ECSE 543
Assignment 1

Sean Stappas
260639512

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Introduction

The programs for this assignment were created in Python 2.7. The source code is provided as listings in Appendix A. To perform the required tasks in this assignment, a custom matrix package was created, with useful methods such as add, multiply, transpose, etc. This package can be seen in Listing 1. The structure of the rest of the code will be discussed as appropriate for each question. In addition, logs of the output of the programs are provided in Appendix B.

1 Choleski Decomposition

The source code for the Question 1 main program can be seen in Listing 4.

1.a Choleski Program

The code relating specifically to Choleski decomposition can be seen in Listing 2.

1.b Constructing Test Matrices

The matrices were constructed with the knowledge that, if A is positive-definite, then $A = LL^T$ where L is a lower triangular non-singular matrix. The task of choosing valid A matrices then boils down to finding non-singular lower triangular L matrices. To ensure that L is non-singular, one must simply choose nonzero values for the main diagonal.

1.c Test Runs

The matrices were tested by inventing x matrices, and checking that the program solves for that x correctly. The output of the program, comparing expected and obtained values of x, can be seen in Listing 8.

1.d Linear Networks

As can be seen in Listing 3, the `csv_to_network_branch_matrices` method of the `linear_networks.py` script reads from a CSV file where row k contains J_k, R_k and E_k. It then converts the resistances to a diagonal admittance matrix Y and produces the J and E column vectors. The incidence matrix A is also read directly from file, as seen in Listing 4.

First, the program was tested on the circuits provided on MyCourses. These circuits are labeled 1 to 5 and have corresponding incidence matrix and network branch CSV files, located in the `network_data` directory. The program obtains the expected voltages, as seen in the output in Listing 8. Then, some additional simple test circuits were created. Circuit 6 can be seen in Figure 1 and the SPICE analysis output in Table 1. These voltages match the ones calculated by the program, as seen in Listing 8.

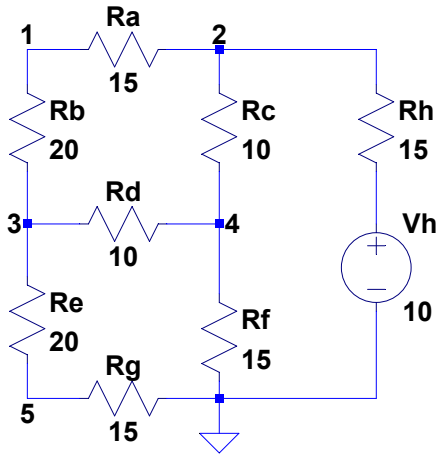

Figure 1: Test circuit 6 with nodes labeled 1 to 4.

Table 1: Output of SPICE operating point analysis of circuit 6.

Node	Voltage (V)
1	4.443
2	5.498
3	3.036
4	3.200
5	1.301

2 Finite Difference Mesh

The source code for the Question 2 main program can be seen in Listing 5.

2.a Equivalent Resistance

The code for creating all the network matrices and for finding the equivalent resistance of an N by $2N$ mesh can be seen in Listing 3. The resistances found by the program for values of N from 2 to 10 can be seen in Table 2.

The resistance values returned by the program for small meshes were validated using simple SPICE circuits. The voltage found at the V_{test} node for the 2x4 mesh is 1.875 V and the equivalent resistance is therefore 1875 Ω. Similarly, for the 3x6 mesh, V_{test} = 2.379 55 V and the equivalent resistance is

Table 2: Mesh equivalent resistance R versus mesh size N.

N	R (Omega)
2	1875.000
3	2379.545
4	2741.025
5	3022.819
6	3253.676
7	3449.166
8	3618.675
9	3768.291
10	3902.189

2379.55 Ω. These match the results found by the program, as seen in Table 2.

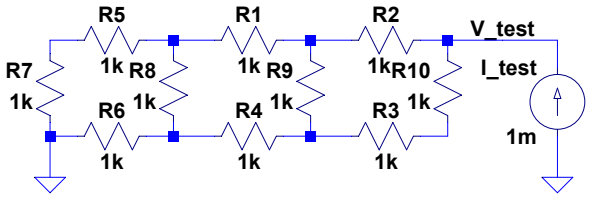

Figure 2: SPICE circuit used to test the 2x4 mesh.

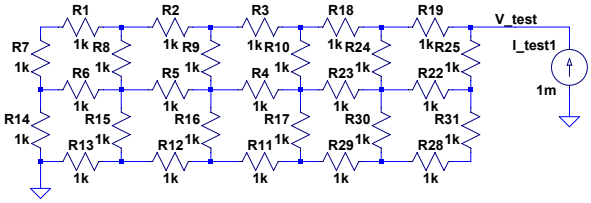

Figure 3: SPICE circuit used to test the 3x6 mesh.

2.b Time Complexity

The runtime data for the mesh resistance solver is tabulated in Table 3 and plotted in Figure 4. Theoretically, the time complexity of the program should be $O(N^6)$, and this matches the obtained data.

2.c Sparsity Modification

The runtime data for the banded mesh resistance solver is tabulated in Table 4 and plotted in Figure 5. By inspection of the constructed network matrices, a half-bandwidth of $2N+1$ was chosen. Theoretically, the banded version should have a time complexity of $O(N^4)$.

Table 3: Runtime of mesh resistance solver program versus mesh size N.

N	Runtime (s)
2	0.000
3	0.016
4	0.094
5	0.386
6	1.266
7	3.142
8	6.953
9	14.438
10	27.922

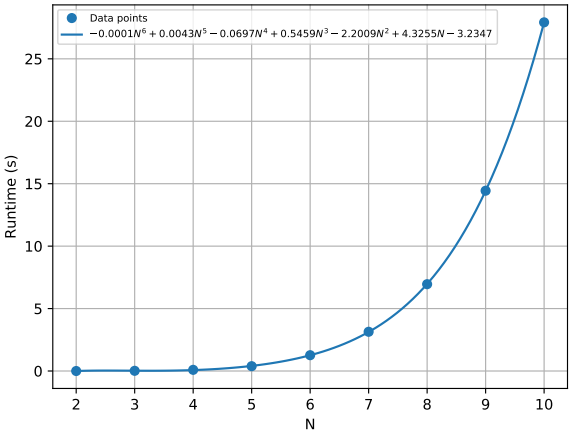

Figure 4: Runtime of mesh resistance solver program versus mesh size N.

Table 4: Runtime of banded mesh resistance solver program versus mesh size N.

N	Runtime (s)
2	0.016
3	0.015
4	0.078
5	0.372
6	1.099
7	2.969
8	6.417
9	13.317
10	25.448

The runtime of the banded and non-banded versions of the program are plotted in Figure 6, showing the benefits of banded elimination.

2.d Resistance vs. Mesh Size

The equivalent mesh resistance R is plotted versus the mesh size N in Figure 7. The function $R(N)$

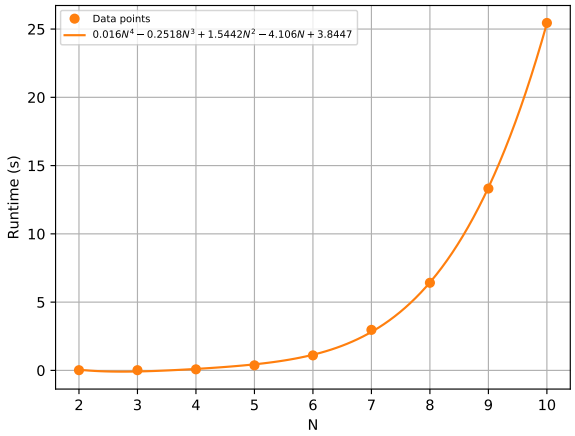

Figure 5: Runtime of banded mesh resistance solver program versus mesh size N.

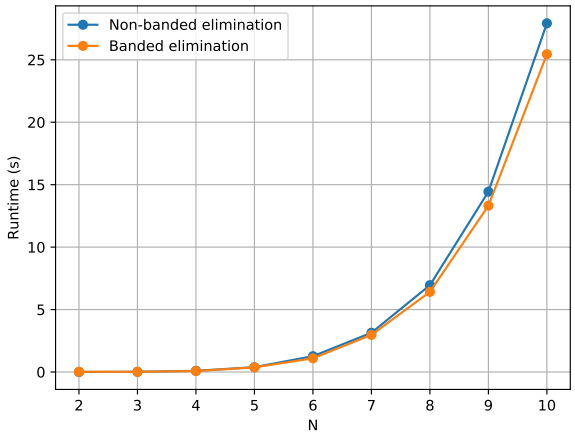

Figure 6: Comparison of runtime of banded and non-banded resistance solver programs versus mesh size N.

appears logarithmic, and a log function does indeed fit the data well.

3 Coaxial Cable

The source code for the Question 2 main program can be seen in Listing 7.

3.a SOR Program

The source code for the finite difference methods can be seen in Listing 6. Horizontal and vertical symmetries were exploited by only solving for a quarter of the coaxial cable, and reproducing the results where necessary.

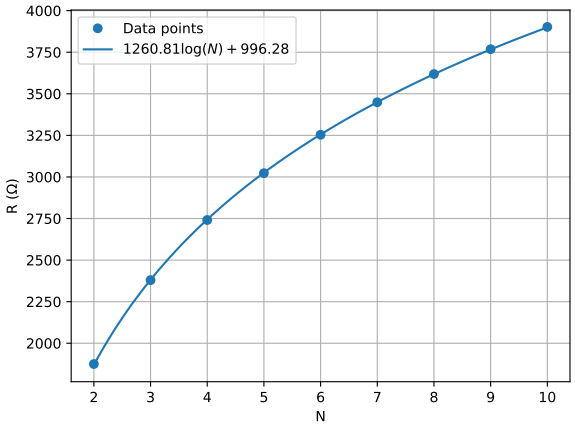

Figure 7: Resistance of mesh versus mesh size N.

3.b Varying ω

The number of iterations to achieve convergence for 10 values of ω between 1 and 2 are tabulated in Table 5 and plotted in Figure 8. Based on these results, the value of ω yielding the minimum number of iterations is 1.3.

Table 5: Number of iterations of SOR versus ω.

