CHAPTER 1

The Definition of K-Theory

K-Theory is a functor from the category of topological spaces to the category of rings. Topological spaces are messy, making it difficult to understand properties about topological spaces and homomorphisms between topological spaces. However, groups and rings are simple algebraic objects with much structure — an easier object to analyze.

The K-Theory functor first considers all possible vector bundles over a topological space. Looking at every possible bundle is too much information, but it turns out that after simplifying the set to equivalence classes of vector bundles, a ring structure emerges.

There are two veins of K-Theory; the difference is in the equivalence classes used to reduce the set of vector bundles. First, there is the *K-Theory* of a topological space X, denoted $K(X)$. In this case, the equivalence classes have a natural semiring structure and the ring is defined through the ring extension. Secondly, there is the *reduced K-Theory* of a topological space X, denoted $\widetilde{K}(X)$, which has bigger equivalence classes. In reduced K-Theory, the equivalence classes themselves can be made directly into a ring. In both K-Theory and reduced K-Theory, the functor is contravariant.

1. The K-Theory Functor K

/*I have a plan on how to motivate this equivalence relation*/

The equivalence relation used in K-Theory to simplify the set of vector bundles is as follows.

DEFINITION 1.1 (Stably Isomorphic). Define the equivalence relation $\approx_s$ on vector bundles over the same base space such that for bundles E_1 and E_2, $E_1 \approx_s E_2$ if $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$ for some n where ε^n denotes the n dimensional trivial bundle. Here, E_1 and E_2 are said to be *stably isomorphic*.

This equivalence relation gives a natural semiring structure on the equivalence classes.

CLAIM 1.2. Take compact Hausdorff base space X. The set of all stably isomorphic equivalence classes over the vector bundles on X forms a commutative semiring with cancellation when taking the direct sum $\oplus$ as the additive operation and the tensor product $\otimes$ as the multiplicative operation. This semiring is denoted $J(X)$.

Proving the above claim takes some work, but the full proof is given in section 3.1 of this chapter. Most of the proof is routine verification, but getting the cancellation property and verifying that multiplication is well-defined appeals to /*ref*/, which is where the compact Hausdorff condition

is used. Because this semiring is commutative with the cancellation property, it is most convenient to consider the unique commutative ring promised through ring completion.

DEFINITION 1.3 (K-Theory of a Topological Object). Take compact Hausdorff base space X and let $J(X)$ denote the commutative semiring with cancellation as described in claim. Then, the ring completion of $J(X)$ is the *K-Theory of X* and is denoted $K(X)$.

Next, consider the following computations of K-Theory on simple topological spaces.

EXAMPLE 1.4 (K-Theory of a point). Consider as a topological space a single point $\{x_0\}$. The only choice of vector bundles on $\{x_0\}$ are the trivial bundles of each dimension. That is, the set $\{\varepsilon^0, \varepsilon^1, \varepsilon^2, \dots\}$. No two trivial bundles will be in the same stable isomorphism class, giving the set of equivalence classes $\{[\varepsilon^0], [\varepsilon^1], [\varepsilon^2], \dots\}$, which is isomorphic to the semiring $\mathbb{N}$. So, the ring $K(\{x_0\})$ is the ring completion of $\mathbb{N}$. That is, $K(\{x_0\}) \cong \mathbb{Z}$.

EXAMPLE 1.5 (K-Theory of n points). Consider the topological space of n disconnected points $\{x_0, x_2, \dots, x_{n-1}\}$. Then, each point can have a fiber of any dimension, and the choice of fibers is independent of one another. This gives that the set of all vector bundles is isomorphic to the set $\mathbb{N}^n$. Then, any arbitrary vector bundle over the space can be denoted $(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}})$ where the first element in the tuple represents the bundle over the first disconnected point, the second element represents the bundle over the second, and so on. The equivalence classes can then be represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}})]$, and in this case every vector bundle is its own equivalence class, and so this is isomorphic to the semiring $\mathbb{N}^n$, which has ring completion $\mathbb{Z}^n$. Thus, $K(\{x_0, x_1, \dots, x_{n-1}\}) \cong \mathbb{Z}^n$.

