CHAPTER 1

Algebra

1. Ring Completion

An example of a category that the reader is likely unfamiliar with is the category of semirings. The objects in these categories are called semirings, which are simply rings without necessarily having an additive inverse. For completion, a formal definition of semiring follows.

DEFINITION 1.1 (Semiring). A *semiring* is a set S paired with the binary operations $(+, \cdot)$ such that the following properties hold:

(i) The operation $+$ is associative and commutative
(ii) The operation $\cdot$ is associative
(iii) The operation $\cdot$ distributes over $+$
(iv) S has both an additive and multiplicative identity.

A simple example of a semiring is the set of nonnegative integers under the usual addition and multiplication operations. The element 0 is the additive identity and 1 is the multiplicative identity. In fact, this example of $\mathbb{N} \cup \{0\}$ has two additional nice properties: commutativity of multiplication and the cancellation property under addition. To be precise, the cancellation property promises that given elements a, b, and s in a semiring, the statement $a+s = b+s$ implies $a = b$. This section will focus on commutative semirings with the additive cancellation property.

To complete the category of semirings, the morphisms of a category must be discussed. In this case, the morphisms are referred to as homomorphisms of semirings and the definition follows.

DEFINITION 1.2 (Homomorphism of Semirings). Take monoids S and R and consider a mapping $\varphi : S \to R$. Then, φ is a *homomorphism of semirings* if:

(i) $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in S$.
(ii) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$
(iii) $\varphi(1) = 1$

Note homomorphisms between semirings follows the same structure as homomorphism between rings; in fact, a homomorphism of rings *is* a homomorphism of semiring, for rings are themselves semirings. In fact, even a mapping from a semiring S to a ring R could be considered a homomorphism of semirings if the mapping satisfies the necessary properties. Overall, the category of semirings is frustratingly close to the category of rings. Luckily, there is a functor from the category of commutative semirings with cancellation to the category of rings called *ring extension* – a way to expand the structure of a monoid into a fully fledged ring. K-Theory heavily relies on this functor, so pay particular attention to it.

The formal definition of ring extension is addressed shortly, but first consider the following example. Take the semiring of nonnegative integers; predictably, the ring extension of this example is the set of all integers. In this example, ring extension hinges on the fact we can map a pair of nonnegative integers (a, b) to an element of $\mathbb{Z}$ via the mapping $a - b$. In a semiring, there is no promise of subtraction, but the pair (a, b) can secretly represent the difference $a-b \in \mathbb{Z}$ through an equivalence relation.

Definition 1.3 (Ring Completion). Take commutative semiring S with additive cancellation. Then, a *ring completion* of of S is a commutative ring R together with an injective homomorphism $i : S \to R$ that satisfies the following property: for any commutative ring R' and corresponding homomorphism of semirings $\varphi : S \to R'$, there exists a unique homomorphism of rings such $\psi : R \to R'$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "i"] & R' \\ R \arrow[ur, dashed, "\psi", "\exists!"'] & \end{tikzcd}$$

Figure 1. The Universal Property

That is, $\psi \circ i = \varphi$.

There is still work to be done with this definition; it must still be verified that the above construction exists and is unique. The requirement that the above triangle commutes is the *universal property*, and throughout this chapter there will be many constructions using the universal property structure.

To get a better feel for this definition, recall the example of semiring of nonnegative integers extending into ring of all integers. In this case, the extension function $i : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ is given by the injective identity function $i(n) = n$. First, observe how the choice of $\mathbb{Z}$ as the extension fulfills the requirement of the definition. For instance, taking $R' = \mathbb{Z}/(2)$ and homomorphism $\varphi : n \mapsto n \mod 2$, then the homomorphism over the integers $\psi : z \mapsto z \mod 2$ satisfies the triangle, and it follows from the restrictions provided by the definition of a ring homomorphism that this is the unique choice of ψ. However, this is only one specific case. The homomorphism ψ will be unique regardless of the choice of R' and φ. This makes $\mathbb{Z}$ a valid group completion for the nonnegative integers. In fact, $\mathbb{Z}$ is the *unique* group completion and the proof of this is given now.

