COMP9020: Set Theory and Logic - Lecture Summary and Notes

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1. Introduction and Review

- The lecture began with a note on how today's topic builds on predicate logic, models, and propositional connectives.
- Set theory introduces new **notation** and **terminology**, but the **core logical ideas remain the same**.

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2. Set Notation and Definitions

2.1 Set Builder Notation

- $\{x \in S \mid P(x)\}$: Set of all elements x in set S such that property P holds.
- Alternative syntax includes semicolons and vertical bars. Semicolons preferred by lecturer due to overuse of vertical bar.

2.2 Subsets and Set Identity

- **Subset**: $A \subseteq B \iff \forall x(x \in A \Rightarrow x \in B)$
- **Not a Subset**: $A \nsubseteq B \iff \exists x(x \in A \wedge x \notin B)$
- **Proper Subset**: $A \subset B \iff A \subseteq B \wedge \exists x(x \in B \wedge x \notin A)$
- **Set Equality**: $A = B \iff A \subseteq B \wedge B \subseteq A$
- **Important**: $A \subset B \wedge B \subset A$ is a contradiction.

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3. Examples with Integer Sets

3.1

Let: - $A = \{m \in \mathbb{Z} \mid m = 2a \text{ for some } a \in \mathbb{Z}\}$ - $B = \{n \in \mathbb{Z} \mid n = 2b - 2 \text{ for some } b \in \mathbb{Z}\}$

Proof of Equality: Show mutual inclusion by algebraic manipulation: - $b = a + 1 \Rightarrow 2b - 2 = 2a \Rightarrow A \subseteq B$ - $a = b - 1 \Rightarrow 2a = 2b - 2 \Rightarrow B \subseteq A$ - $\therefore A = B$

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4. Operations on Sets

4.1 Basic Operations

- **Union**: $A \cup B = \{x \in U \mid x \in A \vee x \in B\}$
- **Intersection**: $A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$
- **Difference**: $B - A = \{x \in B \mid x \notin A\}$

- **Complement**: $\overline{A} = \{x \in U \mid x \notin A\}$

4.2 Indexed Families

- **Union**: $\bigcup_{i=1}^{n} A_i = \{x \in U \mid \exists i (x \in A_i)\}$
- **Intersection**: $\bigcap_{i=1}^{n} A_i = \{x \in U \mid \forall i (x \in A_i)\}$

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5. The Empty Set

- **Notation**: $\emptyset$ or $\{\}$
- **Important**: $\{\emptyset\} \neq \emptyset$
- One and only one empty set, since a set is defined entirely by its elements.

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6. Avoiding Russell's Paradox

- **Naive Set Theory** leads to paradoxes under **unrestricted comprehension**.
- **Russell's Paradox**: $R = \{x \mid x \notin x\}$ → contradiction.
- Modern set theory uses **iterative conception** to avoid such paradoxes.

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7. Partitions and Disjoint Sets

- **Disjoint Sets**: $A \cap B = \emptyset$
- **Pairwise Disjoint**: $A_i \cap A_j = \emptyset$ for all $i \neq j$
- **Partition**: A set of non-empty, pairwise disjoint subsets whose union is the original set.

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8. Power Set

- **Definition**: $\mathcal{P}(A) = \{X \mid X \subseteq A\}$
- **Size**: If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$

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9. Algebra of Sets

9.1 Key Properties

- **Commutativity**:
- $A \cup B = B \cup A$

- $A \cap B = B \cap A$

- **Associativity**:

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

- **Distributivity**:

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- **Identity Laws**:

- $A \cup \emptyset = A$
- $A \cap U = A$

- **Domination Laws**:

- $A \cup U = U$
- $A \cap \emptyset = \emptyset$

- **Complement Laws**:

- $A \cup \overline{A} = U$
- $A \cap \overline{A} = \emptyset$

- **Double Complement**:

- $\overline{\overline{A}} = A$

- **Idempotent Laws**:

- $A \cup A = A$
- $A \cap A = A$

- **Absorption Laws**:

- $A \cup (A \cap B) = A$
- $A \cap (A \cup B) = A$

- **De Morgan's Laws**:

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$

- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- **Set Difference**:

- $A - B = A \cap \overline{B}$

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10. Partial Orders and Logic Analogy

- $\subseteq$ behaves like logical consequence ($\Rightarrow$)
- $A \subseteq B \wedge B \subseteq A \Rightarrow A = B$
- $\subseteq$ is a **partial order**:
- Reflexive: $A \subseteq A$
- Transitive: $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$
- Antisymmetric: $A \subseteq B, B \subseteq A \Rightarrow A = B$

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11. Ordered Pairs and Cartesian Products

- $\langle x, y \rangle \in A \times B \iff x \in A \wedge y \in B$
- Angle brackets used to denote ordered pairs
- Contrast: sets are unordered, tuples are ordered

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12. Functions and Relations Preview

- **Functions** are a special case of **relations**
- Functions: one input maps to exactly one output
- Relations: general, may have multiple outputs per input

12.1 Extensions:

- 1-place predicate → set of objects
- 2-place predicate → set of ordered pairs
- n-place predicate → set of ordered n-tuples

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13. Exam and Assessment Info

- No midterm, just 8 problem sets
- Problem Set 4 due after Flex week
- Final exam is **open book**, pen-and-paper, centrally scheduled
- Practice exams: past papers + supplementary exam
- Practice under timed conditions (2 hours)

- Exam includes: propositional logic, predicate logic, set theory, functions, relations, probability, graph theory
- Use strategy: answer easy questions first, return to harder ones later

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14. Final Encouragement

- Hardest part of the course (predicate logic) is behind you
- Everything builds on that foundation
- Practice and engagement more important than memorisation
- Stay calm and keep problem-solving