However, defining the K-theory on topological objects only brings the operation half way to being a functor. Functors map objects to objects but also morphisms to morphism. Just as K-Theory brings topological spaces to rings, K-Theory must bring continuous functions between topological spaces to homomorphisms of rings. In this case, K-Theory is a contravariant functor and so reverses the direction of the mapping.

CLAIM 1.6. Take topological spaces X and Y with a continuous function $f : X \to Y$. Let $J(X)$ denote the semiring as described in claim 1.2 and let $K(Y)$ be the K-Theory of Y. Further, define the function $J(f) : J(X) \to K(Y)$ defined on an equivalence class $[E] \in J(X)$ by

$$J(f) : [E] \mapsto [f^*(E)]$$

where f^* denotes the pullback as defined in /*ref*/. Then, $J(f)$ is a well-defined homomorphisms of semirings.

Verifying the above follows easily from the properties of pullback given in /*ref*/. The full proof is given in section 3.2. Note that this is the point where the contravariant property emerges. Because the elements of the semiring is equivalence classes of vector bundles, a homomorphism consists of mappings from one vector bundle to another vector bundle. This is best done through the induced vector bundle by the pullback which, as discussed previously, must be done in the reverse direction.

DEFINITION 1.7 (K-Theory of a Topological Morphism). For compact Hausdorff spaces X and Y with a continuous function $f : X \to Y$, let $J(f) : J(X) \to J(Y)$ denote the homomorphism of semirings as described in claim 1.6. Further, let $i_X : J(X) \to K(X)$ and $i_Y : J(Y) \to K(Y)$ denote ring completions. Then, the *K-Theory of f* is the unique homomorphism of rings $K(f) : K(X) \to$

$K(Y)$ such that the following diagram commutes as promised by the universal property. That is, that $K(f) \circ i_X = i_Y \circ J(f)$.

$$
\begin{array}{ccc}
J(Y) & \xleftarrow{\quad J(f) \quad} & J(X) \\
\downarrow{\scriptstyle i_Y} & \swarrow{\scriptstyle i_Y \circ J(f)} & \downarrow{\scriptstyle i_X} \\
K(Y) & \xleftarrow{\quad K(f) \quad} & K(X)
\end{array}
$$

FIGURE 1. Definition of $K(f)$ through Universal Property

And that is the definition of K-Theory! Figure 2 denotes a diagram of the K-Theory functor. Next, observe how the K-Theory functor on the following examples of concrete topological spaces.

EXAMPLE 1.8. This example examines the K-Theory of the inclusion from the space with one point to the space with n points. So, Take the topological space of n points $\{x_0, x_1, \ldots, x_{n-1}\}$ and consider the subspace $\{x_0\} \subset \{x_0, \ldots, x_{n-1}\}$. Then, let $i : \{x_0\} \to \{x_0, \ldots, x_{n-1}\}$ be the inclusion map.

Recall that the equivalence classes of vector bundles over x_0 can be represented $[\varepsilon^n]$ and the equivalence classes over $\{x_0, x_1, \ldots, x_{n-1}\}$ can be represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})]$. Then, by definition the function $J(i) : J(\{x_0, \ldots, x_{n-1}\}) \to J(\{x_0\})$ is given by

$$
J(i) : [(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})] \mapsto [i^*(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})]
$$

Recall that the pullback of the inclusion is simply the restriction to the space in the domain. In this case, that is the restriction to the point x_0 and so function $J(i)$ is

$$
J(i) : [(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})] \mapsto [\varepsilon^{k_0}]
$$

Now step away from the vector bundles themselves and let $J(i)$ denote the semiring homomorphism $J(i) : \mathbb{N}^n \to \mathbb{N}$ given by $J(i) : (k_0, k_1, \ldots, k_{n-1}) \mapsto k_0$. Finally, consider the ring completions $\mathbb{Z}^n$ and $\mathbb{Z}$. The universal property then promises a unique ring homomorphism $K(i) : \mathbb{Z}^n \to \mathbb{Z}$. In this case, the ring homomorphism is given by $K(i) : (k_0, k_1, \ldots, k_{n-1}) \mapsto k_0$, but this time, each k_i can take on the value of any integer. See figure 3 for a visual of this construction.

