ECSE 543
Assignment 1

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Introduction

The programs for this assignment were created in Python 2.7. The source code is provided as listings in Appendix A. To perform the required tasks in this assignment, a custom matrix package was created, with useful methods such as add, multiply, transpose, etc. This package can be seen in Listing 1. In addition, logs of the output of the programs are provided in Appendix B.

1 Choleski Decomposition

The source code for the Question 1 main program can be seen in Listing 4.

1.a Choleski Program

The code relating specifically to Choleski decomposition can be seen in Listing 2.

1.b Constructing Test Matrices

The matrices were constructed with the knowledge that, if A is positive-definite, then $A = LL^T$ where L is a lower triangular non-singular matrix. The task of choosing valid A matrices then boils down to finding non-singular lower triangular L matrices. To ensure that L is non-singular, one must simply choose nonzero values for the main diagonal.

1.c Test Runs

The matrices were tested by inventing x matrices, and checking that the program solves for that x correctly. The output of the program, comparing expected and obtained values of x, can be seen in Listing 8.

1.d Linear Networks

As can be seen in Listing 3, the `csv_to_network_branch_matrices` method of the `linear_networks.py` script reads from a CSV file where row k contains J_k, R_k and E_k. It then converts the resistances to a diagonal admittance matrix Y and produces the J and E column vectors. The incidence matrix A is also read directly from file, as seen in Listing 4.

First, the program was tested on the circuits provided on MyCourses. These circuits are labeled 1 to 5 and have corresponding incidence matrix and network branch CSV files, located in the `network_data` directory. The program obtains the expected voltages, as seen in the output in Listing 8.

Then, some additional simple test circuits were created. Circuit 6 can be seen in Figure 1 and the SPICE analysis output in Table 1. These voltages match the ones calculated by the program, as seen in Listing 8.

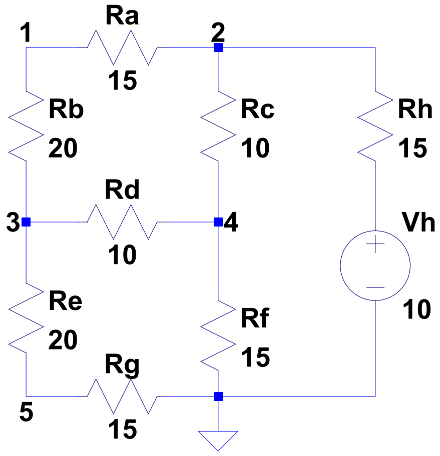

Figure 1: Test circuit 6 with nodes labeled 1 to 4.

Table 1: Output of SPICE operating point analysis of circuit 6.

Node	Voltage (V)
1	4.443
2	5.498
3	3.036
4	3.200
5	1.301

2 Finite Difference Mesh

The source code for the Question 2 main program can be seen in Listing 5.

2.a Equivalent Resistance

The code for creating all the network matrices and for finding the equivalent resistance of an N by $2N$ mesh can be seen in Listing 3. The resistances found by the program for values of N from 2 to 10 can be seen in Table 2.

The resistance values returned by the program for small meshes were validated using simple SPICE circuits. The voltage found at the V_{test} node for the 2x4 mesh is 1.875 V and the equivalent resistance is therefore 1875 Ω. Similarly, for the 3x6 mesh, $V_{test} = 2.379\,55$ V and the equivalent resistance is 2379.55 Ω. These match the results found by the program, as seen in Table 2.

Table 2: Mesh equivalent resistance R versus mesh size N.

N	R (Omega)
2	1875.000
3	2379.545
4	2741.025
5	3022.819
6	3253.676
7	3449.166
8	3618.675
9	3768.291
10	3902.189

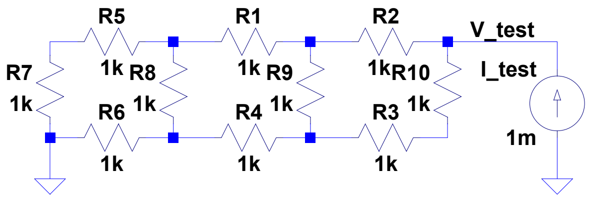

Figure 2: SPICE circuit used to test the 2x4 mesh.

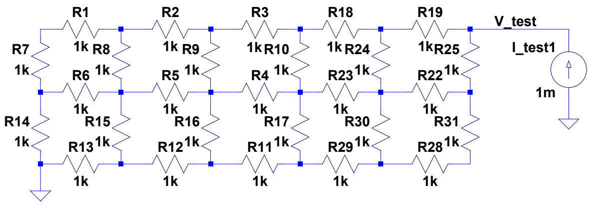

Figure 3: SPICE circuit used to test the 3x6 mesh.

2.b Time Complexity

The runtime data for the mesh resistance solver is tabulated in Table 3 and plotted in Figure 4. Theoretically, the time complexity of the program should be $O(N^6)$, and this matches the obtained data.

Table 3: Runtime of mesh resistance solver program versus mesh size N.

N	Runtime (s)
2	0.000
3	0.016
4	0.094
5	0.386
6	1.266
7	3.142
8	6.953
9	14.438
10	27.922

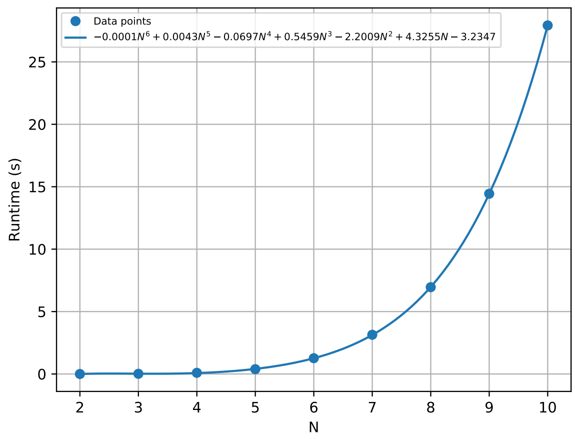

Figure 4: Runtime of mesh resistance solver program versus mesh size N.

2.c Sparsity Modification

The runtime data for the banded mesh resistance solver is tabulated in Table 4 and plotted in Figure 5. By inspection of the constructed network matrices, a half-bandwidth of $2N+1$ was chosen. Theoretically, the banded version should have a time complexity of $O(N^4)$.

Table 4: Runtime of banded mesh resistance solver program versus mesh size N.

N	Runtime (s)
2	0.016
3	0.015
4	0.078
5	0.372
6	1.099
7	2.969
8	6.417
9	13.317
10	25.448

The runtime of the banded and non-banded versions of the program are plotted in Figure 6, showing the benefits of banded elimination.

2.d Resistance vs. Mesh Size

The equivalent mesh resistance R is plotted versus the mesh size N in Figure 7. The function $R(N)$ appears logarithmic, and a log function does indeed fit the data well.

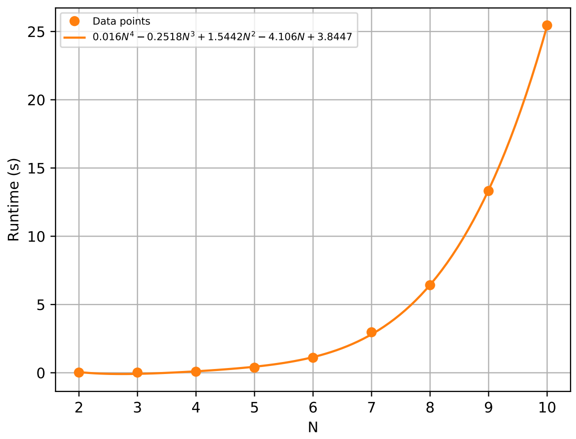

Figure 5: Runtime of banded mesh resistance solver program versus mesh size N.

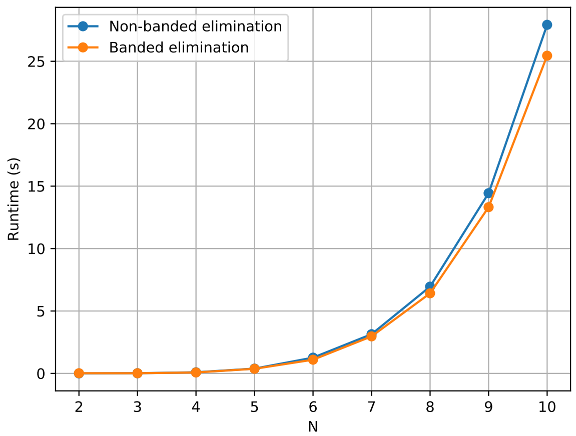

Figure 6: Comparison of runtime of banded and non-banded resistance solver programs versus mesh size N.

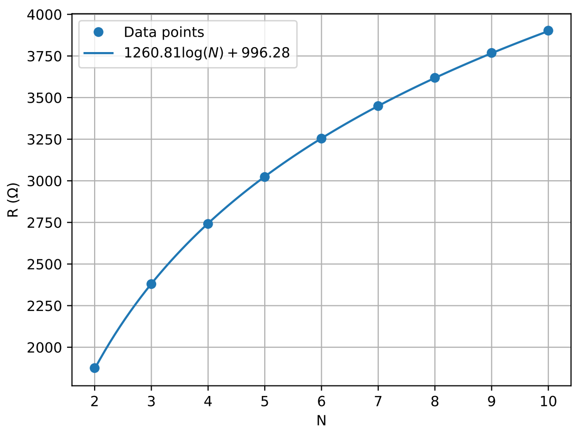

Figure 7: Resistance of mesh versus mesh size N.

