

# Calculus 3 Packet

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Spring 2018

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# **Part I**

# **Syllabus**

## Educational Goals

**Goal 1:** To become fluent in the language of vector (fields), coordinatizations (affine and curvilinear coordinatizations included) and infinitesimal line/area/volume elements. The corresponding assessment will primarily be based on student performance on related (midterm, final) exam problems and homework.

**Goal 2:** To develop an ability to articulate and evaluate cumulative effect in higher dimensional context, by means of integral calculus. The extent to which this goal is achieved will be determined by student performance on related (midterm, final) exam problems and homework.

**Goal 3:** To master the connections between the differential and the integral calculus as in the Fundamental Theorems of Calculus. The extent to which this goal is achieved will be determined by student performance on related (midterm, final) exam problems and homework.

**Goal 4:** To become fluent in differential calculus computations such as those involving Taylor approximations, the Chain Rule and/or (constrained) optimization. The corresponding assessment will be based on student performance on related (midterm, final) exam problems and homework.

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## Grading scheme

For each rubric under Educational Goals above you will receive a letter grade determined by your performance on the corresponding portion(s) of exams, homework assignments etc. The final course grade will be the weighted average of the above, with weights as given below:

**Goal 1:** 30%

**Goal 2:** 30%

**Goal 3:** 30%

**Goal 4:** 10%

For the description of letter grades and their numerical equivalents please refer to our College Catalog. Please note that a professor has a right to withdraw a student for the reasons of non-attendance. *I reserve the right to fail any student who exhibits an extreme lack of understanding of multivariable and vector calculus on the final exam. I also reserve the right to boost the course grade of any student who presents an impressive amount of progress throughout the semester.*

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## Textbook, calculators etc.

Our main reference will be the 4th edition of *Calculus: early transcendental functions* by Robert T. Smith and Roland B. Minton. **Do note that much of the course content will not be directly based on the textbook material.** I expect you to rely on your own lecture notes. The textbook is highly optional; several copies are available in the department and/or SQRC.

I expect you to have access to a calculator for basic homework and exam problems. In addition, I will expect you to develop an ability to use a more substantial computing software. Although I do not have a strong preference as to what you use, I will give you some basic introduction to a computing software called Mathematica. For further information see the chapter on Mathematica in this packet.

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## Homework

Homework problems will be assigned daily, and can be found later on in this packet. Problems involving material covered on Mondays and Tuesdays will be due the following Thursday, while the problems involving material covered on Thursdays and Fridays will be due the following Monday. The second week of class and the weeks of the Fall and Thanksgiving Breaks are clear exceptions.

I understand that you might find yourself in a situation where you cannot complete a homework assignment on time. *Assuming you are making a good faith effort to stay current in the class,* you will be given an opportunity to make up an incomplete assignment. Do note that grading of homework assignment which are turned in late will be *much postponed*. Please do not expect any timely feedback on those assignments; instead, come talk to me (or at least an SQRC tutor) in person.

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## Exams and such

**MIDTERM EXAMS:** There will be two midterm exams. **Both exams will be evening exams:** so that students can have extra time to work on the problems. The first midterm exam will take place on **Monday, February 27th** and the second midterm exam will take place on **Monday, April 16th**. Students with class conflicts will be given an opportunity to take the exam at a different time. I recommend setting aside two hours for the exams, just in case.

**FINAL EXAM:** Cumulative exam on **Wednesday, May 2nd, from 1PM to 4PM.** Expect the exam to take the full three hours.

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## Day-to-day schedule

**Tuesday, January 16th:** Cartesian/rectangular coordinates in plane.

**Thursday, January 18th:** Cartesian/rectangular coordinates in space.

**Friday, January 19th:** Practice.

**Monday, January 22nd:** Contour maps.

**Tuesday, January 23rd:** First examples of quadratic surfaces.

**Thursday, January 25th:** On vectors.

**Friday, January 26th:** On matrices and determinants.

**Monday, January 29th:** The dot product, angles, lengths, areas and volumes, day 1.

**Tuesday, January 30th:** The dot product, angles, lengths, areas and volumes, day 2.

**Thursday, February 1st:** Orthogonality and the cross product.

**Friday, February 2nd:** Practice of vector algebra.

**Monday, February 5th:** Polar coordinates.

**Tuesday, February 6th:** Cylindrical coordinates.

**Thursday, February 8th:** Spherical coordinates.

**Friday, February 9th:** Curvilinear coordinatization.

**Monday, February 12th:** Curvilinear transformations, the Jacobi matrix and linearization, day 1.

**Tuesday, February 13th:** Curvilinear transformations the Jacobi matrix and linearization, day 2.

**Thursday, February 15th:** The Chain Rule, day 1.

**Friday, February 16th:** The Chain Rule, day 2.

**Monday, February 19th:** The second derivative, the Hessian and the second order Taylor approximation, day 1.

**Tuesday, February 20th:** The second derivative, the Hessian and the second order Taylor approximation, day 2.

**Thursday, February 22nd:** Unconstrained Optimization.

**Friday, February 23rd:** Practice.

**Monday, February 26th:** Extra practice day / Exam review.

**Tuesday, February 27th:** EVENING EXAM! (No class; Optional class-time office-hour.)

**Thursday, March 1st:** Infinitesimal line, area and volume elements (day 1).

**Friday, March 2nd:** Infinitesimal line, area and volume elements (day 2).

**Monday, March 5th:** Line Integrals.

**Tuesday, March 6th:** Computing integrals: The Fubini Theorem (day 1).

**Thursday, March 8th:** Computing integrals: The Fubini Theorem (day 2).

**Friday, March 9th:** Practice.

**Monday, March 12th:** Changing the area element; polar coordinates.

**Tuesday, March 13th:** Changing the volume element; cylindrical coordinates.

**Thursday, March 15th:** Changing the volume element; spherical coordinates.

**Friday, March 16th:** Integration practice.

**Monday, March 19th:** Circulation (work) and flux (line) integrals, day 1.

**Tuesday, March 20th:** Circulation (work) and flux (line) integrals, day 2.

**Thursday, March 21st:** Surface and flux integrals, day 1.

**Friday, March 22nd:** Surface and flux integrals, day 2.

**Monday, April 2nd:** The concept of divergence.

**Tuesday, April 3rd:** Divergence Theorems.

**Thursday, April 5th:** The concept of scalar curl; Green's Theorem.

**Friday, April 6th:** The concept of curl; Stokes' Theorem.

**Monday, April 9th:** Stokes' Theorem practice.

**Tuesday, April 10th:** Fundamental Theorems of Calculus: Synthesis.

**Thursday, April 12th:** Changing the domain of integration.

**Friday, April 13th:** Fundamental Theorems of Calculus: One more synthesis day.

**Monday, April 16th:** EVENING EXAM! (No class; Optional class-time office-hour.)

**Tuesday, April 17th:** The gradient vector field.

**Thursday, April 19th:** The gradient vector field and constrained optimization.

**Friday, April 20th:** FESTIVAL OF SCHOLARS – NO CLASS!

**Monday, April 23rd:** The Fundamental Theorem of Calculus in gradient form.

**Tuesday, April 24th:** Revision of vector (integral) calculus.

**Thursday, April 26th:** Course evaluations; Revision of differential calculus.

## **Part II**

# **Homework Problems**

## Chapter 1

# Geometry and differential calculus

### 1.1 Cartesian/rectangular coordinates in plane

#### Reference in the book

This material should be a review, but some students may find the “Ellipses” subsection of Section 9.6 helpful.

#### Homework problems, due Monday, January 22nd

1. Sketch the regions of the  $xy$ -plane described by the following:
  - (a)  $x^2 + y^2 > 4$ ;
  - (b)  $4x^2 + (y + 1)^2 \leq 1$ ;
  - (c)  $1 \leq x^2 + 3y^2 \leq 3$ ;
  - (d)  $-3 \leq x \leq \sin(2y)$ .
2. Express the following regions in the format described below. (You just need to present one solution; the other solution presented below is just there for variety.)
  - (a) Triangle with vertices  $(1, 0)$ ,  $(0, 2)$  and  $(1, 2)$ ;  
**Solution(s):**  $x$  variable is allowed to range freely between 0 and 1, but for each such value of  $x$  variable  $y$  is limited between the value of  $2 - 2x$  and 2. Alternatively, we could describe the triangle by letting  $y$  range freely between 0 and 2. For each such value of  $y$  variable  $x$  is limited between  $1 - \frac{y}{2}$  and 1.
  - (b) The region inside the unit circle located in the third quadrant;
  - (c) The square with vertices  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$  and  $(0, -1)$ ;
  - (d) The region bounded by the parabola  $y = 9 - x^2$  and the line  $8x + y = 0$ ;
  - (e) The disk (i.e interior of the circle) of radius 3 centered at  $(1, 1)$ ;
  - (f) The intersection of disks of radius 2 centered at  $(0, 0)$  and  $(2, 0)$ .

## 1.2 Cartesian/rectangular coordinates in space

### Reference in the book

Examples 2.1, 2.2, 2.4 and 2.5 of Section 10.2 and the entire Section 12.1.

### Homework Problems, due Monday, January 22nd

1. What do the following describe in space? Sketch a picture and/or express in words.

- (a)  $x = 2$ ;
- (b)  $y = 2z$ ;
- (c)  $-1 \leq x, y, z \leq 1$ ;
- (d)  $(x - 1)^2 + (y + 2)^2 + z^2 = 1$ ;
- (e)  $x^2 + 4y^2 + z^2 \leq 16$ ;
- (f)  $x^2 + y^2 = 1$ ;
- (g)  $1 \leq y^2 + 4(z - 1)^2 \leq 4$ ;
- (h)  $x + 1 = z^2$ .

2. Express the following volumes in the format described below. (You just need to present one solution; the other solution presented below is just there for variety.)

- (a) The interior of the sphere centered at the origin and passing through  $(1, 1, 1)$ ;

**Solution 1 – the pancake / potato chips method:** The equation for the sphere is

$$x^2 + y^2 + z^2 = 3,$$

and the radius is  $\sqrt{3}$ . So,  $z$  ranges freely from  $-\sqrt{3}$  and  $\sqrt{3}$ . For each such  $z$  we have a pancake given by  $x^2 + y^2 \leq 3 - z^2$ ; this pancake has radius  $\sqrt{3 - z^2}$ . So, for each  $z$  between  $-\sqrt{3}$  and  $\sqrt{3}$ ,  $x$  variable is limited between  $-\sqrt{3 - z^2}$  and  $\sqrt{3 - z^2}$ . Once such pair of  $z$  and  $x$  is specified,  $y$  variable is limited between  $-\sqrt{3 - z^2 - x^2}$  and  $\sqrt{3 - z^2 - x^2}$ .

**Solution 2 – the French fries method:** (Recall the equation  $x^2 + y^2 + z^2 = 3$  for the sphere.) There is a French fry of  $z$ 's for each point  $(x, y)$  in the equatorial disk; the French fry extends from  $-\sqrt{3 - x^2 - y^2}$  on the bottom to  $\sqrt{3 - x^2 - y^2}$  on the top. The equatorial disk is bounded by the circle  $x^2 + y^2 = 3$ . This means we should let  $x$  range freely between  $-\sqrt{3}$  and  $\sqrt{3}$ , limit  $y$  to the set of values between  $-\sqrt{3 - x^2}$  and  $\sqrt{3 - x^2}$ , and finally bound  $z$  between  $-\sqrt{3 - x^2 - y^2}$  and  $\sqrt{3 - x^2 - y^2}$ .

- (b) The interior of the sphere centered at the point  $(0, 0, -3)$  passing through the point  $(-1, 1, -2)$ ;

- (c) The intersection of balls of radius 2 centered at  $(0, 0, 0)$  and  $(0, 0, 2)$ ;
- (d) The portion of the ball of radius 2 centered at the origin and located above the plane  $z = 1$ .
- (e) The volume inside the infinitely long cylinder of unit radius centered along the  $x$ -axis.
- 

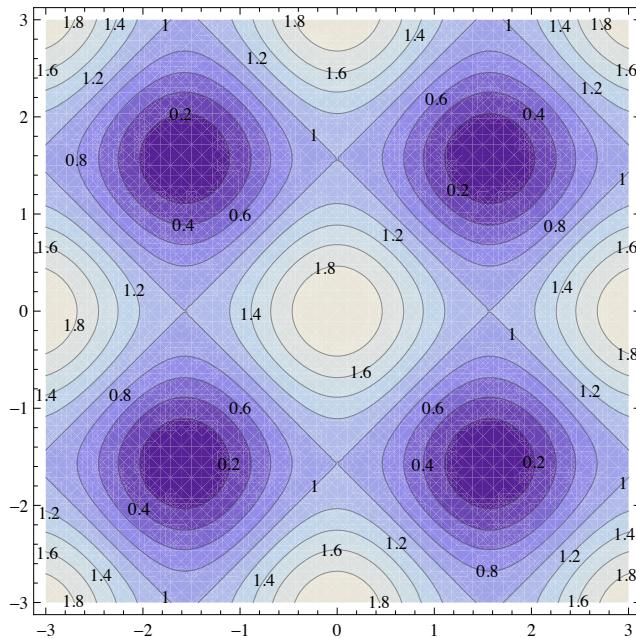
### 1.3 Contour Maps

#### Reference in the book

Section 12.1.

#### Homework Problems, due Thursday, January 25th

- Interpret the following surfaces as graphs of functions of two variables, and use “technology” to visualize them. You are expected to turn-in a print out of both the actual appearance of the graph, and the contour map.
  - $z = \sin(2\pi x) + \sin(2\pi y)$ ;
  - $z = \cos(x^2 + y^2)$ ;
  - $z = e^{-\frac{x^2+y^2}{2}}$ .
- Sketch the graph of the function with the following contour map.



3. Sketch the contour maps and the graphs of the following functions. You need to be able to do this without any help of “technology”.

- (a)  $f(x, y) = -2x + y;$
- (b)  $f(x, y) = 1 - x - y;$
- (c)  $f(x, y) = x^2 + 4y^2;$
- (d)  $f(x, y) = (x - 1)^2 + (y - 1)^2;$
- (e)  $f(x, y) = 4(x - 1)^2 + 9(y + 1)^2;$
- (f)  $f(x, y) = 1 - x^2 - y^2;$
- (g)  $f(x, y) = 1 - 3x^2.$

4. What do level sets for the following functions look like? (A brief explanation in words will suffice.)

- (a)  $f(x, y, z) = x^2 + y^2;$
  - (b)  $f(x, y, z) = x^2 + 2y^2 + (z - 2)^2?$
- 

## 1.4 First examples of quadratic surfaces

### Reference in the book

You may find Sections 9.6, 10.6 helpful.

### Homework Problems, due Thursday, January 25th

1. Sketch the regions of the  $xy$ -plane described by the following. You need to be able to do this without any help of “technology”.

- (a)  $x^2 - y^2 = 9;$
- (b)  $x^2 - y^2 < 9;$
- (c)  $x^2 - 4y^2 < 9;$
- (d)  $4x^2 - y^2 < -9.$

2. Sketch the contour maps and the graphs of the following functions. You need to be able to do this without any help of “technology”.

- (a)  $f(x, y) = x^2 - 4y^2;$
- (b)  $f(x, y) = 4(x - 1)^2 - 9(y + 1)^2;$

- (c)  $f(x, y) = xy + 3$ ;
- (d)  $f(x, y) = 2(x + 1)(y - 1)$ .
3. Identify the shapes of the surfaces  $z = f(x, y)$  where  $f(x, y)$  are given below. You can sketch them or describe them in words. You need to be able to answer this question without using “technology”.
- (a)  $f(x, y) = 2x^2 + 2xy + y^2$ ;
- (b)  $f(x, y) = 2x^2 + 3xy + y^2$ ;
- (c)  $f(x, y) = -2x^2 + 3xy + y^2$ ;
- (d)  $f(x, y) = -2x^2 + 3xy - 2y^2$ ;
- (e)  $f(x, y) = 3xy - 2y^2$ .
4. Identify the shapes of the following regions in space. You can sketch them or describe them in words. You need to be able to answer this question without using “technology”.
- (a)  $x^2 + y^2 \leq z \leq 4$ ;
- (b)  $0 \leq z \leq 1 - x^2 - 4y^2$ ;
- (c)  $x < 4 + y^2 + z^2$ ;
- (d)  $1 < x^2 - z^2 < 4$ .
- 

## 1.5 On vectors

### Reference in the book

Some of this can be found in Sections 10.1– 10.5.

### Homework Problems, due Monday, January 29th

- Let  $A = (-1, 0, 1)$ ,  $B = (0, 3, 6)$  and  $C = (-3, 4, 0)$ . Find  $\overrightarrow{AB}$ ,  $\overrightarrow{BC}$ ,  $\overrightarrow{CA}$ .
  - In this problem let  $\vec{a}$  and  $\vec{b}$  denote the vectors  $\langle 1, -2, 3 \rangle$  and  $\langle 2, -3, 4 \rangle$ , respectively. Find the vectors  $\vec{a} + \vec{b}$ ,  $-\frac{1}{2}\vec{a}$ ,  $\vec{a} - \vec{b}$  and  $-3\vec{a} + 2\vec{b}$ .
  - Sketch two non-collinear 2D vectors  $\vec{a}$  and  $\vec{b}$  which share the same basepoint.
- (a) Sketch vectors  $2\vec{a}$ ,  $\frac{1}{2}\vec{a} + \frac{3}{2}\vec{b}$ ,  $-\frac{1}{2}\vec{a} + 2\vec{b}$ ,  $\vec{a} - 2\vec{b}$ .

- (b) Sketch a different, non-zero 2D vector  $\vec{c}$  which also shares a basepoint with  $\vec{a}$  and  $\vec{b}$ . Based on your sketch, estimate the values of  $\alpha$  and  $\beta$  such that  $\vec{c} = \alpha\vec{a} + \beta\vec{b}$ .
- (c) What shape do the end-points of the vectors  $\alpha\vec{a}$  trace out, if the scalar  $\alpha$  is
- allowed to vary freely throughout the set of all real numbers;
  - only allowed to vary between 0 and 1? How about  $-1$  and  $1$ ?
- (d) What shape do the end-points of the vectors  $\alpha\vec{a} + \beta\vec{b}$  trace out, if
- $\alpha$  and  $\beta$  are allowed to vary freely throughout the set of all real numbers;
  - both  $\alpha$  and  $\beta$  are only allowed to vary between 0 and 1?
  - only  $\alpha$  is allowed to vary freely, but  $\beta$  is limited between  $-1$  and  $1$ ?
  - only  $\alpha$  is allowed to vary freely, but  $\beta$  has to be equal to 1? What is  $\beta$  has to be equal to 2?

4. Consider the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + \alpha + \beta \\ 2 - \alpha + \beta \end{pmatrix}$$

whose inputs are from the  $\alpha\beta$ -plane, and whose outputs are in the  $xy$ -plane.

- (a) Draw side-by-side the  $\alpha\beta$ -plane and the  $xy$ -plane, with the  $\alpha\beta$ -plane on the left.
- (b) Where in the  $\alpha\beta$ -plane and the  $xy$ -plane are the (terminal) points (of the vectors) described by:
- $\alpha = 0$  while  $\beta$  can vary? Color or font code them in the same way. (E.g color them both red, or both blue, or both dashed.)
  - $\alpha = 1$  while  $\beta$  can vary? Color or font code them in the same way, but distinctly from the above.
  - Repeat for  $\alpha = -1, 2, -2$ .
  - Repeat with the roles of  $\alpha$  and  $\beta$  switched.
- (c) What does the unit cell  $0 \leq \alpha, \beta \leq 1$  in the  $\alpha\beta$ -plane correspond to under this transformation? In other words, where is  $0 \leq \alpha, \beta \leq 1$  in the  $xy$ -plane?
- (d) Explain what the transformation of this problem is doing to the graph paper of the  $\alpha\beta$ -plane *in plain words*.

5. Follow the guidelines from Problem 4 to analyze the following. Adjust for the change in dimension when necessary.

(a) 
$$\begin{cases} x = 3\alpha + \beta, \\ y = -1 + \alpha + 3\beta \end{cases}$$

$$(b) \begin{cases} x = 1 + 2\alpha, \\ y = -3\alpha, \\ z = 4 \end{cases}$$

$$(c) \begin{cases} x = 2\alpha, \\ y = \frac{1}{3}\beta, \\ z = \alpha + \beta \end{cases}$$

$$(d) \begin{cases} x = \alpha + \beta + \gamma, \\ y = \beta + \gamma, \\ z = 1 - 2\gamma \end{cases}$$


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## 1.6 More linear algebra: matrices and determinants

### Reference in the book

Very minimal – use your notes.

### Homework Problems, due Monday, January 29th

1. Perform the following matrix multiplications.

$$\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & 9 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & 9 & 9 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

2. Express each of the transformations of Problems 4 and 5 of the last assignment in the matrix notation.

3. Compute the following determinants.

$$\det \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & 9 & 9 \end{pmatrix}, \quad \det \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 4 \\ -1 & 1 & 8 \end{pmatrix}, \quad \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 2 & 1 \\ 4 & 1 & 4 & 1 \\ -8 & -1 & 8 & 1 \end{pmatrix}.$$

4. First discuss (in a sentence or so) what you see as the main difference between vectors and vector fields. Then do the following:

- (a) Plot the vector fields  $\vec{F}(x, y) = \langle x, y \rangle$ ,  $\vec{G}(x, y) = \langle -y, x \rangle$ ,  $\vec{H}(x, y) = \langle -1, 1 \rangle$  by hand. Check your answer using “technology”; no need to include a print-out of your work.

- (b) Use “technology” to plot the vector fields  $\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$  and  $\vec{G}(x, y, z) = \langle x, y, z \rangle$ . Please include the print-out of your work.
-

## 1.7 The dot product, angles, lengths, areas and volumes

### Reference in the book

Section 10.3. **WARNING:** We will go beyond and use a slightly different notation from what the textbook offers.

