Chapter 8

The Sturm-Liouville Theorem

The main idea of Sturm-Liouville theory is that one can systematically characterize boundary value problems whose solutions form collections of functions that can be used to approximate any piecewise smooth function. Before we make a precise statement of the theorem, we first need to carefully describe what type of boundary value problems we are allowed to consider. A Sturm-Liouville problem has three parts:

- a domain Ω and an associated weight function w,
- a differential operator L, and
- a boundary condition.

The main content of this chapter is to discuss precisely what we mean by these three parts, and to state the conditions required by the Sturm-Liouville theory. Finally, at the end of the chapter, we are able to state the main theorem.

Throughout the discussion we track two examples: the periodic boundary value problem that lead to the Fourier series, and a new boundary value problem that we call "Hermite's problem."

8.1 The domain and weight function

For the general case, we work with the domain $\Omega = (a, b)$; typical examples include $\Omega = (-1, 1)$, $\Omega = (0, \infty)$, and $\Omega = (-\infty, \infty)$. On this domain, we use the weighted inner product

$$\langle u, v\rangle_w = \int_a^b u(x)v(x)w(x)\,dx, \tag{8.1}$$ SL-generic-ip

where w is a positive weight function (though we do allow w to vanish at the endpoints of Ω). We denote the corresponding norm with a subscript, so that

$$\|u\|_w^2 = \langle u, u\rangle_w.$$

We restrict our attention to piecewise smooth functions for which $\|u\|_w$ is finite.

Example 8.1 (Periodic BVP)**.** *For the purposes of discussion here, we consider the case of the periodic eigenfunctions with the domain is $\Omega_{periodic} = (-\pi, \pi)$ and the weight function is $w_{periodic}(x) = 1$. Thus*

$$\langle u, v\rangle_{periodic} = \int_{-\pi}^{\pi} u(x)v(x)\,dx \tag{8.2}$$ SL:periodic-ip

and

$$\|u\|_{periodic}^2 = \int_{-\pi}^{\pi} = u(x)^2\,dx.$$

Example 8.2 (Hermite's problem)**.** *Hermite's problem considers the domain $\Omega = (-\infty, \infty)$ and weight function $w_H(x) = e^{-x^2}$. Thus*

$$\langle u, v\rangle_H = \int_{-\infty}^{\infty} u(x)v(x)e^{-x^2}\,dx$$

and

$$\|u\|_H^2 = \int_{-\infty}^{\infty} = u(x)^2 e^{-x^2}\,dx.$$

For the purposes of addressing Hermite's problem we only work with piecewise smooth functions u where $\|u\|_H < \infty$. Notice that means that for all of the functions u we consider we must have $u(x)^2 e^{-x^2} \to 0$ as $x \to \pm\infty$.

8.2 Linear differential operators

The periodic eigenfunctions (7.14) first came to our attention as solutions to the second differential equation appearing in (2.13), together with the boundary condition (2.16). Sturm-Liouville theory makes use of a generalization of this equation, which we now introduce.

On the domain Ω we consider the eigenvalue problem

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = \lambda u, \tag{8.3}$$

`SL-generic-ode`

where a, b, c are functions of x. The left side of (8.3) we write as Lu, where

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \tag{8.4}$$

`SL-generic-L`

is called a ***second-order linear operator***. Using this notation we write (8.3) as

$$Lu = \lambda u. \tag{8.5}$$

`SL-generic`

Example 8.3 (Periodic BVP)**.** *The eigenvalue problem for the periodic eigenfunctions is*

$$\frac{d^2\psi}{dx^2} = \lambda\psi,$$

and thus the corresponding differential operator is

$$L_{periodic} = \frac{d^2}{dx^2}.$$

Example 8.4 (Hermite's problem)**.** *The eigenvalue problem for Hermite's problem is*

$$\frac{d^2\psi}{dx^2} - 2x\frac{d\psi}{dx} = \lambda\psi,$$

and thus the corresponding differential operator is

$$L_H = \frac{d^2}{dx^2} - 2x\frac{d}{dx}.$$

Not every differential operator of the form (8.4) is suitable for generating a list of eigenfunctions appropriate for constructing approximations. In order to define the two properties that characterize a "good" differential operator, we make use of functions with domain Ω that can be differentiated as many times as we like and which vanish near the endpoints of Ω. Such functions are called ***test functions***; the collection of such functions is given the symbol $C_0^\infty(\Omega)$. We use test functions to make two important definitions.

