A Tiny Bit of Fluid Dynamics

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Abstract

A vector field can model the behavior of water molecules. Given such a vector field, we can extract additional information about the behavior of water. Such information includes: the solution curve of an arbitrary water molecule, the phase diagram of solutions, the transformation representing the mapping of water molecules over an interval of time, and the change in volume a portion of molecules occupy after such a transformation. We will then consider a general vector field, revealing a relationship between divergence and the stretch factor of a transformation, allowing for the construction of a vector field that does not stretch.

1 Problem 1

$\vec{U}(\mathbf{y}(t))$ is a well-defined, differentiable vector field that represents the velocity of an arbitrary water molecule at point $\mathbf{y} = (x, y, z)$. Specifically, $\vec{U}(\mathbf{y}(t)) = -2x\hat{\imath} + (-2y + z)\hat{\jmath} + (y - 2z)\hat{k}$. The starting location of the molecule is at an arbitrary point $\mathbf{x} = (a, b, c)$. This information is formally represented with the following IVP.

$$\begin{cases} \frac{d\mathbf{y}}{dt} = \vec{U}(\mathbf{y}(t)) \\ \mathbf{y}(0) = \mathbf{x} \end{cases}$$

We can solve the differential system of equations, which will provide the path a water molecule starting at point $\mathbf{x}$ will take. First, we rewrite the differential equation in its matrix form.

$$\frac{d\mathbf{y}}{dt} = M\mathbf{y}$$
$$\text{Where, } M = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

The solution vectors will be scalar multiples of themselves when multiplied by M. This motivates finding each eigenvalue, λ, of M using $\det(M - \lambda I) = 0$, I being the identity matrix. This can be broken down into the following.

$$(-2 - \lambda)((-2 - \lambda)^2 - 1) = 0$$

The above equation has solutions $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$. Next, we locate the vectors that are scaled by these eigenvalues when multiplied by M. Or, what $\vec{v}$ satisfies $\lambda\vec{v} = M\vec{v}$ for each case of λ? With $\vec{v_1}, \vec{v_2}, \vec{v_3}$ corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively, the below vectors satisfy the requirement.

$$\vec{v_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ \vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \vec{v_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Using each eigenvalue-eigenvector pair, a solution will take the form of $\mathbf{y} = Ce^{\lambda t}\vec{v}$. So, by the superposition principle, the general solution of the system is

$$\mathbf{y} = C_1e^{-t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2e^{-2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3e^{-3t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{1}$$

The initial condition of $\mathbf{y}(0) = (a, b, c)$ provides sufficient information to solve for each constant, C. In applying this specific case of $t = 0$, we arrive at the below.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{2}$$

We can rewrite this equation in matrix form.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

In finding the inverse of the matrix we arrive at the following.

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

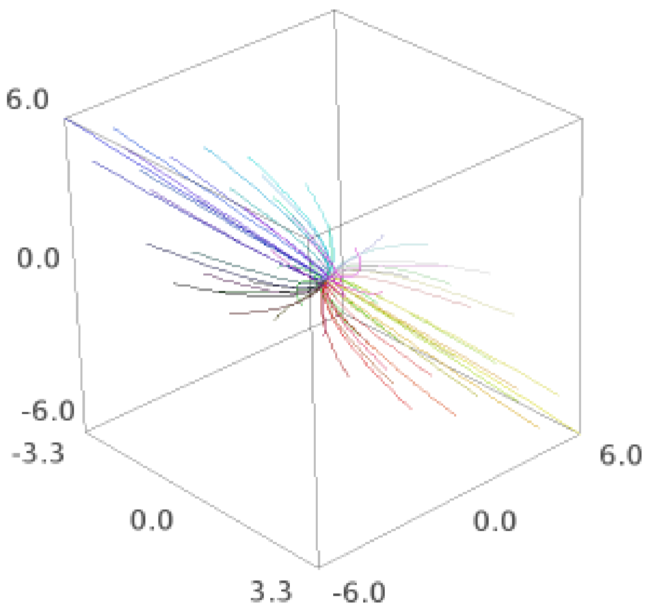
Multiplying $\langle a, b, c \rangle$ into the matrix gives us the solution to the vector containing each C.