Omega	Iterations
1.0	32
1.1	26
1.2	20
1.3	14
1.4	16
1.5	20
1.6	27
1.7	39
1.8	60
1.9	127

The potential values found at (0.06, 0.04) versus ω are tabulated in Table 6. It can be seen that all the potential values are identical to 3 decimal places.

3.c Varying h

With $\omega = 1.3$, the number of iterations of SOR versus $1/h$ is tabulated in Table 7 and plotted in Figure 9. It can be seen that the smaller the node spacing is, the more iterations the program will take to run. Theoretically, the time complexity of the program should be $O(N^3)$, where the finite difference mesh is N by N, and this matches the measured data.

The potential values found at (0.06, 0.04) versus $1/h$ are tabulated in Table 8 and plotted in Fig-

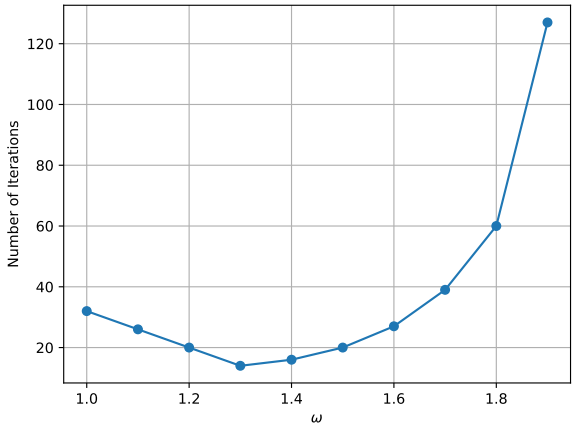

Figure 8: Number of iterations of SOR versus ω.

Table 6: Potential at (0.06, 0.04) versus ω when using SOR.

Omega	Potential (V)
1.0	5.526
1.1	5.526
1.2	5.526
1.3	5.526
1.4	5.526
1.5	5.526
1.6	5.526
1.7	5.526
1.8	5.526
1.9	5.526

Table 7: Number of iterations of SOR versus $1/h$. Note that $\omega = 1.3$.

1/h	Iterations
50.0	14
100.0	59
200.0	189
400.0	552
800.0	1540
1600.0	4507

ure 10. By examining these values, the potential at (0.06, 0.04) to three significant figures is approximately 5.25 V. It can be seen that the smaller the node spacing is, the more accurate the calculated potential is. However, by inspecting Figure 10 it is apparent that the potential converges relatively quickly to around 5.25 V There are therefore diminishing returns to decreasing the node spacing too much, since this will also increase the runtime of the program.

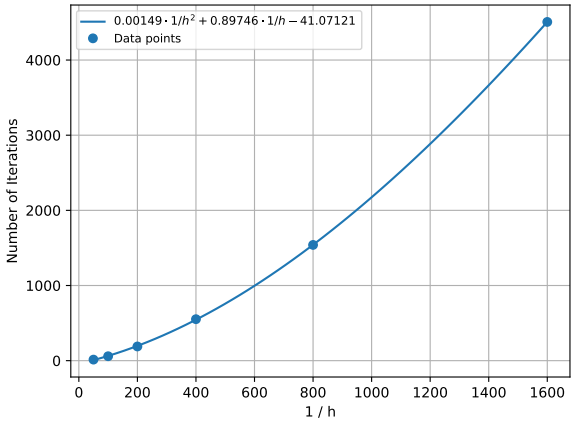

Figure 9: Number of iterations of SOR versus $1/h$. Note that $\omega = 1.3$.

Table 8: Potential at (0.06, 0.04) versus $1/h$ when using SOR.

1/h	Potential (V)
50.0	5.526
100.0	5.351
200.0	5.289
400.0	5.265
800.0	5.254
1600.0	5.247

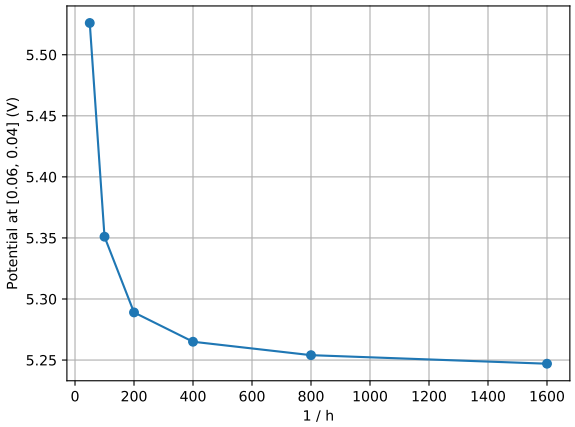

Figure 10: Potential at (0.06, 0.04) found by SOR versus $1/h$. Note that $\omega = 1.3$.

3.d Jacobi Method

The number of iterations of the Jacobi method versus $1/h$ is tabulated in Table 9 and plotted in Figure 11. Similarly to SOR, the smaller the node spacing is, the more iterations the program will take to run. We can see however that the Jacobi method takes a much larger number of iterations to

converge. Theoretically, the Jacobi method should have a time complexity of $O(N^4)$, and this matches the data.

Table 9: Number of iterations versus ω when using the Jacobi method.

1/h	Iterations
50.0	51
100.0	180
200.0	604
400.0	1935
800.0	5836
1600.0	16864

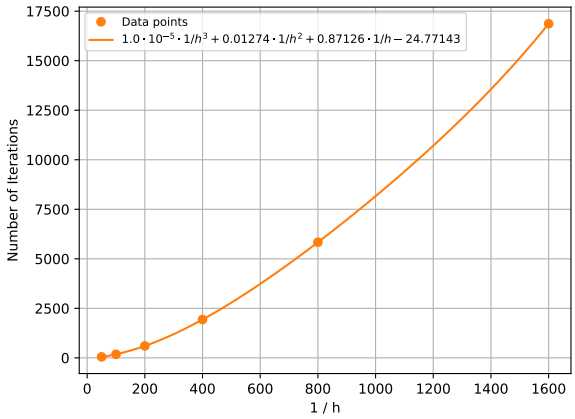

Figure 11: Number of iterations of the Jacobi method versus $1/h$.

The potential values found at (0.06, 0.04) versus $1/h$ with the Jacobi method are tabulated in Table 10 and plotted in Figure 12. These potential values are almost identical to the SOR ones. Similarly to SOR, the smaller the node spacing is, the more accurate the calculated potential is.

Table 10: Potential at (0.06, 0.04) versus $1/h$ when using the Jacobi method.

1/h	Potential (V)
50.0	5.526
100.0	5.351
200.0	5.289
400.0	5.265
800.0	5.254
1600.0	5.246

The number of iterations of both SOR and the Jacobi method can be seen in Figure 13, which shows the clear benefits of SOR.

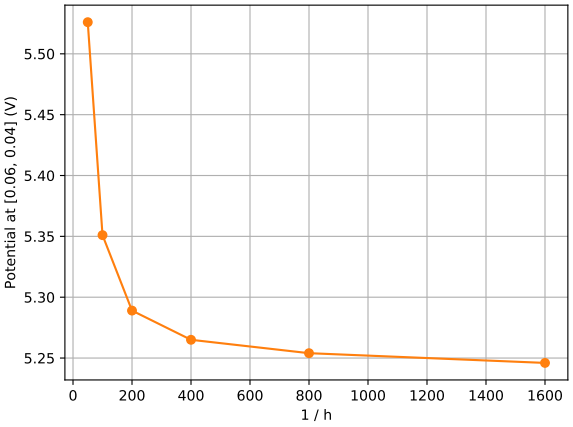

Figure 12: Potential at (0.06, 0.04) versus $1/h$ when using the Jacobi method.

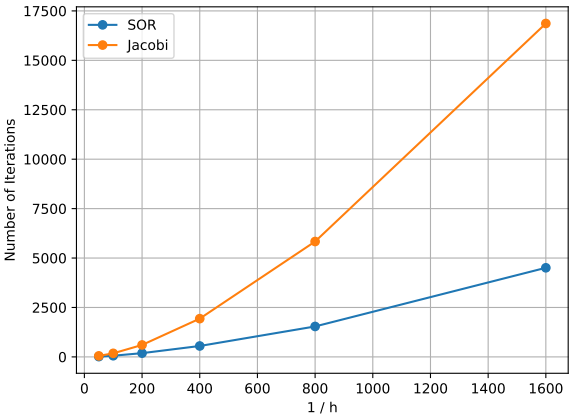

Figure 13: Comparison of number of iterations when using SOR and Jacobi methods versus $1/h$. Note that $\omega = 1.3$ for the SOR program.

3.e Non-uniform Node Spacing

First, we adjust the equation derived in class to set $a_1 = \Delta_x \alpha_1$, $a_2 = \Delta_x \alpha_2$, $b_1 = \Delta_y \beta_1$ and $b_2 = \Delta_y \beta_2$. These values correspond to the distances between adjacent nodes [1], and can be easily calculated by the program. Then, the five-point difference formula for non-uniform spacing can be seen in Equation 1.

$$\phi_{i,j}^{k+1} = \frac{1}{a_1 + a_2}\left(\frac{\phi_{i-1,j}^{k}}{a_1} + \frac{\phi_{i+1,j}^{k}}{a_2}\right) + \frac{1}{b_1 + b_2}\left(\frac{\phi_{i,j-1}^{k}}{b_1} + \frac{\phi_{i,j+1}^{k}}{b_2}\right) \tag{1}$$

[1] Note that, in the program, index i is associated to position x and index j is associated to position y. This is purely for easier printing of the matrices.

This was implemented in the finite difference program, as seen in Listing 6. As can be seen in this code, many different mesh arrangements were tested. The arrangement that was chosen can be seen in Figure 14. The potential at (0.06, 0.04) obtained from this arrangement is 5.243 V, which seems like an accurate potential value. Indeed, as can be seen in Figures 10 and 12, the potential value for small node spacings tends towards 5.24 V for both the Jacobi and SOR methods.

