Chapter 1

Ordinary differential equations

• combine these files in to one

1.1 First-order linear ODEs

der-linear-odes

Let's briefly review some basic features of first-order constant-coefficient systems of linear ODEs taking the form

$$\frac{d}{dt}U = MU, \tag{1.1}$$

generic-first-order-ode

where U is a vector and M is a matrix. Equations of this form give us a chance to discuss three important concepts: linearity, completeness, and independence.

1. We say that an equation is ***linear*** if any linear combination of solutions is also a solution. The system (1.1) is linear in the sense that whenever U_1 and U_2 are solutions to the equation and α, β are constants, then $\alpha U_1 + \beta U_2$ is also a solution. This property is sometimes called the ***principle of superposition.*** It is important that the solutions U_1, U_2, and the constants α, β, can be either real or complex.

2. A collection of solutions to a differential equation is called ***complete*** if all solutions can be constructed as linear combinations of solutions in the collection. This can be verified by making sure that all initial conditions can be obtained by taking linear combinations of solutions in the collection.

3. A list of solutions to a differential equation is called ***independent*** if none of the solutions can be expressed as a linear combination of

the others. The practical consequence of this is that if a collection of solutions is independent, then there is only one way to express a given solution as a linear combination of solutions in the collection.

example:LinearODE1 **Example 1.1.** *Consider the system*

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Here are how the three principles play out for this example:

1. *We can verify by direct computation that*

$$U_1 = e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \text{and} \qquad U_2 = e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

are solutions to our equation. Thus the principle of superposition says that for any α, β the combination

$$\alpha U_1 + \beta U_2 = \alpha e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} + \beta e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

is also a solution to the equation. For example, if we choose $\alpha = \frac{1}{2i}$ and $\beta = -\frac{1}{2i}$ then we see that

$$\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

is a solution. Notice that we were able to construct real solutions from complex solutions by clever choice of α and β.

2. *I claim that the functions U_1, U_2 above form a complete collection. To see this, suppose we want to construct a linear combination to match the initial condition*

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

where u_0, v_0 are constants. This requires

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \alpha\alpha e^{i(0)}\begin{pmatrix} 1 \\ i \end{pmatrix} + \beta e^{-i(0)}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

This reduces to a linear system of two equations with two unknowns

$$\begin{aligned} u_0 &= \alpha + \beta \\ v_0 &= \alpha i - \beta i, \end{aligned} \tag{1.2}$$

ODE-example-lin

which we can easily solve. Since any initial condition can be attained by taking linear combinations of U_1, U_2, we know that U_1, U_2 form a complete collection.

3. *Finally, we can see that the two solutions U_1 and U_2 above are independent because neither is a multiple of the other. Another way to see this is that the linear system (1.2) has only one solution α, β. Thus there is only one way to express a given solution in terms of U_1 and U_2.*

If a list of solutions to a linear equation is complete and independent, then we have a sort of existence and uniqueness: there exists a unique way to build any solution as a linear combination of functions in our collection. Thus finding complete, independent collections of solutions is an important step in understanding a differential equation.

In the ODEs class, we developed a systematic method for constructing collections of solutions to equations of the form (1.1). We did this by looking for solutions of the form

$$e^{\mu t} U_* = e^{\mu t} \begin{pmatrix} u_* \\ v_* \end{pmatrix}, \tag{1.3}$$

LinearODE1-eigen-anzatz

where μ, u_*, v_* are constants. We found that the function in (1.3) is a solution to (1.1) exactly when U_* is an eigenvector of M with eigenvalue μ. Recall also that we were able to understand the behavior of solutions just by knowing the behavior of these eigensolutions. (For instance, we can draw a phase diagram for (1.1) just from knowing the eigenstuff.)

It turns out that the approach that we used in the ODEs course for understanding (1.1) can be applied to many other types of problems. Thus we emphasize the three parts of this approach:

- **Construct a complete, independent collection of eigensolutions.**
- **Use the principle of superposition to build general solutions by taking linear combinations of eigensolutions.**
- **Understand the behavior of general solutions by understanding the behavior of the eigensolutions.**

It is worth coming back to these three points throughout the course.

• Exercise about symmetric matrices having orthogonal eigenvalues... complete, orthogonal, independent collection?

