Homework 7: The Free Particle and step potential (**Due Friday November 2**)

October 26, 2018

1 GAUSSIAN WAVE PACKET (80 POINTS)

An initial Gaussian wave packet of a free particle is given by

$$\Psi(x,0) = \frac{1}{a^{1/2}(2\pi)^{1/4}} e^{ik_0 x} e^{-\frac{x^2}{4a^2}}, \tag{1.1}$$

where a is the spread of the wave packet.

1. **(10 points)** **Find** $\langle\hat{x}\rangle(t=0)$ **and** $\langle\hat{p}\rangle(t=0)$. **Calculate** the uncertainty in momentum and position and show that the uncertainties respect the Heisenberg principle.
2. **(10 points)** **Show** that $\Psi(x,t)$ can be written as

$$\Psi(x,t) = \int_{-\infty}^{\infty} dx' \Psi(x',0) K(x',x,t), \tag{1.2}$$

where

$$K(x',x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{i\left[k(x-x') - \frac{k^2 a^2 t}{\tau}\right]}, \tag{1.3}$$

where $\tau = \frac{2ma^2}{\hbar}$.

3. **(10 points)** **Perform** the integral in (1.3) and **show** that

$$K(x', x, t) = \sqrt{\frac{m}{2\pi i\hbar t}}\, e^{\frac{im(x-x')^2}{2\hbar t}} \tag{1.4}$$

   Hint: you can use the integral $\int_{-\infty}^{\infty} e^{-ay^2} e^{by}\, dy = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}$.

4. **(25 points)** **Perform** the integral in (1.2) and **show** that

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\, a\left(1+\frac{t^2}{\tau^2}\right)^{1/2}} \exp\left[-\frac{\left(x-\frac{\hbar k_0 t}{m}\right)^2}{2a^2\left(1+\frac{t^2}{\tau^2}\right)}\right] \tag{1.5}$$

5. **(10 points)** **Plot** $|\Psi(x,t)|^2$ for $t = 0, t = \tau, t = 2\tau$. τ is called the relaxation time, **why?**

6. **(15 points)** Now substitute $p = \hbar k_0$ in (1.5) and after that take the limit $\hbar \to 0$ (keeping the momentum fixed). **What** you will get as $a \to 0$? This is the classical free particle corresponding to a free wave packet. (**Hint: look up the mathematical definition of the Dirac-delta function**)

2 Step potential (20 points)

A beam of electrons with density $\rho = 10^7$ electrons/m is accelerated through 1000 V potential. Then, the beam is incident on a repulsive step potential with $V_0 = 100$ V. **Draw** the electrostatic configuration that can realize this problem. **Find** the incident, reflected and transmitted currents.