \mathcal{O} if $m > \frac{n}{2}$.

Proof. Immediate from Lemmas 3.10, 3.11(i), and 3.13. \square

The construction of the heat kernel from the parametrix $H^{(m)}$ follows exactly as in [2]. We give only a brief summary.

3.17. NOTATION. For $A, B \in C^0(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$, define the convolution $A * B \in C^0(\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O})$ by

$$A * B(t, x, z) = \int_0^t d\theta \int_{\mathcal{O}} A(t - \theta, x, y) B(\theta, y, z) dy.$$

Note that the convolution operator $*$ is associative.

3.18. LEMMA. Let l be any nonnegative integer, and let $m > \frac{n}{2} + 2l$. Define $F_m(t, x, y) = (\frac{\partial}{\partial t} + \Delta_x) H^{(m)}(t, x, y)$. (See Lemma 3.11 for regularity properties of F_m .) Then for each $T > 0$, the series $\sum_{j=1}^{\infty} (-1)^{j+1} F_m^{*j}(t, x, y)$ converges uniformly on $[0, T] \times \mathcal{O} \times \mathcal{O}$. Let $Q_m : \mathbb{R}_+ \times \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$ be the sum of this series. Then $Q_m \in C^l(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$. Moreover, for any $T > 0$, there exists a constant C such that

$$|Q_m(t, x, y)| \leq C t^{m - \frac{n}{2}}$$

on $[0, T] \times \mathcal{O} \times \mathcal{O}$.

The proof of Lemma 3.18 is identical to that of Lemma E.III.6 of [2] and uses only Lemma 3.11(ii) and Remark 3.12.

3.19. LEMMA. Let $m > \frac{n}{2}$. For $P \in C^0(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$, the function $H^{(m)} * P$, defined formally by the expression in Notation 3.17, exists and is in $C^0(\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O})$. Moreover, if $m > \frac{n}{2} + l$, then $H^{(m)} * P$ is of class C^l in the second variable. For $m > \frac{n}{2} + 2$, $(\frac{\partial}{\partial t} + \Delta_x)(H^{(m)} * P(t, x, y))$ exists and equals $(P + H^{(m)} * P)(t, x, y)$.

Again the proof is identical to that of Lemma E.III.7 of [2] and is based on Lemma 3.13.

Using Lemmas 3.18 and 3.19, we obtain as in Proposition E.III.8 of [2] that:

3.20. PROPOSITION. *Let $m > \frac{n}{2} + 2$ and define Q_m as in Lemma 3.18. Then $K := H^{(m)} - H^{(m)} * Q_m$ is a fundamental solution of the heat equation on \mathcal{O} .*

Note that uniqueness of the heat kernel implies that $H^{(m)} - H^{(m)} * Q_m$ is independent of the choice of $m > \frac{n}{2} + 2$.

3.21. NOTATION. Let

$$\tilde{H}_\alpha(t, \tilde{x}, \tilde{y}) = \sum_{\gamma \in G_\alpha} (4\pi t)^{-n/2} e^{-d(\tilde{x}, \gamma(\tilde{y}))^2/4t} (u_0(\tilde{x}, \gamma(\tilde{y})) + tu_1(\tilde{x}, \gamma(\tilde{y})) + \dots).$$

Observe that \tilde{H}_α is G_α -invariant in both \tilde{x} and \tilde{y} and thus descends to a well-defined function, which we denote by H_α , on $\mathbb{R}_+^* \times W_\alpha \times W_\alpha$.

3.22. THEOREM. *In the notation of Proposition 3.1 and 3.21, the trace of the heat kernel has an asymptotic expansion as $t \rightarrow 0^+$ given by*

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \sim_{t \rightarrow 0^+} \sum_{\alpha=1}^s \int_{\mathcal{O}} \eta_\alpha(x) H_\alpha(t, x, x) dx.$$

Proof. By Corollary 3.3 and Proposition 3.20, we have

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_{\mathcal{O}} (H^{(m)} - H^{(m)} * Q_m)(t, x, x) dx$$

for $m > \frac{n}{2} + 2$. By Lemma 3.18 and the fact that $(4\pi t)^{\frac{n}{2}} H^{(m)}(t, x, x)$ is uniformly bounded for $(t, x) \in (0, T] \times \mathcal{O}$ for any given $T > 0$, it follows that

$$(3.23) \quad \sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_{\mathcal{O}} H^{(m)}(t, x, x) dx + O(t^{m-n})$$

on any interval $(0, T]$. (Aside: When \mathcal{O} is a manifold, then $\int_{\mathcal{O}} H^{(m)}(t, x, x) dx = (4\pi t)^{-\frac{n}{2}} (a_0 + a_1 t + \dots + a_m t^m)$. For general orbifolds, the arguments in the next section will show that $\int_{\mathcal{O}} H^{(m)}(t, x, x) dx$ is of the form $(4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{2m} c_j t^{\frac{j}{2}}$. Thus the error term can be improved to $O(t^{m-\frac{n-1}{2}})$ since $\int_{\mathcal{O}} H^{(m)}(t, x, x) dx = \int_{\mathcal{O}} H^{(m+n)}(t, x, x) dx + O(t^{m-\frac{n-1}{2}})$.)

Since ψ_α is identically one on the support of η_α , Notation 3.8 yields

$$(3.24) \quad H^{(m)}(t, x, x) = \sum_{\alpha} \eta_\alpha(x) H_\alpha^{(m)}(t, x, x).$$

Substituting (3.24) into (3.23), we obtain the theorem. \square

4. COMPUTATION OF THE HEAT ASYMPTOTICS

4.1. NOTATION AND REMARKS. Let γ be a **non-trivial** isometry of a Riemannian manifold M and let $\Omega(\gamma)$ denote the set of components of the fixed point set of γ . Each element of $\Omega(\gamma)$ is a submanifold of M . For each non-negative integer k , Donnelly [10] defined a real-valued function, which we temporarily denote $b_k((M, \gamma), \cdot)$, on the fixed point set of γ . For each $W \in \Omega(\gamma)$, the restriction of $b_k((M, \gamma), \cdot)$ to W is smooth. Two key properties of the b_k are:

- (Locality) For $a \in W$, $b_k((M, \gamma), a)$ depends only on the germs at a of the Riemannian metric of M and of the isometry γ . In particular, if U is a γ -invariant neighborhood of a in M , then $b_k((M, \gamma), a) = b_k((U, \gamma), a)$.
- (Universality) If M and M' are Riemannian manifolds admitting isometries γ and γ' , respectively, and if $\sigma : M \rightarrow M'$ is an isometry satisfying $\sigma \circ \gamma = \gamma' \circ \sigma$, then $b_k((M, \gamma), x) = b_k((M', \gamma'), \sigma(x))$ for all $x \in \text{Fix}(\gamma)$.

In view of the locality property, we will usually delete the explicit reference to M and rewrite these functions as $b_k(\gamma, \cdot)$, as they are written in [10].

4.2. COMPUTATION OF THE b_k . [10]

In the notation of 4.1, let $W \in \Omega(\gamma)$, and let $n = \dim(M)$ and $m = \dim(W)$. For $x \in W$, the orthogonal complement $T_x(W)^\perp$ of $T_x(W)$ in the tangent space $T_x(M)$ is invariant under γ_* . Define $A_\gamma(x) = \gamma_* : T_x(W)^\perp \rightarrow T_x(W)^\perp$, and observe that $A_\gamma(x)$ is nonsingular. Set

$$B_\gamma(x) = (I - A_\gamma(x))^{-1}.$$

Donnelly showed that

$$b_k(\gamma, x) = |\det(B_\gamma(x))| \tilde{b}_k(\gamma, x),$$

where $\tilde{b}_k(\gamma, \cdot)$ is an $O(m) \times O(n - m)$ universal invariant polynomial in the components of B_γ and in the curvature tensor R of M and its covariant derivatives.

Explicit formulae for b_0 and b_1 are given in [10, Thm. 5.1] using the following indexing conventions: $1 \leq \alpha, \beta \leq m$, $m + 1 \leq i, j, k \leq n$ and $1 \leq a, b, c \leq n$. At each point $x \in W$, choose an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(M)$ so that the first m vectors are tangent to W . The sign convention on the curvature tensor R of M is chosen so that R_{abab} is the sectional curvature of the plane spanned by e_a and e_b . Set

$$\tau = \sum_{a,b=1}^n R_{abab}$$

and

$$\rho_{ab} = \sum_{c=1}^n R_{acbc}.$$

Thus τ is the scalar curvature and ρ the Ricci tensor of M . Then

$$(4.3) \quad b_0(\gamma, x) = |\det(B_\gamma(x))|$$

and, summing over repeated indices,

$$(4.4) \quad b_1(\gamma, x) = |\det(B_\gamma(x))| \left(\frac{1}{6}\tau + \frac{1}{6}\rho_{kk} + \frac{1}{3}R_{iksh}B_{ki}B_{hs} \right. \\ \left. + \frac{1}{3}R_{ikth}B_{kt}B_{hi} - R_{kaha}B_{ks}B_{hs} \right).$$

4.5. NOTATION. Let \mathcal{O} be an orbifold and let (\tilde{U}, G_U, π_U) be an orbifold chart.

In the notation of 2.12 and 2.14, let \tilde{N} be a **singular** \tilde{U} -stratum and let $\gamma \in \text{Iso}^{\max}(\tilde{N})$. Then \tilde{N} is an open subset of a component of $\text{Fix}(\gamma)$ and thus by 4.1, $b_k(\gamma, \cdot) (= b_k((\tilde{U}, \gamma), \cdot))$ is smooth on \tilde{N} for each nonnegative integer k . Define a function $b_k(\tilde{N}, \cdot)$ on \tilde{N} by

$$b_k(\tilde{N}, x) = \sum_{\gamma \in \text{Iso}^{\max}(\tilde{N})} b_k(\gamma, x).$$

Note that $b_k(\tilde{N}, x) = 0$ if \tilde{N} is not a primary stratum in the language of 2.14.

