Complex Analysis

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1 Basics

1.1 Definitions

Definition 1.1 (Complex Number)**.** We define a complex number z to be of the form $z = x + iy$. Where $x, y \in \mathbb{R}$ and i is defined such that $i^2 = -1$. We say $z \in \mathbb{C}$.

Definition 1.2 (Conjugate)**.** If $z \in \mathbb{C}$ such that $x = x+iy$ then z *conjugate*, written $\bar{z}$, is defined $\bar{z} = x - iy$.

Definition 1.3. Some notation definitions

- $|z|^2 \coloneqq z\bar{z}$
- If $z = x + iy$ then $\mathrm{Re}(z) \coloneqq x$.
- If $z = x + iy$ then $\mathrm{Im}(z) \coloneqq y$.
- /*$\arg(z) = \theta$*/

/*Complex Plane and def reps*/

1.2 Properties

Proposition 1.4. Take $z_1, z_2 \in \mathbb{C}$

- $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$
- $\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Proposition 1.5 (Triangle Inequality)**.** $|z_1 + z_2| \leq |z_1| + |z_2|$

Proposition 1.6 (/*?*/)**.** $e^{i\theta} = \cos\theta + i\sin\theta$

Proposition 1.7. $z^n = |z|^n e^{in\theta}$

Proposition 1.8. $\sqrt[n]{z}$ is anything such that $(\sqrt[n]{z})^n = z$. So,

$$\sqrt[n]{z} = \{ \sqrt[n]{|z|} e^{\frac{i(\theta+2\pi k)}{n}} \mid 0 \leq k < n \}$$

2 Transformations

2.1 Intro

Example 2.1.

- $R(z) = e^{i\alpha} z$ defines a rotation by α
- $T(z) = z + z_0$ translates 0 to z_0
- $S(z) = \alpha z$ scales/dilates the plane by α centered at 0

Example 2.2. $z \to z^2$ transformation /*explanation and visual*/

2.2 Continuity

Definition 2.3 (Continuity)**.** If we have some function $f : \mathbb{C} \to \mathbb{C}$. We say f is continuous at $z \in \mathbb{C}$ if for every sequence $\{z_k\}$, we have $z_k \to z \implies f(z_k) \to f(z)$. Or, intuitively, if $z \approx a$, we have $f(z) \approx f(a)$.

2.3 Branches

If we have some function f that maps from one value to a set of values, $f : \mathbb{C} \to \{z \in \mathbb{C}\}$. Then, we want to restrict the function such that f is $1-1$ and continuous.

Definition 2.4. If we have a function $f : \mathbb{C} \to z \in \mathbb{C}$

/*Principle Branch*/
/*Riemann Surface*/