### Homework Problems, due Thursday, February 1st

1. Let  $P$ ,  $Q$  and  $R$  be the points  $(0, 1, 1)$ ,  $(1, 0, -1)$ ,  $(-1, -1, 1)$  respectively.
    - (a) The lengths of  $PQ$ ,  $QR$  and  $PR$ .
    - (b) The angles  $\angle PQR$ ,  $\angle QRP$  and  $\angle RPQ$ . (It is OK to leave your answer in the arccos-form.)
  2. Find the following areas and volumes.
    - (a) the area of the parallelogram spanned by  $\langle 1, 2 \rangle$  and  $\langle 2, 1 \rangle$ ;
    - (b) the area of the parallelogram spanned by  $\langle -1, 1, 1 \rangle$  and  $\langle 2, 1, 0 \rangle$ ;
    - (c) the volume of the parallelotope spanned by  $\langle 1, 1, 0 \rangle$ ,  $\langle 1, 0, 1 \rangle$ ,  $\langle 0, 1, 1 \rangle$ .
  3. Assume that  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are three unit vectors forming angles  $\frac{\pi}{3}$  with each other.
    - (a) What is the area of the parallelogram spanned by  $\vec{a}$  and  $\vec{b}$ ?
    - (b) What is the area of the parallelogram spanned by  $3\vec{a}$  and  $8\vec{b}$ ?
    - (c) What is the volume of the parallelotope spanned by  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ ?
    - (d) What is the volume of the parallelotope spanned by  $2\vec{a}$ ,  $3\vec{b}$  and  $4\vec{c}$ ?
    - (e) What is the volume of the parallelotope spanned by  $\alpha\vec{a}$ ,  $\beta\vec{b}$  and  $\gamma\vec{c}$ ? You may assume that  $\alpha, \beta, \gamma$  are some positive coefficients.
  4. Consider the transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by
 
$$(x, y) = T(u, v) = (1 + u + 3v, 1 - u + v).$$
    - (a) Express the transformation in matrix notation.
    - (b) What is the unit cell  $0 \leq u, v \leq 1$  in the  $uv$ -plane mapped to under this transformation? (Provide a very rough sketch.)
    - (c) What is the *stretching factor* of this transformation?
  5. Repeat the above for the transformation  $T(u, v, w) = (u + v, v + w, w + u)$ .
-

## 1.8 Orthogonality and the cross product

### Reference in the book

Section 10.4.

### Homework Problems, due Monday, February 5th

1. Consider the line in the  $xy$ -plane passing through the point  $(2, 3)$  spanned by the vector  $\langle 5, 8 \rangle$ .
    - (a) Find a normal vector to this line.
    - (b) Use the normal vector you found to obtain the equation of the line in the form of  $\underline{\hspace{1cm}}x + \underline{\hspace{1cm}}y = \underline{\hspace{1cm}}$ .
  2. Describe the plane  $2x + 5y + 7z = 10$  by addressing the following:
    - $x, y, z$ -intercepts;
    - the normal vectors.
  3. Let  $\vec{u} = \langle 1, 0, 1 \rangle$ ,  $\vec{v} = \langle 1, 1, 0 \rangle$  and  $\vec{w} = \langle 0, 1, 1 \rangle$ . Compute and illustrate:
    - (a)  $\vec{u} \times \vec{v}$ ;
    - (b)  $\vec{v} \times \vec{w}$ ;
    - (c)  $\vec{w} \times \vec{u}$ .
  4. Find the following cross products without ever using determinants.
    - (a)  $(\vec{i} \times \vec{j}) \times (\vec{k} \times \vec{j})$
    - (b)  $(2\vec{k} \times \vec{i}) \times (\vec{i} \times 3\vec{j})$
    - (c)  $\vec{i} \times (\vec{j} \times 3\vec{i})$
    - (d)  $(\vec{j} + 2\vec{k}) \times (\vec{i} - \vec{j})$
    - (e)  $\vec{i} \times (\vec{j} \times \vec{k}) + \vec{j} \times (\vec{k} \times \vec{i}) + \vec{k} \times (\vec{i} \times \vec{j})$ .
  5. A point is rotating around the origin within the plane spanned by vectors  $\langle 1, 1, 1 \rangle$  and  $\langle 0, 1, -1 \rangle$ . Find the axis of rotation, that is, find the line around which the point is rotating.
  6. Consider the plane through the point  $(1, 0, 2)$  spanned by the vectors  $\langle 0, 1, 1 \rangle$  and  $\langle -1, 0, 1 \rangle$ . Write the equation for this plane in the form of  $\underline{\hspace{1cm}}x + \underline{\hspace{1cm}}y + \underline{\hspace{1cm}}z = \underline{\hspace{1cm}}$ .
-

## 1.9 Polar coordinates

### Reference in the book

Section 9.4.

### Homework problems, due Thursday, February 8th

1. Find:

- (a) the polar coordinates of the following Cartesian points:
  - i.  $(-2, 0)$ ;
  - ii.  $(-3, 3)$ ;
  - iii.  $(1, -3)$ ;
  - iv.  $(-2, -1)$ .

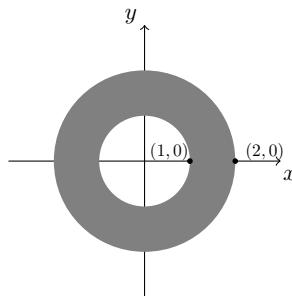
- (b) the Cartesian coordinates of the point whose polar coordinates  $(r, \theta)$  are  $(2, \frac{\pi}{3})$ .

Remember to use radians for all angles.

2. The following equations describe a curve or a region of the Cartesian plane by means of polar coordinates. Identify (and draw) these regions without any help of “technology”.

- (a)  $r \leq 1$ , and any  $\theta$ ;
- (b)  $r = \frac{\pi}{4}$ ,  $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ ;
- (c)  $1 \leq r \leq 4$ ,  $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ ;
- (d)  $r = e^{2\theta}$ ,  $-\infty < \theta < \infty$ ;
- (e)  $r = e^{-\frac{\theta}{2}}$ ,  $-\infty < \theta < \infty$ .

3. Consider the region  $R$  of the  $xy$ -plane described by the following picture:



- (a) Write down the expressions for this region in polar  $r\theta$ -coordinates;

- (b) Draw the region  $D$  in the  $r\theta$ -plane which corresponds to  $R$ ;
- (c) Re-draw the regions  $D$  and  $R$  side-by-side, with  $D$  on the left. Then color-code and label the corresponding points, edges, etc like we did in class.
4. Express the following geometric objects using polar coordinates. Follow the template provided below.
- (a) Washer centered at  $(3, 1)$ , of inner radius 2 and outer radius 4;
- Solution:** We have  $x = 3 + r \cos(\theta)$  and  $y = 1 + r \sin(\theta)$ , with the following restrictions on variables  $r$  and  $\theta$ :
- $$2 \leq r \leq 4 \text{ and } 0 \leq \theta \leq 2\pi.$$
- Since we are describing a region with area, it makes sense that we have two free variables  $r$  and  $\theta$ , indicating two independent directions of motion inside this washer.
- (b) Disk of radius 2 centered at the origin;
- (c) The first quadrant of the Cartesian plane;
- (d) The line  $x = 1$ ;
- (e) The interior of the triangle with the vertices at  $(0, 0)$ ,  $(1, 1)$  and  $(1, -1)$ .
- 

## 1.10 Cylindrical coordinates

### Reference in the book

Section 13.6 (Examples 6.2 and 6.3).

### Homework Problems, due Thursday, February 8th

1. Find:
- (a) the cylindrical coordinates of the Cartesian point  $(0, -1, 0)$ ;
- (b) the cylindrical coordinates of the Cartesian point  $(0, 1, -1)$ ;
- (c) the Cartesian coordinates of the point whose cylindrical coordinates  $(r, \theta, z)$  are  $(2, \frac{2\pi}{3}, \frac{\pi}{4})$ .

Remember to use radians for all angles.

2. Express the following geometric objects using cylindrical coordinates. Follow the template provided below.

- (a) The surface of the infinite upright cylinder of unit radius centered along the  $z$ -axis;

**Solution:** Here  $r = 1$ , while  $\theta$  with  $0 \leq \theta \leq 2\pi$  and  $z$  with  $-\infty < z < \infty$  are free variables. So, we have  $x = \cos(\theta)$ ,  $y = \sin(\theta)$  and  $z = z$ . Since we are describing a surface, it makes sense that we have two free variables  $\theta$  and  $z$ , indicating two independent directions of motion along the surface of the cylinder.

- (b) The interior of the infinite upright cylinder of unit radius centered along the  $z$ -axis;
- (c) The surface of infinite cylinder of radius 1 centered around the  $y$ -axis;
- (d) Unit sphere centered at the origin;
- (e) Unit ball centered at the origin;
- (f) Upper unit hemi-sphere centered at the origin;
- (g) Polar cap of the sphere of radius 2 centered at the origin, located to the “north” of the  $60^0$ -parallel;
- (h) The surface of a circular cone of your choice going around the  $z$ -axis with the tip at the origin.
- 

## 1.11 Spherical coordinates

### Reference in the book

Section 13.7. (Examples 7.1 and 7.2).

### Homework Problems, due Monday, February 12th

1. Find:

- (a) the spherical coordinates of the Cartesian point  $(0, -1, 0)$ ;
- (b) the spherical coordinates of the Cartesian point  $(0, 1, -1)$ ;
- (c) the Cartesian coordinates of the point whose spherical coordinates  $(r, \theta, \phi)$  are  $(2, \frac{2\pi}{3}, \frac{\pi}{4})$ .

Remember to use radians for all angles.

2. The following equations describe regions in spherical coordinates. What regions are those? Explain in words and try to draw. Avoid using “technology”.

- (a)  $r \geq 4$ ;

- (b)  $1 \leq r \leq 4, 0 \leq \phi \leq \frac{\pi}{2}$ ;
- (c)  $r \leq 1, 0 \leq \phi \leq \frac{\pi}{4}$ ;
- (d)  $1 \leq r \leq 4, \frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4}$ .
3. Express the following geometric objects in spherical coordinates. Follow the template provided below.
- (a) Unit sphere centered at the origin;
- Solution:** Here  $r = 1$ , while  $\theta$  with  $0 \leq \theta \leq 2\pi$  and  $\phi$  with  $0 \leq \phi \leq \pi$  are free variables. So, we have  $x = \cos(\theta) \sin(\phi)$ ,  $y = \sin(\theta) \sin(\phi)$  and  $z = \cos(\phi)$ . Since we are describing a surface, it makes sense that we have two free variables  $\theta$  and  $\phi$ , indicating two independent directions of motion along the surface of the sphere:  $\theta$  going from west to east and  $\phi$  going from north to south.
- (b) Upper unit hemi-sphere centered at the origin;
- (c) Unit ball centered at the origin;
- (d) The “north”  $60^0$ -parallel of the sphere of radius 2 centered at the origin;
- (e) Polar cap of the sphere of radius 2 centered at the origin, located to the “north” of the  $60^0$ -parallel;
- (f) The “north-south” meridian passing through the point  $(1, 1, -\sqrt{2})$  of the sphere of radius 2 centered at the origin;
- (g) The “east-west” parallel passing through the point  $(1, 1, -\sqrt{2})$  of the sphere of radius 2 centered at the origin;
- (h) The surface of an infinite circular cone of your choice going around the  $z$ -axis whose tip is at the origin;
- (i) The volume inside of an infinite circular cone of your choice going around the  $z$ -axis whose tip is at the origin.
- 

## 1.12 Curvilinear coordinatization

### Reference in the book

This material is a combination of Sections 9.4, (certain examples from) Sections 13.6 and 13.7, and Section 11.6. **WARNING:** The discussion of vector fields  $\partial_t, \partial_r, \partial_\theta$ , etc is not in your textbook.

**Homework Problems, due Monday, February 12th**

1. Assume  $(r, \theta, z)$  stand for the standard cylindrical coordinates in space.
    - (a) Find formulas for the vector fields  $\partial_r$ ,  $\partial_\theta$  and  $\partial_z$ .
    - (b) Specifically, evaluate and sketch the vector fields  $\partial_r$ ,  $\partial_\theta$  and  $\partial_z$  at the points with Cartesian coordinates  $(1, 0, 0)$  and  $(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$ .
    - (c) Complete a rough sketch of the vector fields  $\partial_r$ ,  $\partial_\theta$  and  $\partial_z$ .
  
  2. Assume  $(r, \theta, \phi)$  denote the standard spherical coordinates in space.
    - (a) Find formulas for vector fields  $\partial_r$ ,  $\partial_\theta$  and  $\partial_\phi$ .
    - (b) Specifically, evaluate and sketch the vector fields  $\partial_r$ ,  $\partial_\theta$  and  $\partial_\phi$  at the points with Cartesian coordinates  $(1, 0, 0)$  and  $(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$ .
    - (c) Complete a rough sketch of the vector fields  $\partial_r$ ,  $\partial_\theta$  and  $\partial_\phi$ .
- 

**1.13 Curvilinear transformations, the Jacobi matrix and linearization****Reference in the book**

NONE!? **WARNING:** The homework problems will make very little sense without class notes. The discussion of vector fields  $\partial_t$ ,  $\partial_u$ ,  $\partial_v$ , etc is not in your textbook. Also, you may find the lectures to be heavy on new notation, which is not in your textbook.

**Homework Problems, due Thursday, February 15th**

1. Use “technology” to plot the following parametrically defined shapes. You are expected to attach a print-out of what you see, and on the print-out make a rough sketch of the coordinate vector fields  $\partial_t$ ,  $\partial_u$ ,  $\partial_w$  etc.
  - (a)  $x(t) = t^2$ ,  $y(t) = t^3$ , where  $-2 \leq t \leq 2$ .
  - (b)  $x(t) = e^{-t/2}$ ,  $y(t) = \sin(4t)$ ,  $z(t) = \cos(4t)$ , where  $0 \leq t \leq 2\pi$ .
  - (c)  $x(u, v) = (2 + \cos(u)) \cos(v)$ ,  $y(u, v) = (2 + \cos(u)) \sin(v)$  and  $z(u, v) = u$  with  $0 \leq u \leq 8\pi$ ,  $0 \leq v \leq 2\pi$ ;
  - (d)  $x(u, v) = (2 + \cos(u)) \cos(v)$ ,  $y(u, v) = (2 + \cos(u)) \sin(v)$  and  $z(u, v) = \sin(u)$  with  $0 \leq u, v \leq 2\pi$ .
  
2. Compute the Jacobi matrices of the following mappings.
  - (a)  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $(x, y) = T(u, v) = (u \sin(\pi v), u \cos(\pi v))$ .

- (b)  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $(x, y, z) = T(u, v) = (u \cos(v), u \sin(v), v)$ .
- (c)  $f : \mathbb{R} \rightarrow \mathbb{R}^3$  given by  $(x, y, z) = f(t) = (t, \cos(\pi t), t)$ .
- (d)  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $x = f(u, v) = v - u$ .
3. Consider the transformation  $(x, y) = T(u, v) = (e^u \sin v, e^u \cos v)$  near the base point  $(u_0, v_0) = (0, 0)$ .
- Compute the Jacobi matrix of  $T$ , and evaluate it at  $(u_0, v_0)$ .
  - Find the linear approximation  $L(\Delta u, \Delta v)$  of  $T$  near  $(u_0, v_0)$ .
  - Please fill out the following table. The last column represents the relative error of the approximation

$$T(u_0 + \Delta u, v_0 + \Delta v) \approx L(\Delta u, \Delta v),$$

so please express those values in the percentage form.

$\langle \Delta u, \Delta v \rangle$	$T(u_0 + \Delta u, v_0 + \Delta v)$	$L(\Delta u, \Delta v)$	$\frac{\ T(u_0 + \Delta u, v_0 + \Delta v) - L(\Delta u, \Delta v)\ }{\ \langle \Delta u, \Delta v \rangle\ }$
$\langle -0.1, 0.1 \rangle$			
$\langle -0.05, 0.05 \rangle$			
$\langle -0.02, 0.02 \rangle$			
$\langle -0.01, 0.01 \rangle$			
$\langle -0.005, 0.005 \rangle$			

- (d) What is the trend in the last column?
- (e) If one were able to utilize an infinitesimal displacement vector  $\langle dx, dy \rangle$ , what would one get for  $T(u_0 + du, v_0 + dv) - L(du, dv)$ ?
4. Go back to the Jacobi matrices from homework problem 2, and answer the following questions:
- What is the linearization of the map  $(x, y) = T(u, v) = (u \sin(\pi v), u \cos(\pi v))$  at the point  $(u_0, v_0) = (1, 1)$ ? Based on your linearization formula, what do you expect  $T(.9, 1.1)$  to be?
  - What is the linearization of the map  $(x, y, z) = T(u, v) = (u \cos(v), u \sin(v), v)$  at the point  $(u_0, v_0) = (1, 0)$ ? Based on your formula, what do you expect  $T(0.9, -0.2)$  to be?
  - What is the linearization of the map  $(x, y, z) = f(t) = (t, \cos(\pi t), t)$  at the point  $t_0 = 2$ ? Based on your linearization formula, what do you expect  $f(2.1)$  to be?
5. (The solutions of the remaining homework problems should look like the examples done in class – please consult the class notes first!) Consider the transformation  $T$  whose domain is the upper half of the  $uv$ -plane, whose range is the  $xy$ -plane and whose rule is given by

$$T(u, v) = (v^2 - u^2, -2uv).$$

- (a) Draw side-by-side the Cartesian  $uv$ -grid and the Cartesian  $xy$ -grid, with the  $uv$ -grid on the left. Emphasize the points  $P(1, 1)$ ,  $Q(0, \sqrt{2})$  and  $R(-1, 1)$  of the  $uv$ -half-plane, and their images  $P'$ ,  $Q'$ ,  $R'$  in the  $xy$ -plane.
- (b) Compute the Jacobi matrix  $DT$  of the transformation  $T$ . Then evaluate the matrix at points  $P(1, 1)$ ,  $Q(0, \sqrt{2})$  and  $R(-1, 1)$ .
- (c) Based on your computation of the Jacobi matrix, please compute the coordinate vector fields  $\partial_u$  and  $\partial_v$  in the image, specifically at points  $P'$ ,  $Q'$  and  $R'$  introduced above. Sketch these vector (fields).
- (d) The transformation  $T$  maps a small neighborhood of  $P$  in the  $uv$ -plane to a small neighborhood of  $P'$  in the  $xy$ -plane. Which linear transformation resembles this particular mapping the most? What is the geometric effect of this mapping? Does it stretch or shrink?
- (e) Repeat the previous question but focus on neighborhoods of
  - $Q$  and  $Q'$ ;
  - $R$  and  $R'$ .
- (f) Use “technology” to plot the mapping  $T$ . You are expected to attach a print-out of what you see.
- (g) Enter the results of the computations you did above on the print-out. Make sure everything makes sense.
- (h) On your print-out sketch the vector fields  $\partial_u$  and  $\partial_v$ .
6. Repeat the previous problem for the following transformations. You should utilize three points  $P$ ,  $Q$  and  $R$  of your choice.
- (a) The transformation  $T(u, v) = (2u \sin(v), u \cos(v))$  of the domain  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2\pi$ .
- (b) The transformation  $T(u, v) = (u \cos(v), u \sin(v), u)$  of the domain  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2\pi$ .
- 

## 1.14 The Chain Rule

### Reference in the book

Section 12.5.

### Homework Problems, due Monday, February 19th

1. Let  $T_1$  and  $T_2$  be two transformations of the Euclidean plane with:

$$T_1(0, 1) = (1, 1), \quad T_1(1, 1) = (0, 1), \quad T_2(0, 1) = (0, 1), \quad T_2(1, 1) = (1, 1).$$

It is furthermore known that the Jacobi matrices of  $T_1$  and  $T_2$  at points  $(0, 1)$  and  $(1, 1)$  are:

$$\begin{aligned} DT_1|_{(0,1)} &= \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}, & DT_1|_{(1,1)} &= \begin{pmatrix} 0 & 3 \\ 5 & 1 \end{pmatrix}, \\ DT_2|_{(0,1)} &= \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, & DT_2|_{(1,1)} &= \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}. \end{aligned}$$

Find the following Jacobi matrices:

(a)  $D(T_2 \circ T_1)|_{(0,1)}$

(b)  $D(T_2 \circ T_1)|_{(1,1)}$

(c)  $D(T_1 \circ T_2)|_{(0,1)}$

(d)  $D(T_1 \circ T_2)|_{(1,1)}$

2. Use the Chain Rule to compute  $\frac{\partial}{\partial u}(f \circ T)$ ,  $\frac{\partial}{\partial v}(f \circ T)$  and  $\frac{\partial}{\partial w}(f \circ T)$  (where applicable) for the following choices of  $f$  and  $T$ . Express your final answer using  $u$ ,  $v$  and  $w$  variables only.

(a)  $f(x, y) = x^2 + y^2$  and  $(x, y) = T(u, v) = (2u + v, u + 2v)$ .

(b)  $f(x, y, z) = xyz$  and  $(x, y, z) = T(u, v, w) = (u + v + w, v + w, w)$ .

(c)  $f(x, y, z) = x^2 + y^2 + z^2$  and  $(x, y, z) = T(u, v) = (u^2, u \sin(v), u \cos(v))$ .

3. Consider the transformation

$$(x, y) = T(u, v) = (u + v, -u + v),$$

and let  $f(x, y)$  be some function.

(a) Express  $\frac{\partial f}{\partial u}$  and  $\frac{\partial f}{\partial v}$  in terms of  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

(b) Express  $\frac{\partial^2 f}{\partial u^2}$ ,  $\frac{\partial^2 f}{\partial u \partial v}$  and  $\frac{\partial^2 f}{\partial v^2}$  in terms of partial derivatives with respect to  $(x, y)$ -variables.

4. Let  $(r, \theta)$  denote the standard polar coordinates in the  $xy$ -plane.

(a) Express  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  in terms of  $\frac{\partial f}{\partial r}$  and  $\frac{\partial f}{\partial \theta}$ .

(b) Express  $\frac{\partial^2 f}{\partial x^2}$ ,  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y^2}$  in terms of  $\frac{\partial^2 f}{\partial r^2}$ ,  $\frac{\partial^2 f}{\partial r \partial \theta}$  and  $\frac{\partial^2 f}{\partial \theta^2}$ .

5. Let  $(r, \theta, \phi)$  denote the standard spherical coordinates in the  $xyz$ -space. Let  $f(x, y, z)$  be some function which depends only on  $r$ :

$$f(x, y, z) = f(r), \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \phi} = 0.$$

- (a) Express  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  and  $\frac{\partial f}{\partial z}$  in terms of  $\frac{\partial f}{\partial r}$ .
- (b) Express  $\frac{\partial^2 f}{\partial x^2}$ ,  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y^2}$  in terms of  $\frac{\partial^2 f}{\partial r^2}$ ,  $\frac{\partial f}{\partial r}$  etc.
- 

## 1.15 The second derivative, the Hessian and the second order Taylor approximation

### Reference in the book

Sections 12.3 (Example 3.4, Theorem 3.1) and 12.5 (Example 5.4); the single variable version of Taylor approximations can be found in Section 8.7. You will need lecture notes for Taylor approximations in the higher dimensional setting.

