Calc III Notes

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1 2D and 3D Space

1.1 Cartesian Coordinates

1.2 Contour Maps

1.3 Quadratic Surfaces

A quadratic surface is any combination of x's, y's, and z's of orders 0, 1, or 2. Here are some simple combinations of these:

$$R^2 = x^2 + y^2 + z^2 \tag{Sphere}$$

The above equation holds for every (x, y, z) coordinate that is a distance R from the origin. So, this expresses a sphere of radius R.

$$R^2 = (ax)^2 = (by)^2 + (cz)^2 \tag{Ellipsoid}$$

The added constants squishes the sphere by a factor of a in the x direction, b in the y direction, and c in the z direction resulting in an ellipsoid.

$$z = x^2 + y^2 \tag{Paraboloid}$$

The 2D equation $R^2 = x^2 + y^2$ traces out a circle of radius R. Equivalently, the above equation traces out a circle of radius $\sqrt{z}$ at each z, resulting in a bowl; specifically, a "Paraboloid".

Add Picture[s]

$$z = x^2 - y^2 \tag{Saddle}$$

The 2D equation $A = x^2 - y^2$

$$z = xy \tag{Saddle}$$

The last three quadratic surfaces have the form of $z = ax^2 + bxy + cy^2$. These will either be a bowl or a saddle, which you can determine with the following rule.

Finish up

For equations of the form,

$$z = ax^2 + bxy + cy^2$$

Consider the quantity $Q = b^2 - 4ac$. The quadratic is a saddle if $Q > 0$, a bowl if $Q < 0$ and a taco if $Q = 0$.

Proof 1 *Do this!*

co

explain 'taco'

1.4 Vectors and Matrices

LA reference

1.5 Vector Fields

1.6 Parallelogram Grid

1.7 Transformations

Previous math classes have addressed simple functions that take 1 variable as an input and 1 variable as an output. We visualized these functions through graphs, where one dimension (the x axis) is reserved for inputs and one direction (the y axis) is reserved for outputs. Functions with two inputs and one output you learned as three dimensional functions — the x and y axes are inputs and the z axis is the output. However, once we reach 2 inputs, (u, v), and 2 outputs, (x, y), we have run out of dimensions to visualize in. The idea of a *transformation* is an alternate visualization technique that instead transforms the input (u, v) plane into the output (x, y) plane. The transformation simply demonstrates how grid lines in the input space are distorted into the output space.

transformation example figure

1.8 Dot Product

The dot product is an operation that takes two vectors as inputs and outputs a scalar.

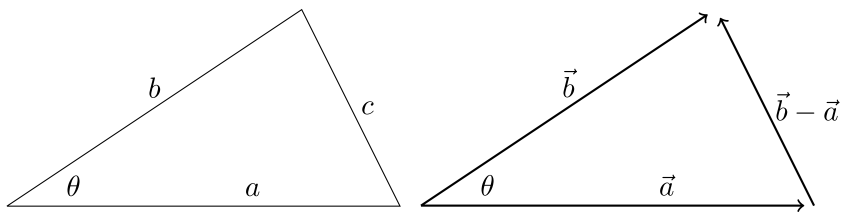

Figure 1: Law of Cosines through Vectors

For two vectors $\vec{a} = \langle a_x, a_y \rangle$ and $\vec{b} = \langle b_x, b_y \rangle$, the dot product is denoted $\vec{a} \cdot \vec{b}$ and is equivalent to:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta = a_x b_x + a_y b_y$$

Where θ is the angle between the two vectors.
In n dimensions, for $a = \langle a_1, a_2, \ldots, a_n \rangle$ and $b = \langle b_1, b_2, \ldots, b_n \rangle$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta = \sum_i a_i b_i$$

Proof 2 *The derivation of the dot product is related to the law of cosines. Consider the left triangle shown in Figure 1. According to the law of cosines,*

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Where a, b, and c refer to the length of each side of the triangle. Equivalently, we can express lengths as the magnitudes of the vectors shown on the right triangle of Figure 1.

$$\|\vec{b} - \vec{a}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\| \cos\theta$$