3 Coaxial Cable

The source code for the Question 2 main program can be seen in Listing 7.

3.a SOR Program

The source code for the finite difference methods can be seen in Listing 6. Horizontal and vertical symmetries were exploited by only solving for a quarter of the coaxial cable, and reproducing the results where necessary.

3.b Varying ω

The number of iterations to achieve convergence for 10 values of ω between 1 and 2 are tabulated in Table 5 and plotted in Figure 8. Based on these results, the value of ω yielding the minimum number of iterations is 1.3.

Table 5: Number of iterations of SOR versus ω.

Omega	Iterations
1.0	32
1.1	26
1.2	20
1.3	14
1.4	16
1.5	20
1.6	27
1.7	39
1.8	60
1.9	127

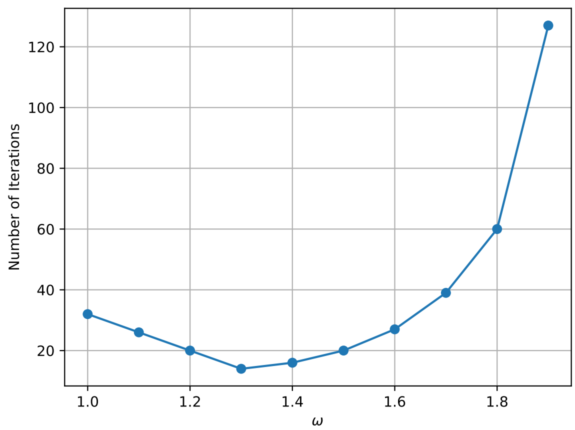

Figure 8: Number of iterations of SOR versus ω.

The potential values found at (0.06, 0.04) versus ω are tabulated in Table 6. It can be seen that all the potential values are identical to 3 decimal places.

Table 6: Potential at (0.06, 0.04) versus ω when using SOR.

Omega	Potential (V)
1.0	5.526
1.1	5.526
1.2	5.526
1.3	5.526
1.4	5.526
1.5	5.526
1.6	5.526
1.7	5.526
1.8	5.526
1.9	5.526

3.c Varying h

With $\omega = 1.3$, the number of iterations of SOR versus $1/h$ is tabulated in Table 7 and plotted in Figure 9. It can be seen that the smaller the node spacing is, the more iterations the program will take to run. Theoretically, the time complexity of the program should be $O(N^3)$, where the finite difference mesh is N by N, and this matches the measured data.

Table 7: Number of iterations of SOR versus $1/h$. Note that $\omega = 1.3$.

1/h	Iterations
50.0	14
100.0	59
200.0	189
400.0	552
800.0	1540
1600.0	4507

The potential values found at (0.06, 0.04) versus $1/h$ are tabulated in Table 8 and plotted in Figure 10. By examining these values, the potential at (0.06, 0.04) to three significant figures is approximately 5.25 V. It can be seen that the smaller the node spacing is, the more accurate the calculated potential is. However, by inspecting Figure 10 it is apparent that the potential converges relatively quickly to around 5.25 V There are therefore diminishing returns to decreasing the node spacing too much, since this will also increase the runtime of the program.

3.d Jacobi Method

The number of iterations of the Jacobi method versus $1/h$ is tabulated in Table 9 and plotted in

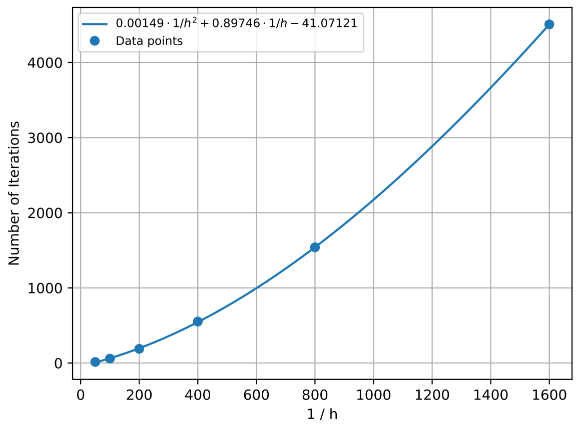

Figure 9: Number of iterations of SOR versus $1/h$. Note that $\omega = 1.3$.

Table 8: Potential at (0.06, 0.04) versus $1/h$ when using SOR.

1/h	Potential (V)
50.0	5.526
100.0	5.351
200.0	5.289
400.0	5.265
800.0	5.254
1600.0	5.247

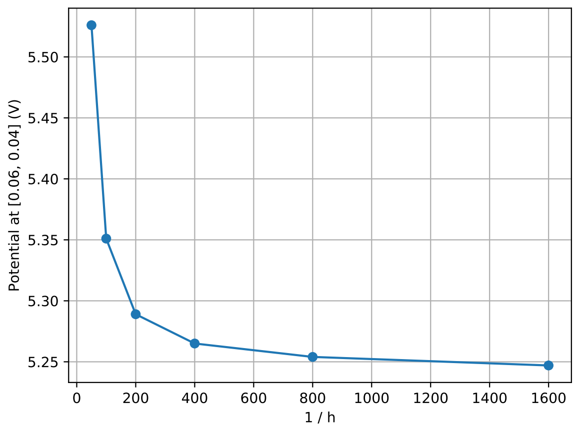

Figure 10: Potential at (0.06, 0.04) found by SOR versus $1/h$. Note that $\omega = 1.3$.

Figure 11. Similarly to SOR, the smaller the node spacing is, the more iterations the program will take to run. We can see however that the Jacobi method takes a much larger number of iterations to converge. Theoretically, the Jacobi method should have a time complexity of $O(N^4)$, and this matches the data.

The potential values found at (0.06, 0.04) ver-

Table 9: Number of iterations versus ω when using the Jacobi method.

1/h	Iterations
50.0	51
100.0	180
200.0	604
400.0	1935
800.0	5836
1600.0	16864

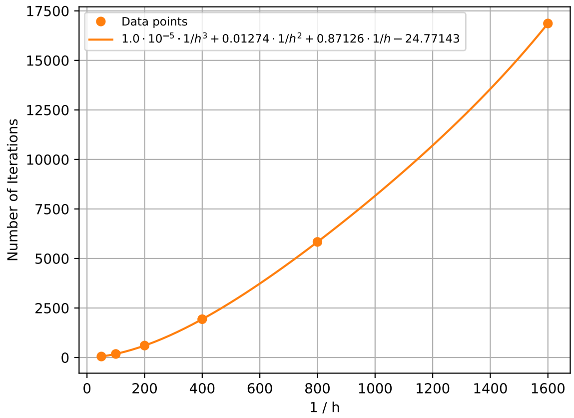

Figure 11: Number of iterations of the Jacobi method versus $1/h$.

sus $1/h$ with the Jacobi method are tabulated in Table 10 and plotted in Figure 12. These potential values are almost identical to the SOR ones. Similarly to SOR, the smaller the node spacing is, the more accurate the calculated potential is.

Table 10: Potential at (0.06, 0.04) versus $1/h$ when using the Jacobi method.

1/h	Potential (V)
50.0	5.526
100.0	5.351
200.0	5.289
400.0	5.265
800.0	5.254
1600.0	5.246

The number of iterations of both SOR and the Jacobi method can be seen in Figure 13, which shows the clear benefits of SOR.

3.e Non-uniform Node Spacing

First, we adjust the equation derived in class to set $a_1 = \Delta_x \alpha_1$, $a_2 = \Delta_x \alpha_2$, $b_1 = \Delta_y \beta_1$ and

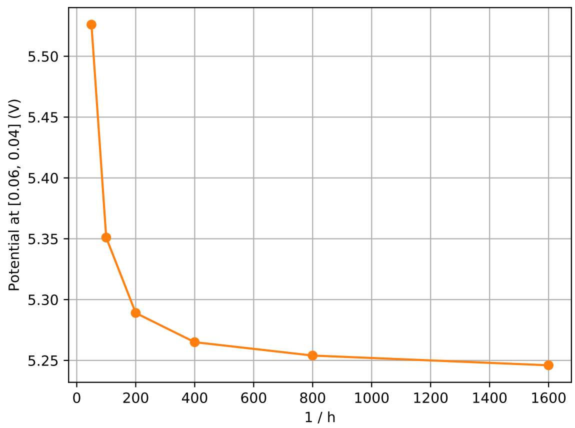

Figure 12: Potential at (0.06, 0.04) versus $1/h$ when using the Jacobi method.

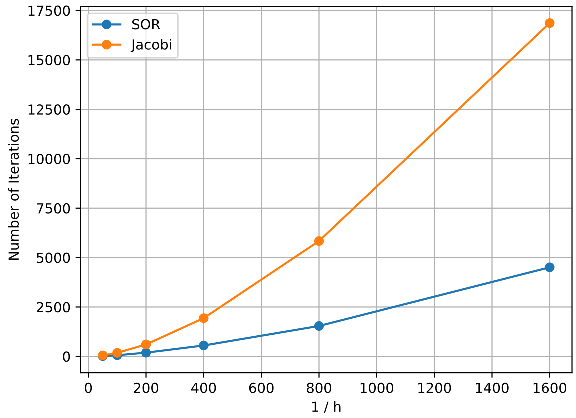

Figure 13: Comparison of number of iterations when using SOR and Jacobi methods versus $1/h$. Note that $\omega = 1.3$ for the SOR program.