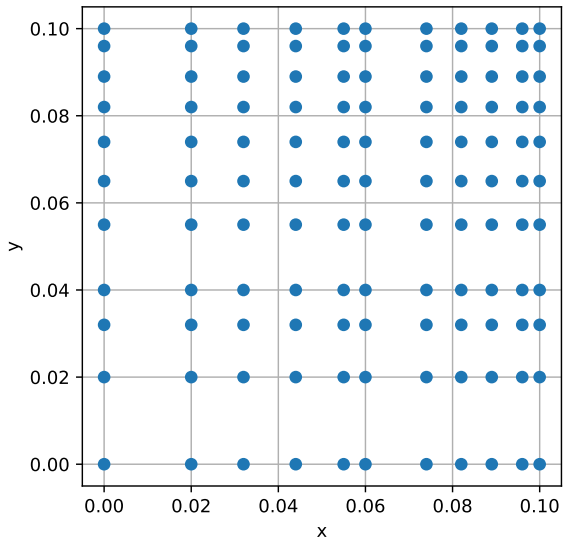

Figure 14: Final mesh arrangement used for non-uniform node spacing. Each point corresponds to a mesh point. Points are positioned closer to the inner conductor, since this is a more difficult area.

A Code Listings

Listing 1: Custom matrix package (`matrices.py`).

```
from __future__ import division

import copy
import csv
from ast import literal_eval

import math


class Matrix:

    def __init__(self, data):
        self.data = data
        self.rows = len(data)
        self.cols = len(data[0])

    def __str__(self):
        string = ''
        for row in self.data:
            string += '\n'
            for val in row:
                string += '{:6.2f} '.format(val)
        return string

    def __add__(self, other):
        if len(self) != len(other) or len(self[0]) != len(other[0]):
            raise ValueError('Incompatible matrix sizes for addition. Matrix A is {}x{}, but matrix B is
             ↪  {}x{}.'
                             .format(len(self), len(self[0]), len(other), len(other[0])))

        return Matrix([[self[row][col] + other[row][col] for col in range(self.cols)] for row in
         ↪  range(self.rows)])

    def __sub__(self, other):
        if len(self) != len(other) or len(self[0]) != len(other[0]):
            raise ValueError('Incompatible matrix sizes for subtraction. Matrix A is {}x{}, but matrix B
             ↪  is {}x{}.'
                             .format(len(self), len(self[0]), len(other), len(other[0])))

        return Matrix([[self[row][col] - other[row][col] for col in range(self.cols)] for row in
         ↪  range(self.rows)])

    def __mul__(self, other):
        if self.cols != other.rows:
            raise ValueError('Incompatible matrix sizes for multiplication. Matrix A is {}x{}, but matrix
             ↪  B is {}x{}.'
                             .format(self.rows, self.cols, other.rows, other.cols))

        # Inspired from https://en.wikipedia.org/wiki/Matrix_multiplication
        product = Matrix.empty(self.rows, other.cols)
        for i in range(self.rows):
            for j in range(other.cols):
                row_sum = 0
                for k in range(self.cols):
                    row_sum += self[i][k] * other[k][j]
                product[i][j] = row_sum
        return product

    def __deepcopy__(self, memo):
        return Matrix(copy.deepcopy(self.data))

    def __getitem__(self, item):
        return self.data[item]

    def __len__(self):
```

```
61          return len(self.data)
62
63      def is_positive_definite(self):
64          """
65          :return: True if the matrix if positive-definite, False otherwise.
66          """
67          A = copy.deepcopy(self.data)
68          for j in range(self.rows):
69              if A[j][j] <= 0:
70                  return False
71              A[j][j] = math.sqrt(A[j][j])
72              for i in range(j + 1, self.rows):
73                  A[i][j] = A[i][j] / A[j][j]
74                  for k in range(j + 1, i + 1):
75                      A[i][k] = A[i][k] - A[i][j] * A[k][j]
76          return True
77
78      def transpose(self):
79          """
80          :return: the transpose of the current matrix
81          """
82          return Matrix([[self.data[row][col] for row in range(self.rows)] for col in range(self.cols)])
83
84      def mirror_horizontal(self):
85          """
86          :return: the horizontal mirror of the current matrix
87          """
88          return Matrix([[self.data[self.rows - row - 1][col] for col in range(self.cols)] for row in
            ↪  range(self.rows)])
89
90      def empty_copy(self):
91          """
92          :return: an empty matrix of the same size as the current matrix.
93          """
94          return Matrix.empty(self.rows, self.cols)
95
96      @staticmethod
97      def multiply(*matrices):
98          """
99          Computes the product of the given matrices.
100
101         :param matrices: the matrix objects
102         :return: the product of the given matrices
103         """
104         n = matrices[0].rows
105         product = Matrix.identity(n)
106         for matrix in matrices:
107             product = product * matrix
108         return product
109
110     @staticmethod
111     def empty(num_rows, num_cols):
112         """
113         Returns an empty matrix (filled with zeroes) with the specified number of columns and rows.
114
115         :param num_rows: number of rows
116         :param num_cols: number of columns
117         :return: the empty matrix
118         """
119         return Matrix([[0 for _ in range(num_cols)] for _ in range(num_rows)])
120
121     @staticmethod
122     def identity(n):
123         """
124         Returns the identity matrix of the given size.
125
126         :param n: the size of the identity matrix (number of rows or columns)
127         :return: the identity matrix of size n
128         """
129         return Matrix.diagonal_single_value(1, n)
```

```
130
131         @staticmethod
132         def diagonal(values):
133             """
134             Returns a diagonal matrix with the given values along the main diagonal.
135
136             :param values: the values along the main diagonal
137             :return: a diagonal matrix with the given values along the main diagonal
138             """
139             n = len(values)
140             return Matrix([[values[row] if row == col else 0 for col in range(n)] for row in range(n)])
141
142         @staticmethod
143         def diagonal_single_value(value, n):
144             """
145             Returns a diagonal matrix of the given size with the given value along the diagonal.
146
147             :param value: the value of each element on the main diagonal
148             :param n: the size of the matrix
149             :return: a diagonal matrix of the given size with the given value along the diagonal.
150             """
151             return Matrix([[value if row == col else 0 for col in range(n)] for row in range(n)])
152
153         @staticmethod
154         def column_vector(values):
155             """
156             Transforms a row vector into a column vector.
157
158             :param values: the values, one for each row of the column vector
159             :return: the column vector
160             """
161             return Matrix([[value] for value in values])
162
163         @staticmethod
164         def csv_to_matrix(filename):
165             """
166             Reads a CSV file to a matrix.
167
168             :param filename: the name of the CSV file
169             :return: a matrix containing the values in the CSV file
170             """
171             with open(filename, 'r') as csv_file:
172                 reader = csv.reader(csv_file)
173                 data = []
174                 for row_number, row in enumerate(reader):
175                     data.append([literal_eval(val) for val in row])
176                 return Matrix(data)
```

Listing 2: Choleski decomposition (`choleski.py`).

```
1   from __future__ import division
2
3   import math
4
5   from matrices import Matrix
6
7
8   def choleski_solve(A, b, half_bandwidth=None):
9       """
10      Solves an Ax = b matrix equation by Choleski decomposition.
11
12      :param A: the A matrix
13      :param b: the b matrix
14      :param half_bandwidth: the half-bandwidth of the A matrix
15      :return: the solved x vector
16      """
17      n = len(A[0])
18      if half_bandwidth is None:
19          elimination(A, b)
```

```
    else:
        elimination_banded(A, b, half_bandwidth)
    x = Matrix.empty(n, 1)
    back_substitution(A, x, b)
    return x


def elimination(A, b):
    """
    Performs the elimination step of Choleski decomposition.

    :param A: the A matrix
    :param b: the b matrix
    """
    n = len(A)
    for j in range(n):
        if A[j][j] <= 0:
            raise ValueError('Matrix A is not positive definite.')
        A[j][j] = math.sqrt(A[j][j])
        b[j][0] = b[j][0] / A[j][j]
        for i in range(j + 1, n):
            A[i][j] = A[i][j] / A[j][j]
            b[i][0] = b[i][0] - A[i][j] * b[j][0]
            for k in range(j + 1, i + 1):
                A[i][k] = A[i][k] - A[i][j] * A[k][j]


def elimination_banded(A, b, half_bandwidth):
    """
    Performs the banded elimination step of Choleski decomposition.

    :param A: the A matrix
    :param b: the b matrix
    :param half_bandwidth: the half_bandwidth to be used for the banded elimination
    """
    n = len(A)
    for j in range(n):
        if A[j][j] <= 0:
            raise ValueError('Matrix A is not positive definite.')
        A[j][j] = math.sqrt(A[j][j])
        b[j][0] = b[j][0] / A[j][j]
        max_row = min(j + half_bandwidth, n)
        for i in range(j + 1, max_row):
            A[i][j] = A[i][j] / A[j][j]
            b[i][0] = b[i][0] - A[i][j] * b[j][0]
            for k in range(j + 1, i + 1):
                A[i][k] = A[i][k] - A[i][j] * A[k][j]


def back_substitution(L, x, y):
    """
    Performs the back-substitution step of Choleski decomposition.

    :param L: the L matrix
    :param x: the x matrix
    :param y: the y matrix
    """
    n = len(L)
    for i in range(n - 1, -1, -1):
        prev_sum = 0
        for j in range(i + 1, n):
            prev_sum += L[j][i] * x[j][0]
        x[i][0] = (y[i][0] - prev_sum) / L[i][i]
```

Listing 3: Linear resistive networks (`linear_networks.py`).

```
from __future__ import division

import csv
```

```python
from matrices import Matrix
from choleski import choleski_solve


def solve_linear_network(A, Y, J, E, half_bandwidth=None):
    """
    Solve the linear resistive network described by the given matrices.

    :param A: the incidence matrix
    :param Y: the admittance matrix
    :param J: the current source matrix
    :param E: the voltage source matrix
    :param half_bandwidth:
    :return: the solved voltage matrix
    """
    A_new = A * Y * A.transpose()
    b = A * (J - Y * E)
    return choleski_solve(A_new, b, half_bandwidth=half_bandwidth)


def csv_to_network_branch_matrices(filename):
    """
    Converts a CSV file to Y, J, E network matrices.

    :param filename: the name of the CSV file
    :return: the Y, J, E network matrices
    """
    with open(filename, 'r') as csv_file:
        reader = csv.reader(csv_file)
        J = []
        Y = []
        E = []
        for row in reader:
            J_k = float(row[0])
            R_k = float(row[1])
            E_k = float(row[2])
            J.append(J_k)
            Y.append(1 / R_k)
            E.append(E_k)
        Y = Matrix.diagonal(Y)
        J = Matrix.column_vector(J)
        E = Matrix.column_vector(E)
        return Y, J, E


def create_network_matrices_mesh(rows, cols, branch_resistance, test_current):
    """
    Create the network matrices needed (A, Y, J, E) to solve the resitive mesh network with the given rows,
    ↪  columns,
    branch resistance and test current.

    :param rows: the number of rows in the mesh
    :param cols: the number of columns in the mesh
    :param branch_resistance: the resistance in each branch
    :param test_current: the test current to apply
    :return: the network matrices (A, Y, J, E)
    """
    num_horizontal_branches = (cols - 1) * rows
    num_vertical_branches = (rows - 1) * cols
    num_branches = num_horizontal_branches + num_vertical_branches + 1
    num_nodes = rows * cols - 1

    A = create_incidence_matrix_mesh(cols, num_branches, num_horizontal_branches, num_nodes,
    ↪  num_vertical_branches)
    Y, J, E = create_network_branch_matrices_mesh(num_branches, branch_resistance, test_current)

    return A, Y, J, E


