COMP9020 Week 8 Lecture 2 Notes

Relations, Equivalence Relations, Partial Orders, and Multiply-Quantified Statements

1 Equivalence Relations

Definition

A relation R on a set A is an **equivalence relation** if it satisfies:

- **Reflexivity:** $\forall x \in A,\ (x, x) \in R$
- **Symmetry:** $\forall x, y \in A,\ (x, y) \in R \Rightarrow (y, x) \in R$
- **Transitivity:** $\forall x, y, z \in A,\ (x, y), (y, z) \in R \Rightarrow (x, z) \in R$

Example

Let $A = \{0, 1, 2, 3, 4\}$ and the partition be $\{\{0, 3, 4\}, \{1\}, \{2\}\}$. The relation induced by this partition includes all pairs within each subset:

$$R = \{(0, 0), (0, 3), (0, 4), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4), (1, 1), (2, 2)\}$$

This relation is reflexive, symmetric, and transitive.

Equivalence Classes

Given $a \in A$, the equivalence class of a under R is:

$$[a] = \{x \in A \mid xRa\}$$

These equivalence classes form a partition of A.

Theorem

A relation on A is an equivalence relation if and only if it is induced by a partition of A.

2 Partial Orders

Definition

A relation R on a set A is a **partial order** if it satisfies:

- **Reflexivity:** $\forall x \in A,\ (x, x) \in R$
- **Antisymmetry:** $\forall x, y \in A,\ (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$
- **Transitivity:** $\forall x, y, z \in A,\ (x, y), (y, z) \in R \Rightarrow (x, z) \in R$

Examples

- Subset relation $\subseteq$ on sets
- Divides relation $a \mid b$ on positive integers
- Less than or equal to $\leq$ on real numbers

Total Orders

A **total order** is a partial order in which every pair of elements is comparable:

$$\forall x, y \in A,\ xRy \text{ or } yRx$$

3 Multiply-Quantified Relations

Key Quantified Statements

Let $R \subseteq A \times A$:

1. $\forall x \forall y\ xRy$: all nodes relate to all others
2. $\exists x \exists y\ xRy$: at least one relation exists
3. $\forall x \exists y\ xRy$: every node has at least one outgoing edge
4. $\exists y \forall x\ xRy$: one node receives from all others
5. $\forall y \exists x\ xRy$: every node has at least one incoming edge
6. $\exists x \forall y\ xRy$: one node points to all others

Examples

- $A = \{1, 2, 3\}$, $R = \{(1, 2), (2, 3), (3, 1)\}$ satisfies $\forall x \exists y\ xRy$ and $\forall y \exists x\ xRy$
- $R = \{(1, 2), (2, 2), (3, 2)\}$ satisfies $\exists y \forall x\ xRy$

4 Logical Relationships Between Statements

Valid Implications

- $\forall x \forall y \; xRy \iff \forall y \forall x \; xRy$ (commutativity of universal quantifiers)
- $\exists x \exists y \; xRy \iff \exists y \exists x \; xRy$ (commutativity of existential quantifiers)
- $\exists y \forall x \; xRy \Rightarrow \forall x \exists y \; xRy$ (single target implies every source has a target)
- $\exists x \forall y \; xRy \Rightarrow \forall y \exists x \; xRy$ (single source implies every target has a source)

Non-Implications and Counterexamples

- $\forall x \exists y \; xRy \not\Rightarrow \exists y \forall x \; xRy$
- $\forall y \exists x \; xRy \not\Rightarrow \exists x \forall y \; xRy$

Counterexample: Let $A = \{1, 2, 3\}$ and:

$$R = \{(1, 2), (2, 3), (3, 1)\}$$

This satisfies:

- $\forall x \exists y \; xRy$: each node has an outgoing edge
- $\forall y \exists x \; xRy$: each node has an incoming edge

But it does not satisfy:

- $\exists y \forall x \; xRy$: no single node is the target of all others
- $\exists x \forall y \; xRy$: no single node has edges to all others

Diagrammatic Insight

- Statements like $\forall x \exists y \; xRy$ imply an *out-degree* of at least 1 for every node.
- $\exists y \forall x \; xRy$ implies a *universal sink* node (in-degree equal to set size).
- Similarly, $\exists x \forall y \; xRy$ implies a *universal source*.
- Graph structure plays a key role in understanding these logical implications.