Proof of Uniqueness of Definition 1.3. Consider two ring completions (R, i) and (R', i'). It must be shown that R and R' are isomorphic. By (R, i) a ring completion and taking (R', i') to be a ring-homomorphism pair, the universal property in the definition of ring completion promises the existence of a unique homomorphism $\psi_1 : R \to R'$ such that $\psi_1 \circ i = i'$. Similarly, by swapping the roles of (R, i) and (R', i'), there exists a a unique homomorphism $\psi_2 \circ i' = i$. But then, the composition $\psi_2 \circ \psi_1 : R \to R$ satisfies the commutativity restriction $(\psi_2 \circ \psi_1) \circ i = i$. Thus $\psi_2 \circ \psi_1$

must be the unique map promised by the universal property by applying the universal property of ring completion (R, i) on (R, i) itself. However, the identity mapping also satisfies the condition $\mathrm{Id} \circ i = i$ and so the uniqueness conditions gives that $\psi_2 \circ \psi_1 = \mathrm{Id}$. See Figure 2 for a visual of this argument. The same argument gives that $\psi_1 \circ \psi_2 = \mathrm{Id}$ and thus ψ_1 and ψ_2 are inverses of one another. This gives that ψ_1 and ψ_2 are isomorphisms and so $R \cong R'$.

$$\begin{tikzcd}
R \arrow[rr, dashed, "\mathrm{Id}"] \arrow[ddr, dashed, "\psi_1"'] & & R \\
& S \arrow[ul, "i"] \arrow[ur, "i"'] \arrow[d, "i'"] & \\
& R' \arrow[uur, dashed, "\psi_2"'] &
\end{tikzcd}$$

FIGURE 2. Uniqueness of Ring Completion Argument

□

The above argument never appeals to the specific properties of rings and semirings; in fact, this argument applies to *all* definitions defines through the universal property. For every additional definition using the universal property in this chapter, uniqueness will follow automatically.

All that needs to be shown to justify a definition using the universal property is existence. For the case of semiring completion, this existence proof is given in section **??** of the Appendix. Here are the important takeaways from the proof. For a semiring S, the proof uses the equivalence relation $\sim$ on $S \times S$ given by $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. The motivation for this equivalence relation is that the integers can be created by all differences of the nonnegative numbers. In fact, one can think of this equivalence relation as "sneaky subtraction" stemming from the wish to express $a_1 - b_1 = a_2 - b_2$ without the explicit use of subtraction. From this equivalence relation, we get a natural addition on the equivalence classes that gives a commutative group structure. However, in order to get a well-defined multiplication, the semiring must have the additive cancellation property.

Rings are nicer than semirings; they have additive inverses and extensive theory. As shown above, every commutative semiring with cancellation extends to a unique ring; therefore, given a semiring with these properties, it is best to ditch the semiring and instead talk about the ring extension. Keep this motivation to extend an "incomplete" object into a nicer object in mind for the following constructions.

2. Packing Together Modules

I am still working on this section

Definition 1.4 (Module). Let M be a set, and let R be a commutative ring with identity. Further, take an additive operation $+ : M \times M \to M$ and a scalar multiplication from $R \times M$ to M. Then, M is a *module over* R if:

(i) $(r+s)m = rm + sm$ for all $r, s \in R$ and $m \in M$
(ii) $r(m+n) = rm + rn$ for all $r \in R$ and $m, n \in M$
(iii) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$
(iv) $1 \cdot m = m$ for all $m \in M$

Modules can be made into a category. Keeping in mind that modules are generalizations of vector spaces, the natural homomorphism to associate with with modules is a linear map as in the following definition.

Definition 1.5 (Module Homomorphism). Let R be a commutative ring and let M and N be R-modules. Then, a *homomorphism of modules* is a mapping $\varphi : M \to N$ such that

(i) $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ for all $m_1, m_2 \in M$.
(ii) $\varphi(rm) = r\varphi(m)$ for all $r \in R$ and $m \in M$.