1. We say that a second order linear operator L is ***self-adjoint*** with respect to the inner product (8.1) if

   $$\langle Lu, v\rangle_w = \langle u, Lv\rangle_w$$

   for all test functions $u, v \in C_0^\infty(\Omega)$.

2. We say that a self-adjoint operator L is ***negative*** with respect to the inner product (8.1) if
$$\langle Lu, u\rangle_w \leq 0$$
for all test functions $u \in C_0^\infty(\Omega)$.

It is straightforward to see that the periodic and Legendre differential operators satisfy both conditions.

Example 8.5 (Periodic BVP)**.** *Suppose that u, v are test functions on the domain $\Omega = (-\pi, \pi)$. This means that they are equal to zero in a tiny neighborhood of the endpoints. Integrating by parts we see that*

$$\begin{aligned}
\langle L_{periodic} u, v\rangle_{periodic} &= \int_{-\pi}^{\pi} \frac{d^2u(x)}{dx^2} v(x)\, dx \\
&= \frac{du(x)}{dx} v(x)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{du(x)}{dx}\frac{dv(x)}{dx}\, dx \qquad (8.6) \\
&= -\int_{-\pi}^{\pi} \frac{du(x)}{dx}\frac{dv(x)}{dx}\, dx,
\end{aligned}$$

SL:periodic-is-

where the boundary term vanishes because u and v are test functions, and thus are equal to zero at and near the endpoints. Integrating by parts once more we easily see that $L_{periodic}$ is self-adjoint with respect to the inner product (8.2).

If we take $u = v$ in (8.6) then we find that

$$\langle L_{periodic} u, u\rangle_{periodic} = -\int_{-\pi}^{\pi} \left(\frac{du(x)}{dx}\right)^2 dx \leq 0$$

and thus $L_{periodic}$ is negative.

Example 8.6 (Hermite's problem)**.** *Suppose that u, v are test functions on the domain $\Omega = (-1, 1)$. Integrating by parts we see that*

$$\begin{aligned}
\langle L_H u, v\rangle_H &= \int_{-\infty}^{\infty} \left(\frac{d^2u(x)}{dx^2} - 2x\frac{du(x)}{dx}\right) v(x) e^{-x^2}\, dx \\
&= \left[\frac{du(x)}{dx} v(x) e^{-x^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{du(x)}{dx}\frac{dv(x)}{dx} e^{-x^2}\, dx \\
&= -\int_{-\infty}^{\infty} \frac{du(x)}{dx}\frac{dv(x)}{dx} e^{-x^2}\, dx.
\end{aligned}$$

Thus it is easy to see that L_H is self-adjoint and negative.

It is straightforward to compute (see Exercise 8.2.1 below) that if we are considering weight function w on domain $\Omega = (a, b)$, then a second-order linear operator L is self-adjoint and negative precisely when L takes the form

$$Lu = \frac{1}{w}\left(\frac{d}{dx}\left[p\frac{du}{dx}\right] + ru\right), \tag{8.7}$$ SL-new-L

where p and r are functions satisfying

$$p(x) \geq 0 \quad \text{and} \quad r(x) \leq 0 \qquad \text{for } a < x < b. \tag{8.8}$$ SL-negative-condition

The relationship between the functions a, b, c and the functions p, r, w is

$$a = \frac{p}{w} \quad b = \frac{1}{w}\frac{dp}{dx} \quad c = \frac{r}{w}.$$

joint-condition **Exercise 8.2.1.** *Suppose that $u, v \in C_0^\infty(\Omega)$. (In particular, we are assuming that u and v are identically zero near the boundary of Ω.)*

1. *Show, using integration by parts ♡, that*

$$\langle Lu, v\rangle_w = \langle u, Lv\rangle_w + \int_a^b u\left\{2\left(\frac{d(aw)}{dx} - bw\right)\frac{dv}{dx} + \frac{d}{dx}\left(\frac{d(aw)}{dx} - bw\right)v\right\}dx.$$

   Conclude that L is self-adjoint only if $b = \dfrac{1}{w}\dfrac{d(aw)}{dx}$.