1.2 Second-order linear ODEs

• re-do this to make it consistent with previou subsection

Now let's recall a some facts about linear, second-order ordinary differential equations.

One can see by direct computation that second-order ordinary differential equations of the form

$$a\frac{d^2u}{dt^2} + b\frac{du}{dt} + cu = 0 \tag{1.4}$$

LinearODE2-gene

are linear; here a, b, c could be constants or could be functions of t.

For the rest of this section, we assume that a, b, c are constants and that $a \neq 0$. We can use linearity to find the most general solution to (1.4) as follows: Notice that $e^{\mu t}$ is a solution to (1.4) precisely when the constant μ satisfies

$$a\mu^2 + b\mu + c = 0. \tag{1.5}$$

LinearODE2-quad

We then consider two cases:

- If (1.5) has two distinct solutions μ_1 and μ_2 then the most general solution to (1.4) is
$$u(t) = \alpha e^{\mu_1 t} + \beta e^{\mu_2 t}.$$
- If (1.5) has only one solution μ, then both $e^{\mu t}$ and $te^{\mu t}$ are solutions to (1.4) and thus the most general solution is
$$u(t) = \alpha e^{\mu t} + \beta t e^{\mu t}.$$

In the case of two distinct solutions to (1.5), it could be that the two solutions μ_1 and μ_2 are complex, in which case we have $\mu_1 = x + iy$ and $\mu_2 = x - iy$ for some real numbers x, y. Thus the most general solution is complex-valued. We can make clever use of linearity to show that both the real part of $e^{\mu_1 t}$ and the imaginary part of $e^{\mu_1 t}$ are solutions to (1.4). The result is that we can construct a real-valued general solution

$$u(t) = Ae^{xt}\cos(yt) + Be^{xt}\sin(yt). \tag{1.6}$$

LinearODE2-real

Finally, recall that the equation (1.4) is equivalent to the first-order system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

The eigenvalues of the matrix in this system are precisely the solutions to (1.5). Thus we may interpret solutions of the form $e^{\mu t}$ as "eigensolutions".

Exercise 1.2.1. *Show how to obtain the formula (1.6) for the real general solution to (1.4) in the case that (1.5) has two complex solutions.*

1.3 The Simple Harmonic Oscillator

Perhaps the most famous ordinary differential equation is the *simple harmonic oscillator* (SHO) equation

$$\frac{d^2u}{dt^2} + \omega^2 u = 0, \tag{1.7}$$ SHO-intro

where $\omega > 0$ is a constant. Solutions u to (1.7) represent the displacement from equilibrium of an object attached to a fixed point by a spring.

There are two ways to obtain the simple harmonic oscillator equation from assumptions arising from physics. The first is to apply Hooke's formula $F = -ku$, which says that the force on the object is proportional to the displacement; the constant k describes the "strength" of the spring. Inserting Hooke's formula in to Newton's formula $ma = F$ yields

$$m\frac{d^2u}{dt^2} = -ku. \tag{1.8}$$ SHO-alt

Setting $\omega^2 = k/m$ leads to (1.7).

The second way to obtain (1.7) is to consider the conservation of energy[1]. Suppose that the displacement from some point of the object of mass m is given by the function u. The *kinetic energy* K, which represents the "energy due to being in motion", is given by

$$K = \frac{1}{2}\left(\frac{du}{dt}\right)^2.$$

The spring that connects the object to some fixed point gives rise to the *potential energy* V, which represents the "energy associated to the physical configuration (or location)" and is given by the formula

$$V = \frac{1}{2}ku^2.$$

The *total energy* $E = K + V$ is *conserved* if it is constant in time. In this case, conservation of energy requires that

$$\begin{aligned} 0 &= \frac{d}{dt}\left[\frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}ku^2\right] \\ &= \frac{du}{dt}\left(m\frac{d^2u}{dt^2} + ku\right), \end{aligned}$$

[1] We won't discuss here what energy *is*–to paraphrase Richard Feynmann (see his *Lectures on Physics*): We don't really know what it is; we just know that we can compute this number... and the number is always the same!

which must hold for all times t. Thus conservation of energy requires that either u is a constant function (which is a possibility, but does not describe most of the interesting physical situations) or u satisfies (1.7).