```

```python
def create_incidence_matrix_mesh(cols, num_branches, num_horizontal_branches, num_nodes, num_vertical_branches):
    """
    Create the incidence matrix given by the resistive mesh with the given number of columns, number of branches,
    number of horizontal branches, number of nodes, and number of vertical branches.

    :param cols: the number of columns in the mesh
    :param num_branches: the number of branches in the mesh
    :param num_horizontal_branches: the number of horizontal branches in the mesh
    :param num_nodes: the number of nodes in the mesh
    :param num_vertical_branches: the number of vertical branches in the mesh
    :return: the incidence matrix (A)
    """
    A = Matrix.empty(num_nodes, num_branches)
    node_offset = -1
    for branch in range(num_horizontal_branches):
        if branch == num_horizontal_branches - cols + 1:
            A[branch + node_offset + 1][branch] = 1
        else:
            if branch % (cols - 1) == 0:
                node_offset += 1
            node_number = branch + node_offset
            A[node_number][branch] = -1
            A[node_number + 1][branch] = 1
    branch_offset = num_horizontal_branches
    node_offset = cols
    for branch in range(num_vertical_branches):
        if branch == num_vertical_branches - cols:
            node_offset -= 1
            A[branch][branch + branch_offset] = 1
        else:
            A[branch][branch + branch_offset] = 1
            A[branch + node_offset][branch + branch_offset] = -1
    if num_branches == 2:
        A[0][1] = -1
    else:
        A[cols - 1][num_branches - 1] = -1
    return A


def create_network_branch_matrices_mesh(num_branches, branch_resistance, test_current):
    """
    Create the Y, J, E network branch matrices of the resistive mesh given by the provided number of branches, branch
    resistance and test current.

    :param num_branches: the number of branches in the mesh
    :param branch_resistance: the resistance of each branch in the mesh
    :param test_current: the test current to apply to the mesh
    :return: the Y, J, E network branch matrices
    """
    Y = Matrix.diagonal([1 / branch_resistance if branch < num_branches - 1 else 0 for branch in range(num_branches)])
    # Negative test current here because we assume current is coming OUT of the test current node.
    J = Matrix.column_vector([0 if branch < num_branches - 1 else -test_current for branch in range(num_branches)])
    E = Matrix.column_vector([0 for branch in range(num_branches)])
    return Y, J, E


def find_mesh_resistance(N, branch_resistance, half_bandwidth=None):
    """
    Find the equivalent resistance of an Nx2N resistive mesh with the given branch resistance and optional
    half-bandwidth

    :param N: the size of the mesh (Nx2N)
    :param branch_resistance: the resistance of each branch of the mesh
    :param half_bandwidth: the half-bandwidth to be used for banded Choleski decomposition (or None to use non-banded)
```

```
    :return: the equivalent resistance of the mesh
    """
    test_current = 0.01
    A, Y, J, E = create_network_matrices_mesh(N, 2 * N, branch_resistance, test_current)
    x = solve_linear_network(A, Y, J, E, half_bandwidth=half_bandwidth)
    test_voltage = x[2 * N - 1 if N > 1 else 0][0]
    equivalent_resistance = test_voltage / test_current
    return equivalent_resistance
```

Listing 4: Question 1 (`q1.py`).

```
from __future__ import division

from linear_networks import solve_linear_network, csv_to_network_branch_matrices
from choleski import choleski_solve
from matrices import Matrix

NETWORK_DIRECTORY = 'network_data'

L_2 = Matrix([
    [5, 0],
    [1, 3]
])
L_3 = Matrix([
    [3, 0, 0],
    [1, 2, 0],
    [8, 5, 1]
])
L_4 = Matrix([
    [1, 0, 0, 0],
    [2, 8, 0, 0],
    [5, 5, 4, 0],
    [7, 2, 8, 7]
])
matrix_2 = L_2 * L_2.transpose()
matrix_3 = L_3 * L_3.transpose()
matrix_4 = L_4 * L_4.transpose()
positive_definite_matrices = [matrix_2, matrix_3, matrix_4]

x_2 = Matrix.column_vector([8, 3])
x_3 = Matrix.column_vector([9, 4, 3])
x_4 = Matrix.column_vector([5, 4, 1, 9])
xs = [x_2, x_3, x_4]


def q1b():
    """
    Question 1(b): Construct some small matrices (n = 2, 3, 4, or 5) to test the program. Remember that the
 ↪  matrices
    must be real, symmetric and positive-definite.
    """
    print('\n=== Question 1(b) ===')
    for count, A in enumerate(positive_definite_matrices):
        n = count + 2
        print('n={} matrix is positive-definite: {}'.format(n, A.is_positive_definite()))


def q1c():
    """
    Question 1(c): Test the program you wrote in (a) with each small matrix you built in (b) in the
 ↪  following way:
    invent an x, multiply it by A to get b, then give A and b to your program and check that it returns x
 ↪  correctly.
    """
    print('\n=== Question 1(c) ===')
    n = 2
    for x, A in zip(xs, positive_definite_matrices):
        b = A * x
        print('Matrix with n={}:'.format(n))
```

```
        print('A: {}'.format(A))
        print('b: {}'.format(b))

        x_choleski = choleski_solve(A, b)
        print('Expected x: {}'.format(x))
        print('Actual x: {}'.format(x_choleski))
        n += 1


def q1d():
    """
    Question 1(d): Write a program that reads from a file a list of network branches (Jk, Rk, Ek) and a reduced
    incidence matrix, and finds the voltages at the nodes of the network. Use the code from part (a) to solve the
    matrix problem.
    """
    print('\n=== Question 1(d) ===')
    for i in range(1, 7):
        A = Matrix.csv_to_matrix('{}/incidence_matrix_{}.csv'.format(NETWORK_DIRECTORY, i))
        Y, J, E = csv_to_network_branch_matrices('{}/network_branches_{}.csv'.format(NETWORK_DIRECTORY, i))
        # print('Y: {}'.format(Y))
        # print('J: {}'.format(J))
        # print('E: {}'.format(E))
        x = solve_linear_network(A, Y, J, E)
        print('Solved for x in network {}:'.format(i))  # TODO: Create my own test circuits here
        for j in range(len(x)):
            print('V{} = {:.3f} V'.format(j + 1, x[j][0]))


def q1():
    """
    Question 1
    """
    q1b()
    q1c()
    q1d()


if __name__ == '__main__':
    q1()
```

Listing 5: Question 2 (`q2.py`).

```
import csv
import time

import matplotlib.pyplot as plt
import numpy as np
import numpy.polynomial.polynomial as poly
import sympy as sp
from matplotlib.ticker import MaxNLocator

from linear_networks import find_mesh_resistance


def find_mesh_resistances(banded):
    branch_resistance = 1000
    points = {}
    runtimes = {}
    for n in range(2, 11):
        start_time = time.time()
        half_bandwidth = 2 * n + 1 if banded else None
        equivalent_resistance = find_mesh_resistance(n, branch_resistance, half_bandwidth=half_bandwidth)
        print('Equivalent resistance for {}x{} mesh: {:.2f} Ohms.'.format(n, 2 * n, equivalent_resistance))
        points[n] = '{:.3f}'.format(equivalent_resistance)
        runtime = time.time() - start_time
```

```
24          runtimes[n] = '{:.3f}'.format(runtime)
25          print('Runtime: {} s.'.format(runtime))
26      plot_runtime(runtimes, banded)
27      return points, runtimes
28
29
30  def q2ab():
31      print('=== Question 2(a)(b) ===')
32      _, runtimes = find_mesh_resistances(banded=False)
33      save_rows_to_csv('report/csv/q2b.csv', zip(runtimes.keys(), runtimes.values()), header=('N', 'Runtime
        ↪  (s)'))
34      return runtimes
35
36
37  def q2c():
38      print('=== Question 2(c) ===')
39      pts, runtimes = find_mesh_resistances(banded=True)
40      save_rows_to_csv('report/csv/q2c.csv', zip(runtimes.keys(), runtimes.values()), header=('N', 'Runtime
        ↪  (s)'))
41      return pts, runtimes
42
43
44  def plot_runtime(points, banded=False):
45      """
46      N^6: non-banded
47      N^4: banded
48
49      :param points:
50      :param banded:
51      """
52      f = plt.figure()
53      ax = f.gca()
54      ax.xaxis.set_major_locator(MaxNLocator(integer=True))
55      x_range = [float(x) for x in points.keys()]
56      y_range = [float(y) for y in points.values()]
57      plt.plot(x_range, y_range, '{}o'.format('C1' if banded else 'C0'), label='Data points')
58
59      x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
60      degree = 4 if banded else 6
61      polynomial_coeffs = poly.polyfit(x_range, y_range, degree)
62      polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
63      N = sp.symbols("N")
64      poly_label = sum(sp.S("{:.4f}".format(v)) * N ** i for i, v in enumerate(polynomial_coeffs))
65      equation = '${}$'.format(sp.printing.latex(poly_label))
66      plt.plot(x_new, polynomial_fit, '{}-'.format('C1' if banded else 'C0'), label=equation)
67
68      plt.xlabel('N')
69      plt.ylabel('Runtime (s)')
70      plt.grid(True)
71      plt.legend(fontsize='x-small')
72      f.savefig('report/plots/q2{}.pdf'.format('c' if banded else 'b'), bbox_inches='tight')
73
74
75  def plot_runtimes(points1, points2):
76      f = plt.figure()
77      ax = f.gca()
78      ax.xaxis.set_major_locator(MaxNLocator(integer=True))
79      x_range = points1.keys()
80      y_range = points1.values()
81      y_banded_range = points2.values()
82      plt.plot(x_range, y_range, 'o-', label='Non-banded elimination')
83      plt.plot(x_range, y_banded_range, 'o-', label='Banded elimination')
84      plt.xlabel('N')
85      plt.ylabel('Runtime (s)')
86      plt.grid(True)
87      plt.legend()
88      f.savefig('report/plots/q2bc.pdf', bbox_inches='tight')
89
90
91  def q2d(points):
```

```
    print('=== Question 2(d) ===')
    f = plt.figure()
    ax = f.gca()
    ax.xaxis.set_major_locator(MaxNLocator(integer=True))
    x_range = [float(x) for x in points.keys()]
    y_range = [float(y) for y in points.values()]
    plt.plot(x_range, y_range, 'o', label='Data points')

    x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
    coeffs = poly.polyfit(np.log(x_range), y_range, deg=1)
    polynomial_fit = poly.polyval(np.log(x_new), coeffs)
    plt.plot(x_new, polynomial_fit, '{}-'.format('CO'), label='${:.2f}\log(N) + {:.2f}$'.format(coeffs[1],
     ↪  coeffs[0]))

    plt.xlabel('N')
    plt.ylabel('R ($\Omega$)')
    plt.grid(True)
    plt.legend()
    f.savefig('report/plots/q2d.pdf', bbox_inches='tight')
    save_rows_to_csv('report/csv/q2a.csv', zip(points.keys(), points.values()), header=('N', 'R (Omega)'))


def q2():
    runtimes1 = q2ab()
    pts, runtimes2 = q2c()
    plot_runtimes(runtimes1, runtimes2)
    q2d(pts)


def save_rows_to_csv(filename, rows, header=None):
    with open(filename, "wb") as f:
        writer = csv.writer(f)
        if header is not None:
            writer.writerow(header)
        for row in rows:
            writer.writerow(row)


if __name__ == '__main__':
    q2()
```