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} \frac{b}{2} + \frac{c}{2} \\ a \\ \frac{b}{2} - \frac{c}{2} \end{pmatrix} \tag{3}$$

We can now substitute each C value in equation 1 with the corresponding function of a, b, and c. This equation provides the exact (x, y, z) position of the particle when given the starting location (a, b, c) and the time t that has passed.

$$\mathbf{y} = \left(\frac{b}{2} + \frac{c}{2}\right) e^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (a) e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{b}{2} - \frac{c}{2}\right) e^{-3t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{4}$$

We now have an equation that describes every solution curve, but how can we visualize these solution curves in x, y, z space? In looking at equation 1, we see that the phase diagram takes the form of a three dimensional sink. The sink nature of the phase diagram is due to every eigenvalue being negative. As time progresses, each $e^{f(t)}$ term will decrease, bringing the curves closer to the origin. The solution vectors that define this sink are centered at the origin. These vectors point in the $\langle 0, 1, 1 \rangle$ direction, the $\langle 1, 0, 0 \rangle$ direction, and the $\langle 0, 1, -1 \rangle$ direction. Close to the origin, when t is large, the vectors with the eigenvalues of the smallest magnitude will have the greatest influence. So, close to the origin, the solution curves hug the $\langle 0, 1, 1 \rangle$ vector. Then, they begin to move into the direction of the $\langle 1, 0, 0 \rangle$ vector. This results in the solutions living mostly in the plane spanned by $\langle 0, 1, 1 \rangle$ and $\langle 1, 0, 0 \rangle$ near the origin. Farther from the origin however, when t becomes largely negative, the $\langle 0, 1, -1 \rangle$ vector will dominate, so the solution curves will parallel this third vector. The graph below is a plot of many solution curves in this described phase diagram.



2 Problem 2

We can rewrite the information within equation 4 as a transformation. $F_t : \mathbf{x} \mapsto \mathbf{y}(t)$ represents the mapping of a water molecule from starting position (a, b, c) to its (x, y, z) position after t seconds. We can add the x, y, and z components of each vector in equation 4 to represent $\mathbf{y}$ as a single vector, which is the location of the water molecule after t seconds. Using the vector $\mathbf{y}$, we craft the following transformation.

$$F_t(a,b,c) = \left(ae^{-2t}, \left(\frac{b}{2}+\frac{c}{2}\right)e^{-t} + \left(\frac{b}{2}-\frac{c}{2}\right)e^{-3t}, \left(\frac{b}{2}+\frac{c}{2}\right)e^{-t} - \left(\frac{b}{2}-\frac{c}{2}\right)e^{-3t}\right) \tag{5}$$

Now that we have the information in transformation form, we can proceed to find properties of the transformation such as the Jacobi matrix, DF_t and the stretch factor, $\det(DF_t)$. The first step to finding the Jacobi matrix is constructing the partial vectors with respect to a, b, and c. These vectors are as follows:

$$\begin{aligned}
\partial_a &= \langle e^{-2t}, 0, 0\rangle \\
\partial_b &= \langle 0, \frac{1}{2}(e^{-t}+e^{-3t}), \frac{1}{2}(e^{-t}-e^{-3t})\rangle \\
\partial_c &= \langle 0, \frac{1}{2}(e^{-t}-e^{-3t}), \frac{1}{2}(e^{-t}+e^{-3t})\rangle
\end{aligned}$$

We assemble these partial vectors vertically to create the Jacobi matrix, DF_t.

