Calc IV: Sturm-Liouville Theory with Physics Applications
Midterm Exam 1, February 2018

You can do it!

Please see the Exam Cover Sheet for the rules for this take-home exam.

Problem 1. This problem relates to the spherical harmonics, functions first discovered in the study of gravity (1782ish by Legendre and Laplace) and later (1900ish) applied in quantum mechanics. We begin by considering the following second order differential equation

$$(1 - x^2)f''(x) - 2xf'(x) + 6f(x) = 0.$$

a. Please assume that this differential equation has a solution of the form $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with some nonzero radius of convergence. Please find a recursion relation that gives the coefficients a_k, $k = 2, 3, 4, \dots$, in terms of free constants a_0 and a_1.

b. Please use your recursion relation to write out a_2, a_3, a_4 and a_5 in terms of a_0 and a_1.

c. Using your recursion relation for part (a), please write solutions $f(x)$ that you have found in the following form

$$f(x) = a_0(\text{stuff1}) + a_1(\text{stuff2}).$$

Note that you do not need to find a closed form for the expressions being added in stuff1 and stuff2, its OK to list a few terms and then $+\dots$ as needed.

d. Determine the radius of convergence of stuff1 and stuff2. (This will require use of the Ratio Test.) The smaller of the two radii of convergence is the radius of convergence of the full series.

e. Celebrate because what you have found is in fact a general solution to the differential equation on the interval around zero given by the radius of convergence. Consider the particular solution when $a_0 = a_1 = 1$. Please graph a partial sum of this particular solution and turn it in with your exam.

Problem 2. This problem probes the results of Problem 1 and serves to stretch our minds a bit. Observe stuff1 and stuff2 once more. Something surprising should have happened with one of these. The surprising thing is called the 'second degree Legendre polynomial.' To obtain a 'degree p Legendre polynomial' (here p is any natural number such as $1, 2, 3, \dots$) we follow the same steps as Problem 1, but we replace the number 6 in the differential equation with the number $p(p+1)$. Please compute the degree 4 Legendre polynomial. Legendre polynomials have a pleasant property when viewed as elements of $L^2([-1, 1])$. What is this property? Please don't prove this in general (we don't have the tools yet), but can you give an example?

Problem 3. This problem seeks standing wave solutions $u(t, x)$ of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

under the assumption of Neumann boundary conditions:

$$\frac{\partial u}{\partial x}(t, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t, L) = 0.$$

Note that we are assuming $0 \leq x \leq L$, so we are modeling a vibrating string that lies on the x-axis over the interval $[0, L]$. Given this problem please complete the following.

a Write down the eigenvalues of the 1-dimensional Laplace operator that you obtain as you complete this problem. Please show the work that you do to get these eigenvalues.

b. Write down the eigenfunctions, also called standing waveforms, corresponding to each of the eigenvalues you found in part (a). Please show the work that you do to get these eigenfunctions.

c. Use a graph to argue why a few of the standing waveforms found in part (b) appear to be correct. Please turn in both the graph and the argument why the graph implies your standing waveforms appear to be correct.

d. Write down all of the standing wave solutions, also called scaling solutions, of this problem. Please show the work that you do to get these standing wave solutions.

e. Please write the general solution to this IBVP.

f. Suppose we are given an initial position $u_0(x)$ and an initial velocity $v_0(x)$ for the motion of this string. Please do two separate computations (using the general solution from part (e)) in order to show how we can use these initial conditions to find a particular solution to our IBVP. Specifically, how do the unknown constants in the general solution relate to $u_0(x)$ and $v_0(x)$? (Note that because you are not given specific functions for the initial conditions, you are not expected to actually find the values of these unknown constants.)