*Listing 6: Finite difference method (**finite_diff.py**).*

```
from __future__ import division

import math
import random
from abc import ABCMeta, abstractmethod

from matrices import Matrix

MESH_SIZE = 0.2


class Relaxer:
    """
    Performs the relaxing stage of the finite difference method.
    """
    __metaclass__ = ABCMeta

    @abstractmethod
    def relax(self, phi, i, j):
        """
        Perform a relaxation iteration on a given (i, j) point of the given phi matrix.

        :param phi: the phi matrix
        :param i: the row index
        :param j: the column index
        """
```

```
27          raise NotImplementedError
28
29      def reset(self):
30          """
31          Optional method to reset the relaxer.
32          """
33          pass
34
35      def residual(self, phi, i, j):
36          """
37          Calculate the residual at the given (i, j) point of the given phi matrix.
38
39          :param phi: the phi matrix
40          :param i: the row index
41          :param j: the column index
42          :return:
43          """
44          return abs(phi[i + 1][j] + phi[i - 1][j] + phi[i][j + 1] + phi[i][j - 1] - 4 * phi[i][j])
45
46
47  class GaussSeidelRelaxer(Relaxer):
48      def relax(self, phi, i, j):
49          return (phi[i + 1][j] + phi[i - 1][j] + phi[i][j + 1] + phi[i][j - 1]) / 4
50
51
52  class JacobiRelaxer(Relaxer):
53      def __init__(self, num_cols):
54          self.num_cols = num_cols
55          self.prev_row = [0] * (num_cols - 1)  # Don't need to copy entire phi, just previous row
56
57      def relax(self, phi, i, j):
58          left_val = self.prev_row[j - 2] if j > 1 else 0
59          top_val = self.prev_row[j - 1]
60          self.prev_row[j - 1] = phi[i][j]
61          return (phi[i + 1][j] + top_val + phi[i][j + 1] + left_val) / 4
62
63      def reset(self):
64          self.prev_row = [0] * (self.num_cols - 1)
65
66
67  class NonUniformRelaxer(Relaxer):
68      def __init__(self, mesh):
69          self.mesh = mesh
70
71      def get_distances(self, i, j):
72          a1 = self.mesh.get_y(i) - self.mesh.get_y(i - 1)
73          a2 = self.mesh.get_y(i + 1) - self.mesh.get_y(i)
74          b1 = self.mesh.get_x(j) - self.mesh.get_x(j - 1)
75          b2 = self.mesh.get_x(j + 1) - self.mesh.get_x(j)
76          return a1, a2, b1, b2
77
78      def relax(self, phi, i, j):
79          a1, a2, b1, b2 = self.get_distances(i, j)
80
81          return ((phi[i - 1][j] / a1 + phi[i + 1][j] / a2) / (a1 + a2)
82                  + (phi[i][j - 1] / b1 + phi[i][j + 1] / b2) / (b1 + b2)) / (1 / (a1 * a2) + 1 / (b1 * b2))
83
84      def residual(self, phi, i, j):
85          a1, a2, b1, b2 = self.get_distances(i, j)
86
87          return abs(((phi[i - 1][j] / a1 + phi[i + 1][j] / a2) / (a1 + a2)
88                      + (phi[i][j - 1] / b1 + phi[i][j + 1] / b2) / (b1 + b2))
89                     - phi[i][j] * (1 / (a1 * a2) + 1 / (b1 * b2)))
90
91
92  class SuccessiveOverRelaxer(Relaxer):
93      def __init__(self, omega):
94          self.gauss_seidel = GaussSeidelRelaxer()
95          self.omega = omega
96
```

```
97      def relax(self, phi, i, j, last_row=None, a1=None, a2=None, b1=None, b2=None):
98          return (1 - self.omega) * phi[i][j] + self.omega * self.gauss_seidel.relax(phi, i, j)
99
100
101 class Boundary:
102     """
103     Constant-potential boundary in the finite difference mesh, representing a conductor.
104     """
105     __metaclass__ = ABCMeta
106
107     @abstractmethod
108     def potential(self):
109         """
110         Return the potential on the boundary.
111         """
112         raise NotImplementedError
113
114     @abstractmethod
115     def contains_point(self, x, y):
116         """
117         Returns true if the boundary contains the given (x, y) point.
118
119         :param x: the x coordinate of the point
120         :param y: the y coordinate of the point
121         """
122         raise NotImplementedError
123
124
125 class OuterConductorBoundary(Boundary):
126     def potential(self):
127         return 0
128
129     def contains_point(self, x, y):
130         return x == 0 or y == 0 or x == 0.2 or y == 0.2
131
132
133 class QuarterInnerConductorBoundary(Boundary):
134     def potential(self):
135         return 15
136
137     def contains_point(self, x, y):
138         return 0.06 <= x <= 0.14 and 0.08 <= y <= 0.12
139
140
141 class PotentialGuesser:
142     """
143     Guesses the initial potential in the finite-difference mesh.
144     """
145     __metaclass__ = ABCMeta
146
147     def __init__(self, min_potential, max_potential):
148         self.min_potential = min_potential
149         self.max_potential = max_potential
150
151     @abstractmethod
152     def guess(self, x, y):
153         """
154         Guess the potential at the given (x, y) point, and return it.
155
156         :param x: the x coordinate of the point
157         :param y: the y coordinate of the point
158         """
159         raise NotImplementedError
160
161
162 class RandomPotentialGuesser(PotentialGuesser):
163     def guess(self, x, y):
164         return random.randint(self.min_potential, self.max_potential)
165
166
```

```
167 class LinearPotentialGuesser(PotentialGuesser):
168     def guess(self, x, y):
169         return 150 * x if x < 0.06 else 150 * y
170 
171 
172 class RadialPotentialGuesser(PotentialGuesser):
173     def guess(self, x, y):
174         def radial(k, x, y, x_source, y_source):
175             return k / (math.sqrt((x_source - x) ** 2 + (y_source - y) ** 2))
176 
177         return 0.0225 * (radial(20, x, y, 0.1, 0.1) - radial(1, x, y, 0, y) - radial(1, x, y, x, 0))
178 
179 
180 class PhiConstructor:
181     """
182     Constructs the phi potential matrix with an outer conductor, inner conductor, mesh points and an inital
    ↪  potential
183     guess.
184     """
185 
186     def __init__(self, mesh):
187         outer_boundary = OuterConductorBoundary()
188         inner_boundary = QuarterInnerConductorBoundary()
189         self.boundaries = (inner_boundary, outer_boundary)
190         self.guesser = RadialPotentialGuesser(0, 15)
191         self.mesh = mesh
192 
193     def construct_phi(self):
194         phi = Matrix.empty(self.mesh.num_rows, self.mesh.num_cols)
195         for i in range(self.mesh.num_rows):
196             y = self.mesh.get_y(i)
197             for j in range(self.mesh.num_cols):
198                 x = self.mesh.get_x(j)
199                 boundary_pt = False
200                 for boundary in self.boundaries:
201                     if boundary.contains_point(x, y):
202                         boundary_pt = True
203                         phi[i][j] = boundary.potential()
204                 if not boundary_pt:
205                     phi[i][j] = self.guesser.guess(x, y)
206         return phi
207 
208 
209 class SquareMeshConstructor:
210     """
211     Constructs a square mesh.
212     """
213 
214     def __init__(self, size):
215         self.size = size
216 
217     def construct_uniform_mesh(self, h):
218         """
219         Constructs a uniform mesh with the given node spacing.
220 
221         :param h: the node spacing
222         :return: the constructed mesh
223         """
224         num_rows = num_cols = int(self.size / h) + 1
225         return SimpleMesh(h, num_rows, num_cols)
226 
227     def construct_symmetric_uniform_mesh(self, h):
228         """
229         Construct a symmetric uniform mesh with the given node spacing.
230 
231         :param h: the node spacing
232         :return: the constructed mesh
233         """
234         half_size = self.size / 2
235         num_rows = num_cols = int(half_size / h) + 2  # Only need to store up to middle
```

```
236             return SimpleMesh(h, num_rows, num_cols)
237
238         def construct_symmetric_non_uniform_mesh(self, x_values, y_values):
239             """
240             Construct a symmetric non-uniform mesh with the given adjacent x coordinates and y coordinates.
241
242             :param x_values: the values of successive x coordinates
243             :param y_values: the values of successive y coordinates
244             :return: the constructed mesh
245             """
246             return NonUniformMesh(x_values, y_values)
247
248
249     class Mesh:
250         """
251         Finite-difference mesh.
252         """
253         __metaclass__ = ABCMeta
254
255         @abstractmethod
256         def get_x(self, j):
257             """
258             Get the x value at the specified index.
259
260             :param j: the column index.
261             """
262             raise NotImplementedError
263
264         @abstractmethod
265         def get_y(self, i):
266             """
267             Get the y value at the specified index.
268
269             :param i: the row index.
270             """
271             raise NotImplementedError
272
273         @abstractmethod
274         def get_i(self, y):
275             """
276             Get the row index of the specified y coordinate.
277
278             :param y: the y coordinate
279             """
280             raise NotImplementedError
281
282         @abstractmethod
283         def get_j(self, x):
284             """
285             Get the column index of the specified x coordinate.
286
287             :param x: the x coordinate
288             """
289             raise NotImplementedError
290
291         def point_to_indices(self, x, y):
292             """
293             Converts the given (x, y) point to (i, j) matrix indices.
294
295             :param x: the x coordinate
296             :param y: the y coordinate
297             :return: the (i, j) matrix indices
298             """
299             return self.get_i(y), self.get_j(x)
300
301         def indices_to_points(self, i, j):
302             """
303             Converts the given (i, j) matrix indices to an (x, y) point.
304
305             :param i: the row index
```

```
306             :param j: the column index
307             :return: the (x, y) point
308             """
309             return self.get_x(j), self.get_y(i)
310 
311 
312     class SimpleMesh(Mesh):
313         def __init__(self, h, num_rows, num_cols):
314             self.h = h
315             self.num_rows = num_rows
316             self.num_cols = num_cols
317 
318         def get_i(self, y):
319             return int(y / self.h)
320 
321         def get_j(self, x):
322             return int(x / self.h)
323 
324         def get_x(self, j):
325             return j * self.h
326 
327         def get_y(self, i):
328             return i * self.h
329 
330 
331     class NonUniformMesh(Mesh):
332         def __init__(self, x_values, y_values):
333             self.x_values = x_values
334             self.y_values = y_values
335             self.num_rows = len(y_values)
336             self.num_cols = len(x_values)
337 
338         def get_i(self, y):
339             return self.y_values.index(y)
340 
341         def get_j(self, x):
342             return self.x_values.index(x)
343 
344         def get_x(self, j):
345             return self.x_values[j]
346 
347         def get_y(self, i):
348             return self.y_values[i]
349 
350 
351     class IterativeRelaxer:
352         """
353         Performs finite-difference iterative relaxation on a phi potential matrix associated with a mesh.
354         """
355 
356         def __init__(self, relaxer, epsilon, phi, mesh):
357             self.relaxer = relaxer
358             self.epsilon = epsilon
359             self.phi = phi
360             self.boundary = QuarterInnerConductorBoundary()
361             self.num_iterations = 0
362             self.rows = len(phi)
363             self.cols = len(phi[0])
364             self.mesh = mesh
365             self.mid_i = mesh.get_i(MESH_SIZE / 2)
366             self.mid_j = mesh.get_j(MESH_SIZE / 2)
367 
368         def relaxation(self):
369             """
370             Performs iterative relaxation until convergence is met.
371 
372             :return: the current iterative relaxer object
373             """
374             while not self.convergence():
375                 self.num_iterations += 1
```

```
            self.relaxation_iteration()
            self.relaxer.reset()
        return self

    def relaxation_iteration(self):
        """
        Performs one iteration of relaxation.
        """
        for i in range(1, self.rows - 1):
            y = self.mesh.get_y(i)
            for j in range(1, self.cols - 1):
                x = self.mesh.get_x(j)
                if not self.boundary.contains_point(x, y):
                    relaxed_value = self.relaxer.relax(self.phi, i, j)
                    self.phi[i][j] = relaxed_value
                    if i == self.mid_i - 1:
                        self.phi[i + 2][j] = relaxed_value
                    elif j == self.mid_j - 1:
                        self.phi[i][j + 2] = relaxed_value

    def convergence(self):
        """
        Checks if the phi matrix has reached convergence.

        :return: True if the phi matrix has reached convergence, False otherwise
        """
        max_i, max_j = self.mesh.point_to_indices(0.1, 0.1)  # Only need to compute for 1/4 of grid
        for i in range(1, max_i + 1):
            y = self.mesh.get_y(i)
            for j in range(1, max_j + 1):
                x = self.mesh.get_x(j)
                if not self.boundary.contains_point(x, y) and self.relaxer.residual(self.phi, i, j) >=
                ↪  self.epsilon:
                    return False
        return True

    def get_potential(self, x, y):
        """
        Get the potential at the given (x, y) point.

        :param x: the x coordinate
        :param y: the y coordinate
        :return: the potential at the given (x, y) point
        """
        i, j = self.mesh.point_to_indices(x, y)
        return self.phi[i][j]


def non_uniform_jacobi(epsilon, x_values, y_values):
    """
    Perform Jacobi relaxation on a non-uniform finite-difference mesh.

    :param epsilon: the maximum error to achieve convergence
    :param x_values: the values of successive x coordinates
    :param y_values: the values of successive y coordinates
    :return: the relaxer object
    """
    mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_non_uniform_mesh(x_values, y_values)
    relaxer = NonUniformRelaxer(mesh)
    phi = PhiConstructor(mesh).construct_phi()
    return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()


def successive_over_relaxation(omega, epsilon, h):
    """
    Perform SOR on a uniform symmetric finite-difference mesh.

    :param omega: the omega value for SOR
    :param epsilon: the maximum error to achieve convergence
    :param h: the node spacing
```

```
    :return: the relaxer object
    """
    mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_uniform_mesh(h)
    relaxer = SuccessiveOverRelaxer(omega)
    phi = PhiConstructor(mesh).construct_phi()
    return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()


def jacobi_relaxation(epsilon, h):
    """
    Perform Jacobi relaxation on a uniform symmetric finite-difference mesh.

    :param epsilon: the maximum error to achieve convergence
    :param h: the node spacing
    :return: the relaxer object
    """
    mesh = SquareMeshConstructor(MESH_SIZE).construct_symmetric_uniform_mesh(h)
    relaxer = GaussSeidelRelaxer()
    phi = PhiConstructor(mesh).construct_phi()
    return IterativeRelaxer(relaxer, epsilon, phi, mesh).relaxation()
```