$$DF_t = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & \frac{1}{2}(e^{-t}+e^{-3t}) & \frac{1}{2}(e^{-t}-e^{-3t}) \\ 0 & \frac{1}{2}(e^{-t}-e^{-3t}) & \frac{1}{2}(e^{-t}+e^{-3t}) \end{pmatrix}$$

The stretch factor of the transformation F_t is given by $\det(DF_t)$. Visually, this is the communicates the change the volume after t seconds in the neighborhood of the water molecule beginning at location (a, b, c).

$$\begin{aligned}
\det(DF_t) &= e^{-2t}\left(\left(\frac{e^{-t}+e^{-3t}}{2}\right)^2 - \left(\frac{e^{-t}-e^{-3t}}{2}\right)^2\right) \\
\det(DF_t) &= e^{-6t} \qquad (6)
\end{aligned}$$

So, the change in volume of the fluid is given by the equation e^{-6t}. This equation means that after t seconds, a portion of the fluid will take up a fraction of its original volume and continue to get denser. This result aligns with the previously described phase diagram. The phase diagram is a sink, so the molecules approach the center. If all the molecules approach the center, the fluid must become more dense, so the volume of a neighborhood of fluid must decrease.

3 Problem e

Next, we generalize the problem to an arbitrary $\vec{U}(\mathbf{y})$. For a general vector field, we must represent the Jacobi matrix as arbitrary partial derivatives. We will now explore the stretch factor $S(t) = \det(DF_t)$ of the general vector field. For simplicity, we will reduce the vector field down to 2 dimensions. We will now evaluate the stretch factor as given by the following equation.

$$S(t) = \det(DF_t) = \det \begin{pmatrix} \frac{\partial x}{\partial a}|_{(x(t),y(t))} & \frac{\partial x}{\partial b}|_{(x(t),y(t))} \\ \frac{\partial y}{\partial a}|_{(x(t),y(t))} & \frac{\partial y}{\partial b}|_{(x(t),y(t))} \end{pmatrix}$$

Taking the determinant results in the following formula for $S(t)$.

$$S(t) = \frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}$$

Now we will take the derivative with respect to time of the stretch factor $S(t)$.

$$\frac{dS}{dt} = \frac{d}{dt}\Big(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}\Big)$$

The application of product rule expands this derivative.

$$\begin{aligned} \frac{dS}{dt} &= \Big[\frac{d}{dt}\Big(\frac{\partial x}{\partial a}\Big)\frac{\partial y}{\partial b} + \frac{\partial x}{\partial a}\frac{d}{dt}\Big(\frac{\partial y}{\partial b}\Big)\Big] - \Big[\frac{d}{dt}\Big(\frac{\partial x}{\partial b}\Big)\frac{\partial y}{\partial a} + \frac{\partial x}{\partial b}\frac{d}{dt}\Big(\frac{\partial y}{\partial a}\Big)\Big] \\ &= \Big[\frac{d}{da}\Big(\frac{\partial x}{\partial t}\Big)\frac{\partial y}{\partial b} + \frac{\partial x}{\partial a}\frac{d}{db}\Big(\frac{\partial y}{\partial t}\Big)\Big] - \Big[\frac{d}{db}\Big(\frac{\partial x}{\partial t}\Big)\frac{\partial y}{\partial a} + \frac{\partial x}{\partial b}\frac{d}{da}\Big(\frac{\partial y}{\partial t}\Big)\Big] \end{aligned}$$

To further simplify this expression, we call upon the relation between the partial derivatives and the original vector field $\vec{U}$.

$$\vec{U} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$

So, anywhere we have a $\frac{dx}{dt}$, we can substitute in $P(x,y)$ and $\frac{dy}{dt}$ can be replaced with $Q(x,y)$. This results in the following simplification.

$$\frac{dS}{dt} = \Big[\frac{\partial P}{\partial a}\frac{\partial y}{\partial b} + \frac{\partial x}{\partial a}\frac{\partial Q}{\partial b}\Big] - \Big[\frac{\partial p}{\partial b}\frac{\partial y}{\partial a} + \frac{\partial x}{\partial b}\frac{\partial Q}{\partial a}\Big]$$

