Heat Expansion of Orientable 3-Orbifold Notes

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1 The Orbifold Formulas

The general formula for the asymptotic expansion of the heat kernel of a compact manifold/orbifold is given by,

$$\mathrm{Tr}(K) = \sum_{j=1}^{\infty} c^{-\lambda_j t} \sim I_0 + \sum_{N \in S(\mathcal{O})} \frac{I_N}{|\operatorname{Iso}(N)|} \tag{1}$$

Equation 2 for I_0. Equation 3 for I_N.

$$I_0 = (4\pi t)^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k(\mathcal{O}) t^k \tag{2}$$

Equation 4 through equation 7 for a_k terms.

$$I_N = (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N, x) dvol_N \tag{3}$$

Equation 8 for $b_k(N, x)$.
$a_k(\mathcal{O})$ is generally given by,

$$a_k(\mathcal{O}) = \int_M u(x, x) dvol_{\mathcal{O}} \tag{4}$$

Where u is some complicated thing. But some specific terms are given by,

$$a_0(\mathcal{O}) = Vol(\mathcal{O}) \tag{5}$$

$$a_1(\mathcal{O}) = \frac{1}{6} \int_M \tau(x) dvol_M(x) \tag{6}$$

Where τ is the scalar curvature. In dimension two we can simplify $a_1(\mathcal{O})$ to,

$$a_1(\mathcal{O}) = \frac{2\pi}{3}\chi(\mathcal{O}) \text{ for } \dim(\mathcal{O}) = 2 \tag{7}$$

Where $\chi(\mathcal{O})$ is the orbifold Euler number.

$$b_k(N,x) = b_k(\tilde{N},\tilde{x}) = \sum_{\gamma\in \mathrm{Iso}^{\max}(\tilde{N})} b_k(\gamma,\tilde{x}) \tag{8}$$

Equation 9 for $\mathrm{Iso}^{\max}(\tilde{N})$. Equation 10 for $b_k(\gamma,\tilde{x})$.

$$\mathrm{Iso}^{\max}(\tilde{N}) = \{\, \gamma \in \mathrm{Iso}(\tilde{N}) \mid \tilde{N} \text{ is open in } \mathrm{Fix}(\gamma)\,\} \tag{9}$$

$$b_k(\gamma,x) = |\det B_\gamma(x)|\tilde{b}_k(\gamma,x) \tag{10}$$

Equation 11 for $B_\gamma(x)$.

$$B_\gamma(x) = (I - A_\gamma(x))^{-1} \tag{11}$$

Where I is the Identity matrix. Equation 12 for A_γ.

$$A_\gamma = \gamma_{*x} : (T_xW)^\perp \mapsto (T_xW)^\perp \tag{12}$$

Where T_xW is the tangent space of a point in the fixed point space.

2 Zero degree term of orientable 3-orbifolds

Lemma 2.1. A singular point in an orientable 3-orbifold has one of the following forms. Also in this lemma introduce linear charts from C. Gordon's expository orbifold article

Proof. Thurston paragraph before Prop 13.3.1 and Artin's Algebra book Theorem 9.1. (all isometries are rotations) □

Theorem 2.1. The degree zero heat invariant of an orientable 3-orbifold is zero.

Proof. Let $\mathcal{O}$ be an orientable 3-orbifold. The heat expansion for $\mathcal{O}$ has the form

$$I_0 + \sum_{N\in S(\mathcal{O})} \frac{I_N}{|\,\mathrm{Iso}(N)|}.$$

Observe that in the expression for I_0, the coefficient $4\pi t$ is raised to the $-3/2$ power, resulting in no zero degree term in I_0.

Now I_N is given by

$$I_N = (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N,x) dvol_N.$$

By Boileau, Maillot, and Porti "Three-Dimensional Orbifolds and their Geometric Structures" page 31-32 we know $\mathcal{O}$ has only 1-dimensional and zero dimensional singular strata. In the case of 1-dimensional strata the coefficient $4\pi t$ in the expression for I_N is raised to the $-1/2$ power, resulting in no zero degree term. In the case of 0-dimensional strata, the coefficient of the sum in the expression for I_N is 1. Thus the constant term in the sum is $\int_N b_0(N,x) dvol_N$. Since N is just a point we just get $b_0(\{p\},x)$.