2. *Let $p = aw$ and choose $b = \dfrac{1}{w}\dfrac{dp}{dx}$ so that L is self-adjoint. Show that L is negative precisely when $p \geq 0$ and $r \leq 0$.*

8.3 Boundary conditions

Once we have a domain Ω and a differential operator L we are ready to pose the corresponding eigenvalue problem. The ***eigenvalue problem for L on domain*** Ω seeks to find a constant λ and a function ψ such that

$$L\psi = \lambda\psi. \tag{8.9}$$ SL:generic-eigenvalue-pro

We require that the function ψ be non-zero.

In order for the eigenvalue problem (8.9) to be well-formulated, we must impose some boundary conditions. The boundary conditions are responsible for ensuring that we obtain a countable list of eigenvalues for our operator. The boundary conditions also ensure that all of the boundary terms vanish when integrating by parts. Before we state the formal requirements, let's consider the two examples to see how these two roles of boundary conditions play out.

Example 8.7 (Periodic BVP)**.** *The eigenvalue problem for the periodic functions on domain $(-\pi,\pi)$ consists of finding λ and ψ such that*

$$\frac{d^2\psi}{dx^2} = \lambda\psi \tag{8.10}$$

ex:sl-periodic-

and such that the periodic boundary conditions

$$\psi(-\pi) = \psi(\pi) \quad \text{and} \quad \psi'(-\pi) = \psi'(\pi)$$

are satisfied.

Note that for every single value of λ we can find a function ψ satistying (8.10).

- *If $\lambda > 0$, then write $\lambda = \omega^2$. The functions $e^{\omega x}$ and $e^{-\omega x}$ both satisfy the eigenvalue problem. The general solution to the eigenvalue problem is*
$$\psi(x) = \alpha e^{\omega x} + \beta e^{-\omega x}$$
for some constants α and β.
- *If $\lambda = 0$, then the functions 1 and x both satisfy the eigenvalue problem. The general solution to the eigenvalue problem is*
$$\psi(x) = \alpha + \beta x$$
for some constants α and β.
- *If $\lambda < 0$, then write $\lambda = -\omega^2$. The functions $e^{i\omega x}$ and $e^{-i\omega x}$ both satisfy the eigenvalue problem. The general solution to the eigenvalue problem is*
$$\psi(x) = \alpha e^{i\omega x} + \beta e^{-i\omega x}$$
for some constants α and β.

If we impose the periodic boundary conditions, however, then the possible values of λ is much restricted.

- *In the case that $\lambda > 0$ imposing the periodic boundary conditions on the general solution $\psi(x) = \alpha e^{\omega x} + \beta e^{-\omega x}$ yields the system*
$$\begin{aligned} \alpha e^{\omega\pi} + \beta e^{-\omega\pi} &= \alpha e^{-\omega\pi} + \beta e^{\omega\pi}, \\ \alpha e^{\omega\pi} - \beta e^{-\omega\pi} &= \alpha e^{-\omega\pi} - \beta e^{\omega\pi}. \end{aligned}$$
From this we deduce that $\alpha = 0$ and $\beta = 0$. Since we are only interested in the situation where ψ is no the zero function, we deduce that $\lambda > 0$ is not a possibility.

- *In the case that $\lambda = 0$, the imposition of the periodic boundary conditions implies that $\psi = \alpha$.*
- *Finally, in the case that $\lambda < 0$, imposing the periodic boundary conditions yields the system*

$$\alpha e^{i\omega\pi} + \beta e^{-i\omega\pi} = \alpha e^{-i\omega\pi} + \beta e^{i\omega\pi},$$
$$\alpha e^{i\omega\pi} - \beta e^{-i\omega\pi} = \alpha e^{-i\omega\pi} - \beta e^{i\omega\pi}.$$

  From this we see that we can choose α and β freely, provided ω is an integer.

