Calculus III

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1 2D and 3D space

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$$f(x,y) = x^2 + y^2 \tag{Paraboloid}$$
$$f(x,y) = x^2 - y^2 \tag{Saddle}$$
$$f(x,y) = xy \tag{Saddle}$$

Theorem 1.1. An equation of the form $f(x,y) = ax^2+bxy+cy^2$ is a saddle if $b^2 - 4ac > 0$ and a bowl if $b^2 - 4ac < 0$.

1.4 Vector Fields

Definition 1.2 (Vector Field)**.** A vector field is some mapping $V : S \to \mathbb{R}^n$ with $S \subset \mathbb{R}^n$. Visually, we represent vector fields by drawing the vector of the output centered at the input point. /*Visual vector field*/

1.5 Transformations

Definition 1.3 (Transformation)**.** A transformation is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$. We represent this mapping $\mathbf{x} \to \mathbf{y}$ with an equation of the form

$$(y^1, \dots, y^m) = T(x^1, \dots, x^n) = (T_1(x^i), \dots, T_m(x^i))$$

Note that *linear-ish* transformations can be written

$$\begin{aligned}
\begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} &= \begin{pmatrix} T_1(x^1, \dots, x^n) \\ \vdots \\ T_m(x^1, \dots, x^n) \end{pmatrix} = \begin{pmatrix} b^1 + \sum a_i^1 x^i \\ \vdots \\ b^m + \sum a_i^m x^i \end{pmatrix} = \begin{pmatrix} b^1 \\ \vdots \\ b^m \end{pmatrix} + \sum \begin{pmatrix} a_i^1 \\ \vdots \\ a_i^m \end{pmatrix} x^i \\
&= \begin{pmatrix} b^1 \\ \vdots \\ b^m \end{pmatrix} + \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_n^m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}$$

1.6 Projection

Theorem 1.4.

$$\mathbf{a_H} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}$$

1.7 Vector Products

Theorem 1.5. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ separated by some angle θ, we have

$$\sum u^i v^i = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$$

Definition 1.6 (Dot Product)**.** The previous equivalency is quite useful, so we define the *dot product* by the mapping $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$u \cdot v = \sum u^i v^i = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$$

Where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ separated by angle θ.

Definition 1.7 (Cross Product)**.** The *cross product* is some mapping $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ such that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $a = \langle a_x, a_y, a_z \rangle$ and $v = \langle b_x, b_y, b_z \rangle$ we have

$$a \times b = \langle (a_y b_z - a_z b_y), (a_z b_x - a_x b_z), (a_x b_y - a_y b_x) \rangle.$$

By abuse of notation, we can write the cross product as:

$$a \times b = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

Note that:

- $\|\mathbf{a} \times \mathbf{b}\|$ = Area of parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a} \times \mathbf{b}$ point in a direction orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

1.8 Volumes

Theorem 1.8 (Volumes of Parallelotopes)**.** Let $V(\mathbf{x}^1, \ldots, \mathbf{x}^n)$ = Volume of the paralleletope spanned by $\mathbf{x}^1, \ldots, \mathbf{x}^n$. Then,

$$V(\mathbf{x}^1, \ldots, \mathbf{x}^n) = \sqrt{\det \begin{pmatrix} \mathbf{x}^1 \cdot \mathbf{x}^1 & \cdots & \mathbf{x}^1 \cdot \mathbf{x}^n \\ \vdots & \ddots & \vdots \\ \mathbf{x}^n \cdot \mathbf{x}^1 & \cdots & \mathbf{x}^n \cdot \mathbf{x}^n \end{pmatrix}}$$

Note if we call the matrix g we have $g_{ij} = \mathbf{x}^{\mathbf{i}} \cdot \mathbf{x}^{\mathbf{j}}$.
If we are working in $\mathbb{R}^n$ and $\mathbf{x}^{\mathbf{i}} = \langle x_1^i, \ldots, x_n^i \rangle$ then,

$$V(\mathbf{x}^1, \ldots, \mathbf{x}^n) = \left| \det \begin{pmatrix} x_1^1 & \cdots & x_n^1 \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{pmatrix} \right|$$

2 ???

2.1 Curvilinear Coordinates

2.1.1 Polar Coordinates

Definition 2.1 (Polar Coordinates)**.** Polar coordinates are defined by a transformation from (r, θ) space to (x, y) space.
Below are transformations $T : (r, \theta) \to (x, y)$ and $T^{-1} : (x, y) \to (r, \theta)$

$$T : (x, y) = (r\cos(\theta), r\sin(\theta))$$
$$T^{-1} : (r, \theta) = (\sqrt{x^2 + y^2}, \arctan(y/x)[+\pi?])$$

Note: In T^{-1} you must add π radians to θ if $x < 0$.