Definition 1.6 (Direct Sum). Take commutative ring R with identity and consider a collection M_λ of R-modules, $\lambda \in I$ where I is an index set. Then, the *direct sum* of the collection M_λ, denoted $\oplus_{\lambda \in I} M_\lambda$, is the unique R-module such that:

(i) For all $\lambda \in I$, there is an inclusion map $i_\lambda : M_\lambda \to \oplus_{\lambda \in I} M_\lambda$.
(ii) The universal property is satisfied. That is, for any R-module N and homomorphisms of R-modules $\varphi_\lambda : M_\lambda \to N$ there exists a unique homomorphism of R-modules $\psi : \oplus_{\lambda \in I} M_\lambda$ such that the following diagram commutes. That is, $\psi \circ i_\lambda = \psi_\lambda$ for all $\lambda \in I$.

$$\begin{tikzcd}
M_\lambda \arrow[r, "\varphi_\lambda"] \arrow[d, "i_\lambda"] & N \\
\bigoplus_{\lambda \in I} M_\lambda \arrow[ur, dashed, "\psi", "\exists!"'] &
\end{tikzcd}$$

Figure 3. Universal Property of Direct Sum

Uniqueness of the direct sum follows directly from the universal property as mentioned in the ring completion section. /*comment on existence*/.

Recall the categories of vector spaces, abelian groups, and commutative rings all are special cases of modules. Then, the functor from each of these categories to the category of modules defines allows each category to borrow the direct sum operation on modules, which in turn defines a direct sum operation on each category. However, borrowing a module operation only promises that the resulting direct sum will be a module — not a vector space, abelian group, or commutative ring. For each individual category, it must be verified that the direct sum construction is an object in the same category. For example, the direct sum of vector spaces with field F promises an F-module which is luckily exactly equivalent to a vector space over F. Similarly, the direct sum of abelian groups promises a $\mathbb{Z}$-module which again is luckily exactly equivalent to an abelian group. However, showing that the direct sum of commutative rings results in a commutative ring takes more work to verify, for there is no predefined multiplication mapping on the direct sum /*prove this works and elaborate*/.

Definition 1.7 (Tensor Product). Take commutative ring R with identity and take M_1 and M_2 to be R-modules. Then, the *tensor product* of M_1 and M_2, denoted $M_1 \otimes M_2$ is the unique R-module such that:

(i) There is a bilinear map $b : M_1 \times M_2 \to M_1 \otimes M_2$
(ii) The universal property is satisfied. That is, for for any R-module N with corresponding bilinear map $\omega : M_1 \times M_2 \to N$, there exists a unique homomorphism of modules $\psi : M_1 \otimes M_2 \to N$ such that the following diagram commutes. That is, $\psi \circ b = \omega$.

$$\begin{tikzcd} M_1 \times M_2 \arrow[r, "\omega"] \arrow[d, "b"] & N \\ M_1 \otimes M_2 \arrow[ur, dashed, "\psi", "\exists!"'] & \end{tikzcd}$$

Figure 4. Universal Property of Tensor Products

Again, uniqueness of the direct sum follows automatically from the universal property. /*comment on existence*/.

/*extend to vector spaces, abelian groups, and commutative rings*/

3. Verifications

Proof of Existence of Definition 1.3. The existence of a ring completion is shown through an explicit construction. Take any commutative semiring with additive cancellation $(S, +, \cdot)$ and consider the equivalence relation $\sim$ on $S \times S$ defined as follows: for (a_1, b_1), (a_2, b_2) in $S \times S$, then let $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. The aim is to make the set of equivalence classes under $\sim$ into a ring.

First, define the additive operation $+$ by

$$[(a, b)] + [(c, d)] = [(a + c, b + d)]$$

Next, define the multiplicative operation $\cdot$ by

$$[(a, b)] \cdot [(c, d)] = [(ac + bd, ad + bc)]$$

This proof aims to verify that the set of equivalence classes $S \times S/\sim$ paired with the operations $(+, \cdot)$ forms a commutative ring that is a ring completion of S.

