

ECSE 543

Assignment 1

Sean Stappas
260639512

October 21st, 2017

Contents

1	Choleski Decomposition	2
1.a	Choleski Program	2
1.b	Constructing Test Matrices	2
1.c	Test Runs	2
1.d	Linear Networks	2
2	Finite Difference Resistive Mesh	3
2.a	Equivalent Resistance	3
2.b	Time Complexity	4
2.c	Sparsity Modification	4
2.d	Resistance vs. Mesh Size	5
3	Coaxial Cable	5
3.a	SOR Program	5
3.b	Varying ω	5
3.c	Varying h	5
3.d	Jacobi Method	6
3.e	Non-uniform Node Spacing	7
	Appendix A Code Listings	9
	Appendix B Output Logs	30

Introduction

The code for this assignment was created in Python 2.7 and can be seen in Appendix A. To perform the required tasks in this assignment, a custom `Matrix` class was created, with useful methods such as add, multiply, transpose, etc. This package can be seen in the `matrices.py` file shown in Listing 1. The structure of the rest of the code will be discussed as appropriate for each question. Output logs of the program are provided in Appendix B.

The only packages used that are not built-in are those for fitting curves and creating the plots for this report. These include `matplotlib` for plotting, `numpy` for curve fitting and `sympy` for printing mathematical symbols on the plots. Curve fitting was used to fit the $R(N)$ function in Question 2 and to fit polynomial complexity functions to the number of iterations or runtime of various parts of the program. For any curve fit, the fitting function is given in the legend of the associated plot.

1 Choleski Decomposition

The source code for the Question 1 program can be seen in the `q1.py` file, shown in Listing 5.

1.a Choleski Program

The code relating specifically to Choleski decomposition can be seen in the `choleski.py` file shown in Listing 3. It is separated into `elimination` and `back_substitution` methods.

1.b Constructing Test Matrices

The matrices were constructed with the knowledge that, if A is positive-definite, then $A = LL^T$ where L is a lower triangular non-singular matrix. The task of choosing valid A matrices then boils down to finding non-singular lower triangular L matrices. To ensure that L is non-singular, one must simply choose nonzero values for the main diagonal. The Choleski decomposition algorithm then validates that the matrix is positive definite during the elimination phase, throwing an error if it is not.

1.c Test Runs

The matrices were tested by inventing x matrices, and checking that the program solves for that x correctly. The output of the program, comparing expected and obtained values of x , can be seen in Listing 9.

1.d Linear Networks

The code relating to solving linear networks can be found in the `linear_networks.py` file and is shown in Listing 4. Here, the `csv_to_network_branch_matrices` method script reads from a CSV file where row k contains the J_k , R_k and E_k values. It then converts the resistances to a diagonal admittance matrix Y and produces the J and E column vectors. The incidence matrix A is also read directly from file, as seen in Listing 5.

First, the program was tested various circuits. These circuits are labeled 1 to 6 and can be seen in Figures 1 to 6. The corresponding voltages solved by SPICE at each node can be seen in Tables 1 to 6. Each circuit has corresponding incidence matrix and network branch CSV files, located in the `network_data` directory. For each circuit, the program obtains the expected voltages, as seen in the output in Listing 9.

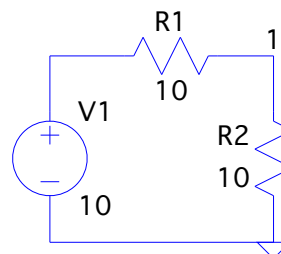


Figure 1: Test circuit 1 with labeled nodes.

Table 1: Voltage at labeled nodes of circuit 1.

Node	Voltage (V)
1	5.000

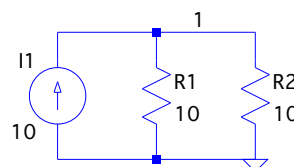


Figure 2: Test circuit 2 with labeled nodes.

Table 2: Voltage at labeled nodes of circuit 2.

Node	Voltage (V)
1	50.000

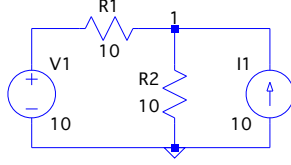


Figure 3: Test circuit 3 with labeled nodes.

Table 3: Voltage at labeled nodes of circuit 3.

Node	Voltage (V)
1	55.000

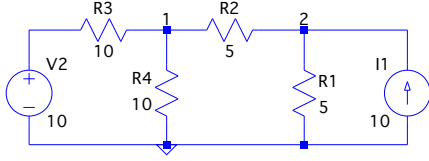


Figure 4: Test circuit 4 with labeled nodes.

Table 4: Voltage at labeled nodes of circuit 4.

Node	Voltage (V)
1	20.000
2	35.000

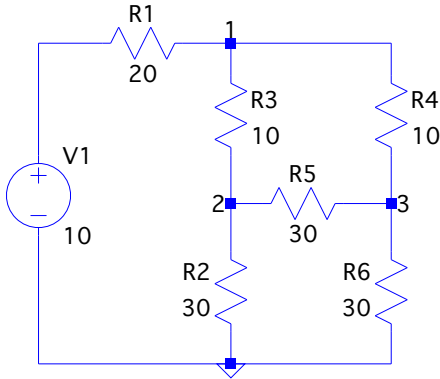


Figure 5: Test circuit 5 with labeled nodes.

Table 5: Voltage at labeled nodes of circuit 5.

Node	Voltage (V)
1	5.000
2	3.750
3	3.750

2 Finite Difference Resistive Mesh

The source code for the Question 2 program can be seen in the q2.py file shown in Listing 6.

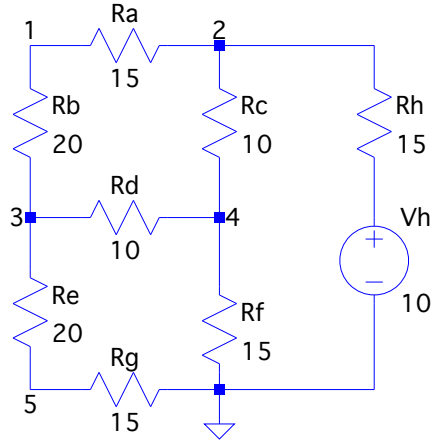


Figure 6: Test circuit 6 with labeled nodes.

Table 6: Voltage at labeled nodes of circuit 6.

Node	Voltage (V)
1	4.443
2	5.498
3	3.036
4	3.200
5	1.301

2.a Equivalent Resistance

The code for creating all the network matrices and for finding the equivalent resistance of an N by $2N$ mesh can be seen in the `linear_networks.py` file shown in Listing 4. The `create_network_matrices_mesh` method creates the incidence matrix A , the admittance matrix Y , the current source matrix J and the voltage source matrix E . The matrix A is created by reading the associated numbered `incidence_matrix` CSV files inside the `network_data` directory. Similarly, the Y , J and E matrices are created by reading the `network_branches` CSV files in the same directory. Each of these files contains a list of network branches (J_k, R_k, E_k) . The resistances found by the program for values of N from 2 to 10 can be seen in Table 7.

The resistance values returned by the program for small meshes were validated using simple SPICE circuits. The voltage found at the V_{test} node for the 2×4 mesh shown in Figure 7 is 1.875 V and the equivalent resistance is therefore 1875Ω . Similarly, for the 3×6 mesh (Figure 8), $V_{test} = 2.37955$ V and the equivalent resistance is 2379.55Ω . These match the results found by the program, as seen in Table 7. Bigger mesh circuits were not tested, but these results give at least some confidence that the program is working correctly.

Table 7: Mesh equivalent resistance R versus mesh size N .

N	R (Omega)
2	1875.000
3	2379.545
4	2741.025
5	3022.819
6	3253.676
7	3449.166
8	3618.675
9	3768.291
10	3902.189

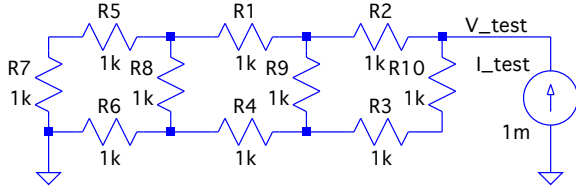


Figure 7: SPICE circuit used to test the 2×4 mesh.

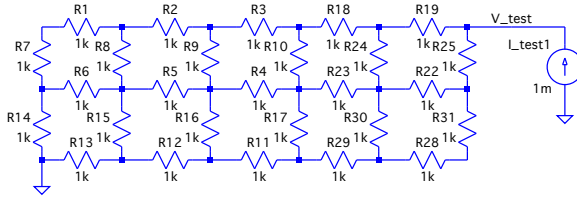


Figure 8: SPICE circuit used to test the 3×6 mesh.

2.b Time Complexity

The runtime data for the mesh resistance solver is tabulated in Table 8 and plotted in Figure 9. Theoretically, the time complexity of the program should be $O(N^6)$. However, as can be seen in Figure 9, $O(N^5)$ more closely matches the obtained data. The simple Choleski program is therefore more efficient than expected. The reasons why are uncertain, but it may be because of successful branch prediction on the repeated zeros of the matrix when performing elimination.

2.c Sparsity Modification

The runtime data for the banded mesh resistance solver is tabulated in Table 9 and plotted in Figure 10. By inspection of the constructed network matrices, a half-bandwidth of $2N + 1$ was chosen. Theoretically, the banded version should have a time complexity of $O(N^4)$, which matches the experimental results.

Table 8: Runtime of mesh resistance solver program versus mesh size N .

N	Runtime (s)
2	0.002
3	0.017
4	0.105
5	0.394
6	1.224
7	3.176
8	7.021
9	14.601
10	27.436

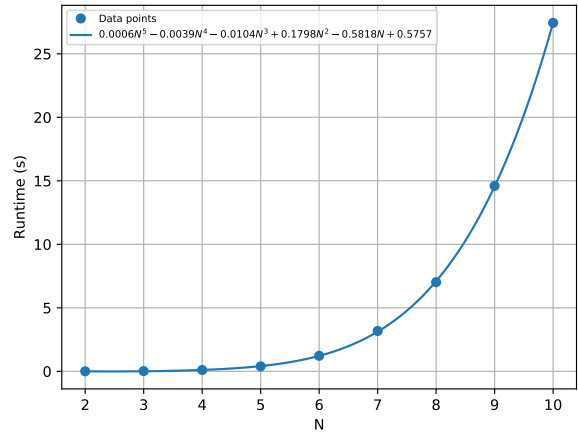


Figure 9: Runtime of mesh resistance solver program versus mesh size N .

Table 9: Runtime of banded mesh resistance solver program versus mesh size N .

N	Runtime (s)
2	0.002
3	0.016
4	0.093
5	0.373
6	1.138
7	2.918
8	6.662
9	13.832
10	26.007

The runtime of the banded and non-banded versions of the program are plotted in Figure 11, showing the marginal benefits of banded elimination. Indeed, the performance increase is not as big as expected, since the non-banded elimination already performed relatively well, possibly because of good branch prediction.

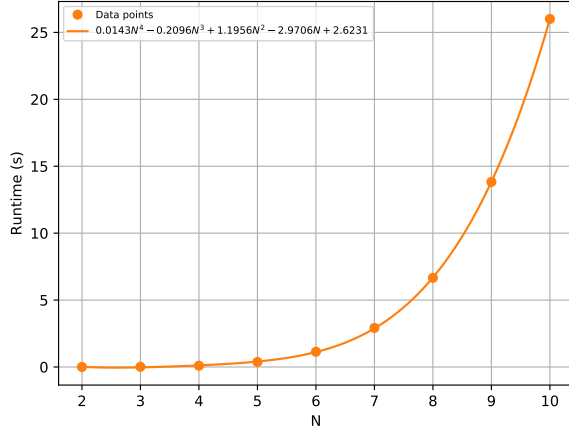


Figure 10: Runtime of banded mesh resistance solver program versus mesh size N .

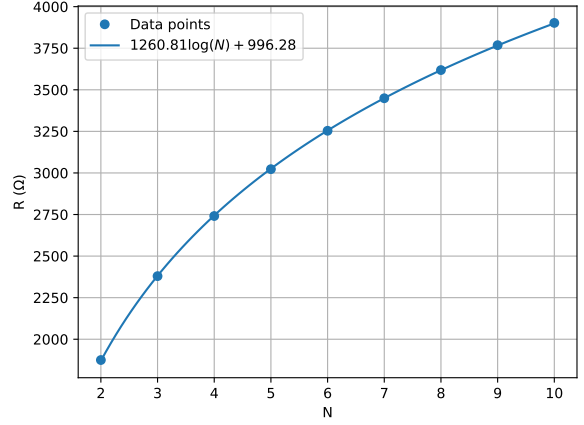


Figure 12: Resistance of mesh versus mesh size N .