Of course, P and Q are not directly functions of a and b. Rather, P and Q are functions of x and y which in themselves are functions of a and b, so we apply chain rule.

$$\begin{aligned} \frac{dS}{dt} = &\Big[\Big(\frac{\partial P}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial a}\Big)\frac{\partial y}{\partial b} + \frac{\partial x}{\partial a}\Big(\frac{\partial Q}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial b}\Big)\Big] \\ -&\Big[\Big(\frac{\partial P}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial b}\Big)\frac{\partial y}{\partial a} + \frac{\partial x}{\partial b}\Big(\frac{\partial Q}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial a}\Big)\Big] \end{aligned}$$

We are now able to factor and rearrange the expression.

$$\begin{aligned} \frac{dS}{dt} = &\Big[\Big(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\Big)\Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big) + \Big(\frac{\partial y}{\partial a}\frac{\partial y}{\partial b}\Big)\Big(\frac{\partial P}{\partial y}\Big) + \Big(\frac{\partial x}{\partial a}\frac{\partial x}{\partial b}\Big)\Big(\frac{\partial Q}{\partial x}\Big)\Big] \\ -&\Big[\Big(\frac{\partial x}{\partial b}\frac{\partial y}{\partial a}\Big)\Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big) + \Big(\frac{\partial y}{\partial a}\frac{\partial y}{\partial b}\Big)\Big(\frac{\partial P}{\partial y}\Big) + \Big(\frac{\partial x}{\partial a}\frac{\partial x}{\partial b}\Big)\Big(\frac{\partial Q}{\partial x}\Big)\Big] \end{aligned}$$

The $\frac{\partial P}{\partial y}$ and the $\frac{\partial Q}{\partial x}$ terms cancel. The $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ term factors from the result, which gives us the following.

$$\frac{dS}{dt} = \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big)\Big(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}\Big)$$

Recall that $\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}$ is the expression for $S(t)$ itself. Furthermore, the function $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is also known as divergence. With this in mind, we can reduce the formula for $\frac{dS}{dt}$ to the following

$$\frac{dS}{dt} = \mathrm{div}(\vec{U}) \cdot S \tag{7}$$

The expression of $\frac{dS}{dt}$ being a product of S and $\text{div}(\vec{U})$ makes sense. The rate of change of S should rely on S itself because the stretch factor is a fractional increase or decrease. Given a specific portion of water, the law for the change in volume is not something like "V decreases by x units" because we are considering an arbitrarily small volume. Instead, it makes more sense to write the law as "V decreases by x percent". The amount of change is fundamentally reliant on the initial volume, explaining why $\frac{dS}{dt}$ is a function of S. The dependence of $\frac{dS}{dt}$ on $\text{div}(\vec{U})$ is also intuitive. $\text{div}(\vec{U})$ measures how much the vectors at a specific point in the field point inwards or outwards. If we were to put our finger on what the causes the volume of the fluid to change, it would be this inwards or outwards pointing nature of the vectors. $\text{div}(\vec{U})$ returns a negative value for inwards pointing vectors and a positive value for outwards pointing vectors. The more extreme the change, the larger the magnitude of the returned value. The specifics of what the divergence returns aligns with what we would expect of our change in volume. If the vectors point inwards, the water molecules move closer together and the change in volume is negative. If the vectors point outwards, the water molecules spread out and the change in volume is positive. The more extreme the vectors point in or out, the greater the magnitude of the rate of change.