Combining the zero and negative cases we conclude that the eigenvalue problem has a countable list of solutions, where $\lambda = -k^2$ for $k = 0, 1, 2, 3, \ldots$. The corresponding functions ψ are e^{ikx} and e^{-ikx}.

We emphasize that it is the imposition of the boundary condition that leads to the "discretizing" (or, in physics-speak, the "quantizing") of the eigenvalues.

ex:hermite-bc

Example 8.8 (Hermite's problem)**.** *We use the example of Hermite's problem to illustrate another role that eigenvalues play, namely the elimination of boundary terms when integrating by parts. In the case of Hermite's problem, the domain is the whole real line $\Omega = (-\infty, \infty)$. Thus we impose the boundary conditions "at infinity"*

$$e^{-x^2/2}\psi(x) \to 0 \text{ and } e^{-x^2/2}\frac{d\psi(x)}{dx} \to 0 \qquad \text{as } x \to \pm\infty.$$

We first apply these boundary conditions to show that if λ and ψ satisfy

$$L_H\psi = \lambda\psi$$

then we must have $\lambda \le 0$. To see this, we compute

$$\begin{aligned}
\lambda\|\psi\|_H^2 &= \langle \lambda\psi, \psi\rangle_H \\
&= \langle L_H\psi, \psi\rangle_H \\
&= \int_{-\infty}^{\infty} \left(\frac{d^2\psi(x)}{dx^2} - 2\frac{d\psi(x)}{dx}\right)\psi(x)e^{-x^2}\,dx \\
&= \int_{-\infty}^{\infty} \frac{d}{dx}\left(e^{-x^2}\frac{d\psi(x)}{dx}\right)\psi(x)\,dx \\
&= \left[\frac{d\psi(x)}{dx}e^{-x^2}\psi(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-x^2}\left(\frac{d\psi(x)}{dx}\right)^2 dx.
\end{aligned}$$

The boundary condition at infinity implies that the square-bracket terms vanishes, which implies that $\lambda\|\psi\|_H^2 \leq 0$. *Since the norm of* ψ *cannot be negative, we conclude that* $\lambda \leq 0$.

As a second application of the boundary conditions we show that eigenfunctions having different eigenvalues must be orthogonal. To see this, suppose that we have two solutions to the Hermite eigenvalue problem, meaning we have constants λ_1, λ_2 *and functions* ψ_1, ψ_2 *such that*

$$L_H\psi_1 = \lambda_1\psi_1 \quad \text{and} \quad L_H\psi_2 = \lambda_2\psi_2.$$

Integrating by parts, and using the boundary condition to deduce that the boundary terms vanish, we find that

$$\begin{aligned}\lambda_1\langle\psi_1,\psi_2\rangle_H &= \langle\lambda_1\psi_1,\psi_2\rangle_H \\ &= \langle L_H\psi_1,\psi_2\rangle_H = \langle\psi_1, L_H\psi_2\rangle_H \\ &= \langle\psi_1,\lambda_2\psi_2\rangle_H = \lambda_2\langle\psi_1,\psi_2\rangle_H\end{aligned}$$

and thus that

$$(\lambda_1 - \lambda_2)\langle\psi_1,\psi_2\rangle = 0.$$

From this we easily conclude that if $\lambda_1 \neq \lambda_2$ *then the corresponding eigenfunctions* ψ_1 *and* ψ_2 *must be orthogonal.*

The previous two examples illustrate the importance of boundary conditions. In order to figure out what precise condition we must impose we repeat the integration-by-parts argument from Example 8.8 in more generality. Working with weight function w and linear operator

$$L = \frac{1}{w}\frac{d}{dx}\left(p\frac{d}{dx}\right) + r$$

we see that

$$\begin{aligned}\langle Lu, v\rangle_w = &\left[p(x)\frac{du(x)}{dx}v(x)\right]_a^b \\ &+ \int_a^b \left(-p(x)\frac{du(x)}{dx}\frac{dv(x)}{dx} + r(x)u(x)v(x)\right)dx.\end{aligned}$$