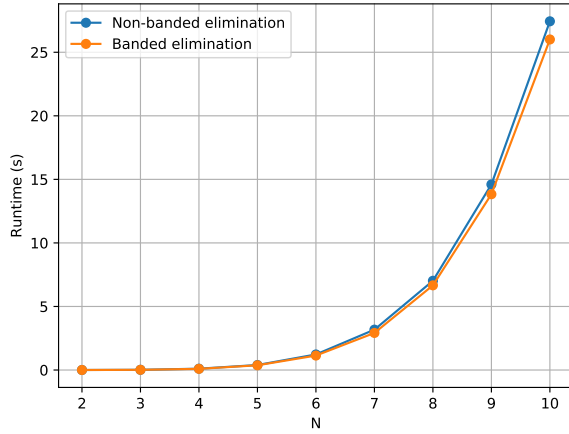


Figure 11: Comparison of runtime of banded and non-banded resistance solver programs versus mesh size N .

2.d Resistance vs. Mesh Size

The equivalent mesh resistance R is plotted versus the mesh size N in Figure 12. The function $R(N)$ appears logarithmic, and a log function does indeed fit the data well. As shown in Figure 12, $R(N) = 1260.81 \log N + 996.28$ is a good fit, where R is in Ω .

3 Coaxial Cable

The source code for the Question 3 program can be seen in the `q3.py` file shown in Listing 8.

3.a SOR Program

The source code for the finite difference methods can be seen in the `finite_diff.py` file shown in

Listing 7. Horizontal and vertical symmetries were exploited by only solving for a quarter of the coaxial cable, and reproducing the results where necessary. The initial potential values are guessed based on a radial function from the conductors.

3.b Varying ω

The number of iterations to achieve convergence for 10 values of ω between 1 and 2 are tabulated in Table 10 and plotted in Figure 13. Based on these results, the value of ω yielding the minimum number of iterations is 1.3.

Table 10: Number of iterations of SOR versus ω .

Omega	Iterations
1.0	32
1.1	26
1.2	20
1.3	14
1.4	16
1.5	20
1.6	27
1.7	39
1.8	60
1.9	127

The potential values found at (0.06, 0.04) versus ω are tabulated in Table 11. It can be seen that all the potential values are identical to 3 decimal places, which shows that the program is converging correctly.

3.c Varying h

With $\omega = 1.3$, the number of iterations of SOR versus $1/h$ is tabulated in Table 12 and plotted in

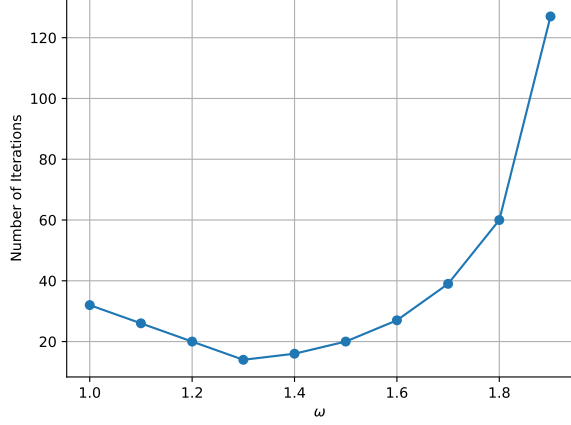


Figure 13: Number of iterations of SOR versus ω .

Table 11: Potential at (0.06, 0.04) versus ω when using SOR.

Omega	Potential (V)
1.0	5.526
1.1	5.526
1.2	5.526
1.3	5.526
1.4	5.526
1.5	5.526
1.6	5.526
1.7	5.526
1.8	5.526
1.9	5.526

Figure 14. It can be seen that the smaller the node spacing is, the more iterations the program will take to run. Theoretically, the time complexity of the program should be $O(N^3)$, where the finite difference mesh is $N \times N$. However, the experimental data shows a complexity closer to $O(1/h^2) = O(N^2)$. The discrepancy can perhaps be because of the relatively small amount of data points.

Table 12: Number of iterations of SOR versus $1/h$. Note that $\omega = 1.3$.

$1/h$	Iterations
50.0	14
100.0	59
200.0	189
400.0	552
800.0	1540
1600.0	4507

The potential values found at (0.06, 0.04) versus $1/h$ are tabulated in Table 13 and plotted in Fig-

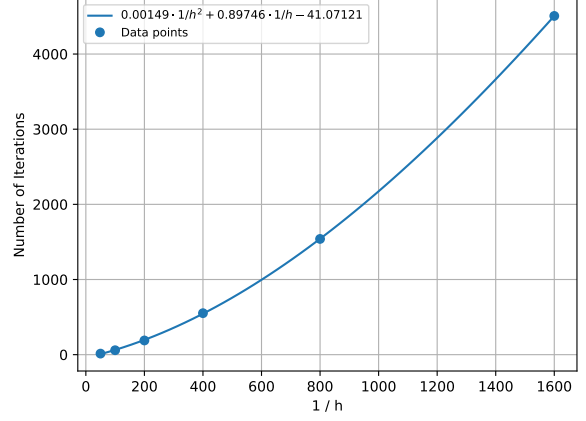


Figure 14: Number of iterations of SOR versus $1/h$. Note that $\omega = 1.3$.

ure 15. By examining these values, the potential at (0.06, 0.04) to three significant figures is approximately 5.25 V. It can be seen that the smaller the node spacing is, the more accurate the calculated potential is. However, by inspecting Figure 15 it is apparent that the potential converges relatively quickly to around 5.25 V. There are therefore diminishing returns to decreasing the node spacing too much, since this will also greatly increase the runtime of the program.

Table 13: Potential at (0.06, 0.04) versus $1/h$ when using SOR.

$1/h$	Potential (V)
50.0	5.526
100.0	5.351
200.0	5.289
400.0	5.265
800.0	5.254
1600.0	5.247

3.d Jacobi Method

The number of iterations of the Jacobi method versus $1/h$ is tabulated in Table 14 and plotted in Figure 16. Similarly to SOR, the smaller the node spacing is, the more iterations the program will take to run. We can see however that the Jacobi method takes a much larger number of iterations to converge. Theoretically, the Jacobi method should have a time complexity of $O(N^4)$. However, the experimental data shows a complexity closer to $O(1/h^3) = O(N^3)$. The discrepancy can perhaps be because of the relatively small amount of data points.

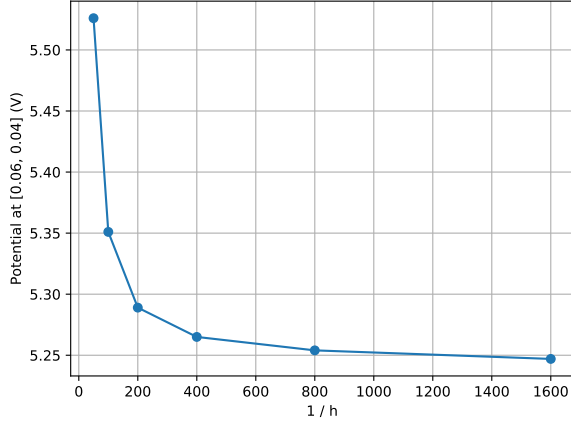


Figure 15: Potential at $(0.06, 0.04)$ found by SOR versus $1/h$. Note that $\omega = 1.3$.

Table 14: Number of iterations versus ω when using the Jacobi method.

1/h	Iterations
50.0	51
100.0	180
200.0	604
400.0	1935
800.0	5836
1600.0	16864

Table 15: Potential at $(0.06, 0.04)$ versus $1/h$ when using the Jacobi method.

1/h	Potential (V)
50.0	5.526
100.0	5.351
200.0	5.289
400.0	5.265
800.0	5.254
1600.0	5.246

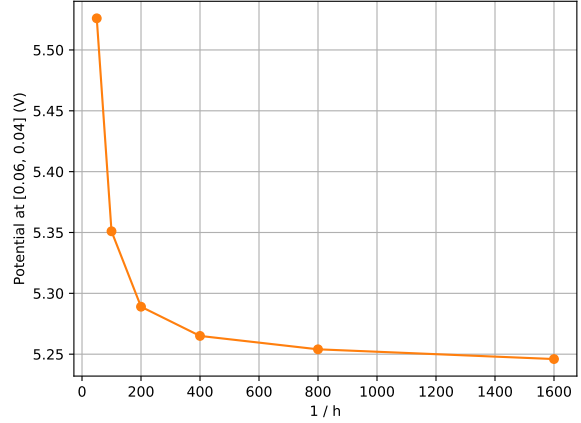


Figure 17: Potential at $(0.06, 0.04)$ versus $1/h$ when using the Jacobi method.

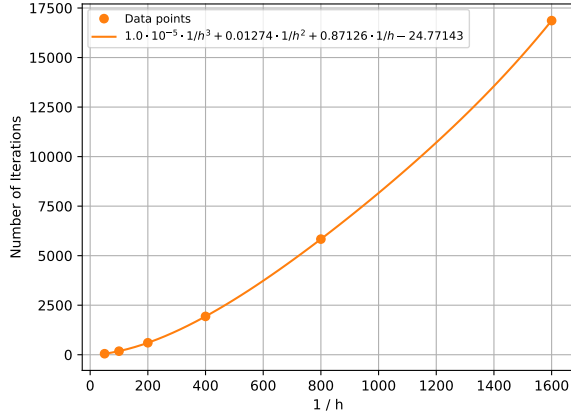


Figure 16: Number of iterations of the Jacobi method versus $1/h$.

The potential values found at $(0.06, 0.04)$ versus $1/h$ with the Jacobi method are tabulated in Table 15 and plotted in Figure 17. These potential values are almost identical to the SOR ones. Similarly to SOR, the smaller the node spacing is, the more accurate the calculated potential is.

The number of iterations of both SOR and the Jacobi method can be seen in Figure 18, which

shows the clear benefits of SOR.

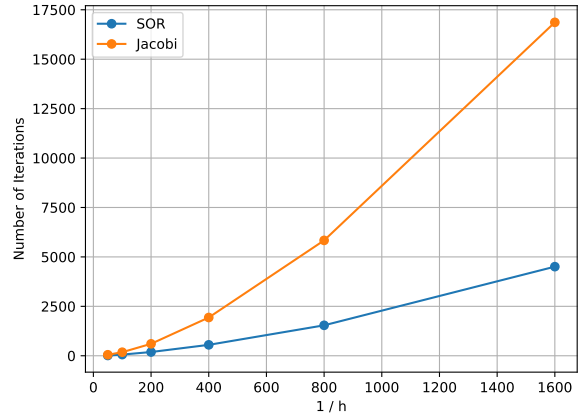


Figure 18: Comparison of number of iterations when using SOR and Jacobi methods versus $1/h$. Note that $\omega = 1.3$ for the SOR program.

3.e Non-uniform Node Spacing

First, we adjust the equation derived in class to set $a_1 = \Delta_x \alpha_1$, $a_2 = \Delta_x \alpha_2$, $b_1 = \Delta_y \beta_1$ and

$b_2 = \Delta_y \beta_2$. These values correspond to the distances between adjacent nodes ¹, and can be easily calculated by the program. Then, the five-point difference formula for non-uniform spacing can be seen in Equation 1.

$$\phi_{i,j}^{k+1} = \frac{1}{a_1 + a_2} \left(\frac{\phi_{i-1,j}^k}{a_1} + \frac{\phi_{i+1,j}^k}{a_2} \right) + \frac{1}{b_1 + b_2} \left(\frac{\phi_{i,j-1}^k}{b_1} + \frac{\phi_{i,j+1}^k}{b_2} \right) \quad (1)$$

This was implemented in the finite difference program, as seen in `NonUniformRelaxer` class in the `finite_diff.py` file shown in Listing 7. As can be seen in this code, many different mesh arrangements were tested. The arrangement that was chosen can be seen in Figure 19. This arrangement was chosen because the “difficult” regions are close to the inner conductor, where there is a higher concentration of nodes. The potential at (0.06, 0.04) obtained from this arrangement is 5.243 V, which seems like an accurate potential value. Indeed, as can be seen in Figures 15 and 17, the potential value for small node spacings tends towards 5.24 V for both the Jacobi and SOR methods.