4 Problem 3

We can apply equation 7 to the specific vector field, $\vec{U}$, described in Problem 1. First, we find the divergence of $\vec{U}$.

$$\begin{aligned}
\text{div}(\vec{U}) &= \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-2y+z) + \frac{\partial}{\partial z}(y-2z) \\
\text{div}(\vec{U}) &= (-2) + (-2) + (-2) = -6.
\end{aligned}$$

So the divergence of $\vec{U}$ is -6. In pairing this fact with equation 7, we get the following differential equation.

$$\frac{dS}{dt} = -6 \cdot S$$

This simple differential equation has the following general solution.

$$S = Ce^{-6t}$$

A sensible initial condition is $S(0) = 1$, which says at $t = 0$, the fluid has not stretched or changed in volume. After all, at $t = 0$ the fluid hasn't had any time to change. We now implement this initial condition to solve for C.

$$1 = S(0) = Ce^{-6(0)} = C$$

So, $C = 1$, which gives us the same stretch factor of $S = e^{-6t}$ that we arrived at in equation 6 with $\det(DF_t)$.

5 Problem π

We redefine the vector field to be $\vec{U}(\mathbf{y}) = \alpha x\hat{\imath} + (-2y+z)\hat{\jmath} + (y-2z)\hat{k}$. For what value of α would make this a *incompressible fluid*? An *incompressible fluid* is a fluid for which the volume goes unchanged. Unchanging volume means that $\frac{dS}{dt} = 0$ for all t.

$$0 = \frac{dS}{dt} = \text{div}(\vec{U}) \cdot S$$

An unchanging volume means S forever remains at a volume of 1. But importantly, S is non-zero, for the molecules will always take up some space. After all, the fluid is meant to be *incompressible*. If $\text{div}(\vec{U}) \cdot S = 0$ and $S \neq 0$, then $\text{div}(\vec{U}) = 0$.

$$\begin{aligned}
\text{div}(\vec{U}) &= 0 \\
\frac{\partial}{\partial x}(\alpha x) + \frac{\partial}{\partial y}(-2y+z) + \frac{\partial}{\partial z}(y-2z) &= 0 \\
\alpha - 2 - 2 &= 0
\end{aligned}$$

So, $\alpha = 4$. In plugging α into $\vec{U}$, we get a new vector field. $\vec{U}(\mathbf{y}(t)) = 4x\hat{\imath}+(-2y+z)\hat{\jmath}+(y-2z)\hat{k}$. We will now continue to solve this new differential equation for an arbitrary starting point, express the movement of molecules as a transformation, and determine the stretch factor as we did previously. The first step is to find the matrix, M, that satisfies the differential equation $\frac{d\mathbf{y}}{dt} = M\mathbf{y}$.

$$\text{In this case, } M = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

The eigenvalues of M are the solutions to the equation $\det(M-\lambda I) = 0$. This determinant provides the equation $(4-\lambda)((-2-\lambda)^2 - 1) = 0$. So, the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 4$, $\lambda_3 = -3$. Each λ corresponds to an eigenvector $\vec{v}$ satisfying $\lambda\vec{v} = M\vec{v}$. The corresponding solution vectors, $\vec{v}$, are:

$$\vec{v_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ \vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \vec{v_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

In crafting each eigenvalue-eigenvector pair into the corresponding solution and applying the superposition principle, we can craft the general solution to the differential equation.

$$\mathbf{y} = C_1 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{-3t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{8}$$