4.6. LEMMA. Let \mathcal{O} be a Riemannian orbifold, let N be a **singular** \mathcal{O} -stratum and let $p \in N$. Let (\tilde{U}, G_U, π_U) and $(\tilde{U}', G_{U'}, \pi_{U'})$ be two orbifold charts with $p \in U \cap U'$. Let $\tilde{p} \in \tilde{U}$ and $\tilde{p}' \in \tilde{U}'$ with $\pi_U(\tilde{p}) = p = \pi_{U'}(\tilde{p}')$, and let \tilde{N} , respectively \tilde{N}' , be the \tilde{U} -stratum through \tilde{p} , respectively \tilde{U}' -stratum through \tilde{p}' . Then for each k , we have $b_k(\tilde{N}, \tilde{p}) = b_k(\tilde{N}', \tilde{p}')$.

Proof. By Definition 2.1, it suffices to consider the case that one chart embeds in the other, say $\lambda : (\tilde{U}, G_U, \pi_U) \rightarrow (\tilde{U}', G_{U'}, \pi_{U'})$ is an isometric embedding with $\lambda(\tilde{p}) = \tilde{p}'$. The associated homomorphism $\tau : G_U \rightarrow G_{U'}$ carries $\text{Iso}_{G_U}(\tilde{p})$ isomorphically onto $\text{Iso}_{G_{U'}}(\tilde{p}')$ and $\text{Iso}^{\max}(\tilde{p})$ to $\text{Iso}^{\max}(\tilde{p}')$. The \tilde{U} -stratum \tilde{N} is carried to an open subset of the \tilde{U}' -stratum \tilde{N}' . The lemma is thus an immediate consequence of the universality of the b_k , as discussed in 4.1. \square

4.7. DEFINITION. Let \mathcal{O} be a Riemannian orbifold and let N be a **singular** \mathcal{O} -stratum.

(i) For each non-negative integer k , define a real-valued function $b_k(N, \cdot)$ by setting $b_k(N, p) = b_k(\tilde{N}, \tilde{p})$ where (\tilde{U}, G_U, π_U) is any orbifold chart about p , $\tilde{p} \in \pi_U^{-1}(p)$ and \tilde{N} is the \tilde{U} -stratum through \tilde{p} . By Lemma 4.6, the function $b_k(N, \cdot)$ is well-defined.

(ii) The Riemannian metric on \mathcal{O} induces a Riemannian metric, and thus a volume element, on the manifold N . Set

$$I_N := (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N, x) d\text{vol}_N(x)$$

where $d\text{vol}_N$ is the Riemannian volume element.

(iii) Also set

$$I_0 = (4\pi t)^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k(\mathcal{O}) t^k$$

where the $a_k(\mathcal{O})$ (which we will usually write simply as a_k) are the familiar heat invariants. More precisely, the invariants u_i in (3.5), which are defined in terms of the curvature and its covariant derivatives on any Riemannian manifold, also make sense on any Riemannian orbifold. The invariants $a_k(\mathcal{O})$ are given by $a_k = \int_{\mathcal{O}} u_k(x, x) d\text{vol}_{\mathcal{O}}(x)$. In particular, $a_0 = \text{vol}(\mathcal{O})$, $a_1 = \frac{1}{6} \int_{\mathcal{O}} \tau(x) d\text{vol}_{\mathcal{O}}(x)$, etc. Note that if \mathcal{O} is finitely covered by a Riemannian manifold M , say $\mathcal{O} = G \backslash M$, then $a_k(\mathcal{O}) = \frac{1}{|G|} a_k(M)$.

4.8. THEOREM. *Let \mathcal{O} be a Riemannian orbifold and let $\lambda_1 \leq \lambda_2 \leq \dots$ be the spectrum of the associated Laplacian acting on smooth functions on \mathcal{O} . The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of \mathcal{O} is asymptotic as $t \rightarrow 0^+$ to*

$$I_0 + \sum_{N \in S(\mathcal{O})} \frac{I_N}{|\text{Iso}(N)|}$$

where $S(\mathcal{O})$ is the set of all *singular* \mathcal{O} -strata and where $|\text{Iso}(N)|$ is the order of the isotropy at each $p \in N$ as defined in Remark 2.7. This asymptotic expansion is of the form

$$(4.9) \quad (4\pi t)^{-\dim(\mathcal{O})/2} \sum_{j=0}^{\infty} c_j t^{\frac{j}{2}}$$

for some constants c_j .

4.10. REMARK. Suppose $\mathcal{O} = G \backslash M$ is a good closed orbifold. Note that M may be noncompact and G may be an infinite group, although the isotropy group at any point of M must be a finite subgroup of G . In this setting, Donnelly [11] proved the existence and uniqueness of the heat kernel K^M on M and of an asymptotic expansion for K^M . He then obtained an asymptotic expansion for the heat trace on \mathcal{O} . Theorem 4.8, in the case of good orbifolds, organizes the information in [11] in a way that clarifies the contribution of each \mathcal{O} -stratum to the asymptotics.

The expression for the heat asymptotics of good orbifolds in [11] differs from that in (4.9) in that the half powers are missing. However, the absence of these powers is apparently a typographical error in transcribing a result of Donnelly's earlier paper [10], stated below as Proposition 4.11.

K. Richardson [25] obtained an asymptotic expansion for the heat trace associated with the basic Laplacian on a Riemannian foliation. Also referring to Donnelly's work on good orbifolds, he showed that the expansion is of the form given in (4.9).

The remainder of this section is devoted to the proof of Theorem 4.8.

4.11. PROPOSITION. [10]. *Let M be a closed Riemannian manifold, let $K(t, x, y)$ be the heat kernel of M , and let γ be a nontrivial isometry of M . Then, in the notation of 4.1, $\int_M K(t, x, \gamma(x)) d\text{vol}_M(x)$ is asymptotic as $t \rightarrow 0^+$ to*

$$\sum_{W \in \Omega(\gamma)} (4\pi t)^{-\frac{\dim(W)}{2}} \sum_{k=0}^{\infty} t^k \int_W b_k(\gamma, a) d\text{vol}_W(a)$$

where $d \operatorname{vol}_W$ is the volume form on W defined by the Riemannian metric induced from M .

4.12. PROOF OF THEOREM 4.8 IN SPECIAL CASE. We prove Theorem 4.8 for $\mathcal{O} = G \backslash M$ a good closed orbifold with G finite (and thus M compact). In particular, (M, G, π) is a global orbifold chart where $\pi : M \rightarrow \mathcal{O}$ is the projection. In this case, the theorem is an easy consequence of Proposition 4.11. Indeed, letting K denote the heat kernel of M and letting $\pi : M \rightarrow \mathcal{O}$ be the projection, then the heat kernel $K^\mathcal{O}$ of \mathcal{O} is given by

$$K^\mathcal{O}(t, x, y) = \sum_{\gamma \in G} K(t, \tilde{x}, \gamma(\tilde{y}))$$

where \tilde{x} , respectively \tilde{y} , are any elements of $\pi^{-1}(x)$, respectively $\pi^{-1}(y)$. Thus

$$\int_{\mathcal{O}} K^\mathcal{O}(t, x, x) d \operatorname{vol}_{\mathcal{O}}(x) = \frac{1}{|G|} \sum_{\gamma \in G} \int_M K(t, \tilde{x}, \gamma(\tilde{x})) d \operatorname{vol}_M(\tilde{x}),$$

so Proposition 4.11 implies that

$$(4.13) \quad \begin{aligned} \int_{\mathcal{O}} K^\mathcal{O}(t, x, x) d \operatorname{vol}_{\mathcal{O}}(x) &\sim_{t \rightarrow 0^+} \frac{1}{|G|} \int_M K(t, \tilde{x}, \tilde{x}) d \operatorname{vol}_M(\tilde{x}) \\ &+ \frac{1}{|G|} \sum_{1 \neq \gamma \in G} \sum_{W \in \Omega(\gamma)} (4\pi t)^{-\frac{\dim(W)}{2}} \sum_{k=0}^{\infty} t^k \int_W b_k(\gamma, a) d \operatorname{vol}_W(a). \end{aligned}$$

The first term on the right-hand-side of (4.13) is given by

$$(4.14) \quad \frac{1}{|G|} \int_M K(t, \tilde{x}, \tilde{x}) d \operatorname{vol}_M(\tilde{x}) = \frac{1}{|G|} (4\pi t)^{-\frac{\dim(M)}{2}} \sum_{k=0}^{\infty} a_k(M) t^k = I_0.$$

(See the final comment in Definition 4.7(iii).)