Listing 7: Question 3 (`q3.py`).

```
from __future__ import division

import csv
import time

import matplotlib.pyplot as plt
import numpy as np
import numpy.polynomial.polynomial as poly
import sympy as sp

from finite_diff import successive_over_relaxation, jacobi_relaxation, \
    non_uniform_jacobi

EPSILON = 0.00001
X_QUERY = 0.06
Y_QUERY = 0.04
NUM_H_ITERATIONS = 6


def q3b():
    print('=== Question 3(b) ===')
    h = 0.02
    min_num_iterations = float('inf')
    best_omega = float('inf')

    omegas = []
    num_iterations = []
    potentials = []

    for omega_diff in range(10):
        omega = 1 + omega_diff / 10
        print('Omega: {}'.format(omega))
        iter_relaxer = successive_over_relaxation(omega, EPSILON, h)
        print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
        print('Num iterations: {}'.format(iter_relaxer.num_iterations))
        potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
        print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))
        if iter_relaxer.num_iterations < min_num_iterations:
            best_omega = omega
        min_num_iterations = min(min_num_iterations, iter_relaxer.num_iterations)

        omegas.append(omega)
        num_iterations.append(iter_relaxer.num_iterations)
        potentials.append('{:.3f}'.format(potential))

    print('Best number of iterations: {}'.format(min_num_iterations))
```

```
    print('Best omega: {}'.format(best_omega))

    f = plt.figure()
    x_range = omegas
    y_range = num_iterations
    plt.plot(x_range, y_range, 'o-', label='Number of iterations')
    plt.xlabel('$\omega$')
    plt.ylabel('Number of Iterations')
    plt.grid(True)
    f.savefig('report/plots/q3b.pdf', bbox_inches='tight')

    save_rows_to_csv('report/csv/q3b_potential.csv', zip(omegas, potentials), header=('Omega', 'Potential (V)'))
    save_rows_to_csv('report/csv/q3b_iterations.csv', zip(omegas, num_iterations), header=('Omega', 'Iterations'))

    return best_omega


def q3c(omega):
    print('=== Question 3(c): SOR ===')
    h = 0.04
    h_values = []
    potential_values = []
    iterations_values = []
    for i in range(NUM_H_ITERATIONS):
        h = h / 2
        print('h: {}'.format(h))
        print('1/h: {}'.format(1 / h))
        iter_relaxer = successive_over_relaxation(omega, EPSILON, h)
        # print(phi.mirror_horizontal())
        potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
        num_iterations = iter_relaxer.num_iterations

        print('Num iterations: {}'.format(num_iterations))
        print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

        h_values.append(1 / h)
        potential_values.append('{:.3f}'.format(potential))
        iterations_values.append(num_iterations)

    f = plt.figure()
    x_range = h_values
    y_range = potential_values
    plt.plot(x_range, y_range, 'o-', label='Data points')

    plt.xlabel('1 / h')
    plt.ylabel('Potential at [0.06, 0.04] (V)')
    plt.grid(True)
    f.savefig('report/plots/q3c_potential.pdf', bbox_inches='tight')

    f = plt.figure()
    x_range = h_values
    y_range = iterations_values

    x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
    polynomial_coeffs = poly.polyfit(x_range, y_range, deg=3)
    polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
    N = sp.symbols("1/h")
    poly_label = sum(sp.S("{:.5f}".format(v)) * N ** i for i, v in enumerate(polynomial_coeffs))
    equation = '${}$'.format(sp.printing.latex(poly_label))
    plt.plot(x_new, polynomial_fit, '{}-'.format('C0'), label=equation)

    plt.plot(x_range, y_range, 'o', label='Data points')
    plt.xlabel('1 / h')
    plt.ylabel('Number of Iterations')
    plt.grid(True)
    plt.legend(fontsize='small')

    f.savefig('report/plots/q3c_iterations.pdf', bbox_inches='tight')
```

```
115
116     save_rows_to_csv('report/csv/q3c_potential.csv', zip(h_values, potential_values), header=('1/h',
        ↪  'Potential (V)'))
117     save_rows_to_csv('report/csv/q3c_iterations.csv', zip(h_values, iterations_values), header=('1/h',
        ↪  'Iterations'))
118
119     return h_values, potential_values, iterations_values
120
121
122 def q3d():
123     print('=== Question 3(d): Jacobi ===')
124     h = 0.04
125     h_values = []
126     potential_values = []
127     iterations_values = []
128     for i in range(NUM_H_ITERATIONS):
129         h = h / 2
130         print('h: {}'.format(h))
131         iter_relaxer = jacobi_relaxation(EPSILON, h)
132         potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
133         num_iterations = iter_relaxer.num_iterations
134
135         print('Num iterations: {}'.format(num_iterations))
136         print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))
137
138         h_values.append(1 / h)
139         potential_values.append('{:.3f}'.format(potential))
140         iterations_values.append(num_iterations)
141
142     f = plt.figure()
143     x_range = h_values
144     y_range = potential_values
145     plt.plot(x_range, y_range, 'C1o-', label='Data points')
146     plt.xlabel('1 / h')
147     plt.ylabel('Potential at [0.06, 0.04] (V)')
148     plt.grid(True)
149     f.savefig('report/plots/q3d_potential.pdf', bbox_inches='tight')
150
151     f = plt.figure()
152     x_range = h_values
153     y_range = iterations_values
154     plt.plot(x_range, y_range, 'C1o', label='Data points')
155     plt.xlabel('1 / h')
156     plt.ylabel('Number of Iterations')
157
158     x_new = np.linspace(x_range[0], x_range[-1], num=len(x_range) * 10)
159     polynomial_coeffs = poly.polyfit(x_range, y_range, deg=4)
160     polynomial_fit = poly.polyval(x_new, polynomial_coeffs)
161     N = sp.symbols("1/h")
162     poly_label = sum(sp.S("{:.5f}".format(v if i < 3 else -v)) * N ** i for i, v in
        ↪  enumerate(polynomial_coeffs))
163     equation = '${}$'.format(sp.printing.latex(poly_label))
164     plt.plot(x_new, polynomial_fit, '{}-'.format('C1'), label=equation)
165
166     plt.grid(True)
167     plt.legend(fontsize='small')
168
169     f.savefig('report/plots/q3d_iterations.pdf', bbox_inches='tight')
170
171     save_rows_to_csv('report/csv/q3d_potential.csv', zip(h_values, potential_values), header=('1/h',
        ↪  'Potential (V)'))
172     save_rows_to_csv('report/csv/q3d_iterations.csv', zip(h_values, iterations_values), header=('1/h',
        ↪  'Iterations'))
173
174     return h_values, potential_values, iterations_values
175
176
177 def q3e():
178     print('=== Question 3(e): Non-Uniform Node Spacing ===')
179
```

```
    print('Jacobi (for reference)')
    iter_relaxer = jacobi_relaxation(EPSILON, 0.01)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    jacobi_potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, jacobi_potential))

    print('Uniform Mesh (same as Jacobi)')
    x_values = [0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.11]
    y_values = [0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.11]
    iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    uniform_potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, uniform_potential))
    print('Jacobi potential: {} V, same as uniform potential: {} V'.format(jacobi_potential,
     ↪  uniform_potential))

    print('Non-Uniform (clustered around (0.06, 0.04))')
    x_values = [0.00, 0.01, 0.02, 0.03, 0.05, 0.055, 0.06, 0.065, 0.07, 0.09, 0.1, 0.11]
    y_values = [0.00, 0.01, 0.03, 0.035, 0.04, 0.045, 0.05, 0.07, 0.08, 0.09, 0.1, 0.11]
    iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

    print('Non-Uniform (more clustered around (0.06, 0.04))')
    x_values = [0.00, 0.01, 0.02, 0.03, 0.055, 0.059, 0.06, 0.061, 0.065, 0.09, 0.1, 0.11]
    y_values = [0.00, 0.01, 0.035, 0.039, 0.04, 0.041, 0.045, 0.07, 0.08, 0.09, 0.1, 0.11]
    iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

    print('Non-Uniform (clustered near outer conductor)')
    x_values = [0.00, 0.020, 0.032, 0.044, 0.055, 0.06, 0.074, 0.082, 0.089, 0.096, 0.1, 0.14]
    y_values = [0.00, 0.020, 0.032, 0.04, 0.055, 0.065, 0.074, 0.082, 0.089, 0.096, 0.1, 0.14]
    iter_relaxer = non_uniform_jacobi(EPSILON, x_values, y_values)
    print('Quarter grid: {}'.format(iter_relaxer.phi.mirror_horizontal()))
    print('Num iterations: {}'.format(iter_relaxer.num_iterations))
    potential = iter_relaxer.get_potential(X_QUERY, Y_QUERY)
    print('Potential at ({}, {}): {:.3f} V'.format(X_QUERY, Y_QUERY, potential))

    plot_mesh(x_values, y_values)


def plot_mesh(x_values, y_values):
    f = plt.figure()
    ax = f.gca()
    ax.set_aspect('equal', adjustable='box')
    x_range = []
    y_range = []
    for x in x_values[:-1]:
        for y in y_values[:-1]:
            x_range.append(x)
            y_range.append(y)
    plt.plot(x_range, y_range, 'o', label='Mesh points')
    plt.xlabel('x')
    plt.ylabel('y')
    plt.grid(True)
    f.savefig('report/plots/q3e.pdf', bbox_inches='tight')


def plot_sor_jacobi(h_values, potential_values, potential_values_jacobi, iterations_values,
 ↪  iterations_values_jacobi):
    f = plt.figure()
    plt.plot(h_values, potential_values, 'o-', label='SOR')
    plt.plot(h_values, potential_values_jacobi, 'o-', label='Jacobi')
```

```
    plt.xlabel('1 / h')
    plt.ylabel('Potential at [0.06, 0.04] (V)')
    plt.grid(True)
    plt.legend()
    f.savefig('report/plots/q3d_potential_comparison.pdf', bbox_inches='tight')

    f = plt.figure()
    plt.plot(h_values, iterations_values, 'o-', label='SOR')
    plt.plot(h_values, iterations_values_jacobi, 'o-', label='Jacobi')
    plt.xlabel('1 / h')
    plt.ylabel('Number of Iterations')
    plt.grid(True)
    plt.legend()
    f.savefig('report/plots/q3d_iterations_comparison.pdf', bbox_inches='tight')


def save_rows_to_csv(filename, rows, header=None):
    with open(filename, "wb") as f:
        writer = csv.writer(f)
        if header is not None:
            writer.writerow(header)
        for row in rows:
            writer.writerow(row)


def q3():
    o = q3b()
    h_values, potential_values, iterations_values = q3c(o)
    _, potential_values_jacobi, iterations_values_jacobi = q3d()
    plot_sor_jacobi(h_values, potential_values, potential_values_jacobi, iterations_values,
    ↪  iterations_values_jacobi)
    q3e()


if __name__ == '__main__':
    t = time.time()
    q3()
    print('Total runtime: {} s'.format(time.time() - t))
```

