1. Recap of Functions vs. Relations

- A function maps each element in a set A to exactly one element in set B.

- A relation is a broader concept: it allows an element in A to relate to multiple or no elements in B.

- Relations are subsets of the Cartesian product $A \times B$.

Example: $R = \{(1, 2), (1, 3), (2, 3)\}$ is a relation, not a function.

2. Definition of a Relation

- A relation R from A to B is any subset of the Cartesian product $A \times B$.

- That is, $R \subseteq A \times B$.

- If $A = B$, R is called a relation on A.

Example: $A = \{1, 2\}$, $B = \{x, y\}$, $R = \{(1, x), (2, y)\} \subseteq A \times B$

3. Representations of Relations

- Ordered Pairs: Set notation $(x, y) \in R$

- Arrow Diagrams: Nodes with arrows indicating relations

- Adjacency Matrix: $M[i][j] = 1$ if $(i, j) \in R$

4. Types of Relations

- Binary: Relates two elements (x, y)

- Ternary: Relates three elements (x, y, z), e.g., multiplication

- N-ary: Relates n elements, used in databases and predicates

5. Properties of Relations – Detailed Explanation

1. Reflexive:

- A relation R on a set A is reflexive if every element relates to itself.

- Formally: $\forall x \in A, (x, x) \in R$

- This means all diagonal elements must be present.

Example: $A = \{1,2,3\}$, $R = \{(1,1), (2,2), (3,3)\}$ is reflexive.

Why it's useful:

- Reflexive relations capture ideas of self-inclusion (e.g., 'is equal to').

2. Irreflexive:

- A relation R on A is irreflexive if no element relates to itself.

- Formally: $\forall x \in A, (x, x) \notin R$

- None of the diagonal elements should exist in the relation.

Example: A = {1,2}, R = {(1,2), (2,1)} is irreflexive.

Note:

- A relation cannot be both reflexive and irreflexive unless A is empty.

3. Symmetric:

- A relation R is symmetric if whenever $(x, y) \in R$, then $(y, x) \in R$.

- Reflects bidirectional or mutual relationships.

Example: R = {(1,2), (2,1)} is symmetric.

Counterexample: R = {(1,2)} is not symmetric (missing (2,1)).

Usage:

- Symmetric relations model concepts like "is a sibling of", "is adjacent to".

4. Antisymmetric:

- R is antisymmetric if $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$.

- That is, two distinct elements cannot relate in both directions.

Example: R = {(1,1), (2,2), (1,2)} is antisymmetric.

Counterexample: R = {(1,2), (2,1)} is not antisymmetric.

Usage:

- Antisymmetric relations appear in "less than or equal to", "subset of" etc.

5. Transitive:

- R is transitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

- Ensures chain-link consistency in relationships.

Example: R = {(1,2), (2,3), (1,3)} is transitive.

Counterexample: R = {(1,2), (2,3)} is not transitive (missing (1,3)).

Application:

- Transitive relations help define orderings, inheritance, or access control chains.

6. Worked Examples

Example 1: Reflexivity

A = {1, 2, 3}

R1 = {(1,1), (2,2), (3,3)} ✔■ Reflexive

R2 = {(1,1), (2,2)} ✘ Not reflexive

Example 2: Symmetry and Antisymmetry

R3 = {(1,2), (2,1)} ✔■ Symmetric ✘ Not antisymmetric

R4 = {(1,2)} ✔■ Antisymmetric ✘ Not symmetric

R5 = {(1,1), (2,2)} ✔■ Both symmetric and antisymmetric

Example 3: Transitivity

R6 = {(1,2), (2,3), (1,3)} ✔■ Transitive

R7 = {(1,2), (2,3)} ✘ Not transitive

Example 4: Transitive Closure

Start: R = {(1,2), (2,3)}

Closure adds (1,3): R■ = {(1,2), (2,3), (1,3)}

7. Summary and Applications

- Reflexive: Every node relates to itself (e.g., 'is equal to')

- Irreflexive: No self-relations (e.g., 'is taller than')

- Symmetric: Mutual relation (e.g., 'is a sibling of')

- Antisymmetric: No bidirectional relation unless identical (e.g., 'is a subset of')

- Transitive: Links carry over (e.g., 'ancestor of')

Real-world examples:

- Graphs, databases, ordering systems, social networks

Reflexivity

A relation R on set A is reflexive if $\forall x \in A, (x, x) \in R$.