2.1.2 Cylindrical Coordinates

Definition 2.2 (Cylindrical Coordinates)**.** Cylindrical coordinates are defined through a transformation from (r, θ, z) space to (x, y, z) space. Below are transformations $T : (r, \theta, z) \to (x, y, z)$ and $T^{-1} : (x, y, z) \to (r, \theta, z)$

$$T : (x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$
$$T^{-1} : (r, \theta, z) = (\sqrt{x^2 + y^2}, \arctan(y/x)[+\pi?], z)$$

Note: In T^{-1} you must add π radians to θ if $x < 0$.

2.1.3 Spherical Coordinates

Spherical coordinates are defined through a transformation from (r, θ, ϕ) space to (x, y, z) space. Below are transformations $T : (r, \theta, \phi) \to (x, y, z)$ and $T^{-1} : (x, y, z) \to (r, \theta, \phi)$

$$T : (x, y, z) = (r\cos\theta\sin\phi, r\cos\theta\sin\phi, r\cos\phi)$$
$$T^{-1} : (x, y, z) = (\sqrt{x^2 + y^2 + z^2}, \arctan(y/x)[+\pi?], \arccos(\frac{z}{\sqrt{x^2 + y^2 + z^2}}))$$

2.2 Partial Vectors

Definition 2.3 (Partial Vector Field)**.** Consider some transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $(y^1, \ldots, y^m) = T(x^1, \ldots, x^n)$. We then define the partial vector field for some coordinate x^i in y space, denoted ∂_{x^i}, by

$$\partial_{x_i} = \left\langle \frac{\partial y^1}{\partial x^i}, \ldots, \frac{\partial y^m}{\partial x^i} \right\rangle$$

Note that ∂_{x_i} points in the direction of increase of x_i at any location $\mathbf{y}$.

2.3 Transformation Approximation

2.3.1 Linear

Definition 2.4 (Jacobi Matrix)**.** Consider some transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $(y^1, \ldots, y^n) = T(x^1, \ldots, x^m)$. We define the Jacobi Matrix DT to be an $n \times m$ matrix such that $DT_{ij} = \frac{\partial y^j}{\partial x^i}$ Or,

$$DT = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^m}{\partial x^1} & \cdots & \frac{\partial y^m}{\partial x^n} \end{pmatrix}$$

Theorem 2.5. If we evaluate DT at a specific point $\mathbf{p}$ in x space, denoted $DT|_{\mathbf{p}}$, then this matrix is a linear approximation of T at $\mathbf{p}$. So if $\mathbf{\Delta x} = \mathbf{x} - \mathbf{p}$ and $\mathbf{\Delta y} = T(\mathbf{x}) - T(\mathbf{p})$ then,

$$\mathbf{\Delta y} \approx DT|_{\mathbf{p}} \cdot \mathbf{\Delta x}$$

2.3.2 Quadratic

Definition 2.6 (Hessian Matrix)**.** Consider some transformation $f : \mathbb{R}^n \to \mathbb{R}$ given by $z = f(x^1, \ldots, x^n)$. Then, we define the Hessian Matrix $Hessf$ to an $n \times n$ matrix given by $Hessf_{ij} = \frac{\partial f}{\partial x^i x^j}$. Or,

$$Hessf = \begin{pmatrix} \frac{\partial f}{\partial x^1 x^1} & \cdots & \frac{\partial f}{\partial x^1 x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x^n x^1} & \cdots & \frac{\partial f}{\partial x^n x^n} \end{pmatrix}$$

Theorem 2.7 (Quadratic Approximation)**.** Take some point $\mathbf{p}$ in x space to center our approximation about. Then if we take $\mathbf{\Delta x} = \mathbf{x} - \mathbf{p}$ and $\Delta y = f(\mathbf{x}) - f(\mathbf{p})$ we have,

$$\Delta y \approx Df|_{\mathbf{p}} \cdot \mathbf{\Delta x} + \frac{1}{2} \cdot \mathbf{\Delta x}^{\mathbf{T}} \cdot Hessf|_{\mathbf{p}} \cdot \mathbf{\Delta x}$$

2.3.3 General?

2.4 Chain Rule

2.5 Optimization

3 Calculus

3.1 Line Integral

Theorem 3.1. Let $\mathcal{L} = (x^1(t), \ldots, x^n(t))$ describe some line in $\mathbb{R}^n$. Then, the length of the line between between $t = t_i$ and $t = t_f$ is given by

$$\int_{t_i}^{t_f} \|\partial_t\| dt$$

If we assign a density ρ