$b_2 = \Delta_y \beta_2$. These values correspond to the distances between adjacent nodes [1], and can be easily calculated by the program. Then, the five-point difference formula for non-uniform spacing can be seen in Equation 1.

$$\begin{aligned} \phi_{i,j}^{k+1} = & \frac{1}{a_1 + a_2} \left(\frac{\phi_{i-1,j}^k}{a_1} + \frac{\phi_{i+1,j}^k}{a_2} \right) + \\ & \frac{1}{b_1 + b_2} \left(\frac{\phi_{i,j-1}^k}{b_1} + \frac{\phi_{i,j+1}^k}{b_2} \right) \end{aligned} \tag{1}$$

This was implemented in the finite difference program, as seen in Listing 6. As can be seen in this code, many different mesh arrangements were

[1] Note that, in the program, index i is associated to position x and index j is associated to position y. This is purely for easier printing of the matrices.

tested. The arrangement that was chosen can be seen in Figure 14. The potential at (0.06, 0.04) obtained from this arrangement is 5.243 V, which seems like an accurate potential value. Indeed, as can be seen in Figures 10 and 12, the potential value for small node spacings tends towards 5.24 V for both the Jacobi and SOR methods.

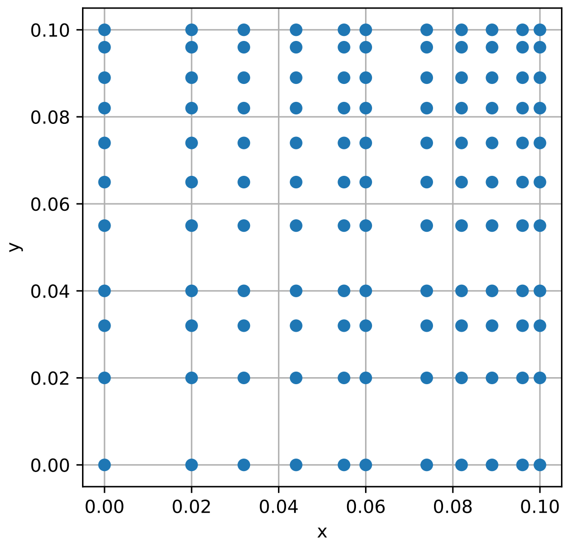

Figure 14: Final mesh arrangement used for non-uniform node spacing. Each point corresponds to a mesh point. Points are positioned closer to the inner conductor, since this is a more difficult area.

A Code Listings

Listing 1: Custom matrix package (`matrices.py`).

```python
from __future__ import division

import copy
import csv
from ast import literal_eval

import math


class Matrix:

    def __init__(self, data):
        self.data = data
        self.rows = len(data)
        self.cols = len(data[0])

    def __str__(self):
        string = ''
        for row in self.data:
            string += '\n'
            for val in row:
                string += '{:10.5f} '.format(val)
        return string

    def __add__(self, other):
        if len(self) != len(other) or len(self[0]) != len(other[0]):
            raise ValueError('Incompatible matrix sizes for addition. Matrix A is {}x{}, but matrix B is {}x{}.'
                             .format(len(self), len(self[0]), len(other), len(other[0])))

        return Matrix([[self[row][col] + other[row][col] for col in range(self.cols)] for row in range(self.rows)])

    def __sub__(self, other):
        if len(self) != len(other) or len(self[0]) != len(other[0]):
            raise ValueError('Incompatible matrix sizes for subtraction. Matrix A is {}x{}, but matrix B is {}x{}.'
                             .format(len(self), len(self[0]), len(other), len(other[0])))

        return Matrix([[self[row][col] - other[row][col] for col in range(self.cols)] for row in range(self.rows)])

    def __mul__(self, other):
        if self.cols != other.rows:
            raise ValueError('Incompatible matrix sizes for multiplication. Matrix A is {}x{}, but matrix B is {}x{}.'
                             .format(self.rows, self.cols, other.rows, other.cols))

        # Inspired from https://en.wikipedia.org/wiki/Matrix_multiplication
        product = Matrix.empty(self.rows, other.cols)
        for i in range(self.rows):
            for j in range(other.cols):
                row_sum = 0
                for k in range(self.cols):
                    row_sum += self[i][k] * other[k][j]
                product[i][j] = row_sum
        return product

    def __deepcopy__(self, memo):
        return Matrix(copy.deepcopy(self.data))

    def __getitem__(self, item):
        return self.data[item]

    def __len__(self):
```

```
        return len(self.data)

    def is_positive_definite(self):
        """
        :return: True if the matrix if positive-definite, False otherwise.
        """
        A = copy.deepcopy(self.data)
        for j in range(self.rows):
            if A[j][j] <= 0:
                return False
            A[j][j] = math.sqrt(A[j][j])
            for i in range(j + 1, self.rows):
                A[i][j] = A[i][j] / A[j][j]
                for k in range(j + 1, i + 1):
                    A[i][k] = A[i][k] - A[i][j] * A[k][j]
        return True

    def transpose(self):
        """
        :return: the transpose of the current matrix
        """
        return Matrix([[self.data[row][col] for row in range(self.rows)] for col in range(self.cols)])

    def mirror_horizontal(self):
        """
        :return: the horizontal mirror of the current matrix
        """
        return Matrix([[self.data[self.rows - row - 1][col] for col in range(self.cols)] for row in range(self.rows)])

    def empty_copy(self):
        """
        :return: an empty matrix of the same size as the current matrix.
        """
        return Matrix.empty(self.rows, self.cols)

    @staticmethod
    def multiply(*matrices):
        """
        Computes the product of the given matrices.

        :param matrices: the matrix objects
        :return: the product of the given matrices
        """
        n = matrices[0].rows
        product = Matrix.identity(n)
        for matrix in matrices:
            product = product * matrix
        return product

    @staticmethod
    def empty(num_rows, num_cols):
        """
        Returns an empty matrix (filled with zeroes) with the specified number of columns and rows.

        :param num_rows: number of rows
        :param num_cols: number of columns
        :return: the empty matrix
        """
        return Matrix([[0 for _ in range(num_cols)] for _ in range(num_rows)])

    @staticmethod
    def identity(n):
        """
        Returns the identity matrix of the given size.

        :param n: the size of the identity matrix (number of rows or columns)
        :return: the identity matrix of size n
        """
        return Matrix.diagonal_single_value(1, n)
```

```python

    @staticmethod
    def diagonal(values):
        """
        Returns a diagonal matrix with the given values along the main diagonal.

        :param values: the values along the main diagonal
        :return: a diagonal matrix with the given values along the main diagonal
        """
        n = len(values)
        return Matrix([[values[row] if row == col else 0 for col in range(n)] for row in range(n)])

    @staticmethod
    def diagonal_single_value(value, n):
        """
        Returns a diagonal matrix of the given size with the given value along the diagonal.

        :param value: the value of each element on the main diagonal
        :param n: the size of the matrix
        :return: a diagonal matrix of the given size with the given value along the diagonal.
        """
        return Matrix([[value if row == col else 0 for col in range(n)] for row in range(n)])

    @staticmethod
    def column_vector(values):
        """
        Transforms a row vector into a column vector.

        :param values: the values, one for each row of the column vector
        :return: the column vector
        """
        return Matrix([[value] for value in values])

    @staticmethod
    def csv_to_matrix(filename):
        """
        Reads a CSV file to a matrix.

        :param filename: the name of the CSV file
        :return: a matrix containing the values in the CSV file
        """
        with open(filename, 'r') as csv_file:
            reader = csv.reader(csv_file)
            data = []
            for row_number, row in enumerate(reader):
                data.append([literal_eval(val) for val in row])
            return Matrix(data)
```

Listing 2: Choleski decomposition (`choleski.py`).

```python
from __future__ import division

import math

from matrices import Matrix


def choleski_solve(A, b, half_bandwidth=None):
    """
    Solves an Ax = b matrix equation by Choleski decomposition.

    :param A: the A matrix
    :param b: the b matrix
    :param half_bandwidth: the half-bandwidth of the A matrix
    :return: the solved x vector
    """
    n = len(A[0])
    if half_bandwidth is None:
        elimination(A, b)
```

```
    else:
        elimination_banded(A, b, half_bandwidth)
    x = Matrix.empty(n, 1)
    back_substitution(A, x, b)
    return x


def elimination(A, b):
    """
    Performs the elimination step of Choleski decomposition.

    :param A: the A matrix
    :param b: the b matrix
    """
    n = len(A)
    for j in range(n):
        if A[j][j] <= 0:
            raise ValueError('Matrix A is not positive definite.')
        A[j][j] = math.sqrt(A[j][j])
        b[j][0] = b[j][0] / A[j][j]
        for i in range(j + 1, n):
            A[i][j] = A[i][j] / A[j][j]
            b[i][0] = b[i][0] - A[i][j] * b[j][0]
            for k in range(j + 1, i + 1):
                A[i][k] = A[i][k] - A[i][j] * A[k][j]


def elimination_banded(A, b, half_bandwidth):  # TODO: Keep limited band in memory, improve time complexity
    """
    Performs the banded elimination step of Choleski decomposition.

    :param A: the A matrix
    :param b: the b matrix
    :param half_bandwidth: the half_bandwidth to be used for the banded elimination
    """
    n = len(A)
    for j in range(n):
        A[j][j] = math.sqrt(A[j][j])
        b[j][0] = b[j][0] / A[j][j]
        max_row = min(j + half_bandwidth, n)
        for i in range(j + 1, max_row):
            A[i][j] = A[i][j] / A[j][j]
            b[i][0] = b[i][0] - A[i][j] * b[j][0]
            for k in range(j + 1, i + 1):
                A[i][k] = A[i][k] - A[i][j] * A[k][j]


def back_substitution(L, x, y):
    """
    Performs the back-substitution step of Choleski decomposition.

    :param L: the L matrix
    :param x: the x matrix
    :param y: the y matrix
    """
    n = len(L)
    for i in range(n - 1, -1, -1):
        prev_sum = 0
        for j in range(i + 1, n):
            prev_sum += L[j][i] * x[j][0]
        x[i][0] = (y[i][0] - prev_sum) / L[i][i]
```