### Homework Problems, due Thursday, February 22nd

1. This problem concerns the function  $f(x) = \ln(x)$ .
  - (a) Find the tangent line to the graph of the function  $f$  at the point  $x_0 = 1$ .
  - (b) Find the quadratic (Taylor) approximation, as well as Taylor approximations of order 3 and 4, of the function  $f$  at  $x_0 = 1$ .
  - (c) Use “technology” to graph the function, its tangent line, and the remaining Taylor approximations you found. Focus on the vicinity of the domain point  $x_0 = 1$ . Provide a print-out which clearly demonstrates the fact that higher order Taylor approximations do a better job of approximating the function  $f$ .
2. Consider the function  $f(x) = \sqrt{1 + 4x}$  near  $x_0 = 0$ .
  - (a) Find the linear approximation of  $f$  near  $x_0 = 0$ ; in what follows please use  $L(\Delta x)$  to denote this approximation.
  - (b) Find the quadratic approximation of  $f$  near  $x_0 = 0$ ; in what follows please use  $Q(\Delta x)$  to denote this approximation.
  - (c) Fill out the following table:

$\Delta x$	$f(x_0 + \Delta x)$	$L(\Delta x)$	$\frac{f(x_0 + \Delta x) - L(\Delta x)}{(\Delta x)^2}$	$Q(\Delta x)$	$\frac{f(x_0 + \Delta x) - Q(\Delta x)}{(\Delta x)^3}$
0.5					
0.1					
0.05					
0.01					

- (d) What is the trend among the numbers in the column labeled by  $\frac{f(x_0+\Delta x)-L(\Delta x)}{(\Delta x)^2}$ ? Use your answer to estimate the size / order of magnitude of  $f(x_0+\Delta x)-L(\Delta x)$  for the following values of  $\Delta x$ :

i.  $\Delta x = 10^{-6}$ ;  
ii.  $\Delta x = 10^{-12}$ .

- (e) What is the trend among the numbers in the column labeled by  $\frac{f(x_0+\Delta x)-Q(\Delta x)}{(\Delta x)^3}$ ? Use your answer to estimate the size / order of magnitude of  $f(x_0+\Delta x)-Q(\Delta x)$  for the following values of  $\Delta x$ :

i.  $\Delta x = 10^{-6}$ ;  
ii.  $\Delta x = 10^{-12}$ .

3. Compute the second derivatives  $\frac{\partial^2 f}{\partial x^2}$ ,  $\frac{\partial^2 f}{\partial x \partial y}$ ,  $\frac{\partial^2 f}{\partial y^2}$  and assemble them into the Hessian (matrix). Don't forget to utilize symmetries when appropriate.

- (a)  $f(x, y) = 2x + 3y$ ;  
(b)  $f(x, y) = e^{-x^2-y^2}$ ;  
(c)  $f(x, y) = \arctan\left(\frac{y}{x}\right)$ ;  
(d)  $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ .

4. This problem concerns the function

$$f_1(x, y) = e^{-\frac{x^2+y^2}{2}}.$$

- (a) Find the tangent plane to the graph of the function  $f_1$  at the point  $(0, 0)$ .  
(b) Find the quadratic (Taylor) approximation of the function  $f_1$  at  $(0, 0)$ .  
(c) Use "technology" to graph the function, its tangent plane (linearization), and its quadratic (Taylor) approximation in the vicinity of the domain point  $(0, 0)$ . Provide a print-out which clearly demonstrates that the quadratic approximation improves the linear approximation of  $f_1$ .  
(d) Based on the quadratic approximation you found estimate the value of  $f_1(0.1, 0.2)$ .  
(e) Repeat the previous question for  $f_1(-0.02, 0.01)$ .

5. This problem concerns the function

$$f_2(x, y) = xy - \frac{x^4}{16} - \frac{y^4}{16}.$$

- (a) Find the tangent plane to the graph of the function  $f_2$  at the point  $(2, 2)$ .  
(b) Find the quadratic (Taylor) approximation of the function  $f_2$  at  $(2, 2)$ .

- (c) Use “technology” to graph the function, its tangent plane (linearization), and its quadratic (Taylor) approximation in the vicinity of the domain point  $(2, 2)$ . Provide a print-out which clearly demonstrates that the quadratic approximation improves the linear approximation of  $f_2$ .
- (d) Based on the quadratic approximation you found estimate the value of  $f_2(2.2, 1.8)$ .
- (e) Repeat the previous question for  $f_2(1.98, 2.04)$ .
6. Find the following quadratic approximations:
- (a) Quadratic approximation of the function  $f(x, y) = \ln(x^2 + y^2)$  near the point  $(1, 0)$ ;
- (b) Quadratic approximation of the function  $f(x, y, z) = x^2y^2z^2$  near the point  $(1, -1, 1)$ .
- 

## 1.16 Unconstrained optimization

### Reference in the book

Section 12.7, up to Definition 7.4.

### Homework Problems, due Monday, February 26th

1. Find all critical points of the following functions and classify them as local maxima, local minima or saddles. Graph the functions involved and see that your answers make sense.

- (a)  $f(x, y) = 3x^2 - 2x^3 + 3y^2 - 2y^3$ ;
- (b)  $f(x, y) = x^3 - 3xy + y^3$ ;
- (c)  $f(x, y) = x^4 + 4xy - xy^2$ ;
- (d)  $f(x, y) = e^{-x^2-y^2}$ .
-

## Chapter 2

# Integration and the Fundamental Theorems of Calculus

### 2.1 Infinitesimal line, area and volume elements

#### Reference in the book

Certain elements of Sections 13.8 and 14.6. **WARNING:** I don't necessarily recommend reading these sections; it will be very difficult to extract relevant information from them. One will have to rely on good lecture notes!

#### Homework Problems, due Monday, March 5th

1. Find the formula for the line element  $ds$ , and use it to compute the total length of the given curve. If the parametric equations are not explicitly given, please parametrize the curve first.
  - (a) The helicoidal spiral  $x(t) = e^t \cos(3t)$ ,  $y(t) = e^t \sin(3t)$  and  $z(t) = 3e^t$  with  $-1 \leq t \leq 1$ ;
  - (b) The helicoidal spiral of your choice which wraps around the  $x$ -axis.
  - (c) The circle in the  $xy$ -plane with center at  $(3, 2)$  and radius 4;
  - (d) The North 45th-parallel of the unit sphere centered at the origin;
  - (e) The  $30^\circ$ -parallel of the southern unit hemi-sphere (centered at the origin).
  - (f) The meridian containing the point  $(\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{2}}{2})$  of the sphere centered at the origin.
2. Find the expressions for the surface area element  $dA$  for the following surfaces. Parametrize the surface first if necessary.
  - (a) The helicoid  $x(u, v) = u \cos(v)$ ,  $y(u, v) = u \sin(v)$ ,  $z(u, v) = v$  with  $0 \leq u \leq 3$ ,  $0 \leq v \leq 2\pi$ .

- (b) The area of ellipsoid  $x^2 + y^2 + 4z^2 = 4$ ;  
 (c) The area of the bowl cut out of the paraboloid  $z = x^2 + y^2$  by the plane  $z = 4$ .
3. Consider the transformation
- $$(x, y, z) = T(u, v, w) = (u \cos(v) \cos(w), u \sin(v) \cos(w), u \sin(w))$$
- on the domain  $u \geq 0$ ,  $0 \leq v \leq 2\pi$ ,  $-\frac{\pi}{2} \leq w \leq \frac{\pi}{2}$ .
- (a) Compute the Jacobi matrix of  $T$  and identify the coordinate tangent vector fields  $\partial_u, \partial_v, \partial_w$  in the  $xyz$ -space.  
 (b) Compute the Jacobian of  $T$ .  
 (c) Compute  $\partial_u \cdot \partial_u, \partial_u \cdot \partial_v, \dots, \partial_w \cdot \partial_w$ .  
 (d) Compute the volume element  $dV$  in  $(u, v, w)$ -coordinates in two different ways.
- 

## 2.2 Line Integrals

### Reference in the book

Section 14.2, up to Definition 2.2.

### Homework Problems, due Thursday, March 8th

1. Let  $C$  be the unit circle centered at the point  $(2, 0)$ . Parametrize  $C$  and find:
    - (a)  $\int_C x \, ds$ ;
    - (b)  $\int_C y \, ds$ .
  2. Let  $C$  be the straight line segment from  $(0, 1, 1)$  to  $(1, 0, 2)$ . Parametrize  $C$  and evaluate  $\int_C xyz \, ds$ .
  3. Let  $C$  be the path consisting of a semi-circle joining  $(1, 0)$  to  $(-1, 0)$  in the upper half-plane, and the diameter joining  $(-1, 0)$  and  $(1, 0)$ . Evaluate  $\int_C x^2 + y^2 \, ds$ .
  4. A thin wire in the shape of circle of radius 2 inches is centered at the origin of the coordinate plane. The linear weight density of the wire at the location with coordinates  $(x, y)$  is  $\varrho(x, y) = 2 + xy$  ounces per inch. (Assume that inches are used as the unit of spatial measurement throughout the problem.)
    - (a) What is the total weight of the wire?
    - (b) What is the average weight density of the wire?
  5. What is the average of  $\varrho(x, y, z) = x^2$  along the North 45th-parallel of the unit sphere centered at the origin?
-

## 2.3 Computing integrals: The Fubini Theorem

### Reference in the book

Sections 13.1 and 13.5.

### Homework Problems, due Monday, March 12th

1. Make a sketch of  $R$  and compute  $\iint_R f(x, y) dA$ , where  $R$  and  $f$  are given as follows. You are expected to use the Fubini Theorem.
  - (a)  $R$  is the rectangle  $-3 \leq x \leq 2$ ,  $0 \leq y \leq 4$  and  $f(x, y) = x^2y - xy^2$ .
  - (b)  $R$  is the rectangle  $0 \leq x, y \leq 1$  and  $f(x, y) = xe^{-x-y}$ .
  - (c)  $R$  is the entire Cartesian plane, and
$$f(x, y) = \frac{1}{(1+x^2)(1+y^2)}.$$
2. A thin  $2 \text{ ft} \times 1 \text{ ft}$  metal sheet covers the rectangle  $[-1, 1] \times [0, 1]$  of the coordinate plane. The weight density of the sheet at the location with coordinates  $(x, y)$  is  $\varrho(x, y) = x^2 + y^2$  lbs per square foot. (Whenever in doubt in this problem, assume that feet are used as the unit of spatial measurement.)
  - (a) What is the total weight of the metal sheet?
  - (b) What is the average weight density of the metal sheet?
3. Make a sketch of  $\Omega$  and compute  $\iiint_{\Omega} f(x, y, z) dV$ , where  $\Omega$  and  $f$  are given as follows. You are expected to use the Fubini Theorem.
  - (a)  $\Omega$  is the box  $0 \leq x, y, z \leq 1$  and  $f(x, y, z) = \sqrt{xyz}$ .
  - (b)  $\Omega$  is the entire first octant (where  $x, y, z \geq 0$ ) and  $f(x, y, z) = e^{-x-y-z}$ .
4. Make a sketch of  $R$  and compute  $\iint_R f(x, y) dA$ , where  $R$  and  $f$  are given as follows. You are expected to use the Fubini Theorem. Also, you may want to set up the integrals in two different ways (both according to the Fubini Theorem) and then choose the easier one to complete.
  - (a)  $R$  is the unit disk centered at the origin and  $f(x, y) = 2x + 3y$ .
  - (b)  $R$  is the triangle enclosed by  $y = |x|$  and  $y = 1$ , and  $f(x, y) = x^2 + y^2$ .
  - (c)  $R$  is the region enclosed by the parabolas  $3y = x^2$  and  $3x = y^2$ , and  $f(x, y) = 1$ .
  - (d)  $R$  is the region outside the unit circle  $x^2 + y^2 = 1$  but inside the square given by  $-1 \leq x, y \leq 1$ , and  $f(x, y) = y^{-2}$ .

5. Make a sketch of  $\Omega$  and compute  $\iiint_{\Omega} f(x, y, z) dV$ , where  $\Omega$  and  $f$  are given as follows. You are expected to use the Fubini Theorem. Also, you may want to set up the integrals in at least two different ways (both according to the Fubini Theorem) and then choose the easier one to complete.
- $\Omega$  is the volume between the  $xy$ -plane and the graph of the paraboloid  $z = x^2 + y^2$  over the domain  $-1 \leq x, y \leq 1$ , and  $f(x, y, z) = z$ .
  - $\Omega$  is the volume between  $z = x^2 + y^2$  and the plane  $z = 1$ , and  $f(x, y, z) = x^3$ .
6. (a) Use integration to find the average value of the function  $f(x, y) = xy$  over the standard unit disk.
- (b) The average value of the function  $f(x, y) = xy$  over the unit disk, which you just computed, could have been foreseen. Figure out how. Then apply what you learned to the following.
- Find the average value of the function  $f(x, y) = x^3 + y^3$  over the unit disk.
  - Find the average value of the function  $f(x, y) = 1 + x^3 + y^3$  over the unit disk.
- 

## 2.4 Changing the area element; polar coordinates

### Reference in the book

Section 13.3.

### Homework Problems, due Thursday, March 15th

- Integrate the following functions over the following regions. *All disks and circles are centered at the origin.*
  - $f(x, y) = e^{-\frac{x^2+y^2}{2}}$  over the unit disk;
  - $f(x, y) = 2\sqrt{9 - x^2 - y^2}$  over the disk of radius 3;
  - $f(x, y) = x + y$  over the region in the first quadrant between the circle of radius 1 and the circle of radius 2;
  - $f(x, y) = x^2$  over the region above the graph of  $y = |x|$ , outside the unit circle, but inside the circle of radius 2;
  - $f(x, y) = \frac{1}{1+x^2+y^2}$  over the entire Cartesian plane.
- Adjust the idea of polar coordinates to compute the following:
  - $\iint_R x + y dA$  where  $R$  is the unit disk centered at  $(2, 2)$ ;

- (b) The area of the ellipse  $4x^2 + 9y^2 = 1$ ;
- (c)  $\iint_R x^2 + y^2 dA$  where  $R$  is the area inside the ellipse  $x^2 + 9y^2 = 4$ .
3. (a) Follow the argument from class to show that  $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ .
- (b) Based on the above evaluate the following:
- $\int_0^{\infty} e^{-x^2} dx$ ;
  - $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$ .
- 

## 2.5 Changing the volume element; cylindrical coordinates

### Reference in the book

Section 13.6.

### Homework Problems, due Thursday, March 15th

1. Use cylindrical coordinates to integrate the following functions over the cylinder of radius 1, centered along the  $z$ -axis between the planes  $z = -1$  and  $z = 1$ .
    - (a)  $f(x, y, z) = x^2 + y^2 + z^2$ ;
    - (b)  $f(x, y, z) = xyz$ .
  2. Integrate the function  $f(x, y, z) = x^2 + y^2 + z^2$  over the region located within the cone  $z = \sqrt{x^2 + y^2}$  but under the plane  $z = 1$ .
  3. Use triple integrals and cylindrical coordinates to compute the following volumes:
    - (a) Volume contained inside the sphere of radius  $R$ .
    - (b) Volume contained between the surfaces  $z = x^2 + y^2$  and  $z = 18 - x^2 - y^2$ .
    - (c) Volume of the region bounded by  $x^2 + y^2 - z^2 = 1$  and the planes  $z = \pm 1$ .
    - (d) Volume of the donut whose overall shape is obtained by the disk  $(x - 3)^2 + y^2 \leq 4$  rotating about the  $y$ -axis.
  4. Adjust the idea of cylindrical coordinates to compute the volume contained between  $z = x^2 + 4y^2$  and  $z = 9$ .
-

## 2.6 Changing the volume element; spherical coordinates

### Reference in the book

Section 13.7.

### Homework Problems, due Monday, March 19th

1. Use spherical coordinates to compute:

- (a) The total weight of the ice cream and the cone in the overall shape of

$$z \geq \sqrt{x^2 + y^2}, \quad x^2 + y^2 + z^2 \leq 1$$

if the weight density is given by  $\varrho(x, y, z) = z$ . (Suppress the units in this problem.)

- (b)  $\iiint_{\Omega} x \, dV$  where  $\Omega$  is the volume enclosed by the front unit hemisphere

$$x \geq 0, \quad x^2 + y^2 + z^2 \leq 1.$$

- (c) Average value of the function  $f(x, y, z) = x^2 + y^2$  over the ball  $x^2 + y^2 + z^2 \leq R^2$ .

2. Compute the volume of the ellipsoid  $x^2 + 4y^2 + 9z^2 \leq 1$ .
- 

## 2.7 Integration practice

### Reference in the book

The entire Chapter 13.

### Homework Problems, due Monday, March 19th

1. Find the most optimal strategy for computing the following integrals. If the strategy is not obvious, then set up the integrals in all of the ways you can: by means of Fubini's Theorem, by means of polar or cylindrical coordinates, by means of spherical coordinates, try a linear change of coordinates. Based on what comes out decide what the best of these methods is, and decide what that is that makes it the best method. **You are not expected to actually compute the integrals, but you are strongly encouraged to pursue the computations until you can freely say "I know how to finish this problem".**

- (a)  $\iint_R x + y \, dA$  where  $R$  is the parallelogram spanned by the vectors  $\langle 1, 1 \rangle$  and  $\langle 0, 2 \rangle$  with the base point at  $(2, 0)$ ;

- (b)  $\iint_R \frac{y}{x} \, dA$  where  $R$  is region between the parabola  $y = x^2 - 4x + 3$  and the line  $y = x - 1$ ;

- (c) The integral which computes the volume enclosed between  $z = \sqrt{x^2 + y^2}$  and  $z = x^2 + y^2$ ;
- (d) The average value of the function  $f(x, y) = x^2 + y^2$  over the upper unit half-ball centered at the origin;
- (e)  $\iint_{\mathbb{R}^2} \frac{dA}{(1+x^2+y^2)^2}$ ;
- (f)  $\iint_{\mathbb{R}^2} \frac{dA}{(1+x^2+4y^2)^2}$ ;
- (g) The integral which computes the volume inside  $x^2 + y^2 + z^2 = R^2$  but above the plane  $z = \frac{R}{2}$ .
- (h)  $\iint_R \frac{1}{\sqrt[3]{x^2+y^2}} dA$  where  $R$  is the portion of the unit disk located in the first quadrant;
- (i) The integral which computes the volume contained in the intersection of two unit balls: one centered at the origin, and one centered at  $(0, 0, 1)$ ;
- (j)  $\iiint_{\Omega} x dV$  where  $\Omega$  is the unit ball centered at  $(1, 0, 0)$ ;
- (k) The integral which computes the volume contained between the planes  $z = \pm 1$  and inside  $x^2 + y^2 - z^2 = 1$ ;
- (l) The integral which computes the volume contained between the planes  $z = \pm 2$  and inside the hyperboloid  $4x^2 + y^2 - z^2 = 4$ ;
- (m) The integral which computes the volume of the donut formed when the circle of radius 2 centered at  $(3, 0)$  rotates around the  $y$ -axis;
- (n)  $\iiint_{\Omega} x + y + z dV$  where  $\Omega$  is the parallelootope spanned by the vectors  $\langle 1, 1, 0 \rangle$ ,  $\langle 1, 0, 1 \rangle$  and  $\langle 0, 1, 1 \rangle$  whose initial point is at the origin;
- (o)  $\iint_R x dA$  where  $R$  is the region within the first quadrant which is inside the ellipse  $x^2 + 4y^2 = 1$ ;
- (p)  $\iiint_{\Omega} x^2 + y^2 + z^2 dV$  where  $\Omega$  is the pyramid whose base is the diamond with vertices  $(\pm 1, 0, 0)$ ,  $(0, \pm 1, 0)$ , and whose tip is at  $(0, 0, 1)$ .
- (q)  $\iint_{\mathbb{R}^2} e^{-(2x-1)^2-(4y+1)^2} dA$ ;
- (r)  $\iiint_{\mathbb{R}^3} e^{-x^2-y^2-z^2} dV$ .
- 

## 2.8 Circulation (work) and flux integrals

### Reference in the book

Section 14.2. **WARNING:** Flux of a vector field across a closed curve in 2D is not in your textbook. Also, we will be using a slightly different notation.

**Homework Problems, due Thursday, March 21st**

1. Let  $C$  denote the unit circle  $x^2 + y^2 = 1$ , oriented counterclockwise, and consider the vector field  $\vec{V}(x, y) = \langle y, 0 \rangle$ .
  - (a) Plot the circle and the vector field on the same diagram. Hand-drawn sketch will suffice.
  - (b) Based on your sketch from the above, verbally / qualitatively describe the extent to which the vector field  $\vec{V}$  flows or circulates along  $C$ . (A couple of sentences will suffice.)
  - (c) Find an explicit formula for the unit tangent vector field  $\vec{T}$  to  $C$ .
  - (d) Find an explicit formula which describes, at an infinitesimal level, the extent to which the vector field  $\vec{V}$  flows along  $C$ . Do make sure that this quantitative information matches with the qualitative description you provided in part (b).
  - (e) What effect does replacing  $C$  with  $-C$  has?
2. Let  $C$  denote the counter-clockwise unit circle centered at the origin. Furthermore, let  $\vec{V}(x, y) = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ .
  - (a) Plot  $C$  and  $\vec{V}$  on the same diagram. Hand-drawn sketch will suffice.
  - (b) Based on your sketch, qualitatively assess (i.e. eyeball)  $\int_C \vec{V} \cdot \vec{T} ds$ . (A sentence or so explaining your logic will suffice.)
  - (c) Parametrize  $C$  and explicitly compute  $\int_C \vec{V} \cdot \vec{T} ds$ . Do make sure that your computation lines up with the description from part (b).
3. Compute  $\int_C \vec{V} \cdot \vec{T} ds$  where:
  - (a)  $C$  is the portion of the curve  $y = \frac{1}{2}(e^x + e^{-x})$  between  $x = -1$  and  $x = 1$ , oriented from left to right, and  $\vec{V}(x, y) = \langle -y, 1 \rangle$ .
  - (b)  $C$  is the top half of the ellipse  $4x^2 + y^2 = 1$ , oriented counter-clockwise, while  $\vec{V}(x, y) = \langle x + y, -x + y \rangle$ .
  - (c)  $C$  is the line segment between  $(1, 0, 1)$  and  $(-1, -2, -1)$ , while the vector field  $\vec{V}$  is given by  $\vec{V}(x, y, z) = \langle x + 2y, x + 2y, 1 \rangle$ .
4. Let  $C$  denote the unit circle  $x^2 + y^2 = 1$  and let  $\vec{V}(x, y) = \langle x^2, 0 \rangle$ .
  - (a) Plot the circle and the vector field on the same diagram. Hand-drawn sketch will suffice.
  - (b) Based on your sketch from the above, qualitatively assess (i.e. eyeball) the extent to which the vector field  $\vec{V}$  flows outside of the region bounded  $C$ . Do so both on a local (e.g. infinitesimal) and global level.