Next let $1 \neq \gamma \in G$, let $W \in \Omega(\gamma)$, and let \tilde{N} be an M -stratum contained in W . Then either \tilde{N} has measure zero in W (in which case $\gamma \notin \operatorname{Iso}^{\max}(\tilde{N})$) or else \tilde{N} is open in W and $\gamma \in \operatorname{Iso}^{\max}(\tilde{N})$. Thus by replacing the integral over W with the integrals over the M -strata that are open in W , reordering the summations in (4.13), and taking note of (4.14), we obtain

$$(4.15) \quad \int_{\mathcal{O}} K^\mathcal{O}(t, x, x) d \operatorname{vol}_{\mathcal{O}}(x) \sim_{t \rightarrow 0^+} I_0 + \frac{1}{|G|} \sum_{\tilde{N} \in \tilde{S}(M)} \tilde{I}_{\tilde{N}}$$

where $\tilde{S}(M)$ denotes the set of all **singular** M -strata and where

$$\tilde{I}_{\tilde{N}} = (4\pi t)^{-\frac{\dim(\tilde{N})}{2}} \sum_{k=0}^{\infty} t^k \int_{\tilde{N}} b_k(\tilde{N}, a) d \operatorname{vol}_{\tilde{N}}(a).$$

Let N be an \mathcal{O} -stratum. Then $\pi^{-1}(N)$ is a union of finitely many mutually isometric M -stratum $\tilde{N}_1, \dots, \tilde{N}_k$ and $\pi : \pi^{-1}(N) \rightarrow N$ is a covering map of

degree $\frac{|G|}{|\text{Iso}(N)|}$. We have

$$\tilde{I}_{\tilde{N}_1} + \cdots + \tilde{I}_{\tilde{N}_k} = \frac{|G|}{|\text{Iso}(N)|} I_N$$

and thus Theorem 4.8, in the case of orbifolds finitely covered by manifolds, follows from (4.15).

The proof in the general case will apply the argument in 4.12 to orbifold charts and then piece the computations together via a partition of unity. We first generalize Proposition 4.11 slightly. The manifolds in the two lemmas below do not have boundaries but could, for example, be bounded domains in a larger manifold.

4.16. LEMMA. *We use the notation of 4.1. Let M be an n -dimensional Riemannian manifold (without boundary) of finite volume and let $\gamma : M \rightarrow M$ be a nontrivial isometry. Assume that the distance $d(\tilde{x}, \gamma(\tilde{x}))$ remains bounded away from zero off arbitrarily small tubular neighborhoods of the fixed point set of γ and that each component of the fixed point set of γ has finite volume. Then as $t \rightarrow 0^+$,*

$$\begin{aligned} & \int_M (4\pi t)^{-n/2} e^{-d(\tilde{x}, \gamma(\tilde{x}))^2/4t} (u_0(\tilde{x}, \gamma(\tilde{x})) + tu_1(\tilde{x}, \gamma(\tilde{x})) + \dots) d\tilde{x} \\ & \sim \sum_{W \in \Omega(\gamma)} (4\pi t)^{-\dim(W)/2} \sum_{k=0}^{\infty} t^k \int_W b_k(\gamma, \tilde{x}) d\text{vol}_W(\tilde{x}) \end{aligned}$$

This result is proven in Donnelly [10, Thm. 4.1], in case M is closed. In that case, of course, the hypotheses on the distance function and on the fixed point set of γ are automatic, and the lemma is a restatement of Proposition 4.11. The proof goes through verbatim in the more general setting of Lemma 4.16.

4.17. LEMMA. *With the notation and hypotheses of the previous lemma, let $\tilde{\eta}$ be a smooth bounded γ -invariant function on M . Then there exists a family of functions $c_k(\gamma, \tilde{\eta}, \cdot)$, $k = 0, 1, 2, \dots$, defined on the fixed point set of γ and smooth on each component $W \in \Omega(\gamma)$, such that as $t \rightarrow 0^+$,*

$$\begin{aligned} & \int_M \tilde{\eta}(\tilde{x}) (4\pi t)^{-n/2} e^{-d(\tilde{x}, \gamma(\tilde{x}))^2/4t} (u_0(\tilde{x}, \gamma(\tilde{x})) + tu_1(\tilde{x}, \gamma(\tilde{x})) + \dots) d\tilde{x} \\ & \sim \sum_{W \in \Omega(\gamma)} (4\pi t)^{-\dim(W)/2} \sum_{k=0}^{\infty} t^k \int_W c_k(\gamma, \tilde{\eta}, \tilde{x}) d\text{vol}_W(\tilde{x}). \end{aligned}$$

Moreover, $c_k(\gamma, \tilde{\eta}, \cdot)$ satisfies the following:

- (i) (Locality) $c_k(\gamma, \tilde{\eta}, \tilde{x})$ depends only on the germs of γ , $\tilde{\eta}$, and the Riemannian metric of M at $\tilde{x} \in W$;
- (ii) $c_k(\gamma, \tilde{\eta}, \cdot)$ is zero off $\text{supp}(\tilde{\eta}) \cap W$;
- (iii) the dependence of $c_k(\gamma, \tilde{\eta}, \cdot)$ on $\tilde{\eta}$ is linear;
- (iv) $c_k(\gamma, 1, \cdot) = b_k(\gamma, \cdot)$ where 1 denotes the constant function $\tilde{\eta} \equiv 1$;
- (v) (Universality) if M' is another Riemannian manifold, γ' is an isometry of M' and $\sigma : M \rightarrow M'$ is an isometry satisfying $\sigma \circ \gamma = \gamma' \circ \sigma$, then $c_k(\gamma', \tilde{\eta} \circ \sigma^{-1}, \sigma(\tilde{x})) = c_k(\gamma, \tilde{\eta}, \tilde{x})$ for all \tilde{x} in the fixed point set of γ .

The proof requires only minor changes in the proof of Theorem 4.1 of [10]. In the proof of that theorem, the functions $b_k(\gamma, \cdot)$ are expressed as linear combinations of certain derivatives of explicitly defined functions h_j , $j = 0, \dots, k$. To obtain the functions $c_k(\gamma, \tilde{\eta}, \cdot)$, one replaces the functions h_j by the functions $\tilde{\eta}h_j$.

4.18. NOTATION AND REMARKS. Let \mathcal{O} be a closed orbifold and consider the charts $\tilde{U}_\alpha, \tilde{V}_\alpha, \tilde{W}_\alpha$ and partition of unity $\{\eta_\alpha\}$ given in Notation 3.8. Let \tilde{V}_α play the role of M in Lemmas 4.16 and 4.17, and let $\tilde{\eta}_\alpha = \eta_\alpha \circ \pi_\alpha$ play the role of $\tilde{\eta}$. Since \tilde{V}_α has compact closure inside the larger Riemannian manifold \tilde{W}_α on which G_α acts by isometries and since the fixed point set of γ in \tilde{W}_α is connected (in fact, it is the union of a collection of geodesics radiating from the center point \tilde{p}_α), one easily verifies for each $\gamma \in G_\alpha$ that the hypothesis concerning the distance function in the two lemmas holds.

- (i) For each **singular** \tilde{V}_α -stratum \tilde{N} , define a smooth function $c_{k,\alpha}(\tilde{N}, \cdot)$ on \tilde{N} by

$$c_{k,\alpha}(\tilde{N}, \tilde{x}) = \sum_{\gamma \in \text{Iso}^{\max}(\tilde{N})} c_k(\gamma, \tilde{\eta}_\alpha, \tilde{x}).$$

- (ii) Let N be a **singular** \mathcal{O} -stratum. For each non-negative integer k and each $\alpha = 1, \dots, s$, define a continuous (in fact, smooth) function $c_{k,\alpha}(N, \cdot)$ on N as follows: First for $x \in N \cap V_\alpha$, set $c_{k,\alpha}(N, x) = c_k(\tilde{N}, \tilde{x})$ where \tilde{x} is any element of $\pi_\alpha^{-1}(x)$ and \tilde{N} is the \tilde{V}_α -stratum through \tilde{x} . By an argument analogous to that of Lemma 4.6, this definition is independent of the choice of \tilde{x} in $\pi_\alpha^{-1}(x)$. Since $\tilde{\eta}_\alpha$ is supported in \tilde{U}_α , Lemma 4.17(ii) implies that $c_{k,\alpha}(N, \cdot)$ is zero off $N \cap \overline{U_\alpha}$ and thus extends to a continuous function on N which is zero off $N \cap U_\alpha$.