Listing 3: Linear resistive networks (`linear_networks.py`).

```
from __future__ import division

import csv
from matrices import Matrix
```

```python
from choleski import choleski_solve


def solve_linear_network(A, Y, J, E, half_bandwidth=None):
    """
    Solve the linear resistive network described by the given matrices.

    :param A: the incidence matrix
    :param Y: the admittance matrix
    :param J: the current source matrix
    :param E: the voltage source matrix
    :param half_bandwidth:
    :return: the solved voltage matrix
    """
    A_new = A * Y * A.transpose()
    b = A * (J - Y * E)
    return choleski_solve(A_new, b, half_bandwidth=half_bandwidth)


def csv_to_network_branch_matrices(filename):
    """
    Converts a CSV file to Y, J, E network matrices.

    :param filename: the name of the CSV file
    :return: the Y, J, E network matrices
    """
    with open(filename, 'r') as csv_file:
        reader = csv.reader(csv_file)
        J = []
        Y = []
        E = []
        for row in reader:
            J_k = float(row[0])
            R_k = float(row[1])
            E_k = float(row[2])
            J.append(J_k)
            Y.append(1 / R_k)
            E.append(E_k)
        Y = Matrix.diagonal(Y)
        J = Matrix.column_vector(J)
        E = Matrix.column_vector(E)
        return Y, J, E


def create_network_matrices_mesh(rows, cols, branch_resistance, test_current):
    num_horizontal_branches = (cols - 1) * rows
    num_vertical_branches = (rows - 1) * cols
    num_branches = num_horizontal_branches + num_vertical_branches + 1
    num_nodes = rows * cols - 1

    A = create_incidence_matrix_mesh(cols, num_branches, num_horizontal_branches, num_nodes,
    ↪   num_vertical_branches)
    Y, J, E = create_network_branch_matrices_mesh(num_branches, branch_resistance, test_current)

    return A, Y, J, E


def create_incidence_matrix_mesh(cols, num_branches, num_horizontal_branches, num_nodes,
↪   num_vertical_branches):
    A = Matrix.empty(num_nodes, num_branches)
    node_offset = -1
    for branch in range(num_horizontal_branches):
        if branch == num_horizontal_branches - cols + 1:
            A[branch + node_offset + 1][branch] = 1
        else:
            if branch % (cols - 1) == 0:
                node_offset += 1
            node_number = branch + node_offset
            A[node_number][branch] = -1
            A[node_number + 1][branch] = 1
```

```
    branch_offset = num_horizontal_branches
    node_offset = cols
    for branch in range(num_vertical_branches):
        if branch == num_vertical_branches - cols:
            node_offset -= 1
            A[branch][branch + branch_offset] = 1
        else:
            A[branch][branch + branch_offset] = 1
            A[branch + node_offset][branch + branch_offset] = -1
    if num_branches == 2:
        A[0][1] = -1
    else:
        A[cols - 1][num_branches - 1] = -1
    return A


def create_network_branch_matrices_mesh(num_branches, resistance, test_current):
    Y = Matrix.diagonal([1 / resistance if branch < num_branches - 1 else 0 for branch in
     ↪  range(num_branches)])
    # Negative test current here because we assume current is coming OUT of the test current node.
    J = Matrix.column_vector([0 if branch < num_branches - 1 else -test_current for branch in
     ↪  range(num_branches)])
    E = Matrix.column_vector([0 for branch in range(num_branches)])
    return Y, J, E


def find_mesh_resistance(n, branch_resistance, half_bandwidth=None):
    test_current = 0.01
    A, Y, J, E = create_network_matrices_mesh(n, 2 * n, branch_resistance, test_current)
    x = solve_linear_network(A, Y, J, E, half_bandwidth=half_bandwidth)
    test_voltage = x[2 * n - 1 if n > 1 else 0][0]
    equivalent_resistance = test_voltage / test_current
    return equivalent_resistance
```

Listing 4: Question 1 (q1.py).

```
from __future__ import division

from linear_networks import solve_linear_network, csv_to_network_branch_matrices
from choleski import choleski_solve
from matrices import Matrix

NETWORK_DIRECTORY = 'network_data'

L_2 = Matrix([
    [5, 0],
    [1, 3]
])
L_3 = Matrix([
    [3, 0, 0],
    [1, 2, 0],
    [8, 5, 1]
])
L_4 = Matrix([
    [1, 0, 0, 0],
    [2, 8, 0, 0],
    [5, 5, 4, 0],
    [7, 2, 8, 7]
])
matrix_2 = L_2 * L_2.transpose()
matrix_3 = L_3 * L_3.transpose()
matrix_4 = L_4 * L_4.transpose()
positive_definite_matrices = [matrix_2, matrix_3, matrix_4]

x_2 = Matrix.column_vector([8, 3])
x_3 = Matrix.column_vector([9, 4, 3])
x_4 = Matrix.column_vector([5, 4, 1, 9])
xs = [x_2, x_3, x_4]

```

```python

def q1b():
    print('=== Question 1(b) ===')
    for count, A in enumerate(positive_definite_matrices):
        n = count + 2
        print('n={} matrix is positive-definite: {}'.format(n, A.is_positive_definite()))


def q1c():
    print('=== Question 1(c) ===')
    n = 2
    for x, A in zip(xs, positive_definite_matrices):
        b = A * x
        # print('A: {}'.format(A))
        # print('b: {}'.format(b))

        x_choleski = choleski_solve(A, b)
        print('Matrix with n={}:'.format(n))
        print('Expected x: {}'.format(x))
        print('Actual x: {}'.format(x_choleski))
        n += 1


def q1d():
    print('=== Question 1(d) ===')
    for i in range(1, 7):
        A = Matrix.csv_to_matrix('{}/incidence_matrix_{}.csv'.format(NETWORK_DIRECTORY, i))
        Y, J, E = csv_to_network_branch_matrices('{}/network_branches_{}.csv'.format(NETWORK_DIRECTORY, i))
        # print('Y: {}'.format(Y))
        # print('J: {}'.format(J))
        # print('E: {}'.format(E))
        x = solve_linear_network(A, Y, J, E)
        print('Solved for x in network {}:'.format(i))  # TODO: Create my own test circuits here
        for j in range(len(x)):
            print('V{} = {:.3f} V'.format(j + 1, x[j][0]))


def q1():
    q1b()
    q1c()
    q1d()


if __name__ == '__main__':
    q1()
```

Listing 5: Question 2 (q2.py).

```python
import csv
import time

import matplotlib.pyplot as plt
import numpy as np
import numpy.polynomial.polynomial as poly
import sympy as sp
from matplotlib.ticker import MaxNLocator

from linear_networks import find_mesh_resistance


def find_mesh_resistances(banded):
    branch_resistance = 1000
    points = {}
    runtimes = {}
    for n in range(2, 11):
        start_time = time.time()
        half_bandwidth = 2 * n + 1 if banded else None
        equivalent_resistance = find_mesh_resistance(n, branch_resistance, half_bandwidth=half_bandwidth)
```

```python
        print('Equivalent resistance for {}x{} mesh: {:.2f} Ohms.'.format(n, 2 * n, equivalent_resistance))
        points[n] = '{:.3f}'.format(equivalent_resistance)
        runtime = time.time() - start_time
        runtimes[n] = '{:.3f}'.format(runtime)
        print('Runtime: {} s.'.format(runtime))
    plot_runtime(runtimes, banded)
    return points, runtimes


def q2ab():
    print('=== Question 2(a)(b) ===')
    _, runtimes = find_mesh_resistances(banded=False)
    save_rows_to_csv('report/csv/q2b.csv', zip(runtimes.keys(), runtimes.values()), header=('N', 'Runtime (s)'))
    return runtimes


def q2c():
    print('=== Question 2(c) ===')
    pts, runtimes = find_mesh_resistances(banded=True)
    save_rows_to_csv('report/csv/q2c.csv', zip(runtimes.keys(), runtimes.values()), header=('N', 'Runtime (s)'))
    return pts, runtimes


def plot_runtime(points, banded=False):
    """
    N^6: non-banded
    N^4: banded

    :param points:
    :param banded:
    """
    f = plt.figure()
    ax = f.gca()
    ax.xaxis.set_major_locator(MaxNLocator(integer=True))
    x_range = [float(x) for x in points.keys()]
    y_range = [float(y) for y in points.values()]
    plt.plot(x_range, y_range, '{}o'.format('C1' if banded else 'C0'), label='Data points')

    x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
    degree = 4 if banded else 6
    polynomial_coeffs = poly.polyfit(x_range, y_range, degree)
    polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
    N = sp.symbols("N")
    poly_label = sum(sp.S("{:.4f}".format(v)) * N ** i for i, v in enumerate(polynomial_coeffs))
    equation = '${}$'.format(sp.printing.latex(poly_label))
    plt.plot(x_new, polynomial_fit, '{}-'.format('C1' if banded else 'C0'), label=equation)

    plt.xlabel('N')
    plt.ylabel('Runtime (s)')
    plt.grid(True)
    plt.legend(fontsize='x-small')
    f.savefig('report/plots/q2{}.pdf'.format('c' if banded else 'b'), bbox_inches='tight')


def plot_runtimes(points1, points2):
    f = plt.figure()
    ax = f.gca()
    ax.xaxis.set_major_locator(MaxNLocator(integer=True))
    x_range = points1.keys()
    y_range = points1.values()
    y_banded_range = points2.values()
    plt.plot(x_range, y_range, 'o-', label='Non-banded elimination')
    plt.plot(x_range, y_banded_range, 'o-', label='Banded elimination')
    plt.xlabel('N')
    plt.ylabel('Runtime (s)')
    plt.grid(True)
    plt.legend()
```

```python
    f.savefig('report/plots/q2bc.pdf', bbox_inches='tight')


def q2d(points):
    print('=== Question 2(d) ===')
    f = plt.figure()
    ax = f.gca()
    ax.xaxis.set_major_locator(MaxNLocator(integer=True))
    x_range = [float(x) for x in points.keys()]
    y_range = [float(y) for y in points.values()]
    plt.plot(x_range, y_range, 'o', label='Data points')

    x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
    coeffs = poly.polyfit(np.log(x_range), y_range, deg=1)
    polynomial_fit = poly.polyval(np.log(x_new), coeffs)
    plt.plot(x_new, polynomial_fit, '{}-'.format('C0'), label='${:.2f}\log(N) + {:.2f}$'.format(coeffs[1],
    ↪   coeffs[0]))

    plt.xlabel('N')
    plt.ylabel('R ($\Omega$)')
    plt.grid(True)
    plt.legend()
    f.savefig('report/plots/q2d.pdf', bbox_inches='tight')
    save_rows_to_csv('report/csv/q2a.csv', zip(points.keys(), points.values()), header=('N', 'R (Omega)'))


def q2():
    runtimes1 = q2ab()
    pts, runtimes2 = q2c()
    plot_runtimes(runtimes1, runtimes2)
    q2d(pts)


def save_rows_to_csv(filename, rows, header=None):
    with open(filename, "wb") as f:
        writer = csv.writer(f)
        if header is not None:
            writer.writerow(header)
        for row in rows:
            writer.writerow(row)


if __name__ == '__main__':
    q2()
```