By the notes above we know

$$b_0(\{p\},x) = \sum_{\gamma \in \mathrm{Iso}^{\max}(\tilde{N})} b_k(\gamma, \tilde{x})$$

where $\mathrm{Iso}^{\max}(\tilde{N})$ is the subset of $\mathrm{Iso}(\tilde{N})$ given in line 9. By Lemma 2.1 we know that all isometries $\gamma \in \mathrm{Iso}(\tilde{N})$ are rotations. In particular $\mathrm{Fix}(\gamma)$ must be a line in $\tilde{U}$. Because $\mathrm{Fix}(\gamma)$ is a line $\{\tilde{p}\}$ is not open in $\mathrm{Fix}(\gamma)$. Thus by definition of $\mathrm{Iso}^{\max}(\tilde{N})$, we have $\mathrm{Iso}^{\max}(\tilde{p})$ is empty. This implies $b_0(p,x) = \sum_{\gamma \in \mathrm{Iso}^{\max}(\tilde{N})} b_k(\gamma, \tilde{x}) = 0$. □

Theorem 2.2. The degree $-\frac{1}{2}$ heat invariant of an orientable 3-orbifold is

$$(4\pi)^{-3/2} \frac{1}{6} \int_{\mathcal{O}} \tau(x)\ dvol_{\mathcal{O}} + \sum_{i=1}^{l} \frac{1}{2\sqrt{\pi} m_i} \cdot \frac{m_i - 1}{12} length(N_i).$$

Proof. Let $\mathcal{O}$ be an orientable 3-orbifold. The heat expansion for $\mathcal{O}$ has the form

$$I_0 + \sum_{N \in S(\mathcal{O})} \frac{I_N}{|\,\mathrm{Iso}(N)|}.$$

We begin by finding the full coefficient of $t^{-1/2}$ in this expansion. Recall that,

$$I_0 = (4\pi t)^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k(\mathcal{O}) t^k.$$

The degree $-\frac{1}{2}$ term of the I_0 expansion is $(4\pi)^{-3/2}a_1 t^{-1/2}$.

Again I_N is given by

$$I_N = (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N,x) dvol_N.$$

Because there only exist singular strata of dimension zero and dimension one, we only need to consider I_N in these cases. When N is zero dimensional we don't see a $-\frac{1}{2}$ degree term in I_N. So we focus on those $N \in S(O)$ having dimension one. In this case the coefficient of $t^{-1/2}$ in I_N is

$$(4\pi)^{-1/2} t^{-1/2} \int_N b_0(N,x) \ dvol_N.$$

Now let $N_1, N_2, \ldots, N_l$ be the list of connected components of the 1-dimensional singular set of $\mathcal{O}$. Take $m_1, m_2, \ldots, m_l$ to be the orders of the corresponding isotropy groups. So N_1 is fixed by a cyclic group of rotations of order m_1, and so on. Note: by Boileau, Maillot, and Porti "Three-Dimensional Orbifolds and their Geometric Structures" page 31-32 we know the 1-dimensional singular strata in $\mathcal{O}$ must have rotational isotropy only. With this notation the coefficient of the degree $-1/2$ term in the trace (1) is

$$(4\pi)^{-3/2}a_1 + \sum_{i=1}^{l} \frac{(4\pi)^{-1/2}}{|Iso(N_i)|} \int_N b_0(N,x) \ dvol_N.$$

Recall once more that

$$b_0(N,x) = \sum_{\gamma \in Iso^{max}(N_1)} b_0(\gamma, x).$$

Write $Iso(N_i) = \{e, r_1, r_2, \ldots, r_{m_i - 1}\}$. Since $Fix\{e\}$ is the entire space, e is not in Iso^{max}. Since $Fix\{r_i\}$ is the entire line, all non-trivial rotations are in Iso^{max}.

Now $b_0(\gamma, x) = |\det(I - A_{\gamma *})^{-1}|$. We observe that using a linear coordinate chart (see C. Gordon's orbifold expository paper) that puts the axis of rotation on the x-axis, then the differentials of the elements of Iso^{max} are as follows.

$$\gamma_{ij*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{2\pi j}{m_i}) & -\sin(\frac{2\pi j}{m_i}) \\ 0 & \sin(\frac{2\pi j}{m_i}) & \cos(\frac{2\pi j}{m_i}) \end{bmatrix}$$