$$
\begin{array}{ccc}
Y & \xrightarrow{\quad f \quad} & X \\
\downarrow{\scriptstyle K} & & \downarrow{\scriptstyle K} \\
K(Y) & \xleftarrow{\quad K(f) \quad} & K(X)
\end{array}
$$

FIGURE 2. The K-Theory Functor

$$
\begin{array}{ccc}
\{x_0\} & \xrightarrow{\quad i \quad} & \{x_0, \ldots, x_{n-1}\} \\
\downarrow & & \downarrow \\
\mathbb{N} & \xleftarrow{\quad J(i) \quad} & \mathbb{N}^n \\
\downarrow & & \downarrow \\
\mathbb{Z} & \xleftarrow[\quad K(i) \quad]{k_0 \leftarrowtail (k_0, \ldots, k_{n-1})} & \mathbb{Z}^n
\end{array}
$$

FIGURE 3. K-Theory of a Morphism Example

2. The Reduced K-Theory Functor $\widetilde{K}$

There is another closely related vein of K-Theory called *reduced K-Theory*. Reduced K-Theory is a is a functor from the category of topological spaces to the category of abelian groups. However, with as assumption discussed later, this functor can be extended to the category of commutative rings (but not necessarily with identity). Reduced K-Theory uses a stronger equivalence relation, which gives fewer equivalence classes.

DEFINITION 1.9. Define the equivalence relation $\sim$ on vector bundles E_1 and E_2 over the same base space such that $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^m$ for some n and m.

Then, this equivalence class immediately gives rise to the desired group.

DEFINITION 1.10. Take a compact Hausdorff topological space X and let $\widetilde{K}(X)$ denote the set of all equivalence classes under the relation $\sim$ as described in definition 1.9. Then, define the group operation by the direct sum $\oplus$ operation on the elements. This forms a well-defined abelian group and is called the *reduded K-Theory* of X.

The verification that $\widetilde{K}(X)$ indeed forms a well-defined group is straight-forward, but it is worth noting that the existence of inverses uses /*ref*/, which requires the compact Hausdorff condition.

Consider some simple computation of reduced K-Theory.

EXAMPLE 1.11 (Reduced K-Theory of a Point). Again as a topological space a single point $\{x_0\}$. The only choice of vector bundles on $\{x_0\}$ is the set trivial bundles $\{\varepsilon^0, \varepsilon^1, \varepsilon^2, \ldots\}$. In this case, however, each trivial bundle is in the same isomorphism class, so the set of equivalence classes has only the identity element ε^0. So, the reduced K-Theory of a point $\widetilde{K}(\{x_0\})$ is the trivial group.

It must still be addressed how reduced K-Theory maps continuous topological maps to group morphisms, which again makes use of the pullback.

DEFINITION 1.12 (Reduced K-Theory of a Topological Morphism). Let $f : Y \to X$ denote a continuous function between topological spaces. Then the induced mapping $\widetilde{K}(f) : \widetilde{K}(X) \to \widetilde{K}(Y)$

is defined by

$$\widetilde{K}(f) : [E] \mapsto [f^*(E)]$$

where the equivalence classes are with respect to the relation $\sim$ as in definition 1.9.

/*verify well-defined*/

It first must be verified that the above mapping of functions is well-defined, which is done in /*ref*/. But once well-defined is out of the way, the properties of pullback $\mathrm{Id}^*(E) \approx E$ and $(f \circ g)^*(E) \approx g^*(f^*(E))$ immediately give that $\widetilde{K}$ obeys the rules for functors.

Now consider the following example, which demonstrates an important relationship between the functors K and $\widetilde{K}$.

