Chapter 9

Waves on the sphere

In this chapter we study the Legendre Sturm-Liouville problem. This problem appears when considering the problem of waves on the surface of a sphere.

9.1 The wave equation on the sphere

In order to derive the wave equation on the sphere we first construct the energy for the wave equation in spherical coordinates

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi. \end{aligned}$$

Using this we can compute that for any function u we have

$$\begin{aligned} \|\operatorname{grad} u\|^2 &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \\ &= \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2\sin^2\phi}\left(\frac{\partial u}{\partial \theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \phi}\right)^2 \end{aligned}$$

If we want to consider the wave equation on the surface of the sphere, then the radius is a constant. For convenience we set $r = 1$. Furthermore, the function u is no longer a function of r. Thus on the surface of the unit sphere we have

$$\|\operatorname{grad} u\|^2 = \frac{1}{\sin^2\phi}\left(\frac{\partial u}{\partial \theta}\right)^2 + \left(\frac{\partial u}{\partial \phi}\right)^2$$

The potential energy for a wave given by $u(t, \theta, \phi)$ on the surface of the sphere is

$$V = \frac{1}{2} \int_{S^2} \| \operatorname{grad} u \|^2 dA.$$

(The mathematical symbol for the two-dimensional sphere is S^2.) In order to compute this in coordinates we need to compute the area element dA. We do this using the methods from calculus 3. The sphere is parametrized by

$$\begin{aligned} x &= \cos\theta \sin\phi \\ y &= \sin\theta \sin\phi \\ z &= \cos\phi, \end{aligned}$$

where

$$-\pi \le \theta \le \pi \quad \text{and} \quad 0 \le \phi \le \pi.$$

We have

$$\partial_\theta = \begin{pmatrix} -\sin\theta\sin\phi \\ \cos\theta\sin\phi \\ 0 \end{pmatrix} \quad \text{and} \quad \partial_\phi = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{pmatrix}$$

Thus

$$dA = \sqrt{\det \begin{pmatrix} \partial_\theta \cdot \partial_\theta & \partial_\theta \cdot \partial_\phi \\ \partial_\phi \cdot \partial_\theta & \partial_\phi \cdot \partial_\phi \end{pmatrix}} = \sin\phi \, d\theta \, d\phi.$$

From this we see that the potential energy for waves on the surface of the sphere is

$$V = \frac{1}{2} \int_0^\pi \int_{-\pi}^{\pi} \left(\frac{1}{\sin^2\phi} \left(\frac{\partial u}{\partial \theta} \right)^2 + \left(\frac{\partial u}{\partial \phi} \right)^2 \right) \sin\phi d\theta d\phi \tag{9.1}$$

Thus the total energy (kinetic plus potential) for a wave $u(t, \theta, \phi)$ on the surface of the unit sphere is

$$E = \frac{1}{2} \int_0^\pi \int_{-\pi}^{\pi} \left(\left(\frac{\partial u}{\partial t} \right)^2 + \frac{1}{\sin^2\phi} \left(\frac{\partial u}{\partial \theta} \right)^2 + \left(\frac{\partial u}{\partial \phi} \right)^2 \right) \sin\phi d\theta d\phi$$

In order to obtain the wave equation, we set $\frac{d}{dt} E = 0$. Using integration-by-parts, we see that the resulting wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\sin\phi} \frac{\partial}{\partial \phi} \left(\sin\phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\sin^2\phi} \frac{\partial^2 u}{\partial \theta^2}. \tag{9.2}$$ `Sphere:wave-eqn`

The computations in Exercise 9.1.1 below show that $u(t,\theta,\phi) = A(t)\psi(\theta,\phi)$ precisely when

$$\frac{d^2A}{dt^2} = -\omega^2$$

and

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\psi}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2\psi}{\partial\theta^2} = -\omega^2\psi \tag{9.3}$$

Sphere:standing-wave-eqn

for some constant $\omega \geq 0$. Then in Exercise 9.1.2 you show that the spherical eigenfunctions Ψ take the form

$$\Phi(\phi)\cos(n\theta) \quad \text{or} \quad \Phi(\phi)\sin(n\theta)$$

where $n = 1, 2, 3, \ldots$ and where $\Phi(\phi)$ must satisfy

$$\sin\phi\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) = \left(-\omega^2\sin^2\phi + n^2\right)\Phi. \tag{9.4}$$

Sphere:physical-Legendre-

anding-wave-eqn★**Exercise 9.1.1.**

1. *Suppose that $u(t,\theta,\phi) = A(t)\psi(\theta,\phi)$ is a standing wave on the sphere. Show that u satisfies (9.2) precisely when A and ψ satisfy*

$$\frac{d^2A}{dt^2} = \lambda A \quad \textit{and} \quad \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\psi}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2\psi}{\partial\theta^2} = \lambda\psi$$

   for some constant λ.