- (iii) Set

$$I_{N,\alpha} := (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N c_{k,\alpha}(N, x) d\text{vol}_N(x).$$

4.19. LEMMA. Let \mathcal{O} be a closed Riemannian orbifold and let N be a **singular** \mathcal{O} -stratum. Then for each non-negative integer k , we have

$$\sum_{\alpha=1}^s c_{k,\alpha}(N, \cdot) = b_k(N, \cdot)$$

and

$$\sum_{\alpha=1}^s I_{N,\alpha} = I_N.$$

Proof. Let $x \in N$, and let $\alpha_1, \dots, \alpha_r$ be those $\alpha \in \{1, \dots, s\}$ for which $x \in V_{\alpha_i}$. Then we can find a coordinate chart (\tilde{U}, G_U, π_U) such that $U \subset V_{\alpha_1} \cap \dots \cap V_{\alpha_r}$ and such that the chart (\tilde{U}, G_U, π_U) embeds in each of the charts $(\tilde{V}_{\alpha_i}, G_{\alpha_i}, \pi_{\alpha_i})$. Let $\lambda_i : \tilde{U} \rightarrow \tilde{V}_{\alpha_i}$ be the embedding. Let $\tilde{x} \in \pi_U^{-1}(x)$, let \tilde{N} be the \tilde{U} -stratum through \tilde{x} , and let $\tilde{x}_i = \lambda_i(\tilde{x})$. As in the proof of Lemma 4.6, $\lambda_i(\tilde{N})$ is an open subset of

the \tilde{V}_{α_i} -stratum \tilde{N}_i through \tilde{x}_i . Using the universality property (v) of Lemma 4.17, Notation 4.18, and an argument analogous to that of Lemma 4.6, we see that

$$c_{k,\alpha_i}(N, x) := c_{k,\alpha_i}(\tilde{N}_i, \tilde{x}_i) = \sum_{\gamma \in \text{Iso}^{\max}(\tilde{N})} c_k(\gamma, \eta_{\alpha_i} \circ \pi_U, \tilde{x}).$$

Thus, since $c_{k,\alpha}(N, x) = 0$ when α is not one of $\alpha_1, \dots, \alpha_r$, we have

$$(4.20) \quad \sum_{\alpha=1}^s c_{k,\alpha}(N, x) = \sum_{\gamma \in \text{Iso}^{\max}(\tilde{N})} \sum_{i=1}^r c_k(\gamma, \eta_{\alpha_i} \circ \pi_U, \tilde{x}).$$

From properties (iii) and (iv) of Lemma 4.17 and the fact that $\sum_{i=1}^r \eta_{\alpha_i} \equiv 1$ on U , we see that

$$(4.21) \quad \sum_{i=1}^r c_k(\gamma, \eta_{\alpha_i} \circ \pi_U, \tilde{x}) = b_k(\gamma, \tilde{x})$$

on \tilde{N} . By Definition 4.7,

$$\sum_{\gamma \in \text{Iso}^{\max}(\tilde{N})} b_k(\gamma, \tilde{x}) = b_k(N, x)$$

and thus the first equation in the lemma follows from (4.20) and (4.21). The second equation is then immediate. \square

We now prove Theorem 4.8.

Proof. Let $n = \dim(\mathcal{O})$. By Theorem 3.22 and the fact that the support of η_α is contained in V_α , we have

$$(4.22) \quad \sum_{j=1}^{\infty} e^{-\lambda_j t} \sim_{t \rightarrow 0^+} \sum_{\alpha=1}^s \int_{V_\alpha} \eta_\alpha(x) H_\alpha(t, x, x) d \text{vol}(\mathcal{O})$$

where $H_\alpha(t, x, x)$ is defined in Notation 3.21. By Notation 3.21,

$$(4.23) \quad \begin{aligned} & \sum_{\alpha=1}^s \int_{V_\alpha} \eta_\alpha(x) H_\alpha(t, x, x) d \text{vol}_{\mathcal{O}}(x) = \\ & \sum_{\alpha=1}^s \frac{1}{|G_\alpha|} \int_{\tilde{V}_\alpha} \tilde{\eta}_\alpha(\tilde{x}) (4\pi t)^{-\frac{n}{2}} (u_0(\tilde{x}, \tilde{x}) + t u_1(\tilde{x}, \tilde{x}) + \dots) d \text{vol}_{\tilde{V}_\alpha}(\tilde{x}) \\ & + \sum_{\alpha=1}^s \frac{1}{|G_\alpha|} \sum_{1 \neq \gamma \in G_\alpha} \int_{\tilde{V}_\alpha} \tilde{\eta}_\alpha(\tilde{x}) (4\pi t)^{-n/2} e^{-d(\tilde{x}, \gamma(\tilde{x}))^2/4t} (u_0(\tilde{x}, \gamma(\tilde{x})) \\ & \quad + t u_1(\tilde{x}, \gamma(\tilde{x})) + \dots) d \text{vol}_{\tilde{V}_\alpha}(\tilde{x}) \end{aligned}$$

where $\tilde{\eta}_\alpha = \eta_\alpha \circ \pi_\alpha$.

Consider the first sum on the right-hand-side of (4.23). Since η_α is supported in V_α , we have

$$\begin{aligned}
 & \sum_{\alpha=1}^s \frac{1}{|G_\alpha|} (4\pi t)^{-\frac{n}{2}} \int_{\tilde{V}_\alpha} \tilde{\eta}_\alpha(\tilde{x}) (u_0(\tilde{x}, \tilde{x}) + tu_1(\tilde{x}, \tilde{x}) + \dots) d\text{vol}_{\tilde{V}_\alpha}(\tilde{x}) \\
 (4.24) \quad &= \sum_{\alpha=1}^s (4\pi t)^{-\frac{n}{2}} \int_{\mathcal{O}} \eta_\alpha(x) (u_0(x, x) + tu_1(x, x) + \dots) d\text{vol}_{\mathcal{O}}(x) \\
 &= (4\pi t)^{-\frac{n}{2}} \int_{\mathcal{O}} (u_0(x, x) + tu_1(x, x) + \dots) d\text{vol}_{\mathcal{O}}(x) = I_0.
 \end{aligned}$$

Next by Lemma 4.17 and the remarks in 4.18, we have for each $1 \neq \gamma \in G_\alpha$,

$$\begin{aligned}
 (4.25) \quad & \frac{1}{|G_\alpha|} \sum_{1 \neq \gamma \in g_\alpha} \int_{\tilde{V}_\alpha} \tilde{\eta}_\alpha(\tilde{x}) (4\pi t)^{-n/2} e^{-d(\tilde{x}, \gamma(\tilde{x}))^2/4t} (u_0(\tilde{x}, \gamma(\tilde{x})) + \\
 & \quad tu_1(\tilde{x}, \gamma(\tilde{x})) + \dots) d\text{vol}_{\tilde{V}_\alpha}(\tilde{x}) \\
 & \sim \frac{1}{|G_\alpha|} \sum_{1 \neq \gamma \in G_\alpha} \sum_{W \in \Omega(\gamma)} (4\pi t)^{-\dim(W)/2} \sum_{k=0}^{\infty} t^k \int_W c_k(\gamma, \tilde{\eta}_\alpha, \tilde{x}) d\text{vol}_W(\tilde{x}).
 \end{aligned}$$

By 4.18 and an argument identical to that in 4.12, the right-hand-side of (4.25) is equal to

$$\sum_{N \in S(\mathcal{O})} \frac{1}{|\text{Iso}(N)|} I_{N, \alpha}.$$

Consequently, Lemma 4.19 and (4.25) imply that the second sum in the right-hand-side of (4.23) is equal to

$$\sum_{N \in S(\mathcal{O})} \frac{1}{|\text{Iso}(N)|} I_N.$$

Thus in view of (4.22), (4.23), and (4.24), the theorem is proved. \square

5. APPLICATIONS

5.1. THEOREM. *Let \mathcal{O} be a Riemannian orbifold with singularities. If \mathcal{O} is even dimensional (respectively, odd dimensional) and if there exists an odd-dimensional (respectively, even-dimensional) **primary singular** \mathcal{O} -stratum N , then \mathcal{O} cannot be isospectral to a Riemannian manifold.*

Recall that the primary singular strata are those singular strata N for which $\text{Iso}^{\max}(N)$ is nontrivial. Note that all singular strata N that have maximal dimension within any given component of the singular set are primary.

Proof. It is clear from (4.3) that if N is any **primary singular** \mathcal{O} -stratum, then the function $b_0(\gamma, \cdot)$ is strictly positive on N for each $\gamma \in \text{Iso}^{\max}(N)$. Thus, in the two cases, the fact that \mathcal{O} is an orbifold can be gleaned from the presence of half-integer powers, respectively integer powers, of t in the asymptotic expansion in Theorem 4.8. \square

5.2. REMARK. In [19] this theorem was stated for good orbifolds, but here we also include bad orbifolds. The statement in [19], and the statement of Theorem 5.1 in [14], did not include the hypothesis that the singular stratum be primary, **as we were under the mistaken impression that all singular strata are primary. Thanks to a question from Naveed Bari, we realized that $\text{Iso}^{\max}(N)$ may be trivial, as we have illustrated with Example 2.16.**

We now restrict our attention to closed two-dimensional orbifolds (2-orbifolds). The singularities which may occur in 2-orbifolds are cone points, dihedral corner reflectors, and mirror reflectors. Recall that dihedral corner reflectors and mirror reflectors both appeared in Example 2.15. A cone point p of order n is an isolated singularity; an orbifold chart for a neighborhood of p is $(\mathbb{D}^2, \mathbb{Z}_n, \pi)$ where \mathbb{D}^2 is an open 2-disk in \mathbb{R}^2 and \mathbb{Z}_n is the cyclic group of order n . The Euler characteristic of a 2-orbifold is 2 minus the sum of the related values: each cone point of order n has value $\frac{n-1}{n}$; each dihedral corner reflector has value $\frac{n-1}{2n}$; each handle has value 2; each cross-cap has value 1; and each mirror reflector has value 1. Every good 2-orbifold admits a (metrically) spherical, Euclidean or hyperbolic structure depending on whether the Euler characteristic is positive, zero or negative, respectively [31]. In addition, all bad 2-orbifolds have positive Euler characteristic.