- (c) Find an explicit formula for the outward pointing unit normal vector field  $\vec{N}$ .
- (d) Find an explicit formula which describes the infinitesimal out-flux of  $\vec{V}$  across  $C$ . Do make sure that this quantitative information matches with the qualitative description you provided in part (b).
- (e) What is the (total) out-flux of  $\vec{V}$  across  $C$ ? What is the (total) in-flux?
5. Let  $C$  denote the unit circle centered at the origin and let  $\vec{V}(x, y) = \langle -x, -y \rangle$ .
- Plot  $C$  and  $\vec{V}$  on the same diagram. Hand-drawn sketch will suffice.
  - Based on your sketch, try to qualitatively assess (i.e. eyeball) the outward flux  $\int_C \vec{V} \cdot \vec{N} ds$ .
  - Parametrize  $C$  and explicitly compute  $\int_C \vec{V} \cdot \vec{N} ds$ . Do make sure that your computation lines up with the description from part (b).
6. Compute the out-flux and the in-flux of the vector field  $\vec{V}(x, y) = \langle y, x \rangle$  across the unit circle centered at the origin.
- 

## 2.9 Surface and flux integrals

### Reference in the book

Section 14.6.

### Homework Problems, due Monday, April 2nd

- Compute the following:
  - $\iint_S z^2 dA$  where  $S$  is the surface of the cylinder of radius 2 centered along the  $z$ -axis, contained between the planes  $z = \pm 1$ . The top and bottom of the cylinder are not included.
  - $\iint_S x^2 + y^2 dA$  where  $S$  is the surface of the sphere of radius 2 centered at the point  $(2, 0, 0)$ .
- An earring is made out of a very thin material and is made in the shape of a helicoid with the parametric equations  $x(u, v) = u \cos(v)$ ,  $y(u, v) = u \sin(v)$ ,  $z(u, v) = v$ , where  $0 \leq u \leq \frac{1}{4}$ ,  $0 \leq v \leq 6\pi$ . (Here  $u$  is measured in inches, while  $v$  is measured in radians.) The weight density of the material at the location with coordinates  $(u, v)$  is  $\varrho = \frac{1}{4\sqrt{u^2+1}}$  ounces per square inch. How heavy is the earring?
- Let  $S$  denote the unit sphere  $x^2 + y^2 + z^2 = 1$  and let  $\vec{V}(x, y, z) = \langle 1, 0, 0 \rangle$ .
  - Sketch the sphere and the vector field on the same diagram.

- (b) Based on your sketch, qualitatively assess (i.e. eyeball) the extent to which the vector field  $\vec{V}$  flows outside of the unit ball. Do so both on a local (e.g. infinitesimal) and global level.
- (c) Find an explicit formula for the outward pointing unit normal vector field  $\vec{N}$ , the area element  $dA$  and for  $\vec{N} dA$ .
- (d) Find an explicit formula which describes the infinitesimal out-flux of  $\vec{V}$  across  $S$ . Do make sure that this quantitative information matches with the qualitative description you provided in part (b).
- (e) What is the (total) out-flux of  $\vec{V}$  across  $S$ ? What is the (total) in-flux?

4. Compute the following:

- (a) Let  $S$  be the upper dome of the sphere of radius 2 centered at the origin. (The equatorial base is not included.) Find the outward flux of  $\vec{V}(x, y, z) = \langle 0, y, z \rangle$  across  $S$ .
- (b) Let  $S$  be the surface of the cone  $z = \sqrt{x^2 + y^2}$  located below the  $z = 1$ . (Note: the circular base / top of the cone is not included.) Find the inward flux of the vector field  $\vec{V} = \langle x, y, 2z \rangle$  across the surface  $S$ .
- (c) Let  $S$  be the surface of the solid formed as an intersection of the paraboloids  $z = x^2 + y^2$  and  $z = 8 - x^2 - y^2$ . Find the outward flux of the vector field  $\vec{V} = \langle 0, 0, z \rangle$  across  $S$ . (Hint: break the flux integral into the sum of two.)
- 

## 2.10 The concept of divergence

### Reference in the book

Definition 5.2 of Section 14.5, and the accompanying examples. **WARNING:** The approach we will be taking in class will be considerably more in depth than what is in your text. Make sure you take good lecture notes!!

### Homework Problems, due Thursday, April 5th

- Let  $B_r$  denote the solid ball of radius  $r$  centered at the point  $(a, b, c)$ . Let  $S_r$  denote the sphere of radius  $r$  which bounds  $B_r$ , let  $\vec{N}$  denote the outward pointing unit normal to  $S_r$  and let  $\vec{V}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$  be some vector field defined near  $(a, b, c)$ . Show that

$$\lim_{r \rightarrow 0} \frac{\iint_{S_r} \vec{V} \cdot \vec{N} dA}{\text{Volume}(B_r)} = \frac{\partial P}{\partial x}(a, b, c) + \frac{\partial Q}{\partial y}(a, b, c) + \frac{\partial R}{\partial z}(a, b, c).$$

Discuss the intuitive meaning of this result in a sentence or so.

2. Consider the vector field given in spherical coordinates by

$$\vec{V} = \frac{1}{r^2} \partial_r = \left\langle \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle$$

and concentric spheres  $S_r$  of radius  $r$  centered at the origin.

- (a) Compute the outward flux  $\iint_{S_r} \vec{V} \cdot \vec{N} dA$  using the definition of the flux integral;

- (b) Find the flux rate

$$\lim_{r \rightarrow 0} \frac{\iint_{S_r} \vec{V} \cdot \vec{N} dA}{\text{Volume}(B_r)}$$

where  $B_r$  denotes the ball enclosed by  $S_r$ .

- (c) Compute the divergence of the vector field  $\vec{V}$ .

- (d) What is the outward flux rate of  $\vec{V}$  at the origin? What about anywhere other than the origin?
- 

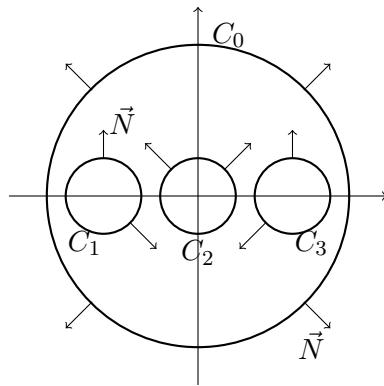
## 2.11 Divergence Theorems

### Reference in the book

Sections 14.4 and 14.7. **WARNING:** The narrative in class regarding the material from Section 14.4 will be rather different than what is in your text. Take good lecture notes!!

### Homework Problems, due Thursday, April 5th

1. Consider the circles  $C_0$ ,  $C_1$ ,  $C_2$ ,  $C_3$ , and the outward pointing vector fields  $\vec{N}$  as in the following picture.



Suppose  $\vec{V}(x, y)$  is a vector field defined (and smooth) throughout the  $xy$ -plane. Furthermore, suppose that  $\vec{V}$  has the following outward fluxes:

$$\int_{C_0} \vec{V} \cdot \vec{N} ds = 1, \quad \int_{C_1} \vec{V} \cdot \vec{N} ds = 3, \quad \int_{C_2} \vec{V} \cdot \vec{N} ds = -2, \quad \int_{C_3} \vec{V} \cdot \vec{N} ds = 5.$$

Use Green's Theorem to evaluate  $\iint_D \operatorname{div}(\vec{V}) dA$  where  $D$  denotes

- (a) The area inside  $C_0$  but outside  $C_2$ ;
- (b) The area inside  $C_0$  but outside both  $C_1$  and  $C_3$ .
2. Evaluate the outward flux integrals  $\int_C \vec{V} \cdot \vec{N} ds$  for the following choices of the vector fields  $\vec{V}$  and the curves  $C$ . You are expected to use Green's Theorem in divergence form.
- (a)  $\vec{V}(x, y) = \langle 3y, x \rangle$ , while  $C$  is the counter-clockwise unit circle centered at the origin.
- (b)  $\vec{V}(x, y) = \langle x^2 + xy + y^2, x^2 - xy + y^2 \rangle$ , while  $C$  is the boundary of the rectangle with vertices at  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 2)$ ,  $(0, 2)$ .
- (c)  $\vec{V}(x, y) = \langle x, y \rangle$ , while  $C$  is the boundary of the triangle with vertices at  $(-1, 0)$ ,  $(0, 1)$  and  $(1, 0)$ .
3. Use Gauss' Theorem to evaluate the following flux integrals.
- (a) The outward flux of  $\vec{V}(x, y, z) = \langle x^3 + y^3, y^3 + z^3, z^3 + x^3 \rangle$  across the unit sphere centered at the origin;
- (b) The inward flux of  $\vec{V}(x, y, z) = \langle xy, yz, xz \rangle$  across the bowl of the elliptic paraboloid  $z = x^2 + y^2$
- i. up to and including the lid at  $z = 4$ ;
  - ii. up to but *not* including the lid at  $z = 4$ ;
- (c) The inward flux of the vector field  $\vec{V}(x, y, z) = \langle x^2 + x, y^2 + y, -z^2 - z \rangle$  across the cone  $z = \sqrt{x^2 + y^2}$
- i. up to and including the lid at  $z = 2$ ;
  - ii. up to but *not* including the lid at  $z = 2$ ;
- (d) The outward flux of  $\vec{V}(x, y, z) = \langle x, y, z \rangle$  across the cube  $-1 \leq x, y, z \leq 1$ .
- 

## 2.12 The concept of scalar curl; Green's Theorem

### Reference in the book

**WARNING:** Although Sections 14.4 and 14.5 cover the same ideas (more-or-less) I do not recommend them. Take good lecture notes!!

**Homework Problems, due Monday, April 9th**

1. Let  $S_r$  denote the square of side  $2r$

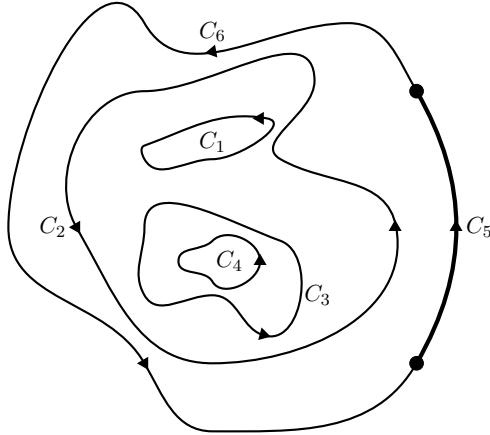
$$a - r \leq x \leq a + r, \quad b - r \leq y \leq b + r$$

centered at the point  $(a, b)$ . Let  $C_r$  be the boundary of that square, and let  $\vec{V}(x, y) = \langle P(x, y), Q(x, y) \rangle$  be some vector field defined near  $(a, b)$ . Show that

$$\lim_{r \rightarrow 0} \frac{\int_{C_r} \vec{V} \cdot \vec{T} ds}{\text{Area}(S_r)} = \frac{\partial Q}{\partial x}(a, b) - \frac{\partial P}{\partial y}(a, b).$$

Discuss the intuitive meaning of this result in a sentence or so.

2. Consider the configuration on the following diagram. Note that  $C_5$  is the bold path with distinct starting and ending points, and that  $C_6$  denotes the “left-over”.



Suppose  $\vec{V}(x, y)$  is a vector field defined (and smooth) throughout the  $xy$ -plane. Furthermore, suppose the following:

$$\int_{C_1} \vec{V} \cdot \vec{T} ds = -1, \quad \int_{C_2} \vec{V} \cdot \vec{T} ds = 1, \quad \int_{C_3} \vec{V} \cdot \vec{T} ds = -2, \quad \int_{C_4} \vec{V} \cdot \vec{T} ds = 2, \quad \int_{C_5} \vec{V} \cdot \vec{T} ds = -3,$$

while  $\text{curl}(\vec{V})$  vanishes outside of  $C_2$ . Use Green’s Theorem to evaluate

- (a)  $\iint_D \text{curl}(\vec{V}) dA$  where  $D$  denotes the area inside  $C_3$  but outside  $C_4$ ;
- (b)  $\iint_D \text{curl}(\vec{V}) dA$  where  $D$  denotes the area inside  $C_2$  but outside  $C_1$ ;
- (c)  $\iint_D \text{curl}(\vec{V}) dA$  where  $D$  denotes the area inside  $C_2$  but outside  $C_4$ ;
- (d)  $\iint_D \text{curl}(\vec{V}) dA$  where  $D$  denotes the area inside  $C_2$  but outside both  $C_1$  and  $C_4$ ;
- (e)  $\int_{C_6} \vec{V} \cdot \vec{T} ds$ .

3. Use Green's Theorem to evaluate the circulation integrals  $\int_C \vec{V} \cdot \vec{T} ds$  for the following choices of  $\vec{V}$  and  $C$ .
- $\vec{V}(x, y) = \langle 3y, x \rangle$ , while  $C$  is the counter-clockwise unit circle centered at the origin.
  - $\vec{V}(x, y) = \langle x^2 + xy + y^2, x^2 - xy + y^2 \rangle$ , while  $C$  is the boundary of the rectangle with vertices at  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 2)$ ,  $(0, 2)$  oriented counter-clockwise.
  - $\vec{V}(x, y) = \langle 3x^2y + y^3, -x^3 - 3xy^2 \rangle$ , while  $C$  is the boundary of the unit circle centered at the origin oriented counter-clockwise.
  - $\vec{V}(x, y) = \langle x, y \rangle$ , while  $C$  is the boundary of the triangle with vertices at  $(-1, 0)$ ,  $(0, 1)$  and  $(1, 0)$  oriented clock-wise.
- 

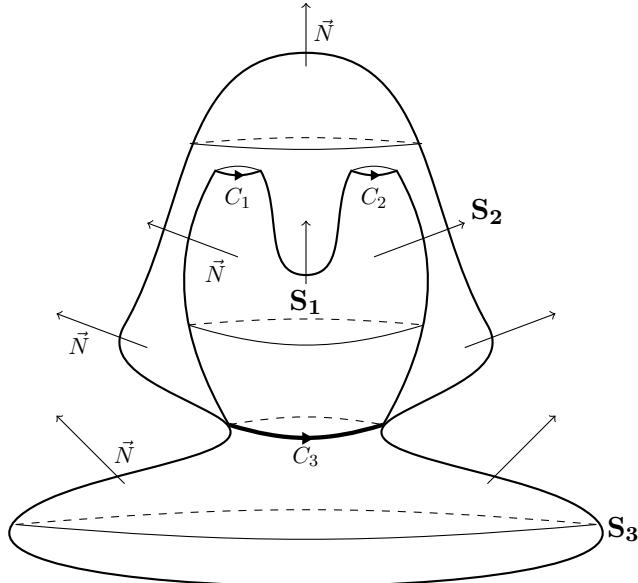
## 2.13 The concept of curl; Stokes' Theorem

### Reference in the book

Sections 14.5 and 14.8. **WARNING:** The narrative in class regarding the concept of  $\overrightarrow{\text{curl}}$  will differ quite a bit from the one in your textbook. Take good lecture notes!!

### Homework Problems, due Thursday, April 12th

- Let  $\vec{V}(x, y, z) = \langle yz, -xz, xy \rangle$ .
  - What is the circulation rate of  $\vec{V}$  around the vector  $\vec{N} = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$  at the following points:
    - $(1, 0, 0)$ ;
    - $(0, 1, 0)$ ;
    - $(0, 0, -1)$ .
  - Repeat the above for the circulation around the vector  $\vec{N} = \langle 0, 1, 0 \rangle$ .
  - Around which vector (or in other words, in which plane) does  $\vec{V}$  circulate the most? What is the maximum circulation rate?
- Consider the configuration on the following diagram. Note: the surface  $S_1$  is just the “volcanos” up to  $C_3$ , the surface  $S_2$  is just the “head” and the surface  $S_3$  is just the bowl on the bottom. The normal vector field  $\vec{N}$  is outward pointing, as indicated. Suppose  $\vec{V}(x, y, z)$  is a vector field defined (and smooth) throughout the  $xyz$ -space. Use Stokes' Theorem to evaluate
  - $\int_{C_3} \vec{V} \cdot \vec{T} ds$  if it is known that  $\iint_{S_3} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA = 2$ ;



- (b)  $\iint_{S_2} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA$  if it is known that  $\iint_{S_3} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA = 1$ ;
- (c)  $\iint_{S_1} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA$  if it is known that  $\int_{C_1} \vec{V} \cdot \vec{T} ds = \int_{C_2} \vec{V} \cdot \vec{T} ds = \int_{C_3} \vec{V} \cdot \vec{T} ds = 3$ ;
- (d)  $\int_{C_2} \vec{V} \cdot \vec{T} ds$  if it is known that  $\iint_{S_1} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA = \iint_{S_2} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA = 3$   
and  $\int_{C_1} \vec{V} \cdot \vec{T} ds = -2$ .
3. Let  $\vec{V}(x, y, z) = \langle -y, x, 0 \rangle$ , let  $S$  be the dome of the upper unit hemisphere centered at the origin, and let  $C$  be the equator of the hemisphere in the  $xy$ -plane, oriented counter-clockwise. Compute:
- The outward flux  $\iint_S \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA$  using the definition of the flux integral;
  - The circulation  $\int_C \vec{V} \cdot \vec{T} ds$  using the definition of the circulation integral.
  - In a sentence or so, comment on the relationship between the answers to parts (a) and (b), and the Stokes' Theorem.
4. Use Stokes' Theorem to evaluate the circulation integrals  $\int_C \vec{V} \cdot \vec{T} ds$  for the following choices of  $\vec{V}$  and  $C$ .
- $\vec{V}(x, y, z) = \langle z, x, y \rangle$  while  $C$  is the contour of the triangle with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$  in that order.
  - $\vec{V}(x, y, z) = \langle x^3, y^3, z^3 \rangle$  while  $C$  is the contour of a triangle with vertices  $(0, 0, 0)$ ,  $(1, 0, 1)$  and  $(0, 1, 1)$  traversed in that order.
  - $\vec{V}(x, y, z) = \langle 0, -z, y \rangle$  while  $C$  is the North 45-th parallel of the unit sphere centered at the origin, oriented counterclockwise when viewed from the above.

## 2.14 Fundamental Theorems of Calculus: Synthesis, part 1

### Reference in the book

The entire Chapter 14. (OK, maybe with the exception of Section 14.9.)

### Homework Problems, due Thursday, April 12th

1. Please compute the following.

- (a)  $\int_C \vec{V} \cdot \vec{T} ds$  where  $\vec{V} = \langle y, 3x \rangle$  and where  $C$  is the counter-clockwise unit circle centered at the origin.
  - (b) The inward flux of  $\vec{V}(x, y) = \langle x, y - y^2 \rangle$  across the unit circle centered at  $(1, 0)$ .
  - (c) The outward flux of  $\vec{V} = \langle 3x, y \rangle$  across the top half of the unit circle centered at  $(1, 0)$ .
  - (d) The outward flux of  $\vec{V} = \langle x, -y, z \rangle$  across the surface of the sphere of radius 2 centered at the origin.
  - (e)  $\int_C \vec{V} \cdot \vec{T} ds$  where  $\vec{V} = \langle x, y - z, y + z \rangle$  and where  $C$  is the North 60-th parallel of the unit sphere centered at the origin, oriented counterclockwise when viewed from the above.
- 

## 2.15 Changing the domain of integration

### Reference in the book

NONE!? Only portions of this material can be found in your textbook (see Example 4.5 of Section 14.4 and Theorem 8.2 of Section 14.8). Please take good lecture notes!!

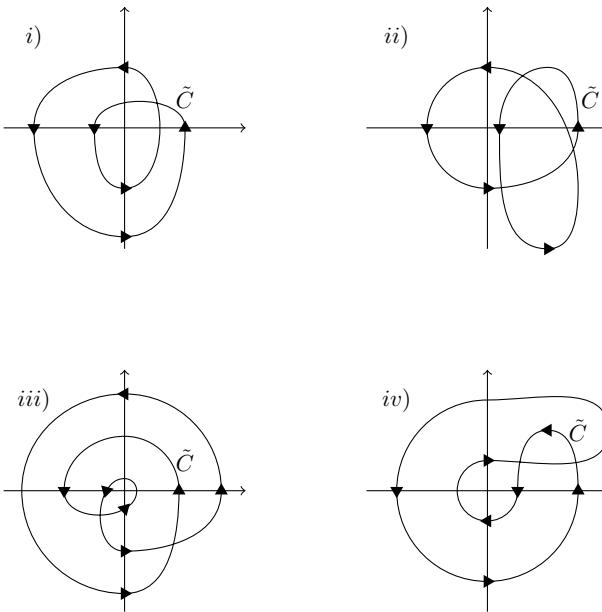
### Homework Problems, due Monday, April 16th

1. A recent example involved the vector field

$$\vec{V}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

along a simple counter-clockwise closed curve (contour)  $C$ .

- (a) In a sentence or so, what was the point of that example?
- (b) What is the value of  $\int_C \vec{V} \cdot \vec{T} ds$  if  $C$  does not contain the origin in its interior?  
Provide a computation / justification for your claim.
- (c) What is the value of  $\int_C \vec{V} \cdot \vec{T} ds$  if  $C$  does contain the origin in its interior?  
Provide a computation / justification for your claim.

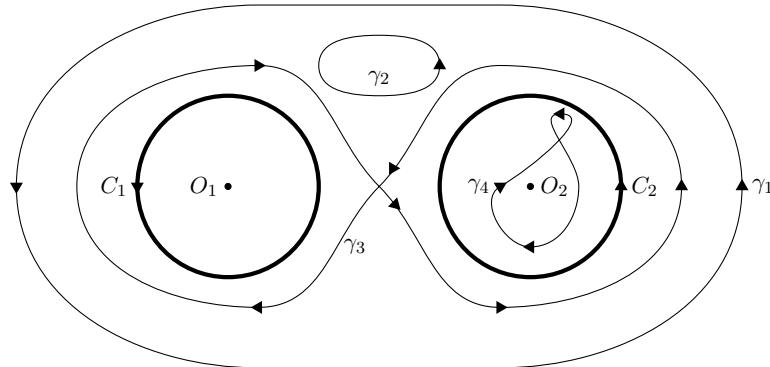


- (d) Furthermore, what is the value of  $\int_{\tilde{C}} \vec{V} \cdot \vec{T} ds$  for the following contours  $\tilde{C}$ :
2. Let  $C_1$  be the ellipse  $4x^2 + 9y^2 = 1$  and let  $C_2$  be the standard unit circle. Assume both are oriented counterclockwise. Furthermore, let

$$\vec{V}(x, y) = \left\langle \frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right\rangle.$$

Use Green's theorem to argue that  $\int_{C_1} \vec{V} \cdot \vec{T} ds = \int_{C_2} \vec{V} \cdot \vec{T} ds$ . Based on this find the value of  $\int_{C_1} \vec{V} \cdot \vec{T} ds$ .

