Hearing the Local Orientability of Orbifolds

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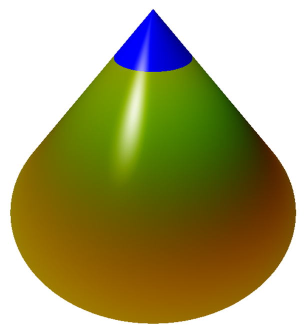
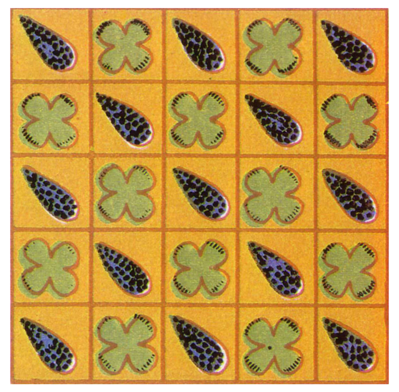
Symmetries of Space

A pattern with symmetry has a group of actions Γ we can perform on the pattern without change. For instance, we can rotate Figure 1a by any element of $\Gamma = \{0°, 90°, 180°, 270°\}$ and still preserve the pattern. Similarly, Figure 1c is preserved by doing nothing and reflecting along the diagonal. Notice these actions do not alter the size of the patterns, making them *isometries*.


(a) Rotational Symmetry


(b) Cone Point


(c) Reflectional Symmetry

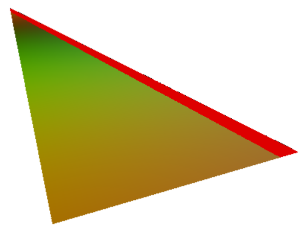

(d) Mirror Edge

Figure 1: Folding Symmetries

To visualize symmetry we imagine folding patterns on pieces of paper such that each piece of the pattern occurs only once. In Figure 1c, we can fold the pattern across the diagonal, creating half plane with a sharp edge. In Figure 1a, we can roll up the pattern into a party hat shape, creating a sharp point. We denote these resulting shapes $\mathbb{R}^n/\Gamma$ where Γ is the symmetry group of the pattern. The sharp parts of the shapes are *strata*.

Orbifolds

An orbifold is a generalization of a simple n dimensional surface, which we call a manifold. If you were to zoom into a manifold, you will see flat space, for it has local structure $\mathbb{R}^n$. However, an orbifold allows local structure of $\mathbb{R}^n/\Gamma$ (such as the cone point or mirror edge in Figure 1). Figure 2 is a visual representation of a two dimensional orbifold with a cone point and a mirror edge.

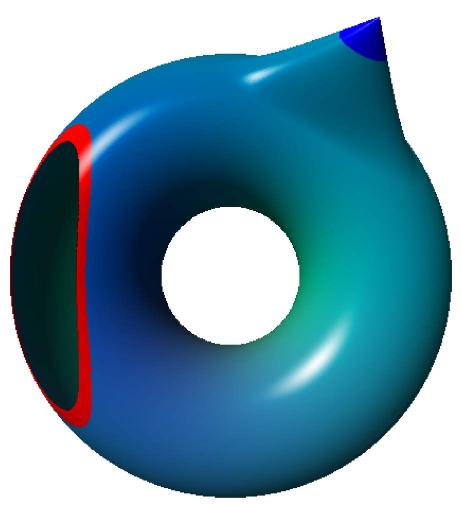


Figure 2: Orbifold Representation

Laplace Spectra

A string on the guitar produces a sound determined by a specific frequency it can vibrate at. Actually, the string can vibrate at a *spectrum* of resonance frequencies. All objects have such a vibrational spectrum — drums will vibrate at only certain sounds. Mathematically, we express these frequencies through the *Laplace Spectra*.
The Laplace Spectra follows from the wave equation, $\Delta u = \frac{\partial^2 u}{\partial t^2}$, which describes the amplitude $u(t, \mathbf{x})$ at any location $\mathbf{x}$ and time t on an orbifold. In assuming $u(t,\mathbf{x}) = A(t)\psi(x)$ for standing wave solutions, we find $\Delta u = -\lambda\psi(\mathbf{x})$. The eigenvalue solutions $\lambda_1, \lambda_2, \ldots$ are called the Laplace Spectra where each $\sqrt{\lambda_i}$ is a valid fundamental frequency.