EXAMPLE 1.13 (Reduced K-Theory of n points). Consider the topological space of n disconnected points $\{x_0, x_2, \ldots, x_{n-1}\}$. Again, each fiber is free to have a vector space of any dimension, so any arbitrary vector bundle over the space can be denoted $(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})$. The equivalence classes under $\sim$ are then represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})]_\sim$. In this case, every equivalence class has more than one element. In particular, $(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}}) \sim (\varepsilon^{l_0}, \varepsilon^{l_1}, \ldots, \varepsilon^{l_{n-1}})$ if there exists bundles ε^k and ε^l such that

$$(\varepsilon^{k_0} \oplus \varepsilon^k, \varepsilon^{k_1} \oplus \varepsilon^k, \ldots, \varepsilon^{k_{n-1}} \oplus \varepsilon^k) \approx (\varepsilon^{l_0} \oplus \varepsilon^l, \varepsilon^{l_1} \oplus \varepsilon^l, \ldots, \varepsilon^{l_{n-1}} \oplus \varepsilon^l)$$

By simply ditching the "ε" symbol, there is an isomorphism to equivalence classes of n tuples of integers $[(k_0, k_1, \ldots, k_{n-1})]$ where $(k_0, k_1, \ldots, k_{n-1}) \sim (l_0, l_1, \ldots, l_{n-1})$ if there exists integers k and l such that

$$(k_0 + k, k_1 + k, \ldots, k_{n-1} + k) = (l_0 + l, l_1 + l, \ldots, l_{n-1} + l)$$

Additionally, the group operation is element wise addition as is taken from the representation with "ε". Note that all of the k's and l's are allowed to be any integer but they originally represented the dimension of a trivial bundle, so it seems they should only be nonnegative integers. However, expanding the elements to integers does not change the group, for every new element containing a negative integer will land in a preexisting equivalence class.

And so, elements in reduced K-Theory are of the form of equivalence classes $[(k_0, k_1, \ldots, k_{n-1})]$ with the relation as defined earlier.

However, this does not clear up what this reduced K-Theory is isomorphic to. The bottom line is that the reduced K-Theory of n points is isomorphic to the group $\mathbb{Z}^{n-1}$. There are a few ways to see this, but the most educational is with the following.

Fix the point $x_0 \in \{x_0, \ldots, x_{n-1}\}$ and consider the K-Theory groups $K(\{x_0, \ldots, x_{n-1}\})$ and $K(\{x_0\})$. The goal will be to construct a homomorphism $\varphi : \widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \to K(\{x_0, \ldots, x_{n-1}\})$ and use φ with the mapping $K(i) : K(\{x_0, \ldots, x_{n-1}\})$ as defined previously. Overall, this will give the chain of mappings as shown in figure 4. Additionally takes note of the isomorphisms $K(\{x_0, \ldots, x_{n-1}\}) \cong \mathbb{Z}^n$ and $K(\{x_0\}) \cong \mathbb{Z}$.

Now, define φ on the discussed representations on the K-Theory and reduced K-Theory of n points as follows.

$$\varphi : [(k_0, k_1, \ldots, k_{n-1})] \to (0, k_1 - k_0, \ldots, k_{n-1} - k_0)$$

The verification that φ is indeed a group homomorphism follows easily. /*ref*/. Now recall that $K(i)$ is given by the following.

$$K(i) : (l_0, l_1, \ldots, l_{n-1}) \mapsto l_0$$

Now make a two observations. Firstly, $\mathrm{Im}(\varphi) = \ker(K(i))$. Secondly, φ is injective. The proofs for both of these are brief and given in /*ref*/. φ injective gives $\widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \cong \mathrm{Im}(\varphi)$ which then gives the relationship

$$\widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \cong \ker(K(i))$$

Lastly note that element of the kernel are of the form $(0, l_1, \ldots, l_{n-1})$ for any choice of l's, so the kernel is isomorphic to $\mathbb{Z}^{n-1}$. Overall, this gives $\widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \cong \mathbb{Z}^{n-1}$.

$$\widetilde{K}(\{x_0, \ldots, x_{n-1}\}) \xrightarrow{\ \varphi\ } K(\{x_0, \ldots, x_{n-1}\}) \xrightarrow{\ K(i)\ } K(\{x_0\})$$

FIGURE 4. Chain of Homomorphisms for n-points example

The above example found the reduced K-Theory by demonstrating that $\widetilde{K}(\{x_0, \ldots, x_{n-1}\})$ is isomorphic to $\ker(K(i))$. In fact, a relationship like this exists in general. For any topological space X with a fixed point $x_0 \in X$ and inclusion $i : x_0 \to X$, the relationship $\widetilde{K}(X) \cong \ker(K(i))$ holds. The proof of this uses tools developed in the next chapter, but the above example gives a taste of the proof. A consequence of this, however, is that $K(i)$ is a ring homomorphism, so $\ker(K(i))$ is an ideal. Then $\widetilde{K}(X)$ is isomorphic to this ideal and thus can be given a multiplication. Thus, we can consider $\widetilde{K}(X)$ to be a ring, but not necessarily with identity.