3. Suppose that a vector field  $\vec{V}(x, y) = \langle P(x, y), Q(x, y) \rangle$  is defined and smooth everywhere except at points  $O_1$  and  $O_2$ , and that  $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$  (i.e. that  $\vec{V}$  is “curl-free”). Let  $C_1$  and  $C_2$  be circles centered at points  $O_1$  and  $O_2$  respectively, and let  $\gamma_1, \gamma_2, \gamma_3$  and  $\gamma_4$  be the closed curves sketched below. (Orientation is also indicated on the sketch.) (Also,  $\gamma_3$  looks like a figure-eight.)



Finally, suppose that

$$\int_{C_1} \vec{V} \cdot \vec{T} ds = 1 \text{ and } \int_{C_2} \vec{V} \cdot \vec{T} ds = 3.$$

Compute the circulations  $\int_{\gamma_1} \vec{V} \cdot \vec{T} ds$ ,  $\int_{\gamma_2} \vec{V} \cdot \vec{T} ds$ ,  $\int_{\gamma_3} \vec{V} \cdot \vec{T} ds$  and  $\int_{\gamma_4} \vec{V} \cdot \vec{T} ds$ .

4. (a) Let  $C$  denote the upper half of the ellipse  $4x^2 + 9y^2 = 16$ . Consider the vector field  $\vec{V}(x, y) = \langle 1+2x+3y, -x-2y \rangle$ ; it is easy to see that  $\operatorname{div}(\vec{V}) = 0$ . Compute the outward flux integral  $\int_C \vec{V} \cdot \vec{N} ds$ . (Note: the “diameter” at the bottom is not included in  $C$ .)

- (b) Let  $C$  denote the circle of radius 2 centered at the origin. Consider the vector field

$$\vec{V}(x, y) = \left\langle \frac{1-x}{(x-1)^2 + (y-1)^2}, \frac{1-y}{(x-1)^2 + (y-1)^2} \right\rangle;$$

a computation shows that  $\operatorname{div}(\vec{V}) = 0$ . Find the outward flux integral  $\int_C \vec{V} \cdot \vec{N} ds$ .

5. Consider the vector field given in spherical coordinates by

$$\vec{V} = \frac{1}{r^2} \partial_r;$$

in an earlier homework assignment you computed that  $\operatorname{div}(\vec{V}) = 0$ .

- (a) Given the fact that  $\vec{V}$  is undefined at the origin, discuss the appropriate ways of applying the Gauss' Theorem to (the flux of) the vector field  $\vec{V}$ .
- (b) What is the outward flux of the vector field  $\vec{V}$  across the unit sphere centered at the origin?
- (c) What is the outward flux of the vector field  $\vec{V}$  across some other sphere centered at the origin?
- (d) What is the outward flux of the vector field  $\vec{V}$  across some other surface which contains the origin in its interior?
- (e) What is the outward flux of the vector field  $\vec{V}$  across some other surface which contains the origin in its exterior?
6. Let  $S$  be the surface of the cylinder of radius 1 centered along the  $z$ -axis, bounded between the  $xy$ -plane on the bottom and the plane  $x+y+z = 20$  on the top. Note that the bases (bottom, lid) are not included in  $S$ . Let  $C_1$  be the bottom boundary of the cylinder  $S$  (i.e. the unit circle in the  $xy$ -plane), and let  $C_2$  be the top boundary of the cylinder  $S$  (i.e. the ellipse in the slanted pane  $x+y+z = 20$ ). Let both  $C_1$  and  $C_2$ , when viewed from above, be oriented clockwise. Finally, let  $\vec{V}(x, y, z) = \langle x-y, x+y, z \rangle$ .
- (a) Make a rough sketch of  $S$ ,  $C_1$  and  $C_2$ . Please include orientation arrows.

- (b) Apply Stokes' Theorem to the situation. You should get an equality involving flux and circulation integrals.
  - (c) Compute the flux integral appearing in the equality you learned about in part (b).
  - (d) Compute the easier one of the two circulation integrals appearing in the equality you learned about in part (b).
  - (e) What is the value of  $\int_{C_2} \vec{V} \cdot \vec{T} ds$ ?
- 

## 2.16 Fundamental Theorems of Calculus: Synthesis, part 2

### Reference in the book

The entire Chapter 14. (OK, maybe with the exception of Section 14.9.)

### Homework Problems, due Monday, April 16th

1. Please compute  $\int_C \vec{V} \cdot \vec{T} ds$  where
  - (a)  $\vec{V}(x, y) = \langle -y^3, x^3 \rangle$  and where  $C$  is the counter-clockwise unit circle centered at the origin.
  - (b)  $\vec{V}(x, y) = \langle y, x \rangle$  and where  $C$  is the counter-clockwise ellipse  $4x^2 + y^2 = 1$ .
  - (c)  $\vec{V}(x, y) = \langle \frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \rangle$  and where  $C$  is the ellipse  $4x^2 + 9y^2 = 1$  oriented counter-clockwise.
  - (d)  $\vec{V}(x, y, z) = \langle -z, -z, x + y \rangle$  and where  $C$  is the North 45-th parallel of the unit sphere centered at the origin, oriented counterclockwise when viewed from the above.
  - (e)  $\vec{V}(x, y, z) = \langle z + x, x + y, y + z \rangle$  and where  $C$  is the contour of the triangle with edges  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$  in that order.
  - (f)  $\vec{V}(x, y, z) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \rangle$  and where  $C$  is a slanted ellipse centered somewhere on the  $z$ -axis.
2. Please compute the following flux integrals.
  - (a) The outward flux of  $\vec{V}(x, y) = \langle 1, 1 \rangle$  across the boundary of some closed contour  $C$ ;
  - (b) The inward flux of  $\vec{V}(x, y) = \langle 2xy, x^2 + y^2 \rangle$  across the boundary of the rectangle with vertices at  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 2)$ ,  $(0, 2)$  oriented counter-clockwise.
  - (c) The outward flux of the vector field  $\vec{V}(x, y) = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle$  across the ellipse  $x^2 + 4y^2 = 1$ ;

- (d) The outward flux of the vector field  $\vec{V}(x, y) = \langle x^3, y^3 \rangle$  across the top half of the unit circle centered at the origin;
- (e) The outward flux of  $\vec{V}(x, y, z) = \langle x^3 + y^3, y^3 + z^3, z^3 + x^3 \rangle$  across the unit sphere centered at the origin;
- (f) The inward flux of  $\vec{V}(x, y, z) = \langle xy, yz, xz \rangle$  across the bowl of the paraboloid  $z = x^2 + y^2$  up to and including the lid at  $z = 4$ ;
- (g) The same as the previous problem except with the lid removed;
- (h) The inward flux of  $\vec{V}(x, y, z) = \langle x^2 + x, y^2 + y, -z^2 - z \rangle$  across the surface of the cone  $z = \sqrt{x^2 + y^2}$  up to but not including the lid at  $z = 2$ ;
- (i) The outward flux of  $\vec{V}(x, y, z) = \langle x, y, z \rangle$  across the cube  $-1 \leq x, y, z \leq 1$ .
- (j) The outward flux of  $\vec{V}(x, y, z) = \langle 1, 1, 1 \rangle$  across the upper hemisphere of the unit sphere centered at the origin.
- 

## 2.17 The gradient vector field

### Reference in the book

Section 12.6.

### Homework Problems, due Thursday, April 19th

1. Let  $f(x, y)$  be some function whose gradient is

$$\overrightarrow{\text{grad}}(f) = \langle -x + y, x - y \rangle.$$

- (a) Find the rate the change of  $f$  in the direction of:
- the unit vector  $\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$  based at  $(x, y) = (-1, 1)$ ;
  - the radial coordinate vector  $\partial_r$  based at  $(x, y) = (-1, 0)$ ;
  - the unit vector  $\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$  based at  $(x, y) = (0, -1)$ .
- (b) Based at  $(-1, 1)$ , find the unit direction in which the rate of change of  $f$  is maximum possible. What is the value of this rate of change?
- (c) Based at  $(-1, 1)$ , find the unit direction in which the rate of change of  $f$  is minimum possible. What is the value of this rate of change?
- (d) Find a normal vector to the level set of  $f$  passing through  $(-1, 1)$ .
- (e) Find a tangent vector to the level set of  $f$  passing through  $(-1, 1)$ .

2. Let  $f(x, y, z)$  be some function whose gradient is

$$\overrightarrow{\text{grad}}(f) = \langle 1, 2y, 3z^2 \rangle.$$

Find the rate the change of  $f$  in the direction of:

- (a) the unit vector  $\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$  based at  $(x, y, z) = (-1, 1, 0)$ ;
- (b) the unit vector  $\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$  based at  $(x, y, z) = (1, 1, 1)$ .

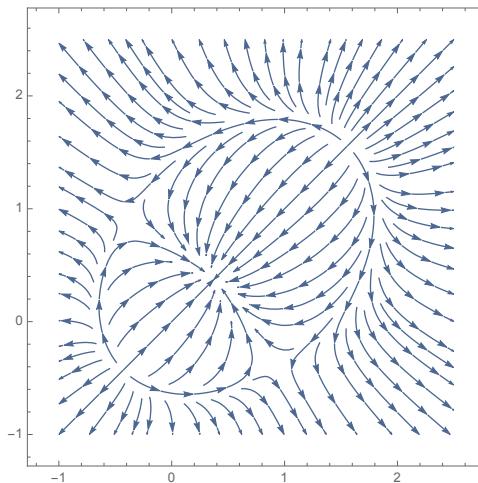
3. Consider the function

$$f(x, y) = 3x + 3y - x^3 - y^3$$

on the domain  $-2 \leq x, y \leq 2$ .

- (a) Find the gradient vector field of the function.
- (b) Use “technology” to show the gradient vector field on top of the contour map.  
Please include a print-out.
- (c) On your contour map draw (by hand) the path of steepest ascent starting at  $(x, y) = (0, 0)$ . How steep is this path at  $(0, 0)$ ?
- (d) On your contour map draw (by hand) the path of steepest ascent starting at  $(x, y) = (-1, 0)$ . How steep is this path at  $(-1, 0)$ ?

4. The following vector field is the gradient vector field for a function  $f(x, y)$ . Sketch the contour map of  $f(x, y)$ .



5. Utilize the gradient of the function  $f(x, y) = x^2 - y^2$  to compute a normal vector field to the hyperbola  $x^2 - y^2 = 1$ . Evaluate the normal vector field you found at six different points on the hyperbola. Use your computations to provide a hand-drawn sketch.

6. Utilize the gradient of the function  $f(x, y, z) = x^2 + y^2 + 2z^2$  to compute a normal vector field to the ellipsoid

$$x^2 + y^2 + 2z^2 = 4.$$

Based on your answer address the following:

- (a) Evaluate and sketch (by hand) the normal vector you found at the point  $(1, -1, 1)$ ;
  - (b) Find the equation of the tangent plane to the ellipsoid at  $(1, -1, 1)$ .
- 

## 2.18 The gradient vector field and constrained optimization

### Reference in the book

Section 12.8.

### Homework Problems, due Monday, April 23rd

1. Find the minimum and the maximum of
  - (a) The function  $f(x, y) = x + y$  subject to the constraint  $x^2 + y^2 = 1$ ;
  - (b) The function  $f(x, y) = x^2 - y^2$  subject to the constraint that  $x^2 + y^2 = 1$ .
2. Find the point on the ellipse  $x^2 + xy + y^2 = 1$  which is
  - (a) The closest to the origin;
  - (b) The furthest from the origin.

Based on your findings make a precise sketch of the ellipse. (*Hint:* You need to find the maximum / minimum value of  $x^2 + y^2$ , which measures the square of the distance to the origin.)
3. Find the minimum and the maximum of the function  $f(x, y) = x + y$  subject to the constraint  $x^2 + y^2 \leq 1$ .
4. Consider the function  $f(x, y) = x^2 - 2x + y^2$ . Find the absolute maximum and the absolute minimum of the function  $f$  over the right half-disk

$$x^2 + y^2 \leq 4, \quad x \geq 0.$$


---

## 2.19 The Fundamental Theorem of Calculus in Gradient Form

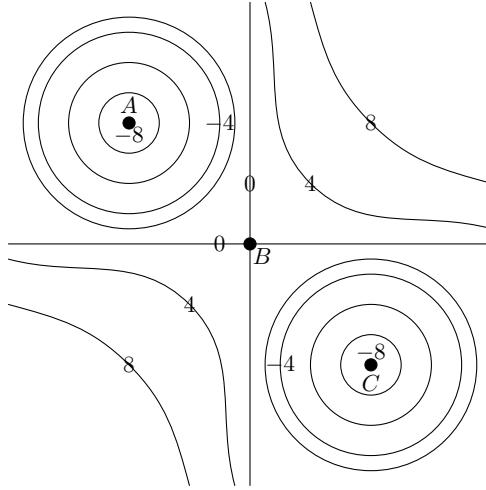
### Reference in the book

Section 14.3. You will benefit from good lecture notes!

**Homework Problems, due Thursday, April 26th**

1. Are the following vector fields  $\vec{V}$  gradient vector fields? If so, find their potential.
- $\vec{V}(x, y) = \langle x + y, x + y \rangle;$
  - $\vec{V}(x, y) = \langle x + 2y, x + 2y \rangle;$
  - $\vec{V}(x, y, z) = \langle x + y + z, x + y + z, x + y + z \rangle.$
2. Compute the following integrals. Use the Fundamental Theorem of Vector Calculus in gradient form whenever appropriate.
- $\int_C \vec{V} \cdot \vec{T} ds$  where  $\vec{V}(x, y) = \langle x, y \rangle$  and where  $C$  is the line segment going from  $(1, 0)$  to  $(1, 1)$ .
  - $\int_C \vec{V} \cdot \vec{T} ds$  where  $\vec{V}(x, y, z) = \langle x + y + z, x + y + z, x + y + z \rangle$  and where  $C$  is a counter-clockwise helix going from  $(1, 0, 0)$  to  $(1, 0, 4\pi)$ .
  - $\int_C \vec{V} \cdot \vec{T} ds$  where  $\vec{V} = \langle \cos(y) - z \sin(x), \cos(z) - x \sin(y), \cos(x) - y \sin(z) \rangle$  and where  $C$  is the counter-clockwise helix going from  $(1, 0, 0)$  to  $(1, 0, 4\pi)$ .
  - $\int_C \vec{V} \cdot \vec{T} ds$  where  $\vec{V}(x, y, z) = \langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \rangle$  and where  $C$  is a counter-clockwise helix going from  $(1, 0, 0)$  to  $(1, 0, 4\pi)$ .
  - $\int_C \vec{V} \cdot \vec{T} ds$  where  $\vec{V}(x, y, z) = \langle x, x + y, x + y + z \rangle$  where  $C$  is the line segment joining  $(1, 1, 0)$  and  $(0, 0, 1)$ .
  - $\int_C \vec{V} \cdot \vec{T} ds$  where  $\vec{V}(x, y) = \langle y, x \rangle$  and where  $C$  is the top half of the counter-clockwise ellipse  $4x^2 + y^2 = 1$ .

The following is the contour map of  $f(x, y)$ . Please note the contour labels, particularly at the points  $A$ ,  $B$  and  $C$ .



- (a) Sketch  $\overrightarrow{\text{grad}}(f)$ .

- (b) Evaluate  $\int_{\gamma} \overrightarrow{\text{grad}}(f) \cdot \vec{T} ds$  where the curve  $\gamma$
- goes from  $A$  to  $B$ ;
  - goes from  $A$  to  $C$ .
- (c) Estimate the  $\pm$  sign of  $\text{div}(\overrightarrow{\text{grad}}(f))$  at points  $A$  and  $C$ .
3. Consider the vector field  $\vec{V}(x, y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ .
- Verify that both  $f_1(x, y) = \arctan\left(\frac{y}{x}\right)$  and  $f_2(x, y) = -\arctan\left(\frac{x}{y}\right)$  serve as potentials for the vector field  $\vec{V}$ .
  - What are the domains for  $f_1$  and  $f_2$ ?
  - Let  $P_1, P_2, P_3$  and  $P_4$  denote the points  $(1, -1)$ ,  $(1, 1)$ ,  $(-1, 1)$  and  $(-1, -1)$ , respectively. Furthermore, let  $P_1P_2$  denote the line segment from  $P_1$  to  $P_2$ , let  $P_2P_3$  denote the line segment from  $P_2$  to  $P_3$ , etc. Which one of the following is true, and why?
    - $\int_{P_1P_2} \vec{V} \cdot \vec{T} ds = f_1(P_2) - f_1(P_1)$  or  $\int_{P_1P_2} \vec{V} \cdot \vec{T} ds = f_2(P_2) - f_2(P_1)$ ?
    - $\int_{P_2P_3} \vec{V} \cdot \vec{T} ds = f_1(P_3) - f_1(P_2)$  or  $\int_{P_2P_3} \vec{V} \cdot \vec{T} ds = f_2(P_3) - f_2(P_2)$ ?
    - $\int_{P_3P_4} \vec{V} \cdot \vec{T} ds = f_1(P_4) - f_1(P_3)$  or  $\int_{P_3P_4} \vec{V} \cdot \vec{T} ds = f_2(P_4) - f_2(P_3)$ ?
    - $\int_{P_4P_1} \vec{V} \cdot \vec{T} ds = f_1(P_1) - f_1(P_4)$  or  $\int_{P_4P_1} \vec{V} \cdot \vec{T} ds = f_2(P_1) - f_2(P_4)$ ?
  - Based on part (c) alone, find the value of the circulation integral  $\int_{P_1P_2P_3P_4} \vec{V} \cdot \vec{T} ds$  along the (counterclockwise) rectangular path  $P_1P_2P_3P_4$  from above?
-

# **Part III**

# **Resources**

## Chapter 3

# A little something about using Mathematica

### 3.1 First steps with Mathematica

LC subscribes to a technical computing software called Mathematica. Mathematica should be installed on all LC owned computers on campus (library, labs, classrooms, Math Department student lounge, SQRC, etc). For homework purposes I expect you to use one of these public computers. You should know though that there is such a thing as a Mathematica “student license”, which allows you to use temporarily use Mathematica on your own machine. Personally, I think that getting that to work is more effort than necessary but if you’d like the license nonetheless then please talk to Math Department secretary Anne Boal. Note that temporary outages of Mathematica are possible due to extremely high use. (This kind of things happens one or two times a semester.)

Once you open Mathematica, you need to open a Notebook. If a Notebook does not open up on its own, you can open it from a drop-down menu (look for “File” on the top of the screen). Your Notebooks will be saved as .nb files.

Under Help drop-down-menu you can also find an item called Wolfram Documentation. *I highly recommend keeping your Notebook and the Documentation open at all times!* Whenever you need to know the syntax/command for something, just search the Documentation. Make use of the examples in the Documentation – I certainly did when I first started using Mathematica.

*Big important note:* Mathematica distinguishes between [Return], which is a line break, and [Enter] = [Shift]+[Return] which tells the computer to compute. You want [Enter].

Another important note: You have to pay attention to case, commas, braces & brackets, etc. For example, Cos[Pi] is Mathematica-speak for  $\cos(\pi)$ .

If you are brand new to computing softwares of this sort I recommend going over a tutorial written by one of former students, Colin Gavin. Both the .nb and .pdf files of the tutorial are on the course website. Alternatively, there are many online tutorials but they typically cover a wide range of topics – well beyond what you need right now.

## 3.2 Graphing in Mathematica

### 3.2.1 Plotting functions

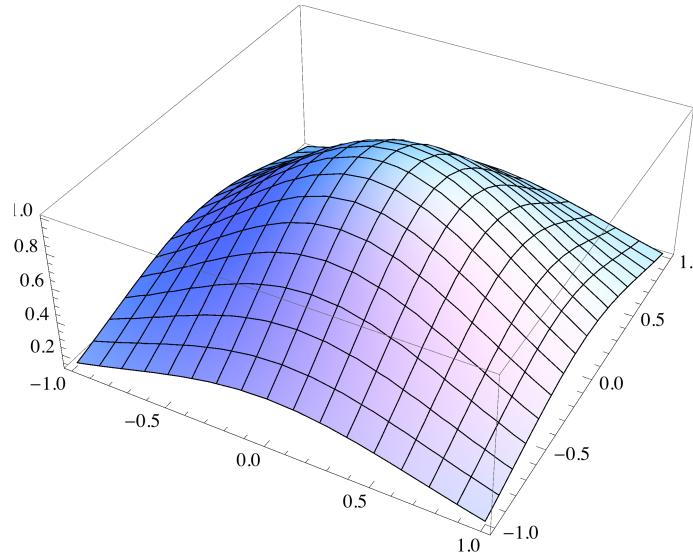
The basic plot command takes the form

```
Plot[function of variable, {variable, starting value, ending value}]
```

or, in the multivariable setting,

```
Plot3D[function of 2 variables, {variable, start, end}, {variable, start, end}]
```

For example, the input `Plot3D[Exp[-x^2 - y^2], {x, -1, 1}, {y, -1, 1}]` should yield the following plot



Note: in the Documentation you will find the Visualization and Graphics section. There, under Function Visualization, you will find all sorts of ways of embellishing your graphs. Take a look!

We can also plot multiple functions on the same plot. Try, for example, the following:

```
Plot3D[{x^2 + y^2, 1 - x^2 - y^2}, {x, -1, 1}, {y, -1, 1}]
```

You should see two intersecting curved sheets. Note that the box containing the graph can be moved around.

### 3.2.2 Contour maps

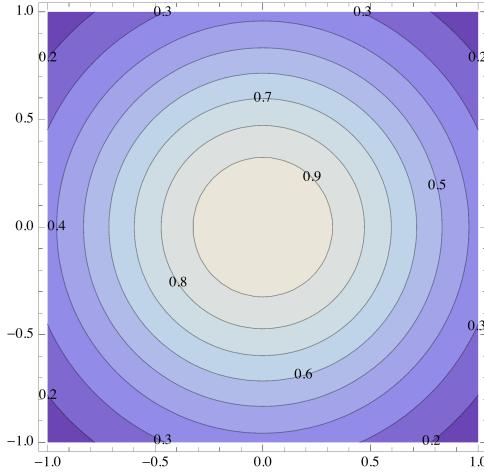
Sometimes we might want to obtain a contour or “topographic map” of a surface in space. This is done using `ContourPlot`. For example, to get the contour map for

$$z = f(x, y) = e^{-x^2-y^2}$$

we use the code

```
ContourPlot[Exp[-x^2 - y^2], {x, -1, 1}, {y, -1, 1}, ContourLabels -> True]
```

The code should yield the following map:



By default lighter regions correspond to higher output values. (One can change that – look up the Documentation!)