5.3. EXAMPLE. Let \mathcal{O} be a 2-orbifold and let p be a cone point of order m . If $N = \{p\}$, then $\text{Iso}(N)$ is a cyclic group of order m and $\text{Iso}^{\max}(N)$ contains all of the nontrivial elements. Letting γ be the generator, then for $j = 1, \dots, m-1$ we have that

$$A_{\gamma^j} = \gamma_*^j = \begin{bmatrix} \cos(\frac{2j\pi}{m}) & -\sin(\frac{2j\pi}{m}) \\ \sin(\frac{2j\pi}{m}) & \cos(\frac{2j\pi}{m}) \end{bmatrix},$$

where A_{γ^j} is as defined in 4.2. Thus

$$b_0(\gamma^j) = |\det((I - A_{\gamma^j})^{-1})| = \frac{1}{2 - 2\cos(\frac{2j\pi}{m})} = \frac{1}{4\sin^2(\frac{j\pi}{m})}.$$

(We are writing $b_0(\gamma^j)$ rather than using the function notation $b_0(\gamma^j, \cdot)$ since N consists of a single point.)

5.4. LEMMA.

$$\sum_{j=1}^{m-1} \frac{1}{\sin^2(\frac{j\pi}{m})} = \frac{m^2 - 1}{3}.$$

Proof. A well-known formula (see, for example, [24, Example 7.9.1]), proven by the calculus of residues, states that

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(k - z)^2}.$$

Thus

$$\sum_{j=1}^{m-1} \frac{1}{\sin^2(\frac{j\pi}{m})} = \frac{m^2}{\pi^2} \sum_{j=1}^{m-1} \sum_{k=-\infty}^{\infty} \frac{1}{(mk - j)^2}.$$

Since

$$\sum_{j=1}^{m-1} \sum_{k=-\infty}^{\infty} \frac{1}{(mk-j)^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \frac{1}{m^2 n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, the lemma follows. \square

5.5. PROPOSITION. *Let \mathcal{O} be a 2-orbifold, let p be a cone point in \mathcal{O} of order m and let $N = \{p\}$. Then in the notation of Theorem 4.8, we have*

$$I_N = \frac{m^2 - 1}{12} + O(t).$$

Proof. By 4.7(ii) and 5.3,

$$I_N = \sum_{j=1}^{m-1} \frac{1}{4 \sin^2(\frac{j\pi}{m})} + O(t).$$

Thus Proposition 5.5 follows from Lemma 5.4. \square

5.6. EXAMPLE. Calculating heat invariants for 2-orbifolds.

Degree zero term for orientable 2-orbifolds. An orientable 2-orbifold \mathcal{O} can have only isolated singularities, i.e., cone points. Suppose \mathcal{O} has k cone points of orders m_1, \dots, m_k . In the notation of 4.7(iii),

$$I_0 = \frac{1}{4\pi} (a_0 t^{-1} + a_1 + O(t)).$$

Thus by Theorem 4.8 and Proposition 5.5, the term of degree zero in the asymptotic expansion in Theorem 4.8 is given by

$$\frac{a_1}{4\pi} + \sum_{i=1}^k \frac{1}{m_i} \frac{m_i^2 - 1}{12}.$$

By the Gauss-Bonnet Theorem (valid also for orbifolds; see [27, 31]), we have

$$a_1 = \frac{2\pi}{3} \chi(\mathcal{O}).$$

Hence the degree zero term is

$$(5.7) \quad \frac{\chi(\mathcal{O})}{6} + \sum_{i=1}^k \frac{m_i^2 - 1}{12m_i}.$$

Degree zero term for nonorientable 2-orbifolds. For a 2-orbifold \mathcal{O} , the dimension zero singular locus is the only portion which contributes to the degree zero term of the asymptotic expansion in Theorem 4.8, aside from the $\frac{a_1}{4\pi} = \frac{\chi(\mathcal{O})}{6}$ component. A nonorientable 2-orbifold \mathcal{O} can have cone points and/or dihedral corner reflector points that contribute in the following ways. As in the computation of the degree zero term for orientable 2-orbifolds, here a simple cone point of order m contributes $\frac{1}{m} \frac{m^2 - 1}{12}$. Let $N = \{p\}$, where p is a corner reflector point created by a rotation of order n and a reflection. Then $|\text{Iso}(N)| = 2n$. By Notation 2.14(ii), $\text{Iso}^{\max}(N)$ contains only the nontrivial elements of the rotation group, since the

reflection fixes one-dimensional strata of the mirror locus, a higher dimensional stratum of the singular set. Hence the computations in Example 5.3 and Proposition 5.5 remain the same, but the difference in $|\text{Iso}(N)|$ is seen as an extra $\frac{1}{2}$ factor in Theorem 4.8. The point contributes $\frac{1}{2n} \frac{n^2-1}{12}$ to the degree zero term.

Thus, for a 2-orbifold \mathcal{O} with cone points p_1, \dots, p_k of orders m_1, \dots, m_k , and dihedral corner reflector points q_1, \dots, q_r of orders n_1, \dots, n_r , the term of degree 0 in the asymptotic expansion of the heat trace is

$$(5.8) \quad \frac{\chi(\mathcal{O})}{6} + \sum_{i=1}^k \frac{1}{m_i} \frac{m_i^2 - 1}{12} + \sum_{j=1}^r \frac{1}{2n_j} \frac{n_j^2 - 1}{12}.$$

Degree one term for 2-orbifolds. Aside from a_2 , only dimension zero strata of the singular set contribute to the t term of the asymptotic expansion in Theorem 4.8. We first note that the last term in (4.4) is zero; this follows from the summation convention and the fact that our singular set is zero-dimensional. Using symmetry properties of the curvature, we can further simplify (4.4) as

$$b_1(\gamma^j) = \frac{1}{4 \sin^2(\frac{j\pi}{m})} \left(\frac{\tau}{6} + \frac{\rho_{kk}}{6} + \frac{2}{3} (R_{1212}(B_{21}^2 + B_{12}^2 - B_{12}B_{21} - B_{22}B_{11})) \right),$$

where R_{1212} is evaluated in the local covering manifold. By the definitions of scalar and Ricci curvature, the preceding equation becomes

$$b_1(\gamma^j) = \frac{R_{1212}(1 + B_{21}^2 + B_{12}^2 - B_{12}B_{21} - B_{22}B_{11})}{6 \sin^2(\frac{j\pi}{m})}.$$

In general, straightforward calculations show that

$$B_{\gamma^j} = (I - A_{\gamma^j})^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{\sin(\frac{2j\pi}{m})}{2-2\cos(\frac{2j\pi}{m})} \\ \frac{\sin(\frac{2j\pi}{m})}{2-2\cos(\frac{2j\pi}{m})} & \frac{1}{2} \end{bmatrix}$$

which implies

$$b_1(\gamma^j) = \frac{R_{1212}}{8 \sin^4(\frac{j\pi}{m})},$$

for $j = 1, \dots, m-1$.

Thus, for a 2-orbifold \mathcal{O} with cone points p_1, \dots, p_k of orders m_1, \dots, m_k , and dihedral corner reflector points q_1, \dots, q_r of orders n_1, \dots, n_r , the coefficient of the term of degree 1 in the asymptotic expansion of the heat trace is

$$(5.9) \quad \frac{a_2}{4\pi} + \sum_{i=1}^k \frac{1}{m_i} \left(\sum_{j=1}^{m_i-1} \frac{R_{1212}}{8 \sin^4(\frac{j\pi}{m_i})} \right) + \sum_{i=1}^r \frac{1}{2n_i} \left(\sum_{j=1}^{n_i-1} \frac{R_{1212}}{8 \sin^4(\frac{j\pi}{n_i})} \right).$$

Recall that $a_2(\mathcal{O}) = \frac{1}{360} \int_{\mathcal{O}} (2|R|^2 - 2|\rho|^2 + 5\tau^2) d\text{vol}_{\mathcal{O}}(g)$, where R is the curvature, ρ is the Ricci curvature, and τ is the scalar curvature of \mathcal{O} (e.g. [2]).

We can further simplify (5.9) by making the substitution

$$\sum_{j=1}^{m_i-1} \frac{1}{\sin^4\left(\frac{j\pi}{m_i}\right)} = \frac{m_i^4 + 10m_i^2 - 11}{45}.$$

See [3] and the references therein or [7, 17] for evaluations of this and similar finite trigonometric sums.