Conclusion

A Choleski matrix solver, finite difference mesh generator, and finite difference potential solver were created with positive results. The trade-offs between the various implementations of the algorithms associated with these programs were explored and analyzed.

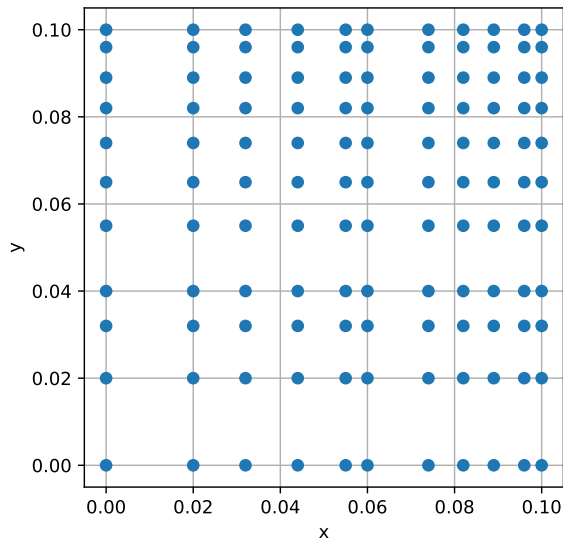


Figure 19: Final mesh arrangement used for non-uniform node spacing. Each point corresponds to a mesh point. Points are positioned closer to the inner conductor, since this is a more difficult area.

¹Note that, in the program, index i is associated to position x and index j is associated to position y . This is purely for easier handling of the matrices.

A Code Listings

Listing 1: Custom matrix package (*matrices.py*).

```
1  from __future__ import division
2
3  import copy
4  import csv
5  from ast import literal_eval
6
7  import math
8
9
10 class Matrix:
11
12     def __init__(self, data):
13         self.data = data
14         self.rows = len(data)
15         self.cols = len(data[0])
16
17     def __str__(self):
18         string = ''
19         for row in self.data:
20             string += '\n'
21             for val in row:
22                 string += '{:6.2f} '.format(val)
23         return string
24
25     def __add__(self, other):
26         if len(self) != len(other) or len(self[0]) != len(other[0]):
27             raise ValueError('Incompatible matrix sizes for addition. Matrix A is {}x{}, but matrix B is
28                 ↳ {}x{}.'.format(len(self), len(self[0]), len(other), len(other[0])))
29
30         return Matrix([[self[row][col] + other[row][col] for col in range(self.cols)] for row in
31             ↳ range(self.rows)])
32
33     def __sub__(self, other):
34         if len(self) != len(other) or len(self[0]) != len(other[0]):
35             raise ValueError('Incompatible matrix sizes for subtraction. Matrix A is {}x{}, but matrix B
36                 ↳ is {}x{}.'.format(len(self), len(self[0]), len(other), len(other[0])))
37
38         return Matrix([[self[row][col] - other[row][col] for col in range(self.cols)] for row in
39             ↳ range(self.rows)])
40
41     def __mul__(self, other):
42         if self.cols != other.rows:
43             raise ValueError('Incompatible matrix sizes for multiplication. Matrix A is {}x{}, but matrix
44                 ↳ B is {}x{}.'.format(self.rows, self.cols, other.rows, other.cols))
45
46         # Inspired from https://en.wikipedia.org/wiki/Matrix_multiplication
47         product = Matrix.empty(self.rows, other.cols)
48         for i in range(self.rows):
49             for j in range(other.cols):
50                 row_sum = 0
51                 for k in range(self.cols):
52                     row_sum += self[i][k] * other[k][j]
53                 product[i][j] = row_sum
54         return product
55
56     def __deepcopy__(self, memo):
57         return Matrix(copy.deepcopy(self.data))
58
59     def __getitem__(self, item):
60         return self.data[item]
```

```

61         return len(self.data)
62
63     def is_positive_definite(self):
64         """
65         :return: True if the matrix is positive-definite, False otherwise.
66         """
67         A = copy.deepcopy(self.data)
68         for j in range(self.rows):
69             if A[j][j] <= 0:
70                 return False
71             A[j][j] = math.sqrt(A[j][j])
72             for i in range(j + 1, self.rows):
73                 A[i][j] = A[i][j] / A[j][j]
74                 for k in range(j + 1, i + 1):
75                     A[i][k] = A[i][k] - A[i][j] * A[k][j]
76         return True
77
78     def transpose(self):
79         """
80         :return: the transpose of the current matrix
81         """
82         return Matrix([[self.data[row][col] for row in range(self.rows)] for col in range(self.cols)])
83
84     def mirror_horizontal(self):
85         """
86         :return: the horizontal mirror of the current matrix
87         """
88         return Matrix([[self.data[self.rows - row - 1][col] for col in range(self.cols)] for row in
89             ↪ range(self.rows)])
90
91     def empty_copy(self):
92         """
93         :return: an empty matrix of the same size as the current matrix.
94         """
95         return Matrix.empty(self.rows, self.cols)
96
97     @staticmethod
98     def multiply(*matrices):
99         """
100         Computes the product of the given matrices.
101
102         :param matrices: the matrix objects
103         :return: the product of the given matrices
104         """
105         n = matrices[0].rows
106         product = Matrix.identity(n)
107         for matrix in matrices:
108             product = product * matrix
109         return product
110
111     @staticmethod
112     def empty(num_rows, num_cols):
113         """
114         Returns an empty matrix (filled with zeroes) with the specified number of columns and rows.
115
116         :param num_rows: number of rows
117         :param num_cols: number of columns
118         :return: the empty matrix
119         """
120         return Matrix([[0 for _ in range(num_cols)] for _ in range(num_rows)])
121
122     @staticmethod
123     def identity(n):
124         """
125         Returns the identity matrix of the given size.
126
127         :param n: the size of the identity matrix (number of rows or columns)
128         :return: the identity matrix of size n
129         """
130         return Matrix.diagonal_single_value(1, n)

```

```

130
131     @staticmethod
132     def diagonal(values):
133         """
134         Returns a diagonal matrix with the given values along the main diagonal.
135
136         :param values: the values along the main diagonal
137         :return: a diagonal matrix with the given values along the main diagonal
138         """
139         n = len(values)
140         return Matrix([[values[row] if row == col else 0 for col in range(n)] for row in range(n)])
141
142     @staticmethod
143     def diagonal_single_value(value, n):
144         """
145         Returns a diagonal matrix of the given size with the given value along the diagonal.
146
147         :param value: the value of each element on the main diagonal
148         :param n: the size of the matrix
149         :return: a diagonal matrix of the given size with the given value along the diagonal.
150         """
151         return Matrix([[value if row == col else 0 for col in range(n)] for row in range(n)])
152
153     @staticmethod
154     def column_vector(values):
155         """
156         Transforms a row vector into a column vector.
157
158         :param values: the values, one for each row of the column vector
159         :return: the column vector
160         """
161         return Matrix([[value] for value in values])
162
163     @staticmethod
164     def csv_to_matrix(filename):
165         """
166         Reads a CSV file to a matrix.
167
168         :param filename: the name of the CSV file
169         :return: a matrix containing the values in the CSV file
170         """
171         with open(filename, 'r') as csv_file:
172             reader = csv.reader(csv_file)
173             data = []
174             for row_number, row in enumerate(reader):
175                 data.append([literal_eval(val) for val in row])
176             return Matrix(data)

```

Listing 2: CSV manipulation utilities (*csv_saver.py*).

```

1  import csv
2
3
4  def save_rows_to_csv(filename, rows, header=None):
5      with open(filename, "wb") as f:
6          writer = csv.writer(f)
7          if header is not None:
8              writer.writerow(header)
9          for row in rows:
10             writer.writerow(row)

```

Listing 3: Choleski decomposition (*choleski.py*).

```

1  from __future__ import division
2
3  import math
4
5  from matrices import Matrix

```

```

6
7
8 def choleski_solve(A, b, half_bandwidth=None):
9     """
10     Solves an  $Ax = b$  matrix equation by Choleski decomposition.
11
12     :param A: the A matrix
13     :param b: the b matrix
14     :param half_bandwidth: the half-bandwidth of the A matrix
15     :return: the solved x vector
16     """
17     n = len(A[0])
18     if half_bandwidth is None:
19         elimination(A, b)
20     else:
21         elimination_banded(A, b, half_bandwidth)
22     x = Matrix.empty(n, 1)
23     back_substitution(A, x, b)
24     return x
25
26
27 def elimination(A, b):
28     """
29     Performs the elimination step of Choleski decomposition.
30
31     :param A: the A matrix
32     :param b: the b matrix
33     """
34     n = len(A)
35     for j in range(n):
36         if A[j][j] <= 0:
37             raise ValueError('Matrix A is not positive definite.')
38         A[j][j] = math.sqrt(A[j][j])
39         b[j][0] = b[j][0] / A[j][j]
40         for i in range(j + 1, n):
41             A[i][j] = A[i][j] / A[j][j]
42             b[i][0] = b[i][0] - A[i][j] * b[j][0]
43             for k in range(j + 1, i + 1):
44                 A[i][k] = A[i][k] - A[i][j] * A[k][j]
45
46
47 def elimination_banded(A, b, half_bandwidth):
48     """
49     Performs the banded elimination step of Choleski decomposition.
50
51     :param A: the A matrix
52     :param b: the b matrix
53     :param half_bandwidth: the half_bandwidth to be used for the banded elimination
54     """
55     n = len(A)
56     for j in range(n):
57         if A[j][j] <= 0:
58             raise ValueError('Matrix A is not positive definite.')
59         A[j][j] = math.sqrt(A[j][j])
60         b[j][0] = b[j][0] / A[j][j]
61         max_row = min(j + half_bandwidth, n)
62         for i in range(j + 1, max_row):
63             A[i][j] = A[i][j] / A[j][j]
64             b[i][0] = b[i][0] - A[i][j] * b[j][0]
65             for k in range(j + 1, i + 1):
66                 A[i][k] = A[i][k] - A[i][j] * A[k][j]
67
68
69 def back_substitution(L, x, y):
70     """
71     Performs the back-substitution step of Choleski decomposition.
72
73     :param L: the L matrix
74     :param x: the x matrix
75     :param y: the y matrix

```

```

76     """
77     n = len(L)
78     for i in range(n - 1, -1, -1):
79         prev_sum = 0
80         for j in range(i + 1, n):
81             prev_sum += L[j][i] * x[j][0]
82             x[i][0] = (y[i][0] - prev_sum) / L[i][i]

```

Listing 4: Linear resistive networks (*linear_networks.py*).

```

1  from __future__ import division
2
3  import csv
4  from matrices import Matrix
5  from choleski import choleski_solve
6
7
8  def solve_linear_network(A, Y, J, E, half_bandwidth=None):
9      """
10     Solve the linear resistive network described by the given matrices.
11
12     :param A: the incidence matrix
13     :param Y: the admittance matrix
14     :param J: the current source matrix
15     :param E: the voltage source matrix
16     :param half_bandwidth:
17     :return: the solved voltage matrix
18     """
19     A_new = A * Y * A.transpose()
20     b = A * (J - Y * E)
21     return choleski_solve(A_new, b, half_bandwidth=half_bandwidth)
22
23
24  def csv_to_network_branch_matrices(filename):
25      """
26     Converts a CSV file to Y, J, E network matrices.
27
28     :param filename: the name of the CSV file
29     :return: the Y, J, E network matrices
30     """
31     with open(filename, 'r') as csv_file:
32         reader = csv.reader(csv_file)
33         J = []
34         Y = []
35         E = []
36         for row in reader:
37             J_k = float(row[0])
38             R_k = float(row[1])
39             E_k = float(row[2])
40             J.append(J_k)
41             Y.append(1 / R_k)
42             E.append(E_k)
43         Y = Matrix.diagonal(Y)
44         J = Matrix.column_vector(J)
45         E = Matrix.column_vector(E)
46         return Y, J, E
47
48
49  def create_network_matrices_mesh(rows, cols, branch_resistance, test_current):
50      """
51     Create the network matrices needed (A, Y, J, E) to solve the resistive mesh network with the given rows,
52     ↪ columns,
53     branch resistance and test current.
54
55     :param rows: the number of rows in the mesh
56     :param cols: the number of columns in the mesh
57     :param branch_resistance: the resistance in each branch
58     :param test_current: the test current to apply
59     :return: the network matrices (A, Y, J, E)