2. *Multiply the second equation above by ψ, integrate over the sphere, and then integrate by parts in order to show that $\lambda = -\omega^2$ for some constant $\omega \geq 0$.*

-eigenfunctions★**Exercise 9.1.2.** *Suppose that the eigenfunction ψ is of the form $\psi(\theta,\phi) = \Theta(\theta)\Phi(\phi)$.*

1. *Explain what boundary conditions Φ and Θ must satisfy.*

2. *Show that*

$$\frac{1}{\Phi}\sin\phi\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + \omega^2\sin^2\phi = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}.$$

   Explain why this implies that

$$\sin\phi\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) = \left(-\omega^2\sin^2\phi + \mu\right)\Phi \quad \textit{and} \quad \frac{d^2\Theta}{d\theta^2} = -\mu\Theta$$

   for some constant μ.

3. *Explain why $\mu = n^2$ for some integer n and that Θ must be one of $\cos(n\theta)$ or $\sin(n\theta)$.*

9.2 The Legendre Sturm-Liouville problem

The equation (9.4), together with the boundary condition that requires $\Phi(0)$ and $\Phi(\pi)$ to be finite, can be transformed to a Sturm-Liouville problem. To do this, we look for solutions of the form

$$\Phi(\phi) = P(\cos\phi) \tag{9.5}$$

`Sphere:cosine-s`

and introduce the new variable $\xi = \cos\phi$. Since $0 \le \phi \le \pi$ we have $-1 \le \xi \le 1$. The function P must satisfy differential equation

$$\frac{d}{d\xi}\left((1-\xi^2)\frac{dP}{d\xi}\right) - \frac{n^2}{1-\xi^2}P = \lambda P, \tag{9.6}$$

`Sphere:Associat`

where $\lambda = -\omega^2$, as well as the boundary conditions

$$P(-1) \text{ and } P(1) \text{ finite.} \tag{9.7}$$

`Sphere:Associat`

The eigenvalue problem (9.5)-(9.6) is called the ***Associated Legendre Problem***. In the case that $n = 0$, the problem is simply called the ***Legendre Problem***.

It is straightforward to verify that for each fixed value of n the Associated Legendre Problem is an admissible Sturm-Liouville problem on the domain $\Omega = (-1, 1)$ with weight function is $w = 1$. The operator L takes the form (8.7) with $p = 1 - \xi^2$ and $r = -n^2/(1-\xi^2)$. Thus the negative condition (8.8) and the boundary condition (8.11) are satisfied.

Since the Associated Legendre Problem satisfies the criteria of the Sturm-Liouville Theorem, we know that there exists a sequence of eigenvalues $0 \ge \lambda_1 > \lambda_2 > \lambda_3 > \ldots$ with $\lambda_k \to -\infty$ and corresponding eigenfunctions $P_1, P_2, P_3, \ldots$ satisfying (9.5)-(9.6). Since we technically have a new problem for each value of n, it is common to decorate the symbols for the eigenfunctions P with the n as well. Thus the eigenfunctions are given the symbol P_1^n, P_2^n, $\ldots$, P_k^n,$\ldots$. These functions are called the ***associated Legendre functions***. The Sturm-Liouville theorem tells us that these eigenvalues and eigenfunctions exist, that they are orthogonal, and that they are complete.

In fact, it is possible to deduce exact formulas for the associated Legendre functions. To do this, we first figure out the leading order behavior by writing the equation (9.5) as

$$(1-\xi^2)\frac{d}{d\xi}\left((1-\xi^2)\frac{dP}{d\xi}\right) - n^2P = \lambda(1-\xi^2)P. \tag{9.8}$$

`AL-ode`

We then suppose that the leading order behavior of P is $(1-\xi^2)^p$ for some power p. That is, we suppose that

$$P = (1-\xi^2)^p + \text{higher powers of } (1-\xi^2).$$

Inserting this in to (9.7) yields

$$4p^2(1-\xi^2)^p - n^2(1-\xi^2)^p + \text{higher powers} = \text{higher powers}.$$

From this we conclude that we must have $4p^2 = n^2$, meaning that

$$P(\xi) = (1-\xi^2)^{n/2} Q(\xi) \tag{9.9}$$ AL-ansatz

for some function Q that is bounded at $\xi = \pm 1$.