Listing 6: Finite difference method (`finite_diff.py`).

```python
from __future__ import division

import math
import random
from abc import ABCMeta, abstractmethod

from matrices import Matrix

MESH_SIZE = 0.2


class Relaxer:
    """
    Performs the relaxing stage of the finite difference method.
    """
    __metaclass__ = ABCMeta

    @abstractmethod
    def relax(self, phi, i, j):
        """
        Perform a relaxation iteration on a given (i, j) point of the given phi matrix.

```

```python
        :param phi: the phi matrix
        :param i: the row index
        :param j: the column index
        """
        raise NotImplementedError

    def reset(self):
        """
        Optional method to reset the relaxer.
        """
        pass

    def residual(self, phi, i, j):
        """
        Calculate the residual at the given (i, j) point of the given phi matrix.

        :param phi: the phi matrix
        :param i: the row index
        :param j: the column index
        :return:
        """
        return abs(phi[i + 1][j] + phi[i - 1][j] + phi[i][j + 1] + phi[i][j - 1] - 4 * phi[i][j])


class GaussSeidelRelaxer(Relaxer):
    def relax(self, phi, i, j):
        return (phi[i + 1][j] + phi[i - 1][j] + phi[i][j + 1] + phi[i][j - 1]) / 4


class JacobiRelaxer(Relaxer):
    def __init__(self, num_cols):
        self.num_cols = num_cols
        self.prev_row = [0] * (num_cols - 1)  # Don't need to copy entire phi, just previous row

    def relax(self, phi, i, j):
        left_val = self.prev_row[j - 2] if j > 1 else 0
        top_val = self.prev_row[j - 1]
        self.prev_row[j - 1] = phi[i][j]
        return (phi[i + 1][j] + top_val + phi[i][j + 1] + left_val) / 4

    def reset(self):
        self.prev_row = [0] * (self.num_cols - 1)


class NonUniformRelaxer(Relaxer):
    def __init__(self, mesh):
        self.mesh = mesh

    def get_distances(self, i, j):
        a1 = self.mesh.get_y(i) - self.mesh.get_y(i - 1)
        a2 = self.mesh.get_y(i + 1) - self.mesh.get_y(i)
        b1 = self.mesh.get_x(j) - self.mesh.get_x(j - 1)
        b2 = self.mesh.get_x(j + 1) - self.mesh.get_x(j)
        return a1, a2, b1, b2

    def relax(self, phi, i, j):
        a1, a2, b1, b2 = self.get_distances(i, j)

        return ((phi[i - 1][j] / a1 + phi[i + 1][j] / a2) / (a1 + a2)
                + (phi[i][j - 1] / b1 + phi[i][j + 1] / b2) / (b1 + b2)) / (1 / (a1 * a2) + 1 / (b1 * b2))

    def residual(self, phi, i, j):
        a1, a2, b1, b2 = self.get_distances(i, j)

        return abs(((phi[i - 1][j] / a1 + phi[i + 1][j] / a2) / (a1 + a2)
                    + (phi[i][j - 1] / b1 + phi[i][j + 1] / b2) / (b1 + b2))
                   - phi[i][j] * (1 / (a1 * a2) + 1 / (b1 * b2)))


class SuccessiveOverRelaxer(Relaxer):
```

```
    def __init__(self, omega):
        self.gauss_seidel = GaussSeidelRelaxer()
        self.omega = omega

    def relax(self, phi, i, j, last_row=None, a1=None, a2=None, b1=None, b2=None):
        return (1 - self.omega) * phi[i][j] + self.omega * self.gauss_seidel.relax(phi, i, j)


class Boundary:
    """
    Constant-potential boundary in the finite difference mesh, representing a conductor.
    """
    __metaclass__ = ABCMeta

    @abstractmethod
    def potential(self):
        """
        Return the potential on the boundary.
        """
        raise NotImplementedError

    @abstractmethod
    def contains_point(self, x, y):
        """
        Returns true if the boundary contains the given (x, y) point.

        :param x: the x coordinate of the point
        :param y: the y coordinate of the point
        """
        raise NotImplementedError


class OuterConductorBoundary(Boundary):
    def potential(self):
        return 0

    def contains_point(self, x, y):
        return x == 0 or y == 0 or x == 0.2 or y == 0.2


class QuarterInnerConductorBoundary(Boundary):
    def potential(self):
        return 15

    def contains_point(self, x, y):
        return 0.06 <= x <= 0.14 and 0.08 <= y <= 0.12


class PotentialGuesser:
    """
    Guesses the initial potential in the finite-difference mesh.
    """
    __metaclass__ = ABCMeta

    def __init__(self, min_potential, max_potential):
        self.min_potential = min_potential
        self.max_potential = max_potential

    @abstractmethod
    def guess(self, x, y):
        """
        Guess the potential at the given (x, y) point, and return it.

        :param x: the x coordinate of the point
        :param y: the y coordinate of the point
        """
        raise NotImplementedError


class RandomPotentialGuesser(PotentialGuesser):
```

```
    def guess(self, x, y):
        return random.randint(self.min_potential, self.max_potential)


class LinearPotentialGuesser(PotentialGuesser):
    def guess(self, x, y):
        return 150 * x if x < 0.06 else 150 * y


class RadialPotentialGuesser(PotentialGuesser):
    def guess(self, x, y):
        def radial(k, x, y, x_source, y_source):
            return k / (math.sqrt((x_source - x) ** 2 + (y_source - y) ** 2))

        return 0.0225 * (radial(20, x, y, 0.1, 0.1) - radial(1, x, y, 0, y) - radial(1, x, y, x, 0))


class PhiConstructor:
    """
    Constructs the phi potential matrix with an outer conductor, inner conductor, mesh points and an inital potential
    guess.
    """

    def __init__(self, mesh):
        outer_boundary = OuterConductorBoundary()
        inner_boundary = QuarterInnerConductorBoundary()
        self.boundaries = (inner_boundary, outer_boundary)
        self.guesser = RadialPotentialGuesser(0, 15)
        self.mesh = mesh

    def construct_phi(self):
        phi = Matrix.empty(self.mesh.num_rows, self.mesh.num_cols)
        for i in range(self.mesh.num_rows):
            y = self.mesh.get_y(i)
            for j in range(self.mesh.num_cols):
                x = self.mesh.get_x(j)
                boundary_pt = False
                for boundary in self.boundaries:
                    if boundary.contains_point(x, y):
                        boundary_pt = True
                        phi[i][j] = boundary.potential()
                if not boundary_pt:
                    phi[i][j] = self.guesser.guess(x, y)
        return phi


class SquareMeshConstructor:
    """
    Constructs a square mesh.
    """

    def __init__(self, size):
        self.size = size

    def construct_uniform_mesh(self, h):
        """
        Constructs a uniform mesh with the given node spacing.

        :param h: the node spacing
        :return: the constructed mesh
        """
        num_rows = num_cols = int(self.size / h) + 1
        return SimpleMesh(h, num_rows, num_cols)

    def construct_symmetric_uniform_mesh(self, h):
        """
        Construct a symmetric uniform mesh with the given node spacing.

        :param h: the node spacing
```

```
        :return: the constructed mesh
        """
        half_size = self.size / 2
        num_rows = num_cols = int(half_size / h) + 2  # Only need to store up to middle
        return SimpleMesh(h, num_rows, num_cols)

    def construct_symmetric_non_uniform_mesh(self, x_values, y_values):
        """
        Construct a symmetric non-uniform mesh with the given adjacent x coordinates and y coordinates.

        :param x_values: the values of successive x coordinates
        :param y_values: the values of successive y coordinates
        :return: the constructed mesh
        """
        return NonUniformMesh(x_values, y_values)


class Mesh:
    """
    Finite-difference mesh.
    """
    __metaclass__ = ABCMeta

    @abstractmethod
    def get_x(self, j):
        """
        Get the x value at the specified index.

        :param j: the column index.
        """
        raise NotImplementedError

    @abstractmethod
    def get_y(self, i):
        """
        Get the y value at the specified index.

        :param i: the row index.
        """
        raise NotImplementedError

    @abstractmethod
    def get_i(self, y):
        """
        Get the row index of the specified y coordinate.

        :param y: the y coordinate
        """
        raise NotImplementedError

    @abstractmethod
    def get_j(self, x):
        """
        Get the column index of the specified x coordinate.

        :param x: the x coordinate
        """
        raise NotImplementedError

    def point_to_indices(self, x, y):
        """
        Converts the given (x, y) point to (i, j) matrix indices.

        :param x: the x coordinate
        :param y: the y coordinate
        :return: the (i, j) matrix indices
        """
        return self.get_i(y), self.get_j(x)

    def indices_to_points(self, i, j):
```

```
        """
        Converts the given (i, j) matrix indices to an (x, y) point.

        :param i: the row index
        :param j: the column index
        :return: the (x, y) point
        """
        return self.get_x(j), self.get_y(i)


class SimpleMesh(Mesh):
    def __init__(self, h, num_rows, num_cols):
        self.h = h
        self.num_rows = num_rows
        self.num_cols = num_cols

    def get_i(self, y):
        return int(y / self.h)

    def get_j(self, x):
        return int(x / self.h)

    def get_x(self, j):
        return j * self.h

    def get_y(self, i):
        return i * self.h


class NonUniformMesh(Mesh):
    def __init__(self, x_values, y_values):
        self.x_values = x_values
        self.y_values = y_values
        self.num_rows = len(y_values)
        self.num_cols = len(x_values)

    def get_i(self, y):
        return self.y_values.index(y)

    def get_j(self, x):
        return self.x_values.index(x)

    def get_x(self, j):
        return self.x_values[j]

    def get_y(self, i):
        return self.y_values[i]


class IterativeRelaxer:
    """
    Performs finite-difference iterative relaxation on a phi potential matrix associated with a mesh.
    """

    def __init__(self, relaxer, epsilon, phi, mesh):
        self.relaxer = relaxer
        self.epsilon = epsilon
        self.phi = phi
        self.boundary = QuarterInnerConductorBoundary()
        self.num_iterations = 0
        self.rows = len(phi)
        self.cols = len(phi[0])
        self.mesh = mesh
        self.mid_i = mesh.get_i(MESH_SIZE / 2)
        self.mid_j = mesh.get_j(MESH_SIZE / 2)

    def relaxation(self):
        """
        Performs iterative relaxation until convergence is met.