Since (1.7) is a linear equation, the ideas of the previous sections can applied. Seeking solutions of the form $e^{\mu t}$ leads to two complex solutions

$$e^{i\omega t} \quad \text{and} \quad e^{-i\omega t}.$$

Thus the most general complex solution is

$$u(t) = \alpha e^{i\omega t} + \beta e^{-i\omega t}.$$

Splitting the solution $e^{i\omega t}$ in to real and imaginary parts, we see that the most general real solution is

$$u(t) = A\cos(\omega t) + B\sin(\omega t).$$

From these formulas we see that the constant ω represents the frequency at which solutions to (1.7) oscillate.

Alternatively, we can write (1.7) as the first-order system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

The eigenvalues of the matrix

$$M = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

and $\mu = \pm i\omega$ and thus the straight-line solutions are

$$e^{i\omega t}\begin{pmatrix} 1 \\ i\omega \end{pmatrix} \quad \text{and} \quad e^{-i\omega t}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}.$$

Thus the general solution is

$$U(t) = \alpha e^{i\omega t}\begin{pmatrix} 1 \\ i\omega \end{pmatrix} + \beta e^{-i\omega t}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}.$$

Notice how the first component agrees with the previously found general solution u. Notice also that by viewing the simple harmonic oscillator as a first-order system we can interpret the eigenvalues as the frequencies at which the solutions oscillate.

1.4 Power series solutions to ODEs

We say that a function $f(x)$ can be ***represented by a power series*** (centered at zero) if there exists constants a_0, a_1, ... and constant $R > 0$ such that for $|x| < R$ the sum

$$\sum_{k=0}^{\infty} a_k x^k, \tag{1.9}$$

converges to $f(x)$. In other words

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{for } |x| < R. \tag{1.10}$$ GenericPowerSeries

We want the function with a power series representation (1.10) to be differentiable. We can ensure this by requiring that the series converge absolutely on the interval of convergence. The following is proved rigorously in a real analysis course.

Theorem 1.2 (Properties of power series)**.**

1. *(Uniqueness) Suppose that*

$$\sum_{k=0}^{\infty} a_k x^k = 0 \tag{1.11}$$

for all $|x| < R$ *with* $R > 0$*. Then* $a_k = 0$ *for each* k*.*

2. *(Differentiablity) Suppose that*

$$\sum_{k=0}^{\infty} a_k x^k \tag{1.12}$$

converges absolutely to $f(x)$ *for* $|x| < R$*. Then* $f(x)$ *is differentiable on the interval* $|x| < R$ *and*

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}. \tag{1.13}$$

Let's now see how to use power series representations in order to solve differential equations. Consider the equation

$$\frac{dA}{dt} = \lambda A, \tag{1.14}$$ ODE-Exp

where λ is some fixed number. (Of course, we know from our differential equations course that A should be a multiple of $e^{\lambda t}$, but – for the sake of the discussion – let's temporarily put that "knowledge" on hold.)

Let's also assume that A has a power series representation

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad |t| < R. \tag{1.15}$$

Inserting this representation in to (1.14) we find that

$$\sum_{k=1}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty} \lambda a_k t^k. \tag{1.16}$$

We rearrange this to obtain

$$\sum_{k=0}^{\infty} \left[(k+1)a_{k+1} - \lambda a_k\right] t^k = 0 \tag{1.17}$$

for all $|t| < R$.

Thus from the uniqueness property of power series we must have

$$(k+1)a_{k+1} - \lambda a_k = 0 \qquad \text{for } k = 0, 1, 2, 3 \ldots. \tag{1.18}$$

Exp-recurrence

Thus the differential equation (1.14) gives rise to a *recurrence relation* that must be satisfied by the coefficients a_k.

The recurrence relation (1.18) means that

$$\begin{aligned}
a_0 &= \text{anything} \\
a_1 &= \lambda a_0 \\
a_2 &= \frac{1}{2}\lambda a_1 = \frac{1}{2}\lambda^2 a_0 \\
a_3 &= \frac{1}{3}\lambda a_2 = \frac{1}{3 \cdot 2}\lambda^3 a_0 \\
&\vdots \\
a_k &= \frac{1}{k!}\lambda^k a_0.
\end{aligned} \tag{1.19}$$

Because a_0 can have any value, we say that a_0 is *free*.