Our task is now to find a formula for Q. To do this we first derive an equation that Q must satisfy. Inserting (9.8) into (9.7) yields

$$(1-\xi^2)\frac{d^2Q}{d\xi^2} - 2(n+1)\xi\frac{dQ}{d\xi} - n(n+1)Q = \lambda Q. \tag{9.10}$$ AL:reduced-ode

Next, we assume that Q has a power series expansion

$$Q(\xi) = \sum_{i=0}^{\infty} a_i \xi^i. \tag{9.11}$$ AL:reduced-series

Inserting this in to (9.9) we find that

$$\sum_i \left(a_{i+2}(i+2)(i+1) - a_i\left((n+i)(n+i+1)+\lambda\right)\right)\xi^i = 0$$

Thus the coefficients a_i must satisfy the recursion relation

$$a_{i+2} = \frac{(n+i)(n+i+1)+\lambda}{(i+2)(i+1)}\, a_i \tag{9.12}$$ AL:recursion-relation

This recursion relation implies that all of the even-indexed coefficients depend on a_0 and all of the odd-indexed coefficients depend on a_1. That is

$$Q(\xi) = \sum_{j=0}^{\infty} a_{2j}\xi^{2j} + \sum_{j=0}^{\infty} a_{2j+1}\xi^{2j+1}. \tag{9.13}$$ AL:split-series

In order for the function $Q(\xi)$ to be bounded at $\xi = \pm 1$ we need both the series of even powers and the series of odd powers to converge absolutely. Using the Ratio Test, we get absolute convergence at $\xi = \pm 1$ if

$$\lim_{i\to\infty}\left|\frac{a_{i+2}}{a_i}\right| = \lim_{i\to\infty}\left|\frac{(n+i)(n+i+1)+\lambda}{(i+2)(i+1)}\right|$$

converges to some number less than 1. It is easy to see that the above limit is equal to 1 .

Suppose, however, that $\lambda = -k(k+1)$ for some integer $k \geq n$. Then when $i = k-n$ the recursion relation (9.11) indicates that $a_{k-n+2} = 0$. This means that one of the series in (9.12) is actually a finite sum. We can make the entire series a finite sum by choosing the free coefficients a_0, a_1 in a clever manner. Suppose $k-n$ is even. Then the sum of even terms is already finite. Choosing $a_1 = 0$ makes all of the odd terms zero, leaving us with a finite sum. Similarly if $k-n$ is odd then choosing $a_0 = 0$ leaves us with a finite sum.

The result of this discussion is that we have a list of negative eigenvalues

$$\lambda_1 = -2, \lambda_2 = -6, \ldots, \lambda_k = -k(k+1), \ldots$$

for which the sum (9.10) is finite, and thus there is a polynomial solving (9.9). We give this polynomial the symbol Q_l^n. (Technically, we still need to decide how to choose whichever of a_0 or a_1 is non-zero. This is addressed below.) Once we have the polynomial Q_k^n then we can form the function P_k^n according to (9.8). That is

$$P_k^n(\xi) = (1-\xi^2)^{n/2} Q_k^n(\xi).$$

The functions P_k^n are called ***associated Legendre functions of the first kind***, and are the eigenfunctions satisfying the Associated Legendre Problem (9.5)-(9.6) that the Sturm-Liouville Theorem tells us exist.

In the next section we analyze the case $n = 0$ in detail. The following exercise shows that once the $n = 0$ case has been analyzed, then the situation of general n is easy to understand.

educe-associated-Legendre **Exercise 9.2.1.**

1. *Suppose $n \geq 0$. Show that*

$$\frac{d^n}{d\xi^n}\left((1-\xi^2)\frac{d^2P}{d\xi^2}\right) = (1-\xi^2)\frac{d^2}{d\xi^2}\left(\frac{d^nP}{d\xi^n}\right) - 2n\xi\frac{d}{d\xi}\left(\frac{d^nP}{d\xi^n}\right) - n(n-1)\frac{d^nP}{d\xi^n}$$

and that

$$\frac{d^n}{d\xi^n}\left(-2\xi\frac{dP}{d\xi}\right) = -2\xi\frac{d}{d\xi}\left(\frac{d^nP}{d\xi^n}\right) - 2n\left(\frac{d^nP}{d\xi^n}\right).$$

2. *Suppose that P satisfies (9.5) with $n = 0$, which we write as*

$$(1-\xi^2)\frac{d^2P}{d\xi^2} - 2\xi\frac{dP}{d\xi} = \lambda P. \qquad (9.14)$$ hw-legendre-ode

Use the formulas from the first part of this exercise to show that $\frac{d^n P}{d\xi^n}$ *satisfies* (9.9).