The computed examples hint at another relationship between K-Theory and reduced K-Theory. Note that for a point $\{x_0\}$, $K(\{x_0\}) \cong \mathbb{Z}$ and $\widetilde{K}(\{x_0\}) \cong \{0\}$. The computations for a collection of points $\{x_0, x_1, \ldots, x_{n-1}\}$ showed $K(\{x_0, x_1, \ldots, x_{n-1}\}) \cong \mathbb{Z}^n$ and $\widetilde{K}(\{x_0, x_1, \ldots, x_{n-1}\}) \cong \mathbb{Z}^{n-1}$. Note that $\mathbb{Z} \cong \{0\} \oplus Z$ and more generally, $\mathbb{Z}^n \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}$. More generally, it is true that $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ for any topological space X, but this proof again uses techniques of the following chapter.

3. Verifications

3.1. Semiring Verification.

PROOF. Take a compact Hausdorff space X. It must be verified that the set of stable isomorphism classes of vector bundles over X with operations defined by the direct sum $\oplus$ and the tensor product $\otimes$ indeed satisfies all the properties of a commutative semiring with additive cancellation.

Before proceeding further, it must be verified that addition is well defined. So, take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X. Then, take nonnegative integers n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Then it follows that $E_1 \oplus F_1 \approx_s E_2 \oplus F_2$ by the following chain of equalities.

$$(E_1 \oplus F_1) \oplus \varepsilon^{n+m} \approx (E_1 \oplus \varepsilon^n) \oplus (F_1 \oplus \varepsilon^m) \approx (E_2 \oplus \varepsilon^n) \oplus (F_2 \oplus \varepsilon^m) \approx (E_2 \oplus F_2) \oplus \varepsilon^{n+m}$$

Where the equivalence $\varepsilon^{n+m} \approx \varepsilon^n \oplus \varepsilon^m$ /*reference*/ was used.

With addition well defined, the associativity and commutativity of addition follows directly from the associativity and commutativity of the direct sum on vector bundles /*reference*/. Further, the result $E \oplus \varepsilon^0 \cong E$ for any vector bundle E /*ref*/ makes the equivalence class $[\varepsilon^0]$ the additive identity.

The additive cancellation follows from /*reference $E \oplus E'$ trivial result*/, which applies here by X compact Hausdorff. Indeed, take bundles E, F, and S over X such that $[E] + [S] = [F] + [S]$. First note that in the case of S trivial, $[E] = [F]$ by definition. Otherwise, by /*ref*/, there exists a bundle S such that $S \oplus S'$ is trivial. Adding $[S']$ to both sides reduces the expression to the first case with $[E] + [S \oplus S'] = [F] + [S \oplus S']$, giving $[E] = [F]$ as desired.

Before proceeding with any multiplicative verifications, it must be verified that the tensor product $\otimes$ gives a well defined multiplicative operation. So, again take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X and nonnegative integers n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Next, define the bundle M by

$$M \approx \varepsilon^n \otimes (F_1 \oplus \varepsilon^m) \oplus \varepsilon^m \otimes (E_1 \oplus \varepsilon^n) \approx \varepsilon^n \otimes (F_2 \oplus \varepsilon^m) \oplus \varepsilon^m \otimes (E_2 \oplus \varepsilon^n)$$