The result is that $A(t)$ has power series representation

$$A(t) = a_0 \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda t)^k. \tag{1.20}$$

free-exponentia

The sum is (of course!) the well-known power series expression for $e^{\lambda t}$, which converges absolutely for all t. The constant a_0 can be chosen freely, representing our choice of initial condition for (1.14). Finally, we can check (using the ratio test) that the radius of convergence for (1.20) is $R = \infty$.

We summarize this discussion as follows.

- Solutions to certain linear ordinary differential equations can be found by looking for power series representations.
- Inserting a generic power series in to the differential equation yields a recurrence relation for the coefficients.
- Some of the coefficients can be chosen freely (corresponding to freedom of choosing initial conditions); the remaining coefficients are determined by the relation.
- It is important to make sure that the resulting power series actually converges absolutely.

Finally, we remark that sometimes there are "standard choices" for the free constants. For example, if we are not given initial conditions for (1.14), then it is standard to choose $a_0 = 1$, which gives us the usual exponential function.

Exercise 1.4.1. *In calculus 2, we learned how to construct a power series representation for a previously known function; we called this Taylor series. Write down the Taylor series expansions for the following functions. Please also include the radius of convergence for the Taylor series.*

1. e^x
2. $\ln(1-x)$ *(Hint: What is the series for* $1/(1-x)$*?)*
3. $\cos x$
4. $\sin x$

Note: In calculus 1 and 2 these functions were defined in some sort of ad hoc manner. A more sophisticated point of view would be to define the functions by their power series expansions.

Exercise 1.4.2. *Show that the series*

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} x^{2k+1}$$

converges absolutely for $|x| < 1$. Then show that the series converges to

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Exercise 1.4.3.

1. *Recall that a function $f(x)$ is called* ***even*** *if $f(-x) = f(x)$ for all x. Show that the power series representation of an even function contains only even powers of x. [Hint: Use the uniqueness property of power series.]*

2. *A function $f(x)$ is called* ***odd*** *if $f(-x) = -f(x)$ for all x. Show that the power series representation of an odd function contains only odd powers of x.*

3. *Any function can be written as the sum of an even function and an odd function. To see this, note that*

$$f(x) = \frac{1}{2}\underbrace{[f(x) + f(-x)]}_{even} + \frac{1}{2}\underbrace{[f(x) - f(-x)]}_{odd}.$$

   What happens when this decomposition is applied to the power series representation of a function?

4. *What are the even and odd parts of the exponential function?*

Exercise 1.4.4.

Use the power series method to construct the most general solution to

$$f'(x) = x\, f(x).$$

Does the series look familiar?

★**Exercise 1.4.5.** *One can also use the power series method to construct solutions to second order differential equations. As an example, consider*

$$\frac{d^2u}{dt^2} = -\omega^2 u, \tag{1.21}$$

SHO2

where $\omega > 0$ is some constant.

1. *Assume that solution u to (1.21) has power series representation $\sum_{k=0}^{\infty} a_k t^k$. Insert this in to (1.21) and obtain a recursion relation for the coefficients.*

2. *Your recursion relation should leave a_0 and a_1 fixed, and should "split" over the even/odd-numbered coefficients in the sense that a_0 depends on a_2, a_4, etc. and a_1 determines a_3, a_5, etc. Rewrite your recursion relations to account for this splitting by finding formulas of the form*

$$a_{2k} = \boxed{\textit{stuff involving } a_{2k-2}} \qquad \textit{and} \qquad a_{2k+1} = \boxed{\textit{stuff involving } a_{2k-1}}$$

3. *Find an even solution to (1.21) by setting $a_1 = 0$ and $a_0 = 1$. The resulting power series should look very familiar!*

4. *Find the odd solution to (1.21) by setting $a_0 = 0$ and $a_1 = 1$. Again, the result should look very familiar!*

5. *Use linearity to construct the general solution to (1.21).*

6. *Verify that $e^{i\omega t}$ is a solution to (1.21). What choices of a_0 and a_1 gives rise to this solution?*