3. *Explain why we can fully understand solutions to* (9.9) *by understanding solutions to* (9.13).

9.3 Legendre polynomials

To get a better understanding of what is happening here, let's focus on the case that $n = 0$. In this case the functions P_k^0, which are just polynomials, are called the ***Legendre polynomials*** and are given the symbol P_k. According to the discussion above, we choose $a_0 = 0$ if k is odd and $a_1 = 0$ if k is even. It is a standard convention to choose whichever of a_0 or a_1 is nonzero in such a way that $P_k(1) = 1$. Let's do this systematically.

- Suppose $k = 0$. Since k is even we set $a_1 = 0$. The recursion relation (9.11) implies that a_i is zero for all odd values of i. Since $k = 0$ the eigenvalue is $\lambda 0 = -(0)(0+1) = 0$. Thus the recursion relation implies that $a_2 = 0$ and consequently all higher even-indexed coefficients are zero as well. The result of all this is that

  $$P_0(\xi) = a_0.$$

  Since we want $P_0(1) = 1$ we choose $a_0 = 1$. Thus we arrive at the zeroeth Legendre polynomial

  $$P_0(\xi) = 1.$$

- Now consider the case that $k = 1$. Since k is odd, we set $a_0 = 0$, which implies that all of the even-indexed coefficients are zero. We have $\lambda_1 = -(1)(1+1) = -2$. Using the recursion relation we find that $a_3 = 0$ and thus

  $$P_1(\xi) = a_1\xi.$$

  In order to have $P_1(1) = 1$ we choose $a_1 = 1$. Thus the first Legendre polynomial is

  $$P_1(\xi) = \xi. \tag{9.15}$$

- Now consider the case $k = 2$. Since k is even, we set $a_1 = 0$, which implies all of the odd-indexed coefficients are zero. We have $\lambda_2 = -2(2+1) = -6$. The recursion relation implies that

  $$a_2 = \frac{-6}{2}a_0 \quad \text{and} \quad a_4 = 0.$$

Thus

$$P_2(\xi) = a_0 - 3a_0\xi^2.$$

In order to have $P_2(1) = 1$ we choose $a_0 = -1/2/$. Thus

$$P_2(\xi) = -\frac{1}{2} + \frac{3}{2}\xi^2.$$

If we continue in this fashion, we obtain the following table.

counter	eigenvalue	Legendre polynomial
$k = 0$	$\lambda_0 = 0$	$P_0(\xi) = 1$
$k = 1$	$\lambda_1 = 0$	$P_1(\xi) = \xi$
$k = 2$	$\lambda_2 = 0$	$P_2(\xi) = -\frac{1}{2} + \frac{3}{2}\xi^2$
$k = 3$	$\lambda_3 = 0$	$P_3(\xi) = -\frac{3}{2}\xi + \frac{5}{2}\xi^3$
$k = 4$	$\lambda_4 = 0$	$P_4(\xi) = \frac{3}{8} - \frac{15}{4}\xi^2 + \frac{35}{8}\xi^4$

Exercise 9.3.1. *Fine formulas for P_5 and P_6.*

Exercise 9.3.2. *The Sturm-Liouville Theorem tells us that the Legendre polynomials P_k obtained from an admissible boundary value problem are orthogonal (relative to the standard inner product on the domain $(-1, 1)$) and form a complete collection.*

1. *Verify directly that $\langle P_2, P_3\rangle = 0$ and that $\langle P_2, P_4\rangle = 0$.*

2. *It can be proven that $\|P_k\|^2 = \frac{2}{2k+1}$. Verify this formula for $k = 0, 1, 2, 3$.*

3. *The completeness of the Legendre polynomials implies that any smooth function f can be expressed as*

$$f(x) = \sum_{k=0}^{\infty} \frac{\langle f, P_k\rangle}{\|P_k\|^2} P_k(x).$$

This means that for any value of N the finite sum yields the approximation

$$f(x) \approx \sum_{k=0}^{N} \frac{\langle f, P_k\rangle}{\|P_k\|^2} P_k(x).$$

Compute this approximation when $N = 5$ and $f(x) = \sin(\pi x)$. Demonstrate how well the function is approximated by plotting the approximation against f.