Next, observe that M is constructed exactly so that the relation $E_1 F_1 \oplus M \approx E_2 F_2 \oplus M$ holds:

$$E_1 \otimes F_1 \oplus M \approx (E_1 \oplus \varepsilon^n)(F_1 \oplus \varepsilon^m) \oplus \varepsilon^n \varepsilon^m \approx (E_2 \oplus \varepsilon^n)(F_2 \oplus \varepsilon^m) \oplus \varepsilon^n \varepsilon^m \approx E_2 \otimes F_2 \oplus M$$

So, take M' to be the bundle such that $M \oplus M'$ is trivial as promised by /*ref*/. Then, the desired conclusion follows easily, giving that multiplication is well-defined.

$$E_1 \otimes F_1 \oplus (M \oplus M') = E_2 \otimes F_2 \oplus (M \oplus M')$$

With multiplication well defined, the associativity and commutativity of multiplication follows directly from the associativity and commutativity of the tensor product on vector bundles /*reference*/. Similarly, the distributivity of $\otimes$ over $\oplus$ in vector bundles /*ref*/ gives that the defined multiplication distributes over the defined addition. Finally, the result $E \otimes \varepsilon^1 \cong E$ for any vector bundle E /*ref*/ makes the equivalence class $[\varepsilon^1]$ the multiplicative identity.

$\square$

3.2. Homomorphism of Semirings Verification.

PROOF. Let $f : X \to Y$ denote a continuous function between two compact Hausdorff spaces.

First it must be verified that $J(f)$ is well-defined. Specifically, it must be shown that if $[E_1] = [E_2]$, then $J(f)([E_1]) = J(f)([E_2])$. That is, it must be shown that $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$ for some n implies

$f^*(E_1) \approx_s f^*(E_2)$. First, note the following application of the distributivity of pullback over direct sum /*ref*/.

$$f^*(E_1) \oplus f^*(\varepsilon^n) \approx f^*(E_1 \oplus \varepsilon^n) \approx f^*(E_1 \oplus \varepsilon^n) \approx f^*(E_2) \oplus f^*(\varepsilon^n)$$

The result that the pullback of a trivial bundle is trivial combined with the above confirms $f^*(E_1) \approx_s f^*(E_2)$ and so $J(f)$ is well-defined.

With $J(f)$ well-defined, verifying that $J(f)$ is a semiring homomorphism follows easily from the properties of pullback. Specifically, the distributivity of pullback over direct sum directly gives the distributivity of $J(f)$ over the defined addition. Similarly, the distributivity of pullback over tensor product gives that $J(f)$ distributes over the defined multiplication. Lastly, the property that f^* maps the bundle ε^1 over X to the bundle ε^1 over Y implies that $J(f)$ maps the multiplicative identity to the multiplicative identity. □

3.3. K-Theory Functor Satisfies Contravariant Composition Law.

PROOF. Let X, Y, and Z denote compact Hausdorff spaces and take $f : X \to Y$ and $g : Y \to Z$ be continuous functions between them.

Further denote the semirings as defined in claim /*ref*/ by $J(X)$, $J(Y)$, and $J(Z)$. Additionally, let $i_X : J(X) \to K(X)$, $i_Y : J(Y) \to K(Y)$, and $i_Z : J(Z) \to K(Z)$ denote the ring completions of each semiring as in definition /*ref*/. Further, let $J(g) : J(Z) \to J(Y)$ and $J(f) : J(Y) \to J(X)$ denote the homomorphism of semirings as described in claim /*ref*/. Finally, let the functions $K(g) : K(Z) \to K(Y)$, $K(f) : K(Y) \to K(X)$, and $K(f \circ g) : K(Z) \to K(X)$ be the unique functions such that the following composition identities hold.

$$K(f) \circ i_X = i_Y \circ J(f)$$
$$K(g) \circ i_Y = i_Z \circ J(g)$$
$$K(f \circ g) \circ i_X = i_Z \circ J(f \circ g)$$

Additionally note that $J(f \circ g) = J(g) \circ J(f)$ follows from the pullback property of /*ref*/ $(f \circ g)^*(E) = g^*(f^*(E))$ on a bundle E by the following computation on an element $[E] \in J(X)$.

$$J(f \circ g)([E]) = [(f \circ g)^*(E)] = [g^*(f^*(E))] = J(g)([f^*(E)]) = J(g)(J(f)([E]))$$

Substitutions of the preceding result together with the earlier composition identities allows for the following result.

$$(K(g) \circ K(f)) \circ i_X = K(g) \circ (i_Y \circ J(f)) = (i_Z \circ J(Y)) \circ J(f) = i_Z \circ J(f \circ g)$$

And so $K(f) \circ K(g)$ fulfills the defining property of $K(f \circ g)$. Because the function $K(f \circ g)$ is the unique function fulfilling this property, it must be that $K(f) \circ K(g) = K(f \circ g)$. Figure 5 provides a visual aid for this argument.