```

```

59     """
60     num_horizontal_branches = (cols - 1) * rows
61     num_vertical_branches = (rows - 1) * cols
62     num_branches = num_horizontal_branches + num_vertical_branches + 1
63     num_nodes = rows * cols - 1
64
65     A = create_incidence_matrix_mesh(cols, num_branches, num_horizontal_branches, num_nodes,
66     ↪ num_vertical_branches)
67     Y, J, E = create_network_branch_matrices_mesh(num_branches, branch_resistance, test_current)
68
69     return A, Y, J, E
70
71 def create_incidence_matrix_mesh(cols, num_branches, num_horizontal_branches, num_nodes,
72 ↪ num_vertical_branches):
73     """
74     Create the incidence matrix given by the resistive mesh with the given number of columns, number of
75     ↪ branches,
76     number of horizontal branches, number of nodes, and number of vertical branches.
77
78     :param cols: the number of columns in the mesh
79     :param num_branches: the number of branches in the mesh
80     :param num_horizontal_branches: the number of horizontal branches in the mesh
81     :param num_nodes: the number of nodes in the mesh
82     :param num_vertical_branches: the number of vertical branches in the mesh
83     :return: the incidence matrix (A)
84     """
85     A = Matrix.empty(num_nodes, num_branches)
86     node_offset = -1
87     for branch in range(num_horizontal_branches):
88         if branch == num_horizontal_branches - cols + 1:
89             A[branch + node_offset + 1][branch] = 1
90         else:
91             if branch % (cols - 1) == 0:
92                 node_offset += 1
93                 node_number = branch + node_offset
94                 A[node_number][branch] = -1
95                 A[node_number + 1][branch] = 1
96             branch_offset = num_horizontal_branches
97             node_offset = cols
98             for branch in range(num_vertical_branches):
99                 if branch == num_vertical_branches - cols:
100                     node_offset -= 1
101                     A[branch][branch + branch_offset] = 1
102                 else:
103                     A[branch][branch + branch_offset] = 1
104                     A[branch + node_offset][branch + branch_offset] = -1
105             if num_branches == 2:
106                 A[0][1] = -1
107             else:
108                 A[cols - 1][num_branches - 1] = -1
109     return A
110
111 def create_network_branch_matrices_mesh(num_branches, branch_resistance, test_current):
112     """
113     Create the Y, J, E network branch matrices of the resistive mesh given by the provided number of
114     ↪ branches, branch
115     resistance and test current.
116
117     :param num_branches: the number of branches in the mesh
118     :param branch_resistance: the resistance of each branch in the mesh
119     :param test_current: the test current to apply to the mesh
120     :return: the Y, J, E network branch matrices
121     """
122     Y = Matrix.diagonal([1 / branch_resistance if branch < num_branches - 1 else 0 for branch in
123     ↪ range(num_branches)])
124     # Negative test current here because we assume current is coming OUT of the test current node.
125     J = Matrix.column_vector([0 if branch < num_branches - 1 else -test_current for branch in
126     ↪ range(num_branches)])

```

```

123     E = Matrix.column_vector([0 for branch in range(num_branches)])
124     return Y, J, E
125
126
127 def find_mesh_resistance(N, branch_resistance, half_bandwidth=None):
128     """
129     Find the equivalent resistance of an  $N \times 2N$  resistive mesh with the given branch resistance and optional
130     half-bandwidth
131
132     :param N: the size of the mesh ( $N \times 2N$ )
133     :param branch_resistance: the resistance of each branch of the mesh
134     :param half_bandwidth: the half-bandwidth to be used for banded Choleski decomposition (or None to use
    ↪ non-banded)
135     :return: the equivalent resistance of the mesh
136     """
137     test_current = 0.01
138     A, Y, J, E = create_network_matrices_mesh(N, 2 * N, branch_resistance, test_current)
139     x = solve_linear_network(A, Y, J, E, half_bandwidth=half_bandwidth)
140     test_voltage = x[2 * N - 1 if N > 1 else 0][0]
141     equivalent_resistance = test_voltage / test_current
142     return equivalent_resistance

```

Listing 5: Question 1 (q1.py).

```

1  from __future__ import division
2
3  from csv_saver import save_rows_to_csv
4  from linear_networks import solve_linear_network, csv_to_network_branch_matrices
5  from choleski import choleski_solve
6  from matrices import Matrix
7
8  NETWORK_DIRECTORY = 'network_data'
9
10 L_2 = Matrix([
11     [5, 0],
12     [1, 3]
13 ])
14 L_3 = Matrix([
15     [3, 0, 0],
16     [1, 2, 0],
17     [8, 5, 1]
18 ])
19 L_4 = Matrix([
20     [1, 0, 0, 0],
21     [2, 8, 0, 0],
22     [5, 5, 4, 0],
23     [7, 2, 8, 7]
24 ])
25 matrix_2 = L_2 * L_2.transpose()
26 matrix_3 = L_3 * L_3.transpose()
27 matrix_4 = L_4 * L_4.transpose()
28 positive_definite_matrices = [matrix_2, matrix_3, matrix_4]
29
30 x_2 = Matrix.column_vector([8, 3])
31 x_3 = Matrix.column_vector([9, 4, 3])
32 x_4 = Matrix.column_vector([5, 4, 1, 9])
33 xs = [x_2, x_3, x_4]
34
35
36 def q1():
37     """
38     Question 1
39     """
40     q1b()
41     q1c()
42     q1d()
43
44
45 def q1b():

```



```

46     """
47     Question 1(b): Construct some small matrices (n = 2, 3, 4, or 5) to test the program. Remember that the
↪     matrices
48     must be real, symmetric and positive-definite.
49     """
50     print('\n=== Question 1(b) ===')
51     for count, A in enumerate(positive_definite_matrices):
52         n = count + 2
53         print('n={} matrix is positive-definite: {}'.format(n, A.is_positive_definite()))
54
55
56 def q1c():
57     """
58     Question 1(c): Test the program you wrote in (a) with each small matrix you built in (b) in the
↪     following way:
59     invent an x, multiply it by A to get b, then give A and b to your program and check that it returns x
↪     correctly.
60     """
61     print('\n=== Question 1(c) ===')
62     n = 2
63     for x, A in zip(xs, positive_definite_matrices):
64         b = A * x
65         print('Matrix with n={}.'.format(n))
66         print('A: {}'.format(A))
67         print('b: {}'.format(b))
68
69         x_choleski = choleski_solve(A, b)
70         print('Expected x: {}'.format(x))
71         print('Actual x: {}'.format(x_choleski))
72         n += 1
73
74
75 def q1d():
76     """
77     Question 1(d): Write a program that reads from a file a list of network branches (Jk, Rk, Ek) and a
↪     reduced
78     incidence matrix, and finds the voltages at the nodes of the network. Use the code from part (a) to
↪     solve the
79     matrix problem.
80     """
81     print('\n=== Question 1(d) ===')
82     for i in range(1, 7):
83         A = Matrix.csv_to_matrix('{}incidence_matrix_{}.csv'.format(NETWORK_DIRECTORY, i))
84         Y, J, E = csv_to_network_branch_matrices('{}network_branches_{}.csv'.format(NETWORK_DIRECTORY,
↪         i))
85         # print('Y: {}'.format(Y))
86         # print('J: {}'.format(J))
87         # print('E: {}'.format(E))
88         x = solve_linear_network(A, Y, J, E)
89         print('Solved for x in network {}'.format(i)) # TODO: Create my own test circuits here
90         node_numbers = []
91         voltage_values = []
92         for j in range(len(x)):
93             print('V{} = {:.3f} V'.format(j + 1, x[j][0]))
94             node_numbers.append(j + 1)
95             voltage_values.append('{}'.format(x[j][0]))
96         save_rows_to_csv('report/csv/q1_circuit_{}.csv'.format(i), zip(node_numbers, voltage_values),
97                             header=('Node', 'Voltage (V)'))
98
99
100 if __name__ == '__main__':
101     q1()

```

Listing 6: Question 2 (q2.py).

```

1 import time
2
3 import matplotlib.pyplot as plt
4 import numpy as np

```

```

5  import numpy.polynomial.polynomial as poly
6  import sympy as sp
7  from matplotlib.ticker import MaxNLocator
8
9  from csv_saver import save_rows_to_csv
10 from linear_networks import find_mesh_resistance
11
12
13 def q2():
14     """
15     Question 2
16     """
17     runtimes1 = q2ab()
18     pts, runtimes2 = q2c()
19     plot_runtimes(runtimes1, runtimes2)
20     q2d(pts)
21
22
23 def q2ab():
24     """
25     Question 2(a): Using the program you developed in question 1, find the resistance,  $R$ , between the node
    ↪ at the
26     bottom left corner of the mesh and the node at the top right corner of the mesh, for  $N = 2, 3, \dots, 10$ .
27
28     Question 2(b): Are the timings you observe for your practical implementation consistent with this?
29
30     :return: the timings for finding the mesh resistance for  $N = 2, 3 \dots 10$ 
31     """
32     print('\n=== Question 2(a)(b) ===')
33     _, runtimes = find_mesh_resistances(banded=False)
34     save_rows_to_csv('report/csv/q2b.csv', zip(runtimes.keys(), runtimes.values()), header=('N', 'Runtime
    ↪ (s)'))
35     return runtimes
36
37
38 def q2c():
39     """
40     Question 2(c): Modify your program to exploit the sparse nature of the matrices to save computation
    ↪ time.
41
42     :return: the mesh resistances and the timings for  $N = 2, 3 \dots 10$ 
43     """
44     print('\n=== Question 2(c) ===')
45     resistances, runtimes = find_mesh_resistances(banded=True)
46     save_rows_to_csv('report/csv/q2c.csv', zip(runtimes.keys(), runtimes.values()), header=('N', 'Runtime
    ↪ (s)'))
47     return resistances, runtimes
48
49
50 def q2d(resistances):
51     """
52     Question 2(d): Plot a graph of  $R$  versus  $N$ . Find a function  $R(N)$  that fits the curve reasonably well and
    ↪ is
53     asymptotically correct as  $N$  tends to infinity, as far as you can tell.
54
55     :param resistances: a dictionary of resistance values for each  $N$  value
56     """
57     print('\n=== Question 2(d) ===')
58     f = plt.figure()
59     ax = f.gca()
60     ax.xaxis.set_major_locator(MaxNLocator(integer=True))
61     x_range = [float(x) for x in resistances.keys()]
62     y_range = [float(y) for y in resistances.values()]
63     plt.plot(x_range, y_range, 'o', label='Data points')
64
65     x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
66     coeffs = poly.polyfit(np.log(x_range), y_range, deg=1)
67     polynomial_fit = poly.polyval(np.log(x_new), coeffs)
68     plt.plot(x_new, polynomial_fit, '{-}'.format('C0'), label='${:.2f} \log(N) + {:.2f}$'.format(coeffs[1],
    ↪ coeffs[0]))

