Homework 1: Historical Remarks
(**Due Friday September 14**
Total points= 130)

September 7, 2018

1 THE BLACK-BODY RADIATION (110 POINTS)

In this problem you will fill in the gaps and derive a few useful relations for the black-body radiation. As was emphasized in the class, understanding the the black-body radiation within classical mechanics is not possible. In fact, the black-body radiation was one of the first puzzles that led to the birth of quantum mechanics.

1. (**10 points**) A black-body can be thought of as a cavity that is in thermal equilibrium within itself and the surrounding medium, and hence it absorbs and emits radiation at the same rate. A black body is an idealization of many realistic physical systems. The sun is an example of a black body. **Explain** why the sun is considered a black body. **Give two** more examples of black bodies.

2. (**35 points**) In this part of the problem you will count the number of the radiation modes in the range of frequencies between ν and $\nu+d\nu$. Before doing this, let me pause here to explain to you what we mean by the number of radiation modes. When heated up, a black body emits electromagnetic radiation of all frequencies (e.g., the sun light is a continuous spectrum of all colors). Every frequency (color) is called a radiation mode. This is part of the jargon you need to learn. Then, the immediate question is how to compute the density of the radiation modes, or in other words, the number of the radiation modes in the range of frequencies between ν and $\nu+d\nu$. To this end,

think of the radiation inside the cavity (again, the cavity is an idealization of the black-body) as a collection of oscillators (the oscillators are waves) of frequencies ν, where ν is a positive real number, $0 < \nu < \infty$. In addition, take the cavity to be a box of length L. As you may recall from previous courses, the mathematical form of wave is

$$\Phi(\mathbf{x}, t) = \sin\left(\mathbf{q}\cdot\mathbf{x} - \omega t\right) , \tag{1.1}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{q} = (q_x, q_y, q_z)$, and the magnitude of the wave vector $|\mathbf{q}|$ is given by $|\mathbf{q}| = \frac{2\pi\nu}{c}$, where c is the speed of light. Now, we require Φ to satisfy certain conditions at the boundaries of the box. In particular, we want Φ to satisfy periodic boundary conditions, i.e., $\Phi(\mathbf{x} = (0, y, z), t) = \Phi(\mathbf{x} = (L, y, z), t)$, $\frac{\partial\Phi}{\partial x}(\mathbf{x} = (0, y, z), t) = \frac{\partial\Phi}{\partial x}(\mathbf{x} = (L, y, z), t)$. There are also additional conditions we obtain by replacing x with y and z.

Show that the wave vectors that are compatible with the periodic boundary conditions are given by

$$\mathbf{q} = \frac{2\pi\mathbf{n}}{L} , \tag{1.2}$$

where $\mathbf{n} = (n_x, n_y, n_z)$ are integers, e.g., $n_x = ..., -2, -1, 0, 1, 2, ...$ If you have not seen the periodic boundary conditions before you may feel uneasy about them. However, such boundary conditions are very natural. If you are curious to know why we apply these conditions to the black-body problem you may stop by my office. (**Hint**: It is easier to study the problem in one dimension and then to generalize it to three dimensions.)

Now, think of the vector (n_x, n_y, n_z) that lives in a three-dimensional space of integer numbers. **D**raw a sketch of this space and the vector. Now consider a spherical shell in this space. **Show** that a spherical shell of radii $|\mathbf{n}|$ and $|\mathbf{n}| + d|\mathbf{n}|$, where $d|\mathbf{n}|$ is an infinitesimal radius, has infinitesimal volume

$$dV = 4\pi|\mathbf{n}|^2 d|\mathbf{n}| . \tag{1.3}$$

Notice that the spherical shell has nothing to do with the cubic cavity.

Now, using the relation $|\mathbf{q}| = \frac{2\pi\nu}{c}$ **s**how that

$$dV = 4\pi\left(\frac{L}{c}\right)^3 \nu^2 d\nu . \tag{1.4}$$

In fact, this differential formula gives the number of modes between frequencies ν and $\nu + d\nu$. However, this formula was derived assuming that the oscillators are scalars, Φ. As we know, the electromagnetic radiation is carried by vector fields (electric and magnetic fields), and hence, we have to multiply the above formula by 2 to account for the two polarizations of the electromagnetic wave. Thus, we finally have the total number of modes between frequencies ν and $\nu + d\nu$

$$N(\nu)d\nu \equiv 2dV = 8\pi\left(\frac{L}{c}\right)^3 \nu^2 d\nu . \tag{1.5}$$

We can also obtain the number of modes per unit volume (the number density), $n(\nu)$, by dividing the right hand side above by the volume of the box L^3. We then obtain the formula we used in the class

$$n(\nu) = \frac{8\pi}{c^3}\nu^2 . \tag{1.6}$$

Therefore, as you can see the number density is independent of the cavity length L, and hence, is independent of the geometry. The cubic cavity was taken as a simple geometrical shape to derive the above formula.