```

```
        :return: the current iterative relaxer object
        """
        while not self.convergence():
            self.num_iterations += 1
            self.relaxation_iteration()
            self.relaxer.reset()
        return self

    def relaxation_iteration(self):
        """
        Performs one iteration of relaxation.
        """
        for i in range(1, self.rows - 1):
            y = self.mesh.get_y(i)
            for j in range(1, self.cols - 1):
                x = self.mesh.get_x(j)
                if not self.boundary.contains_point(x, y):
                    relaxed_value = self.relaxer.relax(self.phi, i, j)
                    self.phi[i][j] = relaxed_value
                    if i == self.mid_i - 1:
                        self.phi[i + 2][j] = relaxed_value
                    elif j == self.mid_j - 1:
                        self.phi[i][j + 2] = relaxed_value

    def convergence(self):
        """
        Checks if the phi matrix has reached convergence.

        :return: True if the phi matrix has reached convergence, False otherwise
        """
        max_i, max_j = self.mesh.point_to_indices(0.1, 0.1)  # Only need to compute for 1/4 of grid
        for i in range(1, max_i + 1):
            y = self.mesh.get_y(i)
            for j in range(1, max_j + 1):
                x = self.mesh.get_x(j)
                if not self.boundary.contains_point(x, y) and self.relaxer.residual(self.phi, i, j) >=
                 ↪  self.epsilon:
                    return False
        return True

    def get_potential(self, x, y):
        """
        Get the potential at the given (x, y) point.

        :param x: the x coordinate
        :param y: the y coordinate
        :return: the potential at the given (x, y) point
        """
        i, j = self.mesh.point_to_indices(x, y)
        return self.phi[i][j]


def non_uniform_jacobi(epsilon, x_values, y_values):
    """
    Perform Jacobi relaxation on a non-uniform finite-difference mesh.

    :param epsilon: the maximum error to achieve convergence
    :param x_values: the values of successive x coordinates
    :param y_values: the values of successive y coordinates
    :return: the relaxer object
    """
    mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_non_uniform_mesh(x_values, y_values)
    relaxer = NonUniformRelaxer(mesh)
    phi = PhiConstructor(mesh).construct_phi()
    return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()


def successive_over_relaxation(omega, epsilon, h):
    """
    Perform SOR on a uniform symmetric finite-difference mesh.
```

```python

    :param omega: the omega value for SOR
    :param epsilon: the maximum error to achieve convergence
    :param h: the node spacing
    :return: the relaxer object
    """
    mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_uniform_mesh(h)
    relaxer = SuccessiveOverRelaxer(omega)
    phi = PhiConstructor(mesh).construct_phi()
    return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()


def jacobi_relaxation(epsilon, h):
    """
    Perform Jacobi relaxation on a uniform symmetric finite-difference mesh.

    :param epsilon: the maximum error to achieve convergence
    :param h: the node spacing
    :return: the relaxer object
    """
    mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_uniform_mesh(h)
    relaxer = GaussSeidelRelaxer()
    phi = PhiConstructor(mesh).construct_phi()
    return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()
```

Listing 7: Question 3 (q3.py).

```python
from __future__ import division

import csv
import time

import matplotlib.pyplot as plt
import numpy as np
import numpy.polynomial.polynomial as poly
import sympy as sp

from finite_diff import successive_over_relaxation, jacobi_relaxation, \
    non_uniform_jacobi

EPSILON = 0.00001
X_QUERY = 0.06
Y_QUERY = 0.04
NUM_H_ITERATIONS = 6


def q3b():
    print('=== Question 3(b) ===')
    h = 0.02
    min_num_iterations = float('inf')
    best_omega = float('inf')

    omegas = []
    num_iterations = []
    potentials = []

    for omega_diff in range(10):
        omega = 1 + omega_diff / 10
        print('Omega: {}'.format(omega))
        iter_relaxer = successive_over_relaxation(omega, EPSILON, h)
        print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
        print('Num iterations: {}'.format(iter_relaxer.num_iterations))
        potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
        print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))
        if iter_relaxer.num_iterations < min_num_iterations:
            best_omega = omega
        min_num_iterations = min(min_num_iterations, iter_relaxer.num_iterations)

        omegas.append(omega)
```

```python
        num_iterations.append(iter_relaxer.num_iterations)
        potentials.append('{:.3f}'.format(potential))

    print('Best number of iterations: {}'.format(min_num_iterations))
    print('Best omega: {}'.format(best_omega))

    f = plt.figure()
    x_range = omegas
    y_range = num_iterations
    plt.plot(x_range, y_range, 'o-', label='Number of iterations')
    plt.xlabel('$\omega$')
    plt.ylabel('Number of Iterations')
    plt.grid(True)
    f.savefig('report/plots/q3b.pdf', bbox_inches='tight')

    save_rows_to_csv('report/csv/q3b_potential.csv', zip(omegas, potentials), header=('Omega', 'Potential (V)'))
    save_rows_to_csv('report/csv/q3b_iterations.csv', zip(omegas, num_iterations), header=('Omega', 'Iterations'))

    return best_omega


def q3c(omega):
    print('=== Question 3(c): SOR ===')
    h = 0.04
    h_values = []
    potential_values = []
    iterations_values = []
    for i in range(NUM_H_ITERATIONS):
        h = h / 2
        print('h: {}'.format(h))
        print('1/h: {}'.format(1 / h))
        iter_relaxer = successive_over_relaxation(omega, EPSILON, h)
        # print(phi.mirror_horizontal())
        potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
        num_iterations = iter_relaxer.num_iterations

        print('Num iterations: {}'.format(num_iterations))
        print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

        h_values.append(1 / h)
        potential_values.append('{:.3f}'.format(potential))
        iterations_values.append(num_iterations)

    f = plt.figure()
    x_range = h_values
    y_range = potential_values
    plt.plot(x_range, y_range, 'o-', label='Data points')

    plt.xlabel('1 / h')
    plt.ylabel('Potential at [0.06, 0.04] (V)')
    plt.grid(True)
    f.savefig('report/plots/q3c_potential.pdf', bbox_inches='tight')

    f = plt.figure()
    x_range = h_values
    y_range = iterations_values

    x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
    polynomial_coeffs = poly.polyfit(x_range, y_range, deg=3)
    polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
    N = sp.symbols("1/h")
    poly_label = sum(sp.S("{:.5f}".format(v)) * N ** i for i, v in enumerate(polynomial_coeffs))
    equation = '${}$'.format(sp.printing.latex(poly_label))
    plt.plot(x_new, polynomial_fit, '{}-'.format('C0'), label=equation)

    plt.plot(x_range, y_range, 'o', label='Data points')
    plt.xlabel('1 / h')
    plt.ylabel('Number of Iterations')
```

```python
    plt.grid(True)
    plt.legend(fontsize='small')

    f.savefig('report/plots/q3c_iterations.pdf', bbox_inches='tight')

    save_rows_to_csv('report/csv/q3c_potential.csv', zip(h_values, potential_values), header=('1/h', 'Potential (V)'))
    save_rows_to_csv('report/csv/q3c_iterations.csv', zip(h_values, iterations_values), header=('1/h', 'Iterations'))

    return h_values, potential_values, iterations_values


def q3d():
    print('=== Question 3(d): Jacobi ===')
    h = 0.04
    h_values = []
    potential_values = []
    iterations_values = []
    for i in range(NUM_H_ITERATIONS):
        h = h / 2
        print('h: {}'.format(h))
        iter_relaxer = jacobi_relaxation(EPSILON, h)
        potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
        num_iterations = iter_relaxer.num_iterations

        print('Num iterations: {}'.format(num_iterations))
        print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

        h_values.append(1 / h)
        potential_values.append('{:.3f}'.format(potential))
        iterations_values.append(num_iterations)

    f = plt.figure()
    x_range = h_values
    y_range = potential_values
    plt.plot(x_range, y_range, 'C1o-', label='Data points')
    plt.xlabel('1 / h')
    plt.ylabel('Potential at [0.06, 0.04] (V)')
    plt.grid(True)
    f.savefig('report/plots/q3d_potential.pdf', bbox_inches='tight')

    f = plt.figure()
    x_range = h_values
    y_range = iterations_values
    plt.plot(x_range, y_range, 'C1o', label='Data points')
    plt.xlabel('1 / h')
    plt.ylabel('Number of Iterations')

    x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
    polynomial_coeffs = poly.polyfit(x_range, y_range, deg=4)
    polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
    N = sp.symbols("1/h")
    poly_label = sum(sp.S("{:.5f}".format(v if i < 3 else -v)) * N ** i for i, v in enumerate(polynomial_coeffs))
    equation = '${}$'.format(sp.printing.latex(poly_label))
    plt.plot(x_new, polynomial_fit, '{}-'.format('C1'), label=equation)

    plt.grid(True)
    plt.legend(fontsize='small')

    f.savefig('report/plots/q3d_iterations.pdf', bbox_inches='tight')

    save_rows_to_csv('report/csv/q3d_potential.csv', zip(h_values, potential_values), header=('1/h', 'Potential (V)'))
    save_rows_to_csv('report/csv/q3d_iterations.csv', zip(h_values, iterations_values), header=('1/h', 'Iterations'))

    return h_values, potential_values, iterations_values