```

```

69     plt.xlabel('N')
70     plt.ylabel('R ($\Omega$)')
71     plt.grid(True)
72     plt.legend()
73     f.savefig('report/plots/q2d.pdf', bbox_inches='tight')
74     save_rows_to_csv('report/csv/q2a.csv', zip(resistances.keys(), resistances.values()), header=('N', 'R
75     ↪ ($\Omega$')))
76
77
78 def find_mesh_resistances(banded):
79     branch_resistance = 1000
80     points = {}
81     runtimes = {}
82     for n in range(2, 11):
83         start_time = time.time()
84         half_bandwidth = 2 * n + 1 if banded else None
85         equivalent_resistance = find_mesh_resistance(n, branch_resistance, half_bandwidth=half_bandwidth)
86         print('Equivalent resistance for {}x{} mesh: {:.2f} Ohms.'.format(n, 2 * n,
87         ↪ equivalent_resistance))
88         points[n] = '{:.3f}'.format(equivalent_resistance)
89         runtime = time.time() - start_time
90         runtimes[n] = '{:.3f}'.format(runtime)
91         print('Runtime: {} s.'.format(runtime))
92     plot_runtime(runtimes, banded)
93     return points, runtimes
94
95 def plot_runtime(points, banded=False):
96     f = plt.figure()
97     ax = f.gca()
98     ax.xaxis.set_major_locator(MaxNLocator(integer=True))
99     x_range = [float(x) for x in points.keys()]
100    y_range = [float(y) for y in points.values()]
101    plt.plot(x_range, y_range, '{o}'.format('C1' if banded else 'C0'), label='Data points')
102
103    x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
104    degree = 4 if banded else 5
105    polynomial_coeffs = poly.polyfit(x_range, y_range, degree)
106    polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
107    N = sp.symbols("N")
108    poly_label = sum(sp.S("{:.4f}".format(v)) * N ** i for i, v in enumerate(polynomial_coeffs))
109    equation = '${}$'.format(sp.printing.latex(poly_label))
110    plt.plot(x_new, polynomial_fit, '{-}'.format('C1' if banded else 'C0'), label=equation)
111
112    plt.xlabel('N')
113    plt.ylabel('Runtime (s)')
114    plt.grid(True)
115    plt.legend(fontsize='x-small')
116    f.savefig('report/plots/q2{}.pdf'.format('c' if banded else 'b'), bbox_inches='tight')
117
118
119 def plot_runtimes(points1, points2):
120     f = plt.figure()
121     ax = f.gca()
122     ax.xaxis.set_major_locator(MaxNLocator(integer=True))
123     x_range = points1.keys()
124     y_range = points1.values()
125     y_banded_range = points2.values()
126     plt.plot(x_range, y_range, 'o-', label='Non-banded elimination')
127     plt.plot(x_range, y_banded_range, 'o-', label='Banded elimination')
128     plt.xlabel('N')
129     plt.ylabel('Runtime (s)')
130     plt.grid(True)
131     plt.legend()
132     f.savefig('report/plots/q2bc.pdf', bbox_inches='tight')
133
134
135 if __name__ == '__main__':
136     q2()

```

Listing 7: Finite difference method (*finite_diff.py*).

```

1  from __future__ import division
2
3  import math
4  import random
5  from abc import ABCMeta, abstractmethod
6
7  from matrices import Matrix
8
9  MESH_SIZE = 0.2
10
11
12  class Relaxer:
13      """
14      Performs the relaxing stage of the finite difference method.
15      """
16      __metaclass__ = ABCMeta
17
18      @abstractmethod
19      def relax(self, phi, i, j):
20          """
21          Perform a relaxation iteration on a given (i, j) point of the given phi matrix.
22
23          :param phi: the phi matrix
24          :param i: the row index
25          :param j: the column index
26          """
27          raise NotImplementedError
28
29      def reset(self):
30          """
31          Optional method to reset the relaxer.
32          """
33          pass
34
35      def residual(self, phi, i, j):
36          """
37          Calculate the residual at the given (i, j) point of the given phi matrix.
38
39          :param phi: the phi matrix
40          :param i: the row index
41          :param j: the column index
42          :return:
43          """
44          return abs(phi[i + 1][j] + phi[i - 1][j] + phi[i][j + 1] + phi[i][j - 1] - 4 * phi[i][j])
45
46  class GaussSeidelRelaxer(Relaxer):
47      def relax(self, phi, i, j):
48          return (phi[i + 1][j] + phi[i - 1][j] + phi[i][j + 1] + phi[i][j - 1]) / 4
49
50
51  class JacobiRelaxer(Relaxer):
52      def __init__(self, num_cols):
53          self.num_cols = num_cols
54          self.prev_row = [0] * (num_cols - 1) # Don't need to copy entire phi, just previous row
55
56      def relax(self, phi, i, j):
57          left_val = self.prev_row[j - 2] if j > 1 else 0
58          top_val = self.prev_row[j - 1]
59          self.prev_row[j - 1] = phi[i][j]
60          return (phi[i + 1][j] + top_val + phi[i][j + 1] + left_val) / 4
61
62      def reset(self):
63          self.prev_row = [0] * (self.num_cols - 1)
64
65
66  class NonUniformRelaxer(Relaxer):

```

```

68     def __init__(self, mesh):
69         self.mesh = mesh
70
71     def get_distances(self, i, j):
72         a1 = self.mesh.get_y(i) - self.mesh.get_y(i - 1)
73         a2 = self.mesh.get_y(i + 1) - self.mesh.get_y(i)
74         b1 = self.mesh.get_x(j) - self.mesh.get_x(j - 1)
75         b2 = self.mesh.get_x(j + 1) - self.mesh.get_x(j)
76         return a1, a2, b1, b2
77
78     def relax(self, phi, i, j):
79         a1, a2, b1, b2 = self.get_distances(i, j)
80
81         return ((phi[i - 1][j] / a1 + phi[i + 1][j] / a2) / (a1 + a2)
82                + (phi[i][j - 1] / b1 + phi[i][j + 1] / b2) / (b1 + b2)) / (1 / (a1 * a2) + 1 / (b1 * b2))
83
84     def residual(self, phi, i, j):
85         a1, a2, b1, b2 = self.get_distances(i, j)
86
87         return abs(((phi[i - 1][j] / a1 + phi[i + 1][j] / a2) / (a1 + a2)
88                    + (phi[i][j - 1] / b1 + phi[i][j + 1] / b2) / (b1 + b2))
89                    - phi[i][j] * (1 / (a1 * a2) + 1 / (b1 * b2))))
90
91
92 class SuccessiveOverRelaxer(Relaxer):
93     def __init__(self, omega):
94         self.gauss_seidel = GaussSeidelRelaxer()
95         self.omega = omega
96
97     def relax(self, phi, i, j, last_row=None, a1=None, a2=None, b1=None, b2=None):
98         return (1 - self.omega) * phi[i][j] + self.omega * self.gauss_seidel.relax(phi, i, j)
99
100
101 class Boundary:
102     """
103     Constant-potential boundary in the finite difference mesh, representing a conductor.
104     """
105     __metaclass__ = ABCMeta
106
107     @abstractmethod
108     def potential(self):
109         """
110         Return the potential on the boundary.
111         """
112         raise NotImplementedError
113
114     @abstractmethod
115     def contains_point(self, x, y):
116         """
117         Returns true if the boundary contains the given (x, y) point.
118
119         :param x: the x coordinate of the point
120         :param y: the y coordinate of the point
121         """
122         raise NotImplementedError
123
124
125 class OuterConductorBoundary(Boundary):
126     def potential(self):
127         return 0
128
129     def contains_point(self, x, y):
130         return x == 0 or y == 0 or x == 0.2 or y == 0.2
131
132
133 class QuarterInnerConductorBoundary(Boundary):
134     def potential(self):
135         return 15
136
137     def contains_point(self, x, y):

```

```

138         return 0.06 <= x <= 0.14 and 0.08 <= y <= 0.12
139
140
141 class PotentialGuesser:
142     """
143     Guesses the initial potential in the finite-difference mesh.
144     """
145     __metaclass__ = ABCMeta
146
147     def __init__(self, min_potential, max_potential):
148         self.min_potential = min_potential
149         self.max_potential = max_potential
150
151     @abstractmethod
152     def guess(self, x, y):
153         """
154         Guess the potential at the given (x, y) point, and return it.
155
156         :param x: the x coordinate of the point
157         :param y: the y coordinate of the point
158         """
159         raise NotImplementedError
160
161
162 class RandomPotentialGuesser(PotentialGuesser):
163     def guess(self, x, y):
164         return random.randint(self.min_potential, self.max_potential)
165
166
167 class LinearPotentialGuesser(PotentialGuesser):
168     def guess(self, x, y):
169         return 150 * x if x < 0.06 else 150 * y
170
171
172 class RadialPotentialGuesser(PotentialGuesser):
173     def guess(self, x, y):
174         def radial(k, x, y, x_source, y_source):
175             return k / (math.sqrt((x_source - x) ** 2 + (y_source - y) ** 2))
176
177         return 0.0225 * (radial(20, x, y, 0.1, 0.1) - radial(1, x, y, 0, y) - radial(1, x, y, x, 0))
178
179
180 class PhiConstructor:
181     """
182     Constructs the phi potential matrix with an outer conductor, inner conductor, mesh points and an initial
183     ↪ potential
184     guess.
185     """
186
187     def __init__(self, mesh):
188         outer_boundary = OuterConductorBoundary()
189         inner_boundary = QuarterInnerConductorBoundary()
190         self.boundaries = (inner_boundary, outer_boundary)
191         self.guesser = RadialPotentialGuesser(0, 15)
192         self.mesh = mesh
193
194     def construct_phi(self):
195         phi = Matrix.empty(self.mesh.num_rows, self.mesh.num_cols)
196         for i in range(self.mesh.num_rows):
197             y = self.mesh.get_y(i)
198             for j in range(self.mesh.num_cols):
199                 x = self.mesh.get_x(j)
200                 boundary_pt = False
201                 for boundary in self.boundaries:
202                     if boundary.contains_point(x, y):
203                         boundary_pt = True
204                         phi[i][j] = boundary.potential()
205                 if not boundary_pt:
206                     phi[i][j] = self.guesser.guess(x, y)
207         return phi

```

```

207
208
209 class SquareMeshConstructor:
210     """
211     Constructs a square mesh.
212     """
213
214     def __init__(self, size):
215         self.size = size
216
217     def construct_uniform_mesh(self, h):
218         """
219         Constructs a uniform mesh with the given node spacing.
220
221         :param h: the node spacing
222         :return: the constructed mesh
223         """
224         num_rows = num_cols = int(self.size / h) + 1
225         return SimpleMesh(h, num_rows, num_cols)
226
227     def construct_symmetric_uniform_mesh(self, h):
228         """
229         Construct a symmetric uniform mesh with the given node spacing.
230
231         :param h: the node spacing
232         :return: the constructed mesh
233         """
234         half_size = self.size / 2
235         num_rows = num_cols = int(half_size / h) + 2 # Only need to store up to middle
236         return SimpleMesh(h, num_rows, num_cols)
237
238     def construct_symmetric_non_uniform_mesh(self, x_values, y_values):
239         """
240         Construct a symmetric non-uniform mesh with the given adjacent x coordinates and y coordinates.
241
242         :param x_values: the values of successive x coordinates
243         :param y_values: the values of successive y coordinates
244         :return: the constructed mesh
245         """
246         return NonUniformMesh(x_values, y_values)
247
248
249 class Mesh:
250     """
251     Finite-difference mesh.
252     """
253     __metaclass__ = ABCMeta
254
255     @abstractmethod
256     def get_x(self, j):
257         """
258         Get the x value at the specified index.
259
260         :param j: the column index.
261         """
262         raise NotImplementedError
263
264     @abstractmethod
265     def get_y(self, i):
266         """
267         Get the y value at the specified index.
268
269         :param i: the row index.
270         """
271         raise NotImplementedError
272
273     @abstractmethod
274     def get_i(self, y):
275         """
276         Get the row index of the specified y coordinate.

