PROOF. Take a compact Hausdorff space X. It must be verified that the set of stable isomorphism classes of vector bundles over X with operations defined by the direct sum $\oplus$ and the tensor product $\otimes$ indeed satisfies all the properties of a commutative semiring with additive cancellation.

Before proceeding further, it must be verified that addition is well defined. So, take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X. Then, take n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Then it follows that $E_1 \oplus F_1 \approx_s E_2 \oplus F_2$ by the following chain of equalities.

$$(E_1 \oplus F_1) \oplus \varepsilon^{n+m} \approx (E_1 \oplus \varepsilon^n) \oplus (F_1 \oplus \varepsilon^m) \approx (E_2 \oplus \varepsilon^n) \oplus (F_2 \oplus \varepsilon^m) \approx (E_2 \oplus F_2) \oplus \varepsilon^{n+m}$$

Where the equivalence $\varepsilon^{n+m} \approx \varepsilon^n \oplus \varepsilon^m$ /*reference*/ was used.

With $\oplus$ well defined, the associativity and commutativity of addition follows directly from the associativity and commutativity of the direct sum on vector bundles /*reference*/. Further, the additive identity is given by the 0 dimensional trivial bundle ε^0.

The additive cancellation follows from /*reference $E \oplus E'$ trivial result*/, which applies here by X compact Hausdorff. Indeed, take bundles E, F, and S over X such that $[E] + [S] = [F] + [S]$. First note that in the case of S trivial, $[E] = [F]$ by definition. Otherwise, by /*ref*/, there exists a bundle S such that $S \oplus S'$ is trivial. Adding $[S']$ to both sides reduces the expression to the first case with $[E] + [S \oplus S'] = [F] + [S \oplus S']$, giving $[E] = [F]$ as desired.

Before proceeding with any multiplicative verifications, it must be verified that the tensor product $\otimes$ gives a well defined multiplicative operation. So, again take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X. Then, take n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition.

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