It must be verified that the additive operation is well defined, so consider elements (a_1, b_1), (a_2, b_2), (c_1, d_1), (c_2, d_2) in $S \times S$ such that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$. Then, I claim that $(a_1+c_1, b_1+d_1) \sim (a_2+c_2, b_2+d_2)$. Indeed, this satisfies the definition of the equivalence relation, for

$$\begin{aligned}(a_1 + c_1) + (b_2 + d_2) = (a_1 + b_2) + (c_1 + d_2) \\ = (a_2 + b_1) + (c_2 + d_1) = (a_2 + c_2) + (b_1 + d_1)\end{aligned}$$

where the above computation used the substitutions $a_1+b_2 = a_2+b_1$ and $c_1+d_2 = c_2+d_1$ promised by the relations $(m_1, m_2) \sim (m_1', m_2')$ and $(l_1, l_2) \sim (l_1', l_2')$. This confirms that $+$ is well-defined on $(S \times S)/\sim$.

The transitivity and commutativity of $+$ on the equivalence classes follows immediately from the commutativity and transitivity of the operation $+$ on S.

Next, note that the additive identity in $(S \times S)/\sim$ is given by $[(0, 0)]$ where 0 denotes the identity element in S. Indeed, we have $[(a, b)] + [(0, 0)] = [(a, b)]$ for any element $[(a, b)]$.

The proposed ring has an inverse mapping for the addition operation. Consider an element $[(a, b)]$. Then, I claim the element $[(b, a)]$ forms the desired inverse. To see this, consider the sum $[(a+b, b+a)]$ and note that $(a + b) + 0 = 0 + (b + a)$, which shows $[(a + b, b + a)] = [(0, 0)]$.

It must be verified that the multiplicative operation is well-defined before verifying any further properties. Consider the elements $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$ in $S \times S$. It then must be verified that $(a_1c_1 + b_1d_1, a_1d_1 + b_1c_1) \sim (a_2c_2 + b_2d_2, a_2d_2 + b_2c_2)$. To accomplish this, consider the following $M_1, M_2 \in S$:

$$\begin{aligned}M_1 &= c_2(a_1 + b_1) + b_2(c_1 + d_1) + b_2c_2 \\ M_2 &= c_1(a_2 + b_2) + b_1(c_2 + d_2) + b_1c_1\end{aligned}$$

Next, observe that using the relations $a_1 + b_2 = a_2 + b_1$ and $c_1 + d_2 = c_2 + d_1$, it follows that $a_1c_1 + b_1d_1 + M_1 = a_2c_2 + b_2d_2 + M_2$.

$$\begin{aligned}a_1c_1 + b_1d_1 + M_1 &= a_1c_1 + b_1d_1 + c_2a_1 + c_2b_1 + b_2c_1 + b_2d_1 + b_2c_2 \\ &= (a_1 + b_2)(c_1 + c_2) + (d_1 + c_2)(b_1 + b_2) \\ &= (a_2 + b_1)(c_1 + c_2) + (d_2 + c_1)(b_1 + b_2) \\ &= a_2c_2 + b_2d_2 + c_1a_2 + c_1b_2 + b_1c_2 + b_1d_2 + b_1c_1 = a_2c_2 + b_2d_2 + M_2\end{aligned}$$

A similar process shows that $a_1d_1 + b_1c_1 + M_1 = a_2d_2 + b_2c_2 + M_2$. Then, summing the two results gives

$$(a_1c_1 + b_1d_1) + (a_2d_2 + b_2c_2) + (M_1 + M_2) = (a_2c_2 + b_2d_2) + (a_1d_1 + b_1c_1) + (M_1 + M_2)$$

Applying the additive cancellation property of S to the term $(M_1 + M_2)$ gives the desired relation and provides the conclusion $(a_1c_1 + b_1d_1, a_1d_1 + b_1c_1) \sim (a_2c_2 + b_2d_2, a_2d_2 + b_2c_2)$ and so the multiplicative operation is well defined.