4. *Construct an approximation of e^x using Legendre polynomials.*

Exercise 9.3.3. *From Exercise 9.2.1 we know that the polynomials Q_k^n can be obtained from the Legendre polynomials by the formula*

$$A_k^n = \frac{d^k P_k^n}{d\xi^k}$$

and thus that the eigenfunctions solving the Associated Legendre boundary value problem are given by

$$P_k^n(\xi) = (1-\xi^2)^{n/2} \frac{d^k P_k^n}{d\xi^k}(\xi).$$

1. *What are the eigenvalues corresponding to P_k^n?*
2. *Find formulas for $P_1^1, P_2^1, P_3^1, \dots$.*
3. *Find formulas for $P_1^2, P_2^2, P_3^2, \dots$.*

9.4 Standing waves on the sphere

Finally we are ready to return to the problem of standing waves on a sphere. Recall from Exercise 9.1.2 that the shape of the standing waves is given by

$$\psi(\theta, \phi) = \Phi(\phi)\cos(n\theta) \quad \text{or} \quad \psi(\theta, \phi) = \Phi(\phi)\sin(n\theta),$$

where Φ is a solution to (9.4) and the frequency of these standing waves is the parameter ω appearing in that equation. Using the change of variables in (**??**), we see that the standing waves are given by

$$\psi_{nk}^C(\theta, \phi) = P_k^n(\cos\theta)\cos(n\theta) \quad \text{or} \quad \psi_{nk}^S(\theta, \phi) = P_k^n(\cos\theta)\sin(n\theta)$$

where P_k^n are the associated Legendre functions discussed above. The frequency of these standing waves are given by $\omega = \sqrt{k(k+1)}$.

In the physics literature the functions ψ_{nk} are called ***spherical harmonics*** and given the symbol Y_k^n. The Wikipedia article on spherical harmonics has some nice visualizations of these functions...

Chapter 10

More Sturm-Liouville problems

10.1 Bessel functions

Bessel functions provide another example of a Sturm-Liouville problem. A good exercise is to go through the Bessel-function problem and identify what are the parts of the problem. That is, what is the domain and weight function, what is the operator, etc. Note that, just as is the case for the Associated Legendre problem, there is a whole list of Sturm-Liouville problems present.

10.2 Application to SHO waves

This section is written so that it can be used as a take-home exam and/or a longer homework assignment.

In this exploration we consider waves $u(t,x)$ defined on the domain $\Omega = (-\infty, \infty)$. In place of a boundary condition, we require that

$$\lim_{x\to\infty} u(t,x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} u(t,x) = 0.$$

We make use of a modified energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + x^2 u^2 \right\} dx.$$

1. Show that if $u(t,x)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - x^2 u. \qquad (10.1)$$ HermiteWave

then the modified energy is conserved.

2. We now go looking for standing wave solutions to (11.1). Show that $u(t,x) = A(t)e^{-\frac{1}{2}x^2}\psi(x)$ is a standing wave solution if A and ψ that satisfy

$$\frac{d^2A}{dt^2} = \lambda A \tag{10.2}$$

$$\frac{d^2\psi}{dx^2} - 2x\frac{d\psi}{dx} = (\lambda+1)\psi \tag{10.3}$$

`psi-ode`

for some constant λ. For a boundary condition we require

$$\left[e^{-x^2/2}\psi(x)\right]_{-\infty}^{\infty} = 0 \quad \text{and} \quad \left[e^{-x^2/2}\psi'(x)\right]_{-\infty}^{\infty} = 0. \tag{10.4}$$

`Hermite-BC`

3. Show that (11.3) can be written in "Sturm-Liouville form" with $p(x) = e^{-x^2}$, $w(x) = e^{-x^2}$, and eigenvalue $\lambda + 1$. As a result, what does the Sturm-Liouville theorem tell us?

4. It turns out (for reasons that are beyond the scope of this course) that the Sturm-Liouville eigenvalues are such that $\lambda + 1 = -2n$, where $n = 0, 1, 2, 3, \ldots$. Use the power series method to find the corresponding eigenfunctions $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$. For each n, choose the free constants so that (i) the series terminates and thus ψ_n is a polynomial, and (ii) the leading coefficient of the polynomial is 2^n.

5. Write down the standing wave solutions to (11.1). What is the (modified) energy of each solution? Illustrate the standing wave solutions.

Commentary:

1. *The x^2u^2 term in the modified energy can be viewed as a "potential energy" term where the potential energy goes like the square of the distance to the origin. This is the same form that the potential for the simple harmonic oscillator; thus (11.1) can be viewed as the wave version of a simple harmonic oscillator. In what sense are your solutions consistent with this interpretation?*

2. *The polynomials ψ_n are known as the Hermite polynomials.*