B Output Logs

Listing 8: Output of Question 1 program (`q1.txt`).

```
=== Question 1(b) ===
n=2 matrix is positive-definite: True
n=3 matrix is positive-definite: True
n=4 matrix is positive-definite: True

=== Question 1(c) ===
Matrix with n=2:
A:
 25.00   5.00
  5.00  10.00
b:
215.00
 70.00
Expected x:
  8.00
  3.00
Actual x:
  8.00
  3.00
Matrix with n=3:
A:
  9.00   3.00  24.00
  3.00   5.00  18.00
 24.00  18.00  90.00
```

```
b:
165.00
101.00
558.00
Expected x:
  9.00
  4.00
  3.00
Actual x:
  9.00
  4.00
  3.00
Matrix with n=4:
A:
  1.00   2.00   5.00   7.00
  2.00  68.00  50.00  30.00
  5.00  50.00  66.00  77.00
  7.00  30.00  77.00 166.00
b:
 81.00
602.00
984.00
1726.00
Expected x:
  5.00
  4.00
  1.00
  9.00
Actual x:
  5.00
  4.00
  1.00
  9.00

=== Question 1(d) ===
Solved for x in network 1:
V1 = 5.000 V
Solved for x in network 2:
V1 = 50.000 V
Solved for x in network 3:
V1 = 55.000 V
Solved for x in network 4:
V1 = 20.000 V
V2 = 35.000 V
Solved for x in network 5:
V1 = 5.000 V
V2 = 3.750 V
V3 = 3.750 V
Solved for x in network 6:
V1 = 4.443 V
V2 = 5.498 V
V3 = 3.036 V
V4 = 3.200 V
V5 = 1.301 V
```

