COMP9020: Relations — Complete Study Notes

Contents

1. Properties of Relations

Reflexivity

A relation R on set A is reflexive if:

$$\forall x \in A, (x, x) \in R$$

Example: $A = \{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$ is reflexive.
Non-example: $R = \{(1, 1), (2, 2)\}$ is not reflexive (missing (3,3)).

Questions:

- Is $R = \{(a, a), (b, b)\}$ reflexive on $A = \{a, b, c\}$?
- Give an example of a reflexive relation on $A = \{1, 2\}$.
- True/False: A reflexive relation must have exactly $|A|$ elements.

Solutions:

- No, (c,c) is missing.
- $\{(1, 1), (2, 2)\}$.
- False. It can have more than $|A|$ elements.

Irreflexivity

A relation R is irreflexive if:

$$\forall x \in A, (x, x) \notin R$$

Example: $R = \{(1, 2), (2, 1)\}$ on $A = \{1, 2\}$ is irreflexive.
Non-example: $R = \{(1, 1)\}$ is not irreflexive.

Questions and Solutions:

- **Q:** Give an irreflexive relation on $\{1, 2, 3\}$.
  A: $R = \{(1, 2), (2, 3), (3, 1)\}$.
- **Q:** Can a relation be both reflexive and irreflexive?
  A: No, unless A is empty.

Symmetry

R is symmetric if:

$$(x, y) \in R \Rightarrow (y, x) \in R$$

Example: $\{(1, 2), (2, 1)\}$ is symmetric.
Non-example: $\{(1, 2)\}$ is not symmetric.
Q: Is $R = \{(a, b), (b, a), (c, c)\}$ symmetric?
A: Yes.
Q: Can a symmetric relation be irreflexive?
A: Yes, e.g., $\{(1, 2), (2, 1)\}$.

Antisymmetry

R is antisymmetric if:

$$(x, y) \in R \text{ and } (y, x) \in R \Rightarrow x = y$$

Example: $\{(1, 2)\}$ is antisymmetric.
Non-example: $\{(1, 2), (2, 1)\}$ is not antisymmetric.
Q: Let $R = \{(a, a), (a, b), (b, c)\}$. Is R antisymmetric?
A: Yes.
Q: Can a relation be both symmetric and antisymmetric?
A: Yes, if only self-pairs exist.

Transitivity

R is transitive if:

$$(x, y) \in R \text{ and } (y, z) \in R \Rightarrow (x, z) \in R$$

Example: $\{(1, 2), (2, 3), (1, 3)\}$.
Non-example: $\{(1, 2), (2, 3)\}$ missing (1,3).
Q: What must be added to $\{(a, b), (b, c)\}$ to make it transitive?
A: Add (a,c).
Q: Transitive closure of $\{(0, 1), (1, 2)\}$?
A: $\{(0, 1), (1, 2), (0, 2)\}$

Equivalence Relations

Reflexive + Symmetric + Transitive
Example: $\{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$
Q: Prove congruence mod n is equivalence.
A: Reflexive, symmetric, and transitive by definition.
Q: Equivalence classes of $R = \{(0, 0), (1, 1), (1, 3), (3, 1), (3, 3)\}$?
A: $[0] = \{0\}, [1] = [3] = \{1, 3\}$

Partial Orders

Reflexive + Antisymmetric + Transitive
Example: $\leq, \subseteq$
Q: Is $\subseteq$ a partial order?
A: Yes.
Q: Draw Hasse diagram of $(P(\{a, b\}), \subseteq)$
A: Show all subsets ordered by inclusion.

2. Relational Examples Using Mathematical Symbols

Equality (=)

$$R = \{(a, a) \mid a \in A\}$$

An equivalence relation.

Less Than (<)

$$R = \{(a, b) \mid a < b\}$$

Irreflexive, transitive, asymmetric.

Divides (|)

$$R = \{(a, b) \mid a \mid b\}$$

Reflexive, transitive, antisymmetric partial order.

Exam-Style Questions

- Prove that "$\leq$" is a partial order on $\mathbb{Z}$.
- Show that "$|$" is transitive on $\mathbb{N}$.
- Is "$<$" symmetric, antisymmetric, or neither on $\mathbb{R}$?