```

```

277         :param y: the y coordinate
278         """
279         raise NotImplementedError
280
281
282     @abstractmethod
283     def get_j(self, x):
284         """
285         Get the column index of the specified x coordinate.
286
287         :param x: the x coordinate
288         """
289         raise NotImplementedError
290
291     def point_to_indices(self, x, y):
292         """
293         Converts the given (x, y) point to (i, j) matrix indices.
294
295         :param x: the x coordinate
296         :param y: the y coordinate
297         :return: the (i, j) matrix indices
298         """
299         return self.get_i(y), self.get_j(x)
300
301     def indices_to_points(self, i, j):
302         """
303         Converts the given (i, j) matrix indices to an (x, y) point.
304
305         :param i: the row index
306         :param j: the column index
307         :return: the (x, y) point
308         """
309         return self.get_x(j), self.get_y(i)
310
311
312     class SimpleMesh(Mesh):
313         def __init__(self, h, num_rows, num_cols):
314             self.h = h
315             self.num_rows = num_rows
316             self.num_cols = num_cols
317
318         def get_i(self, y):
319             return int(y / self.h)
320
321         def get_j(self, x):
322             return int(x / self.h)
323
324         def get_x(self, j):
325             return j * self.h
326
327         def get_y(self, i):
328             return i * self.h
329
330
331     class NonUniformMesh(Mesh):
332         def __init__(self, x_values, y_values):
333             self.x_values = x_values
334             self.y_values = y_values
335             self.num_rows = len(y_values)
336             self.num_cols = len(x_values)
337
338         def get_i(self, y):
339             return self.y_values.index(y)
340
341         def get_j(self, x):
342             return self.x_values.index(x)
343
344         def get_x(self, j):
345             return self.x_values[j]
346

```



```

347     def get_y(self, i):
348         return self.y_values[i]
349
350
351 class IterativeRelaxer:
352     """
353     Performs finite-difference iterative relaxation on a phi potential matrix associated with a mesh.
354     """
355
356     def __init__(self, relaxer, epsilon, phi, mesh):
357         self.relaxer = relaxer
358         self.epsilon = epsilon
359         self.phi = phi
360         self.boundary = QuarterInnerConductorBoundary()
361         self.num_iterations = 0
362         self.rows = len(phi)
363         self.cols = len(phi[0])
364         self.mesh = mesh
365         self.mid_i = mesh.get_i(MESH_SIZE / 2)
366         self.mid_j = mesh.get_j(MESH_SIZE / 2)
367
368     def relaxation(self):
369         """
370         Performs iterative relaxation until convergence is met.
371
372         :return: the current iterative relaxer object
373         """
374         while not self.convergence():
375             self.num_iterations += 1
376             self.relaxation_iteration()
377             self.relaxer.reset()
378         return self
379
380     def relaxation_iteration(self):
381         """
382         Performs one iteration of relaxation.
383         """
384         for i in range(1, self.rows - 1):
385             y = self.mesh.get_y(i)
386             for j in range(1, self.cols - 1):
387                 x = self.mesh.get_x(j)
388                 if not self.boundary.contains_point(x, y):
389                     relaxed_value = self.relaxer.relax(self.phi, i, j)
390                     self.phi[i][j] = relaxed_value
391                     if i == self.mid_i - 1:
392                         self.phi[i + 2][j] = relaxed_value
393                     elif j == self.mid_j - 1:
394                         self.phi[i][j + 2] = relaxed_value
395
396     def convergence(self):
397         """
398         Checks if the phi matrix has reached convergence.
399
400         :return: True if the phi matrix has reached convergence, False otherwise
401         """
402         max_i, max_j = self.mesh.point_to_indices(0.1, 0.1) # Only need to compute for 1/4 of grid
403         for i in range(1, max_i + 1):
404             y = self.mesh.get_y(i)
405             for j in range(1, max_j + 1):
406                 x = self.mesh.get_x(j)
407                 if not self.boundary.contains_point(x, y) and self.relaxer.residual(self.phi, i, j) >=
408                     ↪ self.epsilon:
409                     return False
410         return True
411
412     def get_potential(self, x, y):
413         """
414         Get the potential at the given (x, y) point.
415
416         :param x: the x coordinate

```

```

416         :param y: the y coordinate
417         :return: the potential at the given (x, y) point
418         """
419         i, j = self.mesh.point_to_indices(x, y)
420         return self.phi[i][j]
421
422
423 def non_uniform_jacobi(epsilon, x_values, y_values):
424     """
425     Perform Jacobi relaxation on a non-uniform finite-difference mesh.
426
427     :param epsilon: the maximum error to achieve convergence
428     :param x_values: the values of successive x coordinates
429     :param y_values: the values of successive y coordinates
430     :return: the relaxer object
431     """
432     mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_non_uniform_mesh(x_values, y_values)
433     relaxer = NonUniformRelaxer(mesh)
434     phi = PhiConstructor(mesh).construct_phi()
435     return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()
436
437
438 def successive_over_relaxation(omega, epsilon, h):
439     """
440     Perform SOR on a uniform symmetric finite-difference mesh.
441
442     :param omega: the omega value for SOR
443     :param epsilon: the maximum error to achieve convergence
444     :param h: the node spacing
445     :return: the relaxer object
446     """
447     mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_uniform_mesh(h)
448     relaxer = SuccessiveOverRelaxer(omega)
449     phi = PhiConstructor(mesh).construct_phi()
450     return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()
451
452
453 def jacobi_relaxation(epsilon, h):
454     """
455     Perform Jacobi relaxation on a uniform symmetric finite-difference mesh.
456
457     :param epsilon: the maximum error to achieve convergence
458     :param h: the node spacing
459     :return: the relaxer object
460     """
461     mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_uniform_mesh(h)
462     relaxer = GaussSeidelRelaxer()
463     phi = PhiConstructor(mesh).construct_phi()
464     return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()

```

Listing 8: Question 3 (q3.py).

```

1  from __future__ import division
2
3  import time
4
5  import matplotlib.pyplot as plt
6  import numpy as np
7  import numpy.polynomial.polynomial as poly
8  import sympy as sp
9
10 from csv_saver import save_rows_to_csv
11 from finite_diff import successive_over_relaxation, jacobi_relaxation, \
12     non_uniform_jacobi
13
14 EPSILON = 0.00001
15 X_QUERY = 0.06
16 Y_QUERY = 0.04
17 NUM_H_ITERATIONS = 6

```

```

18
19
20 def q3():
21     o = q3b()
22     h_values, potential_values, iterations_values = q3c(o)
23     _, potential_values_jacobi, iterations_values_jacobi = q3d()
24     plot_sor_jacobi(h_values, potential_values, potential_values_jacobi, iterations_values,
25                     ↪ iterations_values_jacobi)
26     q3e()
27
28 def q3b():
29     """
30     Question 3(b): With  $h = 0.02$ , explore the effect of varying  $\omega$ .
31
32     :return: the best  $\omega$  value found for SOR
33     """
34     print('\n=== Question 3(b) ===')
35     h = 0.02
36     min_num_iterations = float('inf')
37     best_omega = float('inf')
38
39     omegas = []
40     num_iterations = []
41     potentials = []
42
43     for omega_diff in range(10):
44         omega = 1 + omega_diff / 10
45         print('Omega: {}'.format(omega))
46         iter_relaxer = successive_over_relaxation(omega, EPSILON, h)
47         print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
48         print('Num iterations: {}'.format(iter_relaxer.num_iterations))
49         potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
50         print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))
51         if iter_relaxer.num_iterations < min_num_iterations:
52             best_omega = omega
53         min_num_iterations = min(min_num_iterations, iter_relaxer.num_iterations)
54
55         omegas.append(omega)
56         num_iterations.append(iter_relaxer.num_iterations)
57         potentials.append('{:.3f}'.format(potential))
58
59     print('Best number of iterations: {}'.format(min_num_iterations))
60     print('Best omega: {}'.format(best_omega))
61
62     f = plt.figure()
63     x_range = omegas
64     y_range = num_iterations
65     plt.plot(x_range, y_range, 'o-', label='Number of iterations')
66     plt.xlabel('$\omega$')
67     plt.ylabel('Number of Iterations')
68     plt.grid(True)
69     f.savefig('report/plots/q3b.pdf', bbox_inches='tight')
70
71     save_rows_to_csv('report/csv/q3b_potential.csv', zip(omegas, potentials), header=('Omega', 'Potential
72     ↪ (V)'))
73     save_rows_to_csv('report/csv/q3b_iterations.csv', zip(omegas, num_iterations), header=('Omega',
74     ↪ 'Iterations'))
75
76     return best_omega
77
78 def q3c(omega):
79     """
80     Question 3(c): With an appropriate value of  $w$ , chosen from the above experiment, explore the effect of
81     ↪ decreasing
82      $h$  on the potential.
83
84     :param omega: the  $\omega$  value to be used by SOR
85     :return: the  $h$  values, potential values and number of iterations

```

```

84     """
85     print('\n=== Question 3(c): SOR ===')
86     h = 0.04
87     h_values = []
88     potential_values = []
89     iterations_values = []
90     for i in range(NUM_H_ITERATIONS):
91         h = h / 2
92         print('h: {}'.format(h))
93         print('1/h: {}'.format(1 / h))
94         iter_relaxer = successive_over_relaxation(omega, EPSILON, h)
95         # print(phi.mirror_horizontal())
96         potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
97         num_iterations = iter_relaxer.num_iterations
98
99         print('Num iterations: {}'.format(num_iterations))
100        print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))
101
102        h_values.append(1 / h)
103        potential_values.append('{:.3f}'.format(potential))
104        iterations_values.append(num_iterations)
105
106    f = plt.figure()
107    x_range = h_values
108    y_range = potential_values
109    plt.plot(x_range, y_range, 'o-', label='Data points')
110
111    plt.xlabel('1 / h')
112    plt.ylabel('Potential at [0.06, 0.04] (V)')
113    plt.grid(True)
114    f.savefig('report/plots/q3c-potential.pdf', bbox_inches='tight')
115
116    f = plt.figure()
117    x_range = h_values
118    y_range = iterations_values
119
120    x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
121    polynomial_coeffs = poly.polyfit(x_range, y_range, deg=3)
122    polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
123    N = sp.symbols("1/h")
124    poly_label = sum(sp.S("{:.5f}".format(v)) * N ** i for i, v in enumerate(polynomial_coeffs))
125    equation = '$${}$'.format(sp.printing.latex(poly_label))
126    plt.plot(x_new, polynomial_fit, '{:}-'.format('C0'), label=equation)
127
128    plt.plot(x_range, y_range, 'o', label='Data points')
129    plt.xlabel('1 / h')
130    plt.ylabel('Number of Iterations')
131    plt.grid(True)
132    plt.legend(fontsize='small')
133
134    f.savefig('report/plots/q3c-iterations.pdf', bbox_inches='tight')
135
136    save_rows_to_csv('report/csv/q3c-potential.csv', zip(h_values, potential_values), header=('1/h',
137    ↪ 'Potential (V)'))
138    save_rows_to_csv('report/csv/q3c-iterations.csv', zip(h_values, iterations_values), header=('1/h',
139    ↪ 'Iterations'))
140
141    return h_values, potential_values, iterations_values
142
143 def q3d():
144     """
145     Question 3(d): Use the Jacobi method to solve this problem for the same values of h used in part (c).
146
147     :return: the h values, potential values and number of iterations
148     """
149     print('\n=== Question 3(d): Jacobi ===')
150     h = 0.04
151     h_values = []
152     potential_values = []

```

```

152     iterations_values = []
153     for i in range(NUM_H_ITERATIONS):
154         h = h / 2
155         print('h: {}'.format(h))
156         iter_relaxer = jacobi_relaxation(EPSILON, h)
157         potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
158         num_iterations = iter_relaxer.num_iterations
159
160         print('Num iterations: {}'.format(num_iterations))
161         print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))
162
163         h_values.append(1 / h)
164         potential_values.append('{:.3f}'.format(potential))
165         iterations_values.append(num_iterations)
166
167     f = plt.figure()
168     x_range = h_values
169     y_range = potential_values
170     plt.plot(x_range, y_range, 'C1o-', label='Data points')
171     plt.xlabel('1 / h')
172     plt.ylabel('Potential at [0.06, 0.04] (V)')
173     plt.grid(True)
174     f.savefig('report/plots/q3d_potential.pdf', bbox_inches='tight')
175
176     f = plt.figure()
177     x_range = h_values
178     y_range = iterations_values
179     plt.plot(x_range, y_range, 'C1o', label='Data points')
180     plt.xlabel('1 / h')
181     plt.ylabel('Number of Iterations')
182
183     x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
184     polynomial_coeffs = poly.polyfit(x_range, y_range, deg=4)
185     polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
186     N = sp.symbols("1/h")
187     poly_label = sum(sp.S("{:.5f}".format(v if i < 3 else -v)) * N ** i for i, v in
188         ↪ enumerate(polynomial_coeffs))
189     equation = '${}$'.format(sp.printing.latex(poly_label))
190     plt.plot(x_new, polynomial_fit, '{}-'.format('C1'), label=equation)
191
192     plt.grid(True)
193     plt.legend(fontsize='small')
194
195     f.savefig('report/plots/q3d_iterations.pdf', bbox_inches='tight')
196
197     save_rows_to_csv('report/csv/q3d_potential.csv', zip(h_values, potential_values), header=('1/h',
198         ↪ 'Potential (V)'))
199     save_rows_to_csv('report/csv/q3d_iterations.csv', zip(h_values, iterations_values), header=('1/h',
200         ↪ 'Iterations'))
201
202     return h_values, potential_values, iterations_values
203
204 def q3e():
205     """
206     Question 3(e): Modify the program you wrote in part (a) to use the five-point difference formula
207     ↪ derived in class
208     for non-uniform node spacing.
209     """
210     print('\n=== Question 3(e): Non-Uniform Node Spacing ===')
211
212     print('Jacobi (for reference)')
213     iter_relaxer = jacobi_relaxation(EPSILON, 0.01)
214     print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
215     print('Num iterations: {}'.format(iter_relaxer.num_iterations))
216     jacobi_potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
217     print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, jacobi_potential))
218
219     print('Uniform Mesh (same as Jacobi)')
220     x_values = [0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.11]

