COMPLEX VARIABLES: HOMEWORK DUE MONDAY 1/29/2018

(1) Describe in geometric terms the effect of the following linear mappings.
  (a) $f(z) = -z + 2 - i$;
  (b) $f(z) = (\sqrt{3} - i)z$;
  (c) $f(z) = i(z + 1)$;
  (d) $f(z) = (1 - i)(z - i) + i$.

(2) Express the following as linear transformations of the complex plane.
  - dilation by a factor of 2, centered at $(1, 1)$;
  - counter-clockwise rotation for 60^0 centered at $(1, 1)$.

(3) Express the following as a function of the complex plane.
  - reflection with respect to the y-axis;
  - reflection with respect to the line $y = x$.

(4) Let $w = f(z) = (1 + i)z + i$. Find / draw the image, under f, of
  (a) The disk $|z - 1| < 1$;
  (b) The half-plane $\mathrm{Im}(z) > 1$.

(5) Find the linear transformations $w = f(z)$ that satisfy the following conditions:
  (a) The points $z_1 = 2$ and $z_2 = -3i$ map onto $w_1 = 1 + i$ and $w_2 = 1$.
  (b) The circle $|z| = 1$ maps onto the circle $|w - 3 + 2i| = 5$, and $f(-i) = 3 + 3i$.

(6) Find / draw the images of the following under the mapping $w = z^2$.
  (a) The horizontal line $\mathrm{Im}(z) = 1$.
  (b) The vertical line $\mathrm{Re}(z) = 2$.
  (c) The rectangle $\{z \in \mathbb{C} \,\big|\, 0 < \mathrm{Re}(z) < 2,\ 0 < \mathrm{Im}(z) < 1\}$.
  (d) The region in the right half-plane to the right of the hyperbola $x^2 - y^2 = 1$.

(7) Very roughly sketch the regions of the complex plane given by the following. Please don't bother me with equations.
  (a) $\mathrm{Re}(z^2) > 1$;
  (b) $2 < \mathrm{Im}(z^2) < 6$;
  (c) $|z^2 - 2| = 1$;
  (d) $|z^2 - 2| = 2$;
  (e) $|z^2 - 2| = 4$;
  (f) $|z^2 - 2| = r$ as a function of r. (As in: tell me how the shapes change as r changes).

(8) Find / draw the images of the following under the mapping $w = \text{P.V}\, z^{\frac{1}{2}}$.

(a) The horizontal line $\text{Im}(z) = 1$.

(b) The vertical line $\text{Re}(z) = 2$.

(c) The rectangle $\{z \in \mathbb{C} \,\big|\, 0 < \text{Re}(z) < 2,\ 0 < \text{Im}(z) < 1\}$.

(9) Very roughly sketch the regions of the complex plane given by the following. Please don't bother me with equations.

(a) $\text{Re}(\text{P.V}\, z^{\frac{1}{2}}) > 1$;

(b) $2 < \text{Im}(\text{P.V}\, z^{\frac{1}{2}}) < 6$;

(c) $|\text{P.V}\, z^{\frac{1}{2}} - 2| = 1$;

(d) $|\text{P.V}\, z^{\frac{1}{2}} - 2| = 2$;

(e) $|\text{P.V}\, z^{\frac{1}{2}} - 2| = 4$;

(f) $|\text{P.V}\, z^{\frac{1}{2}} - 2| = r$ as a function of r, as r changes from 0 to 2. (Don't go beyond 2 – it gets confusing.)

(10) Consider the region R of the complex plane given by

$$\tfrac{1}{2} \leq |z| \leq 2, \quad \frac{\pi}{4} \leq \text{Arg}(z) \leq \frac{3\pi}{4}.$$

Find / draw the image of R under the following mappings:

(a) $f(z) = 2z^2 - 1$;

(b) $f(z) = (1 - i)z^3$;

(c) $f(z) = \text{P.V}\,(iz)^{\frac{1}{2}}$.

(11) Examine the existence of the following limits; justify any claims you make.

(a) $\lim_{z \to -1} \text{P.V}\, z^{\frac{1}{3}}$;

(b) $\lim_{z \to 0} \frac{\bar{z}}{z}$;

(c) $\lim_{z \to 1} \text{P.V}\, z^{\frac{1}{3}}$;

(d) $\lim_{z \to i} \frac{\bar{z}}{z}$.

(12) Examine the continuity of the function $f(z) = \text{Arg}(z)$; justify any claims you make.