COMP9020 Week 7 Working with Functions

1 Ways to Define Functions

Using Algebra

Functions can be defined algebraically, e.g.,

$$f(x) = x^2 + 3, \quad g(x) = 2x + 5$$

Using Recursion

$$\text{Factorial:} \quad n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n > 0 \end{cases}$$

$$\text{Fibonacci:} \quad F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

Using Series

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Using Programs

Example: Depth-First Search (DFS) implementation can define a function from a graph to a list of visited nodes.

Using Properties

Example of Linear Function:

$$f(x + y) = f(x) + f(y), \quad f(cx) = cf(x)$$

2 Important Classes of Functions

Exponentials and Logarithms

$$\text{Exponential: } a^x = \sum_{n=0}^{\infty} \frac{(\ln a)^n x^n}{n!}, \quad e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

$$\text{Logarithm: } \log_a(x) = y \Leftrightarrow a^y = x$$

Properties of Logarithms:

$$\log_a(xy) = \log_a(x) + \log_a(y)$$
$$\log_a(x/y) = \log_a(x) - \log_a(y)$$
$$\log_a(x^r) = r \log_a(x)$$

Polynomials

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

Defined over real numbers.

3 Advanced Concepts

Sets of Functions

$$B^A = \text{set of all functions from } A \to B$$

If $|A| = m$ and $|B| = n$, then $|B^A| = n^m$

Currying

$$f : A \times B \to C \quad \text{can be seen as} \quad g : A \to (B \to C)$$
$$g(a)(b) = f(a, b)$$

4 Function Composition

- Associative: $f \circ (g \circ h) = (f \circ g) \circ h$
- Identity: $f \circ \mathrm{id} = f = \mathrm{id} \circ f$
- Not necessarily commutative: $f \circ g \neq g \circ f$

5 Applications of Functions

Complexity Measures

Time Complexity:

$$O(n), \quad O(n^2), \quad O(\log n)$$

Space Complexity:

$$O(1), \quad O(n)$$

Cardinality

$$f(n) = 2n \text{ is a bijection from } \mathbb{N} \to 2\mathbb{N}$$

Two sets have the same cardinality if a bijection exists between them.

6 Exam-Style Questions with Solutions

Q1. Define a function from $\mathbb{N}$ to $\mathbb{N}$ using recursion. Prove correctness by induction.

Solution: Let $f(n) = n!$
Base case: $f(0) = 1$
Inductive step: Assume $f(k) = k!$, show $f(k+1) = (k+1)!$:

$$f(k+1) = (k+1) \cdot f(k) = (k+1) \cdot k! = (k+1)!$$

Q2. Use the Maclaurin series to approximate $\sin(\frac{\pi}{6})$ using 3 terms.

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$x = \frac{\pi}{6} \approx 0.5236 \Rightarrow \sin(x) \approx 0.5236 - \frac{0.1435}{6} + \frac{0.0395}{120} \approx 0.5236 - 0.0239 + 0.00033 \approx 0.500$$

Q3. Explain why function composition is not commutative with an example.

Let $f(x) = x + 2$, $g(x) = 2x$. Then:

$$f(g(x)) = f(2x) = 2x + 2, \quad g(f(x)) = g(x + 2) = 2(x + 2) = 2x + 4$$

$\Rightarrow f \circ g \neq g \circ f$