This motivates us to say that an ***Sturm-Liouville admissible boundary condition*** on domain $\Omega = (a, b)$ is one that implies

$$p(b)u'(b)v(b) - p(a)u'(a)v(a) = 0 \tag{8.11}$$ SL-generic-BC

for all functions u, v satisfying the boundary condition. Note that

- the Dirichlet boundary condition is Sturm-Liouville admissible,
- the Neumann boundary condition is Sturm-Liouville admissible, and
- the periodic boundary condition is Sturm-Liouville admissible.

In the exercise below you should that eigenfunctions that solve a Sturm-Liouville admissible boundary value problem are orthogonal. This means that we can use them to construct approximations of other piecewise smooth functions. The Sturm-Liouville theorem states, in essence, that these approximations behave in a similar way to the Fourier series approximations.

Exercise 8.3.1. *Let w be a weight function on domain Ω. Suppose that L is self-adjoint and negative with respect to w. Suppose also that ψ_1 and ψ_2 are eigenfunctions of L with different eigenvalues, and that they satisfy a Sturm-Liouville admissible boundary condition. Show that $\langle \psi_1, \psi_2 \rangle_w = 0$.*

8.4 The Sturm-Liouville Theorem

Finally, we are ready to state the Sturm-Liouville theorem. We consider the following situation:

- Let $\Omega = (a, b)$ be a spatial domain and suppose that w is a weight function on Ω.
- Suppose that L is a second-order linear differential operator that is self-adjoint and negative with respect to the inner product determined by w.
- Suppose that (BC) is a Sturm-Liouville admissible boundary condition.

In this situation the Sturm-Liouville Theorem tells us that there exists an infinite list of eigenvalues

$$0 \geq \lambda_1 > \lambda_2 > \lambda_3 > \dots$$

with $\lambda_k \to -\infty$ and a corresponding collection of eigenfunctions

$$\psi_1, \psi_2, \psi_3, \dots$$

such that $L\psi_k = \lambda_k \psi_k$ for $k = 1, 2, 3, \dots$. Furthermore, the eigenfunctions have the following properties.

Orthogonality The eigenfunctions ψ_k are mutually orthogonal to one another, meaning that $\langle \psi_j, \psi_k \rangle_w = 0$ whenever $j \neq k$.

Completeness Suppose that u is a piecewise smooth function on domain Ω such that the norm $\|u\|_w$ is finite. Then we can approximate u using combinations of the eigenfunctions by

$$u \approx \sum_{k=1}^{\infty} \alpha_k \psi_k \qquad \text{where} \quad \alpha_k = \frac{\langle \psi_k, u \rangle_w}{\|\psi_k\|_w^2}.$$

Here the symbol $\approx$ means that the sum converges to the value of u at all points where u is continuous.

Unfortunately, the proof of the Sturm-Liouville theorem is beyond the scope of this course. (It makes use of several important theorems developed in Math 341.) If you are interested in some of the details, I recommend the book *Sturm-Liouville Theory and its Applications* by M.A. Al-Gwaiz, published in the Springer Undergraduate Mathematics Series.

Exercise 8.4.1. *Consider the Dirichlet boundary value problem from the beginning of the course:*

$$\frac{d^2\psi(x)}{dx^2} = \lambda \psi(x) \qquad \text{on } \Omega = (0, L)$$
$$\psi(0) = 0 \qquad \text{and} \qquad \psi(L) = 0.$$

1. *Explain why this is an admissible Sturm-Liouville problem. That is, articulate that each of the hypotheses above is satisfied in this situation.*

2. *Explain how the conclusions of the Sturm-Liouville theorem play out in this context. That is, write out what the eigenvalues and eigenfunctions are, and that they have the properties claimed above.*