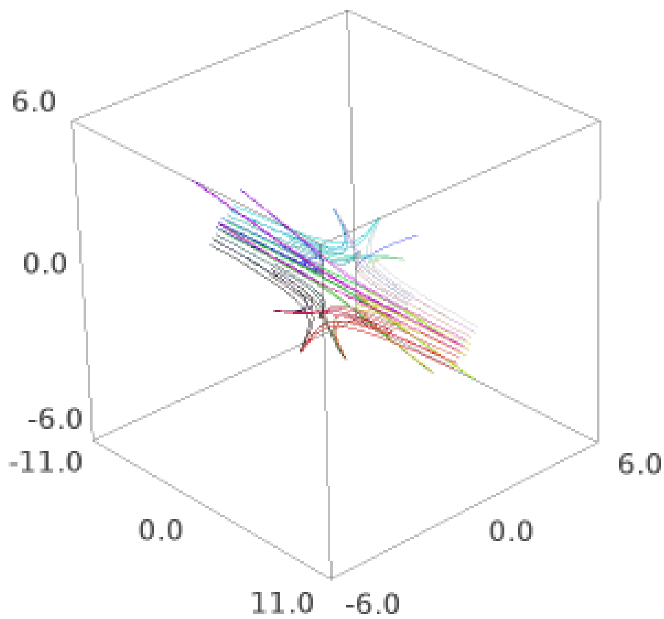
We know that at $\mathbf{y}(0) = \mathbf{x} = \langle a, b, c\rangle$. So, in exploring the specific case of $t = 0$, we arrive at:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{9}$$

Note that equation 9 is identical to equation 2, meaning that the solution of each C in terms of a, b, and c are the same. So, the solution to each C in the current differential equation is the same as described in equation 3. With this information, we can craft the following general solution in terms of a, b, and c.

$$\mathbf{y} = \left(\frac{b}{2} + \frac{c}{2}\right) e^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (a)e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{b}{2} - \frac{c}{2}\right) e^{-3t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{10}$$

Now to address what the phase diagram resulting from equation 8 looks like. Two of the solution vectors, $\vec{v}_1$ and $\vec{v}_3$, approach the origin while the solution spanned by $\vec{v}_2$ gets further from the origin. As t approaches becomes very negative, the portion of the solution in the $\vec{v}_2$ direction becomes irrelevant. So, as t approaches negative infinity, the solution curves live on the plane spanned by $\vec{v}_1$ and $\vec{v}_3$. Furthermore, the solution curves will begin favor $\vec{v}_3$ (λ_3 being of greater magnitude), so the solutions will parallel the $\langle 0, 1, -1\rangle$ vector. As t gets largely positive, $\vec{v}_2$ dominates, and all solutions approach it. So, the phase diagram should consist of a plane of solutions spanned by $\langle 0, 1, 1\rangle$ and $\langle 0, 1, -1\rangle$ that all approach the origin and then funnel upwards or downwards, approaching the solution spanned by $\langle 1, 0, 0\rangle$. This is semi-depicted by the following diagram.



Equation 10 can be rewritten in transformation form by adding the vector components to express $\mathbf{y}$ as a single vector,

$$F_t(a,b,c) = \Big(ae^{4t}, \Big(\frac{b}{2}+\frac{c}{2}\Big)e^{-t} + \Big(\frac{b}{2}-\frac{c}{2}\Big)e^{-3t}, \Big(\frac{b}{2}+\frac{c}{2}\Big)e^{-t} - \Big(\frac{b}{2}-\frac{c}{2}\Big)e^{-3t}\Big) \tag{11}$$

We can break down this transformation into the Jacobi matrix, DF_t.

$$DF_t = \begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & \frac{1}{2}(e^{-t}+e^{-3t}) & \frac{1}{2}(e^{-t}-e^{-3t}) \\ 0 & \frac{1}{2}(e^{-t}-e^{-3t}) & \frac{1}{2}(e^{-t}+e^{-3t}) \end{pmatrix}$$

The value of $\det(DF_t)$ of this transformation represents the stretch factor after t seconds.

$$\det(DF_t) = e^{4t}\Big(\Big(\frac{e^{-t}+e^{-3t}}{2}\Big)^2 - \Big(\frac{e^{-t}-e^{-3t}}{2}\Big)^2\Big)$$
$$\det(DF_t) = e^{4t}\cdot(e^{-4t}) = 1$$

As predicted previously, the stretch factor of the volume will always be 1. This is because the fluid is initially unstretched, taking up exactly 1 of its initial volume. Then, if $\frac{dS}{dt}$ is forever 0, the stretch factor will never change.