CHAPTER 1

Division Algebra Application

Recall from /*ref*/ that a division algebra is a ring with multiplicative inverses. Further recall that a division algebra structure in $\mathbb{R}^n$ induces an H-space structure over the sphere S^{n-1} by considering the subset of $\mathbb{R}^n$ with norm 1. And so if a division algebra structure on $\mathbb{R}^n$ exists, then there is an H-space structure on the sphere S^{n-1}. As Bott periodicity hints at, K-Theory has a close relationship with spheres, making K-Theory a good tool to examine the existence of H-spaces on spheres. This chapter will use K-Theory to show that an H-space structure cannot exist on any sphere other than S^0, S^1, S^3, and S^7. This conclusion regarding H-spaces together with the explicit examples of the reals, the complex numbers, the quaternions, and the octonions gives the following theorem.

THEOREM 1.1. $\mathbb{R}^n$ is a division algebra only when n is 1, 2, 4, or 8.

1. The odd case

The argument will first rule out the possibility of a division algebra structure on odd dimensions, so assume for the purpose of contradiction that there is a division algebra structure on $\mathbb{R}^{2k+1}$ for some nonnegative k. It then follows that S^{2k} is an H-Space, which promises a continuous mapping $\mu : S^{2k} \times S^{2k} \to S^{2k}$. Additionally let p_1 denote the projection from $S^{2k} \times S^{2k}$ to the first factor and let p_2 be the projection to the second factor. Applying the K-Theory functor to the H-space multiplication mapping gives a homomorphism between rings $\mu^* : K(S^{2k} \times S^{2k}) \to K(S^{2k})$ /*ref...do this step as an example earlier*/. This puts the homomorphism in the form $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ such that $\alpha = p_1^*(\gamma)$ and $\beta = p_2^*(\gamma)$. Note that γ is the generator of the ring and in particular, γ is $H - 1$ where H denotes the canonical line bundle over the space.

The contradiction will arise in analyzing the quantity $\mu^*(\gamma)$. To accomplish this, define $i_1 : S^{2k} \to S^{2k} \times S^{2k}$ by $i_1 : x \mapsto (x, e)$ where e is the identity element of S^{2k} as an H-space. Note that $\mu \circ i_1$ is the identity, giving $i_1^* \circ \mu^*$ is the identity, so studying i_1^* will give information about μ^*.

It follows from the definition of i_1 that $p_1 \circ i_1 = \mathrm{Id}$ and so $i_1^* \circ p_1^*$ is identity. Plugging in α to both sides and recalling the definition of α then gives:

$$i_1^*(\alpha) = \gamma$$

In a similar way, it follows that $p_2 \circ i_1$ is a constant function always mapping to the H-space identity point e. Denote this by $p_2 \circ i_1 = \mathrm{const}_e$, which gives $i_1^* \circ p_2^* = \mathrm{const}_e^*$. Again plug in γ to both sides and recall the definition of β and thus $i_1^*(\beta) = \mathrm{const}_e^*(\gamma)$. /*remember to note the notation that will now be used*/ To simplify this further, recall that γ is $H - 1$ where H is the canonical line bundle. And note that because each fiber of H is of dimension 1 and const_e maps to a point, the pullback $\text{const}_e^*(H)$ is the trivial bundle ε^1, which is the multiplicative identity denoted by 1. Thus by the ring homomorphism properties:

$$\text{const}^*(\gamma) = \text{const}^*(H-1) = \text{const}^*(H) - \text{const}^*(1) = 1 - 1 = 0$$

This gives the following crucial piece of information.

$$i_1^*(\beta) = 0$$

Now return to analyzing the quantity $\mu^*(\gamma)$. As an element of $\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$, the quantity is of the form $\mu^*(\gamma) = n + a\alpha + b\beta + m\alpha\beta$ for some integers a, b, n, m. However, now apply $i_1^* \circ \mu^* = \text{Id}$ with the information $i_1^*(\alpha) = \gamma$ and $i_1^*(\beta) = 0$ and keeping ring homomorphism properties in mind:

$$\gamma = i_1^*(\mu^*(\gamma)) = i_1^*(n + a\alpha + b\beta + m\alpha\beta) = n + a \cdot i_1^*(\alpha) + b \cdot i_1^*(\beta) + m \cdot i_1^*(\beta) \cdot i_1^*(\alpha) = n + a\gamma$$

And thus by $\gamma = n + a\gamma$, it follows that $n = 0$ and $a = 1$. Applying the same argument by considering the inclusion $i_2 : S^{2k} \to S^{2k} \times S^{2k}$ by $i_2 : x \mapsto (e, x)$ will give that $b = 1$. And so μ^* can be written in the reduced form

$$\mu^* = \alpha + \beta + m\alpha\beta$$

The contradiction arises from the observation that the relation $\gamma^2 = 0$ gives that $(\mu^*(\gamma))^2 = 0$. However, the derived expression for $\mu^*(\gamma)$ and the relations $\alpha^2 = \beta^2 = 0$ imply a different result.

$$(\mu^*(\gamma))^2 = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0$$

2. The even case