# Hilbert's Problem

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## 1 Algorithm

We take input of a sequence of coefficients $\{a_0, a_1, \ldots, a_n\}$ describing a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$ where the sequence continues until the highest nonzero degree term in the polynomial which we call $n$.
We then have the following proposed algorithm:

1. We define a bound $M \in \mathbb{Z}$ such than $M = \sum_{k=0}^{n-1} |a_k|$.
2. Then, we iterate over the integers from $-M$ to $M$. For each instance $z \in \mathbb{Z}$ we have:
   (a) We evaluate $p(z)$. If $p(z) = 0$, we accept the input for $p$ has a integer root at input $z$. Otherwise, continue.
3. If there are no roots within the interval we can safely reject, for we can guarantee that there are no roots outside of this interval. We have the proof below.

*Proof.* Take some $x \in \mathbb{Z}$ such that $|x| > M$. Then,

$$|a_n x_n| > |x^n| > |x^{n-1}||x| > x^{n-1} \sum_{k=0}^{n-1} |a_k|$$

Note $|x| \geq 1$ and thus $|x^k| < |x^{n-1}|$ for $k \leq n-1$. So,

$$|a_n x^n| > \sum_{k=0}^{n-1} |a_k x^{n-1}| > \sum_{k=0}^{n-1} |a_k x^k|$$

For $a_n x^n > 0$ we then must have

$$a_n x^n > \sum_{k=0}^{n-1} |a_k x^k| > -\sum_{k=0}^{n-1} a_k x^k$$

Thus we have

$$\sum_{k=0}^{n} a_k x^k = a_n x^n + \sum_{k=0}^{n-1} a_k x^k > 0$$

Similarly, for $a_n x^n < 0$ we then must have

$$a_n x^n < -\sum_{k=0}^{n-1} a_k x^k$$

Thus we have

$$\sum_{k=0}^{n} a_k x^k = a_n x^n + \sum_{k=0}^{n-1} a_k x^k > 0$$

So, in both cases $p(x)$ cannot have a root, so it is safe to reject. $\square$

## 2 Note

I noticed that Hilbert's Problem is generalized for a multi-variable equation. Perhaps this is a valid algorithm for one dimensional polynomials, but does not work for more variables.