Research Question

Given a specific drum, it is possible to deduce what sound the drum will make when hit. However, consider the reverse: if you hear a drum in the neighboring room, is it possible to reverse engineer the drum's shape? In other words, "can you hear the shape of a drum?" In this research we ask a similar question: Can you hear the shape of an orbifold?
Formally, consider some Laplace Spectra $\lambda_1, \lambda_2, \ldots$ belonging to some unknown orbifold $\mathcal{O}$. From the Laplace Spectra, what properties can we deduce about $\mathcal{O}$?

Local Orientability

Isometries can preserve or reverse orientation. For instance, a reflection reverses orientation (when you look into a mirror, your reflection is flipped), but a simple rotation preserves orientation.
Some local structure of an orbifold $\mathbb{R}^n/\Gamma$ has a group of isometries Γ associated with it. If Γ contains a single orientation reversing isometry, the local structure is *non-orientable*; otherwise, it is *orientable*.
We define an orbifold to be *locally non-orientable* if it has a single non-orientable local structure; otherwise, the orbifold is *locally orientable*.

Result

We found you can hear the local orientability of an orbifold — there exists no locally orientable orbifold and locally non-orientable orbifold with the same Laplace Spectra. Figure 3 shows two orbifolds guaranteed to have different Laplace Spectra by this result.

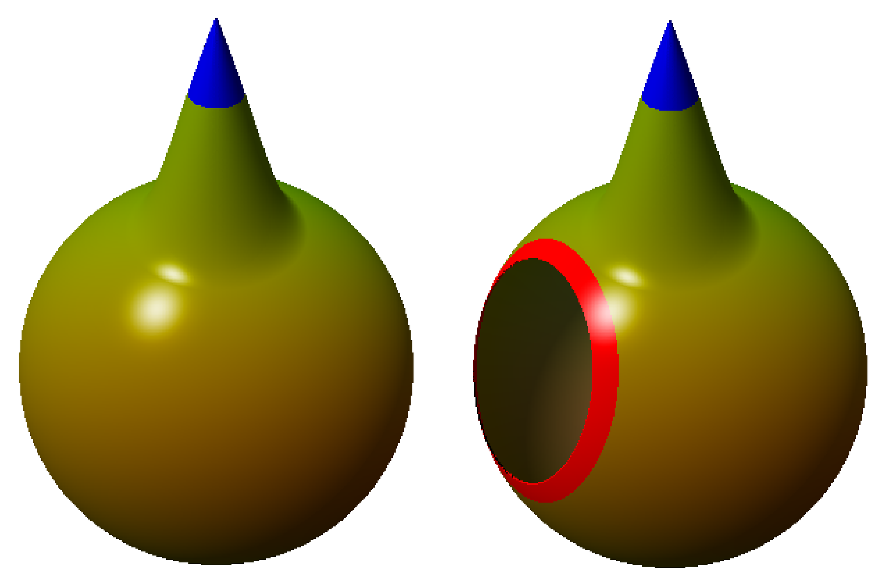


Figure 3: Non-isospectral Orbifolds

Future Work

In addition to this result, we studied symmetry patterns of 3 dimensional space, which are called the crystallographic space groups. Each space group of isometries can be folded into a 3 dimensional orbifold (visualized in higher dimensional space). These orbifolds have a Laplace Spectra we can study. There are 230 such patterns and corresponding orbifolds, so perhaps a computer can automate the process of finding heat expansion coefficients.


Figure 4: Spacial Symmetry from *The Symmetries of Things*

Asymptotic Heat Expansion

Our question requires us to relate the properties of the orbifold to its Laplace Spectra, which can be done by studying how heat disperses throughout the orbifold. This section will touch on the logic used to prove our result.
Consider focusing heat at a single point $\mathbf{x}$ on some orbifold $\mathcal{O}$. Then, allow the heat to disperse around $\mathcal{O}$. The point $\mathbf{x}$ will cool with time. We study a specific function: how a point cools on average for every point in $\mathcal{O}$, which is known to be $\sum_{i=1}^{\infty} e^{-\lambda_i t}$ where λ_i is an element of the Laplace Spectrum of $\mathcal{O}$. Formally, this falls out of the heat equation, $\Delta u = \frac{\partial u}{\partial t}$. The integrand of the solution is the heat kernel K, and we find that $\mathrm{Tr}(K) = \sum_{i=1}^{\infty} e^{-\lambda_i t}$.
If we approximate this function for small values of time, we get a function of the following form (for 2 dimensions)

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} \sim \frac{a}{t} + \frac{b}{\sqrt{t}} + c + d\sqrt{t} + \ldots$$

Where the right side approximates the left for small t as in Figure 6.