□

$$J(Y)$$

$J(g)$ $\quad$ $J(f)$

$J(f \circ g)$

$$J(Z) \longleftarrow \qquad \qquad \qquad J(X)$$

i_Y

i_Z $\qquad\qquad K(Y)$ $\qquad\qquad$ i_X

$K(g)$ $\qquad\qquad$ $K(f)$

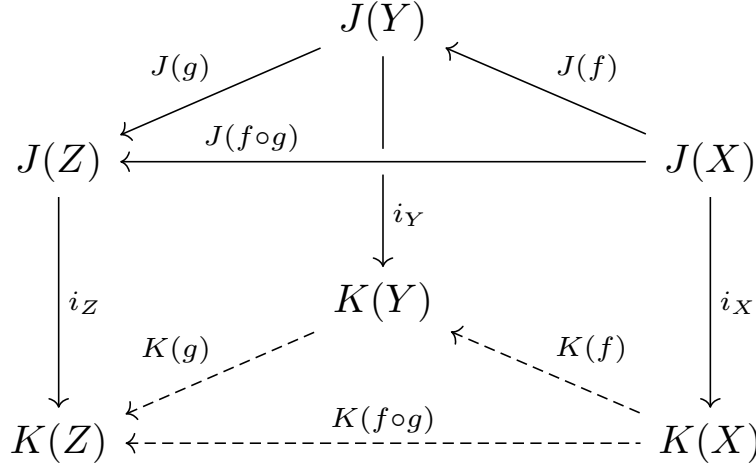$$K(Z) \longleftarrow \qquad K(f \circ g) \qquad K(X)$$

FIGURE 5. K-Theory Contravariant Composition Argument

3.4. Reduced K-Theory forms Group.

PROOF. First it must be verified that the direct sum operation $\oplus$ is well-defined on the equivalence classes. So, consider vector bundles $E_1 \sim E_2$ and $F_1 \sim F_2$. Then let n_1, m_1, n_2, m_2 be the numbers such that $E_1 \oplus \varepsilon^{n_1} \approx E_2 \oplus \varepsilon^{n_2}$ and $F_1 \oplus \varepsilon^{m_1} \approx F_2 \oplus \varepsilon^{m_2}$. It then follows that $E_1 \oplus F_1 \approx E_2 \oplus F_2$

$$(E_1 \oplus F_1) \oplus (\varepsilon^{n_1+m_1}) \approx (E_1 \oplus \varepsilon^{n_1}) \oplus (F_1 \oplus \varepsilon^{m_1}) \approx (E_2 \oplus \varepsilon^{n_2}) \oplus (F_2 \oplus \varepsilon^{m_2}) \approx (E_1 \oplus F_1) \oplus (\varepsilon^{n_2+m_2})$$

Where the above computation used $\varepsilon^{n+m} \approx \varepsilon^n \oplus \varepsilon^m$.

With the group operation well-defined, the associativity and commutativity of the operation follows from direct sum associative and commutative on bundles.

The identity element in the group is given by the equivalence class $[\varepsilon^0]$, which is the set of all trivial bundles. Indeed, $[E] + [\varepsilon^0] = [E \oplus \varepsilon^0] = [E]$.

It only remains to show the existence of inverses, which appeals to /*ref*/. Then take any element $[E]$ and consider the promised bundle E' such that $E \oplus E' \approx \varepsilon^n$ for some trivial bundle of dimension n. Then, the element $[E']$ is the inverse element.

$$[E] + [E'] = [E \oplus E'] = [\varepsilon^n] = [\varepsilon^0]$$

$\square$