With this substitution, (5.9) becomes

$$(5.10) \quad \frac{a_2}{4\pi} + \sum_{i=1}^k \frac{R_{1212}(m_i^4 + 10m_i^2 - 11)}{360m_i} + \sum_{i=1}^r \frac{R_{1212}(n_i^4 + 10n_i^2 - 11)}{720n_i}.$$

Degree $-\frac{1}{2}$ term for 2-orbifolds. The only \mathcal{O} -strata that contribute to the degree $\frac{1}{\sqrt{t}}$ term are those of codimension one in \mathcal{O} . To obtain these \mathcal{O} -strata, remove any dihedral points from the mirror locus and then take the connected components of the remaining set. Let $x \in N$, an \mathcal{O} -stratum of the mirror locus, and note that $\gamma \in \text{Iso}^{\max}(N)$ must act as a reflection.

To compute $b_0(\gamma, x) = |\det((I - A_\gamma)^{-1})|$, notice that on the normal bundle to N , $\gamma_* = [-1]$, and so,

$$b_0(\gamma, x) = |\det((I - A_\gamma)^{-1})| = |[2]^{-1}| = \frac{1}{2}.$$

Applying 4.7(ii),

$$\begin{aligned} I_N &= (4\pi t)^{-1/2} \sum_{\gamma \in \text{Iso}^{\max}(N)} \sum_{k=0}^{\infty} t^k \int_N \frac{1}{2} d\text{vol}_N(x) + O(t) \\ &= \frac{\text{length}(N)}{4\sqrt{\pi}} \frac{1}{\sqrt{t}} + O(\sqrt{t}). \end{aligned}$$

We sum over all \mathcal{O} -strata of the mirror locus to obtain the coefficient of the $\frac{1}{\sqrt{t}}$ term in Theorem 4.8:

$$(5.11) \quad \sum_N \frac{I_N}{|\text{Iso}(N)|} = \sum_N \frac{1}{2} \frac{\text{length}(N)}{4\sqrt{\pi}} = \frac{\text{length}(\text{Mirror Locus}(\mathcal{O}))}{8\sqrt{\pi}}.$$

Degree $\frac{1}{2}$ term for 2-orbifolds. For a 2-orbifold \mathcal{O} , the dimension one singular locus gives the sole contribution to the \sqrt{t} term of the asymptotic expansion in Theorem 4.8.

We have

$$\begin{aligned} b_1(\gamma, x) &= |\det(B_\gamma(x))| \left(\frac{\tau}{6} + \frac{\rho_{kk}}{6} + \frac{1}{3} R_{iksh} B_{ki} B_{hs} \right. \\ &\quad \left. + \frac{1}{3} R_{ikth} B_{kt} B_{hi} - R_{k\alpha h\alpha} B_{ks} B_{hs} \right)(x) \end{aligned}$$

where B_{ij} denotes the i, j entry of $B_\gamma(x)$, τ is the scalar curvature of M at x and ρ is the Ricci curvature.

In the case of a 2-orbifold with a point x in its mirror locus (but not a dihedral corner reflector point), the matrix $B_\gamma(x)$ is one-dimensional, and thus the third and

fourth terms in the sum vanish. Since $B_\gamma(x) = \frac{1}{2}$, the last term is $-\frac{1}{4}R_{1212}$. We also have $\rho_{kk} = R_{1212}$, while $\tau = 2R_{1212}$. Thus $b_1(\gamma, x) = \frac{1}{2}(\frac{1}{4}R_{1212})(x) = \frac{1}{8}R_{1212}(x) = \frac{1}{16}\tau(x)$. Thus for N a stratum in the mirror locus, the term of degree $\frac{1}{2}$ in I_N is given by

$$\frac{\sqrt{t}}{\sqrt{4\pi}} \int_N \frac{1}{16} \tau(x) d\text{vol}_N(x) = \frac{\sqrt{t}}{32\sqrt{\pi}} \int_N \tau(x) d\text{vol}_N(x).$$

The coefficient of the degree $\frac{1}{2}$ term in the asymptotic expansion for the heat trace on \mathcal{O} is thus

$$(5.12) \quad \frac{1}{64\sqrt{\pi}} \int_{\text{MirrorLocus}(\mathcal{O})} \tau$$

where the scalar curvature is the scalar curvature of \mathcal{O} computed at points in the mirror locus and the integral is with respect to the induced Riemannian metric on the one-dimensional mirror locus. See Table 1 for the asymptotic expansions of the heat kernel for orbifolds \mathcal{O} with $\chi(\mathcal{O}) \geq 0$.

Our notation for orbifolds is adapted from Conway's convention [9], with commas added for readability. Namely $\mathcal{O}(a, b, *c, d)$ denotes an orbifold with simple cone points of orders a and b , and dihedral corner reflectors of orders $2c$ and $2d$. In addition, $\mathcal{O}(n \times)$ is a disk with a simple interior cone point and the edge identified via the antipodal map, while $\mathcal{O}(n*)$ is a disk with a simple interior cone point and a mirror edge corresponding to a reflection. A detailed explanation of the orbifold notation can be found in [9] and compared with pictures in [22, pages 80–90]. A proof that these are all of the closed 2-orbifolds with $\chi(\mathcal{O}) \geq 0$ and that the only bad 2-orbifolds are $\mathcal{O}(m)$, $\mathcal{O}(*m)$, $\mathcal{O}(m, n)$ and $\mathcal{O}(*m, n)$ (when $m > 1$ and $m \neq n$), can be found in [31].

Let \mathcal{O} be an orientable 2-orbifold with k cone points of orders m_1, \dots, m_k , denoted $\mathcal{O}(m_1, \dots, m_k)$, and consider the quantity c defined as 12 times the degree zero term:

$$(5.13) \quad c = 2\chi(\mathcal{O}) + \sum_{i=1}^k \left(m_i - \frac{1}{m_i} \right).$$

This quantity is a spectral invariant; note that it depends only on the topology, not on the Riemannian metric. For $\mathcal{O}(m_1, \dots, m_k)$, we denote by $c(m_1, \dots, m_k)$ the associated spectral invariant. We now investigate classes of orientable 2-orbifolds for which c is a complete topological invariant. Although Theorem 5.14 is a special case of Theorem 5.15, we begin with the more restricted class in order to give the reader intuition for the proof techniques used.

5.14. THEOREM. *Within the class of all footballs (good or bad) and all teardrops, the spectral invariant c is a complete topological invariant. I.e., c determines whether the orbifold is a football or teardrop and determines the orders of the cone points.*

Proof. Denote by $\mathcal{O}(m)$ the teardrop with cone point of order m and by $\mathcal{O}(r, s)$ the football with cone points of orders r and s . Let $c(m)$ and $c(r, s)$ denote the

TABLE 1. 2-orbifold expansions with $\chi(\mathcal{O}) \geq 0$

\mathcal{O} with $\chi(\mathcal{O}) > 0$	Asymptotic Expansion
$\mathcal{O}(m)$	$V \frac{1}{t} + \frac{1}{12}(2 + m + \frac{1}{m}) + O(t)$
$\mathcal{O}(*m)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{1}{24}(2 + m + \frac{1}{m}) + O(\sqrt{t})$
$\mathcal{O}(m, n)$	$V \frac{1}{t} + \frac{1}{12}(m + n + \frac{1}{m} + \frac{1}{n}) + O(t)$
$\mathcal{O}(*m, n)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{1}{24}(m + n + \frac{1}{m} + \frac{1}{n}) + O(\sqrt{t})$
$\mathcal{O}(m \times)$	$V \frac{1}{t} + \frac{1}{12}(m + \frac{1}{m}) + O(t)$
$\mathcal{O}(m*)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{1}{12}(m + \frac{1}{m}) + O(\sqrt{t})$
$\mathcal{O}(2, 2, m)$	$V \frac{1}{t} + \frac{1}{12}(3 + m + \frac{1}{m}) + O(t)$
$\mathcal{O}(*2, 2, m), \mathcal{O}(2, *m)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{1}{24}(3 + m + \frac{1}{m}) + O(\sqrt{t})$
$\mathcal{O}(2, 3, 3)$	$V \frac{1}{t} + \frac{43}{72} + O(t)$
$\mathcal{O}(*2, 3, 3), \mathcal{O}(3, *2)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{43}{144} + O(\sqrt{t})$
$\mathcal{O}(2, 3, 4)$	$V \frac{1}{t} + \frac{97}{144} + O(t)$
$\mathcal{O}(*2, 3, 4)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{97}{288} + O(\sqrt{t})$
$\mathcal{O}(2, 3, 5)$	$V \frac{1}{t} + \frac{271}{360} + O(t)$
$\mathcal{O}(*2, 3, 5)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{271}{720} + O(\sqrt{t})$
\mathcal{O} with $\chi(\mathcal{O}) = 0$	Asymptotic Expansion
torus, Klein bottle	$V \frac{1}{t} + O(t)$
*torus, *Klein bottle	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + O(\sqrt{t})$
$\mathcal{O}(2, 2, 2, 2)$	$V \frac{1}{t} + \frac{1}{2} + O(t)$
$\mathcal{O}(*2, 2, 2, 2), \mathcal{O}(2, *2, 2),$ $\mathcal{O}(2, 2*)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{1}{4} + O(\sqrt{t})$
$\mathcal{O}(2, 2 \times)$	$V \frac{1}{t} + \frac{1}{4} + O(t)$
$\mathcal{O}(2, 4, 4)$	$V \frac{1}{t} + \frac{3}{4} + O(t)$
$\mathcal{O}(*2, 4, 4), \mathcal{O}(4, *2)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{3}{8} + O(\sqrt{t})$
$\mathcal{O}(3, 3, 3)$	$V \frac{1}{t} + \frac{2}{3} + O(t)$
$\mathcal{O}(*3, 3, 3), \mathcal{O}(3, *3)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{1}{3} + O(\sqrt{t})$
$\mathcal{O}(2, 3, 6)$	$V \frac{1}{t} + \frac{5}{6} + O(t)$
$\mathcal{O}(*2, 3, 6)$	$V \frac{1}{t} + ML \frac{1}{\sqrt{t}} + \frac{5}{12} + O(\sqrt{t})$