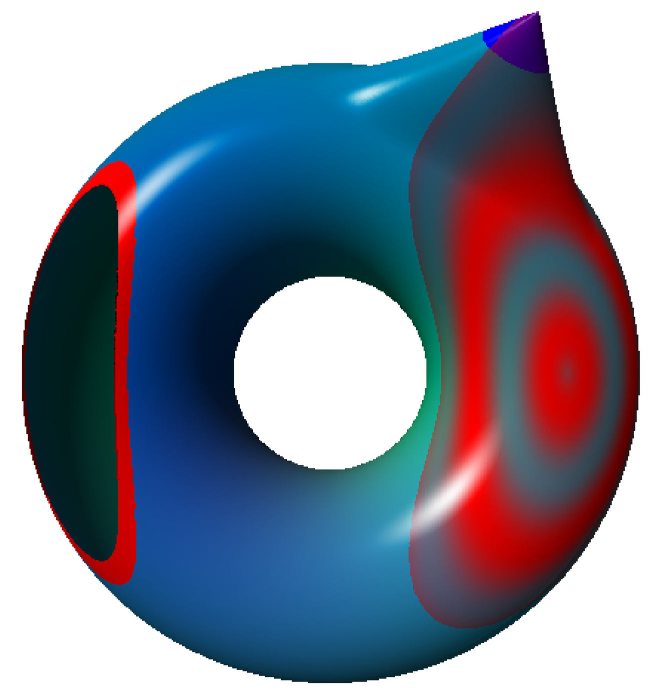


Figure 5: Heat dispersing from $\mathbf{x}$

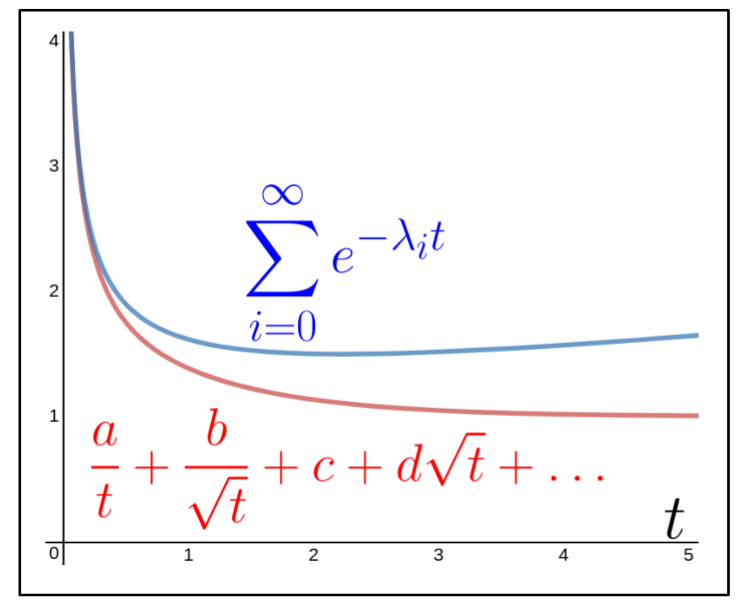


Figure 6: Asymptotic Expansion Approximation

The resulting equation has coefficients $a, b, c, \ldots$. We can find these coefficients from the properties of $\mathcal{O}$. The coefficients are described by

$$(4\pi t)^{-\dim(\mathcal{O})/2}\sum_{k=0}^{\infty} a_k(\mathcal{O})t^k + \sum_{N\in S(\mathcal{O})} \frac{(4\pi t)^{-\dim(N)/2}}{|\mathrm{Iso}(N)|}\sum_{k=0}^{\infty} t^k \int_N \sum_{\gamma\in \mathrm{Iso}^{\max}(\tilde{N})} b_k(\gamma, x)\, dvol_N$$

So, properties of $\mathcal{O}$ determine coefficients $a, b, c, \ldots$ and the coefficients determine the Laplace Spectra $\lambda_1, \lambda_2, \ldots$. If two orbifolds differ in a single coefficient, they will have different Laplace Spectra. In our proof, we use the fact that in odd dimensions, only isometries with even dimensional strata are orientation reversing (and vice versa for even dimensions). With this, we show that the locally orientable and locally non-orientable orbifolds are guaranteed to differ in at least one coefficient, implying different Laplace Spectra.

Acknowledgments

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