Using the distance formula, we expand the expressions for the magnitudes of each vector.

$$\begin{aligned} (b_x - a_x)^2 + (b_y - a_y)^2 &= a_x^2 + a_y^2 + b_x^2 + b_y^2 - 2\|\vec{a}\|\|\vec{b}\| \cos\theta \\ \cancel{b_x^2} - 2a_x b_x + \cancel{a_x^2} + \cancel{b_y^2} - 2a_y b_y + \cancel{a_y^2} &= \cancel{a_x^2} + \cancel{a_y^2} + \cancel{b_x^2} + \cancel{b_y^2} - 2\|\vec{a}\|\|\vec{b}\| \cos\theta \\ a_x b_x + a_y b_y &= \|\vec{a}\|\|\vec{b}\| \cos\theta \end{aligned}$$

The above equality turns out to be useful and often used. So, we name each side of the equality the "dot product" and denote it as $\vec{a} \cdot \vec{b}$. While the above proof is specific to two dimensional vectors, it generalizes to n dimensions.

1.9 Cross Product

In three dimensional space, vectors have a unique property. Given two vectors, there is a single direction that is orthogonal to both initial vectors. It is often useful to find this direction, which we do so through a tool called the "cross product". The cross product takes in two vectors and outputs a vector in the orthogonal direction.

For two vectors $\vec{a} = \langle a_x, a_y, a_z \rangle$ and $\vec{b} = \langle b_x, b_y, b_z \rangle$, the cross product is denoted $\vec{a} \times \vec{b}$ and is equivalent to:

$$\vec{a} \times \vec{b} = \langle a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x \rangle$$

The resulting vector, $\vec{a} \times \vec{b}$, will be orthogonal to both $\vec{a}$ and $\vec{b}$.

Proof 3 *The claim is that $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$. If the dot products between $\vec{a} \times \vec{b}$ and the two vectors is 0, then we will have shown orthogonality. First, we evaluate $\vec{a} \cdot (\vec{a} \times \vec{b})$,*

$$\vec{a} \cdot (\vec{a} \times \vec{b})$$
$$\langle a_x, a_y, a_z \rangle \cdot \langle a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x \rangle$$

Now, we cary out the dot product,

$$a_x a_y b_z - a_x a_z b_y + a_y a_z b_x - a_x a_y b_z + a_x a_z b_y - a_y a_z b_x$$

Every term has a partner of opposite sign to cancel with, so the dot product evaluates to 0. Therefore, $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$. The same argument is applied to $\vec{b}$.

$$\vec{b} \cdot (\vec{a} \times \vec{b})$$
$$\langle b_x, b_y, b_z \rangle \cdot \langle a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x \rangle$$
$$a_y b_x b_y - a_z b_x b_y + a_z b_x b_y - a_x b_y b_z + a_x b_x b_y - a_y b_x b_z = 0$$

And therefore, the cross product $\vec{a} \times \vec{b}$ does indeed output a vector orthogonal to both $\vec{a}$ and $\vec{b}$.

Cross product properties and techniques

2 ???

2.1 Curvilinear Coordinization

2.1.1 Polar Coordinates

In 2D space, we typically communicate the location of a point with (x, y) coordinates; however, there are other ways. One such way is to use "polar coordinates". Consider a vector going from the origin to an unknown point. If we knew the length and direction of this vector, we could find the point. In polar coordinates, we denote the length of this vector with r. The direction is communicated with θ, the angle the vector makes with the positive x axis. Polar coordinates are (r, θ) pairs that specify a point. Using simple trigonometry, we can relate polar coordinates to cartesian coordinates.

Add figures

Translating between polar (r, θ) and cartesian (x, y) coordinates.

$$\begin{aligned} x &= r\cos\theta & \qquad r &= \sqrt{x^2 + y^2} \\ y &= r\sin\theta & \qquad \theta &= \arctan\big(y/x\big)[+\pi?] \end{aligned}$$

In calculating θ, you must add π if (x, y) is in quadrants 3 or 4.

