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1. Functions as a Special Type of Relation

- A **relation** between two sets A and B is any subset of the Cartesian product $A \times B$.
- A **function** $f : A \to B$ is a relation such that:
- $\forall x \in A, \exists! y \in B$ such that $(x, y) \in f$
- Each element of A maps to **exactly one** element in B
- Notation: $f(x) = y$

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2. Terminology

- **Domain**: Set of all input values $x \in A$
- **Codomain**: Target set B , may include elements not used by f
- **Range** (Image): $\{f(x) \mid x \in A\} \subseteq B$
- **Injective (One-to-One)**: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- **Surjective (Onto)**: $\forall y \in B, \exists x \in A : f(x) = y$
- **Bijective**: Both injective and surjective; every $y \in B$ is uniquely hit

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3. Function Composition

- $(g \circ f)(x) = g(f(x))$
- **Associative**: $h \circ (g \circ f) = (h \circ g) \circ f$

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4. Inverse Functions

- A function f has an **inverse** f^{-1} iff it is **bijective**
- $f^{-1}(y) = x \iff f(x) = y$
- $f^{-1} \circ f = id_A, \quad f \circ f^{-1} = id_B$

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5. Characteristic Functions

- For a set $A \subseteq U$, define:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

- Maps from $U \to \{0, 1\}$, useful in logic and probability

6. Sequences and Strings

- A sequence is a function from a finite (or countable) set of indices to values
- Example: $s : \mathbb{N} \to A$
- Finite sequences often indexed by $\{0, 1, ..., n-1\}$
- A string is a sequence over a character set (alphabet)

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7. Relations Revisited

- A relation $R \subseteq A \times B$: set of ordered pairs
- If $(a, b) \in R$, we write aRb
- **Reflexive**: $\forall a \in A, aRa$
- **Symmetric**: $aRb \Rightarrow bRa$
- **Transitive**: $aRb \wedge bRc \Rightarrow aRc$
- **Antisymmetric**: $aRb \wedge bRa \Rightarrow a = b$

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8. Equivalence Relations

- A relation that is:
- Reflexive
- Symmetric
- Transitive
- Partitions the set into **equivalence classes**
- If $a \sim b$, they belong to the same class

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9. Partial and Total Orders

- **Partial order**: relation that is reflexive, antisymmetric, and transitive
- **Total order**: additionally, $\forall a, b \in A, a \leq b \vee b \leq a$
- Not all sets have total orderings

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10. Examples and Diagrams

- Diagrams often used to visualise relations and functions:
- Arrows from domain to codomain
- Graphs or matrices for relations

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11. Summary

- Functions are deterministic relations
- Understanding domain, codomain, and image is critical
- Injectivity and surjectivity define function structure
- Equivalence relations group elements into meaningful clusters
- Order relations allow reasoning about structure

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12. Looking Ahead

- Next: more on equivalence classes and quotient sets
- Practice translating logic into function and relation definitions
- Apply to modelling, logic, and computation