Listing 9: Output of Question 2 program (`q2.txt`).

```
=== Question 2(a)(b) ===
Equivalent resistance for 2x4 mesh: 1875.00 Ohms.
Runtime: 0.000999927520752 s.
Equivalent resistance for 3x6 mesh: 2379.55 Ohms.
Runtime: 0.0169999599457 s.
Equivalent resistance for 4x8 mesh: 2741.03 Ohms.
Runtime: 0.100000143051 s.
Equivalent resistance for 5x10 mesh: 3022.82 Ohms.
Runtime: 0.481999874115 s.
Equivalent resistance for 6x12 mesh: 3253.68 Ohms.
Runtime: 1.46099996567 s.
Equivalent resistance for 7x14 mesh: 3449.17 Ohms.
```

```
Runtime: 3.26600003242 s.
Equivalent resistance for 8x16 mesh: 3618.67 Ohms.
Runtime: 7.53400015831 s.
Equivalent resistance for 9x18 mesh: 3768.29 Ohms.
Runtime: 15.001999855 s.
Equivalent resistance for 10x20 mesh: 3902.19 Ohms.
Runtime: 28.3630001545 s.
=== Question 2(c) ===
Equivalent resistance for 2x4 mesh: 1875.00 Ohms.
Runtime: 0.00100016593933 s.
Equivalent resistance for 3x6 mesh: 2379.55 Ohms.
Runtime: 0.0169999599457 s.
Equivalent resistance for 4x8 mesh: 2741.03 Ohms.
Runtime: 0.0950000286102 s.
Equivalent resistance for 5x10 mesh: 3022.82 Ohms.
Runtime: 0.378000020981 s.
Equivalent resistance for 6x12 mesh: 3253.68 Ohms.
Runtime: 1.19199991226 s.
Equivalent resistance for 7x14 mesh: 3449.17 Ohms.
Runtime: 3.05200004578 s.
Equivalent resistance for 8x16 mesh: 3618.67 Ohms.
Runtime: 6.9430000782 s.
Equivalent resistance for 9x18 mesh: 3768.29 Ohms.
Runtime: 14.2189998627 s.
Equivalent resistance for 10x20 mesh: 3902.19 Ohms.
Runtime: 26.763999939 s.
=== Question 2(d) ===
```

Listing 10: Output of Question 3 program (`q3.txt`).

```
=== Question 3(b) ===
Omega: 1.0
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 32
Potential at (0.06, 0.04): 5.526 V
Omega: 1.1
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 26
Potential at (0.06, 0.04): 5.526 V
Omega: 1.2
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 20
Potential at (0.06, 0.04): 5.526 V
Omega: 1.3
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
```

```
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 14
Potential at (0.06, 0.04): 5.526 V
Omega: 1.4
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 16
Potential at (0.06, 0.04): 5.526 V
Omega: 1.5
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 20
Potential at (0.06, 0.04): 5.526 V
Omega: 1.6
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 27
Potential at (0.06, 0.04): 5.526 V
Omega: 1.7
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 39
Potential at (0.06, 0.04): 5.526 V
Omega: 1.8
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 60
Potential at (0.06, 0.04): 5.526 V
Omega: 1.9
Quarter grid:
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   4.25   9.09  15.00  15.00  15.00  15.00
  0.00   3.96   8.56  15.00  15.00  15.00  15.00
  0.00   3.03   6.18   9.25  10.29  10.55  10.29
  0.00   1.97   3.88   5.53   6.37   6.61   6.37
  0.00   0.96   1.86   2.61   3.04   3.17   3.04
  0.00   0.00   0.00   0.00   0.00   0.00   0.00
```

```
Num iterations: 127
Potential at (0.06, 0.04): 5.526 V
Best number of iterations: 14
Best omega: 1.3
=== Question 3(c): SOR ===
h: 0.02
1/h: 50.0
Num iterations: 14
Potential at (0.06, 0.04): 5.526 V
h: 0.01
1/h: 100.0
Num iterations: 59
Potential at (0.06, 0.04): 5.351 V
h: 0.005
1/h: 200.0
Num iterations: 189
Potential at (0.06, 0.04): 5.289 V
h: 0.0025
1/h: 400.0
Num iterations: 552
Potential at (0.06, 0.04): 5.265 V
h: 0.00125
1/h: 800.0
Num iterations: 1540
Potential at (0.06, 0.04): 5.254 V
h: 0.000625
1/h: 1600.0
Num iterations: 4507
Potential at (0.06, 0.04): 5.247 V
=== Question 3(d): Jacobi ===
h: 0.02
Num iterations: 51
Potential at (0.06, 0.04): 5.526 V
h: 0.01
Num iterations: 180
Potential at (0.06, 0.04): 5.351 V
h: 0.005
Num iterations: 604
Potential at (0.06, 0.04): 5.289 V
h: 0.0025
Num iterations: 1935
Potential at (0.06, 0.04): 5.265 V
h: 0.00125
Num iterations: 5836
Potential at (0.06, 0.04): 5.254 V
h: 0.000625
Num iterations: 16864
Potential at (0.06, 0.04): 5.246 V
Total runtime: 1724.82099986
=== Question 3(e): Non-Uniform Node Spacing ===
Jacobi (for reference)
Quarter grid:
  0.00   1.99   4.06   6.29   8.78  11.66  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.03   4.14   6.41   8.95  11.82  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.99   4.06   6.29   8.78  11.66  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.87   3.81   5.89   8.23  11.04  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.69   3.42   5.24   7.19   9.28  11.33  12.14  12.50  12.66  12.71  12.66
  0.00   1.46   2.95   4.47   6.02   7.55   8.90   9.73  10.20  10.44  10.51  10.44
  0.00   1.22   2.44   3.66   4.87   6.01   6.99   7.69   8.14   8.38   8.45   8.38
  0.00   0.96   1.92   2.87   3.78   4.63   5.35   5.90   6.27   6.48   6.55   6.48
  0.00   0.71   1.42   2.11   2.77   3.37   3.89   4.29   4.57   4.73   4.79   4.73
  0.00   0.47   0.94   1.39   1.81   2.20   2.53   2.80   2.98   3.09   3.13   3.09
  0.00   0.23   0.46   0.69   0.90   1.09   1.25   1.38   1.47   1.53   1.55   1.53
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 106
Potential at (0.06, 0.04): 5.351 V
Uniform Mesh (same as Jacobi)
Quarter grid:
  0.00   1.99   4.06   6.29   8.78  11.66  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.03   4.14   6.41   8.95  11.82  15.00  15.00  15.00  15.00  15.00  15.00
```

```
  0.00   1.99   4.06   6.29   8.78  11.66  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.87   3.81   5.89   8.23  11.04  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.69   3.42   5.24   7.19   9.28  11.33  12.14  12.50  12.66  12.71  12.66
  0.00   1.46   2.95   4.47   6.02   7.55   8.90   9.73  10.20  10.44  10.51  10.44
  0.00   1.22   2.44   3.66   4.87   6.01   6.99   7.69   8.14   8.38   8.45   8.38
  0.00   0.96   1.92   2.87   3.79   4.63   5.35   5.90   6.27   6.48   6.55   6.48
  0.00   0.71   1.42   2.11   2.77   3.37   3.89   4.29   4.57   4.73   4.79   4.73
  0.00   0.47   0.94   1.39   1.81   2.20   2.53   2.80   2.98   3.09   3.13   3.09
  0.00   0.23   0.46   0.69   0.90   1.09   1.25   1.38   1.47   1.53   1.55   1.53
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 209
Potential at (0.06, 0.04): 5.351 V
Jacobi potential: 5.35062156679 V, same as uniform potential: 5.35067998265 V
Non-Uniform (clustered around (0.06, 0.04))
Quarter grid:
  0.00   2.00   4.08   6.33  11.61  13.25  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.04   4.17   6.45  11.80  13.37  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.00   4.08   6.33  11.61  13.25  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.89   3.84   5.93  10.90  12.71  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.71   3.45   5.28   9.27  10.26  11.15  11.74  12.14  12.66  12.71  12.66
  0.00   1.21   2.43   3.66   6.06   6.57   7.03   7.42   7.75   8.38   8.45   8.38
  0.00   1.09   2.18   3.26   5.35   5.78   6.18   6.52   6.81   7.41   7.48   7.41
  0.00   0.96   1.92   2.87   4.66   5.04   5.38   5.67   5.93   6.48   6.55   6.48
  0.00   0.84   1.67   2.48   4.01   4.33   4.62   4.87   5.09   5.59   5.65   5.59
  0.00   0.71   1.42   2.11   3.39   3.65   3.89   4.11   4.29   4.72   4.77   4.72
  0.00   0.23   0.47   0.69   1.10   1.19   1.26   1.33   1.39   1.54   1.56   1.54
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 385
Potential at (0.06, 0.04): 5.378 V
Non-Uniform (more clustered around (0.06, 0.04))
Quarter grid:
  0.00   2.03   4.14   6.41  13.24  14.65  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.07   4.22   6.53  13.40  14.68  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   2.03   4.14   6.41  13.24  14.65  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.92   3.90   6.02  12.55  14.45  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   1.73   3.51   5.36  10.40  11.09  11.24  11.38  11.86  12.65  12.71  12.65
  0.00   1.10   2.19   3.28   5.90   6.21   6.29   6.36   6.62   7.44   7.51   7.44
  0.00   1.00   1.99   2.97   5.28   5.56   5.62   5.69   5.92   6.69   6.75   6.69
  0.00   0.97   1.94   2.89   5.13   5.40   5.46   5.52   5.75   6.50   6.57   6.50
  0.00   0.94   1.88   2.81   4.98   5.24   5.30   5.36   5.58   6.32   6.38   6.32
  0.00   0.84   1.68   2.50   4.39   4.62   4.68   4.73   4.92   5.60   5.66   5.60
  0.00   0.24   0.47   0.70   1.21   1.28   1.29   1.31   1.36   1.56   1.57   1.56
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 1337
Potential at (0.06, 0.04): 5.461 V
Non-Uniform (clustered near outer conductor)
Quarter grid:
  0.00   4.38   7.21  10.30  13.47   7.42   8.97   9.82  10.43  10.80  10.86   7.63
  0.00   4.46   7.34  10.46  13.55  15.00  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   4.38   7.21  10.30  13.47  15.00  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   4.19   6.91   9.94  13.24  15.00  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   3.95   6.50   9.37  12.69  15.00  15.00  15.00  15.00  15.00  15.00  15.00
  0.00   3.61   5.91   8.39  10.87  11.93  12.87  13.10  13.22  13.30  13.33  13.30
  0.00   3.18   5.15   7.16   8.96   9.63  10.73  11.09  11.29  11.43  11.49  11.43
  0.00   2.67   4.27   5.84   7.16   7.66   8.66   9.03   9.27   9.44   9.51   9.44
  0.00   1.89   3.00   4.05   4.91   5.24   5.99   6.29   6.49   6.64   6.71   6.64
  0.00   1.50   2.36   3.17   3.83   4.09   4.69   4.94   5.11   5.23   5.29   5.23
  0.00   0.92   1.44   1.93   2.33   2.49   2.86   3.02   3.13   3.21   3.25   3.21
  0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00   0.00
Num iterations: 222
Potential at (0.06, 0.04): 5.243 V
```