2.1.2 Cylindrical Coordinates

Translating between cylindrical (r_{cyl}, θ, z) and cartesian (x, y, z) coordinates.

$$\begin{aligned} x &= r_{cyl}\cos\theta & \qquad r_{cyl} &= \sqrt{x^2 + y^2} \\ y &= r_{cyl}\sin\theta & \qquad \theta &= \arctan\big(y/x\big)[+\pi?] \\ z &= z & \qquad z &= z \end{aligned}$$

In calculating θ, you must add π if the point is in quadrants 3 or 4.

2.1.3 Spherical Coordinates

Translating between spherical (r_{sph}, θ, ϕ) and cartesian (x, y, z) coordinates.

$$x = r_{sph} \cos\theta \sin\phi \qquad r_{sph} = \sqrt{x^2 + y^2 + z^2}$$
$$y = r_{sph} \sin\theta \sin\phi \qquad \theta = \arctan\left(y/x\right)[+\pi?]$$
$$z = r_{sph} \cos\phi \qquad \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

In calculating θ, you must add π if the point is in quadrants 3 or 4.

2.2 Coordinate Vector Fields

Consider a mapping from curvilinear space to cartesian space. Coordinate vector fields have the job of pointing in the direction curviliear values increase within cartesian space.

Consider some transformation $T(u, v) = (x, y)$. The direction and magnitude in which an arbitrary curvilinear coordinate, u, increases at any given (u, v) is given by the coordinate vector field, $\partial_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \rangle$.

justification?

2.2.1 Polar Vector Field

For the polar coordinate transformation, $(x, y) = T(r, \theta) = (r\cos\theta, r\sin\theta)$, we are interested in finding the direction of increase of

2.2.2 Cylindrical Vector Field

2.2.3 Spherical Vector Field

2.3 Linearization

Take some curvilinear transformation $(x, y) = T(u, v)$. This is a mapping of straight gridlines into bent gridlines. However, if we zoom into the output space enough, the gridlines will begin to appear as straight. The directions of these gridlines are given by the partial vectors, ∂_u and ∂_v, at that point. So, the gridlines are close to straight, we can approximate a short journey

along the u, v gridlines in the x, y plane by following the direction of the partial vectors, ∂_u and ∂_v.
Following the u, v gridlines by small amounts Δu and Δv results in the following changes in x and y:

Effect of following u gridline:	Effect of following v gridline:
$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \approx \partial_u \cdot \Delta u$	$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \approx \partial_u \cdot \Delta u$
$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \approx \frac{\partial x}{\partial u}\Delta u + \frac{\partial y}{\partial u}\Delta u$	$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \approx \frac{\partial x}{\partial u}\Delta u + \frac{\partial y}{\partial u}\Delta u$

Combining the effects of traveling along the two gridlines through addition gives the following.

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \approx \frac{\partial x}{\partial u}\Delta u + \frac{\partial y}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v + \frac{\partial y}{\partial v}\Delta v$$

We now condense this expression with matrix notation.

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \approx \begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v}\end{pmatrix} \begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix} \tag{1}$$

This equation approximates how movement in the input u, v plane translates to movement in the output x, y plane. The matrix appearing in the above equation is widely used, so it gets the name “Jacobian matrix”.

The *Jacobian Matrix* of a transformation T is abreviated as DT and consits of the partial vectors of the transformation in each column. For the transformation $(x, y) = T(u, v)$: $DT = \begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}$. In general, the Jacobian Matrix for a transformation $(x^1, x^2, \ldots, x^n) = T(y^1, y^2, \ldots, y^m)$ is given by,

$DT = \begin{pmatrix}\frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^m}\\ \vdots & \ddots & \vdots\\ \frac{\partial x^n}{\partial y^1} & \cdots & \frac{\partial x^n}{\partial y^m}\end{pmatrix}$ Where the (i, j) entry of DT is $DT_{ij} = \dfrac{\partial x^i}{\partial y^j}$

We can now combine equation 1 with the idea of a Jacobian Matrix to formally introduce linearization.

