Quantum

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1 Historical Remarks

1.1 Ultraviolet Catastrophe

Definition 1.1 (Black Body)**.** A *black body* is some object that satisfies the following conditions:

- Absorbs all incoming radiation
- Emits the entire spectrum
- Is at equilibrium (temperature does not change)

We model Electromagnetic radiation with infinite *harmonic oscillators*. We model a harmonic oscillator of frequency ν mathematically with:

$$\Phi(x,t) = A\cos(\omega t - kx) \qquad \text{(Harmonic Oscillator)}$$

Where the wave number $k \equiv \frac{2\pi}{\nu}$, the angular frequency $\omega \equiv 2\pi\nu$ and we have $\lambda = \frac{c}{\nu}$. The energy of a harmonic oscillator is given by

$$\overline{E_{h.o.}} = k_B T \tag{1}$$

We find that the number density of harmonic oscillators per volume at a specific frequency ν is given by:

$$n(\nu) = \frac{8\pi\nu^2}{c^3} \tag{2}$$

We can then combine (1) and (2) to find the energy density ρ at temperature T and frequency ν:

$$\rho(\nu, T) = \overline{E_{h.o.}} n(\nu) = \frac{8\pi\nu^2 k_B T}{c^3}$$

However, we then have that the total energy of a black body is given by

$$E_{total} = \int_0^\infty \rho(\nu, T) d\nu = \frac{8\pi k_B T}{c^3} \int_0^\infty \nu^2 d\nu = \infty$$

So, the current model predicts infinite energy of a black body. This clearly incorrect result is called the *ultraviolet catastrophe*. So, we must change our model.

Planck solved this in assuming that oscillators of the black body are quantized. They can only be in energy states $E = nh\nu$, $n \in \mathbb{Z}_{\geq 0}$ (Einstein later showed that it is the E.M. radiation itself that is quantized). Under this assumption we find that the energy density is given by

black body curve

$$\rho(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/(k_B T)} - 1} \tag{Plank's Formula}$$

Which fits the experimental black body curve. Note: each $h\nu$ "package" of energy is called a photon.

1.2 Atomic Spectrum

Atomic gases were experimentally found to emit light at only discrete frequencies. Bohr's model offered an explanation for this. Bohr adopted Rutherford's model, of an orbiting electron (this model was rejected because maxwell's equations predict the electron would emit radiation making it unstable). Bohr assumes that angular momentum is quantized. Such that:

$$|\mathbf{L}| = |\mathbf{r} \times \mathbf{p}| = n\hbar$$

Where $n \in \mathbb{N}$. Note $\hbar = \frac{h}{2\pi}$. Then, we find the energy of an electron must be,

$$E_n = \frac{-13.6}{n^2} eV$$

So an emitted photon has energy $E_n - E_m$, $n, m \in \mathbb{N}$, which fits experimental data.
But why is angular momentum quantized? Louis de Broglie answered this with the assumption that an electron (when thought of as a wave) should satisfy periodic boundary conditions. Or, $2\pi r = n\lambda$ where r is the orbiting radius. Next, de Broglie some assumptions:

- For any particle, $E = h\nu$ just as in the case of a photon.
- For a photon, $E = pc$ from relativity.

Then, if we call the wavelength of the photon λ_{dB},

$$\lambda_{dB} = \frac{h}{p}$$

We then extend this from the case of a photon to everything, which works. Additionally, in combining de Broglie's insights: $\lambda_{dB} = \frac{h}{p}$ and $2\pi r = n\lambda$, we find that angular momentum is quantized as Bohr predicted, $|\mathbf{L}| = n\hbar$.

1.3 Schrödinger's Equation

To summarize, we find that particles have wave-like properties. Specifically, $E = h\nu$, and $p = \frac{h}{\lambda_{dB}}$. Or, $E = \hbar\omega$ and $p = \hbar k$. So, we assign a wave equation to particles of the form

$$\Psi(x,t) = e^{i(kx - \omega t)}$$

This is a solution to the following ODE's:

$$\frac{-\hbar}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} = E\Psi(x,t) \tag{ODE 1}$$

$$i\hbar\frac{\partial \Psi(x,t)}{\partial t} = E\Psi(x,t) \tag{ODE 2}$$

This motivates the following PDE, Schrödinger's equation with the additional generalization that $E = \frac{p^2}{2m} + \mathcal{V}(x)$

$$\frac{-\hbar}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + \mathcal{V}(x)\Psi(x,t) = i\hbar\frac{\partial \Psi(x,t)}{\partial t} \tag{Schrödinger's equation}$$

It turns out that the solutions to this equation

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