This means
$$A_{\gamma_{ij*}} = \begin{bmatrix} \cos(\frac{2\pi j}{m_i}) & -\sin(\frac{2\pi j}{m_i}) \\ \sin(\frac{2\pi j}{m_i}) & \cos(\frac{2\pi j}{m_i}) \end{bmatrix}.$$
Thus
$$\det(I - A_{\gamma_{ij*}})^{-1} = \frac{1}{4\sin^2(\frac{j\pi}{m_i})}$$
as in DGGW's computation of heat invariants for a single cone point. Now apply Lemma 5.4 in DGGW to obtain,

$$b_0(N_i, x) = \sum_{j=1}^{m_i-1} \frac{1}{4\sin^2(\frac{j\pi}{m_i})} = \frac{m_i^2 - 1}{12}.$$

Working our way back out of this deep dive into computation we have,

$$\int_{N_i} b_0(N_i, x)\ dvol_{N_i} = \int_{N_i} \frac{m_i^2 - 1}{12}\ dvol_{N_i} = \frac{m_i^2 - 1}{12}\ length(N_i)$$

Recalling that $a_1 = \frac{1}{6}\int_{\mathcal{O}} \tau(x)\ dvol_{\mathcal{O}}$, we write out the full degree $-1/2$ coefficient in the heat expansion of $\mathcal{O}$.

$$(4\pi)^{-3/2} a_1 + \sum_{i=1}^{l} \frac{1}{2\sqrt{\pi} m_i} \cdot \frac{m_i - 1}{12} length(N_i) =$$

$$(4\pi)^{-3/2} \frac{1}{6} \int_{\mathcal{O}} \tau(x)\ dvol_{\mathcal{O}} + \sum_{i=1}^{l} \frac{1}{2\sqrt{\pi} m_i} \cdot \frac{m_i - 1}{12} length(N_i)$$

□

Corollary 2.1. The degree $-\frac{1}{2}$ heat invariant of an flat orientable 3-orbifold is
$$\sum_{i=1}^{l} \frac{1}{2\sqrt{\pi} m_i} \cdot \frac{m_i - 1}{12} length(N_i).$$

Brainstorming: (remember that corollary above requires flat geometry)

- Suppose we assume all dimension 1 singular strata have the same length. Can we say something about the m_i?
- Suppose we assume all of the singular strata have the same isotropy?

- What if we assume same isotropy and same length of components in singular stratification? How many such orbifolds are there?
- How can we get information about the dim zero singular strata? These never appear in isomax in the orientable case.
- Suppose the singular set is a union of disjoint circles. Could we say something within this class of orbifolds. Does this even make sense? (See Thurston's Borromean rings example.) Is there a 3-orbifold whose singular set is just a circle. If so, then is it distinguished spectrally by its heat invariants? (So that no other orbifold could be isospectral to it.)
- How can we use the fact that the 0-dim singular set is not detected by the heat invariants. Could we construct two 3-ofds that have different 0-strata but otherwise identical heat invariants. Then these would be isospectral and non-isometric. OR could we strategically use Sunada's Theorem (or other methods) to create orbifolds that vary only in the 0-dim stratum. This could give candidates of isosp non-isom orbifolds.
- What about the Orbifold Theorem. Could the geometries given there help us? (Yikes... takes decomposing the orbifold which would wreck the spectrum)
- Note in a 3-orbifold the singular set is a 3-regular graph. So the number of edges should tell us the number of vertices.

NEXT: Write up results for deg -1 and deg -3/2. Also 1/2 term for general case (omit full detail on curvature term) and apply to flat case.

Theorem 2.3. The degree -1 term of an orientable 3-orbifold is 0.

Proof. Let $\mathcal{O}$ be an orientable 3-orbifold. The heat expansion of $\mathcal{O}$ has the following form

$$I_0 + \sum_{N \in S(\mathcal{O})} \frac{I_N}{|\operatorname{Iso}(N)|}$$

Because $\dim(\mathcal{O}) = 3$, I_0 consists of only fractional powers of t. So, I_0 contributes nothing to the degree -1 term. We now consider the contribution of I_N to the -1 term.

$$I_N = (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N, x) dvol_N$$

The orbifold $\mathcal{O}$ will have only strata of dimension 0 and 1 due to the lack of reflectional symmetry. Any strata N with $\dim(N) = 0$ and $\dim(N) = 1$ will have not have a degree -1 term in the expansion of I_N. Neither I_0 or I_N for any N contribute to the degree -1 term for our orbifold $\mathcal{O}$, so we have a degree -1 term of 0. $\square$