Given a transformation $(x, y) = T(u, v)$, the *linearization* is the linear approximation of how a small change in u and v affects x and y given by the following equation,

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \approx DT \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

In general, for $(x^1, x^2, \ldots, x^n) = T(y^1, y^2, \ldots, y^m)$:

$$\begin{pmatrix} \Delta x^1 \\ \vdots \\ \Delta x^n \end{pmatrix} \approx DT \begin{pmatrix} \Delta y^1 \\ \vdots \\ \Delta y^m \end{pmatrix}$$

error analysis

2.4 Chain Rule

Consider two Transformations: $(w, z) = T_1(u, v)$ converts the u, v plane into the w, z plane, and $(x, y) = T_2(w, z)$ then converts the w, z plabe to the x, y plane as shown in Figure 2. From our knowledge of linearization, we know how local gridlines change through a single transformation, but how do they change through two transformations? For instance, in Figure 2, how does a small displacement in the u, v direction translate to change in the x, y plane? /*Stuff about Figure 3*/.

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = D(T_2 \circ T_1)|_{(u_0,v_0)} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

/*explanation?*/

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = DT_2|_{(w_0,z_0)} \cdot DT_1|_{(x_0,y_0)} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

So, we have found the following.

$$D(T_2 \circ T_1)|_{(u_0,v_0)} = DT_2|_{(w_0,z_0)} \cdot DT_1|_{(x_0,y_0)} \tag{2}$$

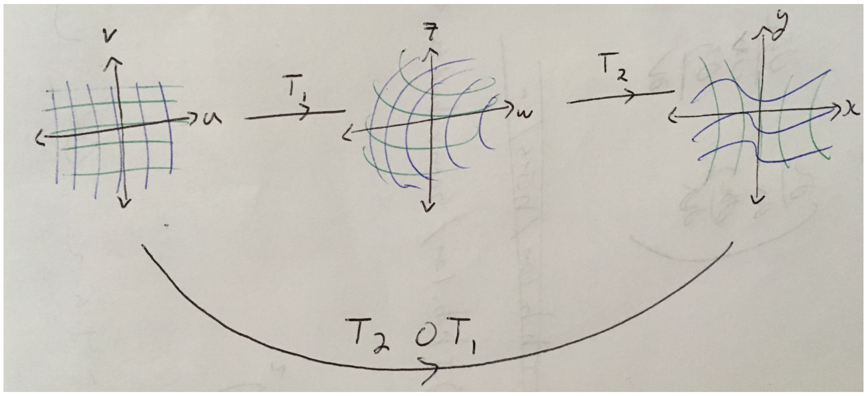

Figure 2: Nested transformations

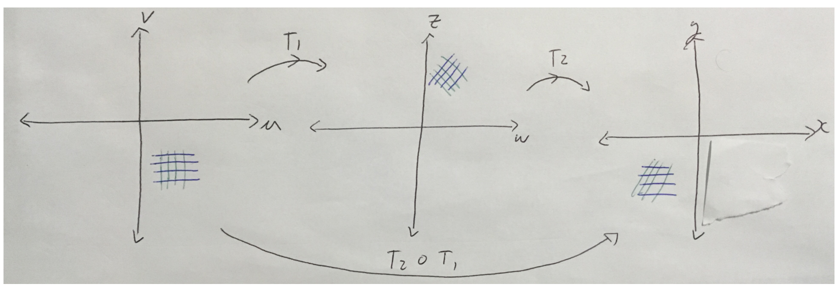

Figure 3: Nested transformations

We have arrived at this expression using the idea that a Jacobian Matrix shifts local gridlines. However, they also have a matrix representation.

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial w}\frac{\partial w}{\partial u} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial u} & \frac{\partial x}{\partial w}\frac{\partial w}{\partial v} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial v} \\ \frac{\partial y}{\partial w}\frac{\partial w}{\partial u} + \frac{\partial y}{\partial z}\frac{\partial z}{\partial u} & \frac{\partial y}{\partial w}\frac{\partial w}{\partial v} + \frac{\partial y}{\partial z}\frac{\partial z}{\partial v} \end{pmatrix}$$

figures are placeholder