Calculus III

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1 2D and 3D space

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1.3 Quadratic Surfaces

$$f(x,y) = x^2 + y^2 \qquad \text{(Paraboloid)}$$
$$f(x,y) = x^2 - y^2 \qquad \text{(Saddle)}$$
$$f(x,y) = xy \qquad \text{(Saddle)}$$

Theorem 1.1. An equation of the form $f(x,y) = ax^2 + bxy + cy^2$ is a saddle if $b^2 - 4ac > 0$ and a bowl if $b^2 - 4ac < 0$.

1.4 Vector Fields

Definition 1.2 (Vector Field)**.** A vector field is some mapping $V : S \to \mathbb{R}^n$ with $S \subset \mathbb{R}^n$. Visually, we represent vector fields by drawing the vector of the output centered at the input point. /*Visual vector field*/

1.5 Transformations

Definition 1.3 (Transformation)**.** A transformation is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$. We represent this mapping $\mathbf{x} \to \mathbf{y}$ with an equation of the form

$$(y^1, \ldots, y^m) = T(x^1, \ldots, x^n) = (T_1(x^i), \ldots, T_m(x^i))$$

Note that *linear-ish* transformations can be written

$$\begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} = \begin{pmatrix} T_1(x^1, \ldots, x^n) \\ \vdots \\ T_m(x^1, \ldots, x^n) \end{pmatrix} = \begin{pmatrix} b^1 + \sum a_i^1 x^i \\ \vdots \\ b^m + \sum a_i^m x^i \end{pmatrix} = \begin{pmatrix} b^1 \\ \vdots \\ b^m \end{pmatrix} + \sum \begin{pmatrix} a_i^1 \\ \vdots \\ a_i^m \end{pmatrix} x^i$$
$$= \begin{pmatrix} b^1 \\ \vdots \\ b^m \end{pmatrix} + \begin{pmatrix} a_1^1 & \ldots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \ldots & a_n^m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

1.6 Vector Products

Theorem 1.4. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ separated by some angle θ, we have

$$\sum u^i v^i = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$$

Definition 1.5 (Dot Product)**.** The previous equivalency is quite useful, so we define the *dot product* by the mapping $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$u \cdot v = \sum u^i v^i = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$$

Where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ separated by angle θ.

Definition 1.6 (Cross Product)**.** The *cross product* is some mapping $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ such that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $a = \langle a_x, a_y, a_z \rangle$ and $v = \langle b_x, b_y, b_z \rangle$ we have

$$a \times b = \langle (a_y b_z - a_z b_y), (a_z b_x - a_x b_z), (a_x b_y - a_y b_x) \rangle.$$

By abuse of notation, we can write the cross product as:

$$a \times b = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

2 ???

2.1 Curvilinear Coordinates

2.1.1 Polar Coordinates

Definition 2.1. Polar coordinates are defined by a transformation from (r, θ) space to (x, y) space.
Below are transformations $T : (r, \theta) \to (x, y)$ and $T^{-1} : (x, y) \to (r, \theta)$

$$T : (x, y) = (r\cos(\theta), r\sin(\theta))$$
$$T^{-1} : (r, \theta) = (\sqrt{x^2 + y^2}, \arctan(y/x)[+\pi?])$$

Note: In T^{-1},

2.1.2 Cylindrical Coordinates

2.1.3 Spherical Coordinates