```

```

218 y_values = [0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.11]
219 iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
220 print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
221 print('Num iterations: {}'.format(iter_relaxer.num_iterations))
222 uniform_potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
223 print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, uniform_potential))
224 print('Jacobi potential: {} V, same as uniform potential: {} V'.format(jacobi_potential,
↪ uniform_potential))

225
226 print('Non-Uniform (clustered around (0.06, 0.04))')
227 x_values = [0.00, 0.01, 0.02, 0.03, 0.05, 0.055, 0.06, 0.065, 0.07, 0.09, 0.1, 0.11]
228 y_values = [0.00, 0.01, 0.03, 0.035, 0.04, 0.045, 0.05, 0.07, 0.08, 0.09, 0.1, 0.11]
229 iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
230 print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
231 print('Num iterations: {}'.format(iter_relaxer.num_iterations))
232 potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
233 print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

234
235 print('Non-Uniform (more clustered around (0.06, 0.04))')
236 x_values = [0.00, 0.01, 0.02, 0.03, 0.055, 0.059, 0.06, 0.061, 0.065, 0.09, 0.1, 0.11]
237 y_values = [0.00, 0.01, 0.035, 0.039, 0.04, 0.041, 0.045, 0.07, 0.08, 0.09, 0.1, 0.11]
238 iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
239 print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
240 print('Num iterations: {}'.format(iter_relaxer.num_iterations))
241 potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
242 print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

243
244 print('Non-Uniform (clustered near outer conductor)')
245 x_values = [0.00, 0.020, 0.032, 0.044, 0.055, 0.06, 0.074, 0.082, 0.089, 0.096, 0.1, 0.14]
246 y_values = [0.00, 0.020, 0.032, 0.04, 0.055, 0.065, 0.074, 0.082, 0.089, 0.096, 0.1, 0.14]
247 iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
248 print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
249 print('Num iterations: {}'.format(iter_relaxer.num_iterations))
250 potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
251 print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))
252
253 plot_mesh(x_values, y_values)
254
255
256 def plot_mesh(x_values, y_values):
257     f = plt.figure()
258     ax = f.gca()
259     ax.set_aspect('equal', adjustable='box')
260     x_range = []
261     y_range = []
262     for x in x_values[:-1]:
263         for y in y_values[:-1]:
264             x_range.append(x)
265             y_range.append(y)
266     plt.plot(x_range, y_range, 'o', label='Mesh points')
267     plt.xlabel('x')
268     plt.ylabel('y')
269     plt.grid(True)
270     f.savefig('report/plots/q3e.pdf', bbox_inches='tight')
271
272
273 def plot_sor_jacobi(h_values, potential_values, potential_values_jacobi, iterations_values,
↪ iterations_values_jacobi):
274     f = plt.figure()
275     plt.plot(h_values, potential_values, 'o-', label='SOR')
276     plt.plot(h_values, potential_values_jacobi, 'o-', label='Jacobi')
277     plt.xlabel('1 / h')
278     plt.ylabel('Potential at [0.06, 0.04] (V)')
279     plt.grid(True)
280     plt.legend()
281     f.savefig('report/plots/q3d_potential_comparison.pdf', bbox_inches='tight')
282
283     f = plt.figure()
284     plt.plot(h_values, iterations_values, 'o-', label='SOR')
285     plt.plot(h_values, iterations_values_jacobi, 'o-', label='Jacobi')

```

```

286     plt.xlabel('1 / h')
287     plt.ylabel('Number of Iterations')
288     plt.grid(True)
289     plt.legend()
290     f.savefig('report/plots/q3d_iterations_comparison.pdf', bbox_inches='tight')
291
292
293 if __name__ == '__main__':
294     t = time.time()
295     q3()
296     print('Total runtime: {} s'.format(time.time() - t))

```

B Output Logs

Listing 9: Output of Question 1 program (q1.txt).

```

1  === Question 1(b) ===
2  n=2 matrix is positive-definite: True
3  n=3 matrix is positive-definite: True
4  n=4 matrix is positive-definite: True
5
6  === Question 1(c) ===
7  Matrix with n=2:
8  A:
9      25.00   5.00
10     5.00  10.00
11  b:
12  215.00
13   70.00
14  Expected x:
15     8.00
16     3.00
17  Actual x:
18     8.00
19     3.00
20  Matrix with n=3:
21  A:
22     9.00   3.00  24.00
23     3.00   5.00  18.00
24    24.00  18.00  90.00
25  b:
26  165.00
27  101.00
28  558.00
29  Expected x:
30     9.00
31     4.00
32     3.00
33  Actual x:
34     9.00
35     4.00
36     3.00
37  Matrix with n=4:
38  A:
39     1.00   2.00   5.00   7.00
40     2.00  68.00  50.00  30.00
41     5.00  50.00  66.00  77.00
42     7.00  30.00  77.00 166.00
43  b:
44     81.00
45    602.00
46    984.00
47   1726.00
48  Expected x:
49     5.00
50     4.00
51     1.00

```

```

52     9.00
53 Actual x:
54     5.00
55     4.00
56     1.00
57     9.00
58
59 === Question 1(d) ===
60 Solved for x in network 1:
61 V1 = 5.000 V
62 Solved for x in network 2:
63 V1 = 50.000 V
64 Solved for x in network 3:
65 V1 = 55.000 V
66 Solved for x in network 4:
67 V1 = 20.000 V
68 V2 = 35.000 V
69 Solved for x in network 5:
70 V1 = 5.000 V
71 V2 = 3.750 V
72 V3 = 3.750 V
73 Solved for x in network 6:
74 V1 = 4.443 V
75 V2 = 5.498 V
76 V3 = 3.036 V
77 V4 = 3.200 V
78 V5 = 1.301 V

```

Listing 10: Output of Question 2 program (q2.txt).

```

1  === Question 2(a)(b) ===
2  Equivalent resistance for 2x4 mesh: 1875.00 Ohms.
3  Runtime: 0.000999927520752 s.
4  Equivalent resistance for 3x6 mesh: 2379.55 Ohms.
5  Runtime: 0.0169999599457 s.
6  Equivalent resistance for 4x8 mesh: 2741.03 Ohms.
7  Runtime: 0.100000143051 s.
8  Equivalent resistance for 5x10 mesh: 3022.82 Ohms.
9  Runtime: 0.481999874115 s.
10 Equivalent resistance for 6x12 mesh: 3253.68 Ohms.
11 Runtime: 1.46099996567 s.
12 Equivalent resistance for 7x14 mesh: 3449.17 Ohms.
13 Runtime: 3.26600003242 s.
14 Equivalent resistance for 8x16 mesh: 3618.67 Ohms.
15 Runtime: 7.53400015831 s.
16 Equivalent resistance for 9x18 mesh: 3768.29 Ohms.
17 Runtime: 15.001999855 s.
18 Equivalent resistance for 10x20 mesh: 3902.19 Ohms.
19 Runtime: 28.3630001545 s.
20 === Question 2(c) ===
21 Equivalent resistance for 2x4 mesh: 1875.00 Ohms.
22 Runtime: 0.00100016593933 s.
23 Equivalent resistance for 3x6 mesh: 2379.55 Ohms.
24 Runtime: 0.0169999599457 s.
25 Equivalent resistance for 4x8 mesh: 2741.03 Ohms.
26 Runtime: 0.0950000286102 s.
27 Equivalent resistance for 5x10 mesh: 3022.82 Ohms.
28 Runtime: 0.378000020981 s.
29 Equivalent resistance for 6x12 mesh: 3253.68 Ohms.
30 Runtime: 1.19199991226 s.
31 Equivalent resistance for 7x14 mesh: 3449.17 Ohms.
32 Runtime: 3.05200004578 s.
33 Equivalent resistance for 8x16 mesh: 3618.67 Ohms.
34 Runtime: 6.9430000782 s.
35 Equivalent resistance for 9x18 mesh: 3768.29 Ohms.
36 Runtime: 14.2189998627 s.
37 Equivalent resistance for 10x20 mesh: 3902.19 Ohms.
38 Runtime: 26.763999939 s.
39 === Question 2(d) ===

```


Listing 11: Output of Question 3 program (q3.txt).

```

1  === Question 3(b) ===
2  Omega: 1.0
3  Quarter grid:
4    0.00  3.96  8.56 15.00 15.00 15.00 15.00
5    0.00  4.25  9.09 15.00 15.00 15.00 15.00
6    0.00  3.96  8.56 15.00 15.00 15.00 15.00
7    0.00  3.03  6.18  9.25 10.29 10.55 10.29
8    0.00  1.97  3.88  5.53  6.37  6.61  6.37
9    0.00  0.96  1.86  2.61  3.04  3.17  3.04
10   0.00  0.00  0.00  0.00  0.00  0.00  0.00
11  Num iterations: 32
12  Potential at (0.06, 0.04): 5.526 V
13  Omega: 1.1
14  Quarter grid:
15    0.00  3.96  8.56 15.00 15.00 15.00 15.00
16    0.00  4.25  9.09 15.00 15.00 15.00 15.00
17    0.00  3.96  8.56 15.00 15.00 15.00 15.00
18    0.00  3.03  6.18  9.25 10.29 10.55 10.29
19    0.00  1.97  3.88  5.53  6.37  6.61  6.37
20    0.00  0.96  1.86  2.61  3.04  3.17  3.04
21    0.00  0.00  0.00  0.00  0.00  0.00  0.00
22  Num iterations: 26
23  Potential at (0.06, 0.04): 5.526 V
24  Omega: 1.2
25  Quarter grid:
26    0.00  3.96  8.56 15.00 15.00 15.00 15.00
27    0.00  4.25  9.09 15.00 15.00 15.00 15.00
28    0.00  3.96  8.56 15.00 15.00 15.00 15.00
29    0.00  3.03  6.18  9.25 10.29 10.55 10.29
30    0.00  1.97  3.88  5.53  6.37  6.61  6.37
31    0.00  0.96  1.86  2.61  3.04  3.17  3.04
32    0.00  0.00  0.00  0.00  0.00  0.00  0.00
33  Num iterations: 20
34  Potential at (0.06, 0.04): 5.526 V
35  Omega: 1.3
36  Quarter grid:
37    0.00  3.96  8.56 15.00 15.00 15.00 15.00
38    0.00  4.25  9.09 15.00 15.00 15.00 15.00
39    0.00  3.96  8.56 15.00 15.00 15.00 15.00
40    0.00  3.03  6.18  9.25 10.29 10.55 10.29
41    0.00  1.97  3.88  5.53  6.37  6.61  6.37
42    0.00  0.96  1.86  2.61  3.04  3.17  3.04
43    0.00  0.00  0.00  0.00  0.00  0.00  0.00
44  Num iterations: 14
45  Potential at (0.06, 0.04): 5.526 V
46  Omega: 1.4
47  Quarter grid:
48    0.00  3.96  8.56 15.00 15.00 15.00 15.00
49    0.00  4.25  9.09 15.00 15.00 15.00 15.00
50    0.00  3.96  8.56 15.00 15.00 15.00 15.00
51    0.00  3.03  6.18  9.25 10.29 10.55 10.29
52    0.00  1.97  3.88  5.53  6.37  6.61  6.37
53    0.00  0.96  1.86  2.61  3.04  3.17  3.04
54    0.00  0.00  0.00  0.00  0.00  0.00  0.00
55  Num iterations: 16
56  Potential at (0.06, 0.04): 5.526 V
57  Omega: 1.5
58  Quarter grid:
59    0.00  3.96  8.56 15.00 15.00 15.00 15.00
60    0.00  4.25  9.09 15.00 15.00 15.00 15.00
61    0.00  3.96  8.56 15.00 15.00 15.00 15.00
62    0.00  3.03  6.18  9.25 10.29 10.55 10.29
63    0.00  1.97  3.88  5.53  6.37  6.61  6.37
64    0.00  0.96  1.86  2.61  3.04  3.17  3.04
65    0.00  0.00  0.00  0.00  0.00  0.00  0.00
66  Num iterations: 20
67  Potential at (0.06, 0.04): 5.526 V