In fact, although the modes are quantized, i.e., $n = ..., -2, -1, 0, 1, 2, ..$ until now there is nothing quantum mechanical since the Planck's constant h has not been used yet. The fact that we have mode quantization is the result of applying certain boundary conditions. This is not different from applying fixed boundary conditions, i.e., $\Phi(x = 0, t) = \Phi(x = L, t) = 0$, when studying the standing modes of a string in a guitar.

3. (**35 points**) Now, it is time to obtain Planck's black-body formula. In order to do that, we need two pieces of information. First, we need to use the fundamental result of statistical mechanics due to William Gibbs (1839-1903) which states that in a system that contains many identical copies of objects (like the oscillators of the electromagnetic radiation), the probability to find an oscillator of energy E at temperature T is proportional to $e^{-\frac{E}{K_B T}}$, where K_B is the Boltzmann constant. **Show** that the probability that an oscillator has energy E is

$$P(E) = \frac{e^{-\frac{E}{K_B T}}}{\sum_E e^{-\frac{E}{K_B T}}} . \tag{1.7}$$

(**Hint**: Think about the total probability, $\sum_E P(E)$, which is the sum of all probabilities. What the total probability should be?).

The second piece of information is that the energy of the oscillators is quantized $E = Nh\nu$, where $N = 0, 1, 2, ...$ This second statement is quantum mechanical since we are using the Planck's constant h. As we know from statistics, the average energy is then given by

$$\bar{E} = \sum_E EP(E) = \frac{\sum_{N=0}^{\infty} Nh\nu e^{-\frac{Nh\nu}{K_B T}}}{\sum_{N=0}^{\infty} e^{-\frac{Nh\nu}{K_B T}}} . \tag{1.8}$$

Perform the sum and **s**how that

$$\bar{E}(\nu) = \frac{h\nu}{e^{\frac{h\nu}{K_B T}} - 1} . \tag{1.9}$$

(**Hint**: The series $\sum_{N=0}^{\infty} Ne^{-Nx}$ is proportional to the derivative of the series $\sum_{N=0}^{\infty} e^{-Nx}$ with respect to x.)

Now, multiplying the average energy at frequency ν, $\bar{E}(\nu)$, by the number density of modes, $n(\nu)$, we obtain the Planck's formula

$$\rho(\nu,T)d\nu \equiv \bar{E}n(\nu) = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{\frac{h\nu}{K_B T}}-1}. \tag{1.10}$$

the quantity $\rho(\nu,T)$ is the energy density of radiation, or the energy per unit volume, in the frequency interval ν and $\nu + d\nu$.

4. **This part is more challenging (30 points)**. Now, **perform** the integral

$$\int_{\nu=0}^{\infty}\rho(\nu,T)d\nu \tag{1.11}$$

and **s**how that

$$\int_{\nu=0}^{\infty}\rho(\nu,T)d\nu = \frac{8\pi^5 k_B^4}{15h^3c^3}T^4. \tag{1.12}$$

(**Hint**: You first need to expand the denominator of equation (1.10) as a Taylor series in $e^{-\frac{h\nu}{K_B T}}$, then integrate every term in the series, and finally sum the results of the integrals of all terms. You will need the sum $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.)

This shows that the energy density is finite and is free from *the ultraviolet catastrophe* that one experiences in classical mechanics.

2 Bohr's atom (20 points)

The total energy of the electron in the Hydrogen atom is its kinetic plus potential energies:

$$E = \frac{mv^2}{2} + V(r), \tag{2.1}$$

where $V(r) = \frac{-k_e e^2}{r}$ is the Coulomb potential between the electron and proton, where k_e is the Coulomb constant. In addition, Bohr applied Newton's second law to an electron orbiting the proton in a circular motion $\mathbf{F} = m\mathbf{a}$, where in this simple case $|\mathbf{a}| = v^2/r$, where v is the electron velocity and r is the radius of the circular orbit. By combining this information along with Bohr's quantization rule of the angular momentum, $mvr = nh/(2\pi)$, **s**how that the energy of the electron is given by

$$E_n = -\frac{k_e^2 e^4 m}{2n^2\hbar^2}, \tag{2.2}$$

where $\hbar = \frac{h}{2\pi}$.

Plug in the numerical values of e, m, k_e, and $\hbar$ and **show that**

$$E_n = -\frac{13.6}{n^2}\text{ev}. \tag{2.3}$$