Here $m, n \geq 1$, $V = \frac{\text{vol}(\mathcal{O})}{4\pi}$ and $ML = \frac{\text{length}(\text{MirrorLocus}(\mathcal{O}))}{8\sqrt{\pi}}$. Note that *torus is an annulus with two mirror reflector edges and *Klein bottle is a Möbius band with one mirror reflector edge.

invariant defined in (5.13) in the two cases. Then $\mathcal{O}(m)$ has Euler characteristic $1 + \frac{1}{m}$ and thus

$$c(m) = 2 + m + \frac{1}{m}.$$

The football $\mathcal{O}(r, s)$ has Euler characteristic $\frac{1}{r} + \frac{1}{s}$, so

$$c(r, s) = r + s + \frac{1}{r} + \frac{1}{s}.$$

TABLE 2. Triangular Pillow Orbifolds

	\mathcal{O}	$\chi(\mathcal{O})$	$c(\mathcal{O})$
$\chi(\mathcal{O}) > 0$	$\mathcal{O}(2, 2, 2)$	$\frac{1}{2}$	$5\frac{1}{2}$
	$\mathcal{O}(2, 2, m)$	$\frac{1}{m}$	$3 + m + \frac{1}{m}$
	$\mathcal{O}(2, 3, 3)$	$\frac{1}{6}$	$7\frac{1}{6}$
	$\mathcal{O}(2, 3, 4)$	$\frac{1}{12}$	$8\frac{1}{12}$
	$\mathcal{O}(2, 3, 5)$	$\frac{1}{30}$	$9\frac{1}{30}$
$\chi(\mathcal{O}) = 0$	$\mathcal{O}(3, 3, 3)$	0	8
	$\mathcal{O}(2, 4, 4)$	0	9
	$\mathcal{O}(2, 3, 6)$	0	10
	$\mathcal{O}(3, 3, 4)$	$-\frac{1}{12}$	$8\frac{11}{12}$
$\chi(\mathcal{O}) < 0$	$\mathcal{O}(3, 4, 4)$	$-\frac{1}{6}$	$9\frac{5}{6}$
	$\mathcal{O}(3, 3, 5)$	$-\frac{2}{15}$	$9\frac{13}{15}$
	$\mathcal{O}(2, 4, 5)$	$-\frac{1}{20}$	$9\frac{19}{20}$
	\vdots	$-1 < \chi(\mathcal{O}) < 0$	$c(\mathcal{O}) > 10$

The invariant is an integer only in the case of $\mathcal{O}(2, 2)$, so the football $\mathcal{O}(2, 2)$ is spectrally distinguishable from the other footballs and teardrops. Thus for the remainder of the proof, all footballs will be assumed to have at least one cone point of order strictly greater than 2.

For teardrops, $c(m)$ trivially determines m . We next claim that footballs are distinguishable from teardrops. Indeed, suppose that $c(m) = c(r, s)$. Then $m + 2 = r + s$ and $\frac{1}{m} = \frac{1}{r} + \frac{1}{s}$. The latter equation implies that $m < \min(r, s)$. Since also $r, s \geq 2$, we have $2 + m < r + s$, a contradiction, thus proving the claim.

It remains only to show that for footballs, $c(r, s)$ determines r and s . From $c(r, s)$, one can read off the quantities $r + s$ and $\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}$. Hence $c(r, s)$ determines both $r + s$ and rs , thus also $|r - s|$, since $(r - s)^2 = (r + s)^2 - 4rs$. Hence (r, s) is determined up to order, completing the proof. \square

5.15. THEOREM. *Let C be the class consisting of all closed orientable 2-orbifolds with $\chi(\mathcal{O}) \geq 0$. The spectral invariant c is a complete topological invariant within C and moreover, it distinguishes the elements of C from smooth oriented closed surfaces.*

Proof. We first consider the 2-orbifolds for which c is an integer. Note that among teardrops, footballs, and triangular pillows, the only integer values are $c(2, 2) = 5$, $c(2, 3, 6) = 10$, $c(2, 4, 4) = 9$, and $c(3, 3, 3) = 8$ (cf. Tables 1 and 2). In addition, $c(2, 2, 2, 2) = 6$, $c(S^2) = 4$, and $c(T^2) = 0$. Let S_g be a Riemann surface of genus $g \geq 2$. Then we also have $c(S_g) = 4 - 4g$. It is clear that the values of c are distinct in each case, so that the spectrum distinguishes these 2-orbifolds.

For the rest of the proof, it suffices to consider orbifolds in C with $\chi(\mathcal{O}) > 0$ since these include all orbifolds within C for which c is not an integer. Table 2

lists the values of c for these triangular pillows. Setting $c(2, 3, 3) = c(2, 2, m)$ and solving for m gives $m = \frac{25 \pm \sqrt{481}}{12}$, which contradicts the assumption that m is an integer. Similar calculations for $c(2, 3, 4)$ and $c(2, 3, 5)$ show that the spectrum distinguishes among triangular pillows with $\chi(\mathcal{O}) > 0$.

By Theorem 5.14, c distinguishes among teardrops and footballs. We next show that c distinguishes both teardrops and footballs from triangular pillows with $\chi(\mathcal{O}) > 0$. We have $c(m) = 2 + m + \frac{1}{m}$, and $c(p, q, r) = -2 + p + q + r + \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$; setting the integer and fractional parts of these equations equal gives

$$\begin{aligned} 2 + m &= -1 + p + q + r \\ \frac{1}{m} &= \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1. \end{aligned}$$

Solving the first equation for m and plugging the result into the second equation yields

$$0 = pr(p + r - 3) + pq(p + q - 3) + qr(q + r - 3) + pqr(5 - p - q - r).$$

None of the possible triples (p, q, r) satisfy this equation, showing that teardrops are distinguished from triangular pillows with $\chi(\mathcal{O}) > 0$. The argument that c distinguishes good footballs from these triangular pillows is analogous.

To see that c distinguishes triangular pillows with $\chi(\mathcal{O}) > 0$ from bad footballs, we compare the respective integer and fractional parts of c in each case. For example, comparing $c(2, 3, 3)$ and $c(r, s, r) \neq s$ gives

$$\begin{aligned} 7 &= r + s \\ \frac{1}{6} &= \frac{1}{r} + \frac{1}{s}, \end{aligned}$$

which implies $s^2 - 7s + 42 = 0$; thus $s = \frac{7 \pm \sqrt{-119}}{2}$, contradicting s being an integer. The calculations are similar for $\mathcal{O}(2, 3, 4)$ and $\mathcal{O}(2, 3, 5)$, while for $\mathcal{O}(2, 2, m)$ we have

$$\begin{aligned} 3 + m &= r + s \\ \frac{1}{m} &= \frac{1}{r} + \frac{1}{s}, \end{aligned}$$

which implies $r(r - 3) + s(s - 3) + rs = 0$. We can assume that one of r and s is strictly greater than 2, say r . Thus $r - 3 \geq 0$ and $s - 3 \geq -1$, which implies

$$r(r - 3) + s(s - 3) + rs \geq s(r - 1) > 0.$$

Hence triangular pillows with $\chi(\mathcal{O}) > 0$ are distinguished from bad footballs. \square

5.16. REMARK. Notably absent from the class C are triangular pillows with $\chi(\mathcal{O}) < 0$. The invariant c does not seem sufficiently strong to distinguish among these triangular pillows. However, as a special case of a result in [13], the spectrum does determine the orders of the cone points in such a 2-orbifold, provided that it is endowed with a metric of constant curvature -1 . On the other hand, to distinguish, say, triangular pillows with $\chi(\mathcal{O}) < 0$ from triangular pillows with $\chi(\mathcal{O}) > 0$, we do not need such a metric assumption.