```

```

68 Omega: 1.6
69 Quarter grid:
70 0.00 3.96 8.56 15.00 15.00 15.00 15.00
71 0.00 4.25 9.09 15.00 15.00 15.00 15.00
72 0.00 3.96 8.56 15.00 15.00 15.00 15.00
73 0.00 3.03 6.18 9.25 10.29 10.55 10.29
74 0.00 1.97 3.88 5.53 6.37 6.61 6.37
75 0.00 0.96 1.86 2.61 3.04 3.17 3.04
76 0.00 0.00 0.00 0.00 0.00 0.00 0.00
77 Num iterations: 27
78 Potential at (0.06, 0.04): 5.526 V
79 Omega: 1.7
80 Quarter grid:
81 0.00 3.96 8.56 15.00 15.00 15.00 15.00
82 0.00 4.25 9.09 15.00 15.00 15.00 15.00
83 0.00 3.96 8.56 15.00 15.00 15.00 15.00
84 0.00 3.03 6.18 9.25 10.29 10.55 10.29
85 0.00 1.97 3.88 5.53 6.37 6.61 6.37
86 0.00 0.96 1.86 2.61 3.04 3.17 3.04
87 0.00 0.00 0.00 0.00 0.00 0.00 0.00
88 Num iterations: 39
89 Potential at (0.06, 0.04): 5.526 V
90 Omega: 1.8
91 Quarter grid:
92 0.00 3.96 8.56 15.00 15.00 15.00 15.00
93 0.00 4.25 9.09 15.00 15.00 15.00 15.00
94 0.00 3.96 8.56 15.00 15.00 15.00 15.00
95 0.00 3.03 6.18 9.25 10.29 10.55 10.29
96 0.00 1.97 3.88 5.53 6.37 6.61 6.37
97 0.00 0.96 1.86 2.61 3.04 3.17 3.04
98 0.00 0.00 0.00 0.00 0.00 0.00 0.00
99 Num iterations: 60
100 Potential at (0.06, 0.04): 5.526 V
101 Omega: 1.9
102 Quarter grid:
103 0.00 3.96 8.56 15.00 15.00 15.00 15.00
104 0.00 4.25 9.09 15.00 15.00 15.00 15.00
105 0.00 3.96 8.56 15.00 15.00 15.00 15.00
106 0.00 3.03 6.18 9.25 10.29 10.55 10.29
107 0.00 1.97 3.88 5.53 6.37 6.61 6.37
108 0.00 0.96 1.86 2.61 3.04 3.17 3.04
109 0.00 0.00 0.00 0.00 0.00 0.00 0.00
110 Num iterations: 127
111 Potential at (0.06, 0.04): 5.526 V
112 Best number of iterations: 14
113 Best omega: 1.3
114 === Question 3(c): SOR ===
115 h: 0.02
116 1/h: 50.0
117 Num iterations: 14
118 Potential at (0.06, 0.04): 5.526 V
119 h: 0.01
120 1/h: 100.0
121 Num iterations: 59
122 Potential at (0.06, 0.04): 5.351 V
123 h: 0.005
124 1/h: 200.0
125 Num iterations: 189
126 Potential at (0.06, 0.04): 5.289 V
127 h: 0.0025
128 1/h: 400.0
129 Num iterations: 552
130 Potential at (0.06, 0.04): 5.265 V
131 h: 0.00125
132 1/h: 800.0
133 Num iterations: 1540
134 Potential at (0.06, 0.04): 5.254 V
135 h: 0.000625
136 1/h: 1600.0
137 Num iterations: 4507

```

```

138 Potential at (0.06, 0.04): 5.247 V
139 === Question 3(d): Jacobi ===
140 h: 0.02
141 Num iterations: 51
142 Potential at (0.06, 0.04): 5.526 V
143 h: 0.01
144 Num iterations: 180
145 Potential at (0.06, 0.04): 5.351 V
146 h: 0.005
147 Num iterations: 604
148 Potential at (0.06, 0.04): 5.289 V
149 h: 0.0025
150 Num iterations: 1935
151 Potential at (0.06, 0.04): 5.265 V
152 h: 0.00125
153 Num iterations: 5836
154 Potential at (0.06, 0.04): 5.254 V
155 h: 0.000625
156 Num iterations: 16864
157 Potential at (0.06, 0.04): 5.246 V
158 Total runtime: 1724.82099986
159 === Question 3(e): Non-Uniform Node Spacing ===
160 Jacobi (for reference)
161 Quarter grid:
162 0.00 1.99 4.06 6.29 8.78 11.66 15.00 15.00 15.00 15.00 15.00 15.00
163 0.00 2.03 4.14 6.41 8.95 11.82 15.00 15.00 15.00 15.00 15.00 15.00
164 0.00 1.99 4.06 6.29 8.78 11.66 15.00 15.00 15.00 15.00 15.00 15.00
165 0.00 1.87 3.81 5.89 8.23 11.04 15.00 15.00 15.00 15.00 15.00 15.00
166 0.00 1.69 3.42 5.24 7.19 9.28 11.33 12.14 12.50 12.66 12.71 12.66
167 0.00 1.46 2.95 4.47 6.02 7.55 8.90 9.73 10.20 10.44 10.51 10.44
168 0.00 1.22 2.44 3.66 4.87 6.01 6.99 7.69 8.14 8.38 8.45 8.38
169 0.00 0.96 1.92 2.87 3.78 4.63 5.35 5.90 6.27 6.48 6.55 6.48
170 0.00 0.71 1.42 2.11 2.77 3.37 3.89 4.29 4.57 4.73 4.79 4.73
171 0.00 0.47 0.94 1.39 1.81 2.20 2.53 2.80 2.98 3.09 3.13 3.09
172 0.00 0.23 0.46 0.69 0.90 1.09 1.25 1.38 1.47 1.53 1.55 1.53
173 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00
174 Num iterations: 106
175 Potential at (0.06, 0.04): 5.351 V
176 Uniform Mesh (same as Jacobi)
177 Quarter grid:
178 0.00 1.99 4.06 6.29 8.78 11.66 15.00 15.00 15.00 15.00 15.00 15.00
179 0.00 2.03 4.14 6.41 8.95 11.82 15.00 15.00 15.00 15.00 15.00 15.00
180 0.00 1.99 4.06 6.29 8.78 11.66 15.00 15.00 15.00 15.00 15.00 15.00
181 0.00 1.87 3.81 5.89 8.23 11.04 15.00 15.00 15.00 15.00 15.00 15.00
182 0.00 1.69 3.42 5.24 7.19 9.28 11.33 12.14 12.50 12.66 12.71 12.66
183 0.00 1.46 2.95 4.47 6.02 7.55 8.90 9.73 10.20 10.44 10.51 10.44
184 0.00 1.22 2.44 3.66 4.87 6.01 6.99 7.69 8.14 8.38 8.45 8.38
185 0.00 0.96 1.92 2.87 3.79 4.63 5.35 5.90 6.27 6.48 6.55 6.48
186 0.00 0.71 1.42 2.11 2.77 3.37 3.89 4.29 4.57 4.73 4.79 4.73
187 0.00 0.47 0.94 1.39 1.81 2.20 2.53 2.80 2.98 3.09 3.13 3.09
188 0.00 0.23 0.46 0.69 0.90 1.09 1.25 1.38 1.47 1.53 1.55 1.53
189 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00
190 Num iterations: 209
191 Potential at (0.06, 0.04): 5.351 V
192 Jacobi potential: 5.35062156679 V, same as uniform potential: 5.35067998265 V
193 Non-Uniform (clustered around (0.06, 0.04))
194 Quarter grid:
195 0.00 2.00 4.08 6.33 11.61 13.25 15.00 15.00 15.00 15.00 15.00 15.00
196 0.00 2.04 4.17 6.45 11.80 13.37 15.00 15.00 15.00 15.00 15.00 15.00
197 0.00 2.00 4.08 6.33 11.61 13.25 15.00 15.00 15.00 15.00 15.00 15.00
198 0.00 1.89 3.84 5.93 10.90 12.71 15.00 15.00 15.00 15.00 15.00 15.00
199 0.00 1.71 3.45 5.28 9.27 10.26 11.15 11.74 12.14 12.66 12.71 12.66
200 0.00 1.21 2.43 3.66 6.06 6.57 7.03 7.42 7.75 8.38 8.45 8.38
201 0.00 1.09 2.18 3.26 5.35 5.78 6.18 6.52 6.81 7.41 7.48 7.41
202 0.00 0.96 1.92 2.87 4.66 5.04 5.38 5.67 5.93 6.48 6.55 6.48
203 0.00 0.84 1.67 2.48 4.01 4.33 4.62 4.87 5.09 5.59 5.65 5.59
204 0.00 0.71 1.42 2.11 3.39 3.65 3.89 4.11 4.29 4.72 4.77 4.72
205 0.00 0.23 0.47 0.69 1.10 1.19 1.26 1.33 1.39 1.54 1.56 1.54
206 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00
207 Num iterations: 385

```

```

208 Potential at (0.06, 0.04): 5.378 V
209 Non-Uniform (more clustered around (0.06, 0.04))
210 Quarter grid:
211 0.00 2.03 4.14 6.41 13.24 14.65 15.00 15.00 15.00 15.00 15.00 15.00
212 0.00 2.07 4.22 6.53 13.40 14.68 15.00 15.00 15.00 15.00 15.00 15.00
213 0.00 2.03 4.14 6.41 13.24 14.65 15.00 15.00 15.00 15.00 15.00 15.00
214 0.00 1.92 3.90 6.02 12.55 14.45 15.00 15.00 15.00 15.00 15.00 15.00
215 0.00 1.73 3.51 5.36 10.40 11.09 11.24 11.38 11.86 12.65 12.71 12.65
216 0.00 1.10 2.19 3.28 5.90 6.21 6.29 6.36 6.62 7.44 7.51 7.44
217 0.00 1.00 1.99 2.97 5.28 5.56 5.62 5.69 5.92 6.69 6.75 6.69
218 0.00 0.97 1.94 2.89 5.13 5.40 5.46 5.52 5.75 6.50 6.57 6.50
219 0.00 0.94 1.88 2.81 4.98 5.24 5.30 5.36 5.58 6.32 6.38 6.32
220 0.00 0.84 1.68 2.50 4.39 4.62 4.68 4.73 4.92 5.60 5.66 5.60
221 0.00 0.24 0.47 0.70 1.21 1.28 1.29 1.31 1.36 1.56 1.57 1.56
222 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00
223 Num iterations: 1337
224 Potential at (0.06, 0.04): 5.461 V
225 Non-Uniform (clustered near outer conductor)
226 Quarter grid:
227 0.00 4.38 7.21 10.30 13.47 7.42 8.97 9.82 10.43 10.80 10.86 7.63
228 0.00 4.46 7.34 10.46 13.55 15.00 15.00 15.00 15.00 15.00 15.00 15.00
229 0.00 4.38 7.21 10.30 13.47 15.00 15.00 15.00 15.00 15.00 15.00 15.00
230 0.00 4.19 6.91 9.94 13.24 15.00 15.00 15.00 15.00 15.00 15.00 15.00
231 0.00 3.95 6.50 9.37 12.69 15.00 15.00 15.00 15.00 15.00 15.00 15.00
232 0.00 3.61 5.91 8.39 10.87 11.93 12.87 13.10 13.22 13.30 13.33 13.30
233 0.00 3.18 5.15 7.16 8.96 9.63 10.73 11.09 11.29 11.43 11.49 11.43
234 0.00 2.67 4.27 5.84 7.16 7.66 8.66 9.03 9.27 9.44 9.51 9.44
235 0.00 1.89 3.00 4.05 4.91 5.24 5.99 6.29 6.49 6.64 6.71 6.64
236 0.00 1.50 2.36 3.17 3.83 4.09 4.69 4.94 5.11 5.23 5.29 5.23
237 0.00 0.92 1.44 1.93 2.33 2.49 2.86 3.02 3.13 3.21 3.25 3.21
238 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00
239 Num iterations: 222
240 Potential at (0.06, 0.04): 5.243 V

```