Problem Set 5

YSC4204 – Statistical Computing

Due on: Tuesday, 1 November, 23:59

Problem 1

A concrete run of Brent's method

Consider Brent's root finding method applied to $f(x) = x^2 - 9$ starting with an initial bracket $(0, 5)$. Which steps (i.e. bisection, secant method or inverse quadratic interpolation) does the algorithm take during the first four iterations? Explain why it takes these steps and confirm your results by showing output from `uniroot()`.

(5 points.)

Problem 2

Maximum-likelihood, comparing root finding methods

The following data are observations from a Cauchy distribution with scale parameter γ.

	1	2	3	4	5	6	7	8	9	10
data	-7.99	24.11	3.41	7.01	22.28	0.54	-1.97	15.97	11.53	237.63

The probability density function of the distribution is

$$f(x) = \frac{\gamma}{\pi(x^2 + \gamma^2)} .$$

(a) Derive a function g whose root equals the maximum-likelihood estimator for γ.

(b) How many iterations does the bisection method need to determine the root with an accuracy $\leq 10^{-6}$ from an initial bracket $(1, 100)$?

(c) How many iterations does Brent's method need for the same accuracy and initial bracket? Include relevant R output in your answer.

(d) Derive g' and carry out the Newton-Raphson algorithm from an initial guess $\gamma = 1$. How many iterations does the algorithm need for an accuracy of $\leq 10^{-6}$? Again include relevant R output in your answer.

(5 points total. 1 each for (a)–(c), 2 for (d).)

Problem 3

The secant method's rate of convergence

Show that the secant method's order of convergence is equal to the golden ratio $q = \frac{1+\sqrt{5}}{2} \approx 1.62$. That is, if ϵ_t is the error of the t-th numerical estimate (i.e. the difference between the estimate and the root), then

$$\lim_{t\to\infty} \frac{|\epsilon_{t+1}|}{|\epsilon_t|^q} = c > 0$$

for some constant c. You can assume that the function f whose root we seek is twice differentiable. For simplicity's sake, you may assume that $\lim_{t\to\infty} \epsilon_t = 0$. Your argument does not need to be a formal mathematical proof, but explain your calculation so that it can be easily followed from one step to the next.

(5 points.)

Problem 4

Logistic regression, multivariate Newton-Raphson method

The data in `face_recog.csv` (on Canvas under "Files") contain test results of a human face recognition algorithm.[1] Each line in the file shows whether the algorithm is able to correctly match an image to a person on a "target" photo (yes = 1, no = 0). The second number "eyediff" on each line records the absolute difference in mean standardized eye region pixel intensity between test image and matching target photo. When "eyediff" is large, recognition is more challenging.

(a) Treating "eyediff" as predictor and "match" as response variable, use R's `glm()` function to fit a logistic regression model,

$$P(\text{match} = 1 \,|\, \text{eyediff} = x) = \frac{1}{1 + e^{-(\beta_0+\beta_1 x)}} \ .$$

What are the maximum-likelihood parameters β_0 and β_1?

(b) Write an R script that performs Newton-Raphson iterations *with explicit matrix multiplications*. If the initial guess is $\beta_0 = 1$ and $\beta_1 = 0$, what are their values after 1, 2, 3 and 4 iteration?

(c) Describe how you confirm numerically that R's `glm()` performs Newton-Raphson iterations behind the scene. Show the relevant R output.

(5 points. 1 for (a), 2 each for (b) and (c).)

[1] Data from `http://www.stat.colostate.edu/computationalstatistics/`