Note that Mathematica can do higher dimensional versions of the same thing. For example,

```
ContourPlot3D[x^2+y^2+z^2==1, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```

yields the level surface for the function  $f(x, y, z) = x^2 + y^2 + z^2$  corresponding to level 1. (Note the two = signs; that was intentional.)

### 3.2.3 Region plots

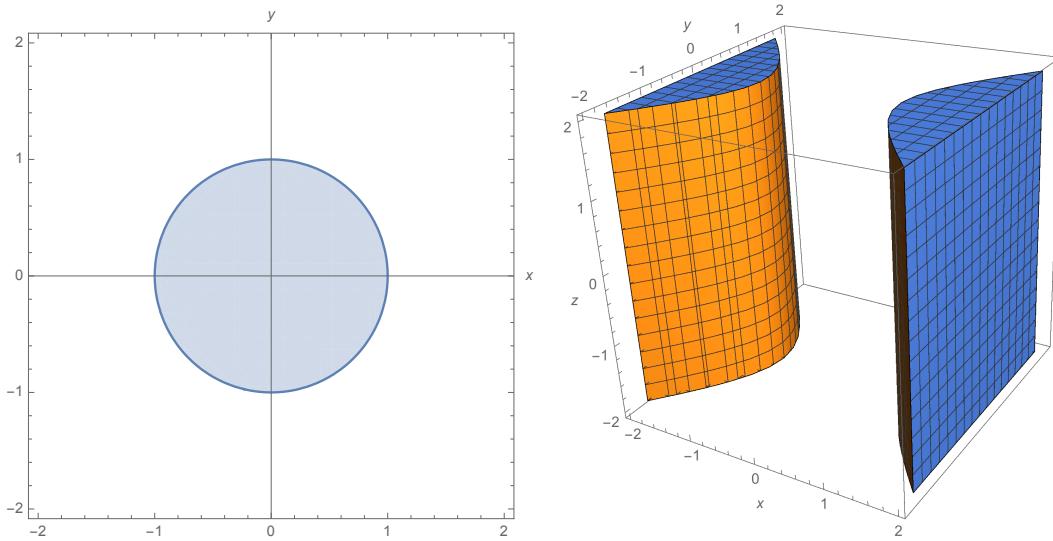
Often times when addressing areas in a plane or volumes in space we end up using inequalities. For example,  $x^2 + y^2 \leq 1$  in a plane describes the standard unit disk, while  $x^2 - y^2 > 1$  in space describes a solid located within two hyperboloidal sheets. To see this in Mathematica we use the command(s) `RegionPlot` and `RegionPlot3D`. So for example,

```
RegionPlot[x^2 + y^2 <= 1, {x, -2, 2}, {y, -2, 2}, Axes -> True, AxesLabel -> {x, y}]
```

produces the left of the two images below and

```
RegionPlot3D[x^2 - y^2 > 1, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}, AxesLabel -> {x, y, z}]
```

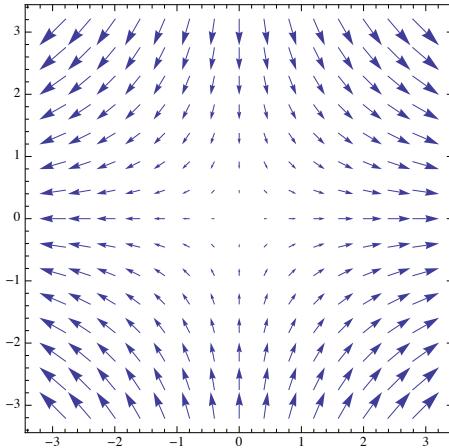
produces the right of the two images below:



### 3.2.4 Plotting vector fields

We can plot a vector field using the command `VectorPlot`.

```
VectorPlot[{x, -y}, {x, -3, 3}, {y, -3, 3}]
```



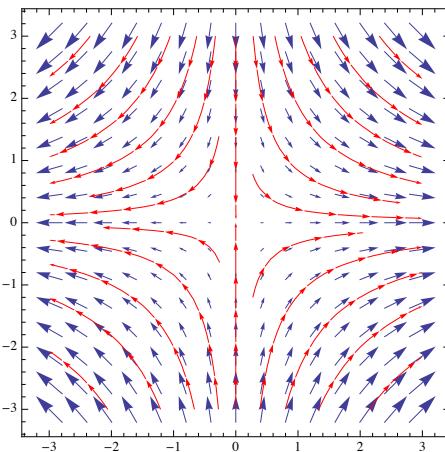
We can adjust the number of sample points with the option `VectorPoints` as in

```
VectorPlot[{x, -y}, {x, -3, 3}, {y, -3, 3}, VectorPoints -> 10]
```

The style of a vector field can be adjusted using `VectorStyle` (see below).

Often times it helps to add a `StreamPlot` to our vector field. This can be done as follows:

```
VectorPlot[{x, -y}, {x, -3, 3}, {y, -3, 3}, StreamPoints -> Coarse, StreamStyle -> Red]
```



Check out this code:

```
VectorPlot[{{x, -y}, {y, x}}, {x, -3, 3}, {y, -3, 3}, StreamStyle -> {Red, Green}]
```

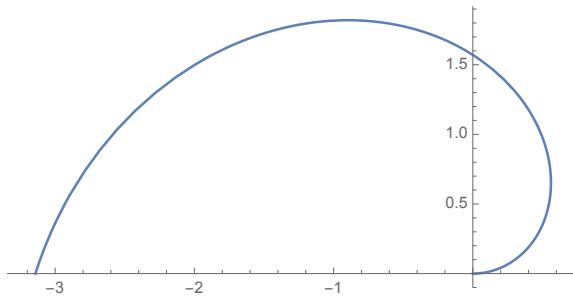
### 3.2.5 Parametric curves

We can plot parametric curves using `ParametricPlot`. To plot

$$x(t) = t \cos t, \quad y(t) = t \sin t, \quad 0 \leq t \leq \pi.$$

we use the following code:

```
ParametricPlot[{t*Cos[t], t*Sin[t]}, {t, 0, Pi}]
```



We can use the `PlotRange` command inside the `ParametricPlot` to adjust the window; for more details / ideas check out the command

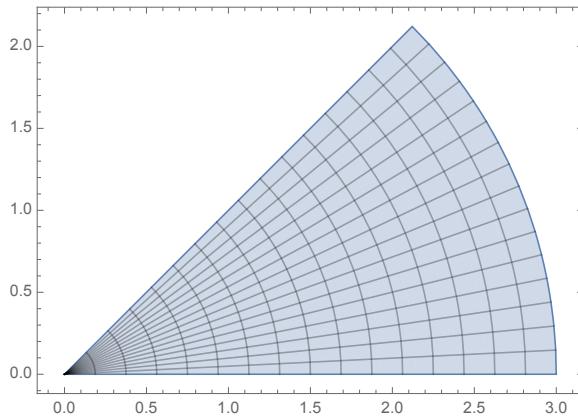
```
ParametricPlot[{t*Cos[t], t*Sin[t]}, {t, 0, Pi}, PlotRange -> {{-10, 4}, {-2, 5}}]
```

Note: In order to plot parametric curves in three dimensions, use the command `ParametricPlot3D`.

### 3.2.6 Regions of the plane described parametrically

The story is very similar here; note that one can get a mesh to show up.

```
ParametricPlot[{u*Cos[2*v], u*Sin[2*v]}, {u, 0, 3}, {v, 0, Pi/8}, Mesh -> True]
```



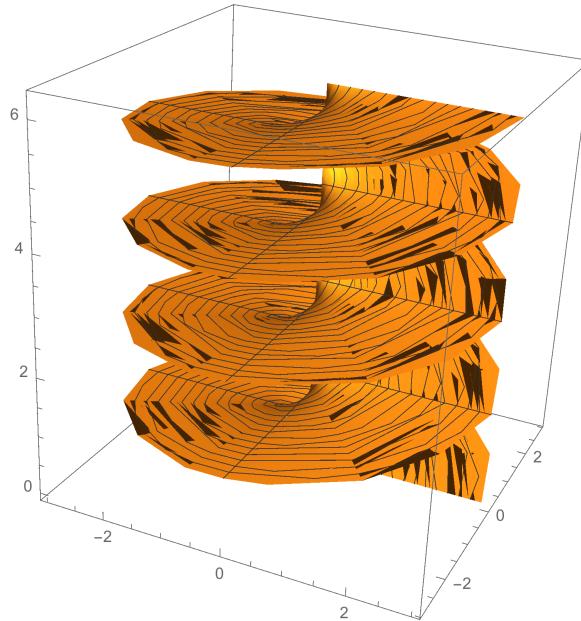
### 3.2.7 Parametric surfaces

We can plot parametric surfaces in space using `ParametricPlot3D`. To plot

$$x(u, v) = u \cos 2v, \quad y(u, v) = u \sin 2v, \quad z(u, v) = \frac{v}{2} \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 4\pi.$$

we use the following code:

```
ParametricPlot3D[{u*Cos[2 v], u*Sin[2 v], v/2}, {u, 0, 3}, {v, 0, 4*Pi}]
```



## 3.3 Combining two graphics

Here's a fun way to combine two graphics. Suppose we made a vector field plot for  $\langle x, -y \rangle$  and  $\langle y, x \rangle$ . We can name it `vectorPlot` as follows:

```
vectorPlot =
VectorPlot[{{x, -y}, {y, x}}, {x, -3, 3}, {y, -3, 3},
```

```
VectorPoints -> 10,
VectorStyle -> {{Red , Arrowheads[0]}, {Purple, Arrowheads[0]}}]
```

We can also make the stream plot, giving it the name `streamPlot` with the code

```
streamPlot =
StreamPlot[{{x, -y}, {y, x}}, {x, -3, 3}, {y, -3, 3},
StreamStyle -> {Black , Blue}]
```

Finally, we can get both pictures at once using

```
Show[vectorPlot, streamPlot]
```

---

### 3.4 Solving algebraic equations

There is a moment in this course when you will have to deal with solving systems of (non-linear) algebraic equations. When that moment comes you should check out the following command. (There are also numerous variations – check out the Documentation.)

So, to solve the system of nonlinear equations

$$\begin{cases} x^2 + y^2 = 1 \\ y = 2\lambda x \\ x = 2\lambda y \end{cases}$$

we enter

```
Solve[{x^2 + y^2 == 1, y == 2*L*x, x == 2*L*y}, {x, y, L}]
```

which produces four different triples  $(x, y, \lambda)$  of solutions, starting with  $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2})$ .

$$\left\{ \left\{ x \rightarrow -\frac{1}{\sqrt{2}}, y \rightarrow -\frac{1}{\sqrt{2}}, L \rightarrow \frac{1}{2} \right\}, \left\{ x \rightarrow -\frac{1}{\sqrt{2}}, y \rightarrow \frac{1}{\sqrt{2}}, L \rightarrow -\frac{1}{2} \right\}, \right. \\ \left. \left\{ x \rightarrow \frac{1}{\sqrt{2}}, y \rightarrow -\frac{1}{\sqrt{2}}, L \rightarrow -\frac{1}{2} \right\}, \left\{ x \rightarrow \frac{1}{\sqrt{2}}, y \rightarrow \frac{1}{\sqrt{2}}, L \rightarrow \frac{1}{2} \right\} \right\}$$


---

### 3.5 Differentiation and integration

- To do  $\frac{\partial}{\partial x}(e^{-x^2-y^2})$  in Mathematica we command

```
D[Exp[-x^2 - y^2], x]
```

To do the second derivative  $\frac{\partial^2}{\partial x \partial y}(e^{-x^2-y^2})$  we do

```
D[Exp[-x^2 - y^2], x,y]
```

Note though that in this class you will be expected to do all differentiation by hand.

- Note that you can also have Mathematica help with you with finding minima and maxima. For example, in the single variable context, the command

```
Maximize[x*Exp[-x], x]
```

find the maximum of the function  $f(x) = xe^{-x}$ . Check out the Documentation for variations on this theme. It is not likely that you will need this in our class.

- To have Mathematica help you with a double or triple integral but you will have to first apply Fubini Theorem yourself. For example if you wanted to compute  $\iint_R x^2 + y^2 dA$  where  $R$  denotes the standard unit disk you command