The transitivity of the multiplicative operation follows directly from $+$ and $\cdot$ transitive in S. Similarly, the commutativity of the multiplicative operation follows directly from the commutativity of $+$ and $\cdot$ in S.

Next, note that the element $[(1,0)]$ acts as an identity element for the multiplicative operation. Indeed, $[(1,0)] \cdot [(a,b)] = [(a,b)]$ for any element $[(a,b)]$.

It only remains to show that $+$ distributes over $\cdot$ to verify that $S \times S/\sim$ forms a ring. Indeed, for elements $[(a,b)]$, $[(c,d)]$, $[(e,f)]$:

$$
\begin{aligned}
&[(e,f)] \cdot ([(a,b)] + [(c,d)]) = [(e,f)] \cdot [(a+c, b+d)] \\
&= [(ea + fb + ec + fd, eb + ed + fa + fe)] \\
&= [(ea + fb, eb + fa)] + [(ec + fd, ed + fc)] = [(e,f)] \cdot [(a,b)] + [(e,f)] \cdot [(e,d)]
\end{aligned}
$$

Thus we have that $(S \times S)/\sim$ forms a commutative ring under the proposed operations. However, it remains to show that $(S \times S)/\sim$ is a valid ring completion. The necessary inclusion map $i : S \to (S \times S)/\sim$ is given by $i(s) = [(s,0)]$. Then, take any ring R' and homomorphism $\varphi : S \to R'$; the existence and uniqueness of a commuting ring homomorphism $\psi : (S \times S)/\sim \to R'$ must be shown.

Uniqueness follows quickly from its homomorphism properties and the commutativity of the universal property. Indeed, take two commuting ring homomorphisms ψ and ψ' from $S \times S/\sim$ to R'. Then, the restrictions $\psi \circ i = \varphi$ and $\psi' \circ i = \varphi$ paired with i injective gives that $\psi = \psi'$ over the image $i(S)$. Then observe that any element $[(a,b)]$ is the composition of elements in $i(S)$ by $[(a,b)] = [(a,0)] - [(b,0)]$. Then, the homomorphism properties of rings extends ψ and ψ' to be equivalent over all of $(S \times S)/\sim$ giving uniqueness.

It only remains to show existence of the homomorphism. The map $\psi : [(a,b)] \mapsto \varphi(a) - \varphi(b)$ works. Commutativity follows easily, for $(\psi \circ i)(s) = \psi([(s,0)]) = \varphi(s)$ for all $s \in S$. Now, it must be verified that ψ is a homomorphism. So, consider elements $[(a,b)]$ and $[(c,d)]$ of the ring completion.

The following equality chain shows that the the additive property of φ gives the additive property of ψ.

$$
\begin{aligned}
&\psi([(a,b) + (c,d)]) = \psi([(a+c, b+d)]) = \varphi(a+c) - \varphi(b+d) \\
&= (\varphi(a) - \varphi(b)) + (\varphi(c) - \varphi(d)) = \psi([(a,b)]) + \psi([(c,d)])
\end{aligned}
$$

Similarly, the additive and multiplicative property of φ gives the multiplicative property of ψ.

$$
\begin{aligned}
&\psi([(a,b)] \cdot [(c,d)]) = \psi([(ac + bd, ad + bc)]) \\
&= \varphi(ac + bd) - \varphi(ad + bc) = \varphi(a)\varphi(c) + \varphi(b)\varphi(d) - \varphi(b)\varphi(c) - \varphi(a)\varphi(d) \\
&= (\varphi(a) - \varphi(b))(\varphi(c) - \varphi(d)) = \psi([(a,b)]) \cdot \psi([(c,d)])
\end{aligned}
$$

Finally $\psi(1) = \psi([(1,0)]) = \varphi(1) = 1$, completing the proof.

$\square$