```

```python
def q3e():
    print('=== Question 3(e): Non-Uniform Node Spacing ===')

    print('Jacobi (for reference)')
    iter_relaxer = jacobi_relaxation(EPSILON, 0.01)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    jacobi_potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, jacobi_potential))

    print('Uniform Mesh (same as Jacobi)')
    x_values = [0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.11]
    y_values = [0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.11]
    iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    uniform_potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, uniform_potential))
    print('Jacobi potential: {} V, same as uniform potential: {} V'.format(jacobi_potential,
      uniform_potential))

    print('Non-Uniform (clustered around (0.06, 0.04))')
    x_values = [0.00, 0.01, 0.02, 0.03, 0.05, 0.055, 0.06, 0.065, 0.07, 0.09, 0.1, 0.11]
    y_values = [0.00, 0.01, 0.03, 0.035, 0.04, 0.045, 0.05, 0.07, 0.08, 0.09, 0.1, 0.11]
    iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

    print('Non-Uniform (more clustered around (0.06, 0.04))')
    x_values = [0.00, 0.01, 0.02, 0.03, 0.055, 0.059, 0.06, 0.061, 0.065, 0.09, 0.1, 0.11]
    y_values = [0.00, 0.01, 0.035, 0.039, 0.04, 0.041, 0.045, 0.07, 0.08, 0.09, 0.1, 0.11]
    iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

    print('Non-Uniform (clustered near outer conductor)')
    x_values = [0.00, 0.020, 0.032, 0.044, 0.055, 0.06, 0.074, 0.082, 0.089, 0.096, 0.1, 0.14]
    y_values = [0.00, 0.020, 0.032, 0.04, 0.055, 0.065, 0.074, 0.082, 0.089, 0.096, 0.1, 0.14]
    iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

    plot_mesh(x_values, y_values)


def plot_mesh(x_values, y_values):
    f = plt.figure()
    ax = f.gca()
    ax.set_aspect('equal', adjustable='box')
    x_range = []
    y_range = []
    for x in x_values[:-1]:
        for y in y_values[:-1]:
            x_range.append(x)
            y_range.append(y)
    plt.plot(x_range, y_range, 'o', label='Mesh points')
    plt.xlabel('x')
    plt.ylabel('y')
    plt.grid(True)
    f.savefig('report/plots/q3e.pdf', bbox_inches='tight')
```

```python
def plot_sor_jacobi(h_values, potential_values, potential_values_jacobi, iterations_values,
    ↪  iterations_values_jacobi):
    f = plt.figure()
    plt.plot(h_values, potential_values, 'o-', label='SOR')
    plt.plot(h_values, potential_values_jacobi, 'o-', label='Jacobi')
    plt.xlabel('1 / h')
    plt.ylabel('Potential at [0.06, 0.04] (V)')
    plt.grid(True)
    plt.legend()
    f.savefig('report/plots/q3d_potential_comparison.pdf', bbox_inches='tight')

    f = plt.figure()
    plt.plot(h_values, iterations_values, 'o-', label='SOR')
    plt.plot(h_values, iterations_values_jacobi, 'o-', label='Jacobi')
    plt.xlabel('1 / h')
    plt.ylabel('Number of Iterations')
    plt.grid(True)
    plt.legend()
    f.savefig('report/plots/q3d_iterations_comparison.pdf', bbox_inches='tight')


def save_rows_to_csv(filename, rows, header=None):
    with open(filename, "wb") as f:
        writer = csv.writer(f)
        if header is not None:
            writer.writerow(header)
        for row in rows:
            writer.writerow(row)


def q3():
    o = q3b()
    h_values, potential_values, iterations_values = q3c(o)
    _, potential_values_jacobi, iterations_values_jacobi = q3d()
    plot_sor_jacobi(h_values, potential_values, potential_values_jacobi, iterations_values,
    ↪  iterations_values_jacobi)
    q3e()


if __name__ == '__main__':
    t = time.time()
    q3()
    print('Total runtime: {} s'.format(time.time() - t))
```

B Output Logs

Listing 8: Output of Question 1 program (`q1.txt`).

```
=== Question 1(b) ===
n=2 matrix is positive-definite: True
n=3 matrix is positive-definite: True
n=4 matrix is positive-definite: True
=== Question 1(c) ===
Matrix with n=2:
Expected x:
  8.00
  3.00
Actual x:
  8.00
  3.00
Matrix with n=3:
Expected x:
  9.00
  4.00
  3.00
Actual x:
  9.00
```

```
  4.00
  3.00
Matrix with n=4:
Expected x:
  5.00
  4.00
  1.00
  9.00
Actual x:
  5.00
  4.00
  1.00
  9.00
=== Question 1(d) ===
Solved for x in network 1:
V1 = 5.000 V
Solved for x in network 2:
V1 = 50.000 V
Solved for x in network 3:
V1 = 55.000 V
Solved for x in network 4:
V1 = 20.000 V
V2 = 35.000 V
Solved for x in network 5:
V1 = 5.000 V
V2 = 3.750 V
V3 = 3.750 V
Solved for x in network 6:
V1 = 4.443 V
V2 = 5.498 V
V3 = 3.036 V
V4 = 3.200 V
V5 = 1.301 V
```

Listing 9: Output of Question 2 program (`q2.txt`).

```
=== Question 2(a)(b) ===
Equivalent resistance for 2x4 mesh: 1875.00 Ohms.
Runtime: 0.000999927520752 s.
Equivalent resistance for 3x6 mesh: 2379.55 Ohms.
Runtime: 0.0169999599457 s.
Equivalent resistance for 4x8 mesh: 2741.03 Ohms.
Runtime: 0.100000143051 s.
Equivalent resistance for 5x10 mesh: 3022.82 Ohms.
Runtime: 0.481999874115 s.
Equivalent resistance for 6x12 mesh: 3253.68 Ohms.
Runtime: 1.46099996567 s.
Equivalent resistance for 7x14 mesh: 3449.17 Ohms.
Runtime: 3.26600003242 s.
Equivalent resistance for 8x16 mesh: 3618.67 Ohms.
Runtime: 7.53400015831 s.
Equivalent resistance for 9x18 mesh: 3768.29 Ohms.
Runtime: 15.001999855 s.
Equivalent resistance for 10x20 mesh: 3902.19 Ohms.
Runtime: 28.3630001545 s.
=== Question 2(c) ===
Equivalent resistance for 2x4 mesh: 1875.00 Ohms.
Runtime: 0.00100016593933 s.
Equivalent resistance for 3x6 mesh: 2379.55 Ohms.
Runtime: 0.0169999599457 s.
Equivalent resistance for 4x8 mesh: 2741.03 Ohms.
Runtime: 0.0950000286102 s.
Equivalent resistance for 5x10 mesh: 3022.82 Ohms.
Runtime: 0.378000020981 s.
Equivalent resistance for 6x12 mesh: 3253.68 Ohms.
Runtime: 1.19199991226 s.
Equivalent resistance for 7x14 mesh: 3449.17 Ohms.
Runtime: 3.05200004578 s.
Equivalent resistance for 8x16 mesh: 3618.67 Ohms.
```

```
Runtime: 6.9430000782 s.
Equivalent resistance for 9x18 mesh: 3768.29 Ohms.
Runtime: 14.2189998627 s.
Equivalent resistance for 10x20 mesh: 3902.19 Ohms.
Runtime: 26.763999939 s.
=== Question 2(d) ===
```