```
Integrate[x^2 + y^2, {x, -1, 1}, {y, -Sqrt[1 - x^2], Sqrt[1 - x^2]}]
```

Similar `Integrate` commands can be used for finding antiderivatives etc. (Check out the Documentation!) I again emphasize that in this class you will be expected to do all integration by hand.

# Chapter 4

## Old Midterm Exams

### 4.1 The First Exam from Spring 2016

1. **Vector algebra:** Let  $\vec{a} = \langle 1, -1, 0 \rangle$  and  $\vec{b} = \langle 0, 0, -1 \rangle$ .
  - (a) Sketch the vectors  $\vec{a}$  and  $\vec{b}$ . Make sure your drawing is legible.
  - (b) Compute *and sketch* the cross product  $\vec{a} \times \vec{b}$ . Make sure your drawing is legible.
  - (c) Find the area of the parallelogram formed by the vectors  $\vec{a}$  and  $\vec{b}$ .
2. **Describing geometric shapes using parametric equations:** Write parametric equations (that is, equations of one of the forms  $(x, y) = T(u)$ ,  $(x, y) = T(u, v)$ ,  $(x, y, z) = T(u)$ ,  $(x, y, z) = T(u, v)$ ,  $(x, y, z) = T(u, v, w)$ , all depending on the problem) for the following. Be clear about the ranges for the parameters  $u$  and  $v$ .
  - (a) The plane through the origin containing vectors  $\vec{a} = \langle 1, -1, 0 \rangle$  and  $\vec{b} = \langle 0, 0, -1 \rangle$  of the previous problem.
  - (b) The half-space located to the right / east of the plane from part 1.
  - (c) The parallel (line of constant latitude) of the sphere centered at the origin which contains the point  $(0, 1, -\sqrt{3})$ .
  - (d) The polar cap located to the south of the line of latitude from part 3.

### 3. Transformations and coordinate vector fields:

Consider the transformation  $(x, y, z) = T(r, \theta)$  given by

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad z = 4 - r.$$

Assume that  $r \geq 0$  and that  $0 \leq \theta < 2\pi$ .

- (a) Explain what this transformation does visually. A sketch is expected.

- (b) Compute the coordinate vector fields  $\partial_r$  and  $\partial_\theta$ , and *indicate them on your sketch*.
- (c) What is the area-stretch factor of the transformation  $T$ ? Where is this transformation stretching? Where is this transformation shrinking?

**4. Curvature of a graph / Taylor approximations:** Consider the function

$$f(x, y) = \frac{x^4}{4} + 4xy + \frac{y^4}{4}.$$

- (a) Find the second order Taylor approximation of  $f$  near the point  $(x_0, y_0) = (2, -2)$ .
- (b) Based on the expression you got estimate the appearance of the graph of  $f(x, y)$  near the point  $(x_0, y_0) = (2, -2)$ .

**5. Chain Rule:** Let  $r$  and  $\theta$  denote the standard polar coordinates in the  $xy$ -plane, and let

$$g(x, y) = 4 - x^2 - y^2.$$

Find  $\frac{\partial g}{\partial r}$  and  $\frac{\partial g}{\partial \theta}$  corresponding the point  $(x_0, y_0) = (1, -1)$ .

---

## 4.2 The First Exam from Fall 2016

**1. Vector algebra:** Let  $\vec{a} = \langle 0, -1, -1 \rangle$  and  $\vec{b} = \langle 1, 1, 0 \rangle$ .

- (a) Find and sketch the vector  $\vec{a} \times \vec{b}$ .
- (b) Find the area of the parallelogram formed by  $\vec{a}$  and  $\vec{b}$  *in two different ways*.
- (c) Find the parametric equations for the plane spanned by the vectors  $\vec{a}$  and  $\vec{b}$  based at the point  $(2, 0, 1)$ . Please use  $u$  and  $v$  as your parameters.
- (d) Express the plane from the above in the form of

$$\underline{\hspace{1cm}}x + \underline{\hspace{1cm}}y + \underline{\hspace{1cm}}z = \underline{\hspace{1cm}}.$$

- (e) Find the parametric equations for the line which is orthogonal to the said plane, and is passing through the point  $(2, 0, 1)$ .

**2. Identifying shapes in space:** Identify the following surfaces **in space**. Provide a drawing or an explanation in words for each surface.

- (a)  $x^2 + 4y^2 = 1$
- (b)  $x^2 + 4y^2 = z$
- (c)  $x^2 + 4y^2 = z^2$ .

**3. Transformations and coordinate vector fields:**

Consider the curvilinear transformation

$$(x, y, z) = T(r, \theta) = (r \cos \theta, r \sin \theta, 2\theta)$$

which maps the rectangle

$$1 \leq r \leq 3, \quad 0 \leq \theta \leq 4\pi$$

in the  $r\theta$ -plane to  $xyz$ -space.

- (a) Describe what  $T$  does visually. You are expected to provide a sketch, complete with grid lines, and write something in words just in case I don't understand your sketch. Please label the best you can.
- (b) Compute and sketch the coordinate vector fields  $\partial_u$  and  $\partial_v$ . Please label the best you can.
- (c) Compute the Jacobi matrix  $DT$  and the linearization of  $T$  at the point  $(r, \theta) = (2, \frac{\pi}{2})$ ; then explain what this linearization is doing visually.

**4. The Chain Rule:**

Consider the change of variables

$$x = u + v, \quad y = u - v.$$

Suppose that  $f(x, y)$  is some function of  $x$  and  $y$  variables.

- (a) Express  $\frac{\partial f}{\partial u}$  and  $\frac{\partial f}{\partial v}$  in terms of  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .
- (b) Express  $\frac{\partial^2 f}{\partial u \partial v}$  in terms of derivatives with respect to  $x$  and  $y$  variables.

5. **Taylor Approximations:** Find the quadratic approximation of  $f(x, y) = x^3 - 3x^2 + y^3 - 3y^2$  at the point  $(x_0, y_0) = (2, 2)$ .
- 
- 

**4.3 The First Exam from Spring 2017**

1. **Vector algebra:** Let  $\vec{a} = \langle 2, 1, 1 \rangle$ ,  $\vec{b} = \langle 1, 1, 0 \rangle$  and  $c = \langle 0, 0, 1 \rangle$ ; assume these vectors are based at the point  $(2, 3, 1)$ .
  - (a) Compute the angle between the vectors  $\vec{a}$  and  $\vec{b}$ .
  - (b) Find the volume of the parallelopiped formed by  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ .

**2. Standard curvilinear coordinates:**

- (a) Find the cylindrical and the spherical coordinates of the point  $(-1, 1, -\sqrt{2})$ .

- (b) Express the unit sphere  $x^2 + y^2 + z^2 = 1$  in both cylindrical and spherical coordinates.
- (c) Express the cone  $z = \sqrt{x^2 + y^2}$  in both cylindrical and spherical coordinates.
- (d) Express the region contained above  $z = \sqrt{x^2 + y^2}$  but inside  $x^2 + y^2 + z^2 = 1$  in both cylindrical and spherical coordinates.

- 3. Curvilinear transformations and coordinate vector fields:** Consider the transformation

$$(x, y, z) = T(u, v) = (u \cos(v), \sqrt{4 - u^2}, u \sin(v))$$

defined on the domain  $0 \leq u < 2$  and  $0 \leq v < 2\pi$ . The transformation defines a surface  $S$  in space.

- (a) The point  $(x, y, z) = (1, \sqrt{3}, 0)$  is located on the surface  $S$ . Which values of  $u$  and  $v$  correspond to this point?
- (b) Compute the coordinate vector fields  $\partial_u$  and  $\partial_v$  on the surface  $S$ . Then evaluate them at the point  $(x, y, z) = (1, \sqrt{3}, 0)$ .
- (c) Compute the normal vector field to  $S$ . Then evaluate it at  $(x, y, z) = (1, \sqrt{3}, 0)$ .
- (d) Illustrate the transformation  $T$  and the surface  $S$ . A picture with grid lines, coordinate vector fields, and the normal vector field as well as an explanation in words are expected. Font-code or color-code some of the corresponding points / shapes / the like.
- (e) **Challenge extra credit question:** Which portions of the uv-plane is the transformation  $T$  shrinking? Stretching?

- 4. Taylor approximations:** Consider the function  $f(x, y) = x^5 + 5xy + y^5$ .

- (a) Find the linearization of  $f(x, y)$  at  $(0, -1)$ . Interpret your linearization result visually. Specifically, address the tangent plane to the graph of  $f(x, y)$  at the point  $(0, -1, -1)$ .
- (b) Find the second order Taylor approximation of  $f(x, y)$  at  $(-1, -1)$ . (Note the change in the base point from the previous parts of the problem!). Interpret your result visually.

- 5. The Chain Rule:** Let  $(r, \theta)$  denote the standard polar coordinates in the  $xy$ -space. Let  $f(x, y)$  be some function which depends only on  $r$ :

$$f(x, y) = f(r), \quad \frac{\partial f}{\partial \theta} = 0.$$

- (a) Express  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  in terms of  $\frac{\partial f}{\partial r}$ .
  - (b) Express  $\frac{\partial^2 f}{\partial x^2}$  and  $\frac{\partial^2 f}{\partial y^2}$  in terms of  $\frac{\partial^2 f}{\partial r^2}$ ,  $\frac{\partial f}{\partial r}$ , etc.
  - (c) Express  $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$  in terms of  $\frac{\partial^2 f}{\partial r^2}$ ,  $\frac{\partial f}{\partial r}$ , etc. Simplify as far as possible.
-

## 4.4 The First Exam from Fall 2017

1. **Vector Algebra:** Let  $\vec{a} = \langle -1, 0, 1 \rangle$ ,  $\vec{b} = \langle 1, 1, 0 \rangle$ ; assume these vectors are based at the origin. In addition, let  $\vec{c}$  be a vector of magnitude 2 based at the origin and forming angles of  $\frac{\pi}{3}$  with both  $\vec{a}$  and  $\vec{b}$ .
  - (a) Compute the dot products  $\vec{a} \cdot \vec{b}$ ,  $\vec{a} \cdot \vec{c}$  and  $\vec{b} \cdot \vec{c}$ .
  - (b) Compute the angle between  $\vec{a}$  and  $\vec{b}$ .
  - (c) Find the volume of the parallelopiped spanned by  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ . (Reflect on your answer.)
2. **Identifying Shapes in Space:** Identify the following spaces given by their equations in Cartesian coordinates. Provide a drawing or an explanation in words for each one of them.
  - (a)  $z = x^2 + y^2$  and  $z^2 = x^2 + y^2$
  - (b)  $y = x^2 + z^2$  and  $y^2 = x^2 + z^2$
  - (c)  $z = x^2 - y^2$  and  $z^2 = x^2 - y^2$

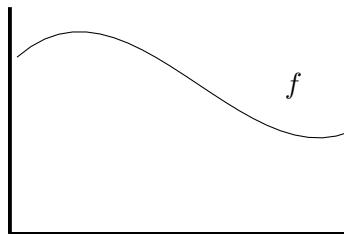
3. **Transformations and Coordinate Vector Fields:** Consider the curvilinear transformation

$$(x, y, z) = T(r, \theta) = (r \cos \theta, r \sin \theta, f(r))$$

which maps the “infinite rectangle”

$$0 < r < \infty, \quad 0 \leq \theta \leq 2\pi$$

in the  $r\theta$ -plane into the  $xyz$ -space. The exact formula for  $f$  is unimportant, but its graph looks like the picture below.



- (a) Describe what  $T$  does visually. You are expected to provide a sketch, complete with grid lines, and write something in words just in case I don't understand your sketch. Please label the best you can.
- (b) Find a formula for and sketch the coordinate vector fields  $\partial_r$  and  $\partial_\theta$ . Please label the best you can. It is totally OK to have  $f$  or some such in your answers.

- (c) Find a formula for  $\partial_r \times \partial_\theta$ . (Again, it is totally OK to have  $f$  or some such in your answers.) Provide a sketch of this vector field, and write something in words regarding the appearance / position of this vector field just in case I don't understand your sketch.
4. **The Chain Rule:** Let  $(r, \theta, \phi)$  denote the standard spherical coordinates for the Cartesian space  $(x, y, z)$ . Suppose that  $f$  is some function with

$$Df(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}.$$

The exact formula for the function  $f$  is unimportant. Use the Chain Rule to find the values of  $\frac{\partial f}{\partial r}$ ,  $\frac{\partial f}{\partial \theta}$  and  $\frac{\partial f}{\partial \phi}$  corresponding to the point  $(x, y, z) = (1, 1, -\sqrt{2})$ . Note: in the process of answering this question you will need to find the spherical coordinates of  $(x, y, z) = (1, 1, -\sqrt{2})$ .

5. **Optimization:** Let  $f(x, y) = x^2 + 6xy - y^4$ . Find the critical points of  $f(x, y)$  and the second order Taylor Approximation of  $f$  at those critical points. Does the function  $f$  reach a local maximum there? Local minimum?
- 

## 4.5 The Second Exam from Spring 2016

1. Compute the value of  $\iint_R (x + y) dA$  where  $R$  is the region bounded by the parabola  $x = 1 - y^2$  and the  $y$ -axis.
2. Set up the integral which finds the total weight of a loaf of bread if it is known that
  - The base of the loaf is in the shape of a rectangle  $-2 \leq x \leq 2, -1 \leq y \leq 1$ .
  - The upper crust of the loaf is described by the surface  $z = 2 - \frac{1}{12}(x^2 + 4y^2)$ .
  - The weight density of the loaf is given by  $\varrho = 2 + \frac{1}{12}(x^2 + 4y^2)$  ounces per cubic inch.

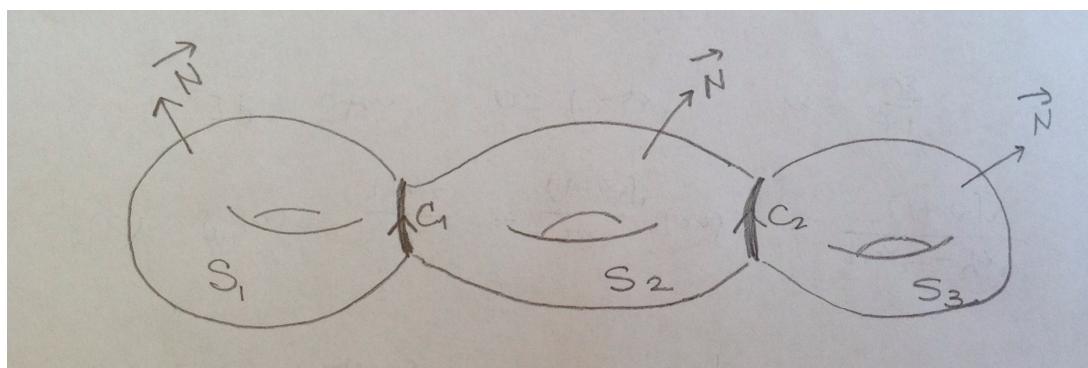
All the axes are calibrated in inches. *You do not have to finish the computation of this integral.*

3. Set up the integral which computes the volume of the region located under the dome of the standard unit sphere (centered at the origin) and above the surface of the bowl  $z = x^2 + y^2$ . *You do not have to finish the computation of this integral.*
4. Set up and compute the integral which finds the *average* value of  $f(x, y, z) = z^2$  over the ball of radius  $R$  centered at the origin.
5. Consider the vector field  $\vec{V}(x, y) = \langle xy, xy \rangle$ .
  - (a) Use the definition (a.k.a brute force) of the circulation (work) integral to compute the value of  $\int_C \vec{V} \cdot \vec{T} ds$ ; here  $C$  is the line segment joining  $(-1, 0)$  and  $(0, 1)$  in that particular direction.

- (b) Use Green's Theorem to compute the circulation (work) integral  $\int_C \vec{V} \cdot \vec{T} ds$  where  $C$  is the counter-clockwise circle of radius 1 centered at  $(1, 1)$ .
6. The curves  $C_1$  and  $C_2$  and surfaces  $S_1$ ,  $S_2$  and  $S_3$  are given in the following diagram; note that  $S_1$  goes only up to  $C_1$ , that  $S_2$  is located between  $C_1$  and  $C_2$  and that  $S_3$  is located to the right of  $C_2$ . It is known that  $\int_{C_1} \vec{V} \cdot \vec{T} ds = 1$  and  $\int_{C_2} \vec{V} \cdot \vec{T} ds = 2$ . Find the values of

$$\iint_{S_1} \operatorname{curl}(\vec{V}) \cdot \vec{N} dA, \quad \iint_{S_2} \operatorname{curl}(\vec{V}) \cdot \vec{N} dA, \quad \iint_{S_3} \operatorname{curl}(\vec{V}) \cdot \vec{N} dA.$$

The orientations of  $C_1$ ,  $C_2$  and  $\vec{N}$  are indicated on the diagram. Note: the entire surface is meant to look like a 3-holed donut.



7. Let  $\vec{V}(x, y, z) = \langle 1, y, z^2 \rangle$  compute the outflux of  $\vec{V}$  across the cone  $z = \sqrt{x^2 + y^2}$  truncated at  $z = 2$  if
- The lid at  $z = 2$  is included.
  - The lid at  $z = 2$  is *not* included.

You are expected to use Gauss' Theorem whenever possible.

## 4.6 The Second Exam from Fall 2016

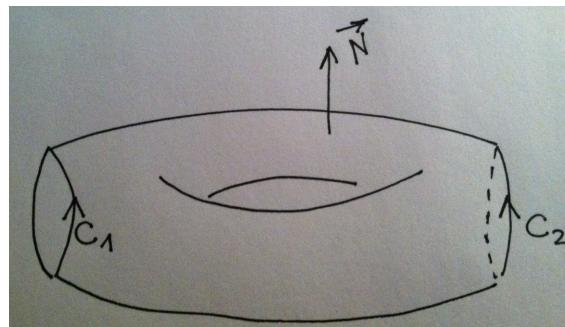
- Consider the region  $R$  bounded by the hyperbola  $x^2 - y^2 = 1$  and the line  $x = 2$ .
  - Set up the integral which computes the area of  $R$  in two different ways. (One integral should be a  $dxdy$ -integral, and the other one should be a  $dydx$ -integral; you do not need to compute the area.)
  - Set up the integral which computes the volume of the object formed when  $R$  rotates around the  $y$ -axis. (You do not need to compute the volume.)
- Find the average value of the function  $f(x, y, z) = \frac{1}{x^2+y^2+z^2}$  over the unit ball centered at the origin.

3. Let  $S$  be the surface of the cylinder of radius 2 centered around the  $x$ -axis, bounded between the planes  $x = -2$  and  $x = 2$ . Set up the integral which computes  $\iint_S \rho dA$  for  $\rho(x, y, z) = y^2$ . Simplify until you reach single-variable integrals; you do not need to finish the computation.
4. Set up the integral which computes  $\int_C \vec{V} \cdot \vec{T} ds$  where  $C$  is the spiral

$$x(t) = e^{-t} \cos(t), \quad y(t) = e^{-t} \sin(t), \quad t \geq 0$$

and where  $\vec{V}$  denotes the vector field  $\vec{V}(x, y) = \langle -y, x \rangle$ . You do not need to finish the integration.

5. Use Green's Theorem to compute  $\int_C \vec{V} \cdot \vec{T} ds$  and the outward flux  $\int_C \vec{V} \cdot \vec{N} ds$  where  $C$  is the counterclockwise unit circle centered at the origin, and where  $\vec{V}(x, y) = \langle 1 + 2x, 3y \rangle$ .
6. Set up Stokes' Theorem for the surface given on the following picture. In broad terms, explain the reasons why you think Stokes' Theorem holds.



7. Consider the vector field

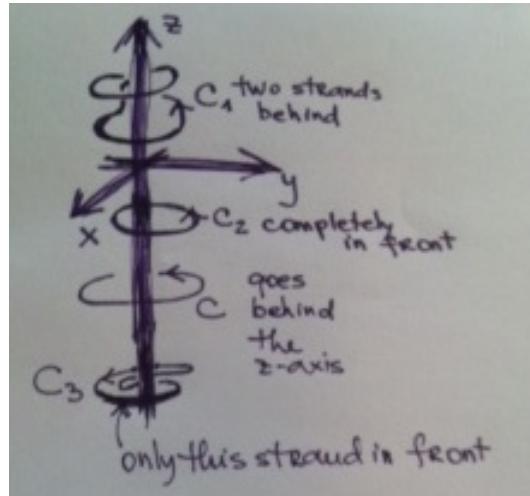
$$\vec{V}(x, y, z) = \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle.$$

- (a) Compute  $\operatorname{div}(\vec{V})$ . (Blunt hint: you need to get zero.)
- (b) Compute the outward flux of  $\vec{V}$  across a sphere of radius  $R$  centered at the origin.
- (c) Compute the outward flux of  $\vec{V}$  across a sphere of radius 2 centered at
- $(0, 0, 3)$ ;
  - $(0, 0, 1)$ .

## 4.7 The Second Exam from Spring 2017

1. Let  $C$  denote the (counterclockwise) outline of the rectangle  $-1 \leq x \leq 2, -2 \leq y \leq 1$ . Use Green's theorem to compute  $\int_C \vec{V} \cdot \vec{T} ds$  and  $\int_C \vec{V} \cdot \vec{N} ds$  where  $\vec{V}(x, y) = \langle x^2y, xy^2 \rangle$ . Do get a numerical value out of this.
2. Consider  $\varrho = x^2 + y^2$ .
  - (a) What is the average value of  $\varrho$  over the unit disk centered at  $(1, 0)$ ? Do get a numerical value.
  - (b) Briefly explain what you would do differently if you were to find the average of  $\varrho$  over the unit ball centered at  $(1, 0, 0)$ . Please do not compute this average.
3. Let  $C$  be the circle of radius 3 centered at  $(5, 0)$ , and let  $S$  be the donut formed when  $C$  rotates around the  $y$ -axis.
  - (a) Find the expression for the line element  $ds$  along  $C$ .
  - (b) Set up, but do not compute, the integral which computes the volume of the donut  $S$ .
  - (c) **Extra Credit:** Find the expression for the area element  $dA$  on  $S$ .
4. Let  $C_1$  denote the arc joining the North Pole  $(0, 0, R)$  of the standard sphere of radius  $R$  to the South Pole  $(0, 0, -R)$  and passing through the “East Pole”  $(0, R, 0)$ . Let  $C_2$  denote the line segment joining the North Pole  $(0, 0, R)$  to the South Pole  $(0, 0, -R)$ . Finally, let  $\vec{V}(x, y, z) = \langle z - y, x - z, y - x \rangle$ .
  - (a) Choose one of the circulation integrals:  $\int_{C_1} \vec{V} \cdot \vec{T} ds$  or  $\int_{C_2} \vec{V} \cdot \vec{T} ds$  and compute it by definition (i.e. by brute force). I want you to go all the way and get a numerical answer.
  - (b) Set up Stokes' Theorem involving  $\int_{C_1} \vec{V} \cdot \vec{T} ds$  and  $\int_{C_2} \vec{V} \cdot \vec{T} ds$ . Then compute the double integral involved in your set-up of Stokes' Theorem. Yes, I want you to go all the way and get a numerical answer.
  - (c) Based on the computation you just did, find the remaining of the circulation integrals  $\int_{C_1} \vec{V} \cdot \vec{T} ds$  or  $\int_{C_2} \vec{V} \cdot \vec{T} ds$ . (Do go all the way and get a numerical answer.)
5. (a) Compute the outward flux of the vector field
 
$$\vec{V}(x, y, z) = \langle x, y, x + y \rangle$$
 across the unit sphere centered at  $(1, 0, 0)$ . Do go all the way and get a numerical answer.
- (b) What if only the front half of the sphere is considered? (Do go all the way and get a numerical answer.)

6. Suppose a vector field  $\vec{V}$  is defined everywhere except along the  $z$ -axis, and that everywhere except for the  $z$ -axis we have  $\operatorname{curl}(\vec{V}) = \vec{0}$ . Consider the set-up as indicated on the picture below. Pretend like it is known that  $\int_C \vec{V} \cdot \vec{T} ds = 4$ . Find the values of  $\int_{C_1} \vec{V} \cdot \vec{T} ds$ ,  $\int_{C_2} \vec{V} \cdot \vec{T} ds$ ,  $\int_{C_3} \vec{V} \cdot \vec{T} ds$ .



## 4.8 The Second Exam from Fall 2017

1. Compute  $\iint_R x dA$  where
  - (a)  $R$  is the triangle carved out of the first quadrant by the line  $x + y = 1$ ;
  - (b)  $R$  is the disk of radius 2 centered at  $(1, 1)$ ;
  - (c)  $R$  is the parallelogram spanned by the vectors  $\langle 1, 2 \rangle$  and  $\langle 2, 1 \rangle$  based at  $(1, 1)$ ;
2. Set up but do not compute the integral which computes the total weight of a pyramid carved out of the first octant by the plane  $x + y + z = 1$ , if the weight density function is given by  $\varrho(x, y, z) = 1 - z$  ounces per cubic inch. All the axes are calibrated in inches. Note: the first octant is that eighth of the Cartesian space where  $x, y, z \geq 0$ .
3. Compute the average value of  $\varrho(x, y, z) = z$  over the volume enclosed by  $x^2 + y^2 - 4z^2 = 1$  and the planes  $z = 0$  and  $z = 1$ .
4. Set up but do not compute the integral for the surface area of the bowl  $z = 1 - x^2 - y^2$  located above the  $xy$ -plane.
5. Compute the counterclockwise circulation  $\int_C \vec{V} \cdot \vec{T} ds$  and the outward flux  $\int_C \vec{V} \cdot \vec{N} ds$  where  $\vec{V}(x, y) = \langle 2x + y, 5x - 3y \rangle$  and where  $C$  is the circle of radius 2 centered at  $(1, 1)$ ;

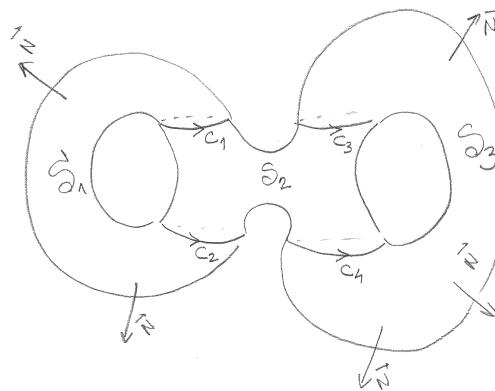
6. The curves  $C_1, C_2, C_3, C_4$ , and the surfaces  $S_1, S_2$  and  $S_3$  are given in the following diagram; note that  $S_1$  goes only up to  $C_1$  and  $C_2$ , that  $S_2$  is located between  $C_1, C_2, C_3$  and  $C_4$ , and that  $S_3$  goes only up to  $C_3$  and  $C_4$ . It is known that

$$\int_{C_1} \vec{V} \cdot \vec{T} ds = 1, \quad \int_{C_2} \vec{V} \cdot \vec{T} ds = 2, \quad \int_{C_3} \vec{V} \cdot \vec{T} ds = -3, \quad \int_{C_4} \vec{V} \cdot \vec{T} ds = -1.$$

Find the values of

$$\iint_{S_1} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA, \quad \iint_{S_2} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA, \quad \iint_{S_3} \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA.$$

The orientations of  $C_1, C_2, C_3, C_4$  and  $\vec{N}$  are indicated on the diagram.



7. Let  $\vec{V}(x, y, z) = \langle x^3 + y^2, y^3 + z^2, z^3 + x^2 \rangle$ . Compute the outflux of  $\vec{V}$  across the upper dome of the standard unit sphere based at the origin. Note: You are expected to use Gauss' Theorem. Also note: the equatorial base is not included.

# Chapter 5

## Old Final Exams

### 5.1 FALL 2015

#### Utilizing Cartesian coordinates

Identify the following. Please provide a rough sketch and an explanation in words for each surface.

$$1. z = 2 - (x - 2)^2 + (y + 2)^2$$

$$2. z = x^2 - xy + y^2$$

$$3. z = x^2.$$

#### Vector and matrix algebra

Let  $\vec{a} = \langle 1, 1, 0 \rangle$  and  $\vec{b} = \langle 0, 0, -1 \rangle$ .

1. Sketch the vectors  $\vec{a}$  and  $\vec{b}$ . Make sure your drawing is legible.
2. Compute *and sketch* the cross product  $\vec{a} \times \vec{b}$ . Make sure your drawing is legible.
3. Find the area of the parallelogram spanned by  $\vec{a}$  and  $\vec{b}$ .

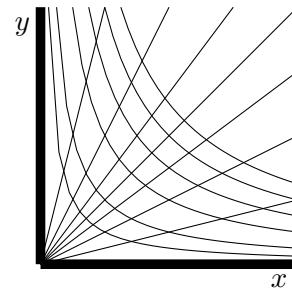
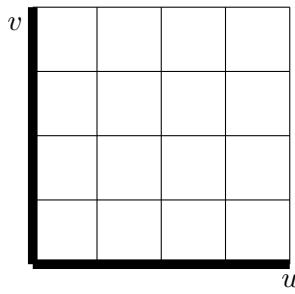
#### Curvilinear coordinatization

Let  $T$  be a transformation from the first quadrant of the  $uv$ -plane to the first quadrant of the  $xy$ -plane given by

$$(x, y) = T(u, v) = \left( \frac{u}{v}, uv \right).$$

With a little bit of work one can convince oneself that the action of  $T$  on coordinate grids looks something like this. (Warning: The sketch is not to scale!)

1. Color or font-code several corresponding points and grid-lines. Please make sure your work is legible.