Examples:

$A = \{1, 2, 3\}$, $R = \{(1,1), (2,2), (3,3)\}$ is reflexive.

$R = \{(1,1), (2,2)\}$ is not reflexive as $(3,3)$ is missing.

Exam-style Questions:

1. Let $A = \{a, b, c\}$. Is $R = \{(a,a), (b,b)\}$ reflexive?
2. Give an example of a reflexive relation on the set $\{1, 2\}$.
3. True/False: If a relation on A is reflexive, then it must have exactly $|A|$ elements.

Solutions:

1. No, since (c, c) is missing.
2. Example: $R = \{(1,1), (2,2)\}$.
3. False. A reflexive relation must include all (x,x), but may include more.

Irreflexivity

R is irreflexive if $\forall x \in A, (x, x) \notin R$.

Examples:

R = {(1,2), (2,1)} is irreflexive on A = {1,2}.

R = {(1,1)} is not irreflexive.

Exam-style Questions:

1. Give an irreflexive relation on {1, 2, 3}.
2. Can a relation be both reflexive and irreflexive?

Solutions:

1. Example: R = {(1,2), (2,3), (3,1)}.
2. No, unless A is empty.

Symmetry

R is symmetric if $\forall x, y \in A, (x,y) \in R \Rightarrow (y,x) \in R$.

Examples:

R = {(1,2), (2,1)} is symmetric.

R = {(1,2)} is not symmetric as (2,1) is missing.

Exam-style Questions:

1. Is R = {(a, b), (b, a), (c, c)} symmetric on A = {a, b, c}?
2. Can a symmetric relation be irreflexive?

Solutions:

1. Yes, all pairs have their symmetric counterparts.
2. Yes, e.g., R = {(1,2), (2,1)} on A = {1,2} is symmetric and irreflexive.

Antisymmetry

R is antisymmetric if $(x,y) \in R$ and $(y,x) \in R \Rightarrow x = y$.

Examples:

R = {(1,2)} is antisymmetric.

R = {(1,2), (2,1)} is not antisymmetric.

Exam-style Questions:

1. Let R = {(a,a), (a,b), (b,c)}. Is R antisymmetric?
2. Can a relation be both symmetric and antisymmetric?

Solutions:

1. Yes, no counterexample to antisymmetry appears.
2. Yes, if only (x,x) pairs exist, e.g., R = {(1,1), (2,2)}.

Transitivity

R is transitive if (x,y) ∈ R and (y,z) ∈ R ⇒ (x,z) ∈ R.

Examples:

R = {(1,2), (2,3), (1,3)} is transitive.

R = {(1,2), (2,3)} is not transitive (missing (1,3)).

Exam-style Questions:

1. Given R = {(a,b), (b,c)}, what must be added to make R transitive?
2. Describe the transitive closure of R = {(0,1), (1,2)} on A = {0,1,2,3}.

Solutions:

1. Add (a,c) to make it transitive.
2. Closure adds (0,2). Final R■ = {(0,1), (1,2), (0,2)}.

Equivalence Relations

R is an equivalence relation if it is reflexive, symmetric, and transitive.

Examples:

R = {(1,1), (2,2), (3,3), (1,3), (3,1)} is an equivalence relation on A = {1,2,3}.

Exam-style Questions:

1. Prove that 'congruent modulo n' is an equivalence relation.
2. What are the equivalence classes of R = {(0,0), (1,1), (1,3), (3,1), (3,3)}?

Solutions:

1. It is reflexive: $a \equiv a \bmod n$; symmetric: if $a \equiv b \bmod n$, then $b \equiv a$; transitive: if $a \equiv b$ and $b \equiv c$, then $a \equiv c$.
2. [{0}], [{1,3}], [{2}] assuming A = {0,1,2,3}.

Partial Orders

R is a partial order if it is reflexive, antisymmetric, and transitive.

Examples:

'$\leq$' on numbers and '$\subseteq$' on sets are partial orders.

Hasse diagrams can represent partial orders efficiently.

Exam-style Questions:

1. Is the subset relation $\subseteq$ a partial order?
2. Draw the Hasse diagram for (P({a,b}), $\subseteq$)

Solutions:

1. Yes, it satisfies reflexivity, antisymmetry, and transitivity.
2. Hasse diagram would include: $\varnothing$ < {a}, {b} < {a,b}, with lines from lower sets to their supersets.