COMP9020 Week 7 Functions

1 Introduction to Functions

A function is a relation from a domain A to a codomain B such that:

- Each input in A is mapped to exactly one output in B.
- Every element in A is assigned a value in B (totality).

Notation: $f : A \to B$, $f(a) = b$ for $a \in A$, $b \in B$
Range: $\{f(a) : a \in A\} \subseteq B$

2 Examples

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
  Range: $[0, \infty)$ (not surjective over $\mathbb{R}$)
- $f : \{1, 2, 3, 4\} \to \{a, b, c, d\}$ with mappings:
  - $f(1) = a$, $f(2) = c$, $f(3) = b$, $f(4) = d$

  Injective and Surjective $\Rightarrow$ Bijective
- $f : \mathbb{N} \to \mathbb{N}$, $f(n) = 2n$
  Injective but not surjective (odd numbers not in range)
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x - 2$
  Surjective and Injective $\Rightarrow$ Bijective
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$
  Not injective (e.g., $f(-2) = f(2)$) and not surjective (odd integers not squares)

3 Function Properties

- **Injective** (one-to-one): $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- **Surjective** (onto): $\forall b \in B, \exists a \in A$ such that $f(a) = b$
- **Bijective**: both injective and surjective

4 Function Operations

- Composition: $(g \circ f)(a) = g(f(a))$
- Iteration: $f^n(x) = f(f^{n-1}(x))$
- Identity function: $\mathrm{Id}_S(x) = x$
- Inverse function: only defined if f is bijective

5 Reversibility

- Function equality: $f = g \iff f(x) = g(x)$ for all x in domain
- Reverse is a function $\iff f$ is injective

6 Summary Table

Property	Description
Functional	$\forall a \in A, \exists \leq 1\, b \in B$
Total	$\forall a \in A, \exists \geq 1\, b \in B$
Injective	$\forall b \in B, \exists \leq 1\, a \in A$
Surjective	$\forall b \in B, \exists \geq 1\, a \in A$
Bijective	Injective and Surjective

7 Exam-Style Questions with Solutions

Q1. Determine if $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2 + 3$ is surjective.

Solution:
To be surjective, every $y \in \mathbb{Z}$ must have some $x \in \mathbb{Z}$ such that $f(x) = y$. But $f(x) \geq 3$ for all x, so integers like 2, 0, -5 are not in the range.
Conclusion: Not surjective.

Q2. Is the function $f(x) = 2x$ from $\mathbb{Z} \to \mathbb{Z}$ bijective?

Solution:

- Injective: $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$
- Surjective: Not all integers are even not surjective

Conclusion: Injective but not bijective.

Q3. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x + 1$, and $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = 2x$. Find $g \circ f$ and $f \circ g$.

Solution:

$$(g \circ f)(x) = g(f(x)) = g(x + 1) = 2(x + 1) = 2x + 2$$
$$(f \circ g)(x) = f(g(x)) = f(2x) = 2x + 1$$

Conclusion: $g \circ f \neq f \circ g$

Q4. Prove or disprove: If $f \circ g$ is injective, then g is injective.

Solution:
Assume $f \circ g$ is injective, but g is not. Then $\exists x_1 \neq x_2$ such that $g(x_1) = g(x_2)$
$\Rightarrow f(g(x_1)) = f(g(x_2)) \Rightarrow f \circ g(x_1) = f \circ g(x_2)$ contradiction.
Conclusion: If $f \circ g$ is injective, g must be injective.