If $\mathcal{O}(p, q, r)$ is any triangular pillow with $\chi(\mathcal{O}) < 0$ and $\text{spec}(\mathcal{O}(p, q, r)) = \text{spec}(\mathcal{O}(2, 2, m))$, then setting the integer and fractional parts of the respective values of c equal, we have

$$\begin{aligned} m + 3 &= p + q + r - 2 \\ \frac{1}{m} &= \frac{1}{p} + \frac{1}{q} + \frac{1}{r}. \end{aligned}$$

Solving the first equation for m and plugging the resulting value into the second equation yields

$$(5.17) \quad 2pqr + pr(p + r - 5) + pq(p + q - 5) + qr(q + r - 5) = 0.$$

Note that for a triangular pillow with $\chi(\mathcal{O}) < 0$, we must have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, which implies that the sum of any two of p, q, r is at least 5. Thus each term on the right-hand side of (5.17) is nonnegative. This contradiction then implies that $\mathcal{O}(2, 2, m)$ cannot be isospectral to such a triangular pillow. For the remaining triples where $\chi(\mathcal{O}) > 0$, we set the integer and fractional parts of the respective value of c equal to those of a triangular pillow with $\chi(\mathcal{O}) < 0$, and note that there are no $\chi(\mathcal{O}) < 0$ triples satisfying these equations. Thus c distinguishes between triangular pillows with $\chi(\mathcal{O}) < 0$ and $\chi(\mathcal{O}) > 0$. A similar argument shows that c distinguishes triangular pillows with $\chi(\mathcal{O}) < 0$ from teardrops; it is clear that c also distinguishes triangular pillows with $\chi(\mathcal{O}) < 0$ from the smooth surfaces and the remaining elements of C , with the exception of footballs. In this last case, it seems that metric assumptions are again necessary.

5.18. REMARK. Other difficulties arise during the consideration of the expanded class which includes nonorientable orbifolds. Metric assumptions are necessary in numerous cases, such as distinguishing nonorientable orbifolds with the same orientable double cover (e.g. $\mathcal{O}(*2, 3, 3)$ and $\mathcal{O}(3, *2)$, see Table 1 and Figure 1).

We now examine classes that include nonorientable orbifolds.

5.19. PROPOSITION. *Within the class of all closed 2-orbifolds with $\chi(\mathcal{O}) \geq 0$, the spectrum distinguishes whether the orbifold has zero or positive Euler characteristic.*

Proof. Note that for 2-orbifolds \mathcal{O} with $\chi(\mathcal{O}) = 0$, c is either an integer or equal to 4.5. Thus c distinguishes all but the following cases: S^2 from the orbifolds $\mathcal{O}(*3, 3, 3)$ and $\mathcal{O}(3, *3)$ (with $c = 4$), the good football $\mathcal{O}(2, 2)$ from $\mathcal{O}(*2, 3, 6)$ (with $c = 5$), and the bad teardrop $\mathcal{O}(2)$ from the orbifolds $\mathcal{O}(*2, 4, 4)$ and $\mathcal{O}(4, *2)$ (with $c = 4.5$). The lack of a mirror locus in the $\chi(\mathcal{O}) > 0$ cases and the presence of a mirror locus in the corresponding $\chi(\mathcal{O}) = 0$ orbifolds can be gleaned from the degree $-\frac{1}{2}$ term, and so they are distinguished. \square

5.20. PROPOSITION. *Within the class of closed 2-orbifolds of constant nonzero curvature R or $-R$ the spectrum determines the sign of the curvature, i.e. whether the orbifold is spherical or hyperbolic.*

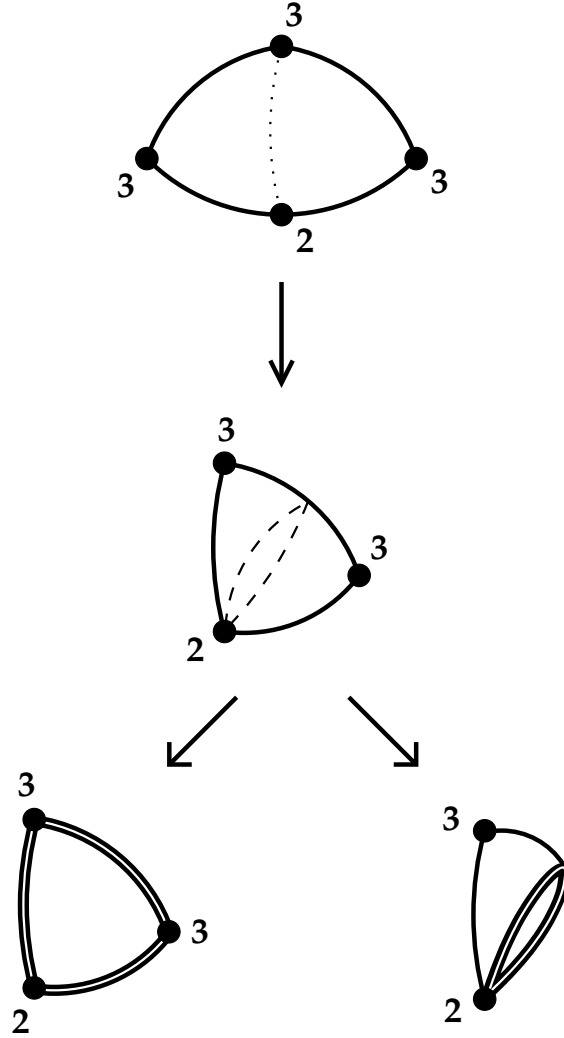


FIGURE 1. The uppermost object is a fundamental domain for $\mathcal{O}(2, 3, 3)$, with its quotient $\mathcal{O}(2, 3, 3)$ below. Here the vertices are cone points labelled with their orders, and the double edges represent reflector edges. On the bottom left we show $\mathcal{O}(*2, 3, 3)$, which is obtained by reflecting $\mathcal{O}(2, 3, 3)$ in the plane of the paper, and on the right is $\mathcal{O}(3, *2)$, obtained by reflecting $\mathcal{O}(2, 3, 3)$ in the plane containing the dashed loop.

Proof. Assume that \mathcal{O} has cone points p_1, \dots, p_k of orders m_1, \dots, m_k , and/or dihedral corner reflector points q_1, \dots, q_r of orders n_1, \dots, n_r . Now look at the coefficient of the t term in the expansion, as in (5.10), which reduces to

$$\frac{a_2}{4\pi} \pm R \left(\sum_{i=1}^k \frac{(m_i^2 + 11)(m_i^2 - 1)}{360m_i} + \sum_{i=1}^r \frac{(n_i^2 + 11)(n_i^2 - 1)}{720n_i} \right)$$

in the presence of constant curvature. The $\frac{1}{t}$ term in the expansion tells us $\text{vol}(\mathcal{O})$ and we know the size of the curvature, so we know the a_2 component and may subtract it off. Notice that the summands are nonnegative, and hence we can read the sign of the curvature unless there were no cone points and no dihedral corner reflector points. In this case, examine the degree zero term, which now has no point contributions and reduces to $\frac{a_1}{4\pi} = \frac{\chi(\mathcal{O})}{6}$, and we can read off the Euler characteristic. \square

5.21. REMARK. In the case of closed 2-orbifolds with a nontrivial mirror locus, (5.12) enables us to make a stronger statement. In particular, among such orbifolds that are endowed with a metric of strictly positive, strictly negative, or zero curvature, the spectrum determines the sign of the curvature. This class includes the bad orbifolds $\mathcal{O}(*m)$ and $\mathcal{O}(*p, q)$ with $p \neq q$ since they admit a metric of strictly positive (but variable) curvature.

5.22. PROPOSITION. *Within the class of spherical 2-orbifolds of constant curvature $R > 0$ the spectrum determines the orbifold.*

Proof. Notice that the metric requirement eliminates teardrops and bad footballs and their quotients from this class. In Table 1, c distinguishes among the remaining spherical orbifolds with the exception that c is unable to distinguish between orbifolds that are nonorientable but have the same orientable double cover: $\mathcal{O}(*m, m)$, $\mathcal{O}(m \times)$, and $\mathcal{O}(m*)$, with double cover $\mathcal{O}(m, m)$; $\mathcal{O}(*2, 2, m)$ and $\mathcal{O}(2, *m)$, with double cover $\mathcal{O}(2, 2, m)$; and $\mathcal{O}(*2, 3, 3)$ and $\mathcal{O}(3, *2)$, with double cover $\mathcal{O}(3, 3, 2)$ (see Figure 1). However, c is able to distinguish each nonorientable class from the remaining orbifolds.

Consider such a class of nonorientable spherical orbifolds with a common orientable double cover. The coefficient of the degree $-\frac{1}{2}$ term, as given in (5.11), distinguishes nonorientable orbifolds with mirror loci from those without, i.e. in this class it distinguishes orbifolds with only crosscaps from those with mirror loci. In the presence of constant curvature, (5.11) also distinguishes among the remaining spherical cases: among $\mathcal{O}(*m, m)$ and $\mathcal{O}(m*)$, the length of the mirror locus of $\mathcal{O}(*m, m)$ is larger in the constant curvature metric; among $\mathcal{O}(*2, 2, m)$ and $\mathcal{O}(2, *m)$, the length of the mirror locus of $\mathcal{O}(*2, 2, m)$ is larger; and among $\mathcal{O}(*2, 3, 3)$ and $\mathcal{O}(3, *2)$, the length of the mirror locus of $\mathcal{O}(*2, 3, 3)$ is larger (see Figure 1). \square

5.23. REMARK. Notice that metric assumptions are only needed to distinguish within each nonorientable class. We cannot make a similar statement for flat 2-orbifolds. For example, it is possible to endow $\mathcal{O}(2, *2, 2)$ and $\mathcal{O}(2, 2*)$ with a metric of zero curvature so that they have the same area and also have mirror loci of the same length. They cannot be distinguished by the asymptotic expansion of the heat trace.

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