2. Compute and sketch the coordinate vector fields  $\partial_u$  and  $\partial_v$  in the  $xy$ -plane. Please make sure your drawing is legible.
3. Relate the area element  $dA$  in the  $xy$ -plane to the area element  $dudv$  of the  $uv$ -plane.
4. What portions of the  $uv$ -plane is  $T$  stretching? What portions of the  $uv$ -plane is  $T$  shrinking?

## **Basics of 1-dimensional integration**

Find the formula for the line element  $ds$  and use it to compute the total length of the spiral

$$x(t) = e^{-t} \cos(t), \quad y(t) = e^{-t} \sin(t), \quad 0 \leq t < \infty.$$

## **Basics of 2-dimensional integration**

Compute  $\iint_R x \, dA$  where  $R$  is (the interior of) the triangle with vertices  $(0,0)$ ,  $(1,1)$  and  $(1,-1)$ .

## **Basics of 3-dimensional integration**

Use integration to compute the volume of the solid ellipsoid

$$x^2 + y^2 + 9z^2 \leq 9.$$

Note: I do want you to show me the details like the coordinatization of the solid ellipsoid, finding the volume element, etc. The value of the volume is  $12\pi$  and is largely beside the point.

## Linearizations and Taylor approximations

Suppose  $f(x, y)$  is a scalar function such that  $f(1, 4) = 0$  and

$$\overrightarrow{\text{grad}}(f)(1, 4) = \langle 2, -1 \rangle \quad \text{and} \quad \text{Hessian}(f)(1, 4) = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

1. Assemble the second order Taylor approximation of  $f$  at  $(1, 4)$ .
2. Approximate the value of  $f(0.99, 4.02)$ . On what order of magnitude do you expect the error of the approximation to be?

## Optimization

Let  $f(x, y) = x - y$  and let the curve  $C$  be given by

$$x^2 + xy + y^2 = 1.$$

What is the absolute maximum and minimum value of  $f$  along  $C$ ?

## The Chain Rule

Let  $f(x, y)$  be a function with

$$\overrightarrow{\text{grad}}(f)(1, -\sqrt{3}) = \langle 1, 2 \rangle.$$

Find  $\frac{\partial f}{\partial r}$  and  $\frac{\partial f}{\partial \theta}$  at  $(1, -\sqrt{3})$ . Assume the standard polar coordinates  $(r, \theta)$ .

## Gradient, curl, divergence and potential

1. Explain the meaning of  $\overrightarrow{\text{curl}}$  of a vector field. *Briefly* outline the reasoning which brought us to the formula for  $\overrightarrow{\text{curl}}$ .
2. Explain the meaning of divergence of a vector field. *Briefly* outline the reasoning which brought us to the formula for divergence.

## The Fundamental Theorems of Calculus

1. State all the Fundamental Theorems of Calculus we covered in class.
2. Pick one of these Fundamental Theorems and explain the overall reasoning behind it. Formal proof is not expected – just an intuitive explanation in paragraph form.

## Evaluating the circulation and flux integrals by means of the Fundamental Theorems of Calculus

1. Find  $\int_C \vec{V} \cdot \vec{T} ds$  where

$$\vec{V}(x, y, z) = \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle$$

and where  $C$  is some path going from  $(1, 0, 0)$  to  $(0, 0, 2)$  which avoids the origin.

2. Find  $\int_C \vec{V} \cdot \vec{T} ds$  if  $C$  is the counterclockwise unit circle centered at  $(1, 1)$  and if

$$\vec{V}(x, y) = \langle x + y, -x - y \rangle.$$

3. Find the outward flux  $\iint_S \vec{V} \cdot \vec{N} dA$  where  $S$  is (the surface of) the ellipsoid

$$x^2 + y^2 + 9z^2 = 9$$

and where  $\vec{V}(x, y, z) = \langle x + y + z, x + y + z, x + y + z \rangle$ .

## 5.2 SPRING 2016

### Utilizing Cartesian coordinates

Identify the following. Please provide a rough sketch and an explanation in words for each surface.

1.  $x + y + z = 1$
2.  $x + y + z^2 = 1$
3.  $x + y^2 + z^2 = 1$
4.  $x^2 + y^2 + z^2 = 1$ .

### Vector and matrix algebra

Let  $\vec{a} = \langle 0, 1, 1 \rangle$  and  $\vec{b} = \langle 1, 0, 1 \rangle$ ; assume these vectors are based at the origin.

1. Write down the formulae for the transformation  $(x, y, z) = T(u, v)$  which maps the unit square  $0 \leq u, v \leq 1$  in the  $uv$ -plane onto the parallelogram spanned by the vectors  $\vec{a}$  and  $\vec{b}$ .
2. Find the area of the parallelogram spanned by  $\vec{a}$  and  $\vec{b}$ .
3. A region  $R$  in the  $uv$ -plane has the area of 3. What is the area of the image of  $R$  under  $T$ ?

### Curvilinear coordinatization

Consider the surface  $S$  formed when the graph of the function  $y = \sqrt{x - 1}$  rotates around the  $y$ -axis.

1. Parametrize  $S$ ; please use  $u, v$  for your parameters.
2. Sketch the surface  $S$  together with the  $uv$  grid-lines.
3. Compute the coordinate vector fields  $\partial_u$  and  $\partial_v$ .
4. Compute a normal vector field to  $S$ .
5. Compute the area element on  $S$ .

### Basics of 1-dimensional integration

Set up the integrals which compute the average value of the function  $f(x, y) = x^4$  over the unit circle  $x^2 + y^2 = 4$ . *Do not evaluate any integrals.*

## Basics of 2-dimensional integration

Set up the integral which computes the total weight of the elliptical washer  $1 \leq x^2 + 4y^2 \leq 4$  if the weight density function is given by  $\varrho(x, y) = x + y$ . *Do not evaluate any integrals.* (Pick units of your choice, if you feel like they are needed.)

## Basics of 3-dimensional integration

Set up (*but do not evaluate*) the integral which computes the volume of the ball of radius 2 centered at the origin in three different ways:

- By using the Fubini Theorem;
- By using the cylindrical coordinates;
- By using the spherical coordinates.

## Linearizations and Taylor approximations

Suppose  $(x, y) = T(u, v)$  is a transformation mapping the point  $(1, -1)$  in the  $uv$ -plane to the point  $(1, 0)$  in the  $xy$ -plane such that the Jacobi matrix  $DT$  at  $(1, -1)$  is equal to

$$DT|_{(1,-1)} = \begin{pmatrix} 2 & 2 \\ -1 & 4 \end{pmatrix}.$$

1. Assemble the following linear approximations:

$$x(1 + \Delta u, -1 + \Delta v) \approx \dots, \quad y(1 + \Delta u, -1 + \Delta v) \approx \dots$$

2. Based on the above, estimate  $T(1.05, -0.95)$ .

## Optimization

Find critical points of  $f(x, y) = x^3 - 3x + y^3 - 3y$  and classify them as minimums, maximums and saddles.

## The Chain Rule

Let  $(x, y) = T(u, v) = (u^2 - v^2, 2uv)$ .

1. Relate  $\frac{\partial f}{\partial u}$  and  $\frac{\partial f}{\partial v}$  to  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

2. Specifically, what are the values of  $\frac{\partial f}{\partial u}$  and  $\frac{\partial f}{\partial v}$  at a point corresponding to  $(u, v) = (\sqrt{3}, \sqrt{3})$  if it is known that  $f(x, y)$  is such that

$$\overrightarrow{\text{grad}}(f) = \left\langle \frac{1}{\sqrt{x^2 + 1}}, \frac{1}{\sqrt{y^2 + 1}} \right\rangle ?$$

3. **Extra Credit:** Relate  $\frac{\partial^2 f}{\partial u^2}$  to derivatives with respect to  $x$  and  $y$  variables.

## Gradient, curl, divergence and potential

1. Explain the meaning of the scalar curl of the vector field  $\vec{V} = \langle P, Q \rangle$  in the plane:

$$\text{curl}(\vec{V}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Specifically, explain where this formula comes from. Be brief.

2. For vector fields  $\vec{V} = \langle P, Q, R \rangle$  in space  $\overrightarrow{\text{curl}}(\vec{V})$  is a vector field. Why does it have to be a vector field, and how did we get the formula for it?

## The Fundamental Theorems of Calculus

1. State all the Fundamental Theorems of Calculus we covered in class.
2. Explain the overall reasoning behind the Divergence Theorems. Formal proof is not expected – just an intuitive explanation in paragraph form.

## Evaluating the circulation and flux integrals by means of the Fundamental Theorems of Calculus

1. Find  $\int_C \vec{V} \cdot \vec{T} ds$  where

$$\vec{V}(x, y) = \left\langle \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right\rangle$$

and where  $C$  is the standard counterclockwise unit circle centered at the origin.

2. Let  $C$  denote the circle obtained as the intersection of the plane  $x - 2y + z = 0$  with the standard unit sphere  $x^2 + y^2 + z^2 = 1$ , and

$$\vec{V}(x, y, z) = \langle -y + z, x - z, y - x \rangle.$$

Use Stokes' Theorem to show that  $\int_C \vec{V} \cdot \vec{T} ds = 0$ .

3. Find the outward flux  $\iint_S \vec{V} \cdot \vec{N} dA$  where  $S$  is (the surface of) the upper dome of

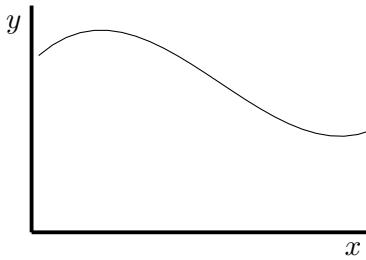
$$x^2 + y^2 + z^2 = 4$$

and where  $\vec{V}(x, y, z) = \langle 1, 2, 3 \rangle$ . Note that the base is not included.

### 5.3 FALL 2016

#### Utilizing Cartesian coordinates

Let  $f(x)$  be a function of single variable whose graph  $y = f(x)$  looks like so:



Illustrate and / or describe in words the appearance of the following in 3-dimensional space.

1.  $z = f(y);$
2.  $z = f(\sqrt{x^2 + y^2}).$

#### Vector and matrix algebra

Let  $\vec{a} = \langle 2, 1, 1 \rangle$  and  $\vec{b} = \langle 1, 1, 0 \rangle$ ; assume these vectors are based at the origin.

1. Compute the angle between  $\vec{a}$  and  $\vec{b}$ .
2. Find the area of the parallelogram spanned by  $\vec{a}$  and  $\vec{b}$  in two different ways.

#### Curvilinear coordinatization

Consider the transformation

$$(x, y, z) = T(u, v) = (u, u \cos(v), u \sin(v))$$

defined on the infinite strip  $u \geq 0$ ,  $0 \leq v \leq 2\pi$ . The transformation describes a surface  $S$  in space.

1. Illustrate the surface  $S$ . A picture with grid lines and coordinate vector fields, as well as an explanation in words is expected.
2. Compute the coordinate vector fields  $\partial_u$  and  $\partial_v$  and evaluate them at the point  $(x, y, z) = (1, 0, 1)$ .
3. Compute the normal vector field to  $S$  and evaluate it at  $(x, y, z) = (1, 0, 1)$ .
4. Find a formula for the area element  $dA$ , and evaluate it at  $(x, y, z) = (1, 0, 1)$ . Explain in words what your result means.

## Basics of 1-dimensional integration

The trajectory of a particle in motion is described by

$$x(t) = \frac{1}{2}(e^t + e^{-t}), \quad y(t) = \frac{1}{2}(e^t - e^{-t}), \quad -1 \leq t \leq 1,$$

with  $t$  denoting time. Set up *but do not evaluate* the integral which computes the distance traveled by the particle. (Choose units as you please, they are besides the point.)

## Basics of 2-dimensional integration

Set up (*but do not finish*) the computation of the average value of the function  $f(x, y) = x^2$  over the disk of radius 2 centered at  $(3, 3)$ .

## Basics of 3-dimensional integration

Set up (*but do not evaluate*) the integral which computes the volume contained inside the bowl  $z = x^2 + y^2$  but under the plane  $z = 9$ .

## Linearizations and Taylor approximations

Find the linearization of the transformation

$$(x, y) = T(u, v) = (e^u \cos(v), e^u \sin(v))$$

at the point  $(u, v) = (0, \frac{\pi}{4})$ . Interpret your result visually.

## Optimization

Classify the critical points of  $f(x, y) = x^5 + 5xy + y^5$  as local maxima, local minima or saddles.

## The Chain Rule

Let  $(r, \theta, \phi)$  denote the standard spherical coordinates for the Cartesian space  $(x, y, z)$ . Suppose that  $f$  is a function with

$$\overrightarrow{\text{grad}}(f)(x, y, z) = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle.$$

Use the Chain Rule to find the values of  $\frac{df}{dr}$ ,  $\frac{df}{d\theta}$  and  $\frac{df}{d\phi}$  corresponding to the point  $(x, y, z) = (1, 1, -\sqrt{2})$ .

## The Fundamental Theorems of Calculus

1. State all the Fundamental Theorems of Calculus we covered in class.
2. Explain the overall reasoning behind the Stokes' Theorem. Formal proof is not expected – just an intuitive explanation in paragraph form.
3. Let  $S$  be the boundary of a region  $\Omega$  in space. (For example, if  $\Omega$  were the ball  $x^2 + y^2 + z^2 \leq 1$  then  $S$  would be the sphere  $x^2 + y^2 + z^2 = 1$ .) Let  $\vec{V}$  be a vector field in space. What can you say about the outward flux

$$\iint_S \overrightarrow{\text{curl}}(\vec{V}) \cdot \vec{N} dA?$$

## Gradient, curl, divergence and potential

1. Explain the concept of divergence. Briefly: how did we in class develop the formula for divergence?
2. What is  $\text{div}(\overrightarrow{\text{curl}}(\vec{V}))$  always equal to? Explain your answer. (You will receive extra credit if you successfully explain your answer in two different ways.)

## Evaluating the circulation and flux integrals by means of the Fundamental Theorems of Calculus

1. Find  $\int_C \vec{V} \cdot \vec{T} ds$  where

$$\vec{V}(x, y, z) = \left\langle \frac{1}{y} - \frac{z}{x^2}, \frac{1}{z} - \frac{x}{y^2}, \frac{1}{x} - \frac{y}{z^2} \right\rangle$$

and where  $C$  is the line segment joining  $(1, 1, 1)$  and  $(2, 2, 2)$ .

2. Let  $C$  denote the counterclockwise unit circle centered at the origin, and let

$$\vec{V}(x, y) = \langle y - xy, -x + xy \rangle.$$

Compute  $\int_C \vec{V} \cdot \vec{T} ds$ .

3. Find the outward flux  $\iint_S \vec{V} \cdot \vec{N} dA$  where  $S$  denotes the sphere

$$(x - 4)^2 + (y - 4)^2 + (z - 4)^2 = 4$$

and where  $\vec{V}(x, y, z) = \langle x, y, z \rangle$ .

## 5.4 SPRING 2017

### Using Cartesian coordinates

Identify the following surfaces, **and** draw their contour maps. Please calibrate the contour maps, and label your contour lines enough so that I can tell that you know what you are doing.

1.  $z = x^2 + 4y^2;$
2.  $z = -x^2 + 4y^2;$
3.  $z = x^2 - 4y^2;$
4.  $z = -x^2 - 4y^2.$

### Vector and matrix algebra

Let  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  be vectors with magnitudes 1, 2 and 4 respectively. Furthermore, assume it is known that the vectors form angles of  $\frac{\pi}{3}$  with one another.

1. Find the values of  $\vec{a} \cdot \vec{b}$ ,  $\vec{b} \cdot \vec{c}$  and  $\vec{a} \cdot \vec{c}$ .
2. Find the volume of the parallelotope spanned by  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ .

### Curvilinear coordinatization

Consider the transformation

$$(x, y, z) = T(u, v) = (u \cos(v), u \sin(v), \cos(\pi u))$$

defined on the infinite strip  $u \geq 0$ ,  $0 \leq v \leq 2\pi$ . The transformation describes a surface  $S$  in space.

1. Illustrate the surface  $S$ . A picture with grid lines and coordinate vector fields, as well as an explanation in words is expected.
2. Compute the coordinate vector fields  $\partial_u$  and  $\partial_v$  and evaluate them at the point  $(x, y, z) = (0, 1, -1)$ .
3. Compute the upward pointing normal vector field to  $S$  and evaluate it at  $(x, y, z) = (0, 1, -1)$ .
4. Find a formula for the area element  $dA$ , and evaluate it at  $(x, y, z) = (0, 1, -1)$ . Explain in words what your result means.

## Basics of 1-dimensional integration

Compute the total length of the infinite spiral

$$x(t) = e^{-t} \cos(t), \quad y(t) = e^{-t} \sin(t), \quad t \geq 0.$$

## Basics of 2-dimensional integration

Compute the average value of  $f(x, y) = x$  over the quarter disk

$$x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0.$$

## Basics of 3-dimensional integration

Set up but (*do not finish*) the integral for the volume located inside the sphere  $x^2 + y^2 + z^2 = 4$  and within the cylinder  $x^2 + y^2 = 1$ .

## Linearizations and Taylor approximations

A function  $f(x, y)$  reaches its local minimum of at  $(x, y) = (1, 1)$ . The value of  $f(1, 1)$  is 2.

1. What are the values of  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  at  $(1, 1)$ ? In other words, what does the linearization of  $f$  at  $(1, 1)$  look like geometrically and / or formulaically?
2. It is (for some random reason) known that  $\frac{\partial^2 f}{\partial x^2}$  and  $\frac{\partial^2 f}{\partial y^2}$  both take on the value of 4 at  $(1, 1)$ .
  - (a) Give two different possible formulas for the second order Taylor approximation of  $f(x, y)$  at  $(1, 1)$ .
  - (b) **Extra Credit:** What is the possible range of values of  $\frac{\partial^2 f}{\partial x \partial y}$  at  $(1, 1)$ ?

## Optimization

Find the absolute maximum and the absolute minimum value of  $f(x, y) = -x + 2y$  along the ellipse  $4x^2 + 9y^2 = 25$ . You are expected to use the method of Lagrange Multipliers.

## The Chain Rule

Let  $f(x, y)$  be a function with

$$\overrightarrow{\text{grad}}(f) = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle$$

and let  $(x, y) = T(u, v)$  be a transformation whose Jacobi matrix is

$$DT = \begin{pmatrix} e^u \cos(v) & -e^u \sin(v) \\ e^u \sin(v) & e^u \cos(v) \end{pmatrix}.$$

It is known that  $T(0, 0) = (1, 0)$ . Use the Chain Rule to find  $\frac{\partial f}{\partial u}$  and  $\frac{\partial f}{\partial v}$  at  $(u, v) = (0, 0)$ .

## Gradient, curl, divergence and potential

Explain the concept of curl. Briefly: how did we in class develop the formula for curl? Feel free to stick to the 2-dimensional situation.

## The Fundamental Theorems of Calculus

1. State all the Fundamental Theorems of Calculus we covered in class.
2. In your personal opinion, what are all these Fundamental Theorems about?
3. Explain the reason why the Fundamental Theorem of Calculus in Gradient Form holds. Formal proof is not expected – just an intuitive explanation in paragraph form.

## Evaluating the circulation and flux integrals by means of the Fundamental Theorems of Calculus

1. Find  $\int_C \vec{V} \cdot \vec{T} ds$  where

$$\vec{V}(x, y, z) = \left\langle \cos(x+y) + \cos(x+z), \cos(x+y) + \cos(y+z), \cos(x+z) + \cos(y+z) \right\rangle$$

and where  $C$  is the line segment joining  $(0, 0, 0)$  and  $(0, \frac{\pi}{4}, \frac{\pi}{2})$ .

2. Let  $C$  be a counterclockwise curve enclosing a region of area 7 in the  $xy$ -plane. Furthermore, let

$$\vec{V}(x, y) = \langle x - y, x - y \rangle.$$

What is the value of  $\int_C \vec{V} \cdot \vec{T} ds$ ?

3. Find the outward flux  $\iint_S \vec{V} \cdot \vec{N} dA$  where  $S$  denotes the standard unit sphere centered at the origin and where

$$\vec{V}(x, y, z) = \langle x^3, y^3, z^3 \rangle.$$

## 5.5 FALL 2017

### Utilizing Cartesian coordinates

Identify the following surfaces in space. Please provide a drawing or an explanation in words for each surface.

1.  $x^2 + y^2 = 1$
2.  $x^2 + y^2 = 2z$
3.  $x^2 + y^2 = 3z^2$
4.  $x^2 + y^2 = 4z^2 + 1$

### Vector algebra

Let  $\vec{a} = \langle 0, 1, 2 \rangle$ ,  $\vec{b} = \langle 2, 0, 1 \rangle$  and  $\vec{c} = \langle 1, 2, 0 \rangle$ .

1. Find the area of the parallelogram spanned by  $\vec{a}$  and  $\vec{b}$ ;
2. Find the volume of the parallelotope spanned by  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ .

### Curvilinear coordinatization

Consider the surface  $S$  of the one-sheet hyperboloid  $x^2 + y^2 - z^2 = 1$ .

1. Parametrize  $S$ ; please use  $u, v$  for your parameters.
2. Sketch the surface  $S$  together with the  $uv$  grid-lines.
3. Compute and sketch the coordinate vector fields  $\partial_u$  and  $\partial_v$ .
4. Compute a normal vector field to  $S$ .

### Basics of 1-dimensional integration

Set up *but do not evaluate* the integral which computes the length of the curve

$$x(t) = (t - 2)e^t, \quad y(t) = (t - 1)e^t, \quad z(t) = te^t, \quad 0 \leq t \leq 1.$$

Stop when you get to a point when you need to compute a single variable integral.

## Basics of 2-dimensional integration

1. Compute  $\iint_R 3y \, dA$  where  $R$  denotes the region of the  $xy$ -plane enclosed between the parabolas  $y = x^2$  and  $y = 8 - x^2$ .
2. Set up *but do not evaluate* the integral(s) which compute(s) the average value of the function  $f(x, y, z) = x^2 + y^2$  over the surface with parametric equations  
 $x(u, v) = e^u \cos(v), \quad y(u, v) = e^u \sin(v), \quad z(u, v) = u, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$   
 Stop when you get to a point when you need to compute single variable integrals.

## Basics of 3-dimensional integration

Set up *but do not evaluate* the integral which computes the volume enclosed inside the cylinder  $x^2 + y^2 = 4$ , and located between the two sheets of the hyperboloid  $x^2 + y^2 - z^2 = -1$ . Stop when you get to a point when you need to compute single variable integrals.

## Linearizations and Taylor approximations

Please find:

1. the linearization
  2. the second order Taylor approximation
- of the function  $f(x, y) = x^3 - 3xy + y^3$  at the point  $(x, y) = (1, 1)$ .

## Optimization

Find the minimum and the maximum value of the function  $f(x, y) = x + y$  along the ellipse  $x^2 + xy + y^2 = 3$ .

## The Chain Rule

Let  $(u, v)$  denote “elliptical” coordinates for the Cartesian plane  $(x, y)$ :

$$x = 2u \cos(v), \quad y = 3u \sin(v).$$

Let  $P$  denote the point with Cartesian coordinates  $(x, y) = (2, 3)$ .

1. Let  $f(x, y)$  be some function. Please relate  $\frac{\partial f}{\partial u}$  and  $\frac{\partial f}{\partial v}$  to  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .
2. Suppose it is known that  $f(x, y)$  is such that

$$\overrightarrow{\text{grad}} f \Big|_P = \langle -1, 5 \rangle.$$

Please find the values of

$$\frac{\partial f}{\partial u} \Big|_P \quad \text{and} \quad \frac{\partial f}{\partial v} \Big|_P.$$

## Gradient, curl, divergence and potential

In class we introduced divergence as “the rate of spread”. We did so by computing

$$\lim_{r \rightarrow 0} \frac{\int_{C_r} \vec{V} \cdot \vec{N} ds}{\pi r^2},$$

where  $C_r$  denotes the circle of radius  $r$  centered at a base point  $(x_0, y_0)$  of our choice. Perform the details of this computation here. Note: the normal vector field  $\vec{N}$  is assumed to be outward pointing. Also note: the goal is for you to show me that you understand the details of where the formula for divergence came from.

## The Fundamental Theorems of Calculus

1. State all the Fundamental Theorems of Calculus you know; please accompany the formulae you write with well-labeled pictures.
2. Give a very brief non-technical explanation of why Stokes’ Theorem holds. A good picture and an informal sentence or two of explanation will totally be enough.

## Evaluating the circulation and flux integrals by means of the Fundamental Theorems of Calculus

1. Let  $\vec{V}(x, y, z) = \langle 6x^2 + 6xy, 3x^2 + 2yz, y^2 + z^2 \rangle$  and let  $C$  denote the arc of the meridian of the standard unit sphere joining the “North Pole”  $(0, 0, 1)$  to the “South Pole”  $(0, 0, -1)$  via the “Front Pole”  $(1, 0, 0)$ . Use the Fundamental Theorem of Calculus in Gradient Form to compute the value of  $\int_C \vec{V} \cdot \vec{T} ds$ .
2. Find the value of  $\int_C \vec{V} \cdot \vec{T} ds$  where  $C$  denotes the counterclockwise unit circle centered at the origin and  $\vec{V}(x, y) = \langle -y^3, x^3 \rangle$ . You are expected to use the Fundamental Theorem(s) as much as possible.
3. Find the outward flux  $\iint_S \vec{V} \cdot \vec{N} dA$  where  $S$  is the upper dome of the unit sphere centered at the origin (base **not** included) and where  $\vec{V}(x, y, z) = \langle x+1, y+2, -2z+3 \rangle$ . You are expected to use the Fundamental Theorem(s) as much as possible.