Listing 10: Output of Question 3 program (`q3.txt`).

```
=== Question 3(b) ===
Omega: 1.0
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 32
Potential at (0.06, 0.04): 5.526 V
Omega: 1.1
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 26
Potential at (0.06, 0.04): 5.526 V
Omega: 1.2
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 20
Potential at (0.06, 0.04): 5.526 V
Omega: 1.3
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 14
Potential at (0.06, 0.04): 5.526 V
Omega: 1.4
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 16
Potential at (0.06, 0.04): 5.526 V
Omega: 1.5
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
```

```
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 20
Potential at (0.06, 0.04): 5.526 V
Omega: 1.6
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 27
Potential at (0.06, 0.04): 5.526 V
Omega: 1.7
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 39
Potential at (0.06, 0.04): 5.526 V
Omega: 1.8
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 60
Potential at (0.06, 0.04): 5.526 V
Omega: 1.9
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 127
Potential at (0.06, 0.04): 5.526 V
Best number of iterations: 14
Best omega: 1.3
=== Question 3(c): SOR ===
h: 0.02
1/h: 50.0
Num iterations: 14
Potential at (0.06, 0.04): 5.526 V
h: 0.01
1/h: 100.0
Num iterations: 59
Potential at (0.06, 0.04): 5.351 V
h: 0.005
1/h: 200.0
Num iterations: 189
Potential at (0.06, 0.04): 5.289 V
h: 0.0025
1/h: 400.0
Num iterations: 552
Potential at (0.06, 0.04): 5.265 V
```

```
h: 0.00125
1/h: 800.0
Num iterations: 1540
Potential at (0.06, 0.04): 5.254 V
h: 0.000625
1/h: 1600.0
Num iterations: 4507
Potential at (0.06, 0.04): 5.247 V
=== Question 3(d): Jacobi ===
h: 0.02
Num iterations: 51
Potential at (0.06, 0.04): 5.526 V
h: 0.01
Num iterations: 180
Potential at (0.06, 0.04): 5.351 V
h: 0.005
Num iterations: 604
Potential at (0.06, 0.04): 5.289 V
h: 0.0025
Num iterations: 1935
Potential at (0.06, 0.04): 5.265 V
h: 0.00125
Num iterations: 5836
Potential at (0.06, 0.04): 5.254 V
h: 0.000625
Num iterations: 16864
Potential at (0.06, 0.04): 5.246 V
Total runtime: 1724.82099986
=== Question 3(e): Non-Uniform Node Spacing ===
Jacobi (for reference)
Quarter grid:
  0.00   1.99   4.06   6.29   8.78  11.66  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.03   4.14   6.41   8.95  11.82  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.99   4.06   6.29   8.78  11.66  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.87   3.81   5.89   8.23  11.04  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.69   3.42   5.24   7.19   9.28  11.33  12.14  12.50  12.66  12.71  12.66
  0.00   1.46   2.95   4.47   6.02   7.55   8.90   9.73  10.20  10.44  10.51  10.44
  0.00   1.22   2.44   3.66   4.87   6.01   6.99   7.69   8.14   8.38   8.45   8.38
  0.00   0.96   1.92   2.87   3.78   4.63   5.35   5.90   6.27   6.48   6.55   6.48
  0.00   0.71   1.42   2.11   2.77   3.37   3.89   4.29   4.57   4.73   4.79   4.73
  0.00   0.47   0.94   1.39   1.81   2.20   2.53   2.80   2.98   3.09   3.13   3.09
  0.00   0.23   0.46   0.69   0.90   1.09   1.25   1.38   1.47   1.53   1.55   1.53
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 106
Potential at (0.06, 0.04): 5.351 V
Uniform Mesh (same as Jacobi)
Quarter grid:
  0.00   1.99   4.06   6.29   8.78  11.66  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.03   4.14   6.41   8.95  11.82  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.99   4.06   6.29   8.78  11.66  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.87   3.81   5.89   8.23  11.04  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.69   3.42   5.24   7.19   9.28  11.33  12.14  12.50  12.66  12.71  12.66
  0.00   1.46   2.95   4.47   6.02   7.55   8.90   9.73  10.20  10.44  10.51  10.44
  0.00   1.22   2.44   3.66   4.87   6.01   6.99   7.69   8.14   8.38   8.45   8.38
  0.00   0.96   1.92   2.87   3.79   4.63   5.35   5.90   6.27   6.48   6.55   6.48
  0.00   0.71   1.42   2.11   2.77   3.37   3.89   4.29   4.57   4.73   4.79   4.73
  0.00   0.47   0.94   1.39   1.81   2.20   2.53   2.80   2.98   3.09   3.13   3.09
  0.00   0.23   0.46   0.69   0.90   1.09   1.25   1.38   1.47   1.53   1.55   1.53
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 209
Potential at (0.06, 0.04): 5.351 V
Jacobi potential: 5.35062156679 V, same as uniform potential: 5.35067998265 V
Non-Uniform (clustered around (0.06, 0.04))
Quarter grid:
  0.00   2.00   4.08   6.33  11.61  13.25  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.04   4.17   6.45  11.80  13.37  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.00   4.08   6.33  11.61  13.25  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.89   3.84   5.93  10.90  12.71  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.71   3.45   5.28   9.27  10.26  11.15  11.74  12.14  12.66  12.71  12.66
  0.00   1.21   2.43   3.66   6.06   6.57   7.03   7.42   7.75   8.38   8.45   8.38
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  0.00   1.09   2.18   3.26   5.35   5.78   6.18   6.52   6.81   7.41   7.48   7.41
  0.00   0.96   1.92   2.87   4.66   5.04   5.38   5.67   5.93   6.48   6.55   6.48
  0.00   0.84   1.67   2.48   4.01   4.33   4.62   4.87   5.09   5.59   5.65   5.59
  0.00   0.71   1.42   2.11   3.39   3.65   3.89   4.11   4.29   4.72   4.77   4.72
  0.00   0.23   0.47   0.69   1.10   1.19   1.26   1.33   1.39   1.54   1.56   1.54
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 385
Potential at (0.06, 0.04): 5.378 V
Non-Uniform (more clustered around (0.06, 0.04))
Quarter grid:
  0.00   2.03   4.14   6.41  13.24  14.65  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.07   4.22   6.53  13.40  14.68  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.03   4.14   6.41  13.24  14.65  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.92   3.90   6.02  12.55  14.45  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.73   3.51   5.36  10.40  11.09  11.24  11.38  11.86  12.65  12.71  12.65
  0.00   1.10   2.19   3.28   5.90   6.21   6.29   6.36   6.62   7.44   7.51   7.44
  0.00   1.00   1.99   2.97   5.28   5.56   5.62   5.69   5.92   6.69   6.75   6.69
  0.00   0.97   1.94   2.89   5.13   5.40   5.46   5.52   5.75   6.50   6.57   6.50
  0.00   0.94   1.88   2.81   4.98   5.24   5.30   5.36   5.58   6.32   6.38   6.32
  0.00   0.84   1.68   2.50   4.39   4.62   4.68   4.73   4.92   5.60   5.66   5.60
  0.00   0.24   0.47   0.70   1.21   1.28   1.29   1.31   1.36   1.56   1.57   1.56
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 1337
Potential at (0.06, 0.04): 5.461 V
Non-Uniform (clustered near outer conductor)
Quarter grid:
  0.00   4.38   7.21  10.30  13.47   7.42   8.97   9.82  10.43  10.80  10.86   7.63
  0.00   4.46   7.34  10.46  13.55  15.00  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   4.38   7.21  10.30  13.47  15.00  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   4.19   6.91   9.94  13.24  15.00  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   3.95   6.50   9.37  12.69  15.00  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   3.61   5.91   8.39  10.87  11.93  12.87  13.10  13.22  13.30  13.33  13.30
  0.00   3.18   5.15   7.16   8.96   9.63  10.73  11.09  11.29  11.43  11.49  11.43
  0.00   2.67   4.27   5.84   7.16   7.66   8.66   9.03   9.27   9.44   9.51   9.44
  0.00   1.89   3.00   4.05   4.91   5.24   5.99   6.29   6.49   6.64   6.71   6.64
  0.00   1.50   2.36   3.17   3.83   4.09   4.69   4.94   5.11   5.23   5.29   5.23
  0.00   0.92   1.44   1.93   2.33   2.49   2.86   3.02   3.13   3.21   3.25   3.21
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